MATH 138 - Numerical Series Tests
Based on *Calculus II for Honours Mathematics* Version 1.51 by *Barbara and Brian Forrest*.
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5.2 Geometric Series Test

$\sum_{n=\mathbf{0}}^{\infty} r^n$ converges to $\dfrac{1}{1-r}$ if $|r| < 1$

$\sum_{n=\mathbf{0}}^{\infty} r^n$ diverges if $|r| \geq 1$

5.3 Divergence Test

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges (e.g. if $\lim_{n\to\infty} a_n$ DNE, then $\sum_{n=1}^{\infty} a_n$ diverges)

5.4 Arithmetic For Series I

Assuming $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then

We have $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ for some $c \in \mathbb{R}$

We have $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

5.4 Arithmetic For Series II

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=\mathbf{j}}^{\infty} a_n$ converges for every $j \in \mathbb{N}$

If $\sum_{n=\mathbf{j}}^{\infty} a_n$ converges for some $j \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ converges

5.5.1 Comparison Test for Series

Assuming $0 \leq a_n \leq b_n$ for every $n \in \mathbb{N}$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

5.5.2 Limit Comparison Test (LCT)

Assuming $a_n > 0$ and $b_n > 0$ for every $n \in \mathbb{N}$

If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$: if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges; if $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges

If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$: if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} b_n$ converges; if $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges

If $\lim_{n\to\infty} \dfrac{a_n}{b_n} \in \mathbb{R}$ then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge

5.6 Integral Test for Convergence

Let $a_k = f(k)$. Assuming $f(k)$ is positive, continuous, and decreasing on $[1, \infty)$

We have that $\displaystyle\int_1^{n+1} f(x)dx \ \leq \ S_n \ \leq \ a_1 + \int_1^n f(x)dx$ for all $n \in \mathbb{N}$

We have that $\sum\limits_{k=1}^{\infty} a_k$ and $\displaystyle\int_1^{\infty} f(x)dx$ either both converge or both diverge

If S converges, then $\displaystyle\int_1^{\infty} f(x)dx \ \leq \ S \ \leq \ a_1 + \int_1^{\infty} f(x)dx$

If S converges, then $\displaystyle\int_{n+1}^{\infty} f(x)dx \ \leq \ S - S_n \ \leq \ \int_n^{\infty} f(x)dx$

5.6 p-series Test

We have $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$

5.7 Alternating Series Test (AST)

Assuming $a_n \geq a_{n+1} > 0$ for every $n \in \mathbb{N}$

If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ converges

We have that $|\, S - S_k \,| \leq a_{k+1}$, the error bound

5.8 Absolute Convergence Theorem

If $\sum\limits_{n=1}^{\infty} |a_n|$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges

5.8 Rearrangement Theorem

Assuming $\sum\limits_{n=1}^{\infty} b_n$ is a rearrangement of $\sum\limits_{n=1}^{\infty} a_n$

If $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely, then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} b_n$

If $\sum\limits_{n=1}^{\infty} a_n$ converges conditionally, then $\sum\limits_{n=1}^{\infty} b_n = \alpha$ for some $\alpha \in \mathbb{R}$ or $\alpha = \pm\infty$

5.9 Ratio Test

If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| < 1$ then $\sum\limits_{n=\mathbf{0}}^{\infty} a_n$ converges absolutely

If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| > 1$ then $\sum\limits_{n=\mathbf{0}}^{\infty} a_n$ diverges

If $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$ then we are inconclusive

5.9 Polynomial vs Factorial Growth

We have that $\lim\limits_{n\to\infty} \dfrac{x_n}{n!} = 0$ for every $x \in \mathbb{R}$

5.10 Root Test

If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$ then $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely

If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ then $\sum\limits_{n=1}^{\infty} a_n$ diverges

If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$ then we are inconclusive