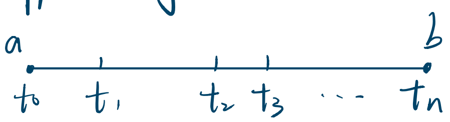
1.1 Areas under curves

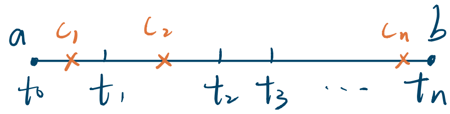
– approximating areas



the increasing sequence $P = \{t_0, t_1, \cdots t_n\}$ is a partition of interval $[a, b]$

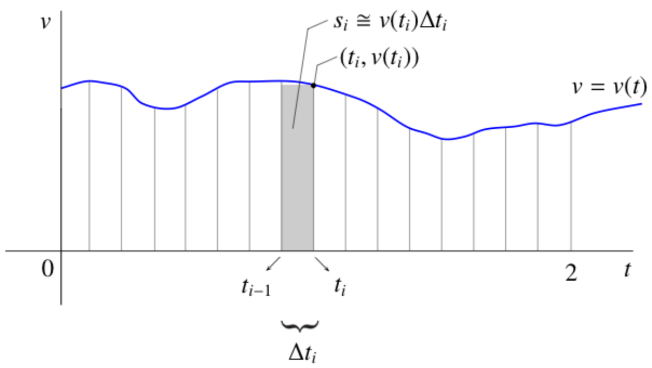
length of ith subinterval is $\Delta t_i = t_i - t_{i-1}$ $i \in 1, 2, \cdots, n$

→ Let $c \in [t_{i-1}, t_i]$



– displacement & velocity

$s = v \Delta t$



$$S_n = \sum_{i=1}^{n} v(t_i) \frac{2}{n}$$

$$\lim_{n \to \infty} \{S_n\} = s$$

1.2 Riemann Sums and Definite Integral

- Partition

Partition P for $[a, b]$ is finite increasing sequence of numbers of the form

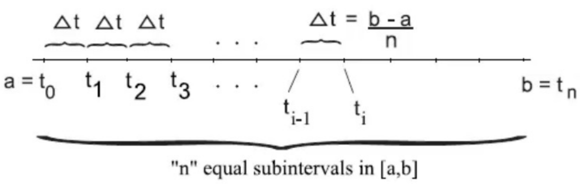
$a = t_0 < t_1 < t_2 < \cdots < t_i < \cdots < t_n = b$

A partition 将 $[a, b]$ 分为 n 份 $[t_0, t_1], [t_1, t_2] \cdots [t_{n-1}, t_n]$

$\Delta t_i = t_i - t_{i-1}$

norm of partition P: $\|P\| = \max\{\Delta t_1, \Delta t_2, \cdots, \Delta t_n\}$

- def. regular n-partition



每个 subinterval 长度相同 $\Delta t_i = \frac{b-a}{n}$

$t_i = a + i \cdot \frac{b-a}{n}$

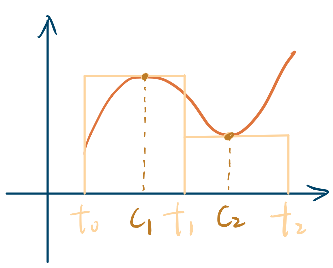
(将 $[a, b]$ 分成相等的 n 份.)

- Riemann Sum

Given a bounded function f and a partition P over the interval $[a, b]$ with $c_i \in [t_{i-1}, t_i]$ a Riemann Sum of f w.r.t P is

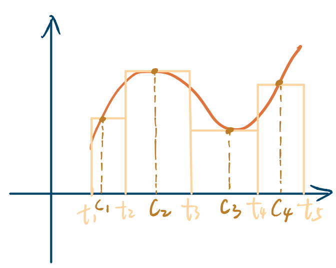
$$S = \sum_{i=1}^{n} f(c_i)\Delta t_i \quad \leftarrow \text{Regular } n\text{-partion } \Delta t_i = \frac{b-a}{n}$$

拆成 2 段



$S = f(c_1)(t_1 - t_0) + f(c_2)(t_2 - t_1)$

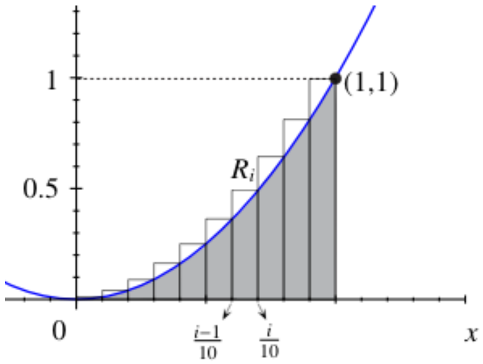
拆成 4 段

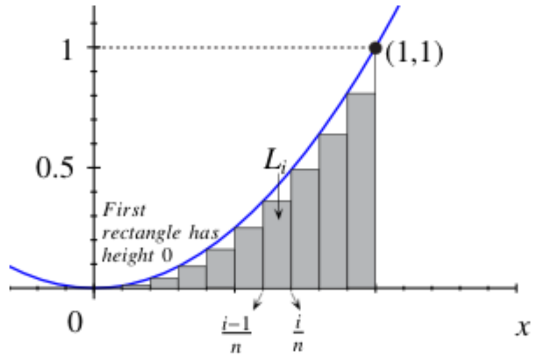


$S = \sum_{i=1}^{4} f(c_i)\Delta t_i$

拆的段数越多, 结果越准确 (Δt_i 足够小时, $\infty \cdot 0$. call I)

R-H 拆成n段 $L_n \le S_n \le R_n$ L-H 拆成n段





$$\int_a^b f(t)\,dt = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\,\frac{b-a}{n}$$

$$\sum_{i=1}^{n} R_i = \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}$$

(increasing function)

right-hand r.s overestimate

$c_i = a + i\Delta t = a + i\cdot\frac{b-a}{n}$

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} L_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_{i-1})\,\frac{b-a}{n}$$

$$\sum_{i=1}^{n} L_i = \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2$$

$$= \frac{1}{n^3}\,\frac{(n-1)(n+1-1)(2(n-1)+1)}{6}$$

$$= \frac{2-\frac{3}{n}+\frac{1}{n^2}}{6}$$

(increasing function)

left-hand r.s underestimate

$c_i = a + (i-1)\Delta t = a + (i-1)\cdot\frac{b-a}{n}$

- def. definite integral 定积分

若∃ unique $I \in \mathbb{R}$.

s.t. 当 $\{P_n\}$ 为 sequence of partitions with $\lim_{n\to\infty}\|P_n\| = 0$.

且 $\{S_n\}$ 为 sequence of Riemann sums associated with P_n's

we have $\underline{\lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta t_i = I}$ $\quad \Delta t_i = \frac{b-a}{n}$

integrand

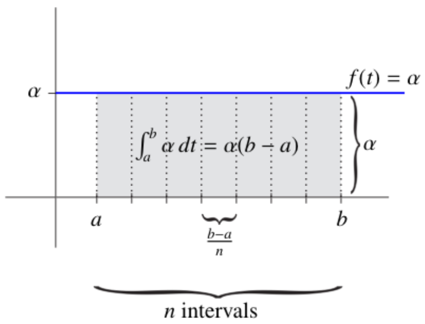
写作 limits of integration $\int_a^b f(t)\,dt$ ← variable of integration

* dummy variable : 可随意更换的字母 (x. y. z ...)

- Integrability Theorem for continuous functions

f is cont on $[a, b]$ $\Rightarrow$ f is integrable on $[a, b]$

$\Rightarrow \int_a^b f(t)\,dt = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta t_i$



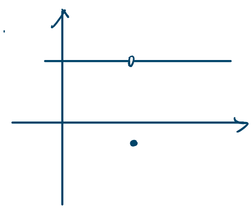
$$R_n = \sum_{i=1}^{n} f(t_i)\Delta t_i = \sum_{i=1}^{n} \alpha \cdot \frac{b-a}{n} = \alpha(b-a)$$

- Integrable condition

cont $\Rightarrow$ integrable.

* If f is bounded with finitely many jump discontinuous then it is also integrable

eg.



Q: Are all bounded functions on $[a, b]$ integrable?

No. counterexample: $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$ f isn't integrable eg. 已知 $f(x)=e^x$ integrable. 求 its integral over $[1,4]$

$$\int_1^4 e^x\,dx = \lim_{x\to\infty}\sum_{i=1}^{n} e^{c_i}\,\underset{\sim}{\Delta t_i} \qquad c\in[t_{i-1},t_i]$$

regular n-pertition : $\Delta t_i = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n} = \Delta t$

c_i 可随意取
- R-H r.s : $c_i = a + i\Delta t = 1+\frac{3i}{n}$
- L-H r.s : $c_i = a+(i-1)\Delta t = 1+\frac{3(i-1)}{n}$

选择 R-H r.s :

$$\begin{aligned}\int_1^4 e^x\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta t\\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(e^{1+\frac{3i}{n}}\right)\left(\frac{3}{n}\right)\\ &= \lim_{n\to\infty}\frac{3e}{n}\sum_{i=1}^{n}\left(e^{\frac{3}{n}}\right)^i \qquad \left(\sum_{i=1}^{n} r^i = \frac{r^{n+1}-r}{r-1}\right)\\ &= \lim_{n\to\infty}\frac{3e}{n}\left(\frac{e^{\frac{3}{n}+3}-e^{\frac{3}{n}}}{e^{\frac{3}{n}}-1}\right)\\ &= \lim_{n\to\infty}\frac{3e\cdot\left(e^{\frac{3}{n}+3}-e^{\frac{3}{n}}\right)}{n\cdot\left(e^{\frac{3}{n}}-1\right)}\\ &= \frac{\lim_{\infty} 3e\cdot\left(e^{\frac{3}{n}+3}-e^{\frac{3}{n}}\right)}{\lim_{\infty} n\cdot\left(e^{\frac{3}{n}}-1\right)} \longrightarrow \text{仅分母用 LHR.}\quad \infty\cdot 0\\ &= \frac{3e\left(e^3-1\right)}{3}\\ &= e\left(e^3-1\right)\end{aligned}$$

Whether $f(x)=e^x$ integrable ?

cont. on $[a,b]$ $\Rightarrow$ integrable on $[a,b]$

1.3 Properties of definite integral

- Properties

Assume that f and g are integrable on the interval $[a, b]$. Then:

i) For any $c \in \mathbb{R}$, $\int_a^b c\, f(t)\, dt = c \int_a^b f(t)\, dt$.

ii) $\int_a^b (f+g)(t)\, dt = \int_a^b f(t)\, dt + \int_a^b g(t)\, dt$.

iii) If $m \le f(t) \le M$ for all $t \in [a, b]$, then $m(b-a) \le \int_a^b f(t)\, dt \le M(b-a)$.

iv) If $0 \le f(t)$ for all $t \in [a, b]$, then $0 \le \int_a^b f(t)\, dt$.

v) If $g(t) \le f(t)$ for all $t \in [a, b]$, then $\int_a^b g(t)\, dt \le \int_a^b f(t)\, dt$.

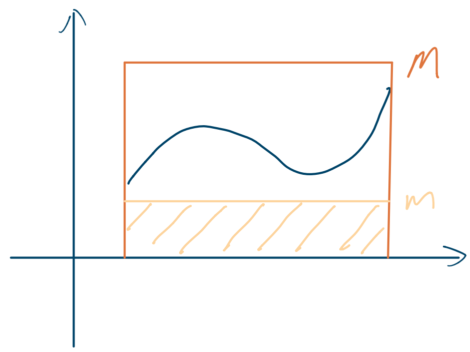
vi) The function $|f|$ is integrable on $[a, b]$ and $\left| \int_a^b f(t)\, dt \right| \le \int_a^b |f(t)|\, dt$.

proof iii) Assume $m \le f(t) \le M \quad \forall t \in [a, b]$

$\because \sum_{i=1}^{n} \Delta t_i = b-a$

$\therefore \sum_{i=1}^{n} m \Delta t_i \le \sum_{i=1}^{n} f(t_i) \Delta t_i \le \sum_{i=1}^{n} M \Delta t_i$

$m(b-a) \le \int_a^b f(t)\, dt \le M(b-a)$

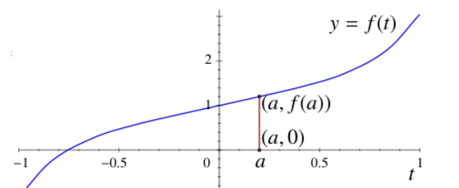


vii) $f(t) \geq 0 \Rightarrow \int_a^b f(t)\, dt \geq 0$

viii) $\int_a^b f(t)\, dt \geq \int_a^b g(t)\, dt$

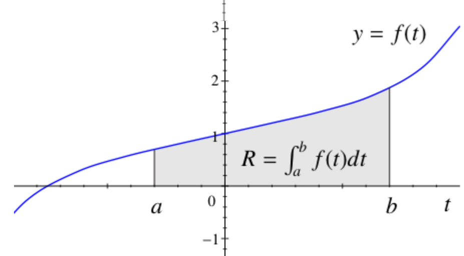
- Identical limits of integration

$$\int_a^a f(t)\, dt = 0$$



- Switching limits of integration

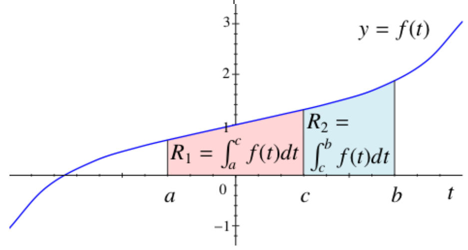
$$\int_b^a f(t)\, dt = -\int_a^b f(t)\, dt$$



- Integrals over subintervals

$$\int_a^b f(t)\, dt = \int_a^c f(t)\, dt + \int_c^b f(t)\, dt \qquad (a < c < b)$$

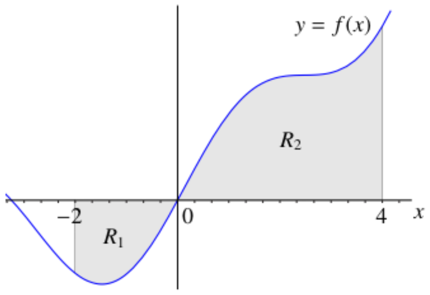
$$= R_1 + R_2$$



- Geometric Interpretation of the Integral

$f \geq 0$. return "expected" area. "signal area"

$f < 0$. return negative area



$$\int_{-2}^{4} f(x)\,dx = \int_{-2}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx$$

$$= R_2 - R_1$$

1.4 The Average Value of a Function

- average value of f.

If f cont. on $[a, b]$.

then f_{av} on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(t)\,dt$

$$f_{av} = \frac{1}{b-a}\int_a^b f(t)\,dt$$

proof: $\lim\limits_{n\to\infty} \frac{\sum_{i=1}^{n} f(t_i)}{n}$ $\quad (t_i = a + \frac{i(b-a)}{n})$

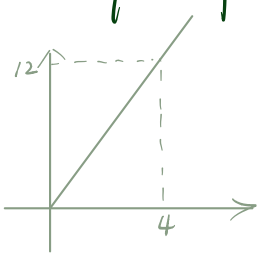
$$= \lim_{\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$

$$= \frac{1}{b-a}\lim_{\infty}\sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$

$$= \frac{1}{b-a}\lim_{\infty} R_n \quad (\text{right-hand r.s})$$

$$= \frac{1}{b-a}\int_a^b f(t)\,dt$$

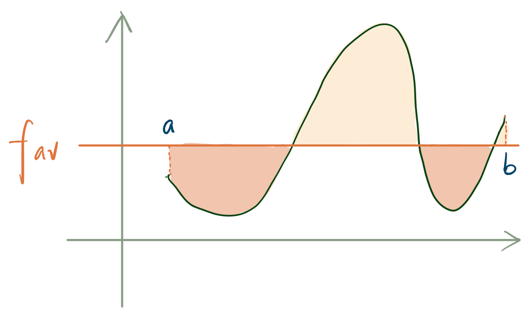
Q. compute f_{av} over $[0,4]$ if $f(x) = 3x$



$$\int_0^4 3x\,dx = \frac{4\times 12}{2} = 24$$

$$f_{av} = \frac{1}{4}\int_0^4 3x\,dx = \frac{24}{4} = 6$$

* f_{av}上方的面积 = f_{av}下方的面积、



proof. → 将x轴向上移到 f_{av}

Let $g(x) = f(x) - f_{av}$

→ Area above $y = f_{av}$ = area below $y = f_{av}$

$$\int_a^b g(x)\,dx = 0$$

$$\to \int_a^b g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b f_{av}\,dx = (b-a)f_{av} - (b-a)f_{av} = 0$$

- average value theorem (AVT)

If f is cont. on $[a, b]$

Then $\exists\, a \le c \le b$. s.t $f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt$ (holds even if $b < a$)

proof. By EVT, $\exists\, p, q \in [a, b]$ s.t $f(p) \le f(x) \le f(q)$

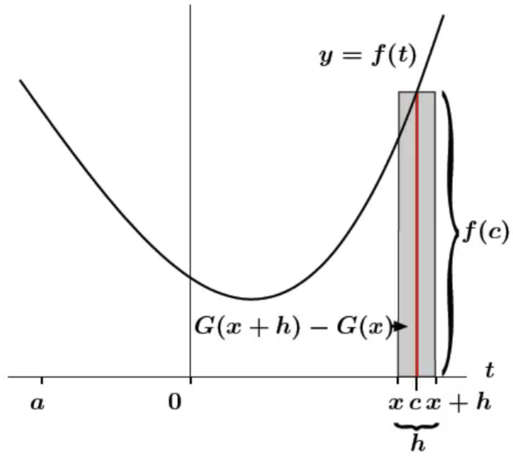
By integral properties $(b-a) f(p) \le \int_a^b f(x)\,dx \le (b-a) f(q)$

$$f(p) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(q)$$

By IVT $\exists\, c \in [a, b]$, $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$f(p) \le f(c) \le f(q)$$

1.5 Fundamental Theorem I



- FTC I

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

observation:

$\exists c \quad x < c < x+h \quad$ (h is small. $h > 0$)

$G(x+h) - G(x) = f(c)\,h.$

$$f(c) = \frac{G(x+h) - G(x)}{h}. \qquad f(c) \text{ 接近于 } f(x)$$

$$\lim_{h\to 0^+}\frac{G(x+h)-G(x)}{h} = \lim_{c\to x^+} f(c) = f(x)$$

$$G'(x) = f(x)$$

proof: $\quad G(x) = \int_a^x f(t)\,dt \qquad$ (f is cont. on $x_0 \in I$)

Let $\varepsilon > 0$.

$\exists \delta > 0$ s.t. $0 < |c - x_0| < \delta \Rightarrow |f(c) - f(x_0)| < \varepsilon$

$$\Rightarrow \frac{G(x) - G(x_0)}{x - x_0} = \frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} = \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt$$

$\because$ AVT.

$\therefore \exists c \in [x, x_0]. \quad f(c) = \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt$

若 $0 < |x - x_0| < \delta.$

$\because 0 < |c - x_0| < \delta$

$$\therefore \left|\frac{G(x) - G(x_0)}{x - x_0} - f(x_0)\right| = |f(c) - f(x_0)| < \varepsilon$$

By def. $\quad G'(x_0) = \lim_{x\to x_0}\frac{G(x) - G(x_0)}{x - x_0} = f(x_0)$

Q: Find $G'(x)$ if $G(x) = \int_3^{x^2} e^{t^2}\,dt$

$$G'(x) = \frac{d}{dx}\int_3^{x^2} e^{t^2}\,dt$$
$$= \frac{d}{dx}\int_3^{u} e^{t^2}\,dt$$
$$= \frac{d}{du}\left(\int_3^{u} e^{t^2}\,dt\right)\cdot\frac{du}{dx} \qquad \text{(by Chain Rule)}$$
$$= e^{u^2}\frac{du}{dx} \qquad \text{(by FTC I)}$$
$$= e^{x^4}\cdot 2x$$

- extended of FTC I

If f is cont. $g \cdot h$ are diff'ble $\quad H(x) = \int_{a(x)}^{b(x)} f(t)\,dt.$

Then $H(x)$ is diff'ble

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$$

Q: Find $H'(x)$ if $H(x) = \int_{\cos x}^{x^2} e^{t^2}\,dt$

$$H'(x) = e^{(x^2)^2}\cdot 2x - e^{\cos^2 x}\cdot(-\sin x)$$
$$= 2x e^{x^4} + \sin x\, e^{\cos^2 x}$$

Q: Compute $\frac{d}{dx}\int_2^{\ln x} \sin t^2\,dt$

Let $u = \ln x \qquad g(u) = \int_2^u \sin t^2\,dt \qquad$ 求 $\frac{d}{dx}g(u)$

By Chain Rule $\frac{d}{dx}[g(u)] = \frac{d}{du}[g(u)]\cdot\frac{du}{dx}$.

$\frac{du}{dx} = \frac{1}{x}$. $\qquad$ By FTC I $\quad \frac{d}{du}[g(u)] = \frac{d}{du}\int_2^u \sin(t^2)\,dt = \sin(u^2)$

$$\frac{d}{dx}[g(u)] = \sin(u^2)\left(\frac{1}{x}\right)$$

$$\frac{d}{dx}\int_2^{\ln x} \sin(t^2)\,dt = \frac{\sin[(\ln x)^2]}{x}$$

1.6 Fundamental Theorem II

- def. antiderivative

$F'(x) = f(x)$ F is antiderivative for f on I.

eg. $F = \frac{x^3}{3}$ & $F = \frac{x^3}{3} + 2$ are antiderivative of $f(x)$

- Indefinite integral

$\int f(x)\,dx$. Due to FTC. $\int f(x)dx$ is antiderivative of $f(x)$

- Constant function theorem

若 $f'(x) = 0 \quad \forall x \in I$. 则 $\exists \alpha \in \mathbb{R}$. s.t $f(x) = \alpha \quad \forall x \in I$

proof: Let $x_1 \in I$. $f(x_1) = \alpha$.

choose $x_2 \in I$.

By MVT, $\exists c \in (x_1, x_2)$. $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

$\because f'(c) = 0 \quad \therefore \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$.

$\therefore f(x_2) = f(x_1) = \alpha$

- The Antiderivative Theorem

为何 $\int = \cdots + C$

If $f'(x) = g'(x)$ for all $x \in I$, then there exists α s.t. $f(x) = g(x) + \alpha \quad \forall x \in I$

proof. Let $H(x) = f(x) - g(x)$

Then $H'(x) = f'(x) - g'(x) = 0 \quad x \in I$.

$\therefore \exists \alpha \in \mathbb{R}$ s.t $H(x) = \alpha \Rightarrow f(x) = g(x) + \alpha \quad \forall x \in I$

- Power rule for antiderivatives

$\int x^\alpha\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C \qquad (\alpha \neq -1)$

- FTC2

f is cont., F is any antiderivative of f.

$\int_a^b f(t)\,dt = F(b) - F(a)$

Q. Compute $\int_1^4 \cos x\,dx$

Using Riemann Sum would require a formula for $\sum_{i=1}^{n} \cos\left(1+\frac{3i}{n}\right)$

and then a limit as $n \to \infty$

Using FTC. $\because \frac{d}{dx}(\sin x) = \cos x$ $\quad \therefore \int_1^4 \cos x\,dx = \sin 4 - \sin 1$.

The expression $g(x)\Big|_a^b = g(b) - g(a)$

Q. $\int_{-1}^{1} |x|\,dx$

$= \int_{-1}^{0} -x\,dx + \int_0^1 x\,dx$

$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_0^1$

$= 0 - \left(-\frac{(-1)^2}{2}\right) + \frac{1}{2} - 0$

$= 1$

- 定积分与不定积分

$\int_a^b f(t)\,dt$	$\int f(t)\,dt$
definite integral	indefinite integral
得到数字	得到 function

Q. Find $\int 2x + \sin x - 1\,dx$

$= \frac{2x^2}{2} - \cos x - x + C$

$= x^2 - \cos x - x + C$

$$\int x^n = \frac{x^{n+1}}{n+1} + c$$

$$\int e^x = e^x + c$$

$$\int \frac{1}{x} = \ln|x| + c$$

$$\int \sin x = -\cos x + c$$

$$\int \cos x = \sin x + c$$

$$\int \tan x = \ln|\sec x| + c$$

$$\int \csc x = -\ln|\csc x + \cot x| + c$$

$$\int \sec x = \ln|\sec x + \tan x| + c$$

$$\int \cot x = \ln|\sin x| + c$$

1.7 Change of Variables

- Change of variables Formula

$$\boxed{u = g(x) \qquad \int f(g(x))\, g'(x)\, dx = \int f(u)\, du}$$

proof:

$\rightarrow \because h'(u) = f(u)$

$\therefore \int f(u)\, du = h(u) + C$

$\rightarrow$ Let $u = g(x)$. $\quad H(x) = h(g(x))$

$H'(x) = h'(g(x))\, g'(x) = f(g(x))\, g'(x)$

$\rightarrow \int f(g(x))\, g'(x)\, dx = H(x) + C = h(g(x)) + C = h(u) + C = \int f(u)\, du$

Q. 求 $\int \sin^3 x \cos x\, dx$

$= \int u^3 \frac{du}{dx}\, dx \qquad (u = \sin x)$

$\frac{d}{dx}[ux^4] = 4u^3 \frac{du}{dx} \qquad u^3 \frac{du}{dx} = \frac{d}{dx}\left[\frac{ux^4}{4}\right]$

$\int u^3 \frac{du}{dx}\, dx = \int \frac{d}{dx}\left[\frac{ux^4}{4}\right] dx = \frac{ux^4}{4} + c \qquad$ (by FTC)

answer $= \frac{\sin x \cdot x^4}{4} + c$

$\int u^3 \cdot \frac{du}{\cancel{dx}}\, \cancel{dx} = \int u^3\, du$

$\int \sin^3 x \cos x\, dx$

let $u = \sin x \qquad \frac{du}{dx} = \cos x$.

$\int u^3 \cos x \frac{du}{\cos x} = \int u^3\, du = \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c$

Q: 求 $\int 2x\, e^{x^2}\, dx = \int f(g(x))\, g'(x)\, dx$

Let $u = g(x) = x^2$ $\quad g'(x) = 2x.$ $\quad f(u) = e^u$

$$\int 2x e^{x^2} dx = \int f(g(x))\, g'(x)\, dx$$
$$= \int f(u)\, du$$
$$= \int e^u\, du = e^u + C$$
$$= e^{x^2} + C$$

Q. 求 $\int x\cos(x^2)\, dx$

$\int f(g(x))\, g'(x)\, dx$

$\frac{1}{2}\int \cos x^2 \cdot 2x\ dx$

$u = x^2 \quad u' = 2x$

$$\int \frac{1}{2}\cos u\ du$$
$$= \frac{1}{2}\int \cos u\, du = \frac{1}{2}\sin u + C$$
$$= \frac{1}{2}\sin(x^2) + C$$

Q. $\int 5x e^{x^2} dx$

$u = x^2 \quad \frac{du}{dx} = 2x \quad dx = \frac{du}{2x}$

$$\int 5x e^u \frac{du}{2x} = \int \frac{5}{2} e^u\, du = \frac{5}{2}e^u + C$$
$$= \frac{5}{2}e^{x^2} + C$$

Q. 求 $\int \sec\theta\, d\theta$

$$\sec\theta = \sec\theta\left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right) = \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}$$

Let $u = \sec\theta + \tan\theta.$ $\quad \frac{du}{d\theta} = \sec\theta\tan\theta + \sec^2\theta$

$$\int \sec\theta\, d\theta = \int \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}\, d\theta$$
$$= \int \frac{du}{u}$$
$$= \ln|u| + C$$
$$= \ln|\sec\theta + \tan\theta| + C$$

* 有时取的 u，无法得出结果

u 取复杂的

eg. $\int x^3 \sqrt{x^2-4}\ dx$

let $u = x^3$

$\int x^3\sqrt{x^2-4}\ dx = \int u\sqrt{x^2-4}\ dx = \int u\sqrt{x^2-4}\ \frac{du}{3x^2} = \frac{1}{3}\int \frac{u}{x^2}\sqrt{x^2-4}\ du$

$(u^{\frac{2}{3}} = x^2)$ $= \frac{1}{3}\int u^{\frac{1}{3}}\sqrt{u^{\frac{2}{3}}-4}\ du$. ...... ×

there will be a lot trial & error

$u = x^2-4$. $\frac{du}{dx} = 2x$. $dx = \frac{du}{2x}$.

$\int x^3\sqrt{u}\ \frac{du}{2x} = \frac{1}{2}\int x^2\sqrt{u}\ du. = \frac{1}{2}\int (u+4)\sqrt{u}\ du. = \frac{1}{2}\int u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\ du.$

$= \frac{1}{2}\left[\frac{2u^{\frac{5}{2}}}{5} - \frac{2\times 4u^{\frac{3}{2}}}{3}\right] + C = \frac{u^{\frac{5}{2}}}{5} - \frac{4u^{\frac{3}{2}}}{3} + C$

$= \frac{(x^2-4)^{\frac{5}{2}}}{5} - \frac{4(x^2-4)^{\frac{3}{2}}}{3} + C$

- Change of variables for the indefinite integral

$g'(x)$ cont on $[a,b]$. $f(u)$ is cont on $g([a,b])$.

$\int_a^b f(g(x))\, g'(x)\ dx = \int_{u=g(a)}^{u=g(b)} f(u)\ du$

Q. 求 $\int_2^4 (5x-6)^3\ dx$

Let $u = g(x) = 5x-6$ $f(u) = u^3$

$\frac{du}{dx} = 5$. $dx = \frac{1}{5}du$

$\int_2^4 (5x-6)^3 dx = \int_{u=g(a)}^{u=g(b)} f(u)\ dx$

$g(b) = 14$
$g(a) = 4$

$= \int_{u=g(2)}^{u=g(4)} u^3 \cdot \frac{1}{5}du$

$= \frac{1}{5}\int_4^{14} u^3\ du$

$= \frac{1}{5}\left[\frac{u^4}{4}\right]_4^{14}$

$= \frac{1}{20}(14^4 - 4^4)$

$= 1908$

Q. $\int_0^1 \frac{x\,dx}{\sqrt{x^2+1}}$

Let $u = g(x) = x^2+1$

$\frac{du}{dx} = 2x. \qquad \frac{1}{2}du = x\,dx.$

$$\int_0^1 \frac{x\,dx}{\sqrt{x^2+1}} = \int_{u=g(0)}^{u=g(1)} \frac{\frac{1}{2}du}{u^{\frac{1}{2}}} \qquad \begin{array}{l} g(1)=2 \\ g(0)=1 \end{array}$$

$$= \frac{1}{2}\int_1^2 u^{-\frac{1}{2}}du$$

$$= \frac{1}{2} \times \left[2u^{\frac{1}{2}}\right]_1^2$$

$$= 2^{\frac{1}{2}} - 1$$

$$= \sqrt{2} - 1$$

Q. $\int e^{5x}dx$

$u = 5x \qquad \frac{du}{dx} = 5 \qquad dx = \frac{du}{5}$

$$\int e^{5x}dx = \int e^u \frac{du}{5} = \frac{1}{5}\int e^u\,du = \frac{e^u}{5} + c = \frac{e^{5x}}{5} + c$$

Q. $\int_1^e \frac{1}{x(2+\ln x)}\,dx$

$u = 2+\ln x \qquad du = \frac{1}{x}dx \qquad x\,du = dx$

$$\int \frac{1}{xu}\,x\,du = \int \frac{1}{u}du = \ln|u| + c = \ln|2+\ln x| + c$$

$$\left[\ln|2+\ln x|\right]_1^e = \ln\frac{3}{2}$$

Q. $\int \frac{1}{\sqrt{1+\sqrt{1+x}}}\,dx$

$u = \sqrt{1+x} \qquad \frac{du}{dx} = \frac{1}{2\sqrt{1+x}} = \frac{1}{2u} \qquad 2u\,du = dx$

$$\int \frac{2u}{\sqrt{1+u}}\,du = \int \frac{2(s-1)}{\sqrt{s}}\,ds = 2\int \sqrt{s} - \frac{1}{\sqrt{s}}\,ds$$

$$= 2\left[\frac{2s^{\frac{3}{2}}}{3} - 2s^{\frac{1}{2}}\right] + c = \frac{4}{3}s^{\frac{3}{2}} - 4s^{\frac{1}{2}} + c$$

$$= \frac{4}{3}(1+u)^{\frac{3}{2}} - 4(1+u)^{\frac{1}{2}} + c$$

$$= \frac{4}{3}\left(1+\sqrt{1+x}\right)^{\frac{3}{2}} - 4\left(1+\sqrt{1+x}\right)^{\frac{1}{2}} + c$$

2.1 Inverse Trigonometric Substitutions

$\sin^2 x + \cos^2 x = 1$ $\qquad 1 + \cot^2 x = \csc^2 x$

$\tan^2 x + 1 = \sec^2 x$

$\cos^2 x = \frac{\cos 2x + 1}{2}$ $\qquad \sin^2 x = \frac{1 - \cos 2x}{2}$

Class of Integrand	Integral	Trig Substitution	Trig Identity
$\sqrt{a^2 - b^2x^2}$	$\int \sqrt{a^2 - b^2x^2}\,dx$	$bx = a\sin(u)$	$\sin^2(x) + \cos^2(x) = 1$
$\sqrt{a^2 + b^2x^2}$	$\int \sqrt{a^2 + b^2x^2}\,dx$	$bx = a\tan(u)$	$\sec^2(x) - 1 = \tan^2(x)$
$\sqrt{b^2x^2 - a^2}$	$\int \sqrt{b^2x^2 - a^2}\,dx$	$bx = a\sec(u)$	$\sec^2(x) - 1 = \tan^2(x)$

$-\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$-\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$0 \le u \le \frac{\pi}{2}$ or $\pi \le u \le \frac{3\pi}{2}$

Q. $\int \frac{1}{1-x^2}\,dx$

let $x = \sin\theta$. $\qquad \frac{dx}{d\theta} = \cos\theta$

$\int \frac{1}{1-\sin^2\theta} \cdot \cos\theta\, d\theta$

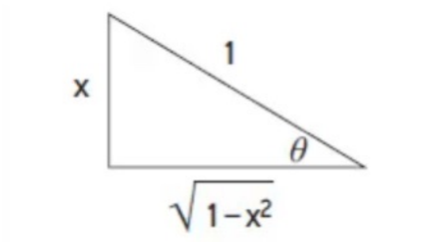
$= \int \frac{\cos\theta}{\cos^2\theta}\, d\theta$

$= \int \frac{1}{\cos\theta}\, d\theta$

$= \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{1+x}{\sqrt{1-x^2}}\right| + C$



$\cos\theta = \sqrt{1-x^2}$

$\sec\theta = \frac{1}{\sqrt{1-x^2}}$

$\tan\theta = \frac{1}{\sqrt{1-x^2}}$

Q. $\int \frac{1}{(x^2+1)^2}\,dx$

let $x = \tan\theta$ $\qquad \frac{dx}{d\theta} = \sec^2\theta$

$\int \frac{1}{(\tan\theta^2+1)^2} \sec^2\theta\, d\theta$

$= \int \frac{\sec^2\theta}{(\sec^2\theta)^2}\, d\theta$

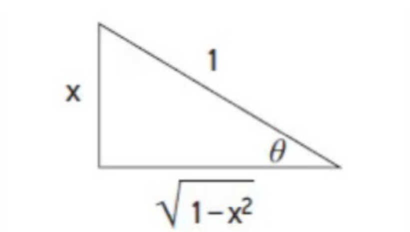
$= \int \cos^2\theta\, d\theta$

$= \frac{1}{2} \cdot \frac{\sin 2\theta}{2} + \frac{\theta}{2} + C$

$= \frac{1}{4}\sin 2\theta + \frac{\theta}{2} + C$

$= \frac{1}{2}(\sin\theta\cos\theta) + \frac{\theta}{2} + C$

$= \frac{1}{2}\left(\frac{x}{1+x^2} + \arctan x\right) + C$



$\cos\theta = \frac{1}{\sqrt{1+x^2}}$

$\sin\theta = \frac{1}{\sqrt{1+x^2}}$

Q. $\int \frac{1}{\sqrt{x^2+4}}\,dx$

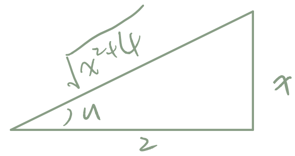
$x = 2\tan u \qquad \frac{dx}{du} = 2\sec^2 u$

$\int \frac{2\sec^2 u}{\sqrt{4\tan u^2 + 4}}\,du$

$= \int \frac{\sec^2 u}{\sqrt{\sec^2 u}}\,du$

$= \int \sec u\,du$

$= \ln|\sec u + \tan u| + c$



$\sec u = \frac{\sqrt{x^2+4}}{2}$

$= \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + c$

Q. $\int \frac{\sqrt{9-4x^2}}{x^2}\,dx$

$= 2\int \frac{\sqrt{\frac{9}{4} - x^2}}{x^2}\,dx$

$x = \frac{3}{2}\sin\theta \qquad \frac{dx}{d\theta} = \frac{3}{2}\cos\theta\,d\theta$

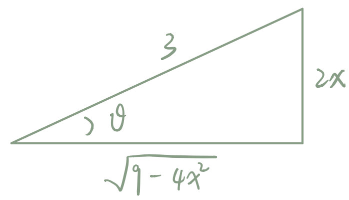
$2\int \frac{\sqrt{\frac{9}{4} - \frac{9}{4}\sin^2\theta}}{\frac{9}{4}\sin^2\theta} \cdot \frac{3}{2}\cos\theta\,d\theta$

$= 2 \times \frac{4}{9} \times \frac{3}{2} \times \frac{3}{2} \int \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta} \cos\theta\,d\theta$

$= 2\int \cot^2\theta\,d\theta$

$= 2\int \csc^2\theta - 1\,d\theta$

$= 2(-\cot\theta - \theta) + c$



$\cos\theta = \frac{\sqrt{9-4x^2}}{2x}$

$\theta = \arccos\frac{2x}{3}$

Q. $\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$

$x = 2\sec\theta \qquad \frac{dx}{d\theta} = 2\sec\theta\tan\theta$

$= \int \frac{2\sec\theta\tan\theta}{4\sec^2\theta\sqrt{4\sec^2\theta - 4}}\,d\theta$

$= \frac{1}{4}\int \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\sec^2\theta - 1}}\,d\theta$

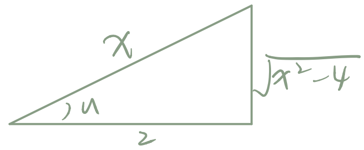
$= \frac{1}{4}\int \frac{1}{\sec\theta}\,d\theta$

$= \frac{1}{4}\int \cos\theta\,d\theta$

$= \frac{1}{4}\sin\theta + c$

$x = \sec\theta$

$\sec\theta = \frac{x}{2} \qquad \cos\theta = \frac{2}{x}$



$\sin\theta = \frac{\sqrt{x^2-4}}{x}$

$= \frac{1}{4} \cdot \frac{\sqrt{x^2-4}}{x} + c$

9. $\int \frac{x}{(3-2x-x^2)^{\frac{3}{2}}}\,dx$

$$= \int \frac{x}{(4-(x+1)^2)^{\frac{3}{2}}}\,dx$$

$x+1 = 2\sin\theta \qquad x = 2\sin\theta - 1 \qquad \frac{dx}{d\theta} = 2\cos\theta$

$$\int \frac{2\sin\theta - 1}{(4-4\sin^2\theta)^{\frac{3}{2}}}\,dx$$

$$= \int \frac{2\sin\theta - 1}{8(1-\sin^2\theta)^{\frac{3}{2}}}\,\frac{dx}{d\theta}\,d\theta$$

$$= \int \frac{2\sin\theta - 1}{8\cos^3\theta} \times 2\cos\theta\,d\theta$$

$$= \int \frac{2\sin\theta - 1}{4\cos^2\theta}\,d\theta$$

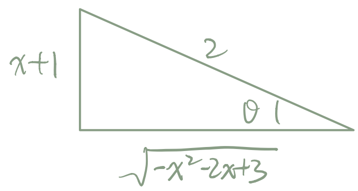
$$= \int \frac{\sin\theta}{2\cos^2\theta} - \frac{1}{4\cos^2\theta}\,d\theta$$

$$= \frac{1}{2}\int \tan\theta\sec\theta\,d\theta - \frac{1}{4}\int \sec^2\theta\,d\theta$$

$$= \frac{1}{2}\sec\theta - \frac{1}{4}\tan\theta + C$$

$$\sin\theta = \frac{x+1}{2}$$

$$\sqrt{4-(x+1)^2} = \sqrt{4-x^2-1-2x} = \sqrt{-x^2-2x+3}$$



$$\cos\theta = \frac{\sqrt{-x^2-2x+3}}{2}$$

$$\tan\theta = \frac{x+1}{\sqrt{-x^2-2x+3}}$$

$$= \frac{1}{\sqrt{-x^2-2x+3}} - \frac{x+1}{4\sqrt{-x^2-2x+3}} + C$$

$$= \frac{4}{4\sqrt{-x^2-2x+3}} - \frac{x+1}{4\sqrt{-x^2-2x+3}} + C$$

$$= \frac{3-x}{4\sqrt{-x^2-2x+3}} + C$$

2.2 Integration by Parts 分部积分

— 分部积分 简单 ino. 碳 e^x 交 $\ln x$.

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$$

proof.

$$\frac{d}{dx}(uv) = \frac{du}{dx} v + u\frac{dv}{dx}$$

$$\int \frac{d}{dx}(uv)\,dx = \int \frac{du}{dx}\cdot v\, dx + \int u \frac{dv}{dx}\, dx$$

$$uv = \int v\, du + \int u\, dv.$$

$$\int u\, dv = uv - \int v\, du.$$

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$$

Q. $\int x^2 \sin x\, dx$.

$u = x^2$ $\quad u' = 2x$

$v = -\cos x$ $\quad v' = \sin x$

$$-x^2 \cos x - \int -2x \cos x \cdot dx$$

$$= -x^2 \cos x + 2\int x \cos x . dx$$

$u = x$ $\quad u' = 1$

$v = \sin x$ $\quad v = \cos x$

$$-x^2 \cos x + 2\left(x \sin x - \int \sin x\, dx\right)$$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + C$$

Q. $\int \ln x\, dx$

$u = \ln x$ $\quad u' = \frac{1}{x}$

$v = x$ $\quad v' = 1$

$$x \ln x - \int x \cdot \frac{1}{x}\, dx$$

$$= x \ln x - x + C$$

2.3 Partial Fraction

$$f(x) = \frac{p(x)}{q(x)}$$

- Proper partial fraction

$\deg(p(x)) < \deg(q(x))$

A proper rational function can be decomposed into "simple" rational functions with linear or irreducible quadratics in denominator

分母化成最简的式

Step 1: factor denominator. $x^4 - 1 = (x^2+1)(x+1)(x-1)$

Step 2: linear factors $\rightarrow \frac{A}{ax+b}$. eg. $\frac{A}{x-1} + \frac{B}{x+1}$

irreducible quadratic constant $\rightarrow \frac{Cx+D}{ax^2+bx+c}$ eg. $\frac{Cx+D}{x^2+1}$

Step 3: If any factor is repeated n times.

construct n terms by increasing exponents.

Q. $$\frac{8x+4}{x^4-1} = \frac{8x+4}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

$$8x+4 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$$

$x=-1$	$-4 = -4B$	$B=1$
$x=1$	$12 = 4A$	$A=3$
$x=0$	$4 = A-B-D$	$D=-2$
$x=2$	$20 = 45+5+6C-6$	$C=-4$

$$\int \frac{3}{x-1} + \frac{1}{x+1} - \frac{4x+2}{x^2+1}\,dx.$$

$$= 3\ln|x-1| + \ln|x+1| - \int \frac{4x}{x^2+1} + \frac{2}{x^2+1}\,dx$$

$$= 3\ln|x-1| + \ln|x+1| - 2\ln|x^2+1| - 2\int \frac{1}{x^2+1}\,dx$$

$$= 3\ln|x-1| + \ln|x+1| - 2\ln|x^2+1| - 2\arctan x + C$$

Q. $\int \frac{1}{x^2-1}\,dx$

distinct linear factor

one constant per factor

repeated linear factors

one constant per power

distinct irreducible quadratics

linear term per quadratic

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

$$1 = A(x-1)+B(x+1)$$

$x=1 \quad 1=2B \quad B=\frac{1}{2}$

$x=-1 \quad 1=-2A \quad A=-\frac{1}{2}$

$$\int \frac{1}{x^2-1}\,dx = -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

Q. $\int \frac{1}{x^2(x+1)}\,dx$

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

$x=0 \quad 1=B$

$x=-1 \quad 1=C$

$x=1 \quad 1=2A+2B+C \quad A=-1$

$$\int \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\,dx$$

$$= -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

☆ $\frac{\cdots}{x(x-4)} = \square + \frac{A}{x} + \frac{B}{x-4}$

$\uparrow$ 前面可能有数！

记得长除!!!

Q. $\int \frac{1}{x^3+x}\,dx$

$$\frac{1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1)+(Bx+C)x$$

$$1 = (A+B)x^2 + Cx + A$$

$x=0 \quad A=1$

$A+B=0 \quad B=-1 \quad C=0$

$$\int \frac{1}{x} - \frac{x}{x^2+1}\,dx$$

$u=x^2+1 \quad \frac{du}{dx}=2x$

$$\int \frac{x}{x^2+1}\,dx = \int \frac{x}{u}\cdot\frac{du}{2x} = \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln(x^2+1) +$$

$$\int \frac{1}{x^3+x}\,dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

- Improper partial fraction

deg (分子) > deg (分母)

① 在前面加一个多项式 (最高次数为：deg (分子) − deg (分母))

② 长除

2.4 Introduction to Improper Integrals.

Improper Integrals：上下限中某一为 $\pm\infty$ $\quad \lim_{b\to\infty}\int_a^b f(x)\,dx \quad \lim_{a\to-\infty}\int_a^b f(x)\,dx$

- def. Type I improper integrals

f be integrable on $[a, b]$, for each $a<b$.

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx$$

$\lim_\infty$ exists $\Rightarrow$ function converge

$\lim_\infty$ doesn't exists $\Rightarrow$ function diverge

不代表 f 逼近 ∞ 与 $-\infty$ 的速度一样

$$\int_{-\infty}^\infty \sin(x)\,dx \neq \lim_{t\to\infty}\int_{-t}^t \sin x\,dx$$
$$= \lim_{t\to\infty} -\cos t + \cos(-t)\,dt = 0$$

ep.

$$\int_1^\infty \frac{1}{x^2}\,dx$$
$$= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx$$
$$= \lim_{b\to\infty}\left[-x^{-1}\right]_1^b$$
$$= \lim_{b\to\infty} 1-\frac{1}{b}$$
$$= 1$$
$$\therefore \int_1^\infty \frac{1}{x^2}\,dx = 1$$
$\therefore$ converge

$$\int_1^\infty \frac{1}{x}\,dx$$
$$= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx$$
$$= \lim_{b\to\infty}\left[\ln x\right]_1^b$$
$$= \lim_{b\to\infty}(\ln b - \ln 1)$$
$$= \lim_{b\to\infty}\ln b$$
$$= \infty$$
$\therefore$ diverge

- def. p-Test for Type I improper integrals.

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ converge to } \frac{1}{p-1} \iff p>1$$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$
$$= \lim_{t\to\infty}\frac{t^{-p+1}}{-p+1} - \frac{1}{1-p}$$
$$= \begin{cases} \frac{1}{p-1} & p>1 \\ \infty & p<1 \end{cases}$$

When $p=1$ $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\ln(t)-\ln(1) = \infty.$

Q. $\int_0^\infty e^{-x}\,dx.$

$$= \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$$
$$= \lim_{b\to\infty}(-e^{-b}+e^0)$$
$$= 1$$

- Properties of type I improper integrals.

$\int_a^\infty f(x)\,dx$ & $\int_a^\infty g(x)\,dx$ both converge.

1. $\int_a^\infty cf(x)\,dx = c\int_a^\infty f(x)\,dx$ converge

2. $\int_a^\infty f(x)+g(x)\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$ converge

3. $f(x) \le g(x)\ \forall\, a \le x. \Rightarrow \int_a^\infty f(x)\,dx \le \int_a^\infty g(x)\,dx$

4. $a < c < \infty$. $\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx$ converge

* 首先将 improper integrals 写成 $\lim\limits_{\infty}$ 的形式

* 不要用换元/分部积分

- Comparison test

(适用于判断很难积分的 f . converge / diverge)

f & g cts on $[a,\infty)$. $0 \le f(x) \le g(x)$

$\int_a^\infty f(x)\,dx$ diverges $\Rightarrow$ $\int_a^\infty g(x)\,dx$ diverge　　证 g diverge . 找 f < g diverge

$\int_a^\infty g(x)\,dx$ converge $\Rightarrow$ $\int_a^\infty f(x)\,dx$ converge　　证 f converge . 找 g > f converge

Q. Does $\int_1^\infty \frac{x^{\frac{2}{3}}+1}{x^3+1}\,dx$. converge?

$$0 \le \frac{x^{\frac{2}{3}}-1}{x^3+1} \le \frac{x^{\frac{2}{3}}}{x^3+1} \le \frac{x^{\frac{2}{3}}}{x^3} = \frac{1}{x^{\frac{7}{3}}}$$

$\because \int_1^\infty \frac{1}{x^{\frac{7}{3}}}\,dx$ converges. $\therefore$ By p-test. $\int_1^\infty \frac{x^{\frac{2}{3}}+1}{x^3+1}\,dx$ converge

Q. Does $\int_3^\infty \frac{1}{x\ln x}\,dx$ converge?

$x\ln(x) \geq x.$　　$\frac{1}{x\ln x} \le \frac{1}{x}$

$$\int_3^\infty \frac{1}{x\ln x}\,dx \le \underset{\substack{\uparrow \\ \text{diverge}}}{\int_3^\infty \frac{1}{x}\,dx}$$

$\therefore$ tells nothing

$$\boxed{\begin{array}{c}\text{较大的 converge} \\ \text{或} \\ \text{较小的 diverge}\end{array}}$$

- Monotone Convergence Theorem for functions

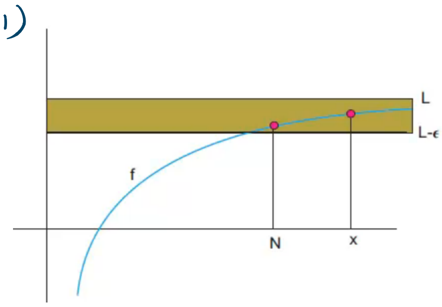
f is non-decreasing on $[a, \infty)$

$S = \{ f(x) : x \in [a, \infty) \}$

1) If S is bounded above, then $\lim_{x\to\infty} f(x) = L = lub(S)$

2) If S isn't bounded above, then $\lim_{x\to\infty} f(x) = \infty$

proof. 1)



Assume S is bounded.

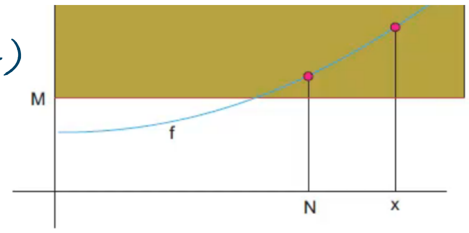
Let $L = lub(S)$ $\quad \varepsilon > 0$.

$L - \varepsilon < f(N) \le L$.

$\because f$ is non-decreasing.

$\therefore x \geq N \Rightarrow L - \varepsilon < f(N) \le f(x) \le L$

2)



Assume S isn't bounded above

Let $M > 0$.

$\because M$ isn't an upper bound for S.

$\therefore \exists N \in [a, \infty)$ $f(N) > M$.

$\because f$ is non-decreasing.

$\therefore x \geq N \Rightarrow f(x) \geq f(N) > M$

- Absolute Convergence Theorem (ACT)

$\int_0^\infty |f(x)|\,dx$ converges $\Rightarrow$ $\int_a^\infty f(x)\,dx$ converges

proof. $0 \le f(x) + |f(x)| \le 2|f(x)|$

$\because \int_0^\infty |f(x)|\,dx$ converge $\therefore \int_a^\infty 2|f(x)|\,dx$ converge

$\therefore \int_0^\infty f(x) + |f(x)|\,dx$ converge

$$\underbrace{\int_a^\infty f(x)\,dx}_{\text{so converge.}} = \underbrace{\int_a^\infty f(x) + |f(x)|\,dx}_{\text{converge}} - \underbrace{\int_a^\infty |f(x)|\,dx}_{\text{converge}}$$

Q: Does $\int_1^\infty \frac{\sin x}{x^3+\sqrt{x}}\,dx$ converge ?

$$\left|\frac{\sin x}{x^3+\sqrt{x}}\right| \le \frac{1}{x^3+\sqrt{x}} \le \frac{1}{x^3}$$

$\because \int_1^\infty \frac{1}{x^3}\,dx$ converges by p-test

$\therefore \int_1^\infty \left|\frac{\sin x}{x^3+\sqrt{x}}\right|$ converges.

$\therefore$ By ACT. $\int_1^\infty \frac{\sin x}{x^3+\sqrt{x}}\,dx$ converge

- Type II. improper integrals

f be integrable on $[a, b]$ where

$x=b$ is vertical asymptote

$\Rightarrow \int_a^b f(x)\,dx$ converge if. $\lim_{t\to b^-} \int_a^t f(x)\,dx$ exists

$x=a$ is vertical asymptote

$\Rightarrow \int_a^b f(x)\,dx$ converge if. $\lim_{t\to a^+} \int_t^b f(x)\,dx$ exists

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

↳ converge if $\int_a^c$ & $\int_c^b$ converge 任何一项 diverge 则 $\int_a^b$ diverge.

Q. $\int_0^1 \frac{1}{\sqrt{x}}\, dx$

$= \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\, dx = \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^1 = \lim_{t\to 0^+} 2-2\sqrt{t} = 2.$

Q. $\int_1^4 \frac{1}{(x-2)^2}\, dx$

$= \left[-\frac{1}{x-2}\right]_1^4$

要检查 vertical asymptote.

$= -\frac{1}{2} - 1$

$= -\frac{3}{2}$ WRONG $\frac{1}{(x-2)^2} > 0 \quad \therefore \int_1^4 \frac{1}{(x-2)^2}\, dx < 0$

need to check $\lim_{t\to 2^-} \int_1^t \frac{1}{(x-2)^2}\, dx.$

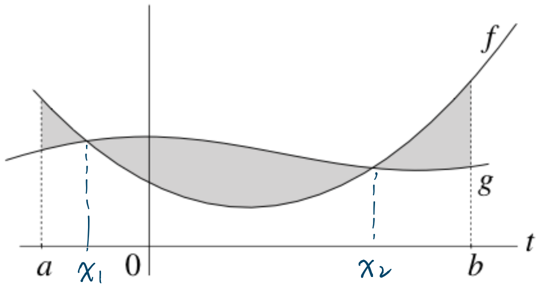
$\int_1^t \frac{1}{(x-2)^2}\, dx = \lim_{t\to 2^-} \left[-\frac{1}{x-2}\right]_1^t = \lim_{t\to 2^-} \frac{1}{2-t} - 1 = \infty$

$\because \int_1^2 \frac{1}{(x-2)^2}\, dx$ diverge $\quad \therefore \int_1^4 \frac{1}{(x-2)^2}\, dx$ diverge

- p-test for Type II.

$\int_0^1 \frac{1}{x^p}$ converge to $\frac{1}{1-p} \iff p<1$

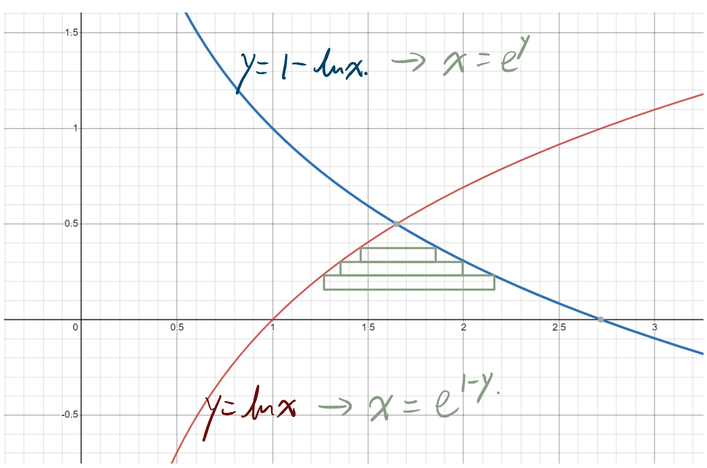
3.1 Area between Curves.



Step 1: 求 f 与 g 的交点. x_1, x_2

Step 2: 分块积分

Q. Find the area of triangular region between $y=\ln x$. $y=1-\ln x$. and x-axis



① 算交点 $x=\sqrt{e}$ $y=\frac{1}{2}$

∴应求 $y=0$ 到 $y=\frac{1}{2}$ 的积分.

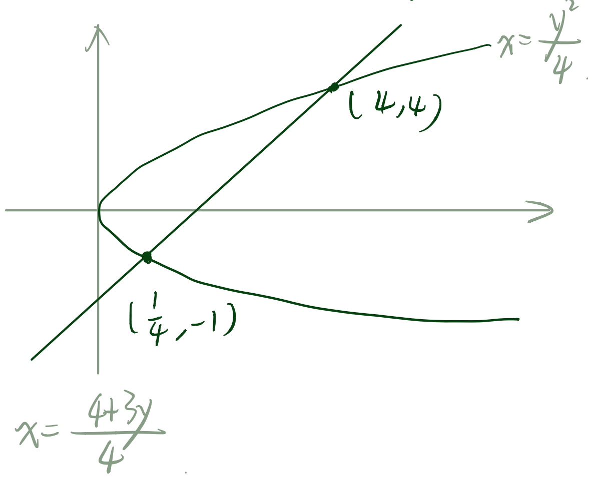
② $\int_0^{\frac{1}{2}} e^{1-y}-e^{y}\,dy$

$= \left[-e^{1-y}-e^{y}\right]_0^{\frac{1}{2}}$

$= -e^{\frac{1}{2}}-e^{\frac{1}{2}}-(-e^{1}-e^{0})$

$=1$

Q. Find area between $y^2=4x$ & $4x-3y=4$



① 交点. $y=4$. $y=-1$.

② $\int_{-1}^{4} \frac{4+3y}{4}-\frac{y^2}{4}\,dy.$

$= \frac{1}{4}\int_{-1}^{4} -y^2+3y+4\,dy$

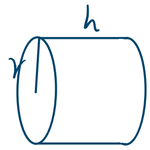
$= \frac{1}{4}\left[-\frac{y^3}{3}+\frac{3y^2}{2}+4y\right]_{-1}^{4}$

$= \frac{1}{4}\left(-\frac{64}{3}+\frac{48}{2}+16-\frac{1}{3}-\frac{3}{2}+4\right)$

$= \frac{125}{24}$

3.2 Disk Method & Washer Method（竖切旋转体积）

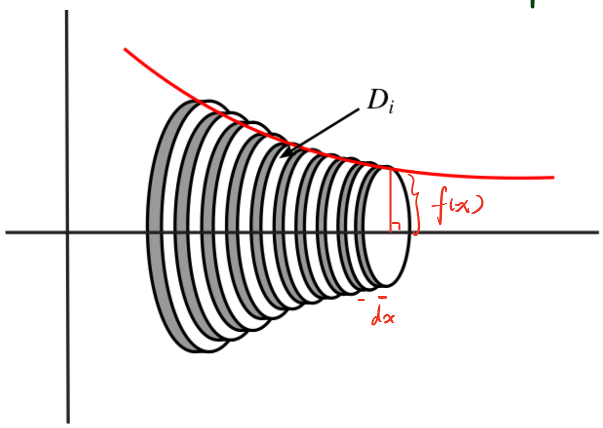
- Volumn of cylinder



$\pi r^2 h$

- 单个体块

Method of Disk



height dx

radius $f(x)$

volume $\pi f(x)^2 dx$.

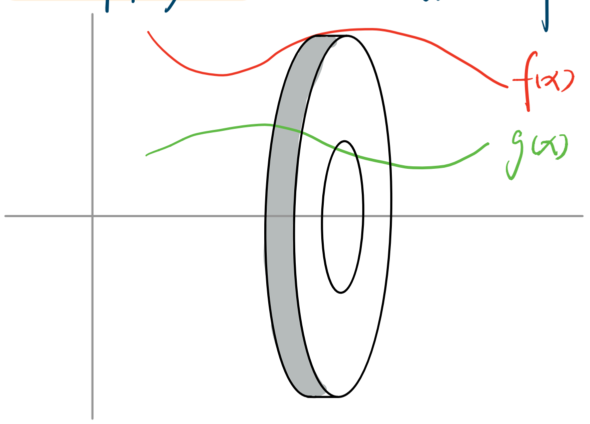
$$V = \int_a^b \pi f^2(x)\, dx$$

Q. Calculate the volume when line $y = \frac{x}{3}$ is rotated around x-axis from $x=0$ to $x=6$.

$$\int_0^6 \pi \left(\frac{x}{3}\right)^2 dx = \frac{\pi}{9}\int_0^6 x^2 dx = \left[\frac{x^3}{3}\cdot\frac{\pi}{9}\right]_0^6 = \frac{216}{27}\pi = 8\pi$$

- 体积差

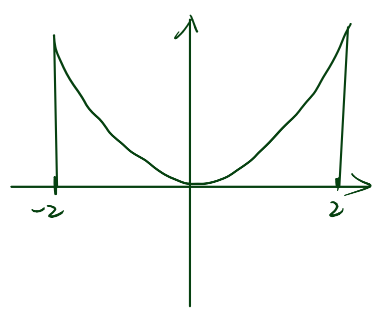
Method of washers / donuts



$$V = \int_a^b \pi f(x)^2 dx - \int_a^b \pi g(x)^2 dx.$$

$$= \int_a^b \pi \left(f(x)^2 - g(x)^2\right) dx$$

Q. Rotate the region between $y = x^2$. $y=0$ & $x=2$.



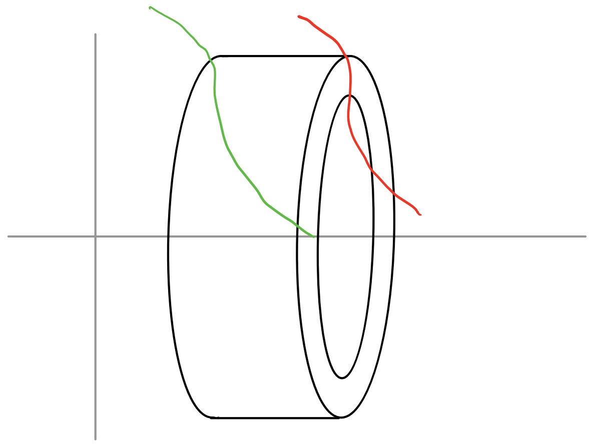
1) V for line rotate around x-axis

$$\int_0^2 \pi (x^2)^2 dx = \pi \int_0^2 x^4 dx = \frac{32}{5}\pi$$

2) V for line rotate around y-axis

$$\int_0^4 \pi (2)^2 dy - \int_0^4 \pi (\sqrt{y})^2 dy = \int_0^4 4\pi\, dy - \int_0^4 y\pi\, dy = 8\pi.$$

3.3 Shell Method (横切旋转体积)



→ 求 individual shell 的体积

find surface area of cylinder. A

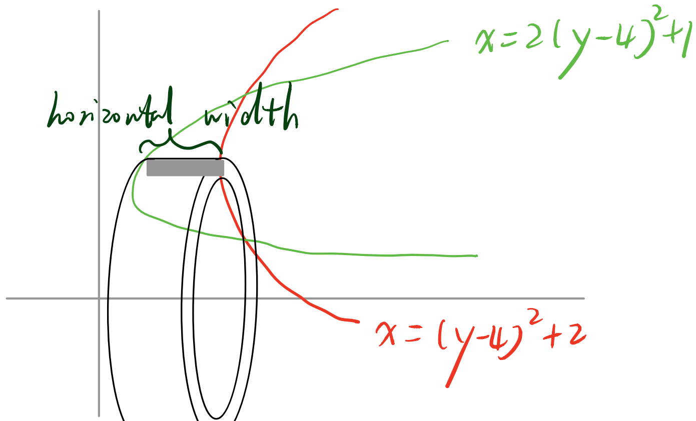
$A \times dy$ (small thickness)

$A = 2\pi y\,[f(y) - g(y)]$

$V = A\,dy = 2\pi y\,[f(y) - g(y)]\,dy$

→ $V = \int_c^d A\,dV = \int_c^d 2\pi y\,[f(y) - g(y)]\,dy$

Q. 求 $x = (y-4)^2 + 2$ $\quad x = 2(y-4)^2 + 1$ 沿x-轴 的旋转体积



horizontal width

$= (x-4)^2 + 2 - [2(y-4)^2 + 1]$

$= 1 - (y-4)^2$

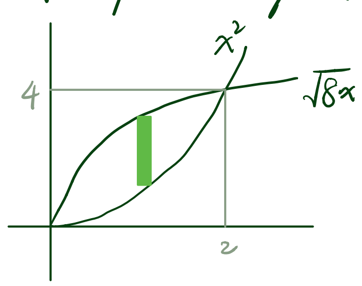
typical volumn

$V = 2\pi y\,[1 - (y-4)^2]\,dy$

total volumn

$V = \int_3^5 2\pi y\,[1 - (y-4)^2]\,dy = \frac{32}{3}\pi$

Q. 求 $y = x^2$ $\quad y = \sqrt{8x}$ 沿 y 轴 的旋转体积.



→ creating vertical rectangles

∴ use shell

→ radius $\quad dx$

height $\quad \sqrt{8x} - x^2$

volumn $\quad dV = 2\pi x\,[\sqrt{8x} - x^2]\,dx$

→ Total volumn:

$V = \int_0^2 2\pi x\,[\sqrt{8x} - x^2]\,dx$

$= \frac{24}{5}\pi$

关于选择 disk / shell

→ 判断 横向 / 纵向切体块

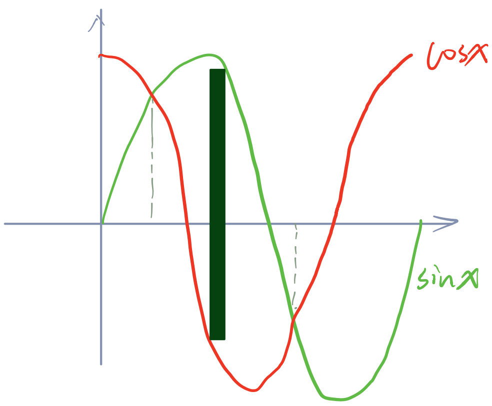
→ 长方形

parallel to 旋转轴 : Shell

perpendicular to 旋转轴 : Disk

→ 尽量选择长方形上下被 f & g bound in

否则需算 $\lim_{a^+} = \lim_{a^-}$



easier to use vertical rectangle

rectangle is parallel to axis of rotation

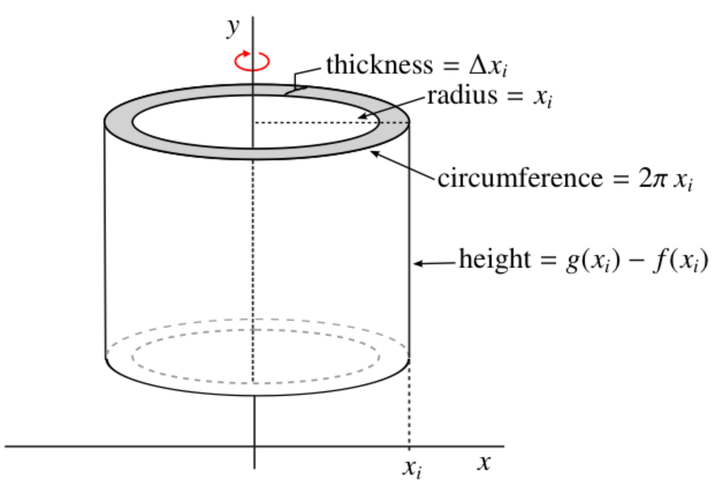
⇒ use shells.

Typical shell: height $= \sin x - \cos x$

radius $= 6 - x$

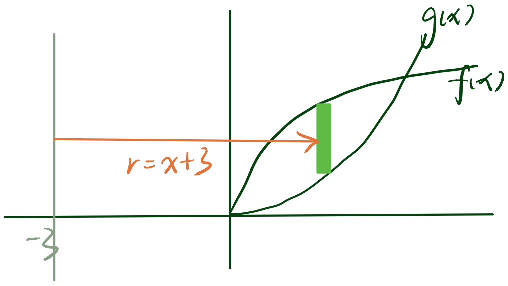
thickness $= dx$

$$V = \int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} 2\pi(6-x)(\sin x - \cos x)\, dx.$$



Volume $= 2\pi x_i (g(x_i) - f(x_i)) \Delta x_i$

Q. 求沿 $x=-3$ 的旋转体积



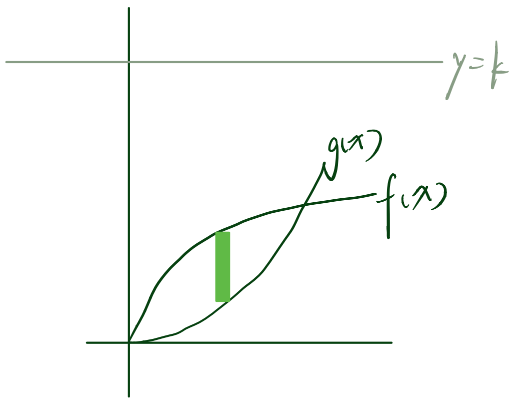
用 Shell method

radius 变成 $x+3$

$$dV = 2\pi(x+3)[f(x)-g(x)]\,dx$$

$$V = \int_a^b 2\pi(x+3)[f(x)-g(x)]\,dx.$$

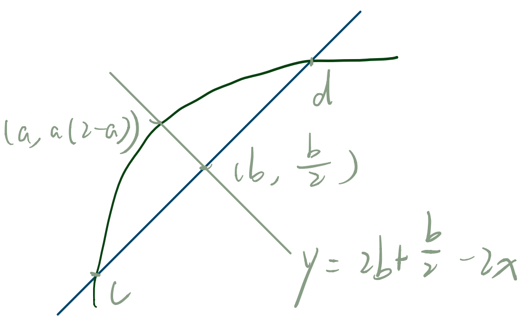
Q. 求沿 $y=k$ 的旋转体积



用 washer method

$$\int_a^b \pi(k-g(x))^2\,dx - \int_a^b \pi(k-f(x))^2\,dx.$$

Q. 求 $y=\frac{x}{2}$. $y=x(2-x)$ 沿 $y=\frac{x}{2}$ 的旋转体积



find a in terms of b.

$$h(b) = \sqrt{(b-a)^2 + (\tfrac{b}{2} - a(2-a))^2}$$

$$V = \pi\sqrt{1+m^2}\int_c^d h(x)^2\,dx.$$

$m = \frac{1}{2}$

$c = \frac{x}{2}$

$d = x(2-x)$

3.4 Arc Length

- arc length (S)

$$S = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

4.1 Introduction to differential equations

- def. differential equation

DE 是一个涉及到很多层 rate of change of f 的 function

nth order DE: $F(x, y, y', y'', \cdots, y^{n}) = 0$

solution to differential equation is function φ.

eg. $F(x, y, y'') = \cos x \cdot y + y'' = 0$ is an differential equation of order 2.

$\because \cos x \, \varphi + \varphi'' = \cos x \cdot 0 + 0 = 0$

$\therefore$ The constant function $\varphi(x) = 0$. is a sol

A DE considered linear if it can be written in the form:

$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_2(x) y'' + a_1(x) y' + a_0(x) y = f(x).$

- First-order derivative

Q. find all f statisfying $f' = f$

$f(x) = 0 \qquad f(x) = e^x$

→ The most basic DE would be $y' = 2y$.

$\therefore$ 求满足 $f' = cf$. 的 f.

→ $\frac{d}{dx} e^x = a e^{ax}$

$\because y'(x) = 2e^{2x} = 2y(x)$

$\therefore$ a sol to $y' = 2y$ is $y(x) = e^{2x}$.

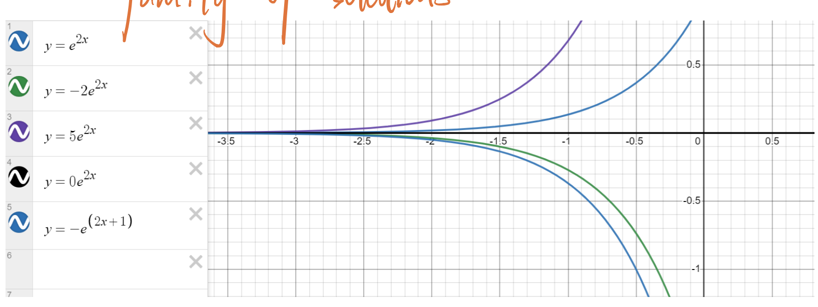
→ $\therefore$ any function of the form $\boxed{y = Ae^{2x}}$ will satisfy the DE

family of solutions

$LHS = y' = 2Ae^{2x}.$

$RHS = 2y = 2(Ae^{2x})$

$LHS = RHS$



- Initial Value Problem (IVP)

def. a DE along with an IC

ep. 求解一阶导 $y'=2y$

step 1: 找 initial condition. (IC) $x=0$ $y=?$

$y(0)=5$.

step 2: 解 DE. : $y(x)=Ae^{2x}$.

$5=y(0)$

$5=Ae^{2\cdot 0} \rightarrow A=5$. $\therefore y(x)=5e^{2x}$

- Direction fields

一般很难求解 DE. 所以用 graphical / numerical method.

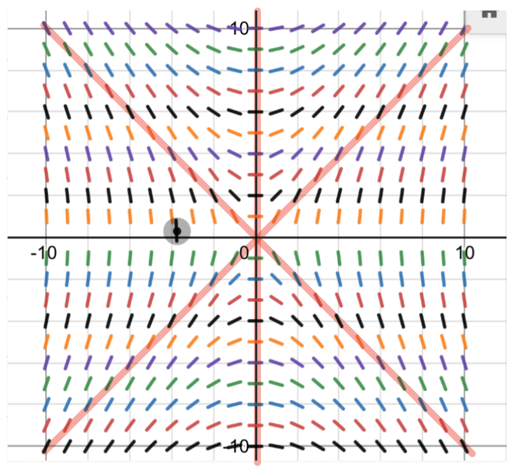
def. a direction field is a collection of line elements on xy plane whose slope correspond with those of solutions through that point

def. isocline is a curve in direction field upon line element have same slope

→ not solution to DE

ep. direction field of $y'=\frac{2x}{y}$

ep. isocline of $y'=\frac{2x}{y}$



ep. 对于 DE $y'=2y-x^2$

$y'=0$ $y=\frac{x^2}{2}$. → 所有在 $\frac{x^2}{2}$ 上 line element 斜率为0

4.2 Separable differential equations

- def. separable differential equation

$\exists f = f(x),\ g = g(y)$ s.t. $y' = f(x)g(y)$

$\Rightarrow$ first-order differential equation is separable.

ep.
- $y' + \sin x \cdot y = 3y^2$ N
- $y' + y^3 = \ln(y^2+1)$ Y $\quad y' = \ln(y^2+1) - y^3 \quad f(x) = 1 \quad g(y) = \ln(y^2+1) - y^3$
- $y' = e^{x-y}$ Y $\quad y' = e^x \cdot e^{-y}$
- $y' = 2y - x$ N

- Equilibrium solution

If constant function $y(x) = b$ statisfies $g(b) = 0$, then $y(x) = b$ is a sol to $y' = f(x)g(y)$

since $LS = y'(x) = 0 \quad RS = f(x)g(y) = 0$

solution: $y(x) = b$ $\leftarrow$ equilibrium solutions

ep. $y' = (y^2-4)(x+3)$ has equilibrium solutions $y(x) = 2 \quad y(x) = -2$.

- Solving equilibrium DEs

1. Find all equilibrium solution
2. Rewrite $h(y)\frac{dy}{dx} = f(x) \qquad h(y) = \frac{1}{g(y)}$
3. 两边积分 $\int h(y)\frac{dy}{dx}dx = \int f(x)dx \rightarrow \int h(y)dy = \int f(x)dx$
4. If possible 提取 $y(x)$. 若不行, remain in implicit form.

ex. 解 $\frac{dy}{dx} = 2x(y-1)$ sol: $y(x) = 1$

Assume $y(x) \neq 1$. $\int \frac{1}{y-1}dy = \int 2x\,dx$

$\ln|y-1| = x^2 + C$

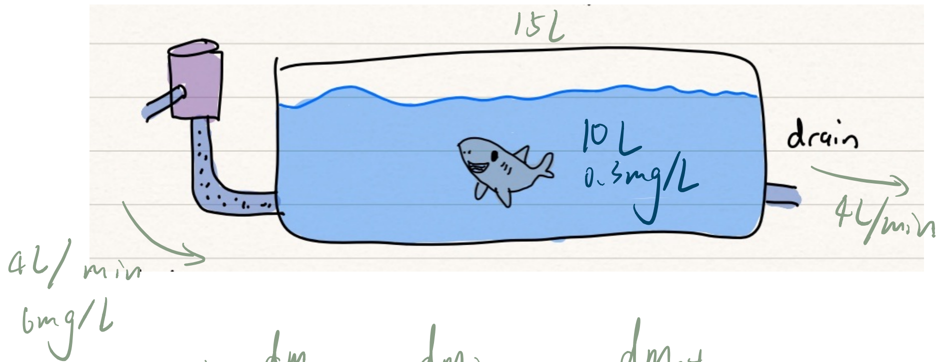
$|y-1| = e^{x^2+C}$

$y - 1 = \pm e^{x^2}e^C \qquad$ let $A = \pm e^C \qquad y = 1 \pm Ae^{x^2} \qquad \because e^C \neq 0 \qquad \therefore A \neq 0$

solution are: $y(x) = 1$ and $y(x) = 1 + Ae^{x}$ $\quad A \neq 0$

- application

Q. Your mini shark can live in water with chlorine concentrations up to $\boxed{5\ \text{mg/L}}$. Your 15L tank is holding 10L of solution with concentration 0.3mg/L. To help clean it out you turn on the in flow pump and set its speed to 4L/min with a concentration of 6mg/L. To avoid overflow you set the drain to 4L/min also. Hoping it won't shoot past 5mg/L you wait 2 min then shut it off. Is your mini shark alive?



$\because \dfrac{dm}{dt} = \dfrac{dm_{in}}{dt} - \dfrac{dm_{out}}{dt}$

$\dfrac{dm_{in}}{dt} = \dfrac{dV_{in}}{dt} \times \dfrac{dm}{dV} = 4\ \dfrac{L}{min} \cdot 6\ \dfrac{mg}{L} = 24\ \dfrac{mg}{min}$

缸内水的浓度？

$\dfrac{dm_{out}}{dt} = \dfrac{dV_{out}}{dt} \times \dfrac{dm}{dV} = 4\ \dfrac{L}{min} \cdot \dfrac{m}{10} = \dfrac{2m}{5}\ \dfrac{mg}{min}$

$\therefore \dfrac{dm}{dt} = 24 - \dfrac{2}{5}m$

$\because$ 最初 0.3 mg/L in 10 L.

$\therefore m(0) = 10 \times 0.3 = 3\ mg$

Solve IVP : $\dfrac{dm}{dt} = 24 - \dfrac{2}{5}m$. $\quad m(0) = 3$

$\dfrac{dm}{dt} = \dfrac{120-2m}{5}$

$\displaystyle\int \frac{dm}{120-2m} = \int \frac{1}{5}\, dt$

$-\frac{1}{2}\ln|120-2m| = \frac{1}{5}t + C_1$

$\ln|120-2m| = -\frac{2}{5}t + C_2 \quad (C_2 = -2C_1)$

$120 - 2m = Ae^{-\frac{2}{5}t} \quad (A = \pm e^{C_2})$

$m(t) = 60 + Be^{-\frac{2}{5}t} \quad (B = -\frac{A}{2})$

$\because m(0) = 3$

$\therefore 3 = 60 + B \quad B = -57$

$m(t) = 60 - 57e^{-\frac{2}{5}t}$

$m(2) = 34.388.$

$\rho = \dfrac{m(2)}{10} \approx 3.44 < 5$

$\therefore$ Shark is safe

4.3 First-Order Linear differential equations

- first-order linear differentiable equation (FOLDE)

y is linear if 能写作 $y' = f(x)y + g(x)$

判断 f 是否为 linear： ① $f, f', f'' \cdots$ 都为一次幂

② $f, f', f'' \cdots$ 前面系数只能有自变量 x^n / constant

③ 不能出现复合函数 ep. $\sin(x)\,y$

- Existence and uniqueness theorem

If $f(x)$ $g(x)$ are cts. on I. then for each (x_0, y_0) where $x_0 \in I$. The IVP. $y' = f(x)y + g(x)$. $y(x_0) = y_0$ has exactly one solution on interval

- Integrating factor. (I)

A tool used to help convert a desired number of expressions into something that can be directly integrated $\quad I = e^{-\int f(x)\,dx}$

- 解 first-order linear differential equations

$$y' = f(x)y + g(x)$$

step 1: Determine whether DE is linear. 写作 $y' - f(x)y = g(x)$

step 2: 计算 $I(x)$ $(I(x) \neq 0)$ $\quad I = e^{-\int f(x)\,dx}$

step 3: solution is $y = \dfrac{\int g(x)\,I(x)\,dx}{I(x)}$

Q. Solve $y' = x - 2y$

$y' + 2y = x$

$I = e^{-2x}$

$y = \dfrac{\int x \cdot e^{-2x}\,dx}{e^{-2x}}$ $\quad u = x \quad u' = 1$, $v = -\frac{1}{2}e^{-2x} \quad v' = e^{-2x}$

$= \dfrac{-\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx + C}{e^{-2x}} = \dfrac{-\frac{1}{2}xe^{-2x} + \frac{1}{2}\cdot(-\frac{1}{2}e^{-2x}) + C}{e^{-2x}} = -\frac{1}{2}x - \frac{1}{4} + Ce^{-2x}$

4.4 Initial Value Problems

- initial value (限制)

找一阶微分方程 $y' = f(x, y)$ 的解时.

寻找 particular solution, 会出现 constrains.

eg. $P' = kP$ for some k.

$P(t) = Ce^{kt}$ C is arbitrary. k is unknown

找 particular solution. $t=0$. $P(t) = P_0$

将 $t=0$. 代入得 $P_0 = P(0) = Ce^{k(0)} = Ce^0 = C$

$\therefore P(t) = P_0 e^{kt}$

$t=1$. $P(t) = P_1$.

$P_1 = P_0 e^{k \cdot 1} = P_0 e^k$

$e^k = \frac{P_1}{P_0}$ $k = \frac{P_1}{P_0}$

∴ 找任意~~两~~ 1 t 所对应的 P. 即可得到 $P(t)$

- Existence & Uniqueness Theorem.

Assume f & g are continuous function on interval I.

Then for each $x_0 \in I$. $y_0 \in \mathbb{R}$.

The initial problem $y' = f(x)y + g(x)$ $y(x_0) = y_0$ has exactly 1 sol on interval I

first-order equation + constrain → unique solution

Q. Solve $y' = xy$ $y(0) = 1$

$I = e^{-\int x dx} = e^{-\frac{x^2}{2}}$ $y = \frac{\int 0 \cdot e^{-\frac{x^2}{2}} dx}{e^{-\frac{x^2}{2}}} = \frac{C}{e^{-\frac{x^2}{2}}} = Ce^{\frac{x^2}{2}}$

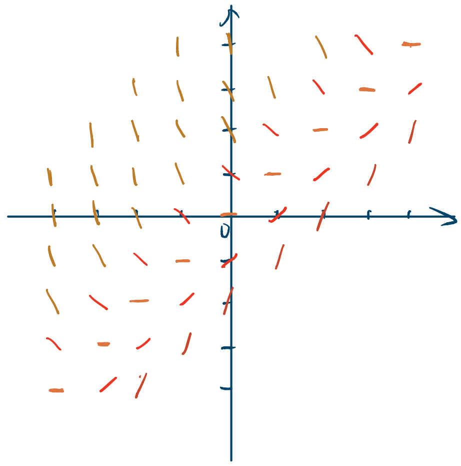
$1 = y(0) = Ce^0 = C$ $\therefore y = e^{\frac{x^2}{2}}$

4.5 Graphical & Numerical Solutions

eg. $y' = x - y$

x	y	y'
0	0	0
1	1	0
2	2	0
1	0	1
2	1	1
0	-1	1
2	0	2

x	y	y'
0	1	-1
1	2	-1
0	2	-2
3	0	-3
⋮	⋮	⋮



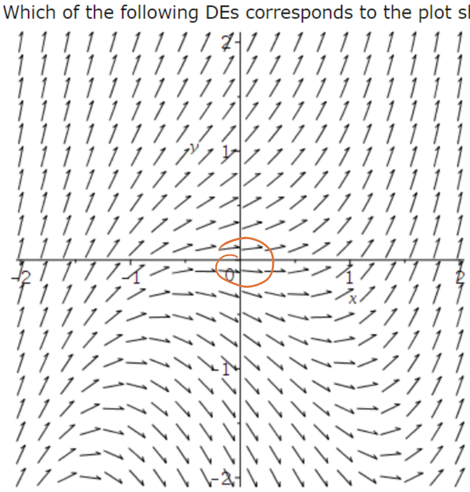
验证: $\frac{dy}{dx} = x - y$

将 $y = x - 1$　$y' = 1$. 代入

$1 = x - (x-1)$

$1 = 1$

Which of the following DEs corresponds to the plot shown below



A good way to examine the plot is to try and find all the points where the slope is flat and estimate the functional representation of the curve that would go through these points.

In this case, it turns out that the curve

$0 = x^2 + y$

corresponds to flat slopes. It is also useful to examine if the slopes are going up or down on either side (or within and without) the curve. Upon doing so, it should be clear that the DE

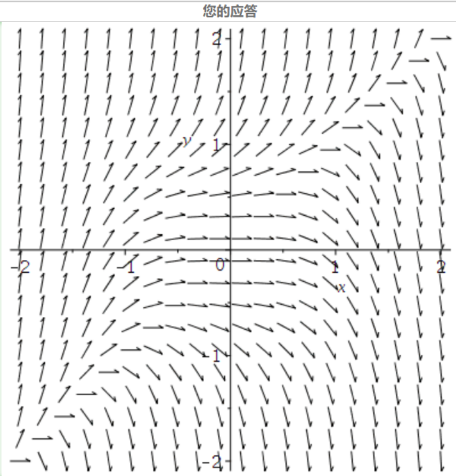
$y' = x^2 + y$

corresponds to the given plot.

您的应答	正确应答
$y' = x^2 + y$	

Which of the following plots represents the directional field for the DE $y' = (-x^3) + y^3$?

您的应答	正确应答



An easy way to determine possible candidates is to look at the equation

$C = (-x^3) + y^3$　$y' = 0$

for different constant values of C, and determine the shape of the curves produced if you consider y as a function of x.

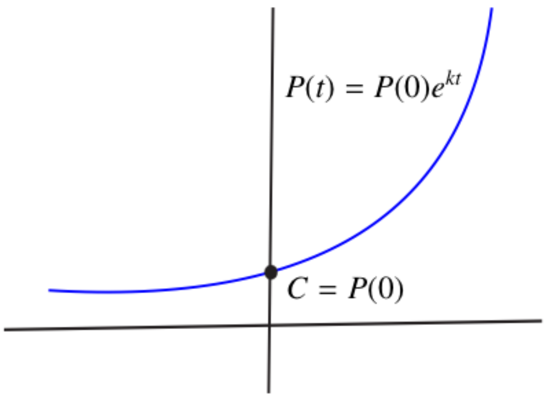
For example, all points along the curve you get when $C = 0$ will have slope 0 (a.k.a be flat).

4.6 Exponential Growth and Decay

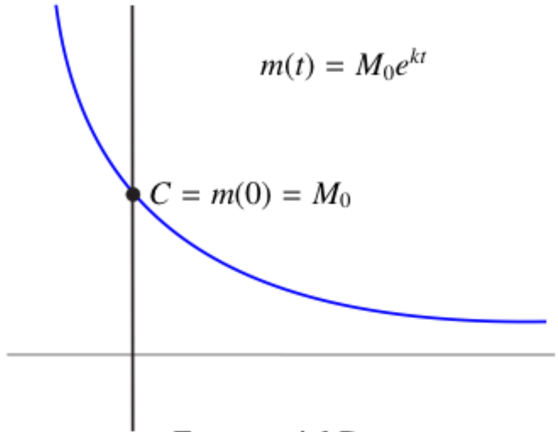
→ Find all function Q satisfying the differential equation $Q' = kQ$ $k \in \mathbb{R}$.

$P(t) = Ce^{kt}$ → $P(t) = P(0)e^{kt}$

initial value $C = P(0)$



Exponential Growth



Exponential Decay

Q. A bacterial colony starts with a population of 1000. After 2hours, the population is estimated to be 3500. What would you expectthe population to be after 7 hours?

$P(t) = Ce^{kt}$

$t=0$ $P(t) = 1000$ $\therefore C = 1000$ $P(t) = 1000e^{kt}$

$3500 = P(2) = 1000e^{2k}$

$e^{2k} = \frac{3500}{1000}$ $k = \frac{\ln 3.5}{2}$

$P(7) = 1000e^{7 \cdot \frac{\ln 3.5}{2}} \approx 80212$

4.7 Newton's Law of Cooling

- An object will cool (or warm) at a rate that is proportional to the difference between T (object) and T_a (surroundings)

$T(t)$: t 时 object 's Temperature.

$T' = k(T - T_a)$

若 $D = D(t) = T(t) - T_a$ $\qquad D = Ce^{kt}$

$\therefore T = Ce^{kt} + T_a$. $\qquad C = D(0) = T_0 - T_a$. $\qquad T_0 = T(0)$

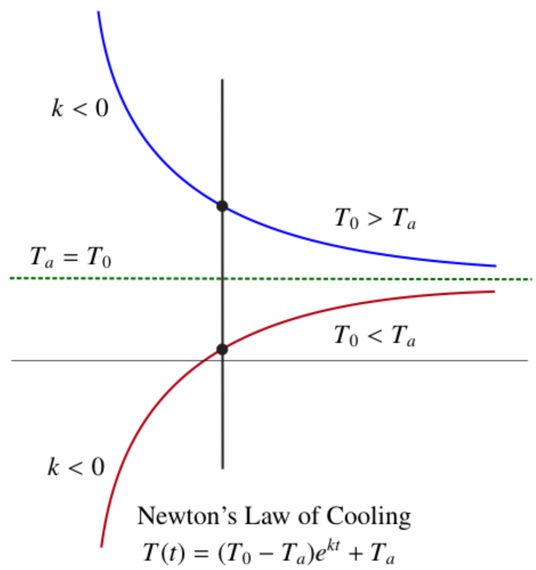
$T(t) = (T_0 - T_a)e^{kt} + T_a$

① $T_0 > T_a$ $\qquad T' = k(T - T_a) < 0$

② $T_0 < T_a$ $\qquad T' = k(T - T_a) > 0$

③ $T_0 = T_a$ $\qquad T' = k(T - T_a) = 0$

$\lim_{t \to \infty} T(t) = T_a$



Newton's Law of Cooling
$T(t) = (T_0 - T_a)e^{kt} + T_a$

4.8 Logistic Growth

- logistic equation

A population with unlimited resources, grows at a rate proportional to its size

$$P' = kP$$

若存在 Maximum population M that resources can support.

then $P' = kP(M-P)$

$\uparrow$

satisfies logistic growth model : $\underline{y' = ky(M-y)}$ (logistic equation)

separable with constant sol : $P(t) = 0$ $P(t) = M$

- Solving logistic equation

$$P' = kP(M-P) \qquad \frac{P'}{P(M-P)} = k$$

$$\rightarrow \int \frac{1}{P(M-P)}\,dP = \int k\,dt = kt + C_1$$

$$\rightarrow \frac{1}{P(M-P)}\,dP = \frac{A}{P} + \frac{B}{M-P}$$

$$1 = A(M-P) + B(P)$$

$$P=0 \quad 1 = A\cdot M \quad A = \frac{1}{M}$$

$$P=M \quad 1 = B\cdot M \quad B = \frac{1}{M}$$

$$\frac{1}{P(M-P)} = \frac{1}{M}\left[\frac{1}{P} + \frac{1}{M-P}\right]$$

$$\int \frac{1}{P(M-P)}\,dP = \frac{1}{M}\left[\int \frac{1}{P}\,dP + \int \frac{1}{M-P}\,dP\right]$$

$$= \frac{1}{M}\left[\ln|P| - \ln|M-P|\right] + C_2$$

$$= \frac{1}{M}\ln\frac{|P|}{|M-P|} + C_2$$

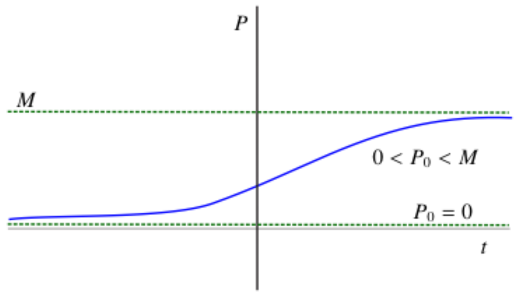
$$\rightarrow \frac{1}{M}\ln\frac{|P|}{|M-P|} + C_2 = kt + C_1 \quad \ln\frac{|P|}{|M-P|} = Mkt + C_3 \qquad \frac{|P(t)|}{|M-P(t)|} = Ce^{Mkt}$$

① $0 < P(t) < M$ $\qquad \frac{|P(t)|}{|M - P(t)|} = \frac{P(t)}{M - P(t)} = Ce^{Mkt}$

$$P(t) = (M - P(t))Ce^{Mkt} = MCe^{Mkt} - P(t)Ce^{Mkt}$$

$$P(t)(1 + Ce^{Mkt}) = MCe^{Mkt}$$

$$P(t) = \frac{MCe^{Mkt}}{1 + Ce^{Mkt}}$$



Logistic Growth

- ∵ $C > 0$ $\quad$ ∴ $0 < \frac{MCe^{Mkt}}{1 + Ce^{Mkt}} < 1$

$0 < P(t) < M$

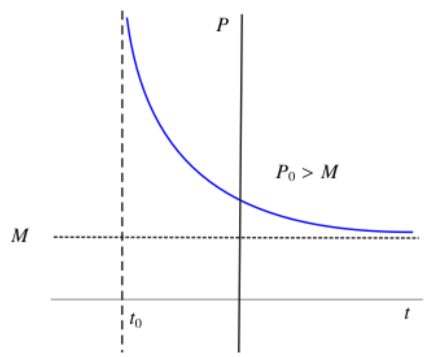
- ∵ $k > 0$ $\quad$ ∴ $\lim_{t \to \infty} P(t) = \lim_{t \to \infty} \frac{MCe^{Mkt}}{1 + Ce^{Mkt}}$

$$= M \lim_{t \to \infty} \frac{Ce^{Mkt}}{1 + Ce^{Mkt}}$$

$$= M$$

② $P(0) > M$. $\qquad \frac{|P(t)|}{|M - P(t)|} = -\frac{P(t)}{M - P(t)} = Ce^{Mkt}$

$$P(t) = M\frac{Ce^{Mkt}}{Ce^{Mkt} - 1}$$



- $P(t) > 0$ if $Ce^{Mkt} > 1$. $\quad$ / $\quad e^{Mkt} > \frac{1}{C}$

5.1 Series

- def. Series

add all terms of the sequence $\{a_n\}$ together $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$

convergence properties. → find what a series adds up to

- def. $S_n = \sum_{k=1}^{n} a_k$ is **nth partial sum** of series $\sum_{k=1}^{\infty} a_k$

eg. $a_n = \frac{2}{3^n}$

$$S_1 = a_1 = \frac{2}{3}$$
$$S_2 = a_1 + a_2 = \frac{2}{3} + \frac{2}{9} = \frac{8}{9}$$
$$S_3 = a_1 + a_2 + a_3 = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} = \frac{26}{27}$$
$$\cdots$$

$\{a_n\}$ the sequence of terms

$\{S_n\}$ the sequence of partial sums

- def. Convergence Series (MCT)

$S_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n$

$\lim_{n\to\infty} S_n$ exists

$\sum_{n=1}^{\infty} a_n$ converges if $\underline{\{S_k\}}$ converges

otherwise, it diverges

若 $L = \lim_{k\to\infty} S_k$, 则 $\sum_{n=1}^{\infty} a_n = L$

$\sum_{n=1}^{\infty} a_n$ is positive series . partial sum: $S_k = \sum_{n=1}^{k} a_n$

MCT ⇒ $\begin{cases} \{S_k\} \text{ converges} & \text{if } \{S_k\} \text{ is bounded.} \\ \{S_k\} \text{ diverges to } \infty & \text{otherwise} \end{cases}$

MCT ⇒ $\begin{cases} \sum_{n=1}^{\infty} a_n \text{ converges} & \text{if } \{S_k\} \text{ is bounded.} \\ \sum_{n=1}^{\infty} a_n \text{ diverge to } \infty & \text{otherwise} \end{cases}$

Q. Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge or diverge?

if $\frac{1}{n^2} < \frac{1}{n^2-n}$ 证 conv. 找比 a_n 大的

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1+\sum_{n=2}^{k} \frac{1}{n^2}$$

$$\leq 1+\sum_{n=2}^{k} \frac{1}{n^2-n}$$

$$\leq 1+1$$

$$=2$$

$\sum_{n=1}^{k} \frac{1}{n^2}$ is bounded above by 2.

$\{T_k\}$ converges

5.2 Geometric Series

- def.

$$\sum_{n=0}^{\infty} Ar^n = A(1 + r + r^2 + \cdots) \qquad r: \text{ratio} \qquad A: \text{constant}$$

For which r does $\sum_{n=0}^{\infty} r^n$ converge?

case 1: $r=1$ $\quad S_k = 1+1+\cdots = k+1$

$\because \{S_k\} = \{k+1\}$ diverges $\quad \therefore \sum_{n=0}^{\infty} 1^n$ diverge to ∞

case 2: $r=-1$ $\quad S_k = 1 + (-1) + 1 + \cdots + (-1)^k = \begin{cases} 1 & k \to \text{even} \\ 0 & k \to \text{odd.} \end{cases}$

$\because \{S_k\} = \{1, 0, 1 \cdots\}$ diverges $\quad \therefore \sum_{n=0}^{\infty} (-1)^n$ diverge.

case 3: $r > 1$ $\quad r^n \to \infty$ as $n \to \infty$

case 4: $r < -1$ $\quad r^n \to \text{DNE}$ as $n \to \infty$

case 5: $-1 < r < 1$ $\quad r^n \to 0$ as $n \to \infty$

$$\begin{aligned} S_k &= A + Ar + Ar^2 + \cdots + Ar^k \\ rS_k &= Ar + Ar^2 + \cdots + Ar^k + Ar^{k+1} \\ (1-r)S_k &= A - Ar^{k+1} \\ S_k &= \frac{A - Ar^{k+1}}{1-r} \end{aligned}$$

$$\therefore |r^{k+1}| \to \begin{cases} 0 & |r| < 0 \\ \infty & |r| > 0 \end{cases} \qquad \lim_{k\to\infty} S_k = \begin{cases} \frac{1}{1-r} & |r| < 0 \\ \text{DNE} & |r| \geq 1 \end{cases}$$

- Geometric Series Test

A geometric series $\sum_{n=0}^{\infty} Ar^n$ converges to $\underline{\sum_{n=0}^{\infty} Ar^n = \frac{A}{1-r}}$, iff $|r| < 1$

otherwise, diverges

$$\lim_{n\to\infty} S_n = \frac{\text{first term}}{1 - \text{common ratio}}$$

5.3 Divergence Test

- def. divergence test. (n^{th} term test)

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=k}^{\infty} a_n$ diverges.

ex. The series $\sum_{n=1}^{\infty} \frac{2n^2}{5n^2+1}$ has to diverge because $\lim_{n\to\infty} \frac{2n^2}{5n^2+1} = \frac{2}{5} \neq 0$

proof: Let $S_n = \sum_{k=1}^{\infty} a_k$

$$S_n - S_{n-1}$$

$$= a_n + a_{n-1} + \cdots + a_3 + a_2 + a_1 - (a_{n-1} + a_{n-2} + \cdots + a_3 + a_2 + a_1)$$

$$= a_n$$

$\because \sum a_n$ converges, then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = L$.

$\therefore \lim_{n\to\infty} a_n = 0$ is a condition for convergence

- Harmonic series

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges despite $\lim_{n\to\infty} \frac{1}{n} = 0$

Q. Does $\sum_{n=1}^{\infty} \frac{1}{n}$ converge?

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2}$

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$\therefore S_{2^n} > 1 + \frac{n}{2}$ $\qquad S_n = 1 + \frac{\log_2 n}{2}$

$\because \lim_{n\to\infty} 1 + \log_2 \sqrt{n} = \infty$ then $\lim_{n\to\infty} S_n = \infty$

$\therefore$ partial sum of $\sum \frac{1}{n}$ diverge

- Thm

Assume a_n is defined for all n.

$\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow$ $\sum_{n=j}^{\infty} a_n$ converges for all j

$\sum_{n=j}^{\infty} a_n$ converges for some j $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges

5.4 Arithmetic of Series

- Arithmetic for Series I.

If $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ both converge. then

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n \text{ for any } c\in\mathbb{R}$$
$$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

* if Σa_n diverge Σb_n converge. then Σa_n+b_n diverge
* if Σa_n & Σb_n diverge. then we can't say anything about Σa_n+b_n

prove by contradiction: $a_n = a_n+b_n-b_n$

ex. The series $\sum_{n=1}^{\infty}\frac{1}{n}$ & $\sum_{n=1}^{\infty}\frac{2}{n}$ both diverge.

$\therefore \sum_{n=1}^{\infty}\frac{1}{n}+\frac{2}{n} = \sum_{n=1}^{\infty}\frac{3}{n}$ diverges

ex. The series $\sum_{n=1}^{\infty}\frac{1}{n}$ & $\sum_{n=1}^{\infty}-\frac{1}{n}$ both diverge

but $\sum_{n=1}^{\infty}\frac{1}{n}+(-\frac{1}{n}) = \sum_{n=1}^{\infty} 0$ converges

$\lim_{n\to\infty} e^{\frac{1}{n}} = 1 \neq 0$

ex. The series $\sum_{n=1}^{\infty}\sqrt[n]{e}$ & $\sum_{n=1}^{\infty}-e^{\frac{1}{n+1}}$ both diverge by divergence test

but $\sum_{n=1}^{\infty}\sqrt[n]{e}-\sqrt[n+1]{e}$ converges

- Arithmetic for Series II.

1. $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \sum_{n=j}^{\infty} a_n$ converges for each j
2. $\sum_{n=j}^{\infty} a_n$ converges for some j $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{j-1} a_n + \sum_{n=j}^{\infty} a_n$$

- Telescoping Series.

When a piece of a_k cancels with a_{k+1}. We call this telescoping series.

ex. $\sum_{k=1}^{n} e^{\frac{1}{k}} - e^{\frac{1}{k+1}} = e^1 - e^{\frac{1}{2}} + (e^{\frac{1}{2}} - e^{\frac{1}{3}}) + (e^{\frac{1}{3}} - e^{\frac{1}{4}}) + \cdots + e^{\frac{1}{n}} - e^{\frac{1}{n+1}}$

$= e - e^{\frac{1}{n+1}}$

$\because \lim_{n\to\infty} S_n = e-1 \quad \therefore \sum_{n=1}^{\infty} e^{\frac{1}{n}} - e^{\frac{1}{n+1}}$ converges

ex. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ diverges

$S_n = \sum_{k=1}^{n} [\ln(k) - \ln(k+1)]$

$= \ln 1 - \cancel{\ln 2} + \cancel{\ln 2} - \cancel{\ln 3} + \cdots - \ln(n+1)$

$= -\ln(n+1)$

$\lim_{n\to\infty} S_n = -\infty$

5.5 Comparison Test

- Comparison Test

Let $0 \le a_n \le b_n \quad \forall n \in \mathbb{N}$.

1. $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges
2. $\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ diverges

大的 converge $\Rightarrow$ 小的 converge
小的 diverge $\Rightarrow$ 大的 diverge

prof: convergence: $S_n = \sum_{N}^{n} a_n$ has upper bound MCT.

divergence: $S_n > t_n$ $\quad S_n = \sum_{N}^{n} b_n. \quad t_n = \sum_{N}^{n} a_n \quad t \to \infty$ since $a_n \geq 0$

Q. Converge / diverge?

a) $\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$

$\frac{n^2+1}{n^3-1} > \frac{n^2}{n^3-1} > \frac{n^2}{n^3} = \frac{1}{n}$

$\because \sum_{n=2}^{\infty} \frac{1}{n}$ diverge

$\therefore \sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$ diverge

b) $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

$\because \ln(n) < n \quad \forall n > 0$

$\therefore \frac{1}{\ln(n)} > \frac{1}{n}$

$\because \sum \frac{1}{n}$ diverge

$\therefore \sum \frac{1}{\ln n}$ diverge

c) $\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$

$\because -1 \le \sin n \le 1$

$\therefore 0 \le \frac{1+\sin n}{n^2} \le \frac{2}{n^2}$

$\because \sum \frac{2}{n^2}$ converge

$\therefore \sum \frac{1+\sin n}{n^2}$ converge.

d) $\sum_{n=1}^{\infty} \frac{5^n+n}{(4-\frac{1}{n})^n}$

$\frac{5^n+n}{(4-\frac{1}{n})^n} > \frac{5^n}{(4-\frac{1}{n})^n} > \frac{5^n}{4^n}$

$\because \sum \left(\frac{5}{4}\right)^n$ diverge

$\therefore \sum \frac{5^n+n}{(4-\frac{1}{n})^n}$ diverge

$\rightarrow$ Consider $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

$\because n^2-1 < n^2 \quad \therefore \frac{1}{n^2-1} > \frac{1}{n^2}$

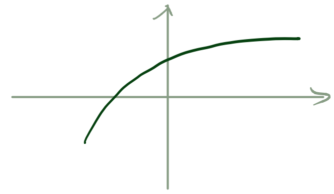
$\because \sum_{n=2}^{\infty} \frac{1}{n^2}$ converge $\therefore$ can't conclude anything.

- Limit Comparison Test (LCT)

Given $a_n \geq 0$. $b_n \geq 0$

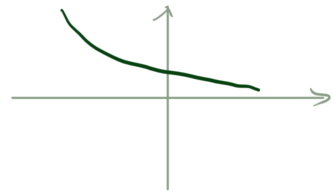
$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

① $0 < L < \infty$ $\quad \sum_{n=1}^{\infty} a_n$ converge $\iff \sum_{n=1}^{\infty} b_n$ converge



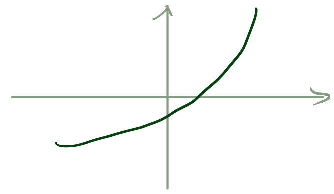
② $L = 0$ $\quad \sum_{n=1}^{\infty} b_n$ converge $\Rightarrow \sum_{n=1}^{\infty} a_n$ converge

$\sum_{n=1}^{\infty} a_n$ diverge $\Rightarrow \sum_{n=1}^{\infty} b_n$ diverge



③ $L = \infty$ $\quad \sum_{n=1}^{\infty} a_n$ converge $\Rightarrow \sum_{n=1}^{\infty} b_n$ converge

$\sum_{n=1}^{\infty} b_n$ diverge $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverge

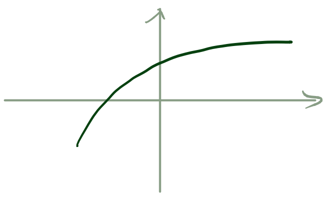


* 若 $\frac{a_n}{b_n} \to 0$. 則 $\frac{a_n}{b_n} < 1$. $a_n < b_n$

若 $\frac{a_n}{b_n} \to \infty$. 則 $\frac{a_n}{b_n} > 1$. $a_n > b_n$

proof: Assume that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

1) $0 < L < \infty$. $\quad L \in (\frac{L}{2}, 2L)$



$\exists$ cutoff $N \in \mathbb{N}$. s.t.

If $n \geq N$, then $\frac{L}{2} < \frac{a_n}{b_n} < 2L$.

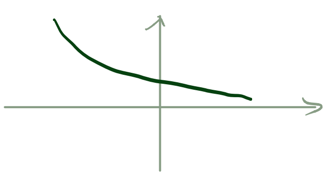
$\frac{L}{2} b_n < a_n < 2L\, b_n$.

If $\sum_{n=1}^{\infty} a_n$ converges, then by comparison test, $\sum_{n=1}^{\infty} \frac{L}{2} \cdot b_n$ converge.

$\therefore \sum_{n=1}^{\infty} b_n$ converges.

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} 2L \cdot b_n$. By comparison test. $\sum_{n=1}^{\infty} a_n$ converges

2) $L = 0$.



$\exists$ cutoff $N \in \mathbb{N}$. s.t.

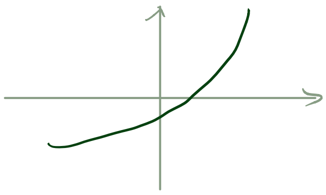
If $n \geq N$, then $0 < \frac{a_n}{b_n} < 1$.

$0 < a_n < b_n$.

If $\sum_{n=1}^{\infty} b_n$ converges, then by comparison test, $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverge

3) $L=\infty$



$\exists$ cutoff $N \in \mathbb{N}$. s.t

if $n \geq N$, then $\frac{a_n}{b_n} > 1$

$b_n < a_n$.

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges by comparison test.

If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverge.

Q. Converge / diverge ?

1) $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

let $a_n = \frac{1}{n^2-1}$ $\quad b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1$

$\because$ By LCT. $\sum \frac{1}{n^2}$ converges

$\therefore \sum \frac{1}{n^2-1}$ converge

2) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ $\qquad$ let $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^2} \cdot n^2 = \infty$

$\because \sum b_n = \sum \frac{1}{n^2}$ converges

$\therefore$ By LCT. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is inconclusive

3) $\sum_{n=1}^{\infty} \frac{2n^2+1}{\sqrt{1+n+n^6}}$

Ignore things that are small as $n \to \infty$

$\therefore \frac{2n^2+1}{\sqrt{1+n+n^6}} \sim \frac{2n^2}{\sqrt{n^6}} = \frac{2}{n}$ for large n.

choose $b_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+1}{\sqrt{1+n+n^6}} \left(\frac{n}{1}\right) = \lim_{n\to\infty} \frac{n^3\left(2+\frac{1}{n^2}\right)}{\sqrt{n^6}\sqrt{\frac{1}{n^6}+\frac{1}{n^5}+1}} = 2$

By LCT. Since $\sum \frac{1}{n}$ diverges. So $\sum \frac{2n^2+1}{\sqrt{1+n+n^6}}$ diverges

5.6 Integral Test for Convergence

- integral test.

(the divergence test can only be used for divergence)

Given $\sum_{n=1}^{\infty} a_n$ if we can find a positive, eventually decreasing continuous function $f(x)$ where $a_n = f(n)$ eventually then

continuous

we can ignore the series until properties hold

$$\boxed{\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges.}}$$

ex. $\sum_{n=1}^{\infty} \frac{1}{n}$ let $f(x) = \frac{1}{x}$.

pos ☑
cont. ☑
decrease on $[1,\infty)$ ☑

Since $\int_1^{\infty} \frac{1}{x}\,dx$ diverges by p-test then $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

ex. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ let $f(x) = \frac{1}{x^2}$

pos ☑
cont. ☑
decrease on $[1,\infty)$ ☑

Since $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[-x^{-1}\right]_1^t$

$= \lim_{t\to\infty} 1 - \frac{1}{t}$

$= 1$ $\therefore$ converges

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

- p-Series test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \iff p > 1$$

Q. Does $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converge?

use $f(x) = \frac{1}{x \ln x}$ (pos. cont. dec on $[2,\infty)$)

$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \left[\ln(\ln(x))\right]_2^{\infty} = \lim_{t\to\infty} \ln(\ln(t)) - \ln(\ln(2)) = \infty$

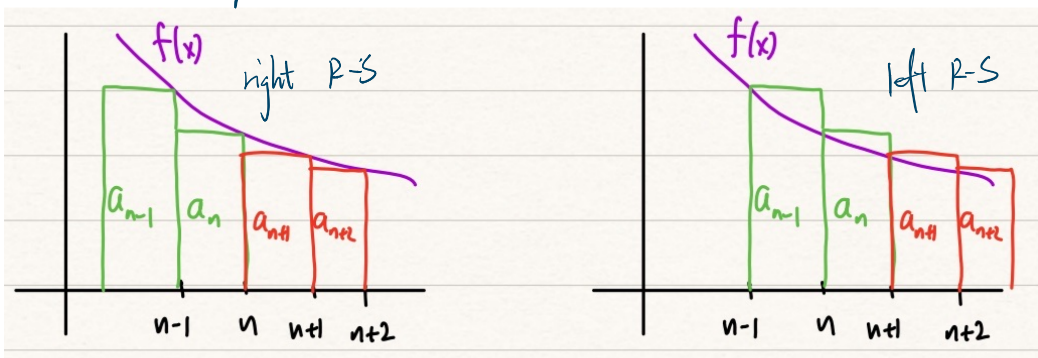
$\therefore \sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges

− Error estimate (for converge series)

Assume $f(x)$ 满足 integral test 的条件.

$\sum_{n=1}^{\infty} a_n$ converge to L.

Let nth partial sum be $S_n = \sum_{i=1}^{n} a_i$.



$$L = \sum_{i=1}^{n} a_i + \sum_{i=n+1}^{\infty} a_i = S_n + \sum_{i=n+1}^{\infty} a_i$$

error : $R_n = L - S_n = \sum_{i=n+1}^{\infty} a_i$ （所有红色面积之和）

图像中： $\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$

$$\boxed{S_n + \int_{n+1}^{\infty} f(x)\,dx \le L \le S_n + \int_{n}^{\infty} f(x)\,dx}$$

$\int_n^\infty$ 真实值　　整个线下面积　　$\int_n^\infty$ 真实值

Q. Find an interval for the value of $L = \sum_{n=1}^{\infty} \frac{1}{n^2}$ if we use 10 terms to approximate it.

$S_n + \int_{n+1}^{\infty} f(x)\,dx \le L \le S_n + \int_{n}^{\infty} f(x)\,dx$　　取 $f(x) = \frac{1}{x^2}$

$S_n + \left[-\frac{1}{x}\right]_{n+1}^{\infty} \le L \le S_n + \left[-\frac{1}{x}\right]_{n}^{\infty}$　　计算 $\int f(x)$ 并代入

$S_n + \frac{1}{n+1} \le L \le S_n + \frac{1}{n}$

$\because S_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100} = 1.5497\cdots$　　计算 S_n　（n : terms 数）

$\therefore S_{10} + \frac{1}{10+1} \le L \le S_{10} + \frac{1}{10}$　　将 S_n 代入

$1.640 \le L \le 1.6497$.

Q. What is the upper bound on error? use S_5 to approximate $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

$\to S_5$ 的最大值.

Let $f(x) = \frac{1}{x^4}$.

$R_5 \le \int_5^{\infty} \frac{1}{x^4} = \left[-\frac{1}{3x^3}\right]_5^{\infty} = 0.0026$

- Proof idea of integral test.

Let $S_m = \underline{a_1 + a_2 + \cdots + a_{n-1} + a_n} + \cdots + a_m$

$= \underline{S_n} + a_{n+1} + \cdots + a_m$

$$S_n + \int_{n+1}^{m} f(x)\,dx \le S_m \le S_n + \int_n^{\infty} f(x)\,dx$$

$\to$ Converge : show $\lim_{m\to\infty} S_m$ exists

思路: $\int_1^{\infty} f(x)\,dx$ converge $\Rightarrow \int_n^{\infty} f(x)\,dx = \alpha.$ $(\neq \pm\infty)$

$\therefore S_m \le \underline{S_n + \alpha}.$ upper bound.

$\{S_m\}$ is increasing sequence.

Since $a_n > 0$. $\therefore$ by MCT. $\lim_{m\to\infty} S_m$ exists. $\sum_{n=1}^{\infty} a_n$ converge.

$\to$ diverge :

思路: $\int_1^{\infty} f(x)$ diverges $\Rightarrow \lim_{m\to\infty} \int_{n+1}^{m} f(x)\,dx = \infty$

$\because f(x) > 0$

$\because S_m \ge \int_{n+1}^{m} f(x)\,dx.$ $\therefore \lim_{m\to\infty} S_m = \infty$ $\sum_{n=1}^{\infty} a_n$ diverge

5.7 Alternating Series

- def. alternating series

A series of the form $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ with $a_n > 0$

ep. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$

- alternating series test (AST)

只能用于证明 converge

If
1. $a_n > 0 \quad \forall n > \text{some } N$
2. $a_{n+1} < a_n \quad \forall n > \text{some } N$
3. $\lim_{n\to\infty} a_n = 0$

每项 >0
decr.
$\lim = 0$

then $\sum_{n=1}^{\infty}(-1)^{n-1} a_n$ converges.

ex. The alternating harmonic series.

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots$ converge

since $a_n = \frac{1}{n}$ satisfies $a_n > 0$. $\lim_{n\to\infty} a_n = 0$. $a_{n+1} < a_n$

Q. 判断 $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{\sqrt{n}}$ 是否 converge

$\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}} = 0$

Let $a_n \geq 0 \quad \forall n \geq 1$.

Let $f(x) = \frac{\ln x}{\sqrt{x}}$. then $f'(x) = \frac{\frac{1}{x}\sqrt{x} - \frac{\ln x}{2\sqrt{x}}}{x} = \frac{2-\ln x}{2x\sqrt{x}}$

$f'(x) < 0$. where $x > e^2 \approx 7.4$

$\therefore a_{n+1} < a_n \quad \forall n \geq 8.$

$\therefore$ By AST $\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{\sqrt{n}}$ converges

Q. 判断 $\sum_{n=1}^{\infty}(-1)^n e^{\frac{1}{n}}$ 是否 converge

if $f(x) = e^{\frac{1}{x}}$, $f'(x) = \frac{-e^{\frac{1}{x}}}{x^2} < 0 \quad \forall x$

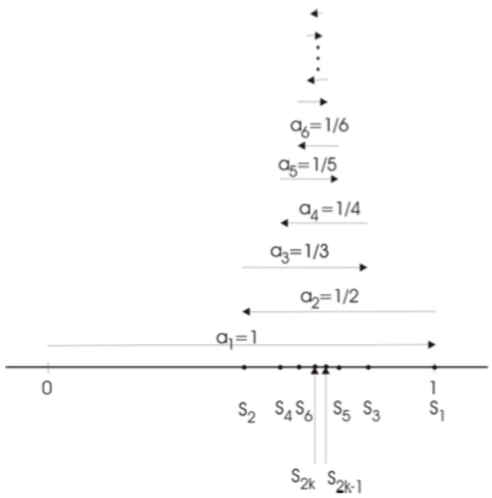
$\therefore e^{\frac{1}{n}}$ is decreasing $a_{n+1} < a_n$.

$\because \lim_{n\to\infty} e^{\frac{1}{n}} = e^0 = 1 \neq 0$ $\therefore$ AST can't used

by divergence test. $\sum_{n=1}^{\infty}(-1)^n e^{\frac{1}{n}}$ diverges

– Proof of AST.

assume we have $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ with $a_n > 0$, $a_{n+1} < a_n$ at $\lim_{n\to\infty} a_n = 0$



eg. $S_6 = \underbrace{a_1 - a_2}_{pos} + \underbrace{a_3 - a_4}_{pos} + \underbrace{a_5 - a_6}_{pos} > a_1 - a_2 + a_3 - a_4 > a_1 - a_2$

In general $S_{2n} > S_{2n-2} > \cdots > S_2$ so $\{S_{2n}\}$ is increasing

$\because S_{2n} < S_1$ $\therefore \{S_{2n}\}$ is bounded. By MCT. $\{S_{2n}\}$ converges

Let $\lim_{n\to\infty} S_{2n} = S$

$S_{2n+1} = S_{2n} + a_{2n+1} \Rightarrow \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1}$

$= S + 0$

$= S$

Since both even and odd partial sums converges to the same value.

then we get $\lim_{n\to\infty} S_n = S$.

$\therefore \sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges

- Error estimate

$$|R_n| = |S - S_n| < a_{n+1}$$

if $a_{n+1} > 0$. then S_n is underestimate

if $a_{n+1} < 0$. then S_n is overestimate

the term when we use n terms to approximate $\sum_{k=1}^{\infty} (-1)^k a_k$ is no longer bigger than first omitted term.

The expression $[a_{n+1}]$ is meant to include the alternating component

eg. $(-1)^n$ or $(-1)^{n+1}$

Q. What is the max error in using S_9 to approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$?

$S_9 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 0.7456\cdots$

error $= |S - S_9| < a_{10} = \frac{1}{10} = 0.1$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 0.7456\cdots \pm 0.1$

$\because$ when $n = 10$. $\frac{(-1)^{11}}{10} = \frac{-1}{10} < 0$.

$\therefore S_9$ is overestimate

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \in [0.6456\cdots, 0.7456\cdots)$.

5.8 Absolute V.S. Conditional Convergence

- def.

$\sum_{n=1}^{\infty}|a_n|$ converges. then $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent

$\sum_{n=1}^{\infty}|a_n|$ diverges. but $\sum a_n$ converges. then $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent

- Absolute convergence theorem. (ACT)

$\sum_{n=1}^{\infty} a_n$ converges absolutely $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges.

$\sum|a_n|$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converge

proof: $-|a_n| \leq a_n \leq |a_n|$. 同时加 $|a_n|$

$0 \leq a_n+|a_n| \leq 2|a_n|$

$\because \sum|a_n|$ converges. $\therefore \sum 2|a_n|$ converges.

$\therefore \sum a_n+|a_n|$ converges by comparison test.

$\sum a_n = \sum a_n+|a_n| - \sum|a_n|$ $\Rightarrow$ $\sum a_n$ converges

ex. $\sum \frac{(-1)^n}{n}$ converges conditionally $\begin{cases} \sum \frac{1}{n} \text{ diverge} \\ \sum \frac{(-1)^n}{n} \text{ converge} \end{cases}$

$\sum \frac{(-1)^n}{n}$ converges absolutely : $\begin{cases} \sum \frac{1}{n} \text{ converge} \\ \sum \frac{(-1)^n}{n} \text{ converge} \end{cases}$

conditionally convergent 的例子 : $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converge by AST.

$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ diverge

$\therefore$ By ACT. it is conditionally convergent.

$\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$.

5.9 Ratio Test

- Ratio Test

Consider the series $\sum_{n=1}^{\infty} a_n$. Let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

If. $L<1$ $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

$L>1$ $\sum_{n=1}^{\infty} a_n$ is divergent

$L=1$. we don't know what happens.

ex. Determine if the following converges

a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right| = \left|2 \cdot \frac{n!}{(n+1)!}\right| = \left|\frac{2}{n+1}\right|$ compute $\left|\frac{a_{n+1}}{a_n}\right|$

$\because \lim_{n\to\infty} \left|\frac{2}{n+1}\right| = 0$ and $0<1$.

$\therefore$ by ratio test, $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges absolutely.

$\sum \frac{b^n}{n!}$ converges for any constant b.

b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-1)^n}\right| = \left|\frac{n^2}{(n+1)^2}\right| = \left|\frac{n^2}{n^2+2n+1}\right|$

$\because \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1$. $\therefore$ ratio test is inconclusive

However, $\sum \frac{(-1)^n}{n^2}$ converges by AST

若 a_n 是 ratio of polynomials. ratio test 不适用. 用 limit comparison

c) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| = \left|(n+1)\frac{n^n}{(n+1)^{n+1}}\right| = \left|\frac{n}{n+1}\right|^n = \left|\frac{1}{(1+\frac{1}{n})}\right|^n = \frac{1}{(1+\frac{1}{n})^n}$

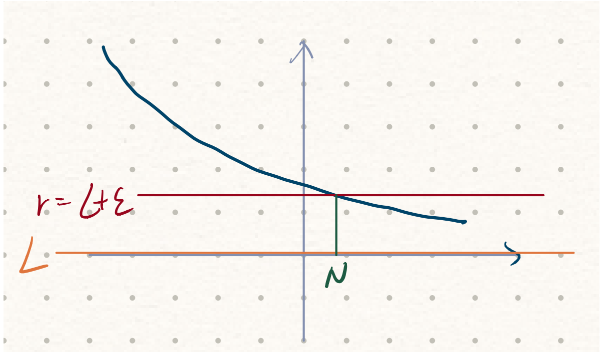
$\because \lim_{n\to\infty} (1+\frac{1}{n})^n = e$. then $\frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e} < 1$

$\therefore$ by ratio test. $\sum \frac{n!}{n^n}$ converges

若 $\sum \frac{n!}{n^n}$ converge. by n^n grows faster than $n!$

- Proof of ratio test

① $L<1$ $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$.



choose $\varepsilon>0$

$\exists\, n\geq N$, s.t. $L-\varepsilon<\left|\frac{a_{n+1}}{a_n}\right|<L+\varepsilon$

$\because L<1$ $\therefore$ choose ε s.t. $L+\varepsilon<1$

Let $r=L+\varepsilon$. $\therefore L<r<1$.

So $\left|\frac{a_{n+1}}{a_n}\right|<r$ for $n\geq N$.

at $n=N$ $\Rightarrow |a_{N+1}|<r|a_N|$

at $n=N+1$ $\Rightarrow |a_{N+2}|<r|a_{N+1}|$

$\vdots$

at $n=N+k-1$ $\Rightarrow |a_{N+k}|<r^k|a_N|$

$\because r<1$ $\therefore \sum_{k=1}^{\infty} r^k|a_N|$ converges since it is geometric series

$\because |a_{N+k}|<r^k|a_N|$ $\therefore \sum_{k=1}^{\infty}|a_{N+k}|$ converges by comparison test

$\sum_{n=1}^{\infty}|a_n| = \underbrace{\sum_{n=1}^{N}|a_n|}_{\text{finite}} + \underbrace{\sum_{k=1}^{\infty}|a_{N+k}|}_{\text{converge}}$ $\to$ converge

$\therefore \sum_{n=1}^{\infty} a_n$ converge absolutely.

② $L>1$

choose $1<r<L$. 得出 $\sum_{k=1}^{\infty}|a_{n+k}| \to \infty$

③ $L=1$

$\underbrace{\sum\frac{1}{n}}_{\text{div}} = \lim_{n\to\infty}\left|\frac{1}{(n+1)}\, n\right| = 1$ $\qquad$ $\underbrace{\sum\frac{1}{n^2}}_{\text{conv.}} = \lim_{n\to\infty}\left|\frac{1}{(n+1)^2}\cdot n^2\right| = 1$

$\therefore$ we don't know what happens at $L=1$

$p>0$ $\quad \ln(x)^p \ll x^p \ll p^x \ll x^x \quad (x\to\infty)$

5.10 Root Test

- Root Test

Given $\sum a_n$ let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$

If $L<1$ then $\sum a_n$ is absolutely convergent

$L>1$ then $\sum a_n$ is divergent.

$L=1$ then we don't know what happens.

$\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$

ex. use the root test to determine whether the following converge or diverge

a) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ b) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{n^2}$ c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ d) $\sum_{n=1}^{\infty} ne^{-n^2}$

a) $a_n = \frac{n}{2^n}$ $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n}{2^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{2} = \frac{1}{2} < 1$

$\therefore$ converges by root test

b) $a_n = \left(\frac{n+1}{n}\right)^{n^2}$, $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \overset{LHR}{=} e > 1$

$\therefore$ it diverges by root test

c) $a_n = \frac{(-1)^n}{\sqrt{n}}$, $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n^{\frac{1}{2}}}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{(n^{\frac{1}{n}})^{\frac{1}{2}}} = 1$

$\therefore$ root test is inconclusive (converges by AST)

d) $a_n = ne^{-n^2}$. then $|a_n|^{\frac{1}{n}} = n^{\frac{1}{n}} e^{-n} \to 1 \cdot 0 = 0$ as $n\to\infty$

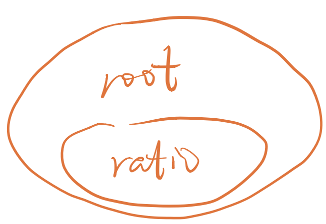
$\therefore$ it converges by the root test

Q: Show that for a rational function $\frac{f(n)}{g(n)}$. the root test fail

let $f(n) = a_p n^p + a_{p-1} n^{p-1} + \cdots + a_1 n + a_0 = a_p n^p \left(1 + \frac{a_{p-1}}{a_p \cdot n} + \frac{a_{p-1}}{a_p \cdot n^2} + \cdots\right)$

Similarly, $g(n) = b_k n^k + b_{k-1} n^{k-1} + \cdots$

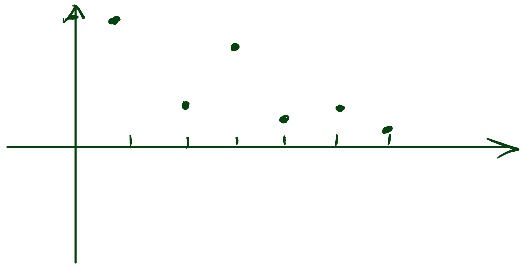
root test 比 ratio test 适用更广



whenever ratio test work, root test also work

Q. Whether $\sum_{n=1}^{\infty} \frac{1}{2^{n+(-1)^n}}$ converge?

$$a_n = \begin{cases} \frac{1}{2^{n+1}} & n \text{ even} \\ \frac{1}{2^{n-1}} & n \text{ odd} \end{cases}$$



ratio test: When n is even $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{2^n}}{\frac{1}{2^{n+1}}}\right| = \frac{2^{n+1}}{2^n} = 2$

When n is odd $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{2^{n+2}}}{\frac{1}{2^{n-1}}}\right| = \frac{2^{n-1}}{2^{n+2}} = \frac{1}{8}$

$\therefore \lim_{\infty} \left|\frac{a_{n+1}}{a_n}\right|$ DNE

root test: When n is even $|a_n|^{\frac{1}{n}} = \left|\frac{1}{2^{n+1}}\right|^{\frac{1}{n}} = \frac{1}{2^{1+\frac{1}{n}}} \to \frac{1}{2}$ as $n \to \infty$

When n is odd. $|a_n|^{\frac{1}{n}} = \left|\frac{1}{2^{n-1}}\right|^{\frac{1}{n}} = \frac{1}{2^{1-\frac{1}{n}}} \to \frac{1}{2}$ as $n \to \infty$

$\therefore \lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{2}$ $\therefore \sum a_n$ converges by root test.

大部分情况 root test 与 ratio 得到的结论相同

两者不同时取 root test （如上）

- Proof of root test

$|a_n|^{\frac{1}{n}} \to L$

① $L < 1$

choose r. s.t $L < r < 1$.

$\exists N$. s.t $\forall n \geq N$ $|a_n|^{\frac{1}{n}} < r$

$\Rightarrow |a_n| < r^n$ for $n \geq N$.

$\because \sum r^n$ converge (by geometric series)

$\therefore$ by comparison $\sum |a_n|$ converges.

② $L > 1$

choose $1 < r < L$.

③ $L = 1$

$\sum \frac{1}{n}$ converge. $\sum \frac{1}{n^2}$ diverge but both $L = 1$

6.1 Power Series

- def.

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ for a given set of coefficients $\{c_n\}$.

For what value of x will the series converge?

let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ domain of f is set of x which let $f(x)$ converge

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

- Fundamental Convergence theorem for power series. (FCT)

Given a power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ centred at $x=a$.

the interval of x over which a power series converges

interval of convergence $I = \{x_0 \in \mathbb{R} : \sum_{n=0}^{\infty} a_n (x_0-a)^n \text{ conv}\}$

radius of convergence. $R = \begin{cases} \text{lub}(\{|x_0-a| : x_0 \in I\}) & I \text{ bounded} \\ \infty & I \text{ isn't bounded.} \end{cases}$

1. $R=0$. $\sum_{n=0}^{\infty} a_n (x-a)^n$ → $x=a$. conv; → other div

2. $0 < R < \infty$. $\sum_{n=0}^{\infty} a_n (x-a)^n$ → $x \in (a-R, a+R)$ conv.; → $|x-a| > R$ div.

3. $R = \infty$ $\sum_{n=0}^{\infty} a_n (x-a)^n$ → $x \in \mathbb{R}$ conv

- Test for the radius of convergence

Let $\sum_{n=0}^{\infty} a_n (x-a)^n$ be a power series, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$

R be radius of convergence of power series

1. $0 < L < \infty$ $\Rightarrow R = \frac{1}{L}$ ⟶ 4 possible interval for conv:
2. $L = 0$ $\Rightarrow R = \infty$
3. $L = \infty$ $\Rightarrow R = 0$

$(a-R, a+R)$
$[a-R, a+R)$
$(a-R, a+R]$
$[a-R, a+R]$

Q. for what value of x do the following converge?

a) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n(n+1)}$ b) $\sum_{0}^{\infty} n!\,(x+2)^n$ c) $\sum_{0}^{\infty} \frac{(x-4)^n}{n^n}$

use ratio test : $\lim_{\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

a) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-1)^{n+1}}{2^{n+1}(n+2)} \cdot \frac{2^n(n+1)}{(x-1)^n}\right| = \left|\frac{x-1}{2} \cdot \frac{n+1}{n+2}\right| \to \left|\frac{x-1}{2}\right|$ as $n \to \infty$

if $\left|\frac{x-1}{2}\right| < 1$, the series converge

$|x-1| < 2$ or $-1 < x < 3$

$\therefore \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n(n+1)}$ converges when $|x-1| < 2$, which is $-1 < x < 3$

now check: $x = 3$ $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n(n+1)} = \sum_{n=0}^{\infty} \frac{2^n}{2^n(n+1)} = \sum_{0}^{\infty} \frac{1}{n+1}$

↳ diverge by integral test

$x = -1$ $\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ conv by AST

$\therefore \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n(n+1)}$ conv for $x \in [-1, 3)$

$\Sigma c_n(x-a)^n$ conv for $|x-a| < R$ div for $|x-a| > R$ rad $= 2$

b) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\,(x+2)^{n+1}}{n!\,(x+2)^n}\right| = |(n+1)(x+2)|$

$\to |(n+1)(x+2)| \to \infty$ for any fixed $x \neq -2$

$\therefore \sum_{0}^{\infty} n!\,(x+2)^n$ only converge when $x = -2$

$\Sigma c_n(x-a)^n$ conv only at $x = a$. rad $= 0$

c) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-4)^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{(x-4)^n}\right| = \left|(x-4)\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}\right|$

$\because n \to \infty$ $(1+\frac{1}{n})^n \to e$

$\therefore \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1+\frac{1}{n}}\right)^n \to \frac{1}{e}$

$\lim_{n\to\infty} \left|(x-4)\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}\right| = (x-4)\,\frac{1}{e} \cdot 0 = 0$ for any x

$\Sigma c_n(x-a)^n$ conv for all $x \in \mathbb{R}$ rad $= \infty$

6.2 Functions Represented by Power Series

- Functions represented by power series.

$\sum_{n=0}^{\infty} a_n (x-a)^n$ be a power series.

radius of conv: $R > 0$. interval of conv: I

$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ for each $x \in I$.

- Continuous of Power Series

If $\sum_{n=0}^{\infty} a_n (x-a)^n$ has interval of conv I.

then $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ is cont on I.

- Addition.

	$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$	$g(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$	$(f \pm g)(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-a)^n$
R	R_f	R_g	$R \geq \min\{R_f, R_g\}$
I	I_f	I_g	$I = I_f \cap I_g$.

- Multiplication by $(x-a)^m$.

$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ $\quad R_f \quad I_f$.

$h(x) = (x-a)^m f(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n+m}$. $\quad R_h = R_f \quad I_h = I_f$.

multiplication of a convergent power series by a constant or finite polynomial doesn't affect convergence. or change radius.

eg. $\alpha \sum_{n=0}^{\infty} b_n (x-a)^n = \sum_{n=0}^{\infty} \alpha b_n (x-a)^n$

$(x-a)^k \sum_{n=0}^{\infty} b_n (x-a)^n = \sum_{n=0}^{\infty} b_n (x-a)^{n+k}$. $\quad$ both have radius convergence R_b.

6.3-6.4. Differentiation & Integration of Power Series.

- Formal

$$\sum_{n=0}^{\infty} n a_n (x-a)^{n-1} = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

$$\sum_{n=0}^{\infty} \int a_n (x-a)^n \, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

- Term-by-term.

$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ $\qquad$ $f'(x) = \sum_{n=0}^{\infty} n a_n (x-a)^{n-1}$

$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ $\qquad$ $\int f(x) = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$

- Thm

(most powerful tool for manipulating pow series)

If $f(x) = \sum_0^{\infty} b_n (x-a)^n$ has radius of conv $R > 0$.

then • f is diff'ble on $(a-R, a+R)$

• $f'(x) = \sum_{n=1}^{\infty} n b_n (x-a)^{n-1}$ $\qquad$ (e.g. $\frac{d}{dx}\sum a_n(x) = \sum \frac{d}{dx} a_n(x)$)

• $\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{b_n (x-a)^{n+1}}{n+1} + C$ $\qquad$ (e.g. $\int \sum a_n(x)\,dx = \sum \int a_n(x)\,dx$)

where both have rad of conv. R.

ex. The series $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$ has radius of convergence of $\underline{R=1}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x|$ as $n \to \infty$ $\qquad$ $\therefore$ ratio test implies $|x| < 1$

$\sum^{\infty} x^{n+3} = x^3 \sum^{\infty} x^n$ also has radius of convergence $R = 1$.

$\ast$ $\sum_{n=0}^{\infty} x^n$ is a geometric series with $A=1$ & $r = x$. It converge to $\frac{A}{1-r} = \frac{1}{1-x}$

$\therefore$ $\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n} = 1 + x + x^2 + \cdots$ $\quad$ for $|x| < 1$

ex. a) find a power series representation of $\frac{x}{(1-x)^2}$

b) compute the exact sum of $\sum_{n=3}^{\infty} \frac{n}{2^n}$

利用 $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$

a) let $f(x) = \frac{1}{1-x}$. $f(x) = \sum_{n=0}^{\infty} x^n$ for $|x|<1$

找与 $\frac{1}{1-x}$ 的关系

求导

by previous thm. $f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$ for $|x|<1$

$x f'(x) = \frac{x}{(1-x)^2} = x \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^n$

i.e $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$ where $|x|<1$.

b) We want $\sum_{n=3}^{\infty} \frac{n}{2^n}$

we found $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n = x + 2x^2 + \sum_{n=3}^{\infty} n x^n$

let $x = \frac{1}{2}$. (在 rad of conv 中).

$\frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = \frac{1}{2} + 2 \times \frac{1}{4} + \sum_{n=3}^{\infty} \frac{n}{2^n} \Rightarrow \sum_{n=3}^{\infty} \frac{n}{2^n} = 1$

$(a-R, a+R) \Rightarrow a=0.\ R=1.$

$\therefore \sum_{n=1}^{\infty} n x^n$ conv for at least $(-1, 1)$

Value at end point should be checked individually:

$x=-1$ $\sum_{n=1}^{\infty} n(-1)^n$ div. since $\lim_{n\to\infty} n(-1)^n$ DNE

$x=1$ $\sum_{n=1}^{\infty} n$ div since $\lim_{n\to\infty} n = \infty$

将 power series 积分/求导 不会改变 R. 但会改变 I. (add or remove endpoints)

Q. Find the ① interval of convergence and the ② actual sum of $\sum_{n=0}^{\infty} (-1)^n (3x)^n$.

① $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(3x)^{n+1}}{(3x)^n} \right| = \lim_{n\to\infty} |3x| < 1$.

$-1 < 3x < 1$

$-\frac{1}{3} < x < \frac{1}{3}$

$\therefore x \in (-\frac{1}{3}, \frac{1}{3})$

② $\sum_{n=1}^{\infty} (-1)^n (3x)^n$

$\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$

$\therefore$ when $a = -3x$

$\text{sum} = \frac{1}{1+3x}$

- Substitution 替换

replace x in a power series 利用 $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

ex. Find a power series representation of $\frac{1}{4+2x}$ centered at 0. What's rad of conv?

$$\frac{1}{4+2x} = \frac{1}{4(1+\frac{x}{2})} = \frac{1}{4(1-(-\frac{x}{2}))}$$

let $u = -\frac{x}{2}$

$$\frac{1}{4}\left(\frac{1}{1-u}\right) = \frac{1}{4}\sum_{n=0}^{\infty} u^n \quad \text{for } |u|<1$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^n}$$

Q. ① Use $\frac{1}{1-x}$ to find a series representation for $\ln(1-x)$.

② What is the interval of convergence?

→ 已知 $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ has interval of convergence $(-1,1)$

$$\frac{d}{dx}\ln(1-x) = -\frac{1}{1-x}$$

→ $\int \frac{1}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\int x^n\,dx$ for $|x|<1$

$\Rightarrow -\ln(1-x)+C = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$ for $|x|<1$.

$x=0$ $C=0$.

$\therefore \ln(1-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$ converges for at least $|x|<1$

→ test endpoint: $x=1$ & $x=-1$

$x=1$ $\sum_{n=0}^{\infty}\frac{1^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{1}{n+1}$ diverge by LCT with $\sum\frac{1}{n}$

$x=-1$ $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$ converge by AST

$\therefore$ Interval of convergence is $[-1,1)$.

ex. Find a power series representation of arctan (x). What's the interval of conv?

$\rightarrow \frac{d}{dx} \arctan(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$

$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ for $|u|<1$.

$\rightarrow$ let $u = -x^2$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$ $\quad |-x^2|<1$

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ $\quad |x^2|<1 \Rightarrow |x|<1$

$\int \frac{1}{1+x^2} dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx$ $\quad |x|<1$

$\arctan x + c = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$x=0 \quad c=0$

$\therefore \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$\rightarrow$ check endpoints: $x=1 \quad x=-1$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converge by AST

$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converge by AST.

$\therefore \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^3}{5} + \cdots$ for $x \in [-1, 1]$

Power Series can be used to compute difficult integrals.

Q. Write $\int_0^1 \arctan(x^4)$ as a power series

$\arctan(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+1}}{2n+1}$

$\Rightarrow \arctan(x^4) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{8n+4}}{2n+1}$

$\Rightarrow \int_0^1 \arctan(x^4) dx = \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+5}}{(2n+1)(8n+5)} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(8n+5)} = \frac{1}{5} - \frac{1}{39} + \frac{1}{105} + \cdots$

6.5 Taylor Polynomial

- Taylor Series.

$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$. can be used to create many other series representations of functions.

给定 $f(x)$, 我们是否可以 create a power series converges to $f(x)$?

Assume $f(x)$ can be written as a power series centred at $x=a$.

and let it converges on the interval $|x-a|<R$ $(R>0)$

Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + \cdots$

at $x=a$. $f(a) = c_0$.

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$

$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + \cdots$

$\vdots$

$f^{(n)}(x) = \frac{n!\,c_n}{0!} + \frac{(n+1)!}{1!} c_{n+1}(x-a) + \frac{(n+2)!}{2!} c_{n+2}(x-a)^2 + \cdots$

$\downarrow$

$f'(a) = c_1$

$f''(a) = 2c_2$

$f'''(a) = 3!\,c_3$

$\vdots$

$f^{(n)}(a) = n!\,c_n$

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!} \quad \text{for } n \geq 0}$$

$$\boxed{(0! = 1 \qquad f^{(0)} = f.)}$$

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a|<R$ $\Rightarrow$ $c_n = \frac{f^{(n)}(a)}{n!}$

$\sum_{n=0}^{\infty} c_n(x-a)^n \Leftrightarrow \exists R>0$ s.t. $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ converge to $f(x)$ for $|x-a|<R$

- nth degree Taylor polynomial.

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

6.6 Taylor's Theorem & errors in approximation

- Taylor Remainder

n times diff'ble at $x=a$

$R_{n,a}(x) = f(x) - T_{n,a}(x)$

↳ n-th degree Taylor remainder function centred at $x=a$.

Error $= |R_{n,a}(x)|$

- Taylor's Theorem

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad c \in (x,a)$$

* $|R_{1,a}(x)| = \left|\frac{f''(c)}{2}(x-a)^2\right|$

* $\forall x \in I.\ \exists c \in (x,a)$

$$f(x) - T_{0,a}(x) = f'(c)(x-a)$$

↳ higher order of MVT. $T_{0,a}(x) = f(a)$.

$$f'(c) = \frac{f(x)-f(a)}{x-a}.$$

* 可通过 $|f^{(n+1)}(c)|$ 得到 c

Q. Find $\lim\limits_{x\to 0} \frac{\sin x - x}{x^2}$

$f(x) = \sin x. \qquad T_{1,0}(x) = T_{2,0}(x) = x.$

$T_{1,0}(x) = \sum\limits_{k=0}^{1} \frac{f^{(k)}(0)}{k!}(x-0)^k = x$

$T_{2,0}(x) = \sum\limits_{k=0}^{2} \frac{f^{(k)}(0)}{k!}(x-0)^k = x + 0 = x.$

$\exists c \in (0,x). \qquad |\sin x - x| = \left|\frac{-\cos(c)}{3!}x^3\right| \le \frac{1}{6}|x|^3$

$\because |-\cos c| \le 1$

$\therefore -\frac{1}{6}|x|^3 \le \sin x - x \le \frac{1}{6}|x|^3 \rightarrow -\frac{|x|}{6} \le \frac{\sin x - x}{x^2} \le \frac{|x|}{6}$

$\because \lim\limits_{x\to 0} -\frac{|x|}{6} = \lim\limits_{x\to 0}\frac{|x|}{6} = 0$

$\therefore$ By Squeeze Theorem $\lim\limits_{0} \frac{\sin x - x}{x^2} = 0$

- Taylor's Approximation Theorem I

Assume $f^{(k+1)}$ cont on $[-1, 1]$.

$\exists\ M > 0$. s.t $|f(x) - T_{k,0}(x)| \le M|x|^{k+1}$

$-M|x|^{k+1} \le f(x) - T_{k,0}(x) \le M|x|^{k+1}$. for each x.

6.7 Taylor Series

- Taylor Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin's and Taylor's Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^r}{r!} f^{(r)}(a) + \ldots$$

$$f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \ldots + \frac{x^r}{r!} f^{(r)}(a) + \ldots$$

✓ $e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots$ for all x $\quad \sum_{k=0}^{n} \frac{x^k}{k!}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots$ $(-1 < x \le 1)$ $\quad \sum_{k=0}^{n} \frac{(-1)^k x^k}{n}$

✓ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots$ for all x $\quad \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

✓ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$ for all x $\quad \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^r \frac{x^{2r+1}}{2r+1} + \ldots$ $(-1 \le x \le 1)$

Q. Find $T_{3,1}(x)$ for $f(x) = \ln x$. Deduce a pattern for $T_{n,1}(x)$

$f(x) = \ln x \qquad f(1) = 0$

$f'(x) = \frac{1}{x} \qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$

$f^{(3)}(x) = \frac{2}{x^3} \qquad f^{(3)}(1) = 2$

$$T_{3,1}(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$$

$$= x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

$f^{(n)}(x) = \frac{(-1)^{n+1}}{x^n}(n-1)! \qquad f^{(n)}(1) = (-1)^{n+1}(n-1)!$

$$T_{n,1}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=1}^{n} \frac{(-1)^{k+1}(k-1)!}{k!}(x-1)^k = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}(x-1)^k$$

$f^{(0)}(1) = 0$

ex. Let $f(x) = \arctan(x^3)$, find $f^{(15)}(0)$ & $f^{(17)}(0)$

sol^n $g(u) = \arctan(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+1}}{2n+1}$ $|u|<1$.

$f(x) = \sum_{k=0}^{\infty} C_k x^k$ $C_k = \frac{f^{(k)}(0)}{k!}$

$f(x) = g(x^3)$ $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \arctan(x^3) = \sum_{n=0}^{\infty} (-1) x^{6n+3}$

by equating coefficients. $k=15 \equiv n=2$ $\frac{f^{(15)}(0)}{15!} = \frac{(-1)^2}{5}$ $f^{(15)}(0) = \frac{15!}{5}$

$k=17 \equiv 6n+3=17$ ← not possible for integer n.

(x^{17} DNE. $\therefore f^{(17)}(0)=0$)

$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots$

$\arctan(x^3) = x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \frac{x^{21}}{7} + \frac{x^{27}}{9} + \cdots$

Q. Find a series representation of $f(x) = \int_0^x \cos(t^2)\,dt$.

$\int \cos(x^2)\,dx$ doesn't have elementary antiderivative.

$\cos(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!}$ $\forall u \in \mathbb{R}$.

$\cos(t^2) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n}}{(2n)!}$

$\int_0^x \cos(t^2)\,dt = \left[\sum_{n=0}^{\infty} \frac{(-1)^n \cdot t^{4n+1}}{(4n+1)(2n)!}\right]_0^x$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}$

Q. $f(10) = \int_0^{10} \cos(t^2)\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n 10^{4n+1}}{(4n+1)(2n)!}$

$\because f(10)$ satisfies AST.

$\therefore$ we can find N. s.t. $\frac{10^{4N+1}}{(4N+1)(2N)!} < 0.001$ $\Leftrightarrow$ $\frac{(4N+1)(2N)!}{10^{4N+1}} > 1000$

$10^{4n+1} >> (2N)!$

$N=136$ $g(136) = 2676\cdots$ $N=135$ $g(135) = 360\cdots$

$\sum_{n=0}^{135} \frac{(-1)^n 10^{4n+1}}{(4n+1)(2n)!}$ have error < 0.001.

验证 $S_{135} = 0.6008\cdots$ $f(10) = 0.6011\cdots$ have error < 0.001

Q. $\lim\limits_{x\to 0} \dfrac{1+3x\ln(1+x^2)-e^{x^2}}{x\sin(x^3)}$

recall: $e^u = 1+u+\dfrac{u^2}{2}+\dfrac{u^3}{3!}+\cdots$

$$\frac{1+3x^2\ln(1+x^2)-e^{x^4}}{x\sin(x^3)} = \frac{1+3x^2\left(x^2-\frac{x^4}{2}+\frac{x^6}{3}+\cdots\right)-\left(1+x^4+\frac{x^8}{2}+\cdots\right)}{x\left(x^3-\frac{x^9}{3!}+\frac{x^{15}}{5!}+\cdots\right)}$$

$$= \frac{3x^4-\frac{3x^6}{2}+x^8-x^4-\frac{x^8}{2}+\cdots}{x^4-\frac{x^{10}}{3!}+\frac{x^{16}}{5!}+\cdots}$$

factor out x^4 let $x\to 0$ to get a limit of 2

6.8 Taylor Convergence

- To determine whether $T_{n,a}(x) \to f(x)$ as $n \to \infty$

  We need to determine $f(x) - T_{n,a}(x)$

$$R_{n,a}(x) = f(x) - T_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad c \in (x, a)$$

若 $\lim_{n\to\infty} R_{n,a}(x) = 0$. 则 $T_{n,a}(x) \to f(x)$ as $n \to \infty$

c is not obtainable $\therefore$ 找 upper bound on $|f^{(n+1)}(c)|$

$|R_{n,a}(x)| \leq \frac{M}{(n+1)!}|x-a|^{x+1}$ whenever $|f^{(n+1)}(c)| \leq M$

- Convergence Theorem for Taylor Series

$\exists M$ s.t. $|f^{(k)}(x)| \leq M$. $\forall k$.

$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \qquad \forall x \in I$

Q. Let $f(x) = \sin x$. Show $T_{n,0}(x) \to \sin(x)$ $\forall x \in \mathbb{R}$.

$\because f(x) = \sin x$

$\therefore |f^{(n+1)}(x)|$ is either $|\cos x|$ or $|\sin x|$

$|f^{(n+1)}(c)| \leq 1 \qquad \forall c.$

$\therefore |R_{n,0}(x)| \leq \frac{|f^{(n+1)}(c)|}{(n+1)!}|x-0|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!}$

$\because b^n \ll n!$

$\therefore \lim_{n\to\infty} \frac{b^n}{n!} = 0$

$\therefore \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \qquad \lim_{n\to\infty} |R_{n,0}(x)| \leq 0$

$\because |\sin x - T_{n,0}(x)| = |R_{n,0}(x)|$

$\therefore \lim_{n\to\infty} T_{n,0}(x) = \sin(x)$ for any $x \in \mathbb{R}$.

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \forall x.$

- def. Taylor Series

Given an infinitely differentiable function f.

We call $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ the Taylor series of f centred at $x=a$

Taylor series isn't $f(x)$. it could be
- $f(x)$ $\forall x \in \mathbb{R}$
- $f(x)$ over $|x-a|<R$
- only equal at $f(a)$

ep. $f(x) = \ln x$ has a Taylor series centred at 1 of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$

by ratio test.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ $\quad\quad \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n}\right| = |x-1| < 1$

$\therefore \forall x \in (0,2)$. $\quad \ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^{n+1}$

check endpoint $x \in (0,2]$

The series diverge outside the interval $(0,2]$

ep. $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x=0 \end{cases}$ satisfy $f^{(n)}(0) = 0$ $\forall n \geq 0$

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}}}{h}$$

let $u = \frac{1}{h}$ $\quad = \lim_{u\to\pm\infty} u e^{-u^2}$

$= 0$

$\therefore \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{0 \cdot x^n}{n!} = 0$ $\quad$ conv $\forall x$

$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x=0 \end{cases}$ has a Taylor series

6.9 Binomial Series

- Binomial Theorem

$$(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k \qquad \binom{n}{k} = \begin{cases} \dfrac{n!}{k!\,(n-k)!} & n \geq k \\ 0 & n < k \end{cases}$$

binomial coefficient

$$f'(x) = b\,(1+x)^{b-1}$$

$$f''(x) = b\,(b-1)\,(1+x)^{b-2}$$

$$\vdots$$

$$f^{(n)}(x) = \underbrace{b \cdot (b-1) \cdots (b-(n-1))}_{n \text{ terms}} \quad (1+x)^{b-n}$$

Q. For what x does this conv? For what val?

Use ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\binom{b}{n+1} x^{n+1}}{\binom{b}{n} x^n}\right| = |x| \left|\frac{b(b-1)(b-2)\cdots(b-n)}{(n+1)!} \cdot \frac{n!}{b(b-1)\cdots(b-(n-1))}\right| = |x| \left|\frac{b-n}{n+1}\right|$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x| \left|\frac{b-n}{n+1}\right| = |x|$$

conv when $|x| < 1$

$b \geq 0 \quad b \notin \mathbb{N}$	$I = [-1, 1]$
$-1 < b < 0$	$I = (-1, 1]$
$b \leq -1$	$I = (-1, 1)$
$b \in \mathbb{N}$	$I = (-\infty, \infty)$

Q. To what does $\sum_0^\infty \binom{b}{n} x^n$ conv?

Assume $|x|<1$ Let $f(x)=\sum_0^\infty \binom{b}{n} x^n$.

$(n+1)\binom{b}{n+1} = \binom{b}{n}(b-n)$

proof: Given $f(x)=\sum_{n=0}^\infty \binom{b}{n} x^n$

$$f'(x) = \sum_0^\infty n\binom{b}{n}x^{n-1} = \sum_{n=0}^\infty (n+1)\binom{b}{n+1}x^n \quad \text{(by reindexing)}$$

$$= \sum_{n=0}^\infty (b-n)\binom{b}{n}x^n \quad \text{(by the aside)}$$

$$= b\sum_{n=0}^\infty \binom{b}{n}x^n - \sum_{n=0}^\infty n\binom{b}{n}x^n$$

$$= bf(x) - x\sum_{n=0}^\infty n\binom{b}{n}x^{n-1}$$

$$= bf(x) - xf'(x)$$

$\therefore f'(x) = bf(x) - xf'(x)$

$$f'(x) = \frac{bf(x)}{1+x}.$$

As it is a separable DE.

$\therefore \frac{df}{dx} = \frac{bf}{1+x}$

$$\int \frac{df}{f} = b\int \frac{1}{1+x}\,dx$$

$$\ln|f| = b\ln|1+x| + c$$

$$\ln|f| = \ln|1+x|^b + c$$

$$f(x) = A(1+x)^b \qquad A = \pm e^c$$

$\because f(x) = \sum_{n=0}^\infty \binom{b}{n}x^n = 1 + bx + \frac{b(b-1)}{2}x^2 + \cdots$

$f(0)=1 \qquad A=1$

$f(x) = (1+x)^b.$

$\therefore (1+x)^b = \sum_{n=0}^\infty \binom{b}{n}x^n \qquad |x|<1$

6.10 Applications of Taylor Series

- 找无理数 in power series representation
- 计算非 geometric / telescoping in 和
- approximate integrals that have no obvious 积分
- evaluate limits

Q. $(1+x)^b = \sum_{n=0}^{\infty} \binom{b}{n} x^n$ when $|x|<1$

Let $b=-\frac{1}{2}$ 将 $x \to x^2$

$$(1-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n \qquad |-x|^2<1$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n x^n$$

$$\int_0^x \frac{1}{\sqrt{1-t^2}}\,dt = \int_0^x \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n t^{2n}\,dt$$

$$\left[\arcsin(t)\right]_0^x = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} \qquad |x|<1$$

$$\arcsin(x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$$

when $x=\frac{1}{2}$. $\quad \frac{\pi}{6} = \arcsin\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \frac{1}{2^{2n+1}(2n+1)}$

$$\pi = 6 \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \frac{(-1)^n}{4^n(2n+1)} = 3\left(1 - \frac{(-\frac{1}{2})}{12}\right) \frac{(-1)^n}{4^n(2n+1)}$$

Q. Find the sum of following series.

a) $\sum_{n=1}^{\infty} \frac{n^3}{e^n}$ b) $\sum_{n=2}^{\infty} \frac{(-1)^n \pi^{2n-4}}{(2n+1)!}$

试着找与 $\frac{1}{1-x}$, e^x, $\sin x$, $\cos x$, $(1+x)^b$ 的联系

a) $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} \quad |x|<1$

两边同乘 x $\quad \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$

now another $\frac{d}{dx}$ $\quad \frac{1-x}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^{n-1} \quad |x|<1$

now another $\frac{d}{dx}$ $\quad \frac{x(1-x)}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n \quad |x|<1$

now another $\frac{d}{dx}$ $\quad \frac{x^2+4x+1}{(1-x)^4} = \sum_{n=1}^{\infty} n^3 x^{n-1} \quad |x|<1$

Let $x = \frac{1}{e}$. $\quad \frac{(\frac{1}{e})^2 + 4(\frac{1}{e}) + 1}{(1-\frac{1}{e})^4} = \sum_{n=1}^{\infty} \frac{n^3}{e^n}$

b) $\sum_{n=2}^{\infty} \frac{(-1)^n \pi^{2n-4}}{(2n-1)!} \qquad \sin x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$

$= \frac{1}{\pi^3} \sum_{n=2}^{\infty} \frac{(-1)^n \pi^{2n-1}}{(2n-1)!}$

$= \frac{1}{\pi^3}\left[\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n-1}}{(2n-1)!}}_{= -\sin\pi} - (-\pi)\right]$

$= \frac{1}{\pi^3}[0 + \pi]$

$= \frac{1}{\pi^2}$