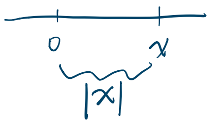
1.1 Absolute Value & Inequalities 不等式

- def. $|x|\ (x \in \mathbb{R}) = \begin{cases} x & (x \geq 0) \\ -x & (x < 0) \end{cases}$

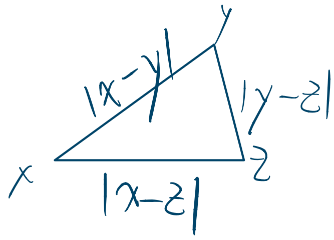
- distance



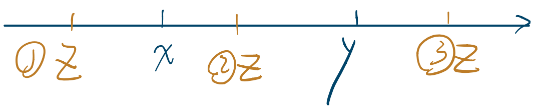
$|b-a| = |a-b|$

- triangle inequality

$|x-y| \leq |x-z| + |z-y| \quad (\forall\ x, y, z \in \mathbb{R})$



proof.



① $z < x < y$ $\quad |x-y| \leq |z-y| \quad |x-y| \leq |x-z| + |z-y|$

② $x < z < y$ $\quad |x-y| = |x-z| + |z-y| \quad |x-y| \leq |x-z| + |z-y|$

③ $x < y < z$ $\quad |x-y| \leq |x-z| + |y-z|$

$\forall\ (a, b \in \mathbb{R})$, $|a+b| \leq |a| + |b|$ $\qquad |a-b| \leq |a| + |b|$

proof. let $x = a$, $y = -b$, $z = 0$

then, $|a-(-b)| \leq |a-0| + |0-(-b)|$

$\Rightarrow |a+b| \leq |a| + |b|$

- reverse inequality

proof. $||a|-|b|| \leq |a+b| \quad (a, b \in \mathbb{R})$

- $|a| = |a+b-b|$

$|a| = |\underbrace{a+b}_{x} + \underbrace{(-b)}_{y}| \leq |a+b| + |-b|$

$|a| \leq |a+b| + |b|$

$|a| - |b| \leq |a+b|$

- $||a|-|b|| = \begin{cases} |a|-|b| \geq 0 & |a|-|b| \\ |a|-|b| < 0 & |b|-|a| \end{cases}$

$||a|-|b|| \leq |a+b|$

proof. $||a|-|b|| \leq |a-b|$

- $|a| = |a-b+b| \leq |a-b| + |b|$

$\therefore |a| - |b| \leq |a-b|$

$|b| = |b-a+a| \leq |b-a| + |a|$

$\therefore |b| - |a| \leq |b-a|$

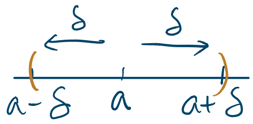
$|a| - |b| \geq -|a-b|$

- $-|a-b| \leq |a| - |b| \leq |a-b|$

which implies $||a|-|b|| \leq |a-b|$

- Important inequalities

$|x-a| < \delta$



$\Rightarrow x \in (a-\delta, a+\delta)$

$|x-a| \leq \delta \qquad -\delta \leq x-a \leq \delta \qquad a-\delta \leq x \leq a+\delta \qquad x \in [a-\delta, a+\delta]$

$0 < |x-a| < \delta \qquad |x-a| > 0 \qquad x \neq a$

$\Downarrow$

$(a-\delta, a) \cup (a, a+\delta)$

ex. $|x-1| - |x-2| \geq 2$

① $x < 1$

$-(x-1)+(x-2) \geq 2$

$\Rightarrow -1 \geq 2$

② $1 \leq x \leq 2$

$(x-1)+(x-2) \geq 2$

$x \geq \frac{5}{2}$

$\frac{5}{2} > 2$ no solution

③ $x > 2$

$(x-1)-(x-2) \geq 2$

$1 \geq 2$ no solution

$|a+b| \geq |a|-|b|$

$|a+b| \leq |a|+|b|$

$|a-b| \leq |a|+|b|$

$|a-b| \geq |a|-|b|$

$|a|+|b|$ 最大 $\qquad |a|-|b|$ 最小

$$|a|-|b| \leq \left\{ \begin{matrix} |a+b| \\ |a-b| \end{matrix} \right\} \leq |a|+|b|$$

解绝对值不等式

- 去绝对值
  - 讨论正负性 $>0 \,/\, <0$
  - 平方
    - $(|x|)^2 = x^2$
    - $(|a|-|b|)\times(|a|+|b|) = a^2-b^2$

解完要检查定义

解分式不等式

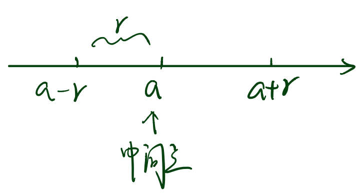
① 分母≠0

② 分式 ⇒ 整式 (同乘分母)

Triangle inequality $|a+b| \le |a|+|b|$

$|x-y| \le |x-z|+|y-z|$

- 证明 ☆ $|a| = |a+b-b|$
  $= |(a+b)+(-b)|$
  $\le |a+b|+|-b|$
- 解不等式 $|x-a|<r$
  - 代数 $-r<x-a<r$
  - 几何



Cosine inequality $2|xy| \le x^2+y^2$

ex. $|x-1|-|x-3| \ge 5$ (*)

case 1. consider $x \in (-\infty, 1)$

(*) ⇒ $-(x-1)+(x-3) \ge 5$

$-2 \ge 5$ (impossible)

So, No solution on $(-\infty, 1)$

case 2. ....

Since $[1,3] \cap [\frac{9}{2}, \infty] = \emptyset$

No solution on $[1,3]$

Overall, no solution for the inequality

ex. $\frac{1+e^x}{1-e^x} < 2$ ($1-e^x \ne 0$ $x \ne 0$)

case 1. $1-e^x>0$, $e^x<1$ $x<0$

(*) ⇒ $1+e^x < 2(1-e^x)$

$e^x < \frac{1}{3}$

$x < \ln\frac{1}{3}$

case 2. $1-e^x<0$ ...... $x > \ln\frac{1}{3}$

Overall, the solution to the inequality is $(-\infty, \ln\frac{1}{3}) \cup (0, \infty)$

ex. 证 $|a|-|b| \le |a+b|$

$|a| = |a+b-b|$

$= |(a+b)+(-b)|$

$\le |a+b|+|-b|$ (due to triangle inequality)

So, $|a|-|b| \le |a+b|$

ex. 解 $|x+1|<1$

法1 (代数) $-1 < x+1 < 1$

法2 (几何)

1.2 Sequence

1.2.1 def.

a set of ordered numbers, be presented as $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \cdots\}$ can be defined explicity or recursively

explicit $a_n = \frac{1}{n}$

$\{(-1)^n\}_{n=1}^{\infty} \Rightarrow -1, 1, -1, 1, \cdots$ $\{\frac{1}{n+1}\}_{n=1}^{\infty} \Rightarrow \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$

每项根据{ }中式子而定 可直接算出第n项

recursive: $a_1=1,\ a_n = 2a_{n-1}\ (n \geq 1)$

fibonacci:
$a_1=1,\ a_n=1,\ a_{n+2}=a_{n+1}+a_n, \cdots$ $\Rightarrow 1, 1, 2, 3, 5 \cdots$

每项根据前一项而定 ex. $a_{n+1}=a_n+b$ 第n项，必须算到n-1项

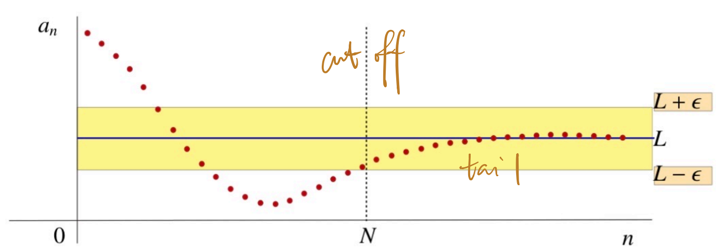
1.2.2 Recursive Sequence 递归

1.2.3 Subsequence and Tails

- given a sequence $\{a_n\}$ and $k \in \mathbb{N}$, the sequence $\{a_k, a_{k+1}, a_{k+2} \cdots\}$ is called **tail** of $\{a_n\}$ with **cut off** N

ex. $a_n = \frac{1}{n} \Rightarrow \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$
tail from $n=10 \Rightarrow \{\frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \cdots\}$

ex. given $\{\frac{1}{n}\}_{n=1}^{\infty}$
subsequence: $\{\frac{1}{2k}\}_{n=1}^{\infty}$ & $\{\frac{1}{2k+1}\}_{n=1}^{\infty}$

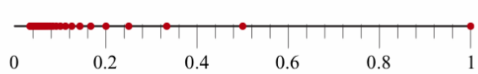


- **Limit (L) 极限**

Sequence 无限趋近于L

def. limit of $\{a_n\}$ is L, if $n \to +\infty$, a_n gets infinitely close to L"

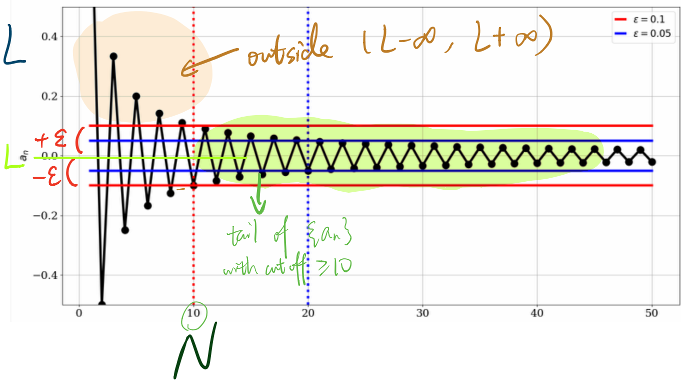
= Asymptotic



1.2.4 Limit of a sequence 极限

收敛 if L exist, then $\{a_n\}$ converges, $\lim_{n\to\infty} a_n = L$
↳ 有极限

发散 if L not exist, then $\{a_n\}$ diverges.
↳ 无极限



– formal definition of limit I

def. For all $\varepsilon > 0$. there exists $N \in \mathbb{N}$, such that if $n \geq N$, then $|a_n - L| < \varepsilon$

- L exists, sequence convergent, $\lim_{n\to\infty} a_n = L$
- L not exists, ~ diverges.

$a_n \leq a_N < \varepsilon$

– formal definition of limit II (Equivalent definitions of the limit)

def $\lim_{n\to\infty} a_n = L$ if, for all $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of the sequence $\{a_n\}$

Since this is true for all $\varepsilon > 0$, if we can pick any open interval (a, b) which contain L, then we can find a small enough ε, such that:

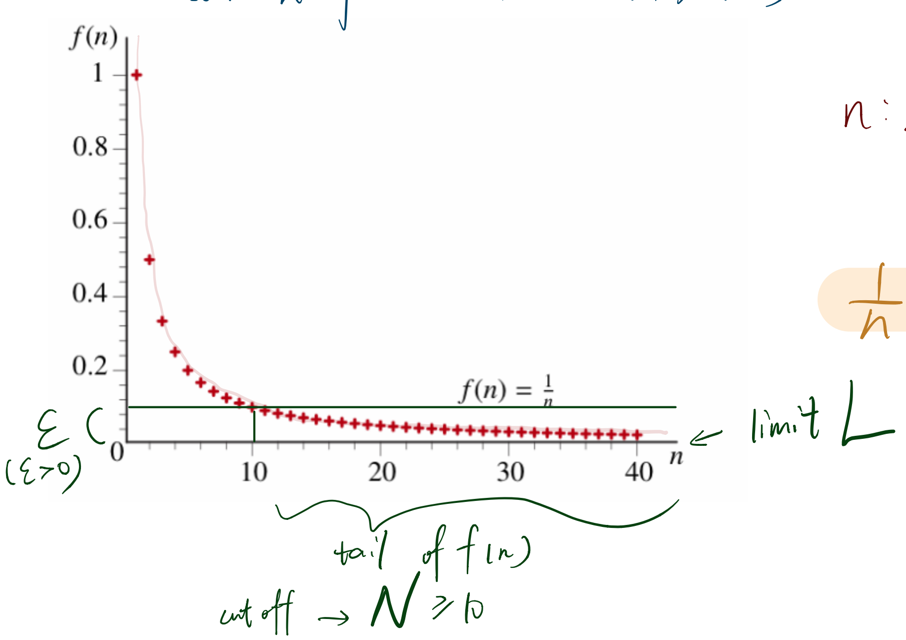
$(L-\varepsilon, L+\varepsilon) \subseteq (a, b)$

a L−ε L L+ε b

– Proposition: the harmonic sequence converges

($a_n = \frac{1}{n}$ for all a, ex. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots$)

期中必考



n: Sequence 中的值

$\frac{1}{n} \leq \frac{1}{N} < \varepsilon$

※ 证明 $\lim_{n\to\infty} a_n = L$

思路

$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geq N \Rightarrow |a_n - L| < \varepsilon$

$a_n \leq a_N < \varepsilon$

Step 1. 打草稿

$|a_n - L| < \varepsilon$ → 解不等式 → $n > \cdots$ → $N > \cdots$

* 判断 $\cdots N < \varepsilon$，再写成 $N > \cdots \varepsilon$ 的形式

Step 2 证明

let $\varepsilon > 0$.
choose $N \in \mathbb{N}$ $\quad N > \cdots$
assume $n \geq N$ $\quad n \geq N > \cdots \quad n > \cdots$

$|a_n - L| = \cdots\cdots \quad (n \in \mathbb{N})$

$\frac{1}{a_n} > \frac{1}{\varepsilon} > 0 \qquad a_n < \varepsilon$

ex. Proof that $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Step 1.

$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ n \geq N$
$\left|\frac{1}{n^2} - 0\right| < \varepsilon$

Step 2.

$\left|\frac{1}{n^2} - 0\right| < \varepsilon \quad 0 < \frac{1}{n^2} < \varepsilon \quad n^2 > \frac{1}{\varepsilon} > 0$

$\Rightarrow n > \frac{1}{\sqrt{\varepsilon}} \qquad N \geq \frac{1}{\sqrt{\varepsilon}}$

Step 3.

let $\varepsilon > 0$
choose $N \in \mathbb{N}$ $\quad N > \frac{1}{\sqrt{\varepsilon}}$
assume $n \geq N$ $\quad n \geq N > \frac{1}{\sqrt{\varepsilon}} \quad n > \frac{1}{\sqrt{\varepsilon}}$

$\Rightarrow \left|\frac{1}{n^2} - 0\right| = \left|\frac{1}{n^2}\right| = \frac{1}{n^2} \quad (n \in \mathbb{N})$

$\frac{1}{n^2} \leq \frac{1}{N^2} < \frac{1}{\left(\frac{1}{\sqrt{\varepsilon}}\right)^2} \qquad \frac{1}{n^2} < \varepsilon \qquad$ QED

Guess : $L = 0$
$|a_n - L| < \varepsilon \qquad \left|\frac{1}{n} - 0\right| < \varepsilon \qquad \frac{1}{n} < \varepsilon \qquad \frac{1}{\varepsilon} < n$

Choose N so that $N > \frac{1}{\varepsilon}$

Proof: Let $\varepsilon > 0$ be arbitrary. Set N to be any positive integer with property that $N > \frac{1}{\varepsilon}$
Then for any integer $n \geq N$. $|a_n - 0| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$
where $\frac{1}{n} \leq \frac{1}{N}$ since $n \geq N$, $\frac{1}{N} < \varepsilon$ since $\frac{1}{\varepsilon} < N$ by construction
Therefore, $\lim_{n\to\infty} \frac{1}{n} = 0$

ex. proof. let $\varepsilon > 0$ be given. Let $N >$ $\quad$, $N \in \mathbb{N}$
Then, if $n \geq N$, $|a_n - L| = \left|\frac{3n+3}{2n+1} - \frac{3}{2}\right| = \left|\frac{6n+6-6n-3}{4n+2}\right|$

$= \frac{3}{4n+2} \leq \frac{3}{4N+2} < \frac{3}{4\left[\frac{1}{4}\left(\frac{3}{\varepsilon} - 2\right)\right] + 2}$

$= \frac{3}{\left(\frac{3}{\varepsilon} - 2\right) + 2} = \varepsilon$

$\frac{3}{4N+2} < \varepsilon$

$N > \frac{1}{4}\left(\frac{3}{\varepsilon} - 2\right)$

ex. Prove $\lim_{n\to\infty} \frac{n^2}{3n^2+7n} = \frac{1}{3}$

- let $\varepsilon > 0$ be given. Let $N > \frac{7}{9\varepsilon}$, $N \in \mathbb{N}$
Then, if $n \geq N$, we have

$$|a_n - L| = \left|\frac{n^2}{3n^2+7n} - \frac{1}{3}\right| = \left|\frac{3n^2-(3n^2+7n)}{9n^2+21n}\right| = \left|\frac{-7n}{9n^2+21n}\right|$$

$$= \frac{7}{9n+21} < \frac{7}{9n} \leq \frac{7}{9N} < \frac{7}{9\times\frac{7}{9\varepsilon}} = \varepsilon \text{ as required}$$

Aside
$\frac{7}{9N} < \varepsilon$
$N > \frac{7}{9\varepsilon}$

✗ 证明极限是否存在

Case 1. Limit exists as L

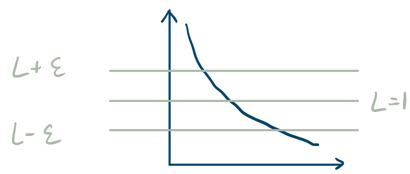
证 $\lim_{n\to\infty} a_n$ 存在且等于题目给的 L 　　　* 找到一定会存在的 N

Case 2. Limit does not exist as L

证 $\lim_{n\to\infty} a_n \neq L$ for some $L \in \mathbb{R}$

ex. $\lim_{n\to\infty} \frac{1}{n} = 0$ 　　$\lim_{n\to\infty} \frac{1}{n} \neq 1$

$\lim_{n=\infty} \frac{1}{n} = 1$ 的定义不成立

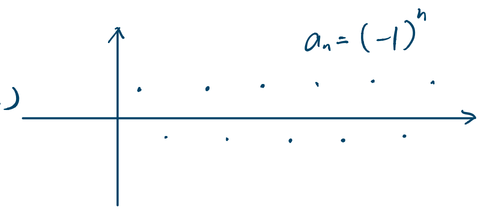


$\Rightarrow$ such cutoff N never exists

$\Rightarrow$ $L=1$ is wrong limit value

Case 3. Limit does not exist

$\forall L \in \mathbb{R}$ 　$\lim_{n\to\infty} a_n \neq L$ (任何 constant 都不是正确的 L)



ex. discrete logistic growth equation $a_{n+1} = ra_n(1-a_n)$
$(1, \frac{1}{2}, 2, \frac{1}{3}, 3, \cdots)$

1.2.5 Formal definition of Divergence 发散

↑ 永远存在 cutoff

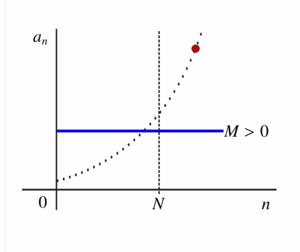
− diverge to infinity 无上下限

1) divergence to $+\infty$ 向∞发散

$\forall M>0$, we can find cutoff $N \in \mathbb{N}$.

So that if $n \geq N$, then $a_n < M$

$$\boxed{\lim_{n\to\infty} a_n = +\infty}$$

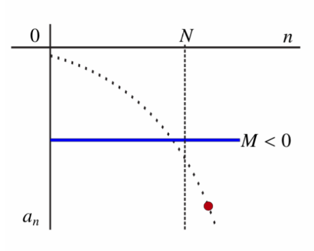


2) divergence to $-\infty$ 向$-\infty$发散

$\forall M<0$, we can find cutoff $N \in \mathbb{N}$.

So that if $n \geq N$, then $a_n < -M$

$$\boxed{\lim_{n\to\infty} a_n = -\infty}$$



ex. Prove $\lim_{n\to\infty}(1-2^n) = -\infty$

− 条件 $\forall M<0$, $\exists N \in \mathbb{N}$ $n \geq N \Rightarrow 1-2^n < M$

− 草稿 $1-2^n < M$ $2^n > 1-M$ $n > \log_2(1-M)$

− 证明

Let $M<0$, and $N > \log_2(1-M)$ $(N \in \mathbb{N})$

Assume $n \geq N$, $n \geq N > \log_2(1-M)$

$2^n > 1-M$

$1-2^n < M$ as required

草稿 $a_n < M$ $N > \cdots M$

Let $M<0$ $(-\infty)$

Assume $n \geq N$ $(N \in \mathbb{N})$ $N > \cdots M$

得出 $a_n < M$ as required

☆ex. If $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty}\sqrt{a_n} = \sqrt{L}$

$\equiv$ Prove $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $n \geq N$ $|\sqrt{a_n} - \sqrt{L}| < \varepsilon$

Case 1. $L=0$ $\lim_{n\to\infty} a_n = 0$. Let $\varepsilon > 0$ be given.

We can find $n > N$ $(N \in \mathbb{N})$, $|a_n - 0| < \varepsilon^2$ $\sqrt{a_n} < \varepsilon$

Case 2. $L \neq 0$ $\lim_{n\to\infty} a_n = L$ Let $\varepsilon > 0$ be given.

We can find $n > N$ $(N \in \mathbb{N})$, $|a_n - L| < \varepsilon\sqrt{L}$

$a_n - L = (\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})$

$$|\sqrt{a_n} - \sqrt{L}| = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} < \frac{\varepsilon\sqrt{L}}{\sqrt{a_n} + \sqrt{L}} < \frac{\varepsilon\sqrt{L}}{\sqrt{L}} = \varepsilon$$

- Subsequence test 存在2个以上极限

For a sequence a_n, if there exists 2 subsequences that converge to different limits, then a_n diverges.

ex. $a_n = (-1)^n$

$a_{2n} = (-1)^{2n} = 1$ $\qquad$ $a_{2n+1} = (-1)^{2n+1} = -1$

$\lim\limits_{n\to\infty} a_{2n} = 1$ $\qquad$ $\lim\limits_{n\to\infty} a_{2n+1} = -1$

$\lim\limits_{n\to\infty} a_{2n} \neq \lim\limits_{n\to\infty} a_{2n+1}$

$\therefore a_n$ is divergent

确定

$\infty + \infty = \infty$

$\infty + c = \infty$

$\frac{c}{0} = +\infty$

$c \times \infty = +\infty$

不确定

$\frac{\infty}{\infty}$, $\infty - \infty$, $\infty \cdot 0$, $\frac{0}{0}$

1.2.6 Arithmetic for limits of Sequence 计算极限

- Rules

$\lim_{n\to\infty} a_n = L \qquad \lim_{n\to\infty} b_n = M$

① For any $a_n = c \ (c\in\mathbb{R})$, $c = L$

② For any $c\in\mathbb{R}$, $\lim_{n\to\infty} c\, a_n = cL$

③ $\lim_{n\to\infty} (a_n \pm b_n) = L \pm M$

⑤ $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} \quad (M\neq 0)$

④ $\lim_{n\to\infty} a_n b_n = LM$

⑥ $a_n \geq 0$ for all n. If $a>0$, then $\lim_{n\to\infty} (a_n)^a = L^a$

⑦ For any $k\in\mathbb{N}$, $\lim_{n\to\infty} a_{n+k} = L$

⑧ If $\alpha > 0$, then $\lim_{n\to\infty} n^\alpha = \infty$. If $\alpha < 0$, then $\lim_{n\to\infty} n^\alpha = 0$.

- Proof.

- Proof if $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$ $(L, m \in \mathbb{R})$, then $\lim_{n\to\infty} (a_n + b_n) = L+M$

Let $\varepsilon > 0$ be given $\quad (|a_n + b_n - (L+M)| < \varepsilon)$

If $n \geq N_1$. $(N_1 \in \mathbb{N})$, then $|a_n - L| < \frac{\varepsilon}{2}$

If $n \geq N_2$. $(N_2 \in \mathbb{N})$, then $|b_n - M| < \frac{\varepsilon}{2}$

Let $N = \max\{N_1, N_2\}$, if $n \geq N$,

We get $|a_n + b_n - (L+M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

- Proof if $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$ $(L, m \in \mathbb{R})$, then $\lim_{n\to\infty} (a_n \cdot b_n) = LM$

$|a_n \cdot b_n - LM| = |a_n \cdot b_n - L\cdot b_n + L\cdot b_n - LM| \leq |a_n - L\cdot b_n| + |L\cdot b_n - M|$

$\exists N_1 > 0$. such that if $0 < |x-a| < N_1$. then $|b_n - M| < \frac{\varepsilon}{2|1+L|}$

$\exists N_2 > 0$. such that if $0 < |x-a| < N_2$, then $|b_n - M| < 1$

$\exists N_3 > 0$. such that $|a_n - L| < \frac{\varepsilon}{2(1+|M|)}$

Let $N = \min\{N_1, N_2, N_3\}$, if $n \geq N$

We get $|a_n \cdot b_n - LM| \leq |a_n - L| \cdot |b_n| + |L| \cdot |b_n - M|$

$< \frac{\varepsilon}{2\times(1+|M|)} \times (1+|M|) + |L| \times \frac{\varepsilon}{2\times(1+|L|)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

- Proof: If $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then $\lim_{n\to\infty} a_n = 0$

suppose $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = k$, for some $k \in \mathbb{R}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n \cdot \frac{b_n}{b_n} = \lim_{n\to\infty} \left(\frac{a_n}{b_n}\right)(b_n) = L \cdot 0 = 0 \qquad \text{QED}$$

相反，If $\lim_{n\to\infty} b_n = 0$ & $\lim_{n\to\infty} a_n \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n}$ DNE.

- 计算 limit

Let

$$a_n = \frac{b_0 + b_1 n + b_2 n^2 + b_3 n^3 + \cdots + b_j n^j}{c_0 + c_1 n + c_2 n^2 + \cdots + c_k n^k}.$$

Consider the sequence $\{a_n\}$. By factoring out n^j from the numerator and n^k from the denominator and rewriting the sequence as

$$a_n = \frac{n^j}{n^k}\left[\frac{\frac{b_0}{n^j} + \frac{b_1}{n^{j-1}} + \frac{b_2}{n^{j-2}} + \frac{b_3}{n^{j-3}} + \cdots + b_j}{\frac{c_0}{n^k} + \frac{c_1}{n^{k-1}} + \frac{c_2}{n^{k-2}} + \cdots + c_k}\right],$$

we can show that

$$\lim_{n\to\infty} a_n = \begin{cases} \frac{b_j}{c_k} & \text{if } j = k \\ 0 & \text{if } j < k \\ \infty & \text{if } j > k \text{ and } \frac{b_j}{c_k} > 0 \\ -\infty & \text{if } j > k \text{ and } \frac{b_j}{c_k} < 0. \end{cases}$$

Then we get that for large n

$$\frac{b_0 + b_1 n + b_2 n^2 + b_3 n^3 + \cdots + b_j n^j}{c_0 + c_1 n + c_2 n^2 + \cdots + c_k n^k} \sim \frac{b_j n^j}{c_k n^k} = \frac{b_j}{c_k} n^{j-k}.$$

* 做题思路

求 $\lim_{n\to\infty} \frac{\cdots}{\cdots}$，上下同除最高系数 $\frac{(n^3+n^2+1)\div n^3}{(n^3+1)\div n^3}$

ex. $\lim_{n\to\infty} \frac{n^3+n^2+1}{2n^3+7n^2-1} = \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^3}}{2+\frac{7}{n}-\frac{1}{n^3}} = \frac{1}{2}$

ex. $\lim_{n\to\infty} \sqrt{n^2+4} - n$

$$= \lim_{n\to\infty} (\sqrt{n^2+4} - n) \times \frac{\sqrt{n^2+4}+n}{\sqrt{n^2+4}+n} = \lim_{n\to\infty} \frac{n^2+4-n^2}{\sqrt{n^2+4}+n} = \frac{4}{\sqrt{n^2+4}+n} = 0$$

Sequence.

- explicitly defined ~ (由式子而定)
- recursively defined ~ (由前一项有关)

专业术语

pricewise 分段的

counterexample 反例

Subsequence (由原 Sequence 中截取)

original 满足的性质，sub 一定满足。 sub 满足的，original 不一定。

- tail of sequence (一定是 subsequence)

ex. $\{\frac{1}{n}\}_{n=1}^{\infty}$ $k=1 \to \{\frac{1}{n}\}_{n=1}^{\infty}$ $k=100 \to \{\frac{1}{n}\}_{n=100}^{\infty}$

性质

- 单调性

当 $n<m$, $a_n<a_m$ 增 sequence ↑

当 $n<m$, $a_n>a_m$ 减 sequence ↓

- 有限性 bundary

lower bound upper bound

There exists $A, B \in \mathbb{R}$ $A \le a_n \le B$ $(\forall n \in \mathbb{N})$, $\{a_n\}$ is bounded

若仅 A 存在，$\{a_n\}$ is lower bounded

证明 limit

Convergence 收敛

草稿 $|a_n - L| < \varepsilon$

$N > \cdots \varepsilon$

Let $\varepsilon > 0$

Assume $n \geq N$ $(N \in \mathbb{N})$ $N > \cdots \varepsilon$

$|a_n - L| = \cdots$ $a_n \le a_N < \varepsilon$

Divergence 发散

草稿 $a_n < M$ $N > \cdots M$

Let $M < 0$ $(-\infty)$

Assume $n \geq N$ $(N \in \mathbb{N})$ $N > \cdots M$

得出 $a_n < M$ as required

1.3 Squeeze Theorem 三明治定理

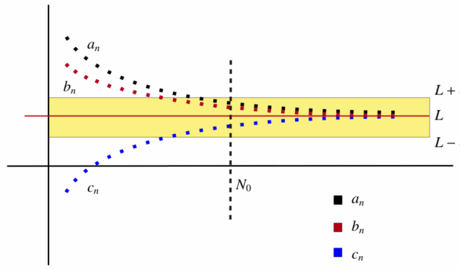
- theorem

If $a_n \le b_n \le c_n$, $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, then $\lim\limits_{n\to\infty} b_n = L$ 存在且相等

proof. let $\varepsilon > 0$ be given. Since $a_n \to L$, $c_n \to L$
we can find $N \in \mathbb{N}$ s.t if $n \ge N_0$, then

$L - \varepsilon < a_n < L + \varepsilon$, $\quad L - \varepsilon < c_n < b + \varepsilon$

Then, for $n \ge N_0$,
$L - \varepsilon < a_n \le b_n \le c_n < L + \varepsilon$



- Proof

$|a_n - L| < \varepsilon$, find $n \ge$?

- Let $\varepsilon > 0$ $\qquad$ Assume $b_n \le a_n \le c_n$.

  $\lim\limits_{n\to\infty} b_n = L$ which means there exists $N_1 \in \mathbb{N}$ s.t. $n \ge N_1 \Rightarrow |b_n - L| < \varepsilon$.

  $\lim\limits_{n\to\infty} c_n = L$ which means there exists $N_2 \in \mathbb{N}$ s.t. $n \ge N_2 \Rightarrow |c_n - L| < \varepsilon$

- choose $N = \max\{N_1, N_2\}$

  Assume $n \in \mathbb{N}$ $\quad n \ge \max\{N_1, N_2\} \Rightarrow \begin{cases} n \ge N_1 & L - \varepsilon < b_n < L + \varepsilon \\ n \ge N_2 & L - \varepsilon < c_n < L + \varepsilon \end{cases}$

- Since $b_n \le a_n \le c_n$ $\quad L - \varepsilon < b_n \le a_n \le c_n < L + \varepsilon$. $\quad L - \varepsilon < a_n < L + \varepsilon$
  $\Rightarrow |a_n - L| < \varepsilon$

ex. 找 $a_n = \dfrac{(-1)^n}{n^2+1}$ in limit

$\dfrac{-1}{n^2+1} \le \dfrac{(-1)^n}{n^2+1} \le \dfrac{1}{n^2+1}$ $\qquad$ 找 lower & upper bound

$\lim\limits_{n\to\infty} \dfrac{-1}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{1}{n^2+1} = 0$. So $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n^2+1} = 0$

* ex. 找 $a_n = \sqrt{n^2+n} - n$ in limit

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{(n^2+n) - n^2}{\sqrt{n^2+n} + n} = \lim\limits_{n\to\infty} \dfrac{n}{\sqrt{n^2+n} + n}$

$= \lim\limits_{n\to\infty} \dfrac{\frac{n}{n}}{\frac{\sqrt{n^2+n}}{n} + 1} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{n}} + 1} = \dfrac{1}{2}$

☆ $\sqrt{a} - \sqrt{b} = \dfrac{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})}{\sqrt{a} + \sqrt{b}}$
$= \dfrac{a - b}{\sqrt{a} + \sqrt{b}}$

1.4 Monotone convergence theorem 单调性

Upper and Lower Bounds

Let $S \subset \mathbb{R}$. We say that α is an *upper bound* of S if

$$x \leq \alpha$$

for every $x \in S$. If S has an upper bound, we say that it is *bounded above*.

We say that β is a *lower bound* of S if

$$\beta \leq x$$

for every $x \in S$. If S has a lower bound, we say that it is *bounded below*.

S is *bounded* if it is bounded both above and below. Note that S is bounded if there exists an M such that

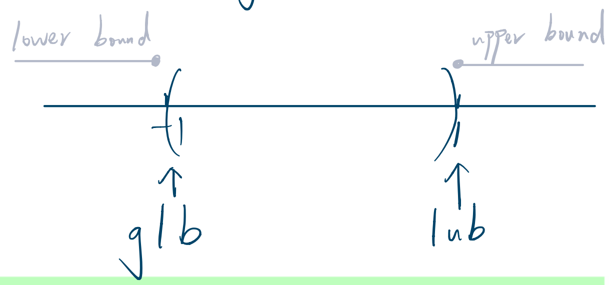
$$S \subseteq [-M, M]$$

ex. $S = (-1, 1)$

lower bound : 所有 ≤ -1 的数

upper bound : 所有 ≥ 1 的数

$glb(S) = -1$ $lub(S) = 1$



Greatest Lower Bound

Let $S \subseteq \mathbb{R}$. Then β is called the *greatest lower bound* of S if

1. β is a lower bound of S.
2. β is the largest such lower bound. That is, if $\gamma \leq x$ for every $x \in S$, then $\gamma \leq \beta$.

We write

$$\boxed{\beta = glb(S).}$$

Note: The greatest lower bound is often called the *infimum* of S and is denoted by

$$inf(S).$$

Least Upper Bound

Let $S \subseteq \mathbb{R}$. Then α is called the *least upper bound* of S if

1. α is an upper bound of S.
2. α is the smallest such upper bound. That is, if $x \leq \gamma$ for every $x \in S$, then $\alpha \leq \gamma$.

We write

$$\boxed{\alpha = lub(S).}$$

Note: The least upper bound is often called the *supremum* of S and is denoted by

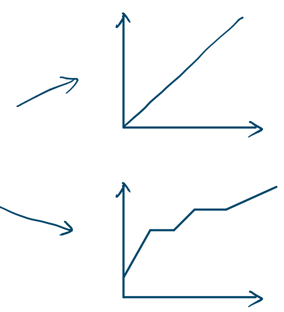
$$sup(S).$$

* $glb(S)$ & $lub(S)$ may or may not be in S

- def. $\{a_n\}$

$a_{n+1} > a_n$ increasing

$a_{n+1} \geq a_n$ non-decreasing



$a_{n+1} < a_n$ decreasing

$a_{n+1} \leq a_n$ non-increasing

$\{a_n\}$ is monotonic if $\{a_n\}$ is either non-decreasing or non-increasing

- Monotone convergence theorem (MCT)

Let $\{a_n\}$ be a non-decreasing

1) if $\{a_n\}$ is bounded above, then $\{a_n\}$ converges to $L = lub(\{a_n\})$

2) if $\{a_n\}$ is not bounded above, then $\{a_n\}$ diverges to ∞

Every bounded and monotonic sequence is convergent.

* 判断

1) If $\{a_n\}$ is convergent, then $\{a_n\}$ is bounded and monotone F $\frac{(-1)^n}{n}$

2) Every convergent sequence is bounded. T

3) Every bounded sequence is convergent F $(-1)^n$

4) Every convergent sequence is monotonic F $\frac{(-1)^n}{n}$

convergent $\Rightarrow$ bounded. (仅表示有上下限)

- Prove $\{a_n\}$ is non-decreasing and not bounded above

assume $\{a_n\}$ is non-decreasing (non-increasing)

1) suppose $\{a_n\}$ is bounded above and let $L = \text{lub}\{a_n\}$ 记 $L = \text{lub}\{a_n\}$

let $\varepsilon > 0$ be given. Then $L - \varepsilon < L$, which means $L - \varepsilon$ is not an upper bound of $\{a_n\}$

So there exist $N \in \mathbb{N}$ s.t. $L - \varepsilon < a_N$.

Then, if $n \geq N$, we have $L - \varepsilon < a_n \leq a_N$ since the sequence is non-decreasing

Therefore, for $n \geq N$, $L - \varepsilon < a_n \leq L < L + \varepsilon$.

So the tail of $\{a_n\}$ $(L-\varepsilon, L+\varepsilon)$ $(\forall \varepsilon > 0)$ which means $\lim_{n\to\infty} a_n = L$

2) Suppose $\{a_n\}$ is not bounded above, Let $M \in \mathbb{R}$. $M > 0$.

We can find $n \in \mathbb{N}$ s.t. $M < a_N$. Then if $n \geq N$, we have $M < a_N \leq a_n$
(since a_n is non-decr)

This shows that $\lim_{n\to\infty} a_n = \infty$

ex. Let $a_1 = 1$. $a_{n+1} = \frac{3+a_n}{2}$. Prove the sequence converges and finds its limit.

Step 1. 检查前几项是否递增/减

$a_1 = 1$, $a_2 = 2$. $a_3 = \frac{5}{2}$. ... look like non-decr.

$a_n \le a_{n+1} \ \forall n \in \mathbb{N}$

proof. base case: is $a_1 \le a_2$? yes. $a_1 = 1 \le a_2 = 2$

Inductive hypothesis (IH): 假设 $a_k \le a_{k+1}$ for some $k \ge 1$

Inductive Step (IS): Prove $a_{k+1} \le a_{k+2}$

Since $a_k \le a_{k+1} \Rightarrow 3 + a_k \le 3 + a_{k+1} \Rightarrow \frac{3+a_k}{2} \le \frac{3+a_{k+1}}{2}$

$\Rightarrow a_{k+1} \le a_{k+2}$

So the sequence is non-decr. by induction.

Step 2. Bounded above 选"合适"的 upper bound

claim: $a_n \le 10 \ \forall n \in \mathbb{N}$

proof: Base case: $a_1 = 1 \le 10$ ✓

IH: Assume $a_k \le 10$ for some $k \in \mathbb{N}$

IS: Since $a_k \le 10 \Rightarrow 3 + a_k \le 13 \Rightarrow \frac{3+a_k}{2} \le \frac{13}{2} \Rightarrow a_{k+1} \le \frac{13}{2} \le 10$

$\therefore$ by induction, $a_n \le 10$. $\forall n \in \mathbb{N}$. So the sequence is bounded above

Step 3. MCT

Since $\{a_n\}$ is bounded above and non-decr, $\{a_n\}$ converges

Step 4. Limit

Since the limit exists, So $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$

Then $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{3+a_n}{2} \Rightarrow L = \frac{3+L}{2}$

Solve, $\lim_{n\to\infty} a_n = 3$

ex. Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{3+2a_n}$ for $n \geq 1$.

a) Show it is non-dec and bounded above

b) Prove that this sequence is convergent and find $\lim_{n\to\infty} a_n$.

a) — Non-dec: Base case: $a_1 = 1$, $a_2 = \sqrt{5}$, so $a_1 \leq a_2$.

Induction: Suppose $a_n \leq a_{n+1}$ for some $n \geq 1$

$$\text{Then } 3+2a_n \leq 3+2a_{n+1}$$
$$\sqrt{3+2a_n} \leq \sqrt{3+2a_{n+1}}$$
$$a_{n+1} \leq a_{n+2}$$

So $\{a_n\}_{n=1}^{\infty}$ is non-decreasing.

通过 a_n $\boxed{<}$ a_{n+1}

a_n 代入式子 $\Rightarrow$ a_{n+1}

证 a_{n+1} $\boxed{<}$ a_{n+2}

— Bounded above: base case: $a \leq 3$

随机选数 "$a = \dots$"

Induction: Suppose $a_n = 3$ for some $n \geq 1$

$$\text{Then } 3+2a \leq 3+2\times 3 = 9$$
$$\Rightarrow \sqrt{3+2a_n} \leq 3$$
$$\Rightarrow a_{n+1} \leq 3$$

So $a_n \leq 3$ for $n \geq 1$

$\{a_n\}_{n=1}^{\infty}$ is bounded above

a 代入式子

求得 a_{n+1}

$a_{n+1} \leq a_n$

b) — By MCT, there is some $L \in \mathbb{R}$, such that $\lim_{n\to\infty} a_n = L$

$$L = \lim_{n\to\infty} a_{n+1} = \underline{\lim_{n\to\infty} \sqrt{3+2a_n} = \sqrt{3+2\lim_{n\to\infty} a_n}} = \sqrt{3+2L}$$

$L^2 - 2L - 3 = 0$ $\quad$ $L = -1$ (✗), 3 $\quad$ So $L = 3$

ex. Prove $a_1 = 2$, $a_{n+1} = \sqrt{7 + a_n}$

Claim: $a_n \le a_{n+1} \le 10 \quad \forall n \in \mathbb{N}$

Proof: Base Case: $a_1 = 2 \le 3 = a_n \le 10$

IH: Assume that $a_k \le a_{k+1} \le 10$ for some $k \in \mathbb{N}$

IS: (Prove $a_{k+1} \le a_{k+2} \le 10$)

Since $a_k \le a_{k+1} \le 10 \Rightarrow a_k + 7 \le a_{k+1} + 7 \le 17$

$\Rightarrow \sqrt{7 + a_k} \le \sqrt{7 + a_{k+1}} \le \sqrt{17}$

$\Rightarrow a_{k+1} \le a_{k+2} \le \sqrt{17} \le 10$

$\therefore$ a_n is non-decreasing and bounded above by induction.

So we conclude that a_n converges by MCT

We have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} = L$

So $L = \sqrt{7+L} \Rightarrow L^2 = 7 + L \qquad L = \frac{1 \pm \sqrt{29}}{2}$

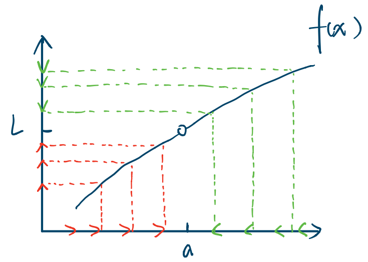
Since $L = \frac{1-\sqrt{29}}{2} < 0$ and a_n is non-decreasing $a_1 = 2$. So it can't be negative

then $L = \frac{1-\sqrt{29}}{2}$ can be limit.

$L = \frac{1+\sqrt{29}}{2}$ is limit of a_n

2.1 Introduction to limits for functions

-



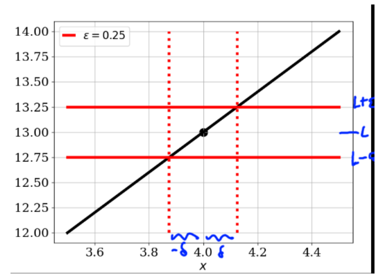
$\lim\limits_{x\to a} f(x) = L \quad (a, L \in \mathbb{R})$

当 x 无限趋近于 a $(x\neq a)$，会使 $f(x)$ 无限趋近于 L

- def. We say $\lim\limits_{x\to a} f(x) = L$ if:

  For all $\varepsilon > 0$, there exists $\delta > 0$.

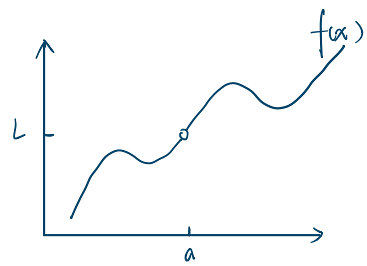
  s.t. if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$



- ① 若 $\lim\limits_{x\to a} f(x)$ 存在，f 在区间 (α, β) 中 $(x \neq a)$

  ② $f(a)$ 不影响 $\lim\limits_{x\to a} f(x)$

  ③ 若 $f(x) = g(x)$ $(x\neq a)$，则 $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$



2.2 Sequential Characterization of limits

- Sequential Characterization of limits.

Let f be defined on an open interval containing $x = a$, except possibly at $x = a$. Then the following two statements are equivalent:

i) $\lim_{x \to a} f(x)$ exists and equals L.

ii) If $\{x_n\}$ is a sequence with $x_n \neq a$ and $x_n \to a$, then

$$\lim_{n\to\infty} f(x_n) = L.$$

1) $\lim_{x \to a} f(x) = L$ $\quad \equiv$

证明是True 需所有seq

证明是False 仅两个即可 $\quad a^+ \quad a^-$

2) if $\{x_n\}$ is any (所有) sequence $x_n \to a$. $x_n \neq a$ then $\lim_{n\to\infty} f(x_n) = L$

ep. 有无数 sequence $\{x_n\}$ converge to 1.

$\{x_n\} \to 1 \quad \{y_n\} \to 1 \quad \lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$ $\Rightarrow$ 2) 不成立 $\Rightarrow$ 1) 不成立

※ 判断

c). If for all $\{x_n\}$ (which tends to a and $x_n \neq a$), $\lim_{n\to\infty} F(x_{2n}) = L$, then $\lim_{x\to a} F(x) = L$. $\quad$ T

而

$x_n = \frac{1}{n} \quad x_{2n} = \frac{1}{2n} \neq x_n$

$x_{2n} \neq a$ 且 $\{x_{2n}\} \to a$.

$\boxed{y_n = x_{2n}}$

$\lim_{n\to\infty} x_n = a \quad \forall n \in \mathbb{N}.\ x_n \neq a$

— The uniqueness of limit　极限存在的唯一性

Assume that $\lim_{x\to a} f(x) = L$ and that $\lim_{x\to a} f(x) = M$. Then $L = M$. That is, the limit of a function is unique.

若 $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} f(x) = M$, 则 $L=M$

$\Leftrightarrow$

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$　存在且相等

当 ① 任何一个 one-sided limit 不存在
② 2个 one-sided limit 不相等　则 $\lim_{x\to a} f(x) = L$ 不存在

— 找 $\lim_{x\to a} f(f(x))$

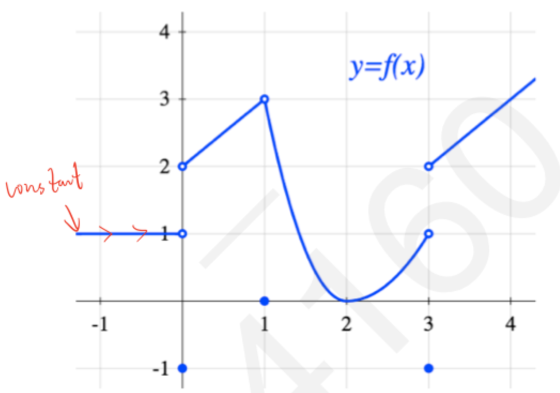
① 由内向外分析

② 若有分段函数，要讨论是否 one-sided

③ 若 $\lim f(x)$ ，则 $\lim f(f(x))$ 也不存在

③ 分析时注意是 approach $(x\to a)$ 还是 equal $(x=a)$

④ 注意分析 $f(x)$ approach 值的方向 (ex. ↑↓)



(a) $\lim_{x\to 0} f(f(x))$

As $x\to 0^+$, $f(x)\to 2$, $f(f(x))\to 0$

$x\to 0^-$, $f(x)\equiv 1$, $f(f(x))=0$

So $\lim_{x\to 0^+} f(f(x)) = \lim_{x\to 0^-} f(f(x)) = 0$

(b) $\lim_{x\to 1} f(f(x))$

As $x\to 1$, $f(x)\to 3^-$

$f(f(x))\to 1$

(c) $\lim_{x\to 2} f(f(x))$

As $x\to 2$, $f(x)\to 0^+$, $f(f(x))\to 2$

(d) $\lim_{x\to 3} f(f(x))$

As $x\to 3^+$, $f(x)\to 2^+$, $f(f(x))\to 0$

As $x\to 3^-$, $f(x)\to 1^-$, $f(f(x))\to 3$

一 Prove limit 不存在

① 找 sequence $\{x_n\}$. $x_n \to a$. $x_n \neq a$, for which $\lim_{x\to\infty} f(x_n)$ DNE

② 找 sequence $\{x_n\}$ & $\{y_n\}$. $x_n \to a$. $x_n \neq a$, $y_n \to a$. $y_n \neq a$

for which $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$

ex. Prove $\lim_{x\to 0} \frac{|x|}{x}$ DNE $\qquad \frac{|x|}{x} = \begin{cases} 1 & (x>0) \\ -1 & (x<0) \end{cases}$

找 function，符合 $n\to\infty$ $f(n)\to 0$

Proof. ① Let $x_n = \frac{1}{n} \to 0$ $\quad x_n \neq 0$.

$y_n = -\frac{1}{n} \to 0$ $\quad y_0 \neq 0$

② So $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{|x_n|}{x_n} = \lim_{n\to\infty} \frac{|\frac{1}{n}|}{\frac{1}{n}} = \lim_{n\to\infty} 1 = 1$

$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \frac{|-\frac{1}{n}|}{-\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{-\frac{1}{n}} = \lim_{n\to\infty} -1 = -1$

③ Hence $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$ then $\lim_{x\to 0} f(x)$ DNE

ex. Prove limit of $x_n = \frac{1}{2\pi n + \frac{\pi}{2}}$ DNE

Proof. ① Let $x_n = \frac{1}{2\pi n} \to 0$ as $n\to\infty$ $\quad x_n \neq 0$ $\quad \forall n \in \mathbb{N}$

$y_n = \frac{1}{2\pi n + \frac{\pi}{2}} \to 0$ as $n\to\infty$ $\quad y_n \neq 0$ $\quad \forall n \in \mathbb{N}$

② So $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin\left(\frac{1}{x_n}\right) = \lim_{n\to\infty} \sin(2\pi n) = 0$

$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \sin\left(2\pi n + \frac{\pi}{2}\right) = \lim_{n\to\infty} \sin\left(\frac{\pi}{2}\right) = 1$

③ Hence $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$ then $\lim_{x\to 0} f(x)$ DNE

ex. Prove $\lim_{x\to 3} \frac{1}{x-3}$ DNE $\qquad$ 记 lim 为 $\infty / -\infty \to$ DNE

Let $x_n = 3 + \frac{1}{n} \to 3$ as $n\to\infty$

So $\lim_{n\to\infty} f(x_n) = \frac{1}{3+\frac{1}{n}-3} = n = \infty$ $\quad$ DNE

2.3 Arithmetic Rules for limit of functions

– Arithmetic rules for limits of functions

Let f and g be functions and let $a \in \mathbb{R}$. Assume that $\lim_{x\to a} f(x) = L$ and that $\lim_{x\to a} g(x) = M$. Then

i) Assume that $f(x) = c$ for every $x \in \mathbb{R}$. Then $\lim_{x\to a} f(x) = c$.
ii) For any $c \in \mathbb{R}$, $\lim_{x\to a} cf(x) = cL$.
iii) $\lim_{x\to a} f(x) + g(x) = L + M$.
iv) $\lim_{x\to a} f(x)g(x) = LM$.
v) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$.
vi) $\lim_{x\to a} (f(x))^\alpha = L^\alpha$ for all $\alpha > 0$, $L > 0$.

ex. Proof $\lim_{x\to a} \alpha_0 + \alpha_1 x = \alpha_0 + \alpha_1 \cdot a$

Let $\varepsilon > 0$ be given. Let $\delta = \frac{\varepsilon}{|\alpha_1|}$

Then, if $0 < |x-a| < \delta$, then we have

$$|\alpha_0 + \alpha_1 x - (\alpha_0 + \alpha_1 a)| = |\alpha_1 (x-a)| = |\alpha_1|\,|x-a| < |\alpha_1|\delta = |\alpha_1| \times \frac{\varepsilon}{|\alpha_1|} = \varepsilon$$

– Limit of polynomials 多项式的极限

If $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n$ is any *polynomial*, then

$$\lim_{x\to a} p(x) = p(a).$$

– Limit of rational functions

consider $\frac{p(x)}{q(x)}$ where p, q are polynomials.

1) $q(a) \neq 0$. $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$

2) $\lim_{x\to a} q(x) = 0$ $\quad \lim_{x\to a} p(x) \neq 0$. $\quad \lim_{x\to a} \frac{p(x)}{q(x)}$ DNE

3) $\lim_{x\to a} q(x) = q(a) = 0 = p(a) = \lim_{x\to a} p(x)$

找 $(x-a)$ 同时为 $p(x)$ 与 $q(x)$ 因子

$p(x) = (x-a)\, p^*(x) \qquad q(x) = (x-a)\, q^*(x)$

$$\lim_{x\to a} \frac{p(x)}{q(x)} = \lim_{x\to a} \frac{(x-a)\, p^*(x)}{(x-a)\, q^*(x)} = \lim_{x\to a} \frac{p^*(x)}{q^*(x)}$$

2.4 One-side Limits

- def.

x比a大

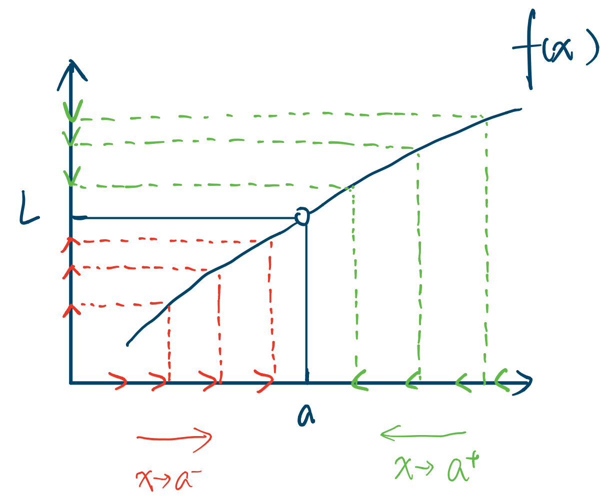
RS: $\lim\limits_{x \to a^+} f(x) = L$ if $\forall \varepsilon > 0. \exists \delta > 0.$ s.t $|x-a| < \delta \wedge x > a \Rightarrow |f(x) - L| < \varepsilon$

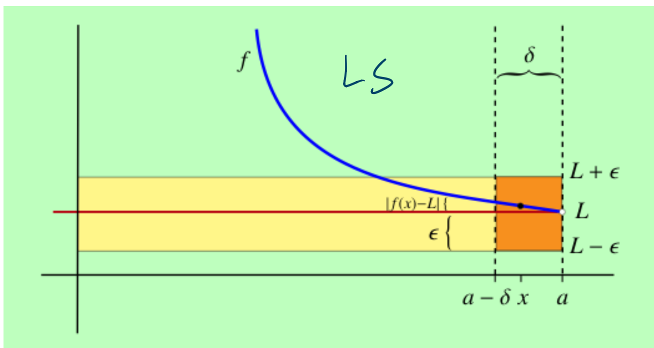
LS: $\lim\limits_{x \to a^-} f(x) = L$ if $\forall \varepsilon > 0. \exists \delta > 0.$ s.t $|x-a| < \delta \wedge x < a \Rightarrow |f(x) - L| < \varepsilon$

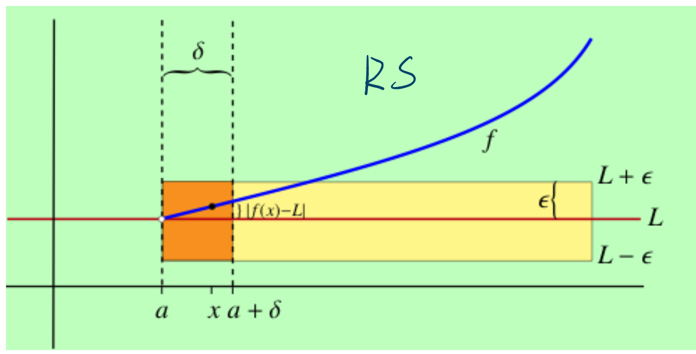
x比a小

$\lim\limits_{x \to a^+} f(x) = L$: Right hand side limit $(x > a)$

$\lim\limits_{x \to a^-} f(x) = L$: Left hand side limit $(x < a)$





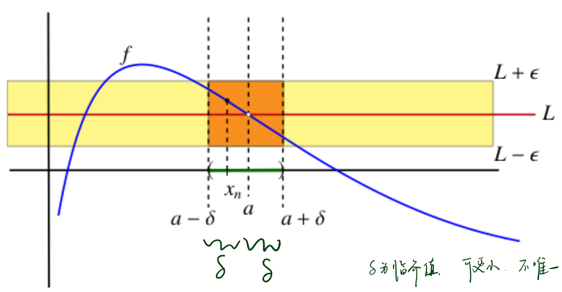


Two-sides limit

$\lim\limits_{x \to a} f(x) = L$ 同时考虑两个方向

* 若f(x)趋近于 $\pm\infty$, 则 limit 不存在

* 对 limit 而言, x是否=a 不影响f(x)



δ为临界值 可更小. 不唯一

For $\forall \varepsilon > 0.$ there exist $\delta > 0.$

s.t. $0 < |x-a| < \delta \rightarrow |f(x) - L| < \varepsilon$

$L - \varepsilon < f(x) < L + \varepsilon$

ex. Proof lim 是否 = constant L

草稿 $|f(x) - L| < \varepsilon \Rightarrow \underbrace{|x-a| < ?}_{\delta}$

找 $\delta = ?$ (临界值)

证明 ① Let $\varepsilon > 0$

② choose $\delta = ?$

③ Assume $0 < |x-a| < \delta$

④ 推 $|f - L| < ? < \varepsilon$.

Let $f(x) = 4 - \frac{x}{2}$. Use ϵ-δ definition of limit, prove that

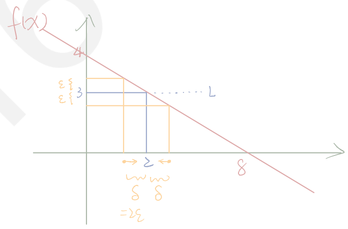
$$\lim_{x \to 2} f(x) = 3$$

草稿 $|4 - \frac{x}{2} - 3| < \varepsilon \Rightarrow |x-2| < ?$

$|1 - \frac{x}{2}| < \varepsilon$

$\frac{1}{2} \times |2-x| < \varepsilon$

$|x-2| < 2\varepsilon$



Proof Let $\varepsilon > 0$

choose $\delta = 2\varepsilon$

Assume $0 < |x-2| < \delta$ $\quad 0 < |x-2| < 2\varepsilon$

$|(4 - \frac{x}{2}) - 3| = |1 - \frac{x}{2}| = \frac{1}{2}|2-x| = \frac{1}{2}|x-2|$

$< \frac{1}{2}\delta = \frac{1}{2} \cdot (2\varepsilon) = \varepsilon$ as desired

ex Show that $\lim_{x \to 2} (x^2 - x - 3) = -1$.

草稿 $|x^2 - x - 3 + 1| < \varepsilon \to |x-2| = ?$

$|x^2 - x - 2| < \varepsilon$

$|x-2| \cdot |x+1| < \varepsilon$

$\therefore |x-2| <$ 任何一个正实数

As $x \to 2$. $|x-2| \to 0 \Rightarrow 0 < |x-2| < 1$

$1 < x < 3$

$2 < x+1 < 4$

$|x+1| < 4$

$|x-2||x+1| < 4|x-2| < \varepsilon$

$|x-2| < \frac{\varepsilon}{4}$ and $|x-2| < 1$

$|x-2| < \min\{1, \frac{\varepsilon}{4}\}$

Proof: Let $\varepsilon > 0$

choose $\delta = \min\{1, \frac{\varepsilon}{4}\}$

Assume $0 < |x-2| < \delta$ $\quad |x-2| < \min\{1, \frac{\varepsilon}{4}\}$

$\Rightarrow \begin{cases} |x-2| < 1 & 1 < x < 3 \\ |x-2| < \frac{\varepsilon}{4} \end{cases}$

$|x^2 - x - 3 + 1| = |x^2 - x - 2| = |(x-2)(x+1)| = |x-2| \cdot |x+1|$

Since $1 < x < 3 \Rightarrow |x+1| < 4$

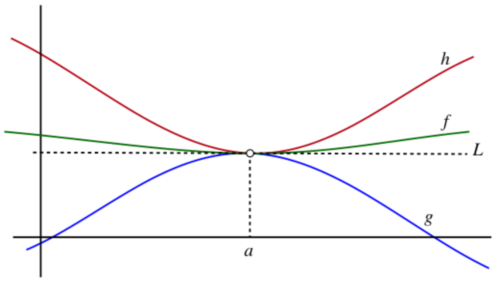
So $|x-2| \cdot |x+1| < 4|x-2| < 4 \times \frac{\varepsilon}{4} = \varepsilon$ as desired

- One-side VS. Two-side

$$\lim_{x \to a} f(x) = L \quad \equiv \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

2.5 The squeeze theorem

- def.



Assume $f(x)$, $g(x)$, $h(x)$ are defined on an open interval I, containing $x=1$. $x \in I$.

except possibly $x=a$, that $g(x) \le f(x) \le h(x)$

and $\lim\limits_{x\to a} g(x) = L = \lim\limits_{x\to a} h(x)$

Then $\lim\limits_{x\to a} f(x)$ exists and $\lim\limits_{x\to a} f(x) = L$

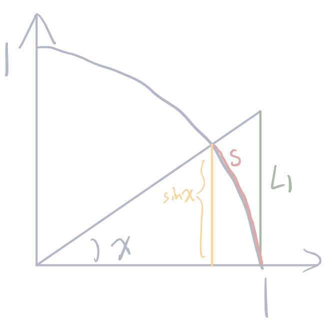
ex. 求 $\lim\limits_{x\to 0} x^2 \cos(e^x+7)$

We know $-1 \le \cos(e^x+7) \le 1 \Rightarrow -x^2 \le x^2\cos(e^x+7) \le x^2$

Since $\lim\limits_{x\to 0} \pm x^2 = 0$, then, by squeeze theorem

we have $\lim\limits_{x\to 0} x^2 \cos(e^x+7) = 0$

ex. 求 $\lim\limits_{x\to 0} \frac{\sin x}{x}$



$\sin x = \frac{\text{opp}}{\text{hyp}} = \frac{L}{1} \Rightarrow L = \sin x$

$\tan x = \frac{\text{opp}}{\text{adj}} = \frac{L_1}{1} \Rightarrow L_1 = \tan x$

$s = r\theta = 1 \cdot x \Rightarrow s = x$

$\sin x \le x \le \tan x$

If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, we have $|\sin x| \le |x| \le |\tan x|$

$1 \le \left|\frac{x}{\sin x}\right| \le \left|\frac{1}{\cos x}\right|$ (if $x \ne 0$)

$\Rightarrow 1 \ge \left|\frac{\sin x}{x}\right| \ge |\cos x|$ but $\frac{\sin x}{x} \ge 0$ and $\cos x > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$

So, $\cos \le \frac{\sin x}{x} \le 1$. Since $\lim\limits_{x\to 0} 1 = 1 = \lim\limits_{x\to 0} \cos x$

then $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ by squeeze theorem

2.6 The Fundamental Trigonometric Limit

- Theorem

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\sin \square}{\square} = 1$$

$$\lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{1}{\frac{\sin x}{x}} = 1$$

ex. 求 $\lim\limits_{x\to 0} \frac{\sin 5x}{\sin 2x}$

$\sin 2x = 2\sin x \cos x$

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 2x} = \lim_{x\to 0} \frac{\sin(5x)\cdot 5x}{5x} \times \frac{2x}{\sin(2x)} \times \frac{1}{2x}$$

$$= \lim_{x\to 0} \frac{\sin(5x)}{5x} \times \lim_{x\to 0} \frac{5}{2} \times \lim_{x\to 0} \frac{2x}{\sin 2x} = \frac{5}{2}$$

ex. 求 $\lim\limits_{x\to 0} \frac{\tan 3x}{\sin x}$

$$\lim_{x\to 0} \frac{\tan 3x}{\sin x} = \lim_{x\to 0} \frac{\sin 3x}{\cos 3x} \times \frac{1}{\sin x} = \lim_{x\to 0} \frac{1}{\cos 3x} \times \frac{\sin 3x}{3x} \times \frac{x}{\sin x} \times \frac{3x}{x}$$

$= 3$ (by FTL)

- 函数极限计算方法 3-Fundamental Trigonometric Limit.

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

✓ **Example 10**: Evaluate the following limits.

a). $\lim\limits_{x \to 0} \frac{x}{\sin(2x)}$

$= \lim\limits_{x\to 0} \frac{1}{2} \times \frac{2x}{\sin 2x}$

As $x \to 0$. $2x \to 0$

$\lim\limits_{x\to 0} \frac{2x}{\sin 2x} = 1$

$L = \frac{1}{2}$

b). $\lim\limits_{x \to 0} \frac{\sin(2x)}{\sin(3x)}$

$= \lim\limits_{x\to 0} \frac{\sin 2x}{2x} \times \lim\limits_{x\to 0} \frac{3x}{\sin 3x} \times 2x \times \frac{1}{3x}$

As $x \to 0$. $2x \to 0$. $3x \to 0$

$\lim\limits_{x\to 0} \frac{\sin(2x)}{2x} = 1$. $\lim\limits_{x\to 0} \frac{3x}{\sin 3x} = 1$

$L = 2 \times \frac{1}{3} \times 1 \times 1 = \frac{2}{3}$

c). $\lim\limits_{x \to 0} \frac{2e^x}{\sin(2e^x)}$

As $x \to 0$, $2e^x \to 2 \neq 0$

$L = \frac{2}{\sin(2)}$

d). $\lim\limits_{x \to 0} \frac{1 - \cos(3x)}{x^2}$

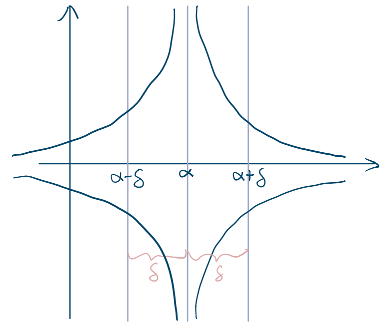
general form.

$\lim\limits_{x\to 0} \frac{\sin \square}{\square} = 1$

$\lim\limits_{x\to 0} \frac{x}{\sin x} = \lim\limits_{x\to 0} \frac{1}{\frac{\sin x}{x}} = 1$

2.7 Limit at infinity and asymptotes

2.7.1 Limits as infinity and asymptotes



- 渐近线

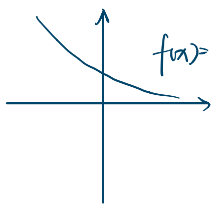
1) limits of infinity $(x \to \pm\infty)$ 水平渐近线

2) infinite limits $(f(x) \to \pm\infty)$ 竖直渐近线

- def.

- $\lim_{x\to\infty} f(x) = L$. $\forall \varepsilon > 0$. there exists $N \in \mathbb{R}$. s.t. if $x > N$. $|f(x) - L| < \varepsilon$

  $\lim_{x\to-\infty} f(x) = L$. $\forall \varepsilon > 0$. there exists $N \in \mathbb{R}$. s.t. if $x < N$. $|f(x) - L| < \varepsilon$

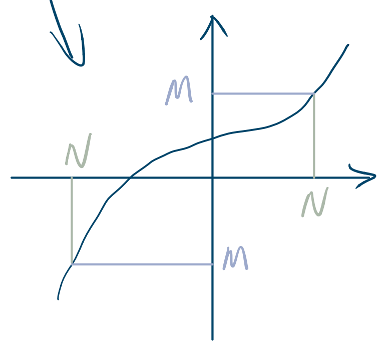
ex. 

$\lim_{x\to\infty} e^{-x} = 0$ 指 $x \to \infty$, $f(x) \to L$, $f(x) = e^{-x}$

- 若 $\lim_{x\to\infty} f(x) = L$. $\lim_{x\to-\infty} f(x) = L$. 则 L 为 f in 水平渐近线 (horizontal asymptote)

- $\lim_{x\to\infty} f(x) = \infty$. $M > 0$. there exists $N \in \mathbb{R}$. s.t. if $x > N$. $f(x) > M$

  $\lim_{x\to\infty} f(x) = -\infty$. $M > 0$. there exists $N \in \mathbb{R}$. s.t. if $x < N$. $f(x) < M$



- Squeeze theorem for limits at $\pm\infty$.

  If $g(x) \le f(x) \le h(x)$ for all $x \ge N$. for some $N \in \mathbb{R}$.

  and $\lim_{x\to\infty} g(x) = L = \lim_{x\to\infty} h(x)$

  Then $\lim_{x\to\infty} f(x) = L$

- 计算 limit

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_n}{b_m x^m + \cdots + b_1 x + b_0}$$

$$\lim_{x\to\pm\infty} f(x) = \begin{cases} 若\ n = m & \frac{a_n}{b_m} \\ 若\ n < m & 0 \\ 若\ n > m & \text{DNE} \end{cases}$$

2.7.2 The fundamental log limit

- $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

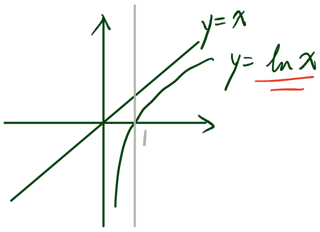
  $\lim_{x\to\infty} \frac{\ln x}{x^p} = 0$ for any $p>0$

  $\lim_{x\to\infty} \frac{\ln(x^p)}{x} = 0$

for $p>0$,
$(\ln x)^p << x^p << p^x << x^x$ (as $x\to\infty$)

证: $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

用 squeeze theorem



① for $x \geq 1$, $\frac{\ln x}{x} \geq 0$

$\because \ln x \leq x$, $\therefore \frac{\ln x}{x} \leq 1$

So, $0 \leq \frac{\ln x}{x} \leq 1$

$\frac{\ln\sqrt{x}}{\sqrt{x}} \leq 1$

② $\frac{\ln x}{x} = \frac{\ln(\sqrt{x}\cdot\sqrt{x})}{\sqrt{x}\cdot\sqrt{x}} = \frac{\ln[(\sqrt{x})^2]}{\sqrt{x}\cdot\sqrt{x}} = \frac{2\ln(\sqrt{x})}{\sqrt{x}\cdot\sqrt{x}} = \frac{2}{\sqrt{x}}\cdot\frac{\ln\sqrt{x}}{\sqrt{x}}$

$0 \leq \frac{\ln x}{x}$

$\leq \frac{2}{\sqrt{x}}$

we have $0 \leq \frac{\ln x}{x} \leq \frac{2}{\sqrt{x}}$ Since $\lim_{u\to\infty} \frac{2}{\sqrt{x}} = 0$ $\lim_{x\to\infty} \frac{\ln x}{x} = 0$ due to squeeze theorem

ex. 求 $\lim_{x\to\infty} \frac{x^p}{e^x}$

Let $u = e^x \Rightarrow x = \ln u$. $x\to\infty \Rightarrow u\to\infty$

$\lim_{u\to\infty} \frac{(\ln u)^p}{u} = \lim_{u\to\infty} \frac{[\ln u]^p}{(u^{\frac{1}{p}})^p} = \lim_{u\to\infty} \left[\frac{\ln u}{u^{\frac{1}{p}}}\right]^p = 0$ → by above limits

ex. Let $p>0$. 求 $\lim_{x\to 0^+} x^p \ln x$

Let $u = \frac{1}{x} \Rightarrow x = \frac{1}{u}$ So $x\to 0^+ \Rightarrow u\to\infty$

$\lim_{x\to 0^+} x^p \ln x = \lim_{u\to\infty} \frac{\ln\frac{1}{u}}{u^p} = \lim_{u\to\infty} \frac{-\ln u}{u^p} = 0$ → by above limits

This shows $x^p \to 0$ faster than $\ln x \to \infty$ as $x \to 0^+$

2.7.3. Vertical asymptotes and infinite limits

- def. infinite limits

$\lim_{x\to a^+} f(x) = \infty$, if $\forall\, m>0$. $\exists\, \delta>0$. So that if $a<x<a+\delta$ We have $f(x)>m$.

$\lim_{x\to a^-} f(x) = \infty$, if $\forall\, m>0$. $\exists\, \delta>0$. So that if $a-\delta<x<a$ We have $f(x)>m$.

$\lim_{x\to a^+} f(x) = \infty = \lim_{x\to a^-} f(x)$

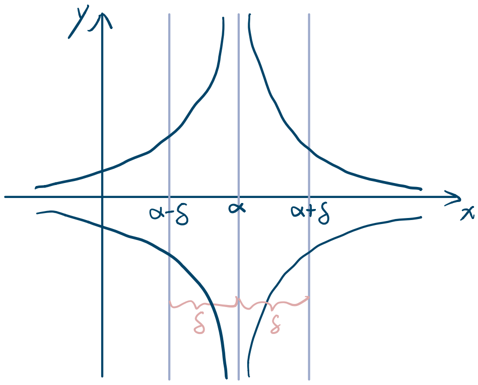
- def. of vertical asymptote

$\lim_{x\to \alpha^-} f(x) = \infty$

$M>0$. $\exists\, \delta>0$
s.t. if $\alpha-\delta<x<\alpha$
then $f(x)>M$

$\lim_{x\to \alpha^-} f(x) = -\infty$

$M<0$. $\exists\, \delta>0$
s.t. if $\alpha-\delta<x<\alpha$
then $f(x)<M$



$\lim_{x\to \alpha^+} f(x) = \infty$

$M>0$ $\exists\, \delta>0$.
s.t. if $\alpha<x<\alpha+\delta$
then $f(x)>M$

$\lim_{x\to \alpha^+} f(x) = -\infty$

$M<0$ $\exists\, \delta>0$.
s.t. if $\alpha<x<\alpha+\delta$
then $f(x)<M$

水平渐近线. → 求 $\lim_{x\to\pm\infty} = ?$

竖直渐近线. → 求 $\lim_{x\to ?} = \pm\infty$

If $\lim_{x\to a^\pm} f(x) = \pm\infty$, then the line $x=a$ is a vertical asymptote of f.

* $\lim f(x)=\infty$ 指 极限不存在. f(x)会变得很大

ex. $\lim_{x\to 3^+} \frac{(x+1)(x-7)}{(x-3)(x-1)}$

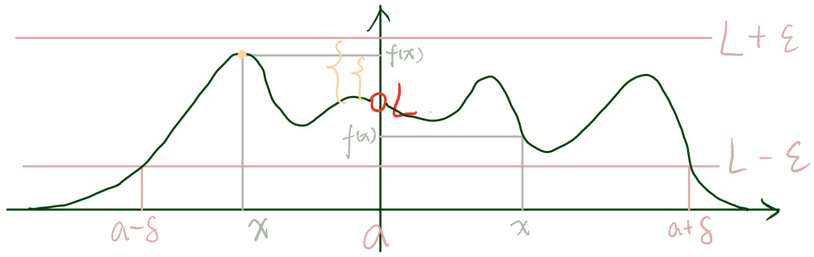
$\frac{4x-4}{0^+ \times 2}$ ↳ $= -\infty$

ex. find all vertical / horizontal asymptotes for $f(x)=\frac{x-3}{x+1}$

$\lim_{x\to\pm\infty} \frac{x-3}{x+1} = 1$. $f(x)$ has horizontal asymptote $y=1$. 水平渐近线. → 求 $\lim_{x\to\pm\infty} = ?$

Also, $\lim_{x\to -1^+} \frac{x-3}{x+1} = -\infty$, So $x=-1$ is a vertical asymptote. 竖直渐近线. → 求 $\lim_{x\to ?} = \pm\infty$

- One-sided



Left-H

$\lim_{x \to a^-} f(x) = L$

$\forall \varepsilon > 0.\ \exists \delta > 0$
s.t $a - \delta < x < a$
$\Rightarrow |f(x) - L| < \varepsilon$

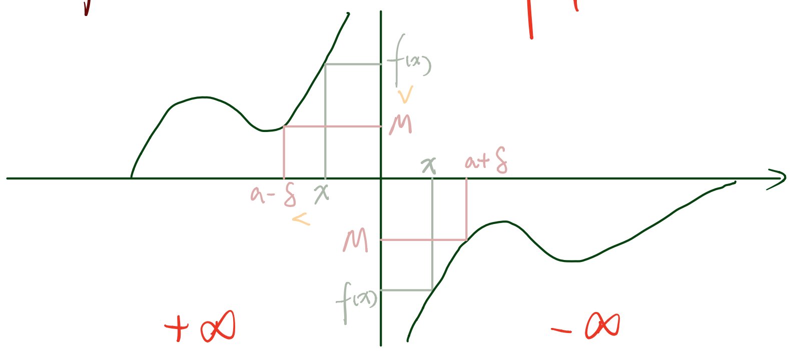
Right-H

$\lim_{x \to a^+} f(x) = L$

$\forall \varepsilon > 0.\ \exists \delta > 0$
s.t $a < x < a + \delta$
$\Rightarrow |f(x) - L| < \varepsilon$

- Infinite Limit

竖直渐近



$+\infty$

$\lim_{x \to a^-} f(x) = +\infty$

$\forall M \in \mathbb{R}.\ \exists \delta > 0$
s.t $a - \delta < x < a$
$\Rightarrow f(x) > M$

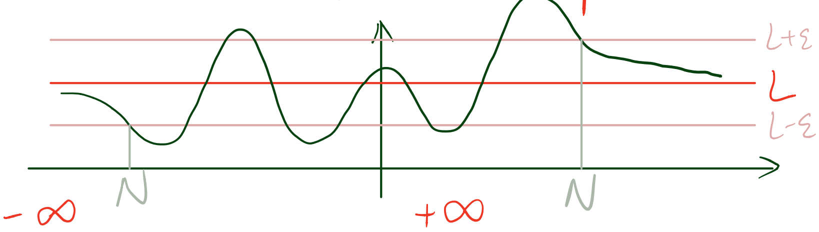
$-\infty$

$\lim_{x \to a^+} f(x) = -\infty$

$\forall M \in \mathbb{R}.\ \exists \delta > 0$
s.t $a < x < a + \delta$
$\Rightarrow f(x) < M$

- Limit at infinity

水平渐近



$-\infty$

$\lim_{x \to -\infty} f(x) = L$

$\forall \varepsilon > 0.\ \exists N \in \mathbb{R}$
s.t $x < N$
$\Rightarrow |f - L| < \varepsilon$

$+\infty$

$\lim_{x \to \infty} f(x) = L$

$\forall \varepsilon > 0.\ \exists N \in \mathbb{R}$
s.t $x > N$
$\Rightarrow |f - L| < \varepsilon$

$x \to \infty \qquad p > 0$

$(\ln x)^p << x^p << p^x << x^x$

show that $\lim_{x \to 2^+} \frac{1}{x-2} = \infty$.

- def. $\forall M > 0.\ \exists \delta > 0.$
s.t $0 < |x - 2| < \delta \Rightarrow \frac{1}{x-2} > M$

- 草稿. $\frac{1}{x-2} > M > 0 \to 2 < x < ?$
$0 < x - 2 < \frac{1}{M}$
$2 < x < \frac{1}{M} + 2 \to \delta = \frac{1}{M}$

- Proof Let $M > 0$.
choose $\delta = \frac{1}{M}$
assume $2 < x < 2 + \delta \Rightarrow 2 < x < 2 + \frac{1}{M}$
$0 < x - 2 < \frac{1}{M}$
$\frac{1}{x-2} > M > 0$ as desired

prove that $\lim_{x \to -\infty} \frac{1}{x-2} = 0$.

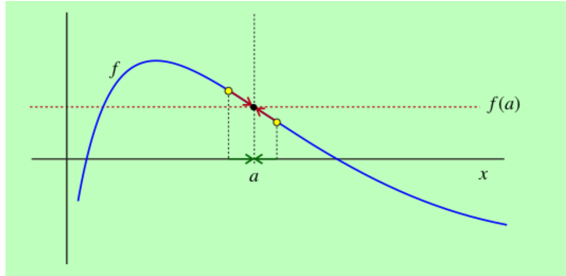
- def. $\forall \varepsilon > 0.\ \exists N \in \mathbb{R}.$
s.t $x < N \Rightarrow \left|\frac{1}{x-2} - 0\right| < \varepsilon$

- 草稿 $\left|\frac{1}{x-2} - 0\right| < \varepsilon \to x < ?$
$-\varepsilon < \frac{1}{x-2} < \varepsilon.$
$-\varepsilon < \frac{1}{x-2} < 0$
$x - 2 < -\frac{1}{\varepsilon}$
$x < \underbrace{2 - \frac{1}{\varepsilon}}_{N}$

- Proof Let $\varepsilon > 0$
choose $N = 2 - \frac{1}{\varepsilon}$
assume $x < N \Rightarrow x < 2 - \frac{1}{\varepsilon}$
$x - 2 < -\frac{1}{\varepsilon} < 0 \quad \because \varepsilon > 0$
$\therefore -\varepsilon < \frac{1}{x-2} < 0 \Rightarrow \left|\frac{1}{x-2} - 0\right| < \varepsilon$
as desired

2.8 Continuity

— Formal definition of continuity I



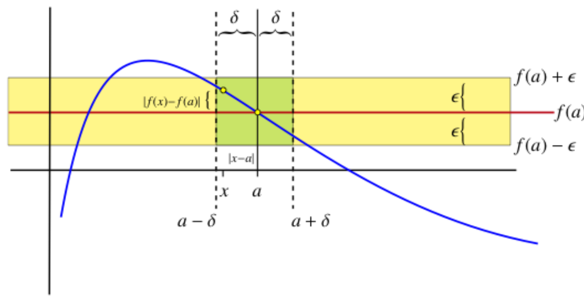
A function is continuous at $x=a$ if.

1) $\lim_{x\to a} f(x)$ exists. 有极限

2) $\lim_{x\to a} f(x) = f(a) = L$ 连续

We say f is continuous at $x=a$. / $x=a$ is a point of discontinuity for f

— Formal definition of continuity II



A function is continuous at $x=a$ if.

$\forall \varepsilon > 0.\ \exists \delta > 0.$ s.t $|x-\alpha| < \delta.$

(<0 没有)

then $|f(x) - f(\alpha)| < \varepsilon$

— sequential characterization of continuity.

$\{x_n\}$ is a seq. $\lim_{n\to\infty} x_n = a$

must have $\lim_{n\to\infty} f(x_n) = f(\alpha) = L$

* Observation

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(\alpha+h) = f(\alpha)$$

$x = \alpha + h \qquad h = x - \alpha$

a function is continuous at $x=a$ $\Leftrightarrow$ $\lim_{h\to 0} f(a+h) = f(a)$

ex. Prove that the function $f(x) = 2x^2 + 9$ is continuous at $x = 2$ using the $\epsilon - \delta$ definition of continuity (the Formal Definition of Continuity II).

- Show $\lim_{x\to a} f(x) = f(a)$ for $a=2$.

$\lim_{x\to 2} (2x^2+9) = 17$

- So for any $\varepsilon > 0$, we can find $0 < |x-2| < \delta$. $\to |2x^2-8| < \varepsilon$
- Let $\varepsilon > 0$, choose $\delta = \min\{\frac{\varepsilon}{8}, 2\}$.

$\because 0 < |x-2| < \delta$ $\therefore \to |x+2| < 4$ $|x-2| < \frac{\varepsilon}{8}$

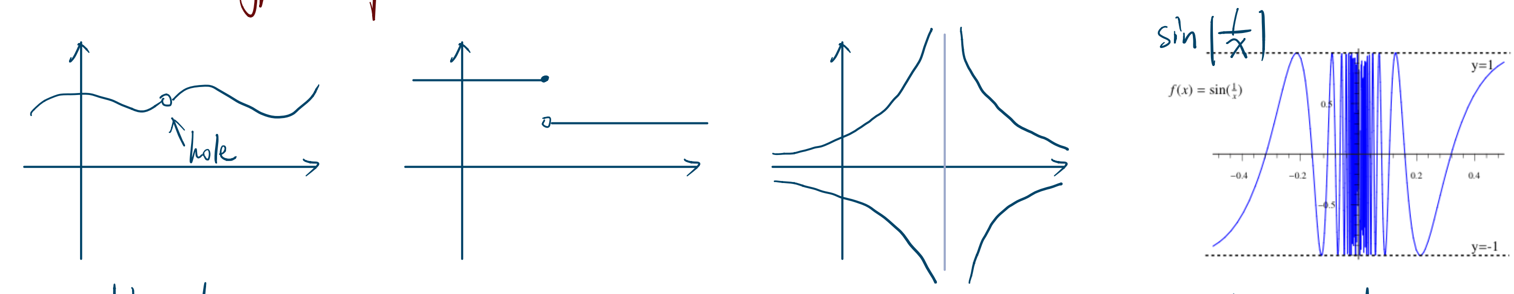
$|x-2||x+2| < \frac{\varepsilon}{8} \times 2$

$|x^2-4| < \frac{\varepsilon}{2}$

$|2x^2-8| < \varepsilon$

Therefore, by ε-δ def of cont. $f(x)$ is continuous at $x=2$

2.8.1 Types of discontinuities



removable discontinuity   finite jump   vertical asymptote   oscillatory discontinuous

2.8.2 Polynomial, $\sin x$, $\cos x$, e^x, $\ln x$ 的连续性

— polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 x^0$$

$$\lim_{x\to\alpha} f(x) = p(\alpha)$$

— $\sin x$ / $\cos x$

- $\lim_{x\to 0} \sin x = 0 = \sin 0$
- $\lim_{x\to 0} \cos x = \lim_{x\to 0}\sqrt{1-\sin^2 x} = \lim_{x\to 0}\sqrt{1-0^2} = 1$
- $\lim_{x\to a} \sin x = \lim_{h\to 0} \sin(\alpha+h) = \lim_{h\to 0}(\sin\alpha\cos h + \cos\alpha\sin h) = \lim_{h\to 0}\sin\alpha\cdot 1 + \cos\alpha\cdot 0$

$$= \lim_{h\to 0}\sin\alpha = f(\alpha)$$

$x = \alpha + h$

$h = x - \alpha$

$f(x)$ is continuous at $x=\alpha \Leftrightarrow \lim_{x\to\alpha} f(x) = \lim_{h\to 0} f(\alpha-h) = f(\alpha)$

$$\lim_{x\to a}\cos x = \lim_{h\to 0}\cos(\alpha+h) = \lim_{h\to 0}\cos\alpha\cosh h - \sin\alpha\sin h$$

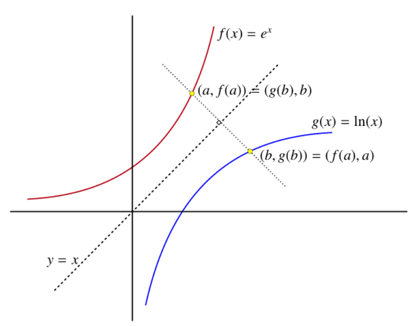
$$= \lim_{h\to 0}\cos\alpha\cdot 1 - \sin\alpha\cdot 0$$

$$= \cos\alpha$$

— e^x / $\ln x$

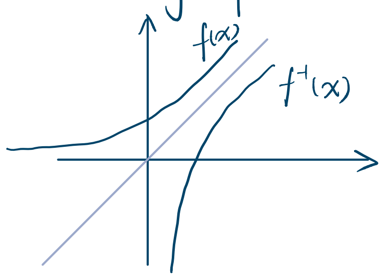
$$\lim_{x\to\alpha} e^x = \lim_{h\to 0} e^{(\alpha+h)} = \lim_{h\to 0} e^\alpha\cdot e^h = \lim_{h\to 0} e^\alpha\cdot 1 = e^\alpha$$

$g(x) = \ln x$ 为 $f(x) = e^x$ 的 inverse function



$\lim_{x\to\alpha}\ln x = \lim_{x\to\infty} e^x = e^\alpha$ (根据图像)

- Continuity of inverse



If $f(x)$ is invertible, $f(a)=b$, and f is continuous at $x=a$, then $f^{-1}(x)$ is continuous at $x=b$.

ex. where does $f(x)=\frac{x^2+x-2}{x^2-4x+3}$ continuous

$f(x)=\frac{(x-1)(x+2)}{(x-3)(x-1)}$ only possible discty's are at $x=1,3$

$x=1$: $\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{(x-1)(x+2)}{(x-3)(x-1)}=\frac{x+2}{x-3}=-\frac{3}{2}$ (limit exists)

But $f(1)$ DNE. so f is not cont. at $x=1$

$x=3$: $\lim_{x\to 3^+} f(x)=\infty$, so f is not cont. at $x=3$

$\therefore f$ is cont on $(-\infty,1)\cup(1,3)\cup(3,\infty)$

2.8.3 Arithmetic rules for continuous functions

- Rules

Let f & g be continuous at $x=a$, then.

1) $f\pm g$ is continuous at $x=a$
2) fg is continuous at $x=a$
3) $\frac{f}{g}$ is continuous at $x=a$ $(g(a)\neq 0)$

- Continuity of compositions

Theorem: If f is continuous at $x=a$. and g is continuous at $x=f(a)$ then $g\circ f=g(f(x))$ is continuous at $x=a$.

Proof: Let $x_n\to a$. Then $f(x_n)\to f(a)$ $\because f(x)$ is continuous at $x=a$

$h(x_n)=g\circ f(x_n)=g(f(x_n))\to g(f(a))=g\circ f(a)=h(a)$

So $h(x)$ is continuous at $x=a$. by sequential characterization of continuity.

2.8.4 Continuity on an interval

- Continuity on an open interval. (a,b)

f is continuous at (a,b) / $\mathbb{R}$., if it is continuous for all $x \in (a,b)$ / $x \in \mathbb{R}$.

- One-side Continuity.

$\lim_{x\to a^+} f(x) = f(a)$

$\lim_{x\to a^-} f(x) = f(a)$

- Continuity on an closed interval. $[a,b]$

f is continuous at $[a,b]$, if

1) f is continuous on (a,b)

2) $\lim_{x\to a^+} f(x) = f(a)$

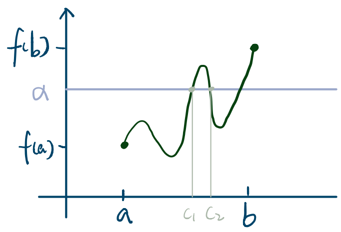
3) $\lim_{x\to b^-} f(x) = f(b)$

$*$ $x \to \infty$ $\quad (\ln x)^p \ll x^p \ll q^x \ll x^x$

$(q>1)$

2.9 The intermediate value theorem

– The intermediate value theorem (IVT)



$f(c_1)=\alpha$

$f(c_2)=\alpha$

If f is cont. at $[a,b]$.

and $f(a)<\alpha<f(b)$ ∨ $f(a)>\alpha>f(b)$

then there exists $c\in(a,b)$. s.t $f(c)=\alpha$

ex. proof $f(x)=x^5-2x^3-2$ has a root between 0 & 2.

Since f is cont. (∵ it's polynomial)

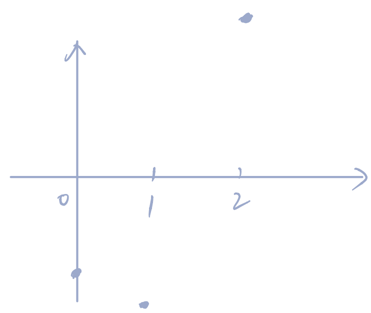
we can use IVT. $f(0)=-2<0$

$f(2)=14>0$ ∴ by IVT, $\exists c\in(0,2)$, s.t $f(c)=0$

* 判断是否连续，需考虑到 interval 中是否有数字取不到

2.9.1 Approximate solutions of equations

– approximate roots of polynomial



ex. $p(x)=x^5-2x^3-2$ has root in $(0,2)$

– recall $p(0)=-2<0$, $p(2)=14>0$

– check midpoint: $p(1)=-3<0$

since $p(1)<0$. $p(2)>0$. IVT concludes there is a root in $(1,2)$

– check new midpoint: $d=\frac{1+2}{2}=\frac{3}{2}$ $p(\frac{3}{2})=-\frac{37}{32}<0$

since $p(-\frac{37}{32})<0$. $p(2)>0$. IVT concludes there is a root in $(-\frac{37}{32},2)$

......

2.9.2 The bisection method

Summary [Bisection Method]

Problem: Given a continuous function f and a positive tolerance $\epsilon > 0$, find a point d so that there exists a point c with $f(c) = 0$ and $|c - d| < \epsilon$.

Algorithm:

Step 1: Find two points $a < b$ with $f(a)f(b) < 0$.

Step 2: Set $\ell = b - a$.

Step 3: Set counter n to equal 0.

Step 4: Let $d = \frac{a+b}{2}$.

Step 5: If $\frac{\ell}{2^{n+1}} < \epsilon$, then STOP.

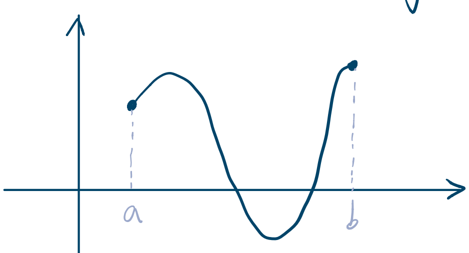
Step 6: If $f(d) = 0$, then STOP.

Step 7: If $f(a)f(d) < 0$, let $b = d$ and $n = n + 1$, then go to Step 4.

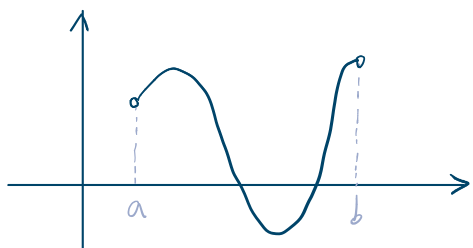
Step 8: Let $a = d$ and $n = n + 1$, then go to Step 4.

2.9.10 Extreme value theorem

- Global maxima & global minima



在[a,b]中一定存在 global max/min

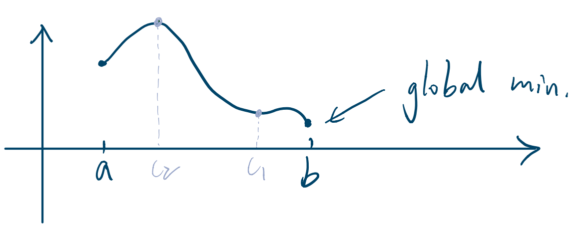


在(a,b)中不一定存在 global max/min

- The extreme value theorem (EVT)

Suppose f is cont. on $[a,b]$. there exist 2 numbers c_1 & $c_2 \in [a,b]$

s.t $f(c_1) \le f(x) \le f(c_2)$ for all $x \in [a,b]$



3.1 Instantaneous velocity

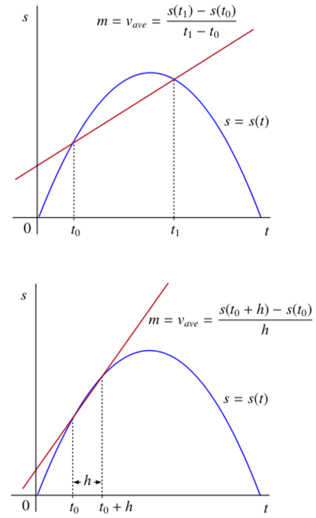
- def.

→ average velocity

V_{avg} from $t=t_0$ to $t=t_1$ is $\quad V_{avg} = \dfrac{s(t_1)-s(t_0)}{t_1-t_0}$

→ instantaneous velocity

V_{ins} at $t=t_0$ $\quad V_{ins} = \lim\limits_{t\to t_0} \dfrac{s(t)-s(t_0)}{t_1-t_0}$

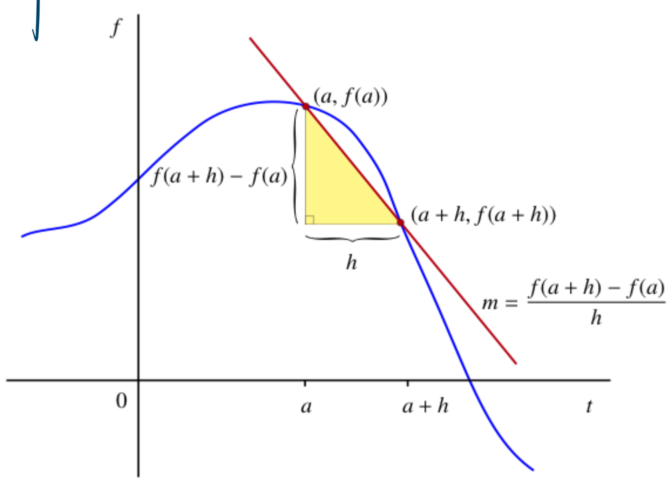


ex. find V_{ins} for $s(t) = t^2+3t$. $t=1$

$$V = \lim_{t\to1}\frac{s(t)-s(1)}{t-1} = \lim_{t\to1}\frac{t^2+3t-4}{t-1} = \lim_{t\to1}\frac{(t-1)(t+4)}{(t-1)} = \lim_{t\to1} t+4 = 5$$

3.2. def. of Derivative

- def.



$$f'(a) = \lim_{h\to0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

We say f is differentiable at $x=a$

证 diff' ① $f(x)$ 连续 ② $f'(x)$ 连续

* diff'ble ⇒ continuous

proof: show $\lim\limits_{x\to a} f(x) = f(a) \Leftrightarrow \lim\limits_{x\to a} f(x)-f(a) = 0$

$\because f$ diff'ble at $x=a$

$\therefore \lim\limits_{x\to a} f(x)-f(a) = \lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}(x-a) = f'(a)\cdot 0 = 0$

So $\lim\limits_{x\to a} f(x)-f(a) = 0 \Rightarrow \lim\limits_{x\to a} f(x) = f(a)$

continuous $\not\Rightarrow$ diff'ble

counter example: $f(x) = \dfrac{|x|}{x}$

$f'(0) = \lim\limits_{h\to0}\dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to0}\dfrac{|h|}{h}$

cont: 能一笔画完

diff'able: smoothly 画完

3.3 The derivative function

$(x+h)^0$			1		
$(x+h)^1$		1		1	
$(x+h)^2$		1	2	1	
$(x+h)^3$	1	3		3	1

- def.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

ex $f(x)=x^3$ 求 $f'(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$
$$= \lim_{h\to 0} 3x^2+3xh+h^2 = 3x^2$$

tangent line at $x=a$: $y = f(a)+f'(a)(x-a)$

3.4 Derivatives of elementary function

- $f(x) = \sin x$ $\quad f'(x) = \cos x$

Theorem : $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ (FLT) $\qquad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0$

$$\frac{\cos x-1}{x} = \lim_{x\to 0} \frac{(\cos x-1)}{x} \times \frac{(\cos x+1)}{(\cos x+1)} = \lim_{x\to 0} \frac{\cos^2 x-1}{x(\cos x+1)} = \lim_{0} \frac{-\sin^2 x}{x(\cos x+1)}$$
$$= \lim_{0} \frac{\sin x}{x} \times \left(-\frac{\sin x}{\cos x+1}\right) = \lim_{0} -\frac{\sin x}{\cos x+1} = 0$$

So, $\lim_{x\to 0} \frac{\cos x-1}{x} = 0$

calculate $f'(\sin x)$

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$
$$= \lim_{h\to 0} \frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$
$$= \lim_{h\to 0} \left(\cos \cdot \frac{\sin h}{h} + \sin x \frac{\cos h-1}{h}\right)$$
$$= \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} + \sin x \lim_{h\to 0} \frac{\cos h-1}{h}$$
$$= \cos x \times 1 + \sin x \cdot 0$$
$$= \cos x.$$

- $f(x) = \cos x \qquad f'(x) = -\sin x$

$$(\cos x)' = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$
$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$
$$= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$
$$= \cos x \cdot 0 - \sin x \cdot 1$$
$$= -\sin x$$

- $f(x) = e^x. \qquad f'(x) = e^x$

先计算 $f'(0)$ 的值

$2 < e < 3 \rightarrow$ $x=2$ $f(x) = 2^x$ $f'(x) = 0.7$; $x=3$ $f(x) = 3^x$ $f'(x) = 1.1$

$\therefore$ tangent line for $f(x) = a^x$ at $x=0$ is 1

So $(e^x)'\Big|_{x=0} = 1$

when $f(x) = e^x$, slope at $x=0$ is 1.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{e^h - e^0}{h} = \lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

$$f'(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$
$$= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h}$$
$$= e^x \times \lim_{h\to 0} \frac{e^h - 1}{h}$$
$$= e^x$$

- $f(x) = a^x \qquad f'(x) = a^x \ln x$

3.5 Tangent lines and linear approximation 无限超越

- Observation

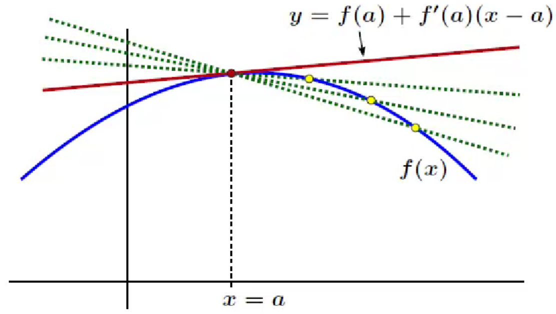
Suppose $f(x)$ is differentiable at $x=a$ with derivative $f'(a)$

Then $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} = f'(a)$

$$\frac{f(x)-f(a)}{x-a} \cong f'(a)$$

$$f(x)-f(a) \cong f'(a)(x-a)$$

$$\underline{f(x) \simeq f(a)+f'(a)(x-a)}$$

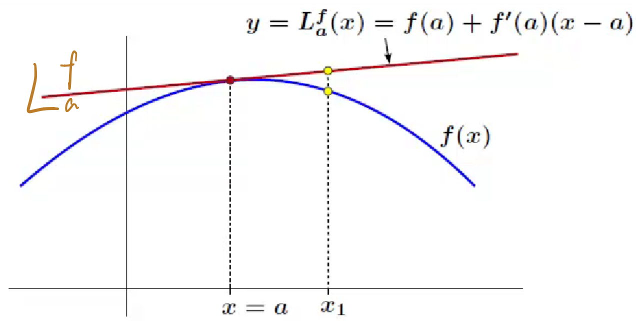


- def. linear approximation

若 $y=f(x)$ diff'able at $x=a$.

则 linear approximation to $f(x)$ at $x=a$ is

$$\underline{L_a^f(x) = f(a)+f'(a)(x-a)}$$



也可叫作 tangent line approx of f at $x=a$

If $L_a^f(x) = f(a)+f'(a)(x-a)$, then when $x \cong a$, $L_a^f(x) \cong f(x)$

- 3 properties of l-a

$f(x)$ is diff' at $x=a$. with $L_a^f(x) = f(a)+f'(a)(x-a)$

Then ① $L_a^f(a) = f(a)$

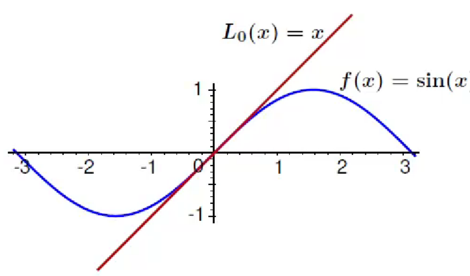
② $L_a^f(x)$ is diff'able at $x=a$ & $L_a^{f'}(a) = f'(a)$

③ $L_a^f(x)$ 基于①和②，是一条直线

ep. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

若 $f(x)=\sin x$, then $f(0)=0$, $f'(0)=\cos 0=1$,

when $x \simeq 0$ $\sin x \simeq L_0(x) = x$.



ex. Find $L_4(x)$ for $f(x)=\sqrt{x}$ as use this to apprx $\sqrt{3.98}$ & $\sqrt{4.05}$

$L_4(x)=f(4)+f'(4)(x-a)$ $\qquad f'(4)=\sqrt{4}=2$ $\qquad f'(4)=\frac{1}{2\sqrt{x}}=\frac{1}{4}$

$L_4(x)=2+\frac{1}{4}(x-4)$

$\sqrt{3.98}\simeq L_4(3.98)=2+\frac{1}{4}(3.98-4)=1.995$

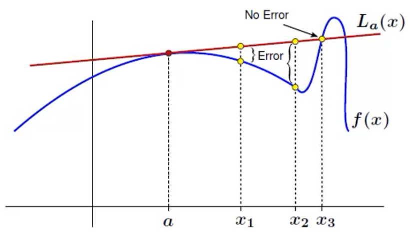
$\sqrt{4.05}\simeq L_4(4.05)=2+\frac{1}{4}(4.05-4)=2.012$

3.5.1 Error in l-a

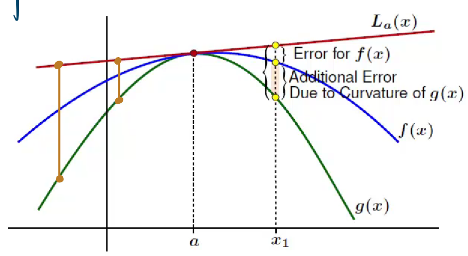
- Property of error

若f(x)未知，无法得出error的值.

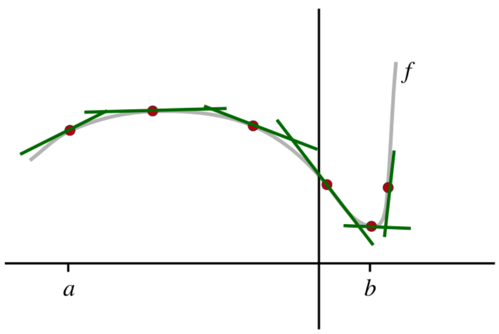
* 有时error不存在



* size of error 由 distance 与 curve 程度决定



- Curvature and Second derivative



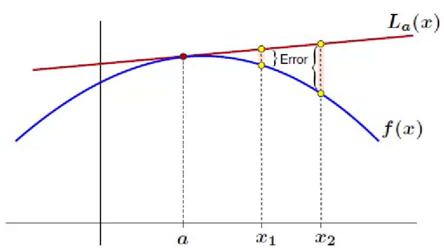
Maclaurin's and Taylor's Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^r}{r!} f^{(r)}(0) + \ldots$$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^r}{r!} f^{(r)}(a) + \ldots$$

$$f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \ldots + \frac{x^r}{r!} f^{(r)}(a) + \ldots$$

- The error in linear approximation



Let $y=f(x)$ be differentiable $x=a$.

The error $= |f(x)-L_a(x)|$

maximum of second derivative

若 x 在包含 a 的 interval I 中满足 $|f''(x)|\le M$

则 $|f(x)-L_a(x)|\le \frac{M}{2}(x-a)^2$ $\quad (x\in I)$

the upper bound for error in l-a

ex. find upper bound on error in using $L_4(x)$ to appx $\sqrt{x}$ on $[1, 6]$

$\rightarrow$ error $\leq \frac{M}{2}(x-4)^2$ on $[1, 6]$

$(x-4)^2$ is largest on $x=1$. $(x-4)^2 = 9$

取x，使$(x-4)^2$最大

$\rightarrow |f''(x)| \leq M$. find $f''(x)$

$|f''(x)| < M$

$f(x) = x^{\frac{1}{2}}$ $\quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$f''(x) = (\frac{1}{2}x^{-\frac{1}{2}})' = -\frac{1}{2} \times \frac{1}{2} x^{-\frac{3}{2}} = -\frac{1}{4}x^{-\frac{3}{2}}$

取x，使$f''(x)$最大

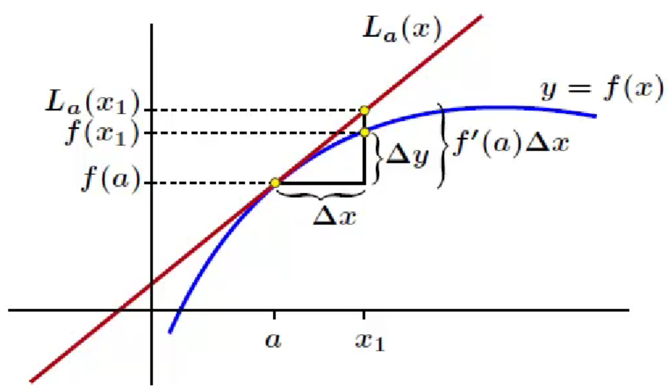
So $|f''(x)| = |-\frac{1}{4}x^{-\frac{3}{2}}| = \frac{1}{4}x^{-\frac{3}{2}} \leq \frac{1}{4} \times 1^{-\frac{3}{2}} = \frac{1}{4}$

$\rightarrow$ error $= |f(x) - L_4^f(x)| \leq \frac{M}{2}(x-4)^2 \leq \frac{M}{2}(1-4)^2 = \frac{\frac{1}{4}}{2}(1-4)^2 = \frac{9}{8}$

Thus, error $\leq \frac{M}{2}(x-4)^2 = \frac{1}{4 \times 2} \times 9 = \frac{9}{8}$

3.5.2 Application of linear approximation

– Estimating change



$\Delta y = f(x_1) - f(a)$

$= L_a(x_1) - f(a)$

$= (f(a) + f'(a)(x_1 - a)) - f(a)$

$= f'(a)(x_1 - a)$

$= f'(a)\Delta x$

ex. 球体从 $r=20m$ 扩到 $20.01m$，求 ΔV.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 3 \times \frac{4}{3}\pi r^2 = 4\pi r^2$

$dV = dr \times 4\pi \times 20^2 = 0.01 \times 1600\pi = 16\pi$

(actural: $\Delta V = V(20.01) - V(20) = \frac{4}{3}\pi(20.01)^3 - \frac{4}{3}\pi(20)^3$)

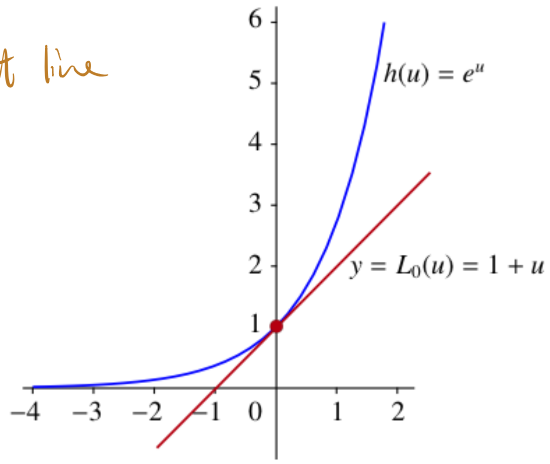
– qualitative analysis of functions

ex. how $f(x) = e^{-x^2}$ behave near $x=0$ 找 tangent line

step 1. Start with $h(u) = e^u$

$\because h(0) = h'(0) = e^0 = 1$

$\therefore e^u \simeq L_0^h(u) = 1+u$ where $u \to 0$



step 2. 已知 $e^u \simeq L_0^h(u) = 1+u$ where $u \to 0$

$x \to 0. \Rightarrow u = -x^2.$

let $u = -x^2$. we get $y = e^{-x^2} \simeq 1 + (-x^2) = 1 - x^2$

3.6 Newton's method

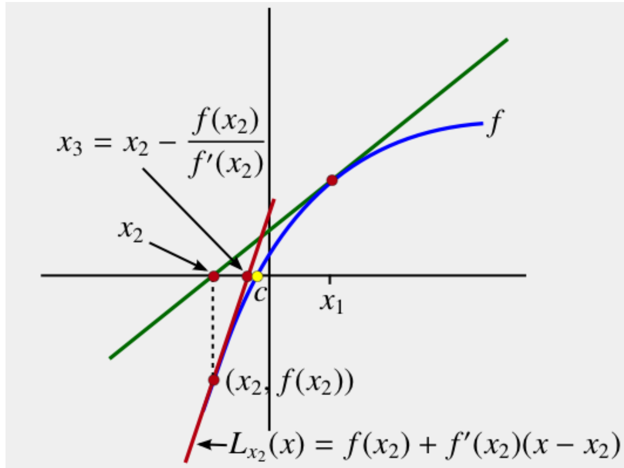
仅限于 converge

- Steps

1. start with an initial guess x_1
2. find tangent line, $L_{x_1}(x)$
3. Since $f(x) \approx L_{x_1}(x)$

   解 $f(x)=0$ 趋近于 $L_{x_1}(x)=0$

   解 $L_{x_1}(x)$ get next estimate x_2
4. 重复 2-4.



general formula: $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

*ex. 证明 starting with $x=a$, Newton's Method is diverge to root of function

→ $x_1 = a$ $\quad x_2 = \cdots$ $\quad x_3 = \cdots$

→ We need to show sequence decreasing / increasing, so that it won't able to converge to the root of the function. Since x_n(求) is already less / greater than the root.

→ 求 x 取值: $x_n > x_{n+1}$ / $x_n < x_{n+1}$

→ from x_n(求) seq is dec / inc, so will never converge to the root.

ex. $f(x) = x^3 - 2x + 2$ find a root start with 0

$$x_{n+1} = x_n - \frac{x^3 - 2x + 2}{3x^2 - 2}$$

$x_0 = 0 \qquad x_1 = 0 - \frac{2}{-2} = 1$

$x_1 = 1 \qquad x_2 = 1 - \frac{1}{1} = 0$

$x_3 = 0 \qquad x_4 = 1 \qquad x_5 = 0 \qquad x_6 = 1 \quad \cdots$

- Heron's Method Revisited

适用于 Newton's M 一直循环的情况

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

ep. estimate $\sqrt{7}$. 取 $a = \sqrt{7}$

3.7. Arithmetic rule of differentiation

↓ All prove by using limit definition of derivative

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

- $h(x) = cf(x)$ $\qquad h'(a) = cf'(a)$

prove: $$(cf)'\Big|_a = \lim_{h\to 0} \frac{cf(a+h)-cf(a)}{h} = c\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = cf'(a)$$

- $h(x) = f(x)+g(x)$ $\qquad h'(a) = f'(a)+g'(a)$

prove: $$\begin{aligned}(f+g)'\Big|_a &= \lim_{h\to 0} \frac{(f+g)(a+h)-(f+g)a}{h} \\ &= \lim_{h\to 0}\frac{f(a+h)+g(a+h)-f(a)-g(a)}{h} \\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} + \frac{g(a+h)-g(a)}{h} \\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} + \lim_{h\to 0}\frac{g(a+h)-g(a)}{h} \\ &= f'(a)+g'(a)\end{aligned}$$

- $h(x) = f(x)g(x)$ $\qquad h'(a) = f'(a)g(a)+f(a)g'(a)$

prove: $$\begin{aligned}(fg)'\Big|_a &= \lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h} \\ &= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a+h)g(a)+f(a+h)g(a)+f(a)g(a)}{h} \\ &= \lim_{h\to 0} f(a+h)\times\underbrace{\frac{g(a+h)-g(a)}{h}}_{g'(a)} + g(a)\times\underbrace{\frac{f(a+h)-f(a)}{h}}_{f'(a)} \\ &= f'(a)g(a)+f(a)g'(a)\end{aligned}$$

- $h(x) = \frac{f(x)}{g(x)}$ $\qquad h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$

prove: $\left(\frac{1}{f}\right)'\Big|_a = \lim_{h\to 0} \frac{\frac{1}{f(a+h)} - \frac{1}{f(a)}}{h}$

$= \lim_{h\to 0} \frac{\frac{f(a)-f(a+h)}{f(a+h)f(a)}}{h}$

$= \lim_{h\to 0} \frac{f(a)-f(a+h)}{h\cdot f(a+h)f(a)}$

$= -\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \times \frac{1}{f(a+h)f(a)}$

$= -\frac{f'(a)}{(f(a))^2}$

$\left(\frac{f}{g}\right)'\Big|_a = \left(f \times \frac{1}{g}\right)'\Big|_a$

$= f'(a)\frac{1}{g(a)} + f(a)\left(\frac{1}{g(a)}\right)'$

$= \frac{f'(a)}{g(a)} + f(a) \times \left(-\frac{g'(a)}{(g(a))^2}\right)$

$= \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}$

- The power rule for differentiation

$f(x) = x^\alpha \ (\alpha \in \mathbb{R},\ \alpha \neq 0)$ $\qquad f'(x) = \alpha x^{\alpha-1}$

3.8 Chain rule

- Chain rule

If $f(x)$ is differentiable at $x=a$. and $g(y)$ is differentiable at $y=f(a)$

Then $h(x) = g\circ f(x) = g(f(x))$ is differentiable at $x=a$

and $\qquad h'(a) = g'(f(a))\, f'(a)$

$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$ $\qquad f(g(h(x))) = f'(g(h(x))) \times g'(h(x)) \times h'(x)$

3.9. More Trignometric derivatives

已知 $f(x)=\sin x$ $f'(x)=\cos x$ ，求证↓

- $\frac{d}{dx}(\cos x) = -\sin x$

$\cos x = \sin(x+\frac{\pi}{2})$

let $y=\sin u$. $u = x+\frac{\pi}{2}$

$y = y(x) = \sin(x+\frac{\pi}{2}) = \cos x$

$$\frac{d}{dx}(\cos x) = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \cos u \times 1$$

$$= \cos(x+\frac{\pi}{2})$$

$$= \cos x \cos\frac{\pi}{2} - \sin x \sin\frac{\pi}{2}$$

$$= -\sin x$$

- $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{(\frac{d}{dx}\sin x)\cos x - \sin x(\frac{d}{dx}\cos x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

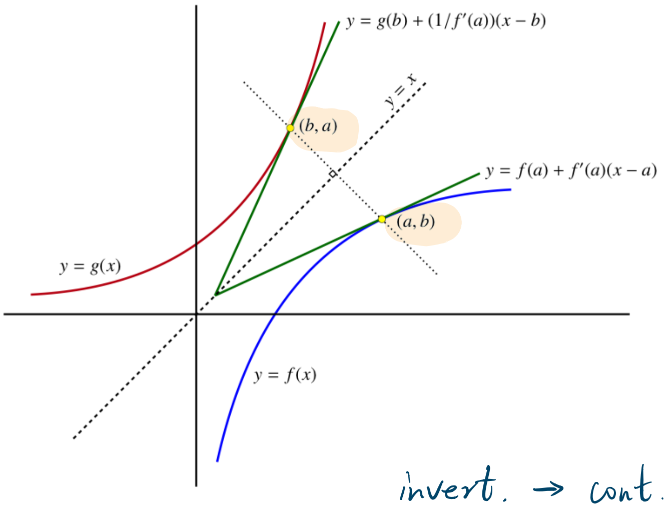
$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

- $\frac{d}{dx}(\sec x) = -\cot x \csc x$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

3.10 Inverse function theorem



- def. Inverse function

条件：① one-to-one

② f in domain = f^{-1} in range

则：$f(g(x)) = x$

- Inverse function theorem (IFT)

invert. → cont.

若 $f(x)$ invertible on $[c, d]$. $f(x)$

且 $f(x)$ diff'able at $a \in (c, d)$, $f'(a) \neq 0$.

→ 则 $f^{-1}(y)$ diff'able at $b = f'(a)$. $\boxed{f^{-1\prime}(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}}$

proof: $f(f^{-1}(x)) = x$

两边求导 $f'(f^{-1}(x)) \cdot \boxed{(f^{-1})'(x)} = 1$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

→ 则 $L^f_a(x)$ is invertible , $(L^f_a)^{-1}(x) = L^{f^{-1}}_b(y) = L^{f^{-1}}_{f(a)}(y)$

proof. tangent line $y = f(a) + f'(a)(x-a)$

inverse line. $x = f(a) + f'(a)(y-a)$

$$x - f(a) = f'(a)y - f'(a)a$$

$$y = a + \frac{1}{f'(a)}(x - f(a))$$

inverse func. $(L^f_a)^{-1}(x) = a + \frac{1}{f'(a)}(x - f(a))$

$$\boxed{y = f^{-1}(b) + \frac{1}{f'(a)}(x - b)}$$

ex. use IFT to calculate derivatives of inverse function. Find $(\ln x)'$

Let $f(x) = e^x$ $\quad f^{-1}(x) = \ln x$. $(x>0)$

$$(f^{-1})'(x) = (\ln x)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\ln x)} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

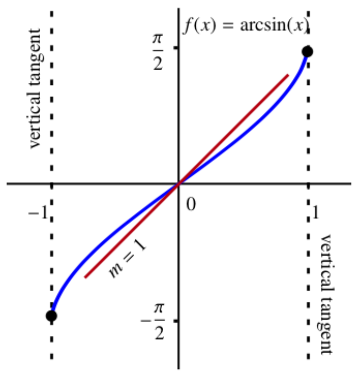
3.11 Derivatives of inverse trigonometric functions

- $f(x) = \arcsin x \qquad f'(x) = \frac{1}{\sqrt{1-x^2}}$

For any $x \in [-1, 1]$, if $y = f(x) = \arcsin x$

and $x = g(y) = \sin y$. $\quad y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

By IFT, $(\arcsin x)' = \frac{1}{(\sin y)'}$



$= \frac{1}{\cos y}$

$= \frac{1}{\sqrt{1-\sin^2 y}}$

$= \frac{1}{\sqrt{1-\sin^2(\arcsin x)}}$

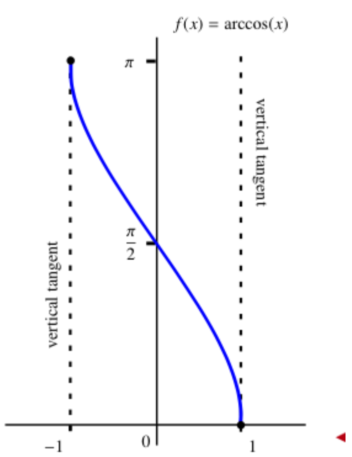
$= \frac{1}{\sqrt{1-x^2}} \qquad \sin(\arcsin x) = x$

- $f(x) = \arccos x \qquad f'(x) = \frac{-1}{\sqrt{1-x^2}}$

For any $x \in [-1, 1]$, if $y = f(x) = \arccos x$

and $x = g(y) = \cos y$. $\quad y \in [0, \pi]$

By IFT, $(\arccos x)' = \frac{1}{(\cos y)'}$



$= \frac{1}{-\sin y}$

$= \frac{-1}{\sqrt{1-\cos^2 y}}$

$= \frac{-1}{\sqrt{1-\cos^2(\arccos x)}}$

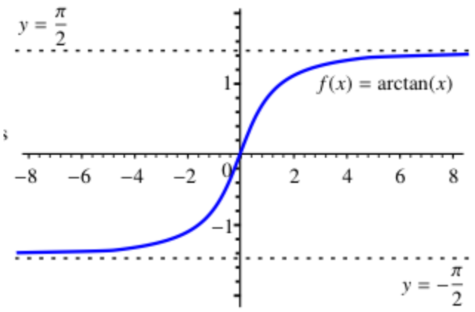
$= \frac{-1}{\sqrt{1-x^2}}$

- $f(x) = \arctan x$. $\quad f'(x) = \frac{1}{1+x^2}$

For any $x \in [-\infty, \infty]$, if $y = f(x) = \arctan x$

and $x = g(y) = \tan y$. $\quad y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

By IFT $\quad (\arctan x)' = \frac{1}{(\tan y)'}$



$$= \frac{1}{\sec^2 y}$$

$$= \frac{1}{1+\tan^2 y} \qquad \sec^2 x = 1 + \tan^2 x$$

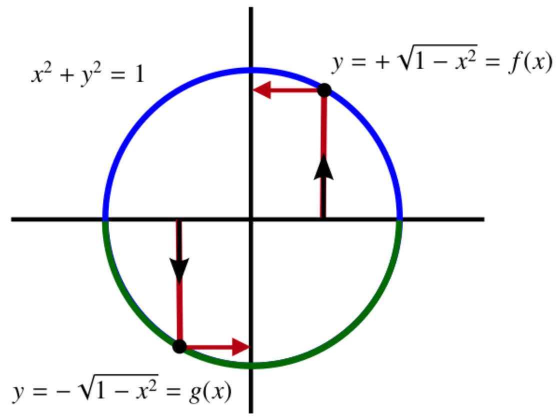
$$= \frac{1}{1+\tan^2(\arctan x)}$$

$$= \frac{1}{1+x^2}$$

3.12 Implicit differentiation

- 求 $\frac{dy}{dx}$ for $x^2+y^2=1$

$y = \pm\sqrt{1-x^2}$



$y = (1-x^2)^{\frac{1}{2}} \qquad y' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times (-2x)$

$\qquad = -x(1-x^2)^{-\frac{1}{2}}$

$\qquad = -\frac{x}{y}$

$y = -(1-x^2)^{\frac{1}{2}} \qquad y' = -\frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times (-2x)$

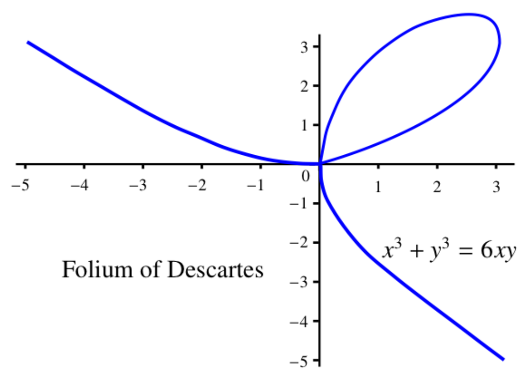
$\qquad = -\frac{x}{y}$

$\therefore \frac{dy}{dx} = \frac{-x}{y}$

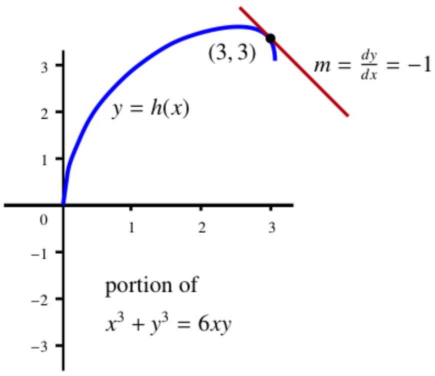
- Folium of descartes

def. graph of relation $\{(x,y) \mid x^3+y^3 = 3a\,xy\}$

$a \in \mathbb{N}$



ex. Find $h'(3)$ if $y=h(x)$ is diff'ble with $h(3)=3$. satisfying $x^3+y^3=6xy$



两边求导 $\quad \frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy)$

$$3x^2+3y^2\frac{dy}{dx} = 6y+6x\frac{dy}{dx}$$

提出 $\frac{dy}{dx}$ $\quad \frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x}$

$$h'(3) = \frac{6\times3-3\times3^2}{3\times3^2-6\times3} = -1$$

ex. Find $\frac{dy}{dx}$ for $3x^3y^3+x^2y+13x=12$

$$(3x^3y^3)'+(x^2y)'+(13x)'=0$$

$$9x^2y^3+9x^3y^2\cdot y'+2xy+x^2y'+13$$

$$y'(9x^3y^2+x^2) = -13-2xy-9x^2y^3$$

$$y' = \frac{-13-2xy-9x^2y^3}{9x^3y^2+x^2}$$

* ex. Find $\frac{dy}{dx}$ for $x^4+y^4=-1-x^2y^2$

LHS $\geqslant 0$ RHS $\leqslant -1$ So No Solution !!

- Logarithmic differentiation.

Find $h'(x)$ of $h(x) = g(x)^{f(x)}$

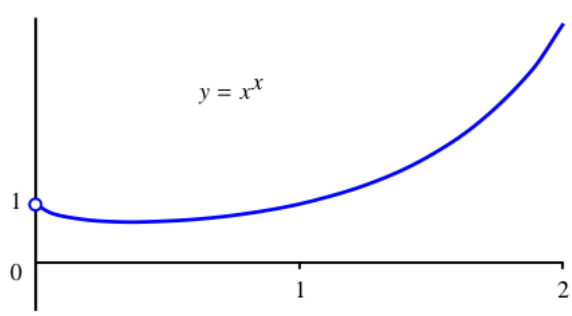
ex. find $\frac{dy}{dx}$ of $y=x^x$

两边加 ln : $\quad \ln y = x\ln x$

两边求导 : $\quad \frac{1}{y}\frac{dy}{dx} = \ln x + x\cdot\frac{1}{x}$

$$\frac{dy}{dx} = y(\ln x+1)$$

x代入 y : $\quad \frac{dy}{dx} = x^x(\ln x+1)$



ex 求 $\frac{dy}{dx}$ for $y=(\ln x)^{\sin x}$ for $x>1$

$[\ln f(x)]' = \frac{1}{f(x)} \cdot f'(x)$

$\ln y = \ln[(\ln x)^{\sin x}] = \sin x \cdot \ln(\ln x)$

两边求导

$\frac{1}{y} \cdot y' = \cos x \cdot \ln(\ln x) + \sin x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$

$y' = (\ln x)^{\sin x} \left[\cos x \ln(\ln x) + \frac{\sin x}{x \ln x} \right]$

ex. 求 y' for $y = x^{\arctan x}$ $(x>0)$

两边加 ln

$\ln y = \ln(x^{\arctan x}) = \arctan x \cdot \ln x$

两边求导

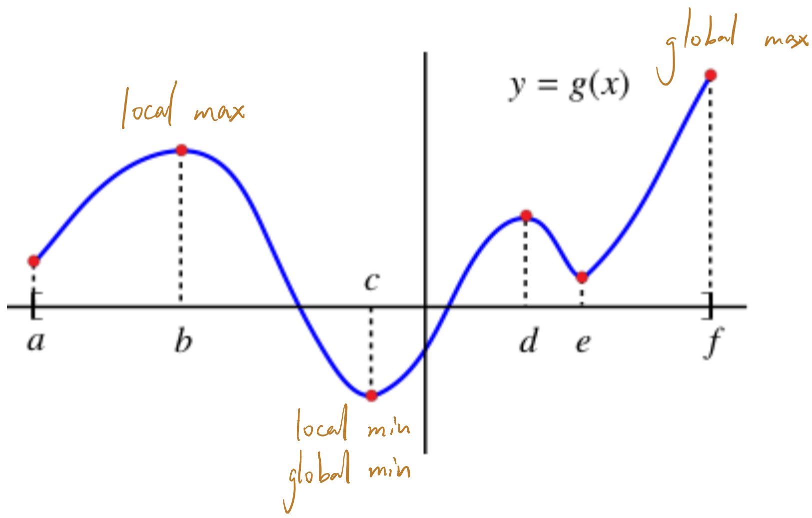
$\frac{1}{y} \cdot y' = \frac{\ln x}{1+x^2} + \frac{\arctan x}{x}$

$y' = x^{\arctan x} \left(\frac{\ln x}{1+x^2} + \frac{\arctan x}{x} \right)$

3.13 Local Extrema

- def.

global max / min :
域中的最高/最低之



local max : $\cap$ 先↑再↓

local min : $\cup$ 先↓再↑

c is local max for function f, if there exists an open interval (a, b) containing c. s.t. $f(x) \le f(c)$

c is local min for function f, if there exists an open interval (a, b) containing c. s.t. $f(x) \ge f(c)$

- local extrema theorem

If c is a local max/min and $f'(c)$ exists then $f'(c)=0$

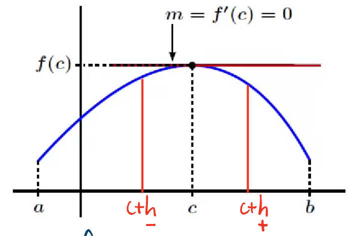
Proof.

Let c be local max

→ Then, interval (a,b) containing c, s.t $f(x) \le f(c) \quad \forall x \in (a,b)$

Also, $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$, and this limit exists.

which means $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$



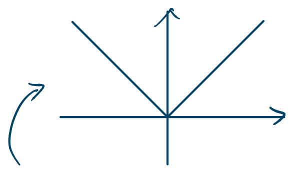
→ if $h>0$ and $a<c+h<b$, then $f(c+h) \le f(c) \Rightarrow \frac{f(c+h)-f(c)}{h} \le 0$

which means $f'(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0$

→ If $h<0$ and $a<c+h<b$, then $f(c+h) \le f(c) \Rightarrow \frac{f(c+h)-f(c)}{h} \ge 0$

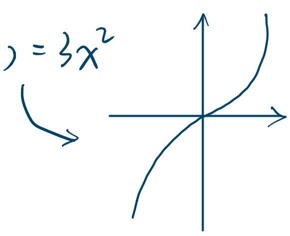
which means $f'(c) = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \ge 0$

$\therefore f'(c) \ge 0$ and $f'(c) \le 0 \Rightarrow f'(c)=0$



* c is a local max/min $\not\Rightarrow f'(c)=0$ counter example : $f(x)=|x|$

* $f'(c)=0 \not\Rightarrow x=c$ is local max/min counter example : $f'(x)=3x^2$



- critical point.

c is critical point of function f, if $f'(c)=0$ or $f'(c)$ DNE

ex. find the global max/min of $f(x)=x^3-3x+2$ on $[-3,3]$

求 $f'(x)$ 不存在 / $=0$

$f'(x)=3x^2-3 = 3(x+1)(x-1)=0$

$x=1 \quad x=-1$ both inside $[-3,3]$

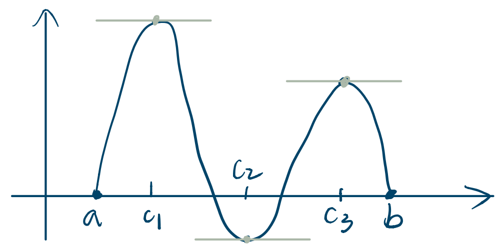
check : $f(-3) = -16$
$f(-1) = 4$
$f(1) = 0$
$f(3) = 20$

global max : $(3,20)$

global min : $(-3,-16)$

4.1 The Mean Value Theorem

- Rolle's Theorem



If $f(x)$ satisfies

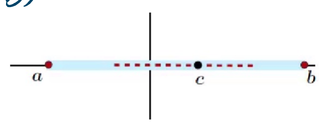
① $f(x)$ cont. for $x \in [a,b]$ 连续
② $f'(x)$ exist for $x \in (a,b)$ 可导
③ $f(a) = f(b) = 0$ 交于0

Then, there exists at least a $c \in (a,b)$. s.t $f'(c) = 0$

Proof. set 3 cases:

case 1: $f(x) = 0$ for all $x \in [a,b] \Rightarrow f'(c) = 0$. $c \in (a,b)$

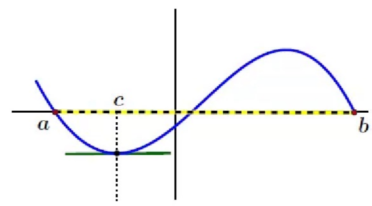
so there's many to choose from. ⟶



case 2: $f(x) \neq 0$ for some $x \in [a,b]$

By EVT, f attains it global max on $[a,b]$ and $f(x_0) > 0$ and $f(a) = f(b) = 0$. global max will occur on $c \in (a,b)$

This means that c is c-p of f.

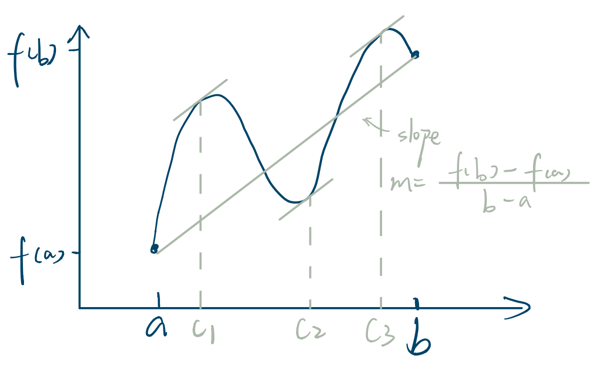


- Mean Value Theorem (MVT)

If $f(x)$ satisfies

① $f(x)$ cont. for $x \in [a,b]$ 连续
② $f'(x)$ exists for $x \in (a,b)$ 可导

Then there exists $c \in (a,b)$. s.t $f'(c) = \dfrac{f(b) - f(a)}{b-a}$ 斜率相同



proof. 斜代

Let $g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ $\quad g'(x) = \frac{f(b)-f(a)}{b-a}$ $\quad x \in (a,b)$

Let $H(x) = f(x) - g(x)$

then $H(a) = f(a) - g(a) = f(a) - [f(a) + \frac{f(b)-f(a)}{b-a}(a-a)] = 0$

$H(b) = f(b) - g(b) = f(b) - [f(a) + \frac{f(b)-f(a)}{b-a}(b-a)] = 0$

Rolle's Theorem ↓

$\exists\, c \in (a,b)$ s.t. $0 = H'(c)$

$= f'(c) - g'(c)$

$= f'(c) - \frac{f(b)-f(a)}{b-a}$

$\therefore f'(c) = \frac{f(b)-f(a)}{b-a}$

ex. Let $f(x) = x^2 + 2x + 1$ and $1 \le x \le 2$. Find x that satisfy MVT.

→ f is cont. on $[1,2]$ and diff'ble on $(1,2)$

→ Due to MVT. $\exists\, c$. s.t:

$$f'(c) = \frac{f(2)-f(1)}{2-1} = \frac{9-4}{1} = 5$$

→ Find c. s.t. $f'(c) = 5$

$f'(x) = 2x+2$ so $f'(c) = 2c+2 = 5 \Rightarrow c = \frac{3}{2}$

4.2 Applications of MVT

4.2.1 Antiderivatives

- def.

If $F'(x) = f(x)$ for all x in interval I, then F is antiderivative for f on I.

ep. $F(x) = \frac{x^3}{3}$ is antiderivative of $f(x)$ $\quad F(x) = \frac{x^3}{3} + 2$ is also anti' of $f(x)$

If $F(x)$ is anti' of $f(x)$, then $F(x) + C$ is also anti' of $f(x)$

- Constant function theorem

If $f'(x) = 0 \quad \forall x \in I$, then $\exists \alpha \in \mathbb{R}$ s.t. $f(x) = \alpha \quad \forall x \in I$

proof.

Let $x_1 \in I$, $f(x) = k$. Let $x_2 \in I$. $x_2 \neq x_1$

$\because f$ is diff'ble & cont. on I

$\therefore$ MVT applies on closed interval with endpoint x_1 & x_2

Due to MVT, $\exists c \in (x_1, x_2)$, $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

$\because f'(c) = 0 \quad \therefore 0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

$\therefore f(x_2) = f(x_1) = \alpha$

- The Antiderivative Theorem 为何 $\int = \cdots + C$

If $f'(x) = g'(x)$ for all $x \in I$, then there exists α s.t. $f(x) = g(x) + \alpha \quad \forall x \in I$

proof.

Let $H(x) = f(x) - g(x)$

Then $H'(x) = f'(x) - g'(x) = 0 \quad x \in I$.

$\therefore \exists \alpha \in \mathbb{R}$ s.t $H(x) = \alpha \Rightarrow f(x) = g(x) + \alpha \quad \forall x \in I$

- Leibniz Notation for Antiderivatives

$$\int f(x)\, dx$$

- Power Rule for Antiderivatives

$$\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

4.2.2 Increasing Function Theorem

- def. of Increasing / decreasing function

i) f is increasing on I if $f(x_1) < f(x_2)$ $\forall x_1, x_2 \in I$. $x_1 < x_2$

ii) f is decreasing on I if $f(x_1) > f(x_2)$ $\forall x_1, x_2 \in I$. $x_1 < x_2$

iii) f is non-increasing on I if $f(x_1) \leq f(x_2)$ $\forall x_1, x_2 \in I$. $x_1 < x_2$

iv) f is non-decreasing on I if $f(x_1) \geq f(x_2)$ $\forall x_1, x_2 \in I$. $x_1 < x_2$

- Increasing / decreasing Function Theorem

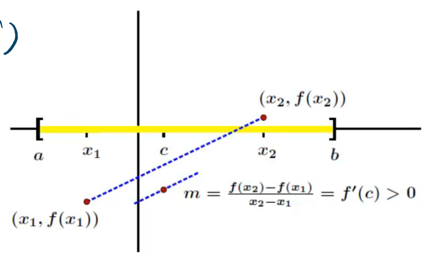
i) $f(x)$ is increasing. $f'(x) > 0$ $\forall x \in I$. $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

ii) $f(x)$ is non-decreasing. $f'(x) \geq 0$ $\forall x \in I$. $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

iii) $f(x)$ is decreasing. $f'(x) < 0$ $\forall x \in I$. $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

iv) $f(x)$ is non-increasing. $f'(x) \leq 0$ $\forall x \in I$. $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

(i)



i) Let I be an interval and assume that $f'(x) > 0$ for all $x \in I$. If $x_1 < x_2$ are two points in I, then

$$f(x_1) < f(x_2).$$

Proof.

Assume $f'(x) > 0$ $\forall x \in I$, $x_1 \cdot x_2 \in I$. $x_1 < x_2$

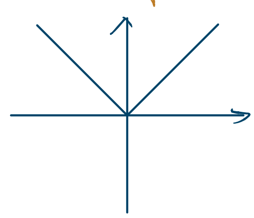
By MVT. $\exists c \in (x_1, x_2)$. s.t $\frac{f(x_2)-f(x_1)}{x_2 - x_1} = f'(c) > 0$

$\because x_2 - x_1 > 0$ $\therefore f(x_2) - f(x_1) > 0$ $f(x_1) < f(x_2)$

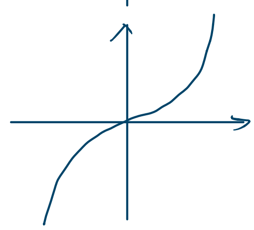
$*$ If f is increasing, then $f'(x) > 0$ $\times$ 判断先考虑 $f'(x)$ 不存在和 $f'(x)=0$

counter example ① $f(x) = |x|$ inc on $[0, 1]$. $f'(0)$ DNE

$\lim_{x \to 0^+} \neq \lim_{x \to 0^-}$



② $f(x) = x^3$ inc on $[-1, 1]$. $f'(0) = 0$



4.2.3 Functions with bounded derivatives

- Observation

Assume f cont. on $[a,b]$. diff'ble on (a,b) and $m \le f'(x) \le M$ $x \in (a,b)$

Pick some $x \in [a,b]$. MVT is true on $[a,x]$.

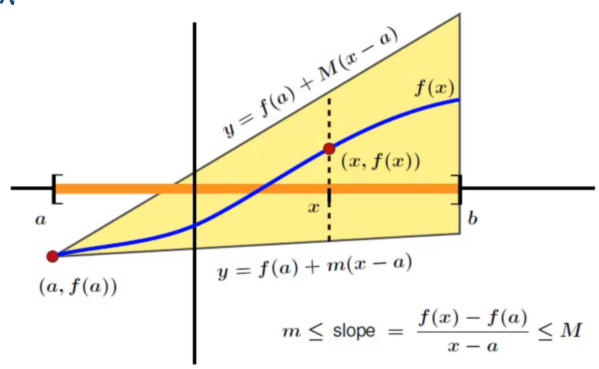
which means $\exists\, c \in [a,x]$ s.t. $f'(c) = \dfrac{f(x)-f(a)}{x-a}$

$\because m \le f'(x) \le M$.

$\therefore m \le \dfrac{f(x)-f(a)}{x-a} \le M$

$m(x-a) \le f(x)-f(a) \le M(x-a)$

$f(a)+m(x-a) \le f(x) \le f(a)+M(x-a)$



- The bounded derivative theorem (BDT)

If f ① cont. on $[a,b]$

② diff'ble on (a,b) with $m \le f'(x) \le M$ for $x \in (a,b)$

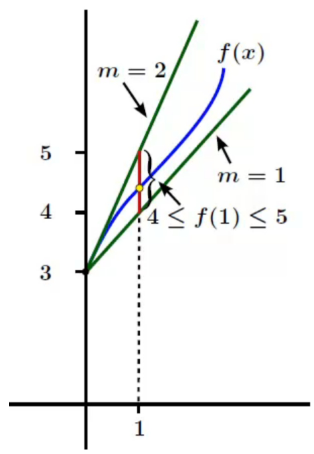
Then $f(a)+m(x-a) \le f(x) \le f(a)+M(x-a)$ $\forall x \in [a,b]$

ex. Assume $f(0)=3$. $1 \le f'(x) \le 2$. $\forall x \in [0,1]$

Show $4 \le f(1) \le 5$.

$f(0)+1(1-0) \le f(1) \le f(0)+2(1-0)$

$4 = 3+1 \le f(1) \le 3+2 = 5$



ex. If $f(12)=2$ and $1 \le f'(x) \le 3$ $\forall x \in \mathbb{R}$. find interval for $f(20)$

BDT. $f(a) + m(x-a) \le f(x) \le f(a) + M(x-a)$

$$f(12) + 1(x-12) \le f(x) \le f(12) + 3(x-12)$$

$$2 + (x-12) \le f(x) \le 2 + 3(x-12)$$

when $x=20$ $\quad 20 - 10 \le f(20) \le 2 + 3(20-12)$

$$10 \le f(20) \le 26$$

ex. Prove $\sqrt{66} \in \left(8+\frac{1}{9}, 8+\frac{1}{8}\right)$

选两个好算的数

Let $f(x)=\sqrt{x}$. $f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f$ is cont on $[64, 81]$ and diff'ble on $(64, 81)$

if $x \in [64, 81]$, then $f'(x) = \frac{1}{2\sqrt{x}} \in \left[\frac{1}{18}, \frac{1}{16}\right]$

By BDT, $f(a) + m(x-a) \le f(x) \le f(a) + M(x-a)$

$$\sqrt{64} + \frac{1}{18}(x-64) \le f(x) \le \sqrt{64} + \frac{1}{16}(x-64)$$

when $x = 66$ $\quad 8 + \frac{1}{18}(66-64) \le \sqrt{66} \le 8 + \frac{1}{16}(66-64)$

$$8 + \frac{1}{9} \le \sqrt{66} \le 8 + \frac{1}{8} \quad \text{as required}$$

4.2.4 Comparing Functions using their derivatives

- Theorem

Let f & g be continuous at $x=a$ and $f(a)=g(a)$

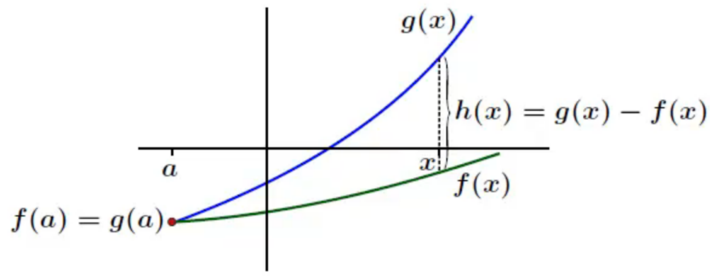
1). f & g diff'ble at $x>a$, $f'(x)\leq g'(x) \Rightarrow f(x)\leq g(x) \quad \forall x>a$

2) f & g diff'ble at $x<a$, $f'(x)\leq g'(x) \Rightarrow f(x)\geq g(x) \quad \forall x<a$

proof: 1)

Assume $f(x)$ & $g(x)$ are diff'ble for $x>a$. $f'(x)\leq g'(x) \quad \forall x>a$.

$f(a)=g(a)$



Let $h(x)=g(x)-f(x)$

$h(a)=g(a)-f(a)=0$

$h'(x)=g'(x)-f'(x)\geq 0 \quad \forall x>a$

By MVT, $\exists c\in(a,x)$ s.t $h'(c)=\dfrac{h(x)-h(a)}{x-a}\geq 0$.

$\because h(a)=0$. $x-a>0$ $\therefore h(x)\geq 0$

$\therefore h(x)=g(x)-f(x)\geq 0$ $\quad g(x)\geq f(x)$ for $x>a$

ex. Show $\underbrace{x-\frac{x^2}{2}}_{f(x)} < \underbrace{\ln(x+1)}_{g(x)} < \underbrace{x}_{h(x)} \quad \forall x>0$

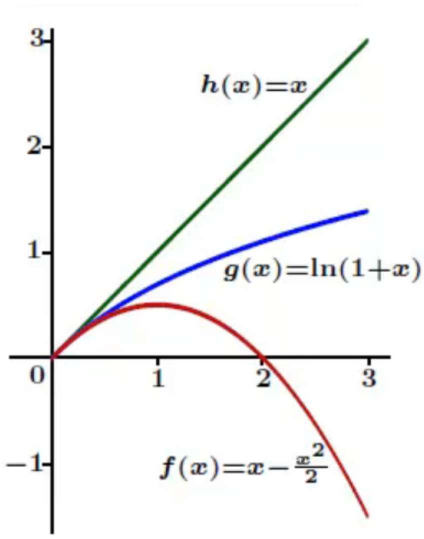
$f'(x)=1-x \qquad g'(x)=\dfrac{1}{x+1} \qquad h'(x)=1$

$x>0 \quad g'(x)<h'(x)=1 \Rightarrow g(x)<h(x)$

$1-x^2<1 \qquad 1-x<\dfrac{1}{x+1} \qquad f'(x)<g'(x) \Rightarrow f(x)<g(x)$

$\therefore$ for $x>0$ $\quad f'(x)<g'(x)<h'(x) \Rightarrow f(x)<g(x)<h(x)$

ex. Show $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$



$\because$ $x-\frac{x^2}{2} < \ln(1+x) < x.$ $\forall x>0$

$1-\frac{x}{2} < \frac{\ln(1+x)}{x} < 1$ $\forall x>0$

$\because$ $\lim_{x\to 0^+} 1-\frac{x}{2} = 1 = \lim_{x\to 0^+} 1$ $\therefore$ $\lim_{x\to 0^+} \frac{\ln(1+x)}{x} = 1$

$\because$ $n\to\infty$, $\frac{1}{n}\to 0$

$\therefore$ $\lim_{x\to\infty} \frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty} \ln(1+\frac{1}{n})^n = 1$

$\lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{n\to\infty} e^{\ln(1+\frac{1}{n})^n} = e^{(\lim_{n\to\infty}\ln(1+\frac{1}{n})^n)} = e$

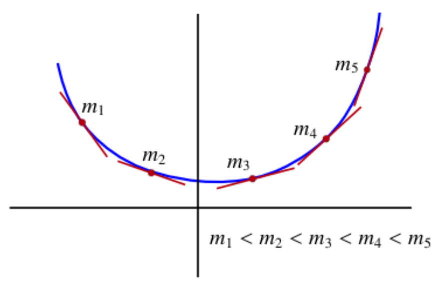
- Theorem

$\alpha\in\mathbb{R}$ $\lim_{n\to\infty}(1+\frac{\alpha}{n})^n = e^{\alpha}$

4.2.5 The second derivative

- Second derivative

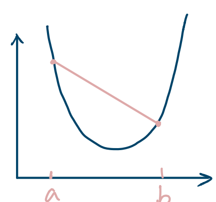
$$f'' = \frac{d}{dx}(f')$$

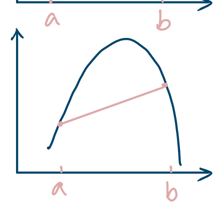


- def. concave up/down

f is concave upwards on I. $\forall a.b\in I$
the secant line $(a, f(a)).(b, f(b))$ lies above the graph

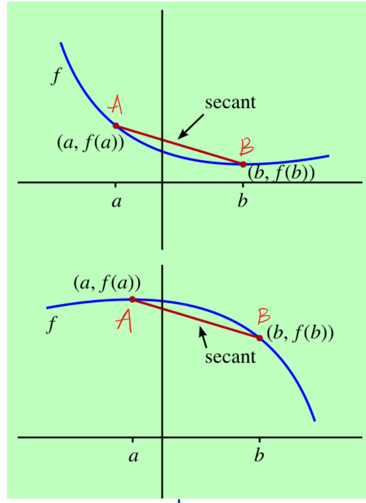
f is concave downwards on I. $\forall a.b\in I$
the secant line $(a, f(a)).(b, f(b))$ lies below the graph





Theorem: if $f''>0$ $\forall x\in I$. then f is concave up on I.
if $f''<0$ $\forall x\in I$. then f is concave down on I.

4.2.6 Formal definition of Concavity



- Concavity.

f concave up. on $I \Rightarrow$ 线段AB在f上方

f concave down on $I \Rightarrow$ 线段AB在f下方

- Second derivative test for concavity

$f''(x) > 0$ for each x in I, then graph of f is concave upward on I.

$f''(x) < 0$ for each x in I, then graph of f is concave downward on I.

- def. inflection point

c is an inflection point for f, if

① f is cont. at $x=c$

② concavity of f changes at $x=c$

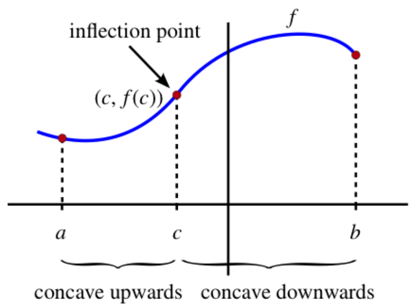
- Test for inflection point

(By IVT)

If ① $f''(x)$ is cont. at $x=c$.

② $(c, f(c))$ is inflection point of f.

Then $f''(c)=0$.



※ $f(c)=0$ 不代表 $f''(c)=0$ eg. $f(x)=x^4$ $f''(x)=12x^2$

ex. find intervals of concavity and inflection points for $f(x)=x^4-6x^2$

→ $f'(x)=4x^3-12x$ $f''(x)=12x^2-12=12(x+1)(x-1)=0$

求$f''(x)=0$时的点

candidate point : $x=1$ $x=-1$

→

		-1		1	
f''	+		−		+
f	∪		∩		∪

列表 $f''(x)>0$ up
$f''(x)<0$ down

$\therefore f$ concave up on $(-\infty,-1]$ & $[-1,\infty)$ and concave down on $(-1,1)$

ex. Find intervals of concavity and inflection points for $f(x)=\frac{1}{x}$

→ $f'(x) = -\frac{1}{x^2}$ $f''(x) = \frac{2}{x^3}$. f'' DNE at $x=0$

→

		0	
f''		$-$	$+$
f		$\frown$	$\smile$

∴ f is concave up on $(0,\infty)$ and concave down on $(-\infty,0)$

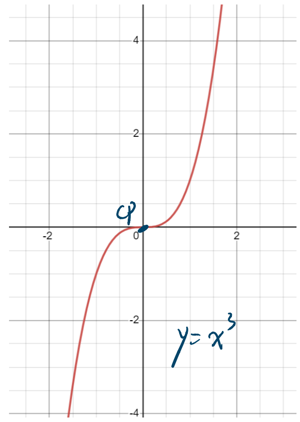
* $x=0$ is not a P.O.I. since f not cont. on $x=0$

4.2.7 Classifying Critical Points.

- def.

If $f'(c)=0$ / $f'(c)$ DNE.

Then c is critical point



* CP 是否是 local max/min?

No. eg. $f(x)=x^3$ $x=0$, $f'(x)=3x^2=0$

(0,0) 在 $f(x)$ 中不是 local，是拐点

- Theorem: The first derivative test.

Assume c is a critical point for f and f is cont. at $x=c$

1). If (a,b) contain c. s.t.

$f'(x)>0$ for $x \in (a,c)$

$f'(x)<0$ for $x \in (c,b)$

Then f has local max at $x=c$

		c
f'	+	−
f	↗	↘

2). If (a,b) contain c. s.t.

$f'(x)<0$ for $x \in (a,c)$

$f'(x)>0$ for $x \in (c,b)$

Then f has local min at $x=c$

		c
f'	−	+
f	↘	↗

ex. Find local max/min for $f(x)=\frac{x^3}{3}-\frac{3x^2}{2}+2x+1$

$f'(x)=x^2-3x+2=(x-1)(x-2)=0$ c.p. $x=1$. $x=2$

		1	2
f'	+	−	+
f	↗	↘	↗

- Theorem: The second derivative test

Assume $f'(c)=0$ $\quad f''(x)$ is cont at $x=c$

1) $f''(c)<0$ $\quad f$ is local max at $x=c$

2) $f''(c)>0$ $\quad f$ is local min at $x=c$

3) $f''(c)=0$ $\quad$ no information

ex. Find local max / min for $f(x)=\frac{x^3}{3}+3x^2-7x+3$

$f'(x)=x^2+6x-7=(x-1)(x+7)=0$ $\quad x=1 \quad x=-7$ are CPs.

$f''(x)=2x+6$ is cont. on $\mathbb{R}$, so we can use f'' at $x=1 \quad x=7$

$\because f''(-7)=-8<0 \Rightarrow x=-7$ is a local max.

$f''(1)=8>0 \Rightarrow x=1$ is a local min.

4.3 L'Hôpital's Rule

- L'Hôpital's Rule

If $f'(x)$ & $g'(x)$ exist near $x=a$, $g'(x) \neq 0$ near $x=a$ except $x=a$

and $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ is $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$

$0\cdot\infty$. $\infty-\infty$. 1^{∞} ∞^{0} 0^{0}

需要转换

Then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$

proof.

Let $h(x) = \frac{f(x)}{g(x)}$ Assume $\lim\limits_{x\to a} f(x) = 0 = \lim\limits_{x\to a} g(x)$

$h(x) = \frac{f(x)}{g(x)} = \frac{0}{0}$ ← indeterminate

$$\frac{f(x)}{g(x)} = \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

$\because f(a) = 0 = g(a)$ f' & g' are continuous

$$\therefore \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

* 有时 L'Hôpital's rule 无法使用

(LHR 与直接解 结果不同时) * 用完 LHR 要代入原式检查

ep. $\lim\limits_{x\to 0^+} \frac{\ln x}{x}$

Lopital: $\lim\limits_{x\to 0^+} \frac{\ln x}{x} = \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{1} = \lim\limits_{x\to 0^+} \frac{1}{x} = \infty$

直接解 $\lim\limits_{x\to 0^+} \frac{\ln x}{x} = \frac{-\infty}{0} = -\infty$

ex. 求 $\lim_{x\to 0} \frac{e^{x^2}-\cos x}{x^2}$ 　　不断使用 LHR 直到能解

→ Let $f(x)=e^{x^2}-\cos x$ 　　$g(x)=x^2$

$\lim_{x\to 0} f(x) = \lim_{x\to 0}(e^{x^2}-\cos x) = e^0-\cos 0 = 0$

$\lim_{x\to 0} g(x) = \lim_{x\to 0} x^2 = 0$ 　　So limit is indeterminat type $\frac{0}{0}$

→ Get $f'(x)=2xe^{x^2}+\sin x$. 　　$g'(x)=2x$.

$\lim_{x\to 0} f'(x) = \lim_{x\to 0} 2xe^{x^2}+\sin x = 0$

$\lim_{x\to 0} g'(x) = \lim_{x\to 0} 2x = 0$ 　　So limit is indeterminate type $\frac{0}{0}$

→ Get $f''(x)=2e^{x^2}+2x\cdot 4xe^{x^2}+\cos x = 2e^{x^2}+4x^2e^{x^2}+\cos x$.

$g''(x)=2$.

$\lim_{x\to 0} f''(x) = \lim_{x\to 0} 2e^{x^2}+2x\cdot 4xe^{x^2}+\cos x = 3$

$\lim_{x\to 0} g''(x) = \lim_{x\to 0} 2 = 2$.

→ Apply L'Hôpital's Rule

$\lim_{x\to 0} \frac{2xe^{x^2}+\sin x}{2x} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \frac{3}{2}$

- Indeterminate form

① $0\cdot\infty$ 　　$f\cdot g = \frac{f}{\frac{1}{g}}$

ex. $\lim_{x\to 0^+} xe^{\frac{1}{x}}$ 　$(0\cdot\infty)$

$= \lim_{x\to 0^+} \frac{x}{e^x}$ 　$\left(\frac{0}{0}\right)$

LHR $= \lim_{x\to 0^+} \frac{1}{e^x}$ 　$\left(\frac{1}{0}\right)$

$= \lim_{x\to 0^+} e^{\frac{1}{x}}$

$= \infty$

ex. $\lim_{x\to 0^+} x\ln x$

$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$ 　$\left(\frac{-\infty}{\infty}\right)$

LHR $= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)}$ 　$\left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to 0^+} -\frac{x^2}{x}$

$= \lim_{x\to 0^+} -x$

$= 0$

② $0 \cdot \pm\infty$

ex. $\lim_{x\to 0^+} x e^{\frac{1}{x}}$ $\quad (0\cdot\infty)$

$= \lim_{x\to 0^+} \frac{e^{x^{-1}}}{x^{-1}} = \lim_{x\to 0^+} \frac{-x^{-2} e^{x^{-1}}}{-x^{-2}} = \lim_{x\to 0^+} e^{\frac{1}{x}} = \infty$

ex. $\lim_{x\to 0^+} x \ln x$ $\quad (0 \cdot -\infty)$

$= \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \overset{LHR}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0$

$(\frac{-\infty}{\infty})$

③ $\infty - \infty$ $\quad$ (通分. 因式分解) $\quad$ (用 log / 三角函数化简)

ex. $\lim_{x\to\infty} \ln x - \ln(3x+1)$

$= \lim_{x\to\infty} \ln\left(\frac{x}{3x+1}\right) = \ln\left[\lim_{x\to\infty} \frac{x}{3x+1}\right]$

$\lim_{x\to\infty} \frac{x}{3x+1} = \lim_{x\to\infty} \frac{1}{3} = \frac{1}{3}$

$= \ln\left[\lim_{x\to\infty} \frac{1}{3+\frac{1}{x}}\right] = \ln\frac{1}{3}$

④ $0^0,\ \infty^0,\ 1^\infty$

(写作 $f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\cdot \ln f(x)}$), $g(x)\ln f(x)$ will be of type $0\cdot\infty$

ex. $\lim_{x\to 0^+} x^x$

$= \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x} = e^0 = 1$

ex. $\lim_{x\to\infty} (1+\frac{1}{x})^x$

$= \lim_{x\to\infty} e^{\ln(1+\frac{1}{x})^x} = e^{\lim_{x\to\infty} x\ln(1+\frac{1}{x})}$

$\lim_{x\to\infty} x\ln(1+\frac{1}{x})$

$= \lim_{x\to\infty} \frac{\ln(1+x^{-1})}{x^{-1}}$ $\quad (\frac{0}{0})$

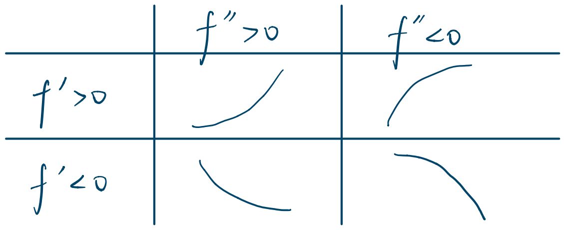
$\overset{LHR}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x^{-1}} \cdot -x^{-2}}{-x^{-2}}$

$= \lim_{x\to\infty} \frac{1}{1+x^{-1}} = 1$

So, $\lim_{x\to\infty} (1+\frac{1}{x})^x$

$= e^{\lim_{x\to\infty} x\ln(1+\frac{1}{x})} = e^1 = e$

4.4 Curve Sketching



ex. Sketch $f(x)=x^4-16x^2$

① domain $\mathbb{R}$.

② x-int $(y=0)$

$x^4-16x^2=x^2(x+4)(x-4)$

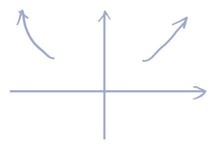
$x=0 \quad x=\pm 4$

y-int $(x=0)$ $\quad y=0$

③ asymptotes

V.A. no

H.A. $\lim\limits_{x\to\pm\infty} x^4-16x^2=\infty$ $\quad\therefore$ no H.A.



④ $f'(x)=4x^3-32x=4x(x^2-8)=4x(x-\sqrt{8})(x+\sqrt{8})$

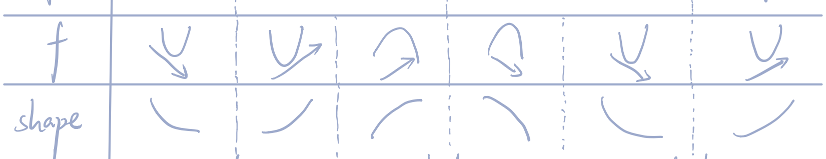
$f'(x)=0 \quad x=0,\ \pm\sqrt{8} \qquad (0,0)\ (\sqrt{8},-64)\ (-\sqrt{8},-64)$

⑤ $f''(x)=12x^2-32=4(3x^2-8)$

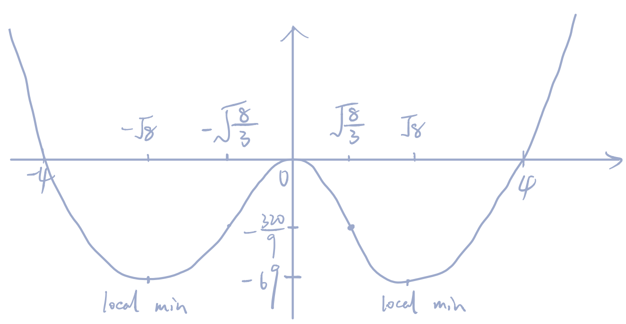
$f''(x)=0 \qquad x=\pm\sqrt{\frac{8}{3}} \qquad \left(\pm\sqrt{\frac{8}{3}},\ -\frac{320}{9}\right)$

⑥

	$-\sqrt{8}$	$-\sqrt{\frac{8}{3}}$	0	$\sqrt{\frac{8}{3}}$	$\sqrt{8}$	
f''	+		−		+	
f'	−	+		−		+



local min ($-\sqrt{8}$) · POI ($-\sqrt{\frac{8}{3}}$) · local max (0) · POI ($\sqrt{\frac{8}{3}}$) · local min ($\sqrt{8}$)



① f 的取值范围

② 找 x & y 轴交点

③ 渐近线

H.A. $\lim\limits_{x\to\pm\infty} f(x)$

V.A. 分母=0 / ln 0

④ 求 $f'(x)$.

CP: $f'(x)=0$ / DNE

⑤ 求 $f''(x)$

inflection point: $f''(x)=0$ / DNE

⑥ 以下所有的 x.

f''

f'

f

shape

plot

5.1. Introduction to Taylor Polynomials

- Recall

→ If f is diff'ble at $x=a$. then $L_a^f(x) = f(a)+f'(a)(x-a)$

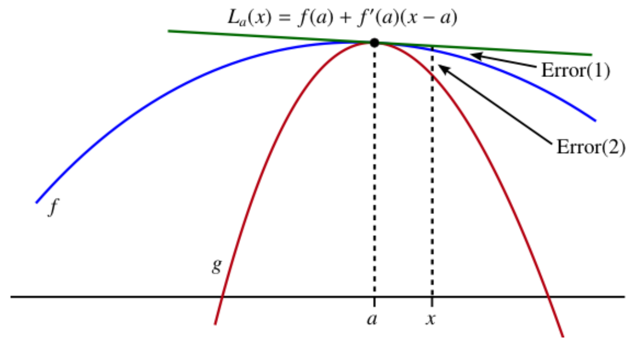
properties: ① $L_a^f(a) = f(a)$

② $(L_a^f)'(a) = f'(a)$

③ $f(a)$ 与 $f'(a)$ 为常数

④ $x \cong a \Rightarrow L_a^f(x) \cong f(x)$

$L_a^f(x)$ is the tangent line



→ The error in L_a depend on:

① $|x-a|$ x到a的距离

② $|f''(x)|$ near $x=a$ a周围的弯度

Q: find polynomial $p(x) = c_0 + c_1(x-a) + c_2(x-a)^2$

s.t $p(a) = f(a)$ $p'(a) = f'(a)$ $p''(a) = f''(a)$

i) $f(a) = p(a) = c_0 + c_1(a-a) + c_2(a-a)^2$ $\qquad c_0 = f(a)$

ii) $f'(a) = p'(a) = c_1 + 2\cdot c_2(a-a)$ $\qquad c_1 = f'(a)$

iii) $f''(a) = p''(a) = 2c_2$ $\qquad c_2 = \frac{f''(a)}{2}$

$$T_{2,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 = L_a^f(x) + \frac{f''(a)}{2}(x-a)^2$$

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{k}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{n}(a)}{n!}(x-a)^n$$

$0 < k < n \qquad T_{n,a}^{(k)}(a) = f^{(k)}(a)$

ex. Find $T_{4,0}(x)$ for $f(x) = \sin x$

$f(0) = \sin x = \sin 0 = 0$

$f'(0) = \cos x = \cos 0 = 1$

$f''(0) = -\sin x = -\sin 0 = 0$

$f^{(3)}(0) = -\cos x = -1$

$f^{(4)}(0) = \sin x = 0.$

$$T_{4,0}(x) = 0 + x + 0 \cdot \frac{x^2}{2!} - \frac{x^3}{3!} + 0 \times \frac{x^4}{4!}$$
$$= x - \frac{x^3}{3!}$$

ex. Find $T_{4,0}(x)$ for $f(x) = \cos x$

$$T_{4,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

ex. Find $T_{3,1}(x)$ for $f(x) = \ln x$

$f(1) = \ln 1 = 0$

$f'(1) = x^{-1} = 1$

$f''(1) = -x^{-2} = -1$

$f^{(3)}(1) = 2x^{-3} = 2$

$$T_{3,1}(x) = 0 + (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$T_{3,1}(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

5.2 Taylor's Theorem and Error

- Taylor Remainder.

Assume f is n times differentiable at $x=a$.

$$\underline{R_{n,a}(x)} = f(x) - T_{n,a}(x).$$

$\uparrow$

with degree Taylor remainder function centred $x=a$

$$\text{Error} = |R_{n,a}(x)|$$

- Taylor's Theorem

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

* if $n=0$, then $\exists c$ between x & a s.t. $f(x) - T_{0,a}(x) = f'(c)(x-a)$

$$f(x) - f(a) = f'(c)(x-a) \qquad f'(c) = \frac{x-a}{f(x)-f(a)}$$

if $n=1$, $R_{1,a}(x) = f(x) - T_{1,a}(x) = \frac{f''(c)}{2}(x-a)^2$

$\uparrow$ $T_{1,a}(x) = f(a) + f'(a)(x-a) = L_a^f(x)$

$$|f(x) - L_a^f(x)| = \left|\frac{f''(c)}{2}\right| (x-a)^2 \leq \frac{M}{2}(x-a)^2 \qquad M \geq |f''(c)|$$

Taylor Thm 无法让人找到 c. 但能找到 upper bound on error.

- Corollary: Taylor's Inequality

if $|f^{(n+1)}c| \leq M$, then

$$\text{error} = |R_{n,a}(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \forall c \in [x,a]/[a,x]$$

ex. $f(x) = \sqrt{1+x}$

1) Find $T_{2,0}(x)$

2) Approximat $\sqrt{1.1}$ using $T_{2,0}(x)$

3) Find upper bound on error

4) Is $T_{2,0}(x)$ an over or underestimate for $f(x)$ if $x \geq 0$?

1) $T_{2,0}(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$

2) $\sqrt{1.1} = f(0.1) \simeq T_{2,0}(0.1) = 1 + \frac{0.1}{2} - \frac{0.01}{8} = \frac{839}{800}$

3) $f'''(x) = \frac{3}{8(1+x)^{\frac{5}{2}}}$ is decreasing on $[0, 0.1]$.

$|f'''(x)| \leq \frac{8}{8 \cdot 1^{\frac{5}{2}}} = \frac{3}{8}$. $M = \frac{3}{8}$

$\therefore$ error $= |R_{2,0}(x)| \leq \frac{\frac{3}{8}|x|^3}{3!}$

$|R_{2,0}(0.1)| \leq \frac{3}{8} \times \frac{(0.1)^3}{3!} = \frac{1}{10} \cdot \frac{1}{1000} = \frac{1}{16000}$

4) $f(x) - T_{2,0}(x) = \frac{f'''(c)}{3!} x^3 = \frac{3}{8(1+c)^{\frac{5}{2}}} \cdot \frac{x^3}{3!} \geq 0$

$\Rightarrow f(x) - T_{2,0}(x) \geq 0$

$T_{2,0}(x) \leq f(x)$

$T_{2,0}(x)$ underestimates $f(x)$ for $x \geq 0$