1.1 Introduction of vectors in $\mathbb{R}^n$

- def. $\mathbb{R}^n$ (set)

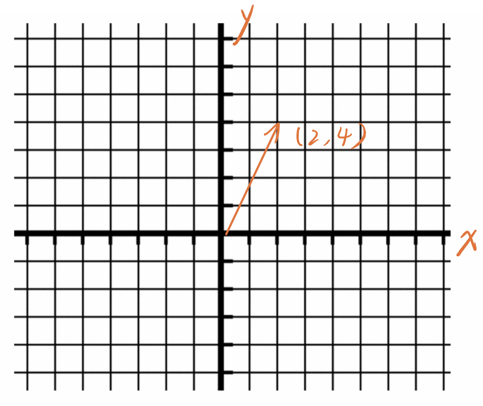
$$\mathbb{R}^n = \Big\{ \underbrace{\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\text{vector: an element } \vec{x}} : x_1 \cdots x_n \in \mathbb{R} \Big\}$$

- row notation

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \text{ is } \vec{v} = [\, v_1 .\ v_2 \ \cdots \ v_n \,]^T$$

$\rightarrow$ transpose

1.2 Algebraic and Geometric Representation



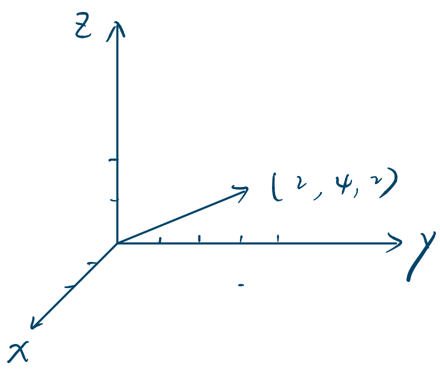
$$\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

① 带方向的线段

② 起始点随意

③ 相同 vector 大小方向相同

In $\mathbb{R}^3$

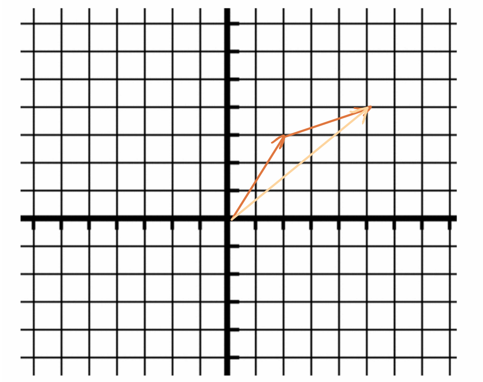


$$\vec{v} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

* $\mathbb{R}^n$ $(n \geq 4)$ 时 无法作图

1.3 Operations on Vectors

- Vector addition



$\begin{bmatrix}2\\4\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}$

symmetry : $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

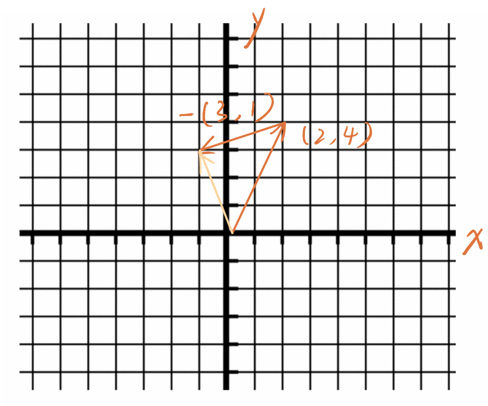
associativity : $\vec{u} + \vec{v} + \vec{w} = (\vec{u} + \vec{v}) + \vec{w}$

zero vector : $\vec{0} = [0, 0 \cdots 0]^T$ in $\mathbb{R}^n$. $\vec{v} + \vec{0} = \vec{v}$

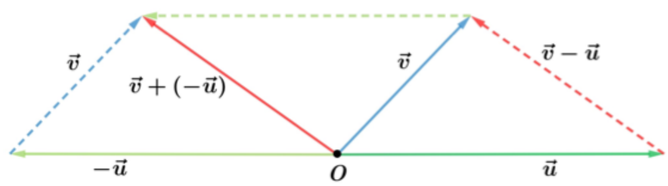
additive inverse property : $\vec{u} + \underbrace{(-\vec{u})} = \vec{0}$

↳ additive inverse of $\vec{u}$.
(大小相同，方向相反)

- Vector Substraction



$\begin{bmatrix}2\\4\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$



- Scalar multiplication

$(c+d)\vec{v} = c\vec{v} + d\vec{v}$

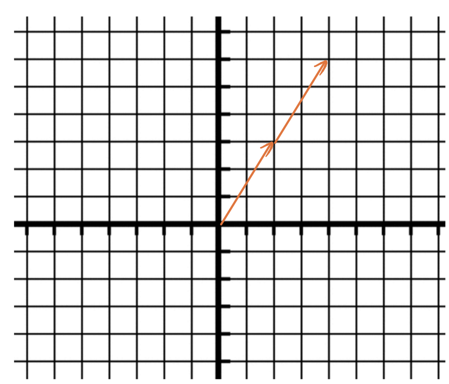
direction是否变化看系数的正负.

$(cd)\vec{v} = c(d\vec{v})$

$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

$0\vec{v} = \vec{0}$

$c\vec{v} = \vec{0} \Rightarrow c = 0 \vee \vec{v} = \vec{0}$ (cancellation law)



$\begin{bmatrix}2\\3\end{bmatrix} \times 2 = \begin{bmatrix}4\\6\end{bmatrix}$

proof:

$LHS = c(\vec{u} + \vec{v})$

$= c\begin{bmatrix}u_1 + v_1\\ \vdots \\ u_n + v_n\end{bmatrix} = \begin{bmatrix}c(u_1 + v_1)\\ \vdots \\ c(u_n + v_n)\end{bmatrix} = \begin{bmatrix}cu_1 + cv_1\\ \vdots \\ cu_n + cv_n\end{bmatrix} = \begin{bmatrix}cu_1\\ \vdots \\ cu_n\end{bmatrix} + \begin{bmatrix}cv_1\\ \vdots \\ cv_n\end{bmatrix} = c\vec{u} + c\vec{v}$

$= RHS$

- Standard basis for $\mathbb{R}^n$

若 $\vec{e_i}$ 除 i^{th} component 为 1 外，其它都为 0.

则 $\mathcal{E} = \{\vec{e_1}, \vec{e_2}, \cdots \vec{e_n}\}$ is standard basis for $\mathbb{R}^n$.

eg. s.b. for $\mathbb{R}^3$ is. $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

每个 vector 都能写成 standard basis in 形式 $\begin{bmatrix}4\\3\\2\end{bmatrix} = 4\vec{e_1} + 3\vec{e_2} + 2\vec{e_3}$

- Components.

$\vec{v} = \begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix} = v_1\vec{e_1} + v_2\vec{e_2} + \cdots + v_n\vec{e_n}$ $v_1 . v_2 \cdots v_n$ is components of $\vec{v}$

eg. $\vec{v} = \begin{bmatrix}5\\3\\-4\end{bmatrix} = 5\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} - 4\begin{bmatrix}0\\0\\1\end{bmatrix} = 5\vec{e_1} + 3\vec{e_2} - 4\vec{e_3}$

components

Q: Determine $\vec{u}, \vec{v} \in \mathbb{R}^3$ $c \in \mathbb{R}$ such that $c[6\vec{u} + 3(3\vec{v} - 2\vec{u})] = 4\vec{v}$

$c(6\vec{u} + 9\vec{v} - 6\vec{u}) = 4\vec{v}$

$6c\vec{u} + 9c\vec{v} - 6c\vec{u} = 4\vec{v}$

$9c\vec{v} = 4\vec{v}$

$\begin{cases} c = \frac{4}{9} \\ \vec{u} \in \mathbb{R}^3 \end{cases}$ or $\begin{cases} \vec{v} = \begin{bmatrix}0\\0\\0\end{bmatrix} \\ \vec{u} \in \mathbb{R}^3 \end{cases}$

1.4 Vectors in $\mathbb{C}^n$

- def. $\mathbb{C}^n$

$\mathbb{R}^n = \left\{ \vec{x} = \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix} : x_1 \cdots x_n \in \mathbb{C} \right\}$

ex. $i\begin{bmatrix}2\\1+i\end{bmatrix} = \begin{bmatrix}2i\\-1+i\end{bmatrix}$ 和实数一样计算

1.5 Dot Product in $\mathbb{R}^n$

- def. Dot Product

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad (\vec{u}.\vec{v} \in \mathbb{R}^n)$$

dot product : $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$ 属于 scalar

- Properties

symmetry : $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

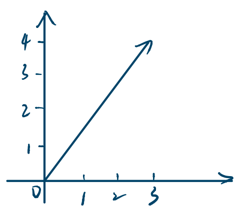
linearity : $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

$(c\vec{u})\vec{v} = c(\vec{u} \cdot \vec{v})$

non-negativity : $\vec{v} \cdot \vec{v} \geq 0 \wedge \vec{v} \cdot \vec{v} = 0 \Leftrightarrow \vec{v} = \vec{0}$

- length (norm / magnitude)

$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$



ep. $\left\|\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\| = \sqrt{\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \sqrt{3 \cdot 3 + 4 \cdot 4} = 5$

* positive square root
* only vector of length 0 is $\vec{0}$

proposition of length : $c \in \mathbb{R}.\ \vec{v} \in \mathbb{R}^n \Rightarrow \|c\vec{v}\| = |c| \cdot \|\vec{v}\|$

ep. $\left\|\begin{bmatrix} -2 \\ -4 \end{bmatrix}\right\| = \left\|-2\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\| = |-2|\left\|\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\| = 2\sqrt{1 \cdot 1 + 2 \cdot 2} = 2\sqrt{5}$

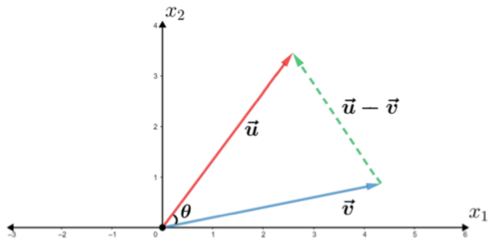
- Unit vector & Normalization

def. unit vector : 满足 $\|\vec{v}\| = 1$ 的 $\vec{v}$

Normalization : $\hat{v} = \dfrac{\vec{v}}{\|\vec{v}\|}$ ($\vec{v} \in \mathbb{R}^n$. $\vec{v}$ is non-zero)

* $\hat{v}$ 与 $\vec{v}$ 方向相同
* $\hat{v}$ is a unit vector $\quad \hat{v} = \left\| \dfrac{1}{\|\vec{v}\|} \vec{v} \right\| = \left| \dfrac{1}{\|\vec{v}\|} \right| \|\vec{v}\| = 1$ (length = 1)

- Angle between vectors



$\vec{u}, \vec{v} \in$ non-zero vectors in $\mathbb{R}^n$

θ : angle between $\vec{u}$ & $\vec{v}$. $(0 \le \theta \le \pi)$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

$*$ $0 \le \theta \le \pi$.

$$\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$$

$*$ $\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} \in [-1, 1]$

$(\because \vec{u}\cdot\vec{v} \le \|\vec{u}\|\|\vec{v}\|)$

proof.

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \underbrace{- 2\|\vec{u}\|\|\vec{v}\|\cos\theta} \quad \text{(cosine law)}$$

$a^2 = b^2 + c^2 - 2bc\cos a$

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - \vec{u}\vec{v} - \vec{u}\vec{v} + \vec{v}\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 \underbrace{-2\vec{u}\vec{v}}$$

$$\therefore \vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

Q : find angle between $\vec{u} = \begin{bmatrix}1\\3\\3\end{bmatrix}$ $\vec{v} = \begin{bmatrix}3\\2\\-3\end{bmatrix}$

$\vec{u}\cdot\vec{v} = 3+6-9 = 0$ $\quad \cos\theta = \dfrac{0}{\|\vec{u}\|\|\vec{v}\|} = 0$ $\quad \theta = \dfrac{\pi}{2}$

- def. Orthogonal in $\mathbb{R}^n$

$\vec{u}\cdot\vec{v} = 0 \rightarrow \vec{u}$ & $\vec{v}$ are orthogonal / perpendicular

Q. Find all non-zero vectors in $\mathbb{R}^2$ which are orthogonal to $\begin{bmatrix}1\\2\end{bmatrix}$

$\begin{bmatrix}x\\y\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = 0$

$x + 2y = 0$

$x = -2y$

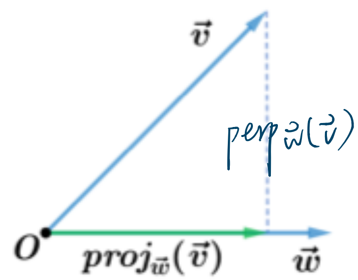
$\left\{\mathbb{R}^2 : c\begin{bmatrix}-2\\1\end{bmatrix}\right\}$

1.6 Projection. Components and Perpendicular

– def. component of $\vec{v}$ along $\vec{w}$

$$\|\vec{v}\|\cos\theta = \vec{v}\cdot\hat{w} \qquad (\vec{v},\ \vec{w}\in\mathbb{R}^n,\ \vec{w}\neq\vec{0})$$

– def. projection of $\vec{v}$ onto $\vec{w}$



$$\vec{v}\cdot\vec{w} = (c\vec{w}+\vec{u})\cdot\vec{w}$$
$$= c(\vec{w}\cdot\vec{w}) + \vec{u}\cdot\vec{w}$$
$$= c\|\vec{w}\|^2$$
$$c = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2} \qquad (\vec{w}\neq\vec{0})$$

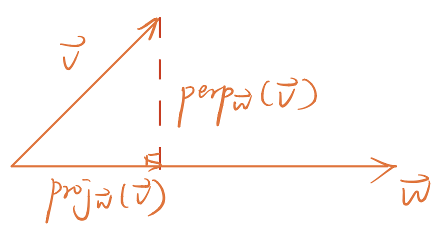
projection of $\vec{v}$ onto $\vec{w}$ 指 $\vec{v}$ 中 $\vec{w}$ 方向（或反方向）的值.

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{(\vec{v}\cdot\vec{w})}{\|\vec{w}\|^2}\vec{w} = \frac{(\vec{v}\cdot\vec{w})}{\vec{w}\cdot\vec{w}}\vec{w} \qquad (\vec{v},\ \vec{w}\in\mathbb{R}^n \quad \vec{w}\neq\vec{0})$$

– def. perpendicular of $\vec{v}$ onto $\vec{w}$

$$\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{w}}(\vec{v}) \qquad (\vec{v},\ \vec{w}\in\mathbb{R}^n,\ \vec{w}\neq\vec{0})$$

– orthogonal 正交. 直角



$$\text{perp}_{\vec{w}}(\vec{v})\cdot\text{proj}_{\vec{w}}(\vec{v}) = 0$$

✲ proof:

→ prove $\text{perp}_{\vec{w}}(\vec{v}) \perp \vec{w}$

$$\text{perp}_{\vec{w}}(\vec{v})\cdot\vec{w}$$
$$= [\vec{v} - \text{proj}_{\vec{w}}(\vec{v})]\cdot\vec{w}$$
$$= \vec{v}\cdot\vec{w} - (\text{proj}_{\vec{w}}(\vec{v}))\cdot\vec{w}$$
$$= \vec{v}\cdot\vec{w} - \frac{(\vec{v}\cdot\vec{w})}{\vec{w}\cdot\vec{w}}\vec{w}\cdot\vec{w}$$
$$= 0$$

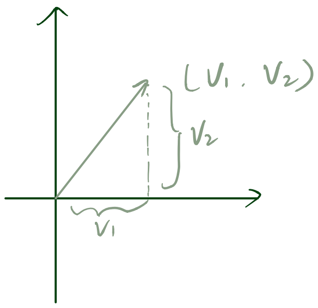
→ $\because$ $\text{perp}_{\vec{w}}(\vec{v}) \perp \vec{w}$

→ LHS $= (\vec{v} - \text{proj}_{\vec{w}}\vec{v})\cdot\text{proj}_{\vec{w}}\vec{v}$

$$= \vec{v}\,\text{proj}_{\vec{w}}\vec{v} - \|\text{proj}_{\vec{w}}\vec{v}\|^2$$
$$= \vec{v}\cdot\frac{(\vec{v}\cdot\vec{w})}{\|\vec{w}\|^2}\vec{w} - \left\|\frac{(\vec{v}\cdot\vec{w})}{\|\vec{w}\|^2}\vec{w}\right\|^2$$
$$= \frac{(\vec{v}\cdot\vec{w})}{\|\vec{w}\|^2}\cdot\vec{v}\,\vec{w} - \left\|\frac{(\vec{v}\cdot\vec{w})}{\|\vec{w}\|^2}\right\|^2\|\vec{w}\|^2$$
$$= \frac{(\vec{v}\cdot\vec{w})^2}{\|\vec{w}\|^2} - \frac{(\vec{v}\cdot\vec{w})^2}{\|\vec{w}\|^2}$$
$$= 0$$

$\therefore$ $\text{perp}_{\vec{w}}(\vec{v}) \perp \text{proj}_{\vec{w}}(\vec{v})$

Q: Compute the projection of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ onto $\vec{e_1}$ & $\vec{e_2}$ (s.b. vectors of $\mathbb{R}^2$)



$$\text{proj}_{\vec{e_1}} \vec{v} = \frac{\vec{v} \cdot \vec{e_1}}{\| \vec{e_1} \|^2} \vec{e_1} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$$

$$\text{proj}_{\vec{e_2}} \vec{v} = \frac{\vec{v} \cdot \vec{e_2}}{\| \vec{e_2} \|^2} \vec{e_2} = v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$$

$$\text{prep}_{\vec{e_1}} \vec{v} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = \vec{v} - \text{proj}_{\vec{e_1}} \vec{v}$$

$$\text{prep}_{\vec{e_2}} \vec{v} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \vec{v} - \text{proj}_{\vec{e_2}} \vec{v}$$

Q. Let $\vec{v_1}, \vec{v_2}, \vec{u} \in \mathbb{R}^n$ where $\vec{u} \neq \vec{0}$ $\quad c_1, c_2 \in \mathbb{R}$

Prove $\text{proj}_{\vec{u}} (c_1 \vec{v_1} + c_2 \vec{v_2}) = c_1 \, \text{proj}_{\vec{u}} (\vec{v_1}) + c_2 \, \text{proj}_{\vec{u}} (\vec{v_2})$

$$\begin{aligned}
LHS &= \text{proj}_{\vec{u}} (c_1 \vec{v_1} + c_2 \vec{v_2}) \\
&= \frac{(c_1 \vec{v_1} + c_2 \vec{v_2}) \cdot \vec{u}}{\| \vec{u} \|^2} \cdot \vec{u} \\
&= \frac{(c_1 \vec{v_1}) \cdot \vec{u}}{\| \vec{u} \|^2} \vec{u} + \frac{(c_2 \vec{v_2}) \vec{u}}{\| \vec{u} \|^2} \vec{u} \\
&= c_1 \cdot \frac{\vec{v_1} \cdot \vec{u}}{\| \vec{u} \|^2} \vec{u} + c_2 \frac{\vec{v_2} \, \vec{u}}{\| \vec{u} \|^2} \vec{u} \qquad \text{(linearity)} \\
&= c_1 \, \text{proj}_{\vec{u}} (\vec{v_1}) + c_2 \, \text{proj}_{\vec{u}} (\vec{v_2}) \\
&= RHS
\end{aligned}$$

Q. Prove $\text{proj}_{\vec{u}} (\text{proj}_{\vec{u}} \vec{v}) = \text{proj}_{\vec{u}} \vec{v}$

$$\begin{aligned}
LHS &= \text{proj}_{\vec{u}} \left(\frac{\vec{v} \cdot \vec{u}}{\| \vec{u} \|^2} \vec{u} \right) \\
&= \frac{\left(\frac{\vec{v} \cdot \vec{u}}{\| \vec{u} \|^2} \vec{u} \right) \vec{u}}{\| \vec{u} \|^2} \vec{u} \\
&= \frac{\vec{v} \cdot \vec{u}}{\| \vec{u} \|^2} \vec{u} \\
&= RHS
\end{aligned}$$

1.7 Standard Inner Product in $\mathbb{C}^n$

- magnitude of a number

$x \in \mathbb{R}$ $\quad |x| = \sqrt{x \cdot x}$

$x \in \mathbb{C}$ $\quad |x| = \sqrt{z(\overline{z})}$

- standard inner product

sip of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{C}^n$ is

$$\langle \vec{v}, \vec{w} \rangle = v_1 \overline{w_1} + v_2 \overline{w_2} + \cdots + v_n \overline{w_n}$$

- properties

$c \in \mathbb{C}$. $\vec{u}, \vec{v}, \vec{w}$ are vectors in $\mathbb{C}^n$

conjugate symmetry $\quad \langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{u}, \vec{v} \rangle}$

linearity in 1st argument $\quad \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

$\langle c\vec{u}, \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$

non-negativity $\quad \langle \vec{v}, \vec{v} \rangle \geq 0$, $\quad \langle \vec{v}, \vec{v} \rangle = 0 \Leftrightarrow \vec{v} = \vec{0}$

- length of $\vec{v} \in \mathbb{C}^n$

$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

properties: $\quad c \in \mathbb{C} \wedge \vec{v} \in \mathbb{C}^n$

$\Rightarrow \|c\vec{v}\| = |c| \cdot \|\vec{v}\|$

$\Rightarrow \|\vec{v}\| \geq 0$, $\quad \|\vec{v}\| = 0 \Leftrightarrow \vec{v} = 0$

- orthogonal & projection 与 $\mathbb{R}$ 一样

(no dot product)

1.8 Fields

- standard inner product

sip of $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{F}^n$ is

$$\langle \vec{v}, \vec{w} \rangle = v_1 \overline{w_1} + v_2 \overline{w_2} + \cdots + v_n \overline{w_n}$$

1.9 The Cross Product in $\mathbb{R}^3$

- def. cross product (only in $\mathbb{R}^3$)

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$

cross product is $\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -(u_1 v_3 - u_3 v_1) \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$

Q. $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0-2 \\ -(0-6) \\ 1+9 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 10 \end{bmatrix}$

- properties

$\vec{u} \cdot \vec{v} = \vec{z}$ $\quad (\vec{u}, \vec{v} \in \mathbb{R}^3)$

a) $\vec{z} \cdot \vec{u} = 0$ $\quad \vec{z} \cdot \vec{v} = 0$ $\quad$ ($\vec{z}$ is orthogonal to both $\vec{u}$ & $\vec{v}$)

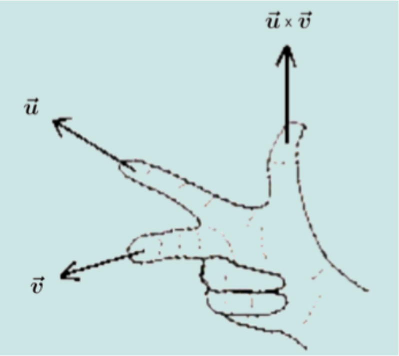
ep. $\vec{u} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ $\vec{u} \times \vec{v} = \begin{bmatrix} -2 \\ 6 \\ 10 \end{bmatrix}$

$\vec{u} \cdot (\vec{u} \times \vec{v}) = -2 - 18 + 20 = 0$

$\vec{v} \cdot (\vec{u} \times \vec{v}) = -6 + 6 + 0 = 0$

b) $\vec{v} \times \vec{u} = -\vec{z} = -\vec{u} \times \vec{v}$ $\quad$ (skew-symmetric)

c) $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$.



proof: $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$

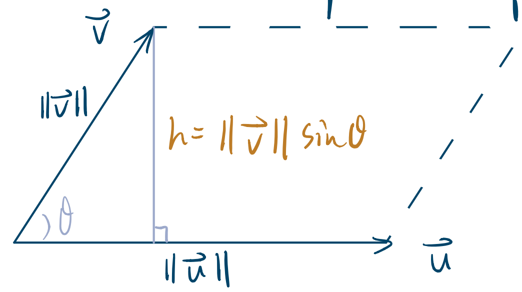
$= \|\vec{u}\|^2 \|\vec{v}\|^2 - \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2\theta$

$= \|\vec{u}\|^2 \|\vec{v}\|^2 (1 - \cos^2\theta)$

$= \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2\theta$

$\because 0 \le \theta \le \pi$ $\quad \sin\theta \ge 0$ $\quad \therefore \|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$

- Geometric Interpretation of Length of Cross Product



$$\text{Area} = \|\vec{u}\| \, \|\vec{v}\| \sin\theta = \|\vec{u} \times \vec{v}\|$$

- linearity of cross product

$c \in \mathbb{R}$. $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$

a) $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$

b) $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$

c) $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

d) $\vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$

proof (c)

$$LHS = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_2(v_3+w_3) - u_3(v_2+w_2) \\ -[u_1(v_3+w_3) - u_3(v_1+w_1)] \\ u_1(v_2+w_2) - u_2(v_1+w_1) \end{bmatrix} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -u_1v_3 + u_3v_1 \\ u_1v_2 - u_2v_1 \end{bmatrix} + \begin{bmatrix} u_2w_3 - u_3w_2 \\ -u_1w_3 + u_3w_1 \\ u_1w_2 - u_2w_1 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$$

proof (d)

$$LHS = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} cv_1 \\ cv_2 \\ cv_3 \end{bmatrix} = \begin{bmatrix} u_2cv_3 - u_3cv_2 \\ -u_1cv_3 + u_3cv_1 \\ u_1cv_2 - u_2cv_1 \end{bmatrix} = c \begin{bmatrix} u_2v_3 - u_3v_2 \\ -u_1v_3 + u_3v_1 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

$$= c(\vec{u} \times \vec{v})$$

- Standard basis $\mathbb{R}^3$ problem

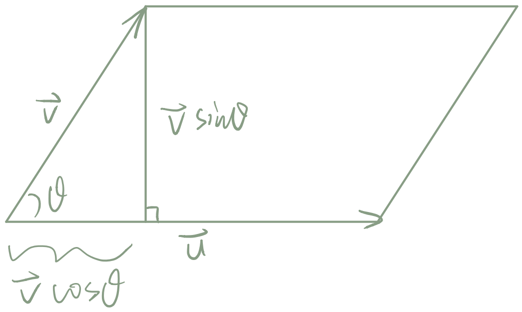
$$\vec{e}_1 \times \vec{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{e}_3$$

$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$

$\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$

$\vec{e}_1 \times \vec{e}_3 = -\vec{e}_1$

Q: Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ where $\vec{u}$ & $\vec{v}$ have integer components. Prove $S_{\square} \in \mathbb{Z}$.



$$S_{\square} = \|\vec{u} \times \vec{v}\|$$

$$= \left\| \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \right\|$$

将$\mathbb{R}^2$变成$\mathbb{R}^3$

$$= \left\| \begin{bmatrix} 0 \\ 0 \\ u_1v_2 - u_2v_1 \end{bmatrix} \right\|$$

$$= \sqrt{(u_1v_2 - u_2v_1)^2}$$

$$= u_1v_2 - u_2v_1$$

$\because \vec{u}, \vec{v} \in \mathbb{R}^2$

$\therefore u_1v_2 - u_2v_1 \in \mathbb{R}$

2.1 Linear Combinations and Span

- def. linear combination

$c_1, c_2 \cdots c_k \in \mathbb{F}$ $\quad \vec{v_1}, \vec{v_2}, \cdots \vec{v_k}$ are vectors in $\mathbb{F}^n$

$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}$ as l-c of $\vec{v_1}, \vec{v_2} \cdots \vec{v_k}$

* Scalars are from $\mathbb{F}$

Q. Write $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ as l-c of $\vec{e_1}$ & $\vec{e_2}$

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- def. span

$$\underline{\operatorname{span}\{\vec{v_1}, \vec{v_2} \cdots \vec{v_k}\}} = \{\overbrace{\underline{c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k}}}^{l\text{-}c} : c_1, c_2 \cdots c_k \in \mathbb{F}\}$$

$\in$ Set

↳ is spanned by $\{\vec{v_1}, \vec{v_2}, \cdots \vec{v_k}\}$

$\{\vec{v_1}, \vec{v_2} \cdots \vec{v_k}\}$ is spanning set for $\operatorname{span}\{\vec{v_1}, \vec{v_2} \cdots \vec{v_k}\}$

→ Span of one vector

What does $S = \operatorname{Span}\{\vec{v_1}\}$ look like geometrically?

- $\vec{v_1} \neq \vec{0}$ $\quad S$ is a line through origin
- $\vec{v_1} = \vec{0}$ $\quad S = \{\vec{0}\}$ just the origin

→ Span of two vectors

What does $S = \operatorname{Span}\{\vec{v_1}, \vec{v_2}\}$ look like geometrically?

- $\vec{v_1} \neq \vec{0}$ $\quad \vec{v_2} \neq \vec{0}$ $\quad$ $\vec{v_1}$ & $\vec{v_2}$ are scalar multiples $\quad \vec{v_1} = c\vec{v_2}$
  S is a line
- $\vec{v_1} = \vec{0}$ $\quad \vec{v_2} = \vec{0}$ $\quad$ point
- $\vec{v_1}$ & $\vec{v_2}$ are not scalar multiples $\quad S$ is a plane in $\mathbb{R}^3$

Q. 判断 every vector in $\mathbb{R}^2$ is one element of Span $\{\vec{e_1}, \vec{e_2}\}$ T

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 \qquad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{e_1}} + v_2 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{e_2}}$$

Q. 判断 every vector in $\mathbb{R}^2$ is an element of Span $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ T

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (v_1 - v_2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\{\vec{v_1}, \vec{v_2}\}$ is a spanning set for $\mathbb{R}^2$ $\qquad \mathbb{R}^2 \leq \text{Span}\{\vec{v_1}, \vec{v_2}\}$

$\text{Span}\{\vec{v_1}, \vec{v_2}\} \leq \mathbb{R}^2 \qquad \therefore \text{Span}\{\vec{v_1}, \vec{v_2}\} = \mathbb{R}^2$

Q. 判断 every vector in $\mathbb{R}^3$ is an element of Span $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ F

$$c_1\vec{w_1} + c_2\vec{w_2} = \begin{bmatrix} c_1 + c_2 \\ c_2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \notin \text{Span}\{\vec{w_1}, \vec{w_2}\}$$

$\{\vec{w_1}, \vec{w_2}\}$ is not a spanning set for $\mathbb{R}^3$

- Equations of line in $\mathbb{R}^2$

→ $y = mx + b$ $\qquad$ m - 斜率 $\qquad$ b - y 轴交点

→ $y - y_1 = m(x - x_1)$ $\qquad$ m - 斜率 $\qquad$ (x_1, y_1) 线上点

→ $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ $\qquad$ $(y_2 \neq y_1,\ x_2 \neq x_1)$ $\qquad$ m - 斜率 $\qquad$ $(x_1, y_1)(x_2, y_2)$ 线上点

↳ (不适用于 vertical / horizontal line)

→ $y - y_1 = \dfrac{p}{q}(x - x_1)$

2.2 Lines in $\mathbb{R}^2$

- def. Parametric equation of a line in $\mathbb{R}^2$

$p, q \in \mathbb{R}$. $q \neq 0$.

parametric equations of a line in $\mathbb{R}^2$ through the point (x_1, y_1) with slope $\frac{p}{q}$:

$$\begin{cases} x = x_1 + qt \\ y = y_1 + pt \end{cases} \quad t \in \mathbb{R}$$

* t不同，线不同

* $q=0$ vertical line with undefined slope $\rightarrow \begin{cases} x = x_1 \\ y = y_1 + pt \end{cases}$

Q: Find parametric equations of the line with equation $y=3x+2$

point $(0, 2)$

slope $= \frac{3}{1}$

$$\begin{cases} x = 0 + t \\ y = 2 + 3t \end{cases}$$

- def. Vector equation of a line in $\mathbb{R}^2$

$\begin{bmatrix} p \\ q \end{bmatrix} \in \mathbb{R}^2$ ^ non-zero vector

$$\vec{l} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t \begin{bmatrix} q \\ p \end{bmatrix} \qquad t \in \mathbb{R}$$

$\vec{l}$ is vector equation of l through $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ with direction $\begin{bmatrix} q \\ p \end{bmatrix}$

* (x_1, y_1) is a point on the line

* $\begin{bmatrix} q \\ p \end{bmatrix}$ is parallel to the line $\qquad (q, p)$ 不一定在线上

Q. Find vector equation for the line through $(3,-1)$ & $(1,7)$

point on the line is $(3,-1)$

direction vector : $\begin{bmatrix} 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$ $\qquad \vec{l} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -8 \end{bmatrix} \quad (t \in \mathbb{R})$

- Line in $\mathbb{R}^2$

$\vec{u}, \vec{v} \in \mathbb{R}^2 \qquad \vec{v} \neq \vec{0}$

$L = \{\vec{u} + t\vec{v} : t \in \mathbb{R}\}$ $\qquad$ line L in $\mathbb{R}^2$ through $\vec{u}$ with direction $\vec{v}$.

Q : 1. Show that the lines $L_1 = \left\{ \begin{bmatrix} 4 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix} : t \in \mathbb{R} \right\}$ and $L_2 = \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$ are the same line

$\begin{bmatrix} 4+t \\ -3-t \end{bmatrix}$ & $\begin{bmatrix} -1-2t \\ 2+2t \end{bmatrix}$ $\quad \begin{cases} 4+t = -1-2t \\ -3-t = 2+2t \end{cases}$ $\quad t = \frac{5}{3}$, $t = \frac{5}{3}$

$\therefore$ there exist a t s.t $\nearrow$. $\therefore$ l_1 & l_2 are the same line

2. Find equation for this line in the form $y = mx + b$

已知 $(4,-3)$. $(-1,2)$ 在线上

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y+3}{2+3} = \frac{x-4}{-1-4}$$

$$\frac{y+3}{5} = \frac{x-4}{-5}$$

$$-y+3 = x-4$$

$$y = -x+7$$

3. Find parametric equations for this line.

slope $= \frac{-1}{1}$ point $(-1, 2)$

$$\begin{cases} x = -1 - t \\ y = 2 + t \end{cases}$$

2.3 Lines in $\mathbb{R}^n$

- def. Vector equation of a line in $\mathbb{R}^n$

$$\vec{l} = \vec{u} + t\vec{v} \quad (t \in \mathbb{R}) \qquad (\vec{u}, \vec{v} \in \mathbb{R}^n \quad \vec{v} \neq \vec{0})$$

$\vec{l}$ is a v-eq of line L in $\mathbb{R}^n$ through $\vec{u}$ with direction $\vec{v}$.

* Parallel $\quad \vec{l_1} = \vec{u_1} + t_1\vec{v_1} \qquad \vec{l_2} = \vec{u_2} + t_2\vec{v_2}$

two vectors are parallel $\Leftrightarrow$ they are scalar multiples ($c\vec{v_1} = \vec{v_2}$ ($c \in \mathbb{R}$ $c \neq 0$)

* 共线

$\vec{u}$ & $\vec{v}$ are on same line $\Leftrightarrow$ $\vec{l_1} \parallel \vec{l_2}$ & share 1 point / share 2 points

- def. Parametric equation of a line in $\mathbb{R}^n$

$\vec{u}, \vec{v} \in \mathbb{R}^n \quad \vec{v} \neq \vec{0}$

v-eq: $\vec{l} = \vec{u} + t\vec{v} \qquad t \in \mathbb{R}$

p-eq:
$$\begin{aligned} l_1 &= u_1 + tv_1 \\ l_2 &= u_2 + tv_1 \\ &\vdots \\ l_n &= u_n + tv_n \end{aligned} \qquad t \in \mathbb{R}$$

Q. $\operatorname{Span}\left\{\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}\right\}$ goes through origin. $(\in \mathbb{R}^5)$

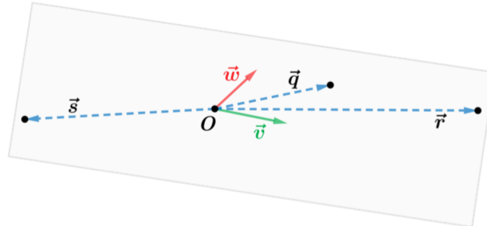
direction vector $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$ parametric equation $\begin{cases} x_1 = t \\ x_2 = 2t \\ x_3 = 3t \\ x_4 = 4t \\ x_5 = 5t \end{cases} \quad t \in \mathbb{R}$

- Line in $\mathbb{R}^n$

$\vec{u}, \vec{v} \in \mathbb{R}^n. \quad \vec{v} \neq \vec{0}.$

$L = \{\vec{u} + t\vec{v} : t \in \mathbb{R}\}$ line L in $\mathbb{R}^n$ through $\vec{u}$ with direction $\vec{v}$.

2.4 Vector Equation of a Plane in $\mathbb{R}^n$



- def. Plane in $\mathbb{R}^n$ through the origin

$\vec{v}, \vec{w}$ be non-zero vector $\quad \vec{v} \neq c\vec{w} \ (c \in \mathbb{R})$

$P = \text{Span}\{\vec{v}, \vec{w}\} = \{s\vec{v} + t\vec{w} : s, t \in \mathbb{R}\}$

↳ plane in $\mathbb{R}^n$ through the origin with direction vectors $\vec{v}$ and $\vec{w}$

Q: $\vec{p} = s\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + t\begin{bmatrix}2\\3\\4\\5\end{bmatrix} \quad s, t \in \mathbb{R}$ $\qquad P = \text{Span}\left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}2\\3\\4\\5\end{bmatrix}\right\}$

* terminal point of $\vec{v}$ & $\vec{w}$ are on the plane $\quad \text{Span}\{\vec{v}, \vec{w}\}$

- def. Vector equation of a plane in $\mathbb{R}^n$ through the origin

$\vec{v}, \vec{w}$ be non-zero vectors in $\mathbb{R}^n$ with $\vec{v} \neq c\vec{w}$ for any $c \in \mathbb{R}$.

$\vec{p} = s\vec{v} + t\vec{w}$.

↳ vector equation of the plane in $\mathbb{R}^n$ through the origin with direction vectors $\vec{v}$ & $\vec{w}$

ep. $P = \text{Span}\left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}5\\6\\7\\8\end{bmatrix}\right\} \qquad \vec{p} = s\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + t\begin{bmatrix}5\\6\\7\\8\end{bmatrix} \quad s, t \in \mathbb{R}$

- def. Plane in $\mathbb{R}^n$

$\vec{u} \in \mathbb{R}^n$. $\vec{v}, \vec{w} \in$ non-zero vectors in $\mathbb{R}^n$ $\quad \vec{v} \neq c\vec{w} \; (c \in \mathbb{R})$

$$P = \{ \vec{u} + s\vec{v} + t\vec{w} \; : s, t \in \mathbb{R} \}$$

↳ plane in $\mathbb{R}^n$ through $\vec{u}$ with direction vector $\vec{v}$ and $\vec{w}$.

* terminal point of $\vec{u}$ is on the plane

* terminal point of $\vec{v}$ & $\vec{w}$ on the plane $\Leftrightarrow$ plane goes through origin

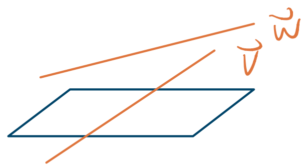
- def. Vector equation of a Plane

$\vec{u} \in \mathbb{R}^n$. $\vec{v}, \vec{w} \in$ non-zero vectors in $\mathbb{R}^n$ $\quad \vec{v} \neq c\vec{w}$. $\quad c \in \mathbb{R}$.

$$\vec{p} = \vec{u} + s\vec{v} + t\vec{w}. \qquad (s, t \in \mathbb{R})$$

↳ v-eq of the plane in $\mathbb{R}^n$ through $\underline{\vec{u}}$ with $\underline{\text{direction vector } \vec{v} \text{ \& } \vec{w}}$

* plane // $\vec{v}$ $\qquad$ plane // $\vec{w}$



Q. Find a vector equation of the plane through $A(1,0,2)$ $B(-3,-2,4)$ $C(1,8,-5)$

$$\vec{v} = \vec{b} - \vec{a} = \begin{bmatrix} -3 \\ -2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix}$$

$$\vec{w} = \vec{c} - \vec{a} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -7 \end{bmatrix}$$

$\vec{v} \nparallel \vec{w}$

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + s\begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ 8 \\ 7 \end{bmatrix} \qquad s, t \in \mathbb{R}$$

Q: Find a non-zero vector orthogonal to the plane. $P=\left\{s\begin{bmatrix}1\\0\\2\end{bmatrix}+t\begin{bmatrix}0\\-1\\1\end{bmatrix}: s,t\in\mathbb{R}\right\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \vec{v} \qquad\qquad \vec{w}$

$\vec{n}=\vec{v}\times\vec{w}=\begin{bmatrix}2\\-1\\-1\end{bmatrix}$ $\qquad$ $\vec{n}\cdot\vec{v}=0$

$\qquad\qquad\qquad\qquad\qquad\qquad n\cdot\vec{w}=0$

Let $\vec{p}=\begin{bmatrix}x\\y\\z\end{bmatrix}\in P$. is $\vec{p}$ parallel to the plane?

yes $\because \begin{bmatrix}0\\0\\0\end{bmatrix}\in P$.

$\therefore \vec{n}\cdot\vec{p}=0$.

scalar equation of the plane $2x-y-z=0$

Q: $\vec{p}=\begin{bmatrix}1\\0\\2\end{bmatrix}+s\begin{bmatrix}-4\\-2\\2\end{bmatrix}+t\begin{bmatrix}0\\8\\-7\end{bmatrix}$ $(s,t\in\mathbb{R})$ Does this P go through origin?

$\qquad\quad \vec{u} \qquad\qquad \vec{v} \qquad\qquad \vec{w}$

$\begin{cases}1-4s=0\\-2s+8t=0\end{cases}$ $\Rightarrow$ $\begin{cases}s=\frac{1}{4}\\t=\frac{1}{16}\end{cases}$

$2+2s-7t=0$ $\qquad$ $2+2s-7t\neq 0$ when $s=\frac{1}{4}$ $t=\frac{1}{16}$

$\therefore$ Doesn't goes through

$\because$ P 不经过 origin $\quad\therefore \vec{p}=\begin{bmatrix}x\\y\\z\end{bmatrix}$ not parallel to the plane

$\qquad\qquad\qquad\qquad\qquad \vec{p}-\begin{bmatrix}x\\y\\z\end{bmatrix}$ parallel to the plane

$\therefore \vec{p}-\vec{u}=\begin{bmatrix}x-1\\y\\z-1\end{bmatrix}$ will parallel to the plane.

$\vec{n}=\vec{v}\times\vec{w}=\begin{bmatrix}-2\\-28\\-32\end{bmatrix}$ will be orthogonal to the plane.

$\vec{n}\,(\vec{p}-\vec{u})=0$ $\qquad$ $\begin{bmatrix}-2\\-28\\-32\end{bmatrix}\cdot\begin{bmatrix}x-1\\y\\z-2\end{bmatrix}=0$

$-2x+2-28y-32z+64=0$

scalar equation $x+14y+16z=33$

2.5 Scalar Equation – a Plane in $\mathbb{R}^3$

– Normal form (Scalar equation of a Plane in $\mathbb{R}^3$)

Let P be a plane in $\mathbb{R}^3$ with direction vectors $\vec{v}$ & $\vec{w}$

vector equation: $\vec{p} = \vec{u} + s\vec{v} + t\vec{w}$ ($\vec{u}$ 为平面上一点)

normal vector $\vec{n} = \vec{v} \times \vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$

normal form of P: $\vec{n} \cdot (\vec{p} - \vec{u}) = 0$

scalar equation of P: $ax + by + cz = d$. $d = (\vec{v} \times \vec{w}) \cdot \vec{u}$

两个平面之间的距离 $\dfrac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

* 仅限 $\mathbb{R}^3$

* Given Scalar equation → easy to determine the normal

Q. Find scalar equation for the plane $\vec{p} = \underset{\vec{u}}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} + \underset{\vec{v}}{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}} s + \underset{\vec{w}}{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}} t$

$\vec{n} = \vec{v} \times \vec{w} = \begin{bmatrix} -4 \\ -7 \\ -2 \end{bmatrix}$ $\quad -4x - 7y - 2z = d$.

代入 $(2, -1, 1)$ $\quad d = -3$ $\quad -4x - 7y - 2z = -3$

Q. $P_1: \vec{p} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + s\begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ $\quad s,t \in \mathbb{R}$.

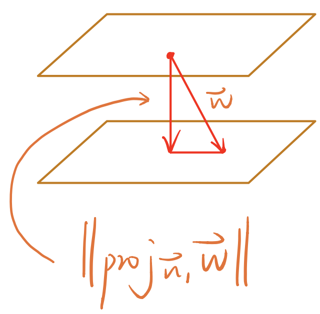
$P_2: 3x - 8y - 2z = 4$.

① Prove $P_1 /\!/ P_2$ → normal of P_1 // normal of P_2

$\vec{n}_{p_1} = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 2 \end{bmatrix}$ $\quad \vec{n}_{p_2} = \begin{bmatrix} 3 \\ -8 \\ -2 \end{bmatrix}$ $\quad \vec{n}_{p_1} = -\vec{n}_{p_2}$

$\therefore \vec{n}_{p_1} /\!/ \vec{n}_{p_2}$ $\quad \therefore P_1 /\!/ P_2$

☆ ② Determine the shortest distance between 2 plane



$d_1 = \vec{n} \cdot \vec{u} = \begin{bmatrix} -3 \\ 8 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = -3 + 24 + 4 = 25$

$d_2 = -4$ $\quad \dfrac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}} = \dfrac{\cancel{25}\ 29}{\sqrt{3^2+8^2+2^2}} = \dfrac{\cancel{25}\ 29}{\sqrt{77}}$ $\quad$ * 符号

3.1 Introduction of linear equations

Q. Find all vectors $\perp$ $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$x_1 + 2x_2 + 3x_3 = 0$

Q. Find the intersection of the 2 planes with equation

$\begin{cases} x_1 + 2x_2 + x_3 = 11 \\ x_1 - x_2 - x_3 = -4 \end{cases}$'s 解

Q. Whether $\begin{bmatrix} 8 \\ -2 \\ 11 \end{bmatrix} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \right\}$

解 $\begin{cases} x_1 + 2x_2 = 8 \\ 2x_1 - 2x_2 = -2 \\ x_1 + 3x_2 = 11 \end{cases}$

3.2 Systems of linear equations

- def. linear equation

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ $\quad a_1 . a_2 \cdots a_n \in \mathbb{F}$.

coefficients : $x_1 \ x_2 \cdots x_n$

constant : b.

- def. system of linear equations

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m.$$

($a_{i,j}$ 指 coefficient of x_j in ith equation)

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is solution to the system

Q. Show $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ $(t \in \mathbb{R})$ is solution to system $\begin{cases} x_1 + 2x_2 + x_3 = 11 \\ x_1 - x_2 - x_3 = -4 \end{cases}$

$\begin{cases} x_1 = 1 + t \\ x_2 = 5 - 2t \\ x_3 = 3t. \end{cases}$

$x_1 + 2x_2 + x_3$
$= 1 + t + 2(5 - 2t) + 3t$
$= 1 + t + 10 - 4t + 3t$
$= 11$

$x_1 - x_2 - x_3$
$= 1 + t - 5 + 2t - 3t$
$= -4$

Q

Is $\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \right\}$

$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$

$\begin{cases} x_1 - x_2 + 4x_3 = 1 \\ x_2 - 2x_3 = 2 \\ 2x_3 = 6 \end{cases}$ $\begin{cases} x_1 = -3 \\ x_2 = 8 \\ x_3 = 3 \end{cases}$

back substitution

- def. equivalent systems.

2 linear systems are equivalent $\Leftrightarrow$ they have same solution set.

set of all solutions

- def. elementary operation.

equation swap. $e_i \leftrightarrow e_j$.

equation scale $e_i \to m e_i$ $m \in \mathbb{F} \setminus \{0\}$

equation addition $\begin{cases} e_j \to c e_i + e_j \\ i \neq j \quad c \in \mathbb{F}. \end{cases}$

Q. Is $(3, 2, 1)$ on the plane P with vector equation $\vec{p} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + t \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix}$.

$\begin{cases} 5 + 2s + 4t = 3 \\ 3s + 6t = 2 \\ 1 + 5s + 7t = 1 \end{cases}$ $\begin{cases} 2s + 4t = -2 \\ 3s + 6t = 2 \\ 5s + 7t = 0 \end{cases}$ $\begin{cases} s + 2t = -1 \\ s + 2t = 1 \\ 5s + 7t = 0 \end{cases}$ $1 \neq -1$ $\therefore$ no solution

$\therefore (3, 2, 1)$ not on the plane

- Inconsistent and consistent system

inconsistent : solution is empty $S = \emptyset$

consistent : unique solution / finitely many solutions $S = \{\vec{v}\}$

infinite number of solutions : 无穷解

Q. Show $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ $(t \in \mathbb{R})$ is a solution to the system $\begin{cases} x_1 + 2x_2 + x_3 = 11 & ① \\ x_1 - x_2 - x_3 = -4 & ② \end{cases}$

① − ②

$$3x_2 + 2x_3 = 15$$
$$x_2 + \frac{2}{3}x_3 = 5$$

Let $x_3 = u$ $(u \in \mathbb{R})$.

$x_2 = 5 - \frac{2}{3}u$ $\quad x_1 = 1 + \frac{1}{3}u$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{3}u \\ 5 - \frac{2}{3}u \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} + \frac{1}{3}u \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad (\frac{1}{3}u \in \mathbb{R})$$

Q. Prove that the only vector in $\mathbb{R}^3$ is orthogonal to $\vec{u} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}$ is $\vec{0}$

let the vector be $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2x - 4y - 6z = 0$$

$$\begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3x + 7y + 2z = 0$$

$$\begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -3x + 5y = 0$$

$$\begin{cases} -2x - 4y - 6z = 0 & ① \\ 3x + 7y + 2z = 0 & ② \\ -3x + 5y = 0 \end{cases}$$

① + 3② $\quad -2x - 4y - 6z + 9x + 21y + 6z = 0$

$$7x + 17y = 0$$

$$\begin{cases} 7x + 17y = 0 \\ -3x + 5y = 0 \end{cases} \qquad \begin{cases} x = 0 \\ y = 0 \end{cases}$$

sol is $\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$

3.3 Solve Systems of Linear Equations

3.4 Use Matrices to Solve

- def. matrix

$m \times n$ matrix $\quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

(i,j)th entry (a_{ij}) 第 i 行，第 j 列

coefficient matrix

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

A 是 ... is coefficient matrix

augment matrix

$$[A \mid \vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$\begin{cases} 2x_1 + 3x_2 = 17 \\ 7x_1 - 2x_2 = 22 \end{cases} \quad A = \begin{bmatrix} 2 & 3 \\ 7 & -2 \end{bmatrix} \quad [A \mid \vec{b}] = \left[\begin{array}{cc|c} 2 & 3 & 17 \\ 7 & -2 & 22 \end{array}\right]$

- Elementary row operations (ERO)

Elementary operation	Equation	Row
Row swap	$e_i \leftrightarrow e_j$	$R_i \leftrightarrow R_j$
Row scale	$e_i \to me_i, m \neq 0$	$R_i \to mR_i, m \neq 0$
Row addition	$e_i \to me_j + e_i, i \neq j$	$R_i \to mR_j + R_i, i \neq j$

Zero row: a row has all 0 entries

row equivalent to A: 由 A 转化得到 matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Q. 解

$$2x_2 + 3x_3 = 4$$
$$2x_1 - 6x_2 + 7x_3 = 15$$
$$x_1 - 2x_2 + 5x_3 = 10$$

$$\begin{array}{c c} R_1 & \\ R_2 & \\ R_3 & \end{array}\left[\begin{array}{ccc|c} 0 & 2 & 3 & 4 \\ 2 & 6 & 7 & 15 \\ 1 & -2 & 5 & 10 \end{array}\right]$$

$$\begin{array}{r} R_1 = R_3 \\ R_2 = R_1 \\ R_3 = R_2 - 2R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -2 & 5 & 10 \\ 0 & 2 & 3 & 4 \\ 0 & 10 & -3 & -5 \end{array}\right]$$

$$\begin{array}{r} R_1 \\ 5R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -2 & 5 & 10 \\ 0 & 10 & 15 & 20 \\ 0 & 10 & -3 & -5 \end{array}\right]$$

$$\begin{array}{r} R_1 \\ \frac{1}{5}R_2 \\ R_2 - R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -2 & 5 & 10 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 18 & 25 \end{array}\right]$$

$$\begin{array}{r} R_1 + R_2 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 8 & 10 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 18 & 25 \end{array}\right]$$

no solution

consistent system

0. 解

$$
\begin{aligned}
x_1 + 2x_2 \qquad\qquad\qquad &= 1 \\
x_1 + 2x_2 + 3x_3 + x_4 &= 0 \\
-x_1 - x_2 + x_3 + x_4 &= -2 \\
x_2 + x_3 + x_4 &= -1 \\
-x_2 + 2x_3 \qquad &= 0
\end{aligned}
$$

$$
\begin{array}{r} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 0 & 0 & -1 \\
1 & 2 & 3 & 1 & 0 \\
-1 & -1 & 1 & 1 & -2 \\
0 & 1 & 1 & 1 & -1 \\
0 & -1 & 2 & 0 & 0
\end{array}\right]
$$

$$
\begin{array}{r} \\ -R_1+R_2 \\ R_1+R_3 \\ \\ \\ \end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 0 & 0 & -1 \\
0 & 0 & 3 & 1 & -1 \\
0 & 1 & 1 & 1 & -1 \\
0 & 1 & 1 & 1 & -1 \\
0 & -1 & 2 & 0 & 0
\end{array}\right]
$$

$$
\begin{array}{r} \\ R_3 \\ R_2 \\ \\ \\ \end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 0 & 0 & -1 \\
0 & 1 & 1 & 1 & -1 \\
0 & 0 & 3 & 1 & -1 \\
0 & 1 & 1 & 1 & -1 \\
0 & -1 & 2 & 0 & 0
\end{array}\right]
$$

$$
\begin{array}{r} \\ \\ \\ -R_2+R_4 \\ R_2+R_5 \end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 0 & 0 & -1 \\
0 & 1 & 1 & 1 & -1 \\
0 & 0 & 3 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 1 & -1
\end{array}\right]
$$

$$
\begin{array}{r} \\ \\ \\ \\ -R_3+R_5 \end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 0 & 0 & -1 \\
0 & 1 & 1 & 1 & -1 \\
0 & 0 & 3 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\begin{array}{r} \\ \\ \frac{1}{3}R_3 \\ \\ \\ \end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 0 & 0 & -1 \\
0 & 1 & 1 & 1 & -1 \\
0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\begin{array}{r} \\ -R_3+R_2 \\ \\ \\ \\ \end{array}
\left[\begin{array}{cccc|c}
1 & 2 & 0 & 0 & -1 \\
0 & 1 & 0 & \frac{2}{3} & -\frac{2}{3} \\
0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\begin{array}{r} -2R_2+R_1 \\ \\ \\ \\ \\ \end{array}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & -\frac{4}{3} & \frac{7}{3} \\
0 & 1 & 0 & \frac{2}{3} & -\frac{2}{3} \\
0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\begin{aligned}
x_1 \qquad - \tfrac{4}{3}x_4 &= \tfrac{7}{3} \\
x_2 \quad + \tfrac{2}{3}x_4 &= -\tfrac{2}{3} \\
x_3 + \tfrac{1}{3}x_4 &= -\tfrac{1}{3} \\
0 &= 0 \\
0 &= 0
\end{aligned}
$$

$$
S = \left\{ \begin{bmatrix} \frac{7}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{4}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}
$$

3.5 Gauss-Jordan Algorithm

- Leading entry

The first non-zero entry in a row of matrix.

leading entry of row 1 → $\begin{bmatrix} 0 & 3 \\ 7 & 0 \end{bmatrix}$

- Row echelon form (REF)

1. leading term 排序 → "stair case"
2. zero rows 在 matrix 的最下方
3. matrix R is a row echelon form of matrix A.

$$\left[\begin{array}{cccc|c} 4 & 0 & 3 & 1 & 4 \\ 0 & 2 & 0 & 5 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

阶梯式

Pivot → a leading term

pivot positions
pivot columns
pivot rows

- Reduced row echelon form (RREF)

matrix R 是 RREF

满足 1. $R \in$ REF
2. all pivots are leading ones 每行由1开头
3. the only non-zero entry in a pivot column is pivot itself.

pivot "1" 上方的数 必须是0

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \times \qquad \left[\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \times \qquad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \times$$

#1 #2 #3

0 0

$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$ augmented matrix isn't RREF.

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ coefficient matrix is RREF

unique RREF eg. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right]$

RREF(A) A 的 RREF

3.6 Rank and Nullity

- $M_{m\times n}(\mathbb{R})$: ~ real entries
  $M_{m\times n}(\mathbb{C})$: ~ complex entries
  $M_{m\times n}(\mathbb{F})$: ~ 未判断

- def. Rank

RRFA / REF 中 pivots in 个数 $\quad \text{rank}(A) = r$

- Rank Bounds

$A \in M_{m\times n}(\mathbb{F}) \Rightarrow \text{rank}(A) \le \min\{m, n\}$

proof: 每行最多有1个 pivots $\rightarrow \text{rank}(A) \le m$
每列最多有1个 pivots $\rightarrow \text{rank}(A) \le n$

- Consistent System Test.

$A \in$ coefficient matrix. of a system

$[A \mid \vec{b}] \in$ augumented matrix of a system

System is consistent $\Leftrightarrow \text{rank}(A) = \text{rank}([A \mid \vec{b}])$

proof. $[A \mid \vec{b}] \longrightarrow$ RREF $[R \mid \vec{c}]$ $\quad (= \text{RREF}(A))$

$\text{rank}(A) \le \text{rank}([A \mid \vec{b}])$

$(\Rightarrow)$ Assume System is consistent.

So $[R \mid \vec{c}]$ 不包含 $[0 \cdots 0 \mid 1]$ 这一列.

$\therefore R$ & $[R \mid \vec{c}]$ share the same pivots.

$\therefore \text{rank}(A) = \text{rank}([A \mid \vec{b}])$

$(\Leftarrow)$ prove by contropositive : inconsistent $\Rightarrow \text{rank}(A) \ne \text{rank}([A \mid \vec{b}])$

Assume the system is inconsistent

Then $[R \mid \vec{c}]$ contains a row of $[0 \cdots 0 \mid 1]$.

So $[R \mid \vec{c}]$ has a pivot which isn't R.

$\therefore \text{rank}(A) < \text{rank}([A \mid \vec{b}])$

ep. An REF of $[A|\vec{b_1}] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 9 \\ 0 & 0 & 5 & 6 & 10 \\ 0 & 0 & 0 & 7 & 11 \\ 0 & 0 & 0 & 0 & 12 \end{array}\right]$

$\text{rank}(A) = 3 \qquad \text{rank}([A|\vec{b_1}]) = 4$

$\because \text{rank}(A) \neq \text{rank}([A|\vec{b_1}]) \qquad \therefore$ inconsistent

An REF of $[A|\vec{b_1}] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 9 \\ 0 & 0 & 5 & 6 & 10 \\ 0 & 0 & 0 & 7 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\text{rank}(A) = 3 \qquad \text{rank}([A|\vec{b_1}]) = 3$

$\because \text{rank}(A) = \text{rank}([A|\vec{b_1}]) \qquad \therefore$ consistent

Q. What values of a. b. c is the system consistent

$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 2 & b \\ 0 & 0 & 0 & -2b+c \end{array}\right] \quad R_3 \to -2R_2 + R_3$

$-2b + c = 0$

$c = 2b. \qquad a.b.c \in \mathbb{F}$

Q. 解

$$\begin{aligned} -3x_1 + (6-9i)x_2 &= -3-21i \\ (1+i)x_1 + (-5+i)x_2 &= -6+8i \end{aligned}$$

$$\left[\begin{array}{cc|c} -3 & 6-9i & -3-21i \\ 1+i & -5+i & -6+8i \end{array}\right]$$

$$-\tfrac{1}{3}R_1 \left[\begin{array}{cc|c} 1 & -2+3i & 1+7i \\ 1+i & -5+i & -6+8i \end{array}\right]$$

$$(-1-i)R_1+R_2 \left[\begin{array}{cc|c} 1 & -2+3i & 1+7i \\ 0 & 0 & 0 \end{array}\right]$$
$$\qquad\quad x_1 \qquad x_2$$

let $x_2 = t. \quad t \in \mathbb{C}$

$x_1 = 1+7i - (-2+3i)t.$

$$S = \left\{ \begin{bmatrix} 1+7i \\ 0 \end{bmatrix} + \begin{bmatrix} 2-3i \\ 1 \end{bmatrix} t : t \in \mathbb{C} \right\}$$

Q. Determine if $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 11 \\ 1 \end{bmatrix} \right\}$ is a spanning set for $\mathbb{R}^3$

$\qquad\qquad\;\; \vec{v_1} \qquad\quad \vec{v_2} \qquad\quad \vec{v_3}$

逻辑: S is spanning set of $\mathbb{R}^3$ $\equiv$ $\operatorname{Span}(S) = \mathbb{R}^3$ $\equiv$ $\mathbb{R}^3 \in \operatorname{Span}(S)$.

Let $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. $\forall \vec{x} \in \mathbb{R}^3$. $\exists c_1 . c_2 . c_3 \in \mathbb{R}$. s.t $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{x}$

$$s\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + v\begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} + t\begin{bmatrix} 5 \\ 11 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 5 & x \\ 2 & 7 & 11 & y \\ 0 & 1 & 1 & z \end{array}\right]$$

$$\begin{array}{r} R_1 \\ R_2 - 2R_1 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 5 & x \\ 0 & 1 & 1 & y-2x \\ 0 & 1 & 1 & z \end{array}\right]$$

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 5 & x \\ 0 & 1 & 1 & y-2x \\ 0 & 0 & 0 & 2x-y+z \end{array}\right]$$

→ The system will be consistent iff. $2x-y+z=0$.

counter example : $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$.

$x=1 \quad y=0 \quad z=0 \qquad 2x-y+z = 2 \neq 0$.

$\therefore$ System isn't consistent when $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

→ $\because \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$. $\quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin \operatorname{Span}(S)$

$\therefore \mathbb{R}^3 \notin \operatorname{Span}(S)$

- System Rank Theorem

Let $A \in M_{m\times n}(\mathbb{F})$ with $\text{rank}(A) = r$. (r 个非0项)

a) Let $\vec{b} \in \mathbb{F}^m$.

nullity

System of l-eq. with augmented matrix $[A|\vec{b}]$ is consistent

$\Rightarrow$ solution set will contain $\boxed{(n-r)}$ parameters.

proof: A has n columns, so we have n variables.

$\text{Rank}(A) = r$ $\quad\therefore$ we have r basic variable

$\therefore$ We have $(n-r)$ free variables

$\therefore$ We have $(n-r)$ parameters

b) $[A|\vec{b}]$ is consistent for every $\vec{b} \in \mathbb{F}^n$. $\Leftrightarrow r = m$

($\Rightarrow$) Prove by contrapositive: $[A|\vec{b}]$ inconsistent $\Rightarrow r \neq m$

Assume $[A|\vec{b}]$ inconsistent

Consistent System test: $RREF([A|\vec{b}])$ 中存在 $[0 \cdots 0 | 1]$ 这一项

$RREF(A)$ 中有 zero-row

$\because$ $RREF(A)$ has m rows. $\quad\therefore \text{rank}(A) < m$

($\Leftarrow$) If $\text{Rank}(A) = m$. then there is a leading one in every row of R

R won't have zero-row.

RREF of $[A|\vec{b}]$ 中不存在 $[0 \cdots 0 | 1]$ 这一项

$\text{rank}(A) = \text{rank}([A|\vec{b}]) \rightarrow$ System is consistent

- def. Nullity

Let $A \in M_{m\times n}(\mathbb{F})$ with $\mathrm{rank}(A) = r$.

$\mathrm{nullity}(A) = n - r$ 　　列 - rank

Q.

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

$\mathrm{rank}(A) = 2$

$\mathrm{nullity}(A) = 3$

\# of parameters $= 3$

$x_2 = s$

$x_4 = t$

$x_5 = u$

$x_1 = -3s - t$

$x_3 = -5t - 2u$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3s-t \\ s \\ -5t-2u \\ t \\ u \end{bmatrix} = s\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$S = \mathrm{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

3.9 Matrix-Vector Multiplication

- def. Row Vector

a matrix with exactly one row.

For matrix $A \in M_{m\times n}(\mathbb{F})$. ith row of A is $\overrightarrow{row_i}(A)$

eg. $A = \begin{bmatrix} 3 & 4 \\ 0 & 1 \\ 6 & -8 \end{bmatrix}$

$\overrightarrow{row_1}(A) = [3 \quad 4]$
$\overrightarrow{row_2}(A) = [0 \quad 1]$
$\overrightarrow{row_3}(A) = [6 \quad -8]$

- Matrix-Vector multiplication

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

※ A的列数与 $\vec{x}$ 的行数相同

eg. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1\times 1 + 2\times(-2) + (-1)\times 6 \\ 3\times 1 + 0\times(-2) + (-5)\times 6 \end{bmatrix} = \begin{bmatrix} -9 \\ -27 \end{bmatrix}$

~ by Columns

If $A \in M_{m\times n}(\mathbb{F})$ $\vec{x} \in \mathbb{F}^n$

$$A\vec{x} = [\vec{a_1} \ . \ \vec{a_2} \ \cdots \ \vec{a_n}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} = 1\cdot\begin{bmatrix} 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -1 \\ -5 \end{bmatrix} = \begin{bmatrix} -9 \\ -27 \end{bmatrix}$$

- Linearity of Matrix-Vector Multiplication

$A \in M_{m\times n}(\mathbb{F})$ $\vec{x}. \vec{y} \in \mathbb{F}^n$ $c \in \mathbb{F}$

then a) $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

b) $A(c\vec{x}) = c(A\vec{x})$

Q. Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^5$.

Prove that there are an infinite number of vectors in $\mathbb{R}^5$ which are orthogonal to $\vec{a}, \vec{b}$ & $\vec{c}$.

let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ orthogonal to $\vec{a}, \vec{b}$ & $\vec{c}$

3.10 Using M-V Product to Express System of L-eq.

3.11 Solution Sets to Systems of Linear Equations

- def. homogenious.

  all right-hand $=0$. consistent.

- def. Nullspace. $\text{Null}(A) := S$

  The solution set of a homogenious system.

  Q. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

  $\text{Null}(A)$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

  $\begin{array}{ccc} x_1 & x_2 & x_3 \\ s & t & -s-t \end{array}$

  $x_2 = s$ $x_3 = t$. $x_1 = -s-t$.

  $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ -s-t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \quad (s, t \in \mathbb{F})$$

  $$\text{Null}(A) = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

- Proposition 3.11.1.

  Let $A\vec{x} = \vec{0}$ be a homogeneous system of l-eq. with sol set S.

  $\vec{x} \in S$. $\vec{y} \in S$. $c \in \mathbb{F}$. $\Rightarrow$ $\vec{x} + \vec{y} \in S$. $\wedge$ $c\vec{x} \in S$

  * S is closed under addition & scalar multiplication.
  * $\vec{x}, \vec{y} \in S$. $c, d \in \mathbb{F}$. $\Rightarrow$ $c\vec{x} + b\vec{y} \in S$
  * ep of subspace

  proof. Let $\vec{x}, \vec{y} \in S$. $c \in \mathbb{F}$.

  $\therefore A\vec{x} = 0$ $A\vec{y} = 0$

  $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$ (linearity)

  $A(c\vec{x}) = c(A\vec{x}) = c\vec{0} = \vec{0}$ (linearity)

- def. Associated homogenious system.

$A\vec{x} = \vec{b}$ ($\vec{b} \neq \vec{0}$.) is non-homogenious system of l-eq.

$A\vec{x} = \vec{0}$ is associated homogenious system.

- def. Particular Solution ($\vec{x_p}$)

sol of $A\vec{x} = \vec{b}$. (consistent system of l-eq). $\vec{x_p}$

- Solutions to $A\vec{x} = \vec{0}$ & $A\vec{x} = \vec{b}$.

$A\vec{x} = \vec{b}$. $\vec{b} \neq \vec{0}$. sol set : $\tilde{S}$

$A\vec{x} = \vec{0}$. sol set : S $S = Null(A)$

$\vec{x_p} \in \tilde{S}$. $\Rightarrow$ $\tilde{S} = \{\vec{x_p} + \vec{x} : \vec{x} \in S\}$

proof: Let $\vec{x_p} \in \tilde{S}$ $A\vec{x_p} = \vec{b}$
$\vec{x} \in S$ $A\vec{x} = \vec{0}$
Prove $\tilde{S} = \{\vec{x_p} + \vec{x} : \vec{x} \in S\}$

Let $\vec{y} = \tilde{S}$ $\therefore A\vec{y} = \vec{b}$ $\vec{y} = \vec{x}_p + \vec{y} - \vec{x_p}$

$A(\vec{y} - \vec{x_p}) = A\vec{y} - A\vec{x_p}. = \vec{b} - \vec{b} = \vec{0}$. $\vec{y} \in \{\vec{x_p} + \vec{x} : \vec{x} \in S\}$.

$\therefore \vec{y} - \vec{x_p} \in S$ $\tilde{S} \subseteq \{\vec{x_p} + \vec{x} : \vec{x} \in S\}$

Let $\vec{z} = \{\vec{x_p} + \vec{x} : \vec{x} \in S\}$

So $S = \vec{z} = \vec{x_p} + \vec{x_1}$ $\vec{x_1} \in S$ $A\vec{x_1} = \vec{0}$

So $A\vec{z} = A(\vec{x_p} + \vec{x_1}) = A\vec{x_p} + A\vec{x_1} = \vec{b} + \vec{0} = \vec{b}$

So $\vec{z} \in \tilde{S}$.

$\therefore \{\vec{x_p} + \vec{x} : \vec{x} \in S\} \subseteq \tilde{S}$

Q. $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 2 & 0 & 2 & 4 \end{bmatrix}$ Solve i) $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ii) $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ iii) $A\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$$\begin{array}{c} \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & -1 & 2 & 0 & 3 \\ 2 & 0 & 2 & 4 & 3 & 0 & 4 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array} \sim \begin{array}{c} \left[\begin{array}{cccc|ccc} 1 & 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & -1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

i) inconsistent.

ii) $x_3 = s$
$x_4 = t$
$x_1 = -s - 2t$
$x_2 = -2s + t$

$$S_2 = \mathrm{Null}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

iii) $x_3 = s$
$x_4 = t$
$x_1 = 2 - s - 2t$
$x_2 = 3 - 2s + t$

$$S_3 = \left\{ \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Q. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Solve the following systems.

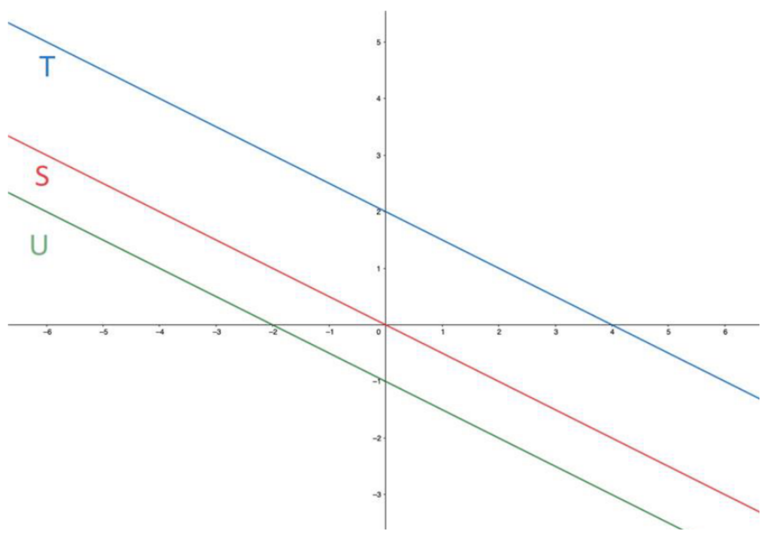
a) $A\vec{x} = \vec{0}$ b) $A\vec{x} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ c) $A\vec{x} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

$$\left[\begin{array}{cc|ccc} 1 & 2 & 0 & 4 & -2 \\ 2 & 4 & 0 & 8 & -4 \end{array}\right] \sim \begin{array}{c} \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cc} x_1 & x_2 \end{array} \end{array}$$

a) $x_2 = t$ $\quad x_1 + 2x_2 = 0$ $\quad x_1 = -2x_2 = -2t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} t$$

$$S = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} t : t \in \mathbb{R} \right\}$$



b) $x_2 = t$ $\quad x_1 + 2t = 4$ $\quad x_1 = 4 - 2t.$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 - 2t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} t$$

$$T = \left\{ \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} t : t \in \mathbb{R} \right\}$$

c) $$U = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} t : t \in \mathbb{R} \right\}$$

- Solutions to $A\vec{x}=\vec{b}$. & $A\vec{x}=\vec{c}$.

$A\vec{x}=\vec{b}$. $A\vec{x}=\vec{c}$. ($\vec{b}\neq\vec{c}$. $A\vec{x}$ → non-homogenious system)

$\tilde{S}_b$ & $\tilde{S}_c$ are respective solution sets, $\vec{x}_b$ & $\vec{x}_c$ are particular solutions.

$$\tilde{S}_c=\{(\vec{x}_c-\vec{x}_b)+\vec{z} : \vec{z}\in\tilde{S}_b\}$$

Q:

$$A=\begin{bmatrix}1&2&3&0\\2&5&1&3\\3&7&-2&3\\4&9&-5&3\end{bmatrix}\qquad \vec{b}=\begin{bmatrix}6\\-1\\-7\\-13\end{bmatrix}\qquad \vec{c}=\begin{bmatrix}0\\3\\3\\3\end{bmatrix}$$

Given sol to $A\vec{x}=\vec{b}$ is $\left\{\begin{bmatrix}1\\-2\\1\\2\end{bmatrix}+s\begin{bmatrix}17\\-7\\1\\0\end{bmatrix}+t\begin{bmatrix}6\\-3\\0\\1\end{bmatrix} : s.t\in\mathbb{F}\right\}$

Determine the solution sets to $A\vec{x}=\vec{0}$ & $A\vec{x}=\vec{c}$

$$\tilde{S}_0=\text{Span}\left\{\begin{bmatrix}17\\-7\\1\\0\end{bmatrix},\begin{bmatrix}6\\-3\\0\\1\end{bmatrix}\right\}$$

particular solution to $A\vec{x}=\vec{c}$ is $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$

$$\tilde{S}_c=\left\{\begin{bmatrix}0\\0\\0\\1\end{bmatrix}+s\begin{bmatrix}17\\-7\\1\\0\end{bmatrix}+t\begin{bmatrix}6\\-3\\0\\1\end{bmatrix}\right\}$$

4.1 The Column and Row Spaces of a Matrix

- Column Space

$A \in M_{m\times n}(\mathbb{F})$. Column Space : $Col(A) = Span\{\vec{a_1}, \vec{a_2} \cdots \vec{a_n}\}$.

ep. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $Col(A) = Span\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$

Consistent System and Column Space

$A \in M_{m\times n}(\mathbb{F})$. $\vec{b} \in \mathbb{F}^n$

$A\vec{x} = \vec{b}$ is consistent $\Leftrightarrow$ $\vec{b} \in Col(A)$

Proof: $(\Rightarrow)$ Assume $A\vec{x} = \vec{b}$. has sol $\vec{x} = [y_1\ y_2 \cdots y_n]^T$.

$[\vec{a_1}\cdot\vec{a_2}\cdots\vec{a_n}][y_1\ y_2 \cdots y_n]^T = \vec{b}$.

$y_1\vec{a_1} + y_2\vec{a_2} + \cdots + y_n\vec{a_n} = \vec{b}$

$\therefore \vec{b}$ is linear combination of A.

$\therefore \vec{b} \in Col(A)$

$(\Leftarrow)$ Assume $\vec{b} \in Col(A)$.

$\vec{b} = s_1\vec{a_1} + s_2\vec{a_2} + \cdots + s_n\vec{a_n}$ $(s \in \mathbb{F})$

Let $\vec{z} = [s_1, s_2 \cdots s_n]^T$

Then $A\vec{z} = s_1\vec{a_1} + s_2\vec{a_2} + \cdots + s_n\vec{a_n} = \vec{b}$.

$\because A\vec{z} = \vec{b}$ $\therefore \vec{z}$ is a sol to the system.

$\therefore A\vec{x} = \vec{b}$ is consistent.

- Transpose (行列交換)

$A \in M_{m\times n}\mathbb{F}$. transpose of A: $(A^T)_{ij} = (A)_{ji}$

ep. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- Row Space

Row (A) is the span of transposed rows of A.

$\text{Row}(A) = \text{Span}\{ (\overrightarrow{row_1}(A))^T, (\overrightarrow{row_2}(A))^T \cdots (\overrightarrow{row_m}(A))^T \}$

ep. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\quad \text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

- Row equivalent

B is equivalent to A $\Rightarrow$ Row (B) = Row (A)

ERO do not change the row space
but change the column space

Q. Let $A \in M_{m \times n}(F)$. Prove $A\vec{x} = \vec{a_i}$ is consistent for $1 \le i \le n$.

proof. $A\vec{e_i} = 0\vec{a_1} + 0\vec{a_2} + \cdots + 1\vec{a_i} + \cdots + 0\vec{a_n} = \vec{a_i}$

4.2 Matrix Equality and Multiplication

- Column Extraction ☆

Let $A = [\vec{a_1} \cdots \vec{a_n}] \in M_{m\times n}(\mathbb{F})$

$\Rightarrow A\vec{e_i} = \vec{a_i} \quad \forall i = 1, \dots, n.$

- Equality of Matrics.

$A = B \iff A\vec{x} = B\vec{x}. \quad \forall x \in \mathbb{F}^n$

proof. $(\Rightarrow)$ Assume $A = B$. Then $A\vec{x} = B\vec{x}$.

$(\Leftarrow)$ Assume $A\vec{x} = B\vec{x}$.

$\therefore A\vec{e_i} = B\vec{e_i}$

$A\vec{e_i} = \vec{a_i} \qquad B\vec{e_i} = \vec{b_i}$

$\therefore \vec{a_i} = \vec{b_i} \qquad \therefore A = B.$

- Matrix Multiplication.

matrix product $AB = C$.

$C = AB = A[\vec{b_1}\ \vec{b_2} \cdots \vec{b_p}] = [A\vec{b_1}\ A\vec{b_2} \cdots A\vec{b_p}]$

$\vec{c_j} = A\vec{b_j}. \quad \forall j = 1, 2 \cdots p$

eg. $\begin{bmatrix} \boxed{1 \quad 7} \\ 2 \quad 0 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 1 \\ -1 \end{matrix}} & 2 & 3 \\ & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot1 + 7\cdot(-1) & 1\cdot2 + 3\cdot7 & 3\cdot1 + 2\cdot7 \\ 2\cdot1 + 0\cdot(-1) & 2\cdot2 + 0.3 & 3\cdot2 + 2\cdot0 \end{bmatrix}$

$= \begin{bmatrix} -6 & 23 & 17 \\ 2 & 4 & 6 \end{bmatrix}$

若两数交换位置，则值 undefined. (2×3 · 2×2 don't work)

$\begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0\cdot1 - 1(-3) & 2\cdot1 + 4\cdot(-3) \\ 0\cdot0 - 1\cdot5 & 2\cdot0 + 4\cdot5 \end{bmatrix} = \begin{bmatrix} 3 & -10 \\ -5 & 20 \end{bmatrix}$

$\begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1\cdot0 + 0\cdot2 & -3\cdot0 + 5\cdot2 \\ 1(-1) + 4\cdot0 & -3\cdot(-1) + 5\cdot4 \end{bmatrix} = \begin{bmatrix} 0 & 10 \\ -1 & 28 \end{bmatrix}$

$AB \neq BA$ even if they are defined

Q. Let $A, C \in M_{m\times n}(\mathbb{F})$ $B \in M_{n\times p}(\mathbb{F})$. Prove or disprove

1) $AB = CB \Rightarrow A = C$

F. B = matrix with all 0 entries. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2) $AB = CB$. $\wedge$ B doesn't have 0 entries $\Rightarrow A = C$

F $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$

$AB = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $AC = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

4.3 Arithmetic Operation on Matrices

- Sum

$A+B=C$ C 的每一项为 $c_{ij} = a_{ij} + b_{ij}$.

* same size
* add corresponding positions

properties:

zero matrix $O + A = A + O = A$. additive identity

additive inverse $A + (-A) = (-A) + A = O$

addition $(A+B)C = AC + BC$

$A(C+D) = AC + AD$

$(AC)E = A(CE) = ACE$.

multiplication $s(A+B) = sA + sB$

$(r+s)A = (rs)A$

$s(AB) = (sA)B = A(sB) = sAB$

* $AB \neq BA$ ∴ order matters.

transpose $(A+B)^T = A^T + B^T$

$(sA)^T = sA^T$

$(AB)^T = B^T A^T$

$(A^T)^T = A$

proof $(A+B)C = AC + BC$

$(A+B)C$

$= \sum_{k=1}^{n} [A+B]_{ik} C_{kj}$

$= \sum_{k=1}^{n} (a_{ik}+b_{ik}) C_{kj}$

$= \sum_{k=1}^{n} a_{ik}C_{kj} + \sum_{k=1}^{n} b_{ik} C_{kj}$

$= [AC]_{ij} + [BC]_{ij}$

$= AC + BC$

$[AB]_{ij}$

$= a_{1i} b_{1j} + a_{2i} b_{2j} + \cdots + a_{mi} + b_{mj}$

$= \sum_{k=1}^{n} a_{ik} + C_{kj}$

proof: $(A+B)^T = A^T + B^T$

$[(A+B)^T]_{ij}$

$= (A+B)_{ji}$

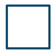
$= A_{ji} + B_{ji}$

$= [A^T]_{ij} + [B^T]_{ij}$

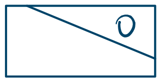
$\therefore (A+B)^T = A^T + B^T$

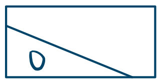
4.4. Square Matrices

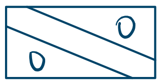
- def. square matrices

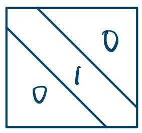
  行与列数量相同 

- upper triangular & lower triangular

  upper triangular  $a_{ij}=0$ for $i>j$

  lower triangular  $a_{ij}=0$ for $i<j$

  diagonal  $a_{ij}=0$ for $i\neq j$

- identity matrix I_n 

Q. T/F $A^2=I_n \Rightarrow A=\pm I_n$

F counter example $\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Q. A square matrix P is called <u>idempotent</u> if $P^2=P$.

Let $A, P \in M_{n\times n}(\mathbb{F})$. Let $Q=P+AP-PAP$.

Prove P is idempotent $\Rightarrow$ Q is idempotent.

$\because P$ is idempotent

$\therefore P^2=P$.

$P^2=P$

$Q^2=(P+AP-PAP)^2$

$=P^2+P(AP)-P(PAP)+(AP)P+(AP)(AP)-(AP)(PAP)-(PAP)P-(PAP)AP+(PAP)(PAP)$

$=P+PAP-PAP+AP+APAP-APAP-PAP-PAPAP+PAPAP$

$=P+AP-PAP$

$=Q$

$\because Q^2=Q$

$\therefore Q$ is idempotent

4.5 Elementary Matrices.

- def. elementary matrix

A matrix that can be obtained by performing a single ERO on identity matrix

一步 ERO 能得到 RREF 的 matrix.

eg. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $R_1 \to R_3$, $R_3 \to R_1$ $\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

- Proposition

Let $A \in M_{m\times n}(\mathbb{F})$ and suppose a single ERO is performed to produce B.

Suppose we perform same ERO on matrix I_m to produce elementary matrix E.

$B = EA$.

proof: Let $A \in M_{m\times n}$

$$A^T = [\vec{r_1}^T \cdots \vec{r_i}^T \cdots \vec{r_j}^T \cdots \vec{r_n}^T] \qquad (\vec{r_i} = \text{Row}_i(A))$$

$$B^T = [\vec{r_1}^T \cdots \vec{r_j}^T \cdots \vec{r_i}^T \cdots \vec{r_n}^T]$$

$$E = [\vec{e_1} \cdots \vec{e_j} \cdots \vec{e_i} \cdots \vec{e_n}]$$

$$\begin{aligned}(EA)^T &= A^T E^T \qquad \hookrightarrow E = E^T \\ &= A^T E \\ &= A^T [\vec{e_1} \cdots \vec{e_j} \cdots \vec{e_i} \cdots \vec{e_n}] \\ &= [A^T\vec{e_1} \cdots A^T\vec{e_j} \cdots A^T\vec{e_i} \cdots A^T\vec{e_n}] \\ &= [\vec{r_1}^T \cdots \vec{r_j}^T \cdots \vec{r_i}^T \cdots \vec{r_n}^T] \\ &= B^T\end{aligned}$$

$\because B^T = (EA)^T \qquad \therefore B = EA$ by properties of matrix transpose

Q. $A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ and the steps involved to row reduce A to I_2.

$$\begin{array}{c} R_1 \\ R_2 \end{array}\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \underset{E_1}{\longrightarrow} \begin{array}{c} R_2 \\ R_1 \end{array}\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \underset{E_2}{\longrightarrow} \begin{array}{c} R_1 \\ -3R_1+R_2 \end{array}\begin{bmatrix} 1 & 2 \\ 0 & -6 \end{bmatrix} \underset{E_3}{\longrightarrow} \begin{array}{c} R_1 \\ -\frac{1}{6}R_2 \end{array}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \underset{E_4}{\longrightarrow} \begin{array}{c} R_1-2R_2 \\ R_2 \end{array}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underbrace{\underset{E_4}{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}} \underset{E_3}{\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{6} \end{bmatrix}} \underset{E_2}{\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}} \underset{E_1}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}} \underset{A}{\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}}$$

$$\begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} = I_2.$$

$$C \quad A \quad = I_2.$$

$\therefore$ C is left inverse of A.

$$\begin{array}{c} R_1 \\ R_2 \end{array}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{array}{c} R_1 \\ R_2 \end{array}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{array}{c} R_2 \\ R_1 \end{array}\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{array}{c} R_1 \\ aR_1+bR_2 \end{array} \qquad \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$$

4.6 Matrix Inverse

- Invertible matrix

若存在 $n\times n$ matrices B, C. $A\underset{\text{right inverse}}{B} = \underset{\text{left inverse}}{C}A = I_n$

则 $n\times n$ matrix A is invertible

* inverse 有时不一定存在

eg. $\because \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \neq I_2$ $\therefore \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ 没有 inverse

Q. Find right inverse of $A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ a+2c & b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{cases} 3a = 1 \\ 3b = 0 \\ a+2c = 0 \\ b+2d = 1 \end{cases} \qquad \begin{cases} a = \frac{1}{3} \\ b = 0 \\ c = -\frac{1}{6} \\ d = \frac{1}{2} \end{cases} \qquad \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

- Equality of matrices (Left inverse = right inverse)

$AB = CA = I_n \Rightarrow B = C$ left & right inverse. 相等

proof: Assume $AB = I_n$ $CA = I_n$

$B = I_nB = (CA)B = C(AB) = CI_n = C$

- Left invertible $\Leftrightarrow$ right invertible

$\exists$ $n\times n$ matrix B. $AB = I_n$ $\Leftrightarrow$ $\exists C$. $CA = I_n$.

left & right inverse. 同时存在 / 不存在

proof: $(\Rightarrow)$ Assume $AB = I_n$

$\rightarrow$ Consider homogenious system $B\vec{x} = \vec{0}$

$AB\vec{x} = A\vec{0}$
$I_n\vec{x} = \vec{0}$
$\vec{x} = \vec{0}$ $\therefore B\vec{x} = \vec{0}$ has unique sol.

By System Rank Theorem, nullity $= 0 = n - r$, rank $= n$

$\rightarrow$ By System Rank Theorem. $B\vec{x} = \vec{b}$ has a sol for every $\vec{b} \in \mathbb{F}^n$

$\forall \vec{b} \in \mathbb{F}^n$. $\exists \vec{x} \in \mathbb{F}^n$ s.t $\vec{b} = B\vec{x}$.

$(BA)\vec{b} = BA(B\vec{x}) = B(AB)\vec{x} = BI_n\vec{x} = B\vec{x} = \vec{b} = I_n\vec{b}$.

$\because (BA)\vec{b} = I_n\vec{b} \quad (\forall \vec{b} \in \mathbb{F}^n)$

$\therefore$ By equality of matrices. $BA = I_n$.

$(\Leftarrow)$ Assume $CA = I_n$.

$I_n = I_n^T = (CA)^T = A^T C^T$

$\because C^T$ is right inverse of A^T.

$\therefore$ By $(\Rightarrow)$ 所证, $C^T A^T = I_n$

$I_n = I_n^T = (C^T A^T)^T = (A^T)^T (C^T)^T = AC$

- Proof : Inverse unique.

Assume $\exists B_1 \& B_2$.

$AB_1 = B_1A = I_n \qquad AB_2 = B_2A = I_n$.

$\therefore B_1$ is left inverse of A. B_2 is right inverse of A.

By equality of left & right inverse. $B_1 = B_2$

- Inverse of matrix. $\qquad (AB = BA = I_n)$

若 $n\times n$ matrix is invertible. 则 B is inverse of A $(AB = I_n)$

$AA^{-1} = A^{-1}A = I_n \qquad (A^{-1}$: inverse of $A)$

* 证明 matrix B is the inverse of A. $\rightarrow$ 求 $AB = I_n$.

Q : Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Prove $A^{-1} = A$.

$AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \therefore A^{-1} = A$

- Invertibility Criteria

The following 3 conditions are equivalent.

a) A is invertible

b) $\operatorname{rank}(A) = n$

c) $\operatorname{RREF}(A) = I_n$

$$\begin{tikzcd} & a) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ b) \arrow[rr, Leftrightarrow] & & c) \end{tikzcd}$$

proof $(a) \Rightarrow (b) \Rightarrow (c)$

$(a) \Rightarrow (b)$ Assume A is invertible

$\exists C$ s.t $CA = I_n$

$\because$ Left invertible $\Leftrightarrow$ Right invertible

$\therefore \operatorname{rank}(A) = n$

$(b) \Rightarrow (c)$ Assume $\operatorname{rank}(A) = n$

$\because A \in M_{n\times n}$

$\therefore$ RREF(A)每行每列都有 pivot

$(c) \Rightarrow (a)$ Assume $\operatorname{RREF}(A) = I_n$

$\therefore \operatorname{rank}(A) =$ number of rows

$\therefore A\vec{x} = \vec{b}$ is consistent $\forall b \in \mathbb{F}^n$ (System Rank theorem)

$\therefore A\vec{x} = \vec{e_i}$ is consistent for $1 \le i \le n$.

Let $\vec{b_i}$ be the sol. to $A\vec{x} = \vec{e_i}$

$\therefore A\vec{b_i} = \vec{e_i}$

Let $B = [\vec{b_1} \cdots \vec{b_n}]$

$$\therefore AB = A[\vec{b_1} \cdots \vec{b_n}] = [A\vec{b_1} \;\; A\vec{b_2} \cdots A\vec{b_n}]$$
$$= [\vec{e_1} \;\; \vec{e_2} \cdots \vec{e_n}]$$
$$= I_n$$

需要解 $A\vec{x} = \vec{e_1} \cdots A\vec{x} = \vec{e_n}$

Solve by super-augmented matrix

$[A \mid \vec{e_1}\ \vec{e_2} \cdots \vec{e_n}] = [A \mid I_n]$

$\therefore AB = I^n$

$B = A^{-1}$ A is invertible

- Algorithm for checking invertibility : 找 I_n.

step 1. construct a super-augmented matrix $[A \mid I_n]$

step 2. 找 RREF. $[R \mid B]$ of $[A \mid I_n]$

step 3. If $R \neq I_n$ A not invertible

If $R = I_n$ A is invertible $A^{-1} = B$

Q. Find the inverse if there exists $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$\therefore A$ is invertible $A^{-1} = \begin{bmatrix} -7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Q. Find the inverse if there exists $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{array}\right]$$

$RREF(A) \neq I_3$ So A isn't invertible

Q. What value of a & b can $A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$ invertible.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \sim \begin{array}{c} R_1 \\ R_2 \\ R_2 - bR_3 \end{array} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1-ab \end{bmatrix}$$

If $1-ab \neq 0$ then $\text{rank}(A) = 3$ and A is invertible.

need $ab \neq 1$

- Inverse of 2x2 matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, A is invertible $\Leftrightarrow$ $ad-bc \neq 0$

$ad-bc \neq 0 \Rightarrow A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

proof: let $B = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\because AB = BC = I_2$ $\therefore A$ is invertible $B = A^{-1}$

Assume $ad-bc=0$. Prove $\text{rank}(A) \neq 2 \Rightarrow A$ isn't invertible

① $d=0$

$A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$

$bc = ad = 0$

$b=0$ or $c=0$.

$\rightarrow c=0$ $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

$\text{rank}(A) \neq 2$.

$\therefore$ not invertible

$\rightarrow b=0$ $A = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$

$\text{rank}(A) \neq 2$

$\therefore$ not invertible

② $d \neq 0$

$\begin{matrix} R_1 \\ R_2 \end{matrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{matrix} dR_1 \\ R_2 \end{matrix} \begin{bmatrix} ad & bd \\ c & d \end{bmatrix}$

$\begin{matrix} -bR_2+R_1 \\ R_2 \end{matrix} \begin{bmatrix} ad-bc & 0 \\ c & d \end{bmatrix}$

$\because ad-bc=0$

$\therefore A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$

$\text{rank}(A) \neq 2$.

$\therefore$ not invertible

Q. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. Find A^{-1}.

Use A^{-1} to solve $A\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$5-6=-1 \neq 0$ $\therefore A^{-1}$ exists

$A^{-1} = \frac{1}{-1} \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

$A\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A^{-1}A\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$I_2\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

5.1 Function determine by a Matrix

- def. function determined by matrix A

$$T_A : \mathbb{F}^n \to \mathbb{F}^m \quad \to \quad T_A(\vec{x}) = A\vec{x}$$

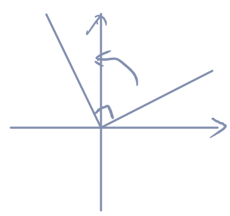
ep. $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$ $\quad T_A : \mathbb{F}^2 \to \mathbb{F}^3$, $\quad T_A(\vec{x}) = A\vec{x}$ "transformation" "mapping"

$$T_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$$

$$T_A\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_1 \\ x_1 \\ 2x_1 + 3x_2 \end{bmatrix}$$

Q. for each matrix A describe the transformation $T_A : R_2 \to R_2$ geometrically

1. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$ - scale vector by 3
2. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ - reflection in $y=x$
3. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$ - rotate of $90°$



4. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$ - projection onto x-axis

5.2 Linear Transformation

- Linear transformation / linear mapping

def. $\mathbb{F}^n \to \mathbb{F}^m$ ($\mathbb{F}^n$ domain $\mathbb{F}^m$ codomain)

Properties:

1. $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ linearity over addition
2. $T(c\vec{x}) = cT(\vec{x})$ linearity over scalar multiplication

T_A : linear transformation determined by A.

- Zero Maps to Zero.

$\mathbb{F}^n \to \mathbb{F}^m$ is linear transformation $\Rightarrow$ $T(\vec{0}_{\mathbb{F}^n}) = \vec{0}_{\mathbb{F}^m}$

proof. $T(\vec{0}_{\mathbb{F}^n}) = T(0 \cdot \vec{x})$ $\vec{x} \in \mathbb{F}^n$

$= 0 \cdot T(\vec{x})$ by linearity

$= \vec{0}_{\mathbb{F}^m}$

判断是否是 Linear Transformation:

① definition $T(c\vec{x}+\vec{y}) = cT(\vec{x}) + T(\vec{y})$

② 排除法. $T(\vec{0}) = \vec{0}$.
(仅判断不是 linear T) $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y+2xy \end{bmatrix}$ (计算中出现高级运算)

Q. Linear or not linear

① $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ $T_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ 2 \end{bmatrix}$

$T_1\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ not linear

② $T_2 : \mathbb{C}^2 \to \mathbb{C}^3$ $\quad T_2 \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2i x_2 \\ 0 \end{bmatrix}$

Let $\vec{x}, \vec{y} \in \mathbb{C}^2$ $\quad c \in \mathbb{C}$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$c\vec{x} + \vec{y} = \begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix}$$

$$T(c\vec{x} + \vec{y}) = T \begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix} = \begin{bmatrix} cx_1 y_1 + cx_2 + y_2 \\ 2icx_2 + 2iy_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} cx_1 + cx_2 \\ 2icx_2 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ 2iy_2 \\ 0 \end{bmatrix} = c \begin{bmatrix} x_1 + x_2 \\ 2ix_2 \\ 0 \end{bmatrix} + T(\vec{y})$$

$$= cT(\vec{x}) + T(\vec{y})$$

∴ linear

③ $T_3 : \mathbb{R}^3 \to \mathbb{R}^3$ $\quad T_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 x_3 \end{bmatrix}$

counter example:

$$T_3 \left(2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = T_3 \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

$$2T_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \neq \qquad T(c\vec{x}) \neq cT(\vec{x})$$

∴ non-linear

Q. 找 d s.t $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3d^2x - 2dxy \\ x + 2d^2 - 2d \end{bmatrix}$ linear ($T: \mathbb{R}^2 \to \mathbb{R}^2$)

$$T(c\vec{x} + \vec{y}) = T\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix} = \begin{bmatrix} 3d^2x - 2dxy \\ x + 2d^2 - 2d \end{bmatrix}$$

当 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2d^2 - 2d \end{bmatrix}$ $d = 0$ or $d = 1$

$\to d = 0$ $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}$

$$T(c\vec{x} + \vec{y}) = T\begin{bmatrix} cx_1 + y_1 \\ cx_2 + y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ cx_1 + y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ cx_1 \end{bmatrix} + \begin{bmatrix} 0 \\ y_1 \end{bmatrix}$$

$$= c\begin{bmatrix} 0 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ y_1 \end{bmatrix} = cT\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + T\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$\to d = 1$ $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - 2xy \\ x \end{bmatrix}$

$$T\left(3\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \end{bmatrix}$$

$$3T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$T\left(3\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \neq 3T\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\therefore$ T isn't linear 不存在 d.

Q. Let $\vec{v}$ be a fixed vector in $\mathbb{R}^3$. Define a mapping $CROSS_{\vec{v}}: \mathbb{R}^3 \to \mathbb{R}^3$ by $CROSS_{\vec{v}}(\vec{x}) = \vec{x} \times \vec{v}$.

Prove $CROSS_{\vec{v}}$ is linear mapping.

$\because$ Let $\vec{x}, \vec{y} \in \mathbb{R}^3$, $c \in \mathbb{R}$

$$CROSS_{\vec{v}}(c\vec{x} + \vec{y}) = c\,CROSS_{\vec{v}}(\vec{x}) + CROSS_{\vec{v}}(\vec{y})$$

$$CROSS_{\vec{v}}(c\vec{x} + \vec{y}) = (c\vec{x} + \vec{y})\vec{v}$$

$$= c(\vec{x} \times \vec{v}) + y \times \vec{v}$$

$$= c\,CROSS_{\vec{v}}(\vec{x}) + CROSS_{\vec{v}}(\vec{y})$$

$\therefore$ $CROSS_{\vec{v}}$ is linear.

5.3 The range of linear transformation

- def. Range

Let $T: \mathbb{F}^n \to \mathbb{F}^m$ be linear transformation

Range $T = \{T(\vec{x}) : \vec{x} \in \mathbb{F}^n\}$ Range $(T) \subseteq \mathbb{F}^m$

$*$ $T(\vec{0}_{\mathbb{F}^n}) = \vec{0}_{\mathbb{F}^m}$ Range$(T) \neq \emptyset$

Q. $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ |x_2| \end{bmatrix}$

$\because T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in$ Range(T)

$\because$ no $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix} \notin$ Range(T)

- Range of linear transformation

Let $A \in M_{m\times n}$. $T_A: \mathbb{F}^n \to \mathbb{F}^m$ be linear transformation by A.

Then Range$(T_A) = $ Col(A)

proof: $\vec{x} \in$ Range(T_A)

$\Leftrightarrow$ $\vec{x} = A\vec{y}$ for some $\vec{y} \in \mathbb{F}^n$.

$\Leftrightarrow$ $\vec{x} \in$ Col(A)

ep. ① $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ $\mathbb{R}^2 \to \mathbb{R}^2$

$T_A(\vec{x}) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$

Range$(T_A) = \{3\vec{x} : \vec{x} \in \mathbb{R}^2\} = \mathbb{R}^2$

scaling

② $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\mathbb{R}^2 \to \mathbb{R}^2$

$T_A(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$

Range$(T_A) = $ Span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

projection

$\boxed{A\vec{x} = \vec{b} \text{ is consistent} \Leftrightarrow \vec{b} \in \text{Range}(T_A) \;/\; \vec{b} \in \text{Col}(A)}$

Q. $A = \begin{bmatrix} i & 1-i & 5 \\ 0 & 2i & 4+i \end{bmatrix}$

$T_A(\vec{x}) = \begin{bmatrix} i & 1-i & 5 \\ 0 & 2i & 4+i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} i \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1-i \\ 2i \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 4+i \end{bmatrix}$

$\text{Range}(T_A) = \text{Span}\left\{ \begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 1-i \\ 2i \end{bmatrix}, \begin{bmatrix} 5 \\ 4+i \end{bmatrix} \right\}$

$\therefore A\vec{x} = \vec{b}$ has solution iff $\vec{b} \in \text{Range}(T_A)$

判断 $A\vec{x}=\vec{b}$ 是否有解

① 求 $T_A(\vec{x}) = A\vec{x} = x_1 \begin{bmatrix} \vdots \end{bmatrix} + x_2 \begin{bmatrix} \vdots \end{bmatrix} + \cdots$

② 求 $\text{Range}(T_A) = \text{Span}\left\{ \begin{bmatrix} \vdots \end{bmatrix}, \begin{bmatrix} \vdots \end{bmatrix} \right\}$.

③ $A\vec{x} = \vec{b}$ has solution iff $\vec{b} \in \text{Range}(T_A)$

- def. Onto.

transformation $T: \mathbb{F}^n \to \mathbb{F}^m$ is onto if $\text{Range}(T) = \mathbb{F}^m$

1) scaling is onto

2) projection isn't onto.

- Onto Criteria

Let $A \in M_{m\times n}(\mathbb{F})$. Let T_A be linear transformation determined by matrix A.

(a) T_A is onto

$\Updownarrow$

(b) $\text{Col}(A) = \mathbb{F}^m$

$\Updownarrow$

(c) $\text{rank}(A) = m$

proof: (a) $\Rightarrow$ (b) Assume T_A is onto $\to$ $\text{Range}(T_A) = \mathbb{F}^m$

$\because \text{Range}(T_A) = \text{Col}(A)$ $\therefore \text{Col}(A) = \mathbb{F}^m$.

(b) $\Rightarrow$ (c) Assume $\text{Col}(A) = \mathbb{F}^m$

$\because \text{Col}(A) = \mathbb{F}^m$ $\therefore A\vec{x} = \vec{b}$ is consistent $\forall \vec{b} \in \mathbb{F}^m$

$\therefore \text{rank}(A) = m$ by System Rank theorem

(c) $\Rightarrow$ (a) Assume $\text{rank}(A) = m$

$A\vec{x} = \vec{b}$ is consistent $\forall \vec{b} \in \mathbb{F}^m$ (system rank theorem)

$\therefore \text{Col}(A) = \mathbb{F}^m$.

$\therefore \text{Range}(T_A) = \mathbb{F}^m$.

Q. Determine whether linear transformation T_A determined by $A = \begin{bmatrix} 1 & 4 \\ -2 & -5 \\ 4 & 6 \end{bmatrix}$ is onto

$\text{Range}(T_A) = \text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} \right\}$ $\rightarrow$ only a plane in $\mathbb{R}^3$.

$\text{Range}(T_A) \neq \mathbb{R}^3$. $\therefore T_A(\vec{x}) = \vec{b}$ 可能无解. eg. $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} u + \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} v.$$

$$\left[\begin{array}{cc|c} 1 & 4 & x_1 \\ -2 & -5 & x_2 \\ 4 & 6 & x_3 \end{array}\right] \sim \left[\quad \right]$$

5.4. Kernel

- def. Kernel.

$\mathrm{Ker}(T) = \{\vec{x} \in \mathbb{F}^n : T(\vec{x}) = \vec{0}_{\mathbb{F}^m}\}$

$\mathrm{Ker}(T)$ will never be $\emptyset$. 因为永远存在 $T(\vec{0}_{\mathbb{F}^n}) = \vec{0}_{\mathbb{F}^m}$ (zero maps to zero)

- Kernel of a Linear Transformation

$A \in M_{m\times n}(\mathbb{F})$ $\quad T_A : \mathbb{F}^n \to \mathbb{F}^m$ is linear transformation determined by A.

$\mathrm{Ker}(T_A) = \mathrm{Null}(A) =$ solution set to $A\vec{x} = \vec{0}$

proof. $\vec{x} \in \mathrm{Ker}(T_A) \Leftrightarrow T_A(\vec{x}) = \vec{0}_{\mathbb{F}^m}$

$\Leftrightarrow A\vec{x} = \vec{0}$

$\Leftrightarrow \vec{x} \in \underline{\mathrm{Null}(A)}$ ⟵ solution set of $A\vec{x} = \vec{0}$

ep. $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ (scaling)

$\left[\begin{array}{cc|c} 3 & 0 & 0 \\ 0 & 3 & 0 \end{array}\right]$ $\quad x = 0,\ y = 0$

$\mathrm{Ker}(T_A) = \mathrm{Span}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (projection)

$\begin{array}{c} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ x_1 \quad x_2 \end{array}$ $\quad x_1 = 0,\ x_2 = t$

$\mathrm{Ker}(T_A) = \mathrm{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

- def. One-to-one.

$T: \mathbb{F}^n \to \mathbb{F}^m$ is <u>one-to-one</u> if whenever $T(\vec{x}) = T(\vec{y}) \Rightarrow \vec{x} = \vec{y}$.

contrapositive: $\vec{x} \neq \vec{y} \Rightarrow T(\vec{x}) \neq T(\vec{y})$

ep. $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ (scaling)

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} 3x \\ 3y \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix}$

$x = a$
$y = b$

$\therefore$ one-to-one

$T_A\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (projection)

$T_A\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- One-to-one test (判断 one-to-one)

$T: \mathbb{F}^n \to \mathbb{F}^m$.

T is one-to-one $\Leftrightarrow$ $\mathrm{Ker}(T) = \{\vec{0}_{\mathbb{F}^n}\}$

proof. $(\Rightarrow)$ Assume T is one-to-one

$\to \{\vec{0}_{\mathbb{F}^n}\} \subseteq \mathrm{Ker}(T)$

$\to$ Let $\vec{x} \in \mathrm{Ker}(T)$ $\quad \therefore T(\vec{x}) = \vec{0}_{\mathbb{F}^m} = T(\vec{0}_{\mathbb{F}^n})$

$\because T$ is one-to-one $\quad \therefore \vec{x} = \vec{0}_{\mathbb{F}^m}$

$\therefore \mathrm{Ker}(T) \subseteq \{\vec{0}_{\mathbb{F}^m}\}$

$(\Leftarrow)$ Assume $\mathrm{Ker}(T) = \{\vec{0}_{\mathbb{F}^n}\}$

$\to$ Let $\vec{x}, \vec{y} \in \mathbb{F}^n$ Assume $T(\vec{x}) = T(\vec{y})$ $\quad$ 证 $\vec{x} = \vec{y}$

$T(\vec{x}) - T(\vec{y}) = \vec{0}_{\mathbb{F}^m}$

$T(\vec{x} - \vec{y}) = \vec{0}_{\mathbb{F}^m}$

$\because T$ is linear $\quad \therefore \vec{x} - \vec{y} \in \mathrm{Ker}(T)$ $\quad \therefore \vec{x} - \vec{y} = \vec{0}_{\mathbb{F}^n}$

$\therefore \vec{x} = \vec{y}$ $\quad$ T is one-to-one

- One-to-one criteria.

$A \in M_{m \times n}(\mathbb{F})$ $\quad$ T_A is linear transformation by matrix A

$\downarrow$ equivalent

(a) T_A is one-to-one

(b) $\mathrm{Null}(A) = \{\vec{0}_{\mathbb{F}^n}\}$

(c) nullity $(A) = 0$

(d) rank $(A) = n$

proof: (a) ⇒ (b) Assume T_A is one-to-one.

$Ker(T_A) = \{\vec{0}_{\mathbb{F}^n}\}$ $\quad Ker(T_A) = Null(A)$

$\therefore Null(A) = \{\vec{0}_{\mathbb{F}^n}\}$

(b) ⇒ (c) Assume $Null(A) = \{\vec{0}_{\mathbb{F}^n}\}$

$\therefore A\vec{x} = \vec{0}$ has unique sol $\longrightarrow \vec{0}_{\mathbb{F}^n}$

$\therefore$ nullity$(A) = 0$

(c) ⇒ (d) Assume nullity$(A) = 0$

$\therefore n - \text{rank}(A) = 0$

rank$(A) = 0$

(d) ⇒ (a) Assume rank$(A) = n$

$A\vec{x} = \vec{0}_{\mathbb{F}^m}$ is consistent. $\quad \vec{x} = \vec{0}_{\mathbb{F}^n}$ is sol.

$\therefore$ rank$(A) = n$.

\# of parameters $= 0$

$\therefore \vec{x} = \vec{0}_{\mathbb{F}^n}$ is the only sol

$\therefore Null(A) = \{\vec{0}_{\mathbb{F}^n}\} = Ker(T_A)$

$\therefore T_A$ is one-to-one by one-to-one test

T one-to-one

$T(x) = b$ 最多一个解

rank(A) = num of cols (每列有解)

range(T) has dimension n

↳ col(A) 为 $n \times 1$ $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

T is onto

$T(x) = b$ 至少一个解

$A\vec{x} = \vec{y} \in \mathbb{R}^m \quad [A \mid \vec{y}]$

每行有解

range(T) has dimension m

↳ col(A) 为 $m \times 1$ $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

Q. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$. Consider $T_A : \mathbb{R}^2 \to \mathbb{R}^3$

1) Is T_A onto?

By Rank Bounds, $\operatorname{rank}(A) \le 2$

$\therefore \operatorname{rank}(A) \ne 3$.

$\therefore$ By onto criteria, T_A isn't one-to-one

2) Is T_A one-to-one?

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \begin{array}{c} R_1 \\ 3R_1 - R_2 \\ 5R_1 - R_3 \end{array}\begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 4 \end{bmatrix} \quad \begin{array}{c} R_1 - R_2 \\ \frac{1}{2}R_2 \\ \frac{1}{4}R_3 \end{array}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{array}{c} R_1 \\ R_2 \\ R_2 - R_3 \end{array}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\operatorname{rank}(A) = 2$

$\therefore T_A$ is one-to-one

Let $A \in M_{n\times n}(\mathbb{F})$. The following three conditions are equivalent:

(a) A is invertible.

(b) $\text{rank}(A) = n$.

(c) $\text{RREF}(A) = I_n$.

Let $A \in M_{\overset{n}{\cancel{m}}\times n}(\mathbb{F})$ and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent.

(a) T_A is onto.

(b) $\text{Col}(A) = \mathbb{F}^m$.

(c) $\text{rank}(A) = m$.

Let $A \in M_{\overset{n}{\cancel{m}}\times n}(\mathbb{F})$ and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent.

(a) T_A is one-to-one.

(b) $\text{Null}(A) = \{\vec{0}_{\mathbb{F}^n}\}$.

(c) $\text{nullity}(A) = 0$.

(d) $\text{rank}(A) = n$.

(Invertibility Criteria – Second Version)

Let $A \in M_{n\times n}(\mathbb{F})$ be a square matrix and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent.

(a) A is invertible.

(b) T_A is invertible

(c) T_A is one-to-one.

(d) T_A is onto.

(e) $\text{Null}(A) = \{\vec{0}\}$. That is, the only solution to the homogeneous system $A\vec{x} = \vec{0}$ is the trivial solution $\vec{x} = \vec{0}$.

(f) $\text{Col}(A) = \mathbb{F}^n$. That is, for every $\vec{b} \in \mathbb{F}^n$, the system $A\vec{x} = \vec{b}$ is consistent.

(g) $\text{nullity}(A) = 0$.

(h) $\text{rank}(A) = n$.

(i) $\text{RREF}(A) = I_n$.

5.5 Every Linear Transformation is determined by a matrix

- def. standard basis vectors in $\mathbb{F}^n$

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\} = \{ \vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n \}$$

- Standard Matrix

Let $T: \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation.

standard matrix of T: $[T]_\mathcal{E}$

$\hookrightarrow$ $m \times n$ matrix. whose col are under T of vectors in standard basis of $\mathbb{F}^n$.

$$[T]_\mathcal{E} = [\, T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; \cdots \;\; T(\vec{e}_n) \,]$$

$$= \left[T\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) \;\; T\left(\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}\right) \cdots T\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right) \right]$$

- every linear transformation is determined by a matrix

$\forall \vec{x} \in \mathbb{F}^n \quad T(\vec{x}) = [T]_\mathcal{E} \vec{x}$.

$T = T_{[T]_\mathcal{E}}$ is linear transformation determined by matrix $[T]_\mathcal{E}$.

* 每个 linear transformation 可用 matrix-vector multiplication 写成
* 可通过 $T(\vec{e}_i)$ 来找.
* 用 matrix-vector multiplication 计算 $T(\vec{x})$

proof. Let $\vec{x} \in \mathbb{F}^n$ $\quad \vec{x} = [x_1 \;\; x_2 \;\; \cdots \;\; x_n]^T$

$$T(\vec{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{bmatrix}\right)$$

$$= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_n T(\vec{e}_n) \qquad \text{linearity}$$

$$= \underbrace{\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}}_{\text{matrix with columns } T(\vec{e}_1)\cdots T(\vec{e}_n)} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

← 叫作 "standard matrix

$$= [T]_{\varepsilon} \cdot \vec{x}$$

Let $T: \mathbb{R} \to \mathbb{R}$ be linear transformation.

Then $\exists\, m \in \mathbb{R}$. s.t $T(x) = mx \quad \forall x \in \mathbb{R}$

proof. $T(x) = [T]_{\varepsilon} \cdot x$. $\quad [T]_{\varepsilon}$ of T is 1×1 matrix $[T(1)]$

Let $m = T(1)$ $\quad T(x) = mx$

- Relationship between matrices & linear transformation

For $T: \mathbb{R}^n \to \mathbb{R}^m$. We can find a $m\times n$ matrix $[T]_{\varepsilon}$ s.t $T(\vec{x}) = [T]_{\varepsilon}\vec{x}$

Given $A \in M_{m\times n}(\mathbb{F})$. We can create $\underline{T_A}: \mathbb{R}^n \to \mathbb{R}^m$. where $T_A(\vec{x}) = A\vec{x}$

← linear

- Properties of standard matrix

$A \in M_{m\times n}(\mathbb{F})$ $\quad T_A: \mathbb{F}^n \to \mathbb{F}^m$ is linear transformation determined by A

$T: \mathbb{F}^n \to \mathbb{F}^m$ is linear transformation

a) $T_{[T]_{\varepsilon}} = T$

b) $[T_A]_{\varepsilon} = A$

c) T is onto $\Leftrightarrow$ $\operatorname{rank}([T]_{\varepsilon}) = m$

d) T is one-to-one $\Leftrightarrow$ $\operatorname{rank}([T]_{\varepsilon}) = n$

proof (a) $T_{[T]_{\varepsilon}}\vec{x} = [T]_{\varepsilon}\vec{x} = T\vec{x} \quad \forall \vec{x} \in \mathbb{F}^n$

(b) $[T_A]_{\varepsilon} = [T_A(\vec{e}_1) \cdots T_A(\vec{e}_n)]$

$= [A\vec{e}_1 \cdots A\vec{e}_n]$

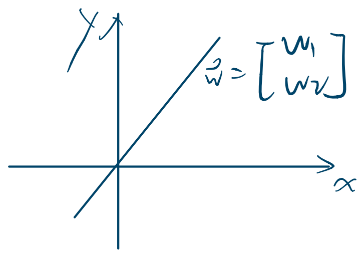
$= [\vec{a}_1 \cdots \vec{a}_n]$ $\qquad$ (column extraction)

$= A$

5.6 Special Linear Trasformation :

projection, perpendicular, rotation, reflection

- projection onto a line through origin



line with direction vector $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $\vec{w} \neq \vec{0}$

$proj_{\vec{w}} : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad proj_{\vec{w}}(\vec{x}) = \dfrac{\vec{x} \cdot \vec{w}}{\|\vec{w}\|^2} \cdot \vec{w}.$

→ Prove $proj_{\vec{w}}$ is linear transformation

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$. $\quad c_1\ c_2 \in \mathbb{R}$

$$proj_{\vec{w}}(c_1\vec{u} + c_2\vec{v}) = c_1 \frac{\vec{u}\cdot\vec{w}}{\|\vec{w}\|^2}\vec{w} + c_2 \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\vec{w}$$

$$= c_1\, proj_{\vec{w}}(\vec{u}) + c_2\, proj_{\vec{w}}(\vec{v})$$

→ Standard matrix $[proj_{\vec{w}}]_{\varepsilon}$ of $proj_{\vec{w}}$

$$[proj_{\vec{w}}]_{\varepsilon} = [proj_{\vec{w}}(\vec{e_1}). \quad proj_{\vec{w}}(\vec{e_2})]$$

$$proj_{\vec{w}}(\vec{e_1}) = \frac{w_1}{w_1^2+w_2^2}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 \\ w_1w_2 \end{bmatrix}$$

$$proj_{\vec{w}}(\vec{e_2}) = \frac{w_2}{w_1^2+w_2^2}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1w_2 \\ w_2^2 \end{bmatrix}$$

$$\boxed{\therefore [proj_{\vec{w}}]_{\varepsilon} = \frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix}}$$

→ Is $proj_{\vec{w}}$ onto?

$$Range(proj_{\vec{w}}) = Col[proj_{\vec{w}}]_{\varepsilon} = Col\left(\frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix}\right)$$

$$= Span\left\{\frac{w_1}{w_1^2+w_2^2}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \frac{w_2}{w_1^2+w_2^2}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right\}$$

$$= Span\left\{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right\} \neq \mathbb{R}^2$$

$\therefore$ not onto

→ Is $proj_{\vec{w}}$ one-to-one?

geometrically. we can find 2 vectors which are distinct that map to same output

- By square standard matrix. not onto ⇒ not one-to-one.
- $Ker(proj_{\vec{w}})$ = sol set to $\frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix} \cdot \vec{x} = \vec{0}$ $\neq \{\vec{0}\}$

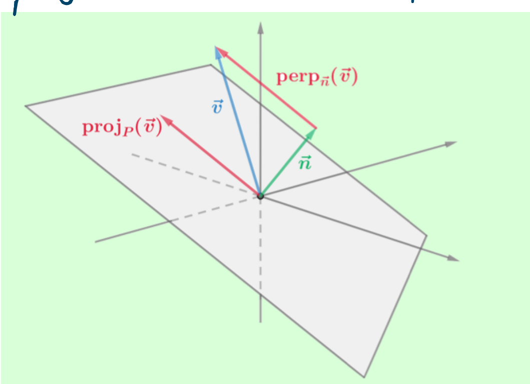
$\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix} \rightarrow \begin{bmatrix} w_1 & w_2 \\ 0 & 0 \end{bmatrix}$ ∴ rank of $\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix} = 1$.

∵ $1 < 2$ ∴ not one-to-one

Q. Find image of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ when projected onto line $\vec{l} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} t$. $(t \in \mathbb{R})$

$[T]_\varepsilon \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$

- projection onto a plane through origin



$\vec{v}, \vec{w} \in \mathbb{R}^3$. $proj_P : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

$proj_P(\vec{v}) = perp_{\vec{n}}(\vec{v})$

eg. P has scalar equation $3x - 4y + 5z = 0$

→ Prove $proj_{\vec{w}}$ is linear transformation

∵ $proj_P(\vec{v}) = perp_{\vec{n}}(\vec{v})$ $perp_{\vec{n}}(\vec{v})$ is linear transformation

∴ $proj_P$ is linear transformation

→ Standard matrix $[proj_P]_\varepsilon$ of $proj_P$.

$[proj_P]_\varepsilon = [proj_P(\vec{e_1}) \quad proj_P(\vec{e_2}) \quad proj_P(\vec{e_3})]$

$proj_P(\vec{e_1}) = perp_{\vec{n}}(\vec{e_1}) = \vec{e_1} - proj_{\vec{n}}(\vec{e_1}) = \vec{e_1} - \frac{\vec{e_1} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n}$

$proj_P(\vec{e_1}) = \begin{bmatrix} \frac{41}{50} \\ \frac{6}{25} \\ -\frac{3}{10} \end{bmatrix}$ $proj_P(\vec{e_2}) = \begin{bmatrix} \frac{6}{25} \\ \frac{17}{25} \\ \frac{2}{5} \end{bmatrix}$ $proj_P(\vec{e_3}) = \begin{bmatrix} -\frac{3}{10} \\ \frac{2}{5} \\ \frac{1}{2} \end{bmatrix}$

$$[proj_P]_{\varepsilon} = \begin{bmatrix} \frac{41}{50} & \frac{6}{25} & -\frac{3}{10} \\ \frac{6}{25} & \frac{17}{25} & \frac{2}{5} \\ -\frac{3}{10} & \frac{2}{5} & \frac{1}{2} \end{bmatrix}$$

→ Is $proj_P$ onto?

找 $\vec{v} \in \mathbb{R}^3$. s.t $[proj_P]_{\varepsilon}\, \vec{x} = \vec{v}$. inconsistent

eg. $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad 1\times 3 + 0(-4) + 0.5 = 3 \neq 0$

$\because \vec{v} \notin Range(proj_P) \quad \therefore Range(proj_P) \neq \mathbb{R}^3$

$\therefore$ $proj_P$ isn't onto

→ Is $proj_P$ one-to-one?

$\because [proj_P]_{\varepsilon}\, \vec{x} = \vec{0} \qquad rank([[proj_P]_{\varepsilon} \mid \vec{0}]) \neq 3$

$\therefore$ $proj_{\vec{w}}$ isn't one-to-one

Q. Find standard matrix for projection onto $2x - 5y + z = 0$

$\vec{n} = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \qquad \|\vec{n}\|^2 = 30$

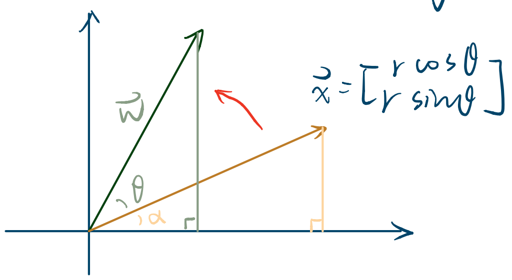
$$proj_P(\vec{e}_1) = \vec{e}_1 - proj_{\vec{n}}(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{30} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 26 \\ 10 \\ -2 \end{bmatrix}$$

$$proj_P(\vec{e}_2) = \vec{e}_2 - proj_{\vec{n}}(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{30} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}$$

$$proj_P(\vec{e}_3) = \vec{e}_3 - proj_{\vec{n}}(\vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{30} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} -2 \\ 5 \\ 29 \end{bmatrix}$$

$$[proj_P]_{\varepsilon} = [proj_P(\vec{e}_1) \quad proj_P(\vec{e}_2) \quad proj_P\, \vec{e}_3] = \frac{1}{30} \begin{bmatrix} 26 & 10 & -2 \\ 10 & 5 & 5 \\ -2 & 5 & 29 \end{bmatrix}$$

- Rotation about the origin.



逆时针转 θ $\quad 0 < \theta < 2\pi$.

$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$.

$$\boxed{[R_\theta]_\varepsilon = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

→ Prove R_θ is linear transformation

$$\begin{aligned}
R_\theta(\vec{x}) &= R_\theta\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{bmatrix} \\
&= \begin{bmatrix} r(\cos\alpha\cos\theta - \sin\alpha\sin\theta) \\ r(\sin\alpha\cos\theta + \cos\alpha\sin\theta) \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta(r\cos\alpha) - \sin\theta(r\sin\alpha) \\ \sin\theta(r\cos\alpha) + \cos\theta(r\sin\alpha) \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix} \\
&= A\vec{x} \qquad \left(A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right)
\end{aligned}$$

$\therefore$ R_θ is linear transformation

→ Standard matrix of R_θ

let $\|\vec{x}\| = r$. $\quad x_1 = r\cos\alpha \quad x_2 = r\sin\alpha$

$\|\vec{w}\| = r$.

$$\begin{aligned}
w_1 &= r\cos(\alpha+\theta) \\
&= r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\
&= x_1\cos\theta - x_2\sin\theta
\end{aligned}
\qquad
\begin{aligned}
w_2 &= r\sin(\alpha+\theta) \\
&= r\sin\alpha\cos\theta + r\cos\alpha\sin\theta \\
&= x_2\cos\theta + x_1\sin\theta
\end{aligned}$$

$$R_\theta\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_2\cos\theta + x_1\sin\theta \end{bmatrix} \qquad R_\theta(\vec{e_1}) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad R_\theta(\vec{e_2}) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$[R_\theta]_\varepsilon = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(∵ n×n matrix ∴ 可直接证 invertible)

→ Is $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ one-to-one? → Is $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ onto ?

case 1. $\cos\theta = 0$.

$\sin\theta = 1$ or $\sin\theta = -1$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

case 2. $\cos\theta \neq 0$

$$\begin{bmatrix} 1 & -\frac{\sin\theta}{\cos\theta} \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{\sin\theta}{\cos\theta} \\ 0 & \frac{\sin^2\theta}{\cos\theta} + \cos\theta \end{bmatrix} \qquad \frac{1}{\cos\theta} \neq 0$$

$\therefore \operatorname{rank}([R_\theta]_\varepsilon) = 2$

By invertibility criteria R_θ is one-to-one & onto.

Q. Find standard matrix of $T: \mathbb{R}^2 \to \mathbb{R}^2$ which is:

① T_1, a $\frac{\pi}{3}$ clockwise rotation

followed by ② T_2, a projection onto $y = 2x$

① $[T_1]_\varepsilon = [R_{-\frac{\pi}{3}}]_\varepsilon = \begin{bmatrix} \cos(-\frac{\pi}{3}) & -\sin(-\frac{\pi}{3}) \\ \sin(-\frac{\pi}{3}) & \cos(-\frac{\pi}{3}) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$

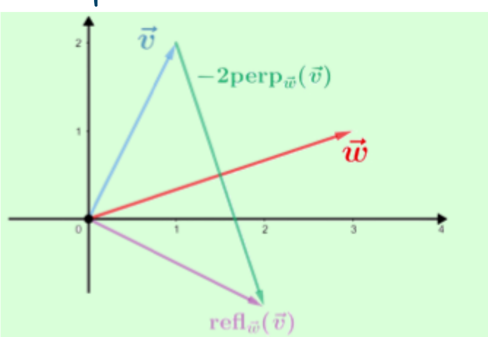
② $y = 2x$ 等价 $\operatorname{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ $\quad \vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\operatorname{proj}_{\vec{w}}(\vec{e}_1) = \frac{1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \operatorname{proj}_{\vec{w}}(\vec{e}_2) = \frac{2}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$[T_2]_\varepsilon = [\operatorname{proj}_{\vec{w}}]_\varepsilon = [\operatorname{proj}_{\vec{w}}(\vec{e}_1) \ \ \operatorname{proj}_{\vec{w}}(\vec{e}_2)] = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$

$[T]_\varepsilon = [T_2]_\varepsilon [T_1]_\varepsilon = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$

- Reflection about a line through origin



$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad \operatorname{refl}: \mathbb{R}^2 \to \mathbb{R}^2$

$\operatorname{refl}_{\vec{w}}(\vec{v}) = \vec{v} - 2\operatorname{perp}_{\vec{w}}(\vec{v})$

旋转后的之

5.7 Composition of Linear Transformations

- composition function $(T_2 \circ T_1)$

$T_1 : \mathbb{F}^n \to \mathbb{F}^m \qquad T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be linear transformation.

$T_2 \circ T_1 : \qquad (T_2 \circ T_1)(\vec{x}) = T_2(T_1(\vec{x}))$

ep. $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 0 \\ x_1 - 2x_2 \end{bmatrix}$

$T_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 + x_2 + x_3$

$$\begin{aligned}
(T_2 \circ T_1)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= T_2\left(T_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\
&= T_2\left(\begin{bmatrix} x_1 + x_2 \\ 0 \\ x_1 - 2x_2 \end{bmatrix}\right) \\
&= x_1 + x_2 + x_1 - 2x_2 \\
&= 2x_1 - x_2
\end{aligned}$$

- Composition of linear transformations is linear

Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m \qquad T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be linear transformations.

Then $T_2 \circ T_1$ is linear transformation.

proof: $\because T_1$ is linear. $\quad \therefore T_1(c\vec{x} + \vec{y}) = c\,T_1(\vec{x}) + T_1(\vec{y})$

$\because T_2$ is linear $\quad \therefore T_2(a\vec{w} + \vec{z}) = a\,T_2(\vec{w}) + T_2(\vec{z})$

$(T_2 \circ T_1)(c\vec{x} + \vec{y}) = T_2(c\,T_1(\vec{x}) + T_1(\vec{y}))$

Let $T_1(\vec{x}) = \vec{w} \qquad T_1(\vec{y}) = \vec{z}$

$$\begin{aligned}
(T_2 \circ T_1)(c\vec{x} + \vec{x}_2) &= T_2(c\vec{w} + \vec{z}) \\
&= c\,T_2(\vec{w}) + T_2(\vec{z}) \\
&= c\,T_2(T_1(\vec{x})) + T_2(T_1(\vec{y})) \\
&= c(T_2 \circ T_1)(\vec{x}) + (T_2 \circ T_1)(\vec{y})
\end{aligned}$$

$\therefore T_2 \circ T_1$ is linear

- Standard Matrix of a composition.

$$[T_2 \circ T_1]_\varepsilon = [T_2]_\varepsilon [T_1]_\varepsilon$$

proof. Let $\vec{x} \in \mathbb{F}^n$. $[T_2 \circ T_1]_\varepsilon \cdot \vec{x}$

$= (T_2 \circ T_1)(\vec{x})$

$= T_2(T_1(\vec{x}))$

$= T_2([T_1]_\varepsilon \cdot \vec{x})$

$= [T_2]_\varepsilon([T_1]_\varepsilon \cdot \vec{x})$

$= ([T_2]_\varepsilon [T_1]_\varepsilon) \cdot \vec{x}.$

$\therefore [T_2 \circ T_1]_\varepsilon \cdot \vec{x} = ([T_2]_\varepsilon [T_1]_\varepsilon) \cdot \vec{x}$ by matrix equality theorem

eg. $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 0 \\ x_1 - 2x_2 \end{bmatrix}$

$T_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 + x_2 + x_3$

$(T_2 \circ T_1)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = 2x_1 - x_2$ 找 $[T_2 \circ T_1]_\varepsilon$

法1: $(T_2 \circ T_1)(\vec{e_1}) = 2$

$(T_2 \circ T_1)(\vec{e_2}) = -1$

$[T_2 \circ T_1]_\varepsilon = [2 \ \ -1]$

法2: $[T_1]_\varepsilon = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -2 \end{bmatrix}$

$[T_2]_\varepsilon = [1 \ \ 1 \ \ 1]$

$[T_2]_\varepsilon [T_1]_\varepsilon = [1 \ \ 1 \ \ 1]\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -2 \end{bmatrix} = [2 \ \ -1]$

Q. Let $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be non-zero vector. $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear transformation which projects a vector onto $\vec{l} = t\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $(t \in \mathbb{R})$.

Find standard matrix of $T \circ T$.

$$[T]_\varepsilon = \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix}$$

$$[T \circ T]_\varepsilon = [T]_\varepsilon [T]_\varepsilon$$

$$= \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix}$$

$$= \frac{1}{(w_1^2 + w_2^2)^2} \begin{bmatrix} w_1^4 + w_1^2 w_2^2 & w_1^3 w_2 + w_1 w_2^3 \\ w_1^3 w_2 + w_1 w_2^3 & w_1^2 w_2^2 + w_2^4 \end{bmatrix}$$

$$= \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix}$$

6.1 Determinant.

- determinant of $M_{1\times1}$ and $M_{2\times2}$

$A = [a_{11}]$ $\quad \det(A) = a_{11}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A) = ad - bc$

$|A| \equiv \det(A)$

ep. $B = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}$. $\det(B) = |B| = 3\times7 - 1\times4 = 17$

- def. (i,j)th submatrix. minor.

Let $A \in M_{n\times n}(\mathbb{F})$ $\quad (i,j)$th submatrix of A is $M_{ij}(A)$

The $(n-1)\times(n-1)$ matrix obtained by removing ith row and jth column from A.

The (i,j)th minor of A is $\det(M_{ij}(A)) = |M_{ij}(A)|$ 去掉A的i行和j列.

ep. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & -2 \\ 5 & 4 & 10 \end{bmatrix}$ $\qquad A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & -2 \\ 5 & 4 & 10 \end{bmatrix}$

$|M_{11}(A)| = \begin{vmatrix} 0 & -2 \\ 4 & 10 \end{vmatrix}$ $\qquad |M_{21}(A)| = \begin{vmatrix} 2 & -1 \\ 4 & 10 \end{vmatrix}$

$= 0\times10 - (-2)\cdot4$ $\qquad = 2\times10 - (-1)\times4$

$= 8$ $\qquad = 24$

- determinant of $M_{n\times n}$

Let $A \in M_{n\times n}(\mathbb{F})$ $\quad n \geq 2$. $\quad$ determinant function: $\det: M_{n\times n}(\mathbb{F}) \to \mathbb{F}$

$$\det(A) = \sum_{j=1}^{n} a_{1j}(-1)^{1+j}\det(M_{1j}(A))$$

lth row expansion $\quad \det(A) = \sum_{j=1}^{n} a_{lj}(-1)^{l+j}|M_{lj}(A)|$

jth column expansion $\quad \det(A) = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}|M_{ij}(A)|$

eg. find $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & -2 \\ 5 & 4 & 10 \end{vmatrix}$ along 1st col.

$$= a_{11}(-1)^{1+1}|M_{11}(A)| + a_{21}(-1)^{2+1}|M_{21}(A)| + a_{31}(-1)^{3+1}|M_{31}(A)|$$

$$= 1 \cdot 1 \cdot 8 + 3(-1) \cdot 24 + 5 \cdot 1 \cdot (-4)$$

$$= -84.$$

find $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & -2 \\ 5 & 4 & 10 \end{vmatrix}$ along 3rd row.

$$= a_{31}(-1)^{3+1}|M_{31}(A)| + a_{32}(-1)^{5}|M_{32}(A)| + a_{33}(-1)^{6}|M_{33}(A)|$$

$$= 5(1)\begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} + 4(-1)\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} + 10(1)\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$$

$$= 5(-4+0) - 4(-2+3) + 10(-6)$$

$$= -84$$

eg. find the determinant of $A = \begin{bmatrix} 3 & 10 & 1 & 2 \\ 0 & 5 & 2 & -7 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ across 2nd row

$$\det A = a_{21}(-1)^{2+1}|M_{21}(A)| + a_{22}(-1)^{2+2}|M_{22}(A)|$$
$$+ a_{23}(-1)^{2+3}|M_{23}(A)| + a_{24}(-1)^{2+4}|M_{24}(A)|$$

$$= 0(-1)\begin{vmatrix} 10 & 1 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \end{vmatrix} + 5(1)\begin{vmatrix} 3 & 1 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \end{vmatrix}$$

$$+ 2(-1)\begin{vmatrix} 3 & 10 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{vmatrix} + (-7)(1)\begin{vmatrix} 3 & 10 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= 5 \cdot [\underline{3(-5+0)} - 0(5-0) + 0(4+2)]$$
$$- 2 \cdot [3(0-0) - 0(50-0) + 0(40-0)]$$
$$- 7\, [3(0-0) - 0(0-0) + 0(-10-0)]$$

$$= -75$$

- Easy determinants

(a) If A has a row consist only 0s. then $\det(A) = 0$

(b) if A has a column of only 0s. then $\det(A) = 0$

(c) if $A = \begin{bmatrix} a_{11} & * & * & \cdots & * \\ 0 & a_{22} & * & \cdots & * \\ 0 & 0 & a_{33} & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is <u>upper triangular</u>

then $\det A = a_{11} a_{12} \cdots a_{nn}$.

(d) If A is <u>lower triangular</u>, then $\det A = a_{11} a_{12} \cdots a_{nn}$.

ep. $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ -7 & 3 & 0 & 0 \\ 0 & 4 & 6 & 0 \\ 2 & 1 & 3 & 1 \end{bmatrix}$, $\det A = (-2) \cdot 3 \cdot 6 \cdot 1 = -36$

(e) $\det(I_n) = I$.

ep. $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$

- Calculation of matrix 对 det 的影响

$\det(A) = \det(A^T)$

$\det(A^{-1}) = \det(A)$ (A is invertible)

$\det(AB) = \det(A)\det(B)$ (A.B has same dimension)

$\det(A^n) = [\det(A)]^n$

$\det(kA) = k^n \det(A)$ (A is $n \times n$)

☆ $\det(A+B) \neq \det(A) + \det(B)$

↳ ep. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\det(A) = 1$

$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\det(B) = (-1)(-1) = 1$

$A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\det(A+B) = 0$

6.2 Computing the determinant in Practice : EROs

- effect of EROs on the determinant

$$A = \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 2 & 3 & -2 \\ 4 & 0 & -6 \\ -1 & 5 & 2 \end{bmatrix} \qquad \det(A) = 14$$

$$\rightarrow B = \begin{array}{c} R_3 \\ R_2 \\ R_1 \end{array} \begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & -6 \\ 2 & 3 & -2 \end{bmatrix} \qquad \det(B) = -14$$

row swap

$$\rightarrow C = \begin{array}{c} 2R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 4 & 6 & -4 \\ 4 & 0 & -6 \\ -1 & 5 & 2 \end{bmatrix} \qquad \det(C) = 28$$

row scale

$$\rightarrow D = \begin{array}{c} R_1 \\ R_2 - 2R_1 \\ R_3 \end{array} \begin{bmatrix} 2 & 3 & -2 \\ 0 & -6 & -2 \\ -1 & 5 & 2 \end{bmatrix} \qquad \det(D) = 14.$$

row addition

Let $A \in M_{n\times n}(\mathbb{F})$

a) Row Swap $\quad A \xrightarrow{\text{interchanging}} B \quad \det(B) = -\det(A)$

b) Row Scale $\quad A \xrightarrow{\text{multiplying a row by } m \neq 0} B. \quad \det(B) = m\det(A)$

c) Row addition $\quad A \xrightarrow{\text{add non-zero multiple of one row to another}} B \quad \det(B) = \det(A)$

- Corollary

Let $A \in M_{n\times n}(\mathbb{F})$. If A has 2 identical rows (columns). then $\det(A) = 0$

proof: substract one of the 2 rows from the other.

A' will have a zero rows.

By Easy determinants. $\det(A') = 0$

By effect of ERO on det. $\det(A)$ didn't change

$\therefore \det(A) = \det(A') = 0$

Q. Use elemantary row operations and triangular form to calculate det A

$$A = \begin{bmatrix} 3 & -4 & 4 \\ 1 & -2 & 0 \\ 2 & -7 & 2 \end{bmatrix}$$

$$\det A = -\det \left(\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & -2 & 0 \\ 3 & -4 & 4 \\ 2 & -7 & 2 \end{bmatrix} \right)$$

$$= -\det \left(\begin{array}{l} R_1 \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 2 & 4 \\ 0 & -3 & 2 \end{bmatrix} \right)$$

$$= -1\left(\tfrac{1}{2}\right) \det \left(\begin{array}{l} R_1 \\ \tfrac{1}{2}R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & -3 & 2 \end{bmatrix} \right)$$

$$= -\tfrac{1}{2} \det \left(\begin{array}{l} R_1 \\ R_2 \\ 3R_2 + R_3 \end{array} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 8 \end{bmatrix} \right)$$

$$= -\tfrac{1}{2} (1 \times 1 \times 8)$$

$$= -4$$

Q.

Calculate $\det(A)$. $A = \begin{bmatrix} 1 & 0 & -4 & 1 \\ -3 & 0 & 2 & 3 \\ -5 & -2 & 10 & 3 \\ -1 & 0 & 4 & 5 \end{bmatrix}$

$$\det(A) = 0 \cdot (-1)^3 \det \begin{bmatrix} -3 & 2 & 3 \\ -5 & 10 & 3 \\ -1 & 4 & 5 \end{bmatrix} + 0 \cdot (-1)^4 \det \begin{bmatrix} 1 & -4 & 1 \\ -5 & 10 & 3 \\ -1 & 4 & 5 \end{bmatrix}$$
$$-2(-1)^5 \det \begin{bmatrix} 1 & -4 & 1 \\ -3 & 2 & 3 \\ -1 & 4 & 5 \end{bmatrix} + 0 \cdot (-1)^6 \det \begin{bmatrix} 1 & -4 & 1 \\ -3 & 2 & 3 \\ -5 & 10 & 3 \end{bmatrix}$$

$$= 2 \det \begin{bmatrix} 1 & -4 & 1 \\ -3 & 2 & 3 \\ -1 & 4 & 5 \end{bmatrix}$$

$$= 2 \det \begin{bmatrix} 1 & -4 & 1 \\ -3 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$ row addition.

$$= 2 \times 6 \det \begin{bmatrix} 1 & -4 \\ -3 & 2 \end{bmatrix}$$

$$= -120$$

- Determinants of elementary matrices

ERO type	determinant	example
Row swap $R_i \leftrightarrow R_j$	$\det(E) = -1$	$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
Row scale $R_i \leftrightarrow mR_i$ $(m \neq 0)$	$\det(E) = m$	$E = \begin{bmatrix} m & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
Row addition $R_i \to R_i + mR_j$	$\det(E) = 1$	$E = \begin{bmatrix} 1 & 0 & m \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Q. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & -2 \\ 5 & 4 & 10 \end{bmatrix}$ $\det(A) = -84$

ERO	E	det(E)	B=EA	det(B)
$R_1 \leftrightarrow R_3$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	-1	$\begin{bmatrix} 5 & 4 & 10 \\ 3 & 0 & -2 \\ 1 & 2 & -1 \end{bmatrix}$	$3(-1)^3(-4-20) - 2(-1)^5(10-4) = 84$ $-84(-1) = 84$
$R_1 \leftrightarrow 2R_1$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	2	$\begin{bmatrix} 2 & 4 & -2 \\ 3 & 0 & -2 \\ 5 & 4 & 10 \end{bmatrix}$	$3(-1)^3(40+8) - 2(-1)^5(8-20) = -168$ $-84(2) = -168$
$R_3 \to R_3 - 2R_1$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$	1	$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & -2 \\ 3 & 0 & 12 \end{bmatrix}$	$2(-1)^3(36+6) = -84$ $-84(1) = -84$

- Determinant after k EROs

$k=1$ $\quad B = EA \quad \to \quad \det(B) = \det(E)\det(A)$

Let $A \in M_{n \times n}(\mathbb{F})$ and suppose we perform k EROs on A to produce B.

$B = E_k \cdots E_2 E_1 A \quad \to \quad \det(B) = \det(E_k \cdots E_2 E_1 A)$

$$= \det(E_k) \cdots \det(E_1) \det(A)$$

Q. Let $A = \begin{bmatrix} 2i & 5 & -i \\ i & 2 & 0 \\ 1 & 3i & 1 \end{bmatrix}$. Evaluate $\det(A)$

$$\det(A) = \begin{vmatrix} 2i & 5 & -i \\ i & 2 & 0 \\ 1 & 3i & 1 \end{vmatrix} \qquad R_1 \to R_1 + iR_3 \quad \text{don't change det}$$

$$= \begin{vmatrix} 3i & 2 & 0 \\ i & 2 & 0 \\ 1 & 3i & 1 \end{vmatrix} \qquad R_1 \to R_1 - R_2 \quad \text{don't change det}$$

$$= \begin{vmatrix} 2i & 0 & 0 \\ i & 2 & 0 \\ 1 & 3i & 1 \end{vmatrix} \qquad \text{lower triangular}$$

$$= 2i(2)(1)$$

$$= 4i$$

6.3 The determinant and invertibility

- invertible iff the determinant is non-zero

$A \in M_{n\times n}(\mathbb{F})$. A is invertible $\Leftrightarrow$ $\det(A) \neq 0$

☆ $\det(A \pm B) \neq \det(A) + \det(B)$

$\det(AB) = \det(A)\det(B)$

理解:

$A \to REF(A)$ $\begin{bmatrix} a_1 & \cdots & 0 \\ & a_2 & \vdots \\ \vdots & & \ddots \\ 0 & \cdots & a_n \end{bmatrix}$

$a_{ii} \neq 0$. → 每个 row 都有 pivot.

$\text{rank}(A) = n$

proof: Let $R = RREF(A) = E_k \cdots E_2 E_1 A$. ($E_i$ is elementary matrix)

$\therefore \det(R) = \det(E_k) \cdots \det(E_2)\det(E_1)\det(A)$

case 1. A is invertible

$R = RREF(A) = I_n$.

$\det(R) = 1$

$\det(A) \neq 0$ $(\because \det(E_i) \neq 0)$

case 2. A isn't invertible

R must have a row of zeros

$\det(R) = 0$

$\det(A) = 0$.

Q. Find k if $A = \begin{bmatrix} 1 & 2 & k \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is invertible.

$\det(A) = 1 \cdot (-1)^5 \det\begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix} + 1 (-1)^6 \det\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$= -3 + k$

$\det(A) \neq 0$ iff $-3 + k \neq 0$

$\therefore k \neq 3$.

- determinant of a product

$A, B \in M_{n\times n}(\mathbb{F})$. $\det(AB) = \det(BA) = \det(A)\det(B)$

$\det(AA^{-1}) = \det(I) = \det(A)\det(A^{-1}) = 1$

Q. $A, B \in M_{n\times n}(\mathbb{F})$ Prove AB is invertible iff BA is invertible

AB is invertible $\Leftrightarrow \det(AB) \neq 0$

$\Leftrightarrow \det(A)\det(B) \neq 0$

$\Leftrightarrow \det(B)\det(A) \neq 0$

$\Leftrightarrow \det(BA) \neq 0$

$\Leftrightarrow BA$ is invertible.

- Determinant of inverse

$A \in M_{n\times n}(\mathbb{F})$ $\quad \det(A^{-1}) = \frac{1}{\det(A)}$

proof. Assume A is invertible. $AA^{-1} = I_n$

$\det(AA^{-1}) = \det(I_n)$

$\det(A)\det(A^{-1}) = 1$

$\det(A^{-1}) = \frac{1}{\det(A)}$

Q. Calculate $\det(A^{-1})$ $\quad A = \begin{bmatrix} 2 & 3 & -2 \\ 4 & 0 & -6 \\ -1 & 5 & 2 \end{bmatrix}$

$\because \det(A) = 14$.

$\therefore \det(A^{-1}) = \frac{1}{14}$

6.4 An Expression for A^{-1}

- Cofactor of A

$A \in M_{n\times n}(\mathbb{F})$

$(i,j)^{th}$ cofactor of A: $\boxed{C_{ij}(A) = (-1)^{i+j}(M_{ij}(A))}$

- Adjugate of A

$A \in M_{n\times n}(\mathbb{F})$

adjugate of A: $\boxed{(\mathrm{adj}(A))_{ij} = C_{ji}(A)}$

transpose: $\mathrm{adj}(A) = C^T$

(adjugate of A is the transpose of matrix of cofactors of A)

Q. Determine the cofactor & adjugate of $A = \begin{bmatrix} 1 & -2 & 0 \\ -3 & 5 & 1 \\ 4 & -3 & 2 \end{bmatrix}$

$(-1)^{even}$ $(-1)^{odd}$

$M_{11} = \begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix}$ $C_{11} = 13$ $M_{12} = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$ $C_{12} = 10$

$M_{13} = \begin{bmatrix} -3 & 5 \\ 4 & -3 \end{bmatrix}$ $C_{13} = -11$ $M_{21} = \begin{bmatrix} -2 & 0 \\ -3 & 2 \end{bmatrix}$ $C_{21} = 4$

$M_{22} = \begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}$ $C_{22} = 2$ $M_{23} = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$ $C_{23} = -5$

$M_{31} = \begin{bmatrix} -2 & 0 \\ 5 & 1 \end{bmatrix}$ $C_{31} = -2$ $M_{32} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $C_{32} = -1$

$M_{33} = \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix}$ $C_{33} = -1$

$$C(A) = \begin{bmatrix} 13 & 10 & -11 \\ 4 & 2 & -5 \\ -2 & -1 & -1 \end{bmatrix}$$

$$\mathrm{adj}(A) = \begin{bmatrix} 13 & 10 & -11 \\ 4 & 2 & -5 \\ -2 & -1 & -1 \end{bmatrix}^T = \begin{bmatrix} 13 & 4 & -2 \\ 10 & 2 & -1 \\ -11 & -5 & -1 \end{bmatrix}$$

- Inverse by adjugate

$A\,\text{adj}(A) = \text{adj}(A)A = \det(A) I_n$

Let $A \in M_{n\times n}(F)$. $\det(A) \neq 0$, then $\boxed{A^{-1} = \frac{1}{\det(A)}\text{adj}(A)}$

两种找 inverse 的方法:

1) $[A \mid I] \rightarrow [I \mid A^{-1}]$

2) $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$

Q. Determine the inverse of $A = \begin{bmatrix} 1 & -2 & 0 \\ -3 & 5 & 1 \\ 4 & -3 & 2 \end{bmatrix}$

已知 $\text{adj}(A) = \begin{bmatrix} 13 & 4 & -2 \\ 10 & 2 & -1 \\ -11 & -5 & -1 \end{bmatrix}$

$\det(A) = 1(-1)^2(10+3) - 2(-1)^3(-6-4) = 13-20 = -7$

$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = -\frac{1}{7}\begin{bmatrix} 13 & 4 & -2 \\ 10 & 2 & -1 \\ -11 & -5 & -1 \end{bmatrix}$

check $-\frac{1}{7}\begin{bmatrix} 1 & -2 & 0 \\ -3 & 5 & 1 \\ 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 13 & 4 & -2 \\ 10 & 2 & -1 \\ -11 & -5 & -1 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = I_3$

Q. If $PQP = I_n$. which one is possible

(a) $\det(P) = -1$ ✓ (b) $\det(Q) = -1$

proof: Let $\det(P) = p$ $\det(Q) = q$.

$\because PQP = I_n$

$\therefore \det(PQP) = \det(P)\det(Q)\det(P) = \det(I_n) = 1$

$\therefore p^2 q = 1$

$\therefore$ (a) is possible

Q. Prove if $A \in M_{n\times n}(\mathbb{F})$ has all integer entries with $\det(A)=1$.

or $\det(A)=-1$. then A is invertible and A^{-1} will have all integer entries.

$\det(A) \neq 0$

proof: $\because \det(A) = \pm 1 \neq 0$ $\therefore A$ is invertible

By inverse of adjugate $A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$

$\because \det(A) = \pm 1$ $\therefore A^{-1} = \pm \operatorname{adj}(A)$

$(\operatorname{adj}(A))_{ji} = C_{ij} = (-1)^{i+j} \det(M_{ij})$

$\because A$ have all integer entries. $\therefore M_{ij}$ have all integer entries.

$\therefore (-1)^{i+j} \det(M_{ij}) \in \mathbb{Z}$

→ n×n 计算方法

1. minor-cofactor：任取一行/一列
2. 拆分 matrix 或 $A_1, A_2 \cdots A_n$
3. $|A| = \det A$
4. $(-1)^{i+j}$
5. 相乘再相加

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & -2 \\ 5 & 4 & 10 \end{vmatrix}$$

-1	2	1
$\begin{bmatrix} 3 & 0 \\ 5 & 4 \end{bmatrix}$	$\begin{bmatrix} 3 & -2 \\ 5 & 10 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 \\ 4 & 10 \end{bmatrix}$
12	40	8
$(-1)^1$	$(-1)^2$	$(-1)^3$

→ 3×3 in 简单算法

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right|$$

$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

$$+ a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

→ triangular matrix

upper $\begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix}$ ↘

lower $\begin{bmatrix} \diagdown & 0 \\ * & \diagdown \end{bmatrix}$ → $\diagdown$ 相乘

diagonal $\begin{bmatrix} \diagdown & 0 \\ 0 & \diagdown \end{bmatrix}$ ↗

→ A 变换得 REF(A)

$$A \xrightarrow[\text{(row operation)}]{E_1 \quad E_2 \quad \cdots \quad E_n} REF(A) \quad \begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix}$$

a) Row Swap $\quad A \xrightarrow{\text{interchanging}} B \quad \det(B) = -\det(A)$

b) Row Scale $\quad A \xrightarrow{\text{multiplying a row by } m \neq 0} B. \quad \det(B) = m \det(A)$

c) Row addition $\quad A \xrightarrow{\text{add non-zero multiple of one row to another}} B \quad \det(B) = \det(A)$

6.5 Cramer's Rule

- Cramer's Rule

若只想求一个 x 的值

$A \in M_{n\times n}(\mathbb{F})$. $A\vec{x}=\vec{b}$ $(\vec{b}\in F^n$ $\det(A)\neq 0)$

solution of x_j : $x_j = \dfrac{\det(B_j)}{\det(A)}$ $j=1,\cdots,n$

Q. $x+2y=7$ $-3x+y=-4$

$A=\begin{bmatrix}1 & 2\\ -3 & 1\end{bmatrix}$ $\vec{b}=\begin{bmatrix}7\\ -4\end{bmatrix}$

$\det(A)=1-(-6)=7$

$A_1=\begin{bmatrix}7 & 2\\ -4 & 1\end{bmatrix}$ $\det(A_1)=7-(-8)=15$

$A_2=\begin{bmatrix}1 & 7\\ -3 & -4\end{bmatrix}$ $\det(A_2)=-4+21=17$

$x=x_1=\dfrac{\det(A_1)}{\det(A)}=\dfrac{15}{7}$ $y=x_2=\dfrac{\det(A_2)}{\det(A)}=\dfrac{17}{7}$

Q. Determine the value of x_3 in the solution to $A\vec{x}=\vec{b}$.

$A=\begin{bmatrix}1 & 3 & 5 & 2\\ 2 & 1 & 2 & 4\\ 3 & 0 & 3 & 6\\ 4 & 5 & 0 & 9\end{bmatrix}$ $\vec{b}=\begin{bmatrix}1\\ 0\\ -1\\ 3\end{bmatrix}$

将 $\vec{b}$ 插入第三列

$\det A=\det\begin{bmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 0 & 5\\ 5 & 2 & 3 & 0\\ 2 & 4 & 6 & 9\end{bmatrix}$ A^T

$=\det\begin{bmatrix}1 & 2 & 3\\ 3 & 1 & 0\\ 5 & 2 & 3\end{bmatrix}\begin{matrix}R_1\\ R_2\\ R_3\end{matrix}$

$=\det\begin{bmatrix}1 & 2 & 3\\ 3 & 1 & 0\\ 4 & 0 & 0\end{bmatrix}$ R_3-R_1

$=4\det\begin{bmatrix}2 & 3\\ 1 & 0\end{bmatrix}$

$=4(-3)$

$=-12.$

$\det(B_3)=\det\begin{bmatrix}1 & 3 & 1 & 2\\ 2 & 1 & 0 & 4\\ 3 & 0 & -1 & 6\\ 4 & 5 & 3 & 9\end{bmatrix}$

$=\det\begin{bmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 0 & 5\\ 1 & 0 & -1 & 3\\ 2 & 4 & 6 & 9\end{bmatrix}\begin{matrix}R_1\\ R_2\\ R_3\\ R_4\end{matrix}$

$=\det\begin{bmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 0 & 5\\ 1 & 0 & -1 & 3\\ 0 & 0 & 0 & 1\end{bmatrix}$ R_4-2R_1

$=\det\begin{bmatrix}1 & 2 & 3\\ 3 & 1 & 0\\ 1 & 0 & -1\end{bmatrix}$

$=\det\begin{bmatrix}2 & 3\\ 1 & 0\end{bmatrix}-\det\begin{bmatrix}1 & 2\\ 3 & 1\end{bmatrix}$

$=-3+5$

$=2$

$x_3=\dfrac{\det(B_3)}{\det(A)}=\dfrac{2}{-12}=-\dfrac{1}{6}$

6·6 The Determinant and Geometry

– Area of parallelogram

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

Area of parallelogram with sides $\vec{v}$ & $\vec{w}$ $= \left| \det\left(\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\right) \right|$

proof.

$$\|\vec{v}_1 \times \vec{w}_1\| = \left\| \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ 0 \\ v_1w_2 - w_1v_2 \end{bmatrix} \right\|$$

$$= | v_1w_2 - w_1v_2 |$$

$$= \left| \det\left(\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\right) \right|$$

Q. Find area of parallelogram with sides $\vec{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\text{area} = \left| \det \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} \right| = |-13| = 13$$

– cross product (memory device)

$$\det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = (a_2b_3 - b_2a_3)\,\vec{e}_1 + (-a_1b_3 + b_1a_3)\,\vec{e}_2 + (a_1b_2 - b_1a_2)\,\vec{e}_3$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ & [a]^T & \\ & [b]^T & \end{bmatrix}$$

$$= \begin{bmatrix} a_2b_3 - b_2a_3 \\ -a_1b_3 + b_1a_3 \\ a_1b_2 - b_1a_2 \end{bmatrix} = \vec{a} \times \vec{b}$$

Q. evaluate $\vec{x} \times \vec{y}$ ($\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$)

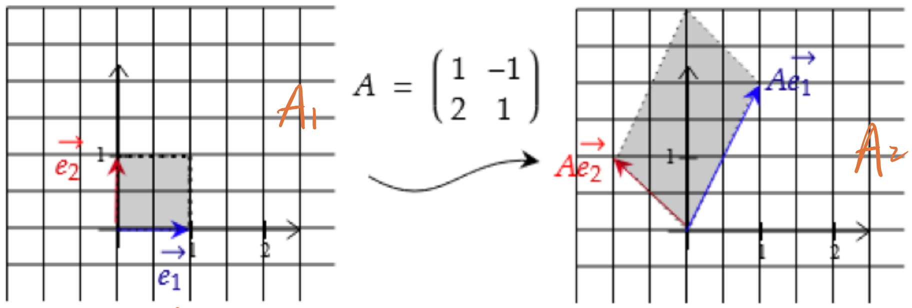
$$\vec{x} \times \vec{y} = \det \left[\begin{array}{ccc|cc} i & j & k & i & j \\ 1 & 2 & 3 & 1 & 2 \\ -2 & 3 & 4 & -2 & 3 \end{array}\right]$$

$$= 8i - 6j + 3k - [-4k + 9i - 4j]$$

$$= -i - 2j + 7k$$

$$\begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix}$$

- Geometric application



$A_1 = 1$

$T_A(\vec{e_1}) = A\vec{e_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$T_A(\vec{e_2}) = A\vec{e_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A_2 = \left| \det \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \right| = 3$

$*$ $|\det(A)|$ = scaling factor

- Effect of ... on Area

? projection onto the line with equation $\vec{l} = t\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $t \in \mathbb{R}$

$$[\text{proj}_{\vec{w}}]_\varepsilon = \frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix}$$

$$\left|\det[\text{proj}_{\vec{w}}]_\varepsilon\right| = \left|\left(\frac{1}{w_1^2+w_2^2}\right)^2 \det\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix}\right|$$

$$= \left|\left(\frac{1}{w_1^2+w_2^2}\right)^2 (\underbrace{w_1^2w_2^2 - w_1w_2w_1w_2}_{0})\right|$$

$$= 0$$

? Rotation θ about origin ? (expect scaling factor of 1)

$$[R_\theta]_\varepsilon = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$|\det[R_\theta]_\varepsilon| = |\cos^2\theta + \sin^2\theta| = 1$$

? Reflection through line $\vec{l} = t\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $t \in \mathbb{R}$.

$$[\text{refl}_{\vec{w}}]_\varepsilon = \frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2-w_2^2 & 2w_1w_2 \\ 2w_1w_2 & w_2^2-w_1^2 \end{bmatrix}$$

$$|\det[\text{refl}_{\vec{w}}]_\varepsilon| = \left|\left(\frac{1}{w_1^2+w_2^2}\right)^2 [(w_1^2-w_2^2)(w_2^2-w_1^2) - 4w_1^2w_2^2]\right|$$

$$= \left|\frac{w_1^2w_2^2 - w_1^4 - w_2^4 + w_1^2w_2^2 - 4w_1^2w_2^2}{w_1^4 + 2w_1^2w_2^2 + w_2^4}\right|$$

$$= |-1|$$

$$= 1$$

7.1 Eigenpair

- projection mapping

mapping $T: \mathbb{R}^2 \to \mathbb{R}^2$ which projects vectors onto line $\vec{w} = t\begin{bmatrix}1\\1\end{bmatrix}$ $(t \in \mathbb{R})$

$$[\text{proj}_{\vec{w}}]_{\varepsilon} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$T(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}1\\1\end{bmatrix} = 1\begin{bmatrix}1\\1\end{bmatrix}$ any vector on the line will map to itself.

$T(\begin{bmatrix}1\\-1\end{bmatrix}) = \begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}1\\-1\end{bmatrix}$

- eigenvector

针对 square matrix.

Let $A \in M_{n\times n}(\mathbb{F})$. A non-zero vector $\vec{x}$ is an eigenvector of A over $\mathbb{F}$ if

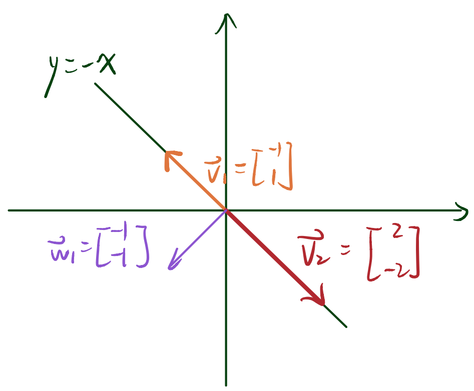
$\exists \lambda \in \mathbb{F}$ s.t. $A\vec{x} = \lambda\vec{x}$. $\leftarrow$ $A\vec{x} \parallel \vec{x}$.

比值 λ: eigenvalue of A over $\mathbb{F}$　　向量 $\vec{x}$: eigenvector corresponding λ

$(\lambda, \vec{x})$ eigenpair of A over $\mathbb{F}$. (不能单独存在)

eg $A = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2}\end{bmatrix}$　$(1, \begin{bmatrix}1\\1\end{bmatrix})$ & $(0, \begin{bmatrix}1\\-1\end{bmatrix})$ are eigenpairs

Q、



若 $T: \mathbb{R}^2 \to \mathbb{R}^2$ reflects over $y=-x$.

$T(\vec{v}_1) = \vec{v}_1$.
$T(\vec{v}_1) = A\vec{v}_1 = \vec{v}_1$　eigenvalue: $\lambda = 1$.

$T(\vec{v}_2) = \vec{v}_2$
$T(\vec{v}_2) = A\vec{v}_2 = \vec{v}_2$　eigenvalue: $\lambda = 1$

$T(\vec{w}_1) = -\vec{w}_1$
$T(\vec{w}_1) = A(-\vec{w}_1)$　eigenvalue: $\lambda = -1$

Q. Let $A \in M_{n\times n}(\mathbb{F})$. Prove 0 is an eigenvalue of A $\Leftrightarrow$ A is not invertible

another invertibility criteria

0 is eigenvalue of A

$\Rightarrow \exists \vec{x} \in \mathbb{F}^n,\ \vec{x} \neq 0$. s.t $A\vec{x} = 0\vec{x} = \vec{0}$

$\Rightarrow \text{Null}(A) \neq \{\vec{0}\}$

$\Rightarrow A$ isn't invertible

- fixed point

$T: \mathbb{F}^n \to \mathbb{F}^n$

$T(\vec{x}) = \vec{x} \quad [T]_{\varepsilon}\vec{x} = \vec{x} = 1\vec{x}$ eigenvalue : 1

Q. $T: \mathbb{R}^2 \to \mathbb{R}^2$ - reflection about line $y=x$

$\vec{l} = t\begin{bmatrix}1\\1\end{bmatrix} \quad t \in \mathbb{R}.$ $\quad [\text{refl}_{\vec{w}}]_{\varepsilon} = \dfrac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2-w_2^2 & 2w_1w_2 \\ 2w_1w_2 & w_2^2-w_1^2\end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix}0 & 2\\2 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

fixed point is every vector on the line, $\begin{bmatrix}t\\t\end{bmatrix}$

$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}t\\t\end{bmatrix} = \begin{bmatrix}t\\t\end{bmatrix}$

Q. Projection onto y-axis. $\quad A = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$

fixed point: vectors on y-axis $\begin{bmatrix}0\\t\end{bmatrix}$

$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0\\t\end{bmatrix} = \begin{bmatrix}0\\t\end{bmatrix}$

Q. Rotation by $\frac{\pi}{2}$

$[R_{\frac{\pi}{2}}]_{\varepsilon} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ impossible to rotate $\vec{x} \neq \vec{0}$ by $\frac{\pi}{2}$ to get $\vec{x}$

$\therefore$ no real eigenvalues.

7.2 Characteristic Polynomial & Find Eigenvalue

- Eigenvalue

def. Let $A \in M_{n\times n}(\mathbb{F})$

$A\vec{x} = \lambda\vec{x}$. $\vec{x} \neq \vec{0}$

λ：相当于在方向上的缩放比例

$\vec{x}$：相当于matrix的特殊方向 $(\neq \vec{0})$

λ : eigenvalue of A

$\vec{x}$: eigenvector of A

$(\lambda, \vec{x})$: eigenpair of A over $\mathbb{F}$.

$\Leftrightarrow A\vec{x} - \lambda\vec{x} = \vec{0}$

$\Leftrightarrow (A-\lambda)\vec{x} = \vec{0}$ $\rightarrow$ homogenious eq $A\vec{x}=\vec{0}$ has non-zero vector

$\Leftrightarrow \boxed{(A-\lambda I_n)\vec{x} = \vec{0}}$

$\because \exists \vec{x} \neq 0$

$\therefore Null(A-\lambda I_n) \neq \{\vec{0}\}$

$\therefore A-\lambda I_n$ isn't invertible

$\therefore \det(A-\lambda I_n) = 0$.

解 eigenvalue：
sol of $\det(A-\lambda I)=0$

Q. 求 $\det(A-\lambda I)=0$. $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$ 的 eigenvalue λ

$\lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$\rightarrow \det(A-\lambda I)=0$

$A-\lambda I = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 4-\lambda \end{bmatrix}$

$(1-\lambda)(4-\lambda)=0$

$\lambda=1 \quad \lambda=4$

$\rightarrow \lambda=1 \quad A\vec{v_1} = 1\cdot\vec{v_1}$

$(A-I)\vec{v_1} = \vec{0}$

$\left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{v_1} = \vec{0}$

$\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}\vec{v_1} = \vec{0}$.

$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{v_1} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad t \in \mathbb{R}$.

$\rightarrow \lambda=4 \quad A\vec{v_1} = 4\vec{v_1}$

$(A-4I)\vec{v_1} = \vec{0}$

$\left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)\vec{v_1} = \vec{0}$

$\begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix}\vec{v_1} = \vec{0}$.

$x_1 = \frac{2}{3}t \quad x_2 = t$

$\vec{v} = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} t \quad t \in \mathbb{R}$.

- eigenvalue equation

Let $A \in M_{n\times n}(\mathbb{F})$.

$A\vec{x} = \lambda\vec{x}$ or $(A-\lambda I)\vec{x} = \vec{0}$. are eigenvalue equation of A over $\mathbb{F}$.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \ddots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad \lambda I = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ \vdots & \lambda & & \vdots \\ & & \ddots & \\ 0 & \cdots & & \lambda \end{bmatrix}$$

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{bmatrix}$$

- Characteristic polynomial ($C_A(\lambda)$)

$$C_A(\lambda) = \det(A-\lambda I)$$

$\det(A-\lambda I) \Rightarrow$ degree 为 n 的多项式

- Characteristic equation

$$C_A(\lambda) = \det(A-\lambda I) = 0$$
$$= k(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$

root: $\lambda_1, \lambda_2 \cdots \lambda_n$ ($A_{n\times n}$ 的 eigenvalue 数量为 n)

Q. Solve eigenvalue problem for $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $A \in M_{2\times 2}(\mathbb{R})$

$\det(A-\lambda I) = 0$

$$A-\lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)^2 - 4 = 0$$
$$(1-\lambda)^2 = 4$$
$$\lambda - 1 = \pm 2$$
$$\lambda_1 = 3 \qquad \lambda_2 = -1$$

分别计算各自的 eigenvectors

$\lambda_1 = 3$ $\quad A\vec{v}_1 = \lambda_1$

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix}\vec{v}_1 = 0$$

$$\sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$$

$\quad x_1 \quad x_2$

$\lambda_2 = -1$ $\quad A\vec{v}_1 = \lambda_2$

$$\begin{bmatrix} 1+1 & 2 \\ 2 & 1+1 \end{bmatrix}\vec{v}_2 = 0$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$$

ep. reflection about $y=x \rightarrow A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$C_A(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1$

$C_A(\lambda) = 0 \Rightarrow \lambda^2 - 1 = 0$

$(\lambda-1)(\lambda+1) = 0$

$\lambda_1 = 1 \quad \lambda_2 = -1$

fixed points. 对应与线垂直的之.

ep. $A = [R_{\frac{\pi}{2}}]_E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$C_A = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$

$C_A(\lambda) = 0 \qquad \lambda^2 + 1 = 0$

no real roots

$\therefore$ no real eigenvalues

has eigenvalues over $\mathbb{C}$. $\quad \lambda^2+1=0$

$\lambda = i \quad \lambda = -i$

$\therefore$ have complex eigenvalues.

Q. Find eigenvalues of $A = \begin{bmatrix} 0 & -2 & -3 \\ -2 & 0 & -3 \\ -2 & 2 & 1 \end{bmatrix}$ over $\mathbb{R}$.

$\det(A - \lambda I_3)$

$= \det \begin{bmatrix} \cancel{-\lambda} & \cancel{-2} & \cancel{-3} \\ -2 & -\lambda & -3 \\ -2 & 2 & 1-\lambda \end{bmatrix}$

$= -\lambda[-\lambda(1-\lambda)+6] + 2[-2(1-\lambda)-6] - 3[-4-2\lambda]$

$= -\lambda^3 + \lambda^2 + 4\lambda - 4$

$= (\lambda-1)(\lambda+2)(\lambda-2) \qquad \lambda = 1 \quad \lambda = -2 \quad \lambda = 2.$

7.3 Characteristic Polynomial

- proposition 7.3.1

$A \in M_{n\times n}(\mathbb{F})$

A is invertible $\Leftrightarrow$ $\lambda = 0$ isn't eigenvalue of A

A isn't invertible $\Leftrightarrow$ $\lambda = 0$ is eigenvalue of A. (反之亦然)

proof: $\because A$ is invertible $\Leftrightarrow \det(A) \neq 0$

$\Leftrightarrow \det(A - 0 I_n) \neq 0$

$\Leftrightarrow$ 0 isn't a root of characteristic polynomial

$\Leftrightarrow$ 0 isn't eigenvalue of A

$\Leftrightarrow \lambda \neq 0$

- Trace

$A \in M_{n\times n}(\mathbb{F})$. $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$

$$\begin{bmatrix} \times & & & \\ & \times & & \\ & & \times & \\ & & & \times \end{bmatrix}$$

对角线之和

- Features of characteristic polynomial

$C_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{(n-1)} + \cdots + c_1\lambda + c_0 = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots$

a) $c_n = (-1)^n$

b) $c_{n-1} = (-1)^{n-1}\operatorname{tr}(A)$

c) $c_0 = \det(A)$

eg. $A = \begin{bmatrix} 5 & -2 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & -3 \end{bmatrix}$.

$\operatorname{tr}(A) = 5 + 2 - 3 = 4$

$\det(A) = 5(2)(-3) = -30$

$$C_A(\lambda) = \begin{vmatrix} 5-\lambda & -2 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & -3-\lambda \end{vmatrix}$$

$= (5-\lambda)(2-\lambda)(-3-\lambda)$

$= (10 - 7\lambda + \lambda^2)(-3-\lambda)$

$= \underbrace{-\lambda^3}_{(-1)^3} + \underbrace{4\lambda^2}_{(-1)^2\operatorname{tr}(A)} + 11\lambda \underbrace{-30}_{\det(A)}$

- Char. Polynomial & Eigenvalue over $\mathbb{C}$.

Let $A \in M_{n\times n}(\mathbb{C})$. $\quad C_A(\lambda) = C_n\lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_1\lambda + C_0$

a) $C_{n-1} = (-1)^{n-1} \sum_{i=1}^{n} \lambda_i = (-1)^{n-1} tr(A)$

b) $C_0 = \prod_{i=1}^{n} \lambda_i = det(A)$

corollary : (a) $\sum_{i=1}^{n} \lambda_i = tr(A)$

(b) $\prod_{i=1}^{n} \lambda_i = det(A)$

eg. $A = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$ over $\mathbb{R}$.

$C_A(\lambda) = \begin{vmatrix} -\lambda & 4 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 4 = (\lambda - 2i)(\lambda + 2i)$

complex eigenvalues $\lambda = 2i \quad \lambda = -2i$.

a) $\sum_{i=1}^{2} \lambda_i = 2i - 2i = 0 \qquad tr(A) = 0 + 0 = 0$

b) $\prod_{i=1}^{2} \lambda_i = 2i(-2i) = 4 \qquad det(A) = 0 + 4 = 4$

proof: $C_A(\lambda) = k(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$

$\because C_n = (-1)^n$ by features of characteristic polynomial.

$\therefore k = (-1)^n$

$C_A = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$

a) $C_{n-1}\lambda^n = (-1)^n [-\lambda_1(\lambda)^{n-1} - \lambda_2(\lambda)^{n-1} - \cdots - \lambda_n(\lambda)^{n-1}]$

$= \left((-1)^{n-1} \sum_{i=1}^{n} \lambda_i\right)\lambda^n$

$C_{n-1} = (-1)^{n-1} \sum_{i=1}^{n} \lambda_i$

b) $C_0 = (-1)^n \prod_{i=1}^{n} (-\lambda_i) = (-1)^{2n} \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} \lambda_i$

Q. $D = \begin{bmatrix} -i & -2 \\ 4 & -3i \end{bmatrix}$ over $\mathbb{C}$ $\quad C_D(\lambda) = \lambda^2 + 4i\lambda + 5. \quad \lambda_1 = -5i. \ \lambda_2 = i$

a) $\sum_{i=1}^{2} \lambda_i = -5i + i = -4i \qquad tr(D) = -i - 3i = -4i$

b) $\prod_{i=1}^{2} \lambda_i = (-5i)(i) = 5 \qquad det(D) = 3i^2 + 8 = 5$

7.4 Finding Eigenvalues

For each eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ over $\mathbb{R}$, find a corresponding eigenpair.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$$\begin{aligned} C_A(\lambda) = \det(A - \lambda I) &= (1-\lambda)^2 - 4 \\ &= 1 + \lambda^2 - 2\lambda - 4 \\ &= \lambda^2 - 2\lambda - 3 \\ &= (\lambda - 3)(\lambda + 1) \end{aligned}$$

$\lambda = 3 \quad \lambda = -1$

$\lambda = 3$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}.$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$x_1 = t \quad x_2 = t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} t : t \in \mathbb{R} \right\}$$

eigenpair : $\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$\lambda = -1$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$x_1 = -t \quad x_2 = t$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} t : t \in \mathbb{R} \right\}$$

eigenpair : $\left(-1, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

For each eigenvalue of $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ over $\mathbb{C}$, find a corresponding eigenpair.

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$$

$$C_A(\lambda) = \det(A - \lambda I) = \lambda^2 + 1 = (\lambda + i)(\lambda - i)$$

$\lambda = i$

$$\begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$x_1 = t \quad x_2 = it \qquad ix_1$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} t : t \in \mathbb{R} \right\}$$

eigenpair : $\left(i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right)$

$\lambda = -i$

$$\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$x_1 = t \quad x_2 = -it$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left\{ \begin{bmatrix} 1 \\ -i \end{bmatrix} t : t \in \mathbb{R} \right\}$$

eigenpair : $\left(-i, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$

For each eigenvalue of $Z = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ over $\mathbb{R}$, find a corresponding eigenpair.

$$A-\lambda I = \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{bmatrix}$$

$$C_A(\lambda) = \det(A)$$

$$= (-1)^2 3-\lambda \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} + (-1)^3 \begin{vmatrix} 1 & 1 \\ 1 & 3-\lambda \end{vmatrix} + (-1)^4 \begin{vmatrix} 1 & 1 \\ 3-\lambda & 1 \end{vmatrix}$$

$$= 3-\lambda\left[(3-\lambda)^2-1\right] - (3-\lambda-1) + (1-3+\lambda)$$

$$= (3-\lambda)^3 - 3+\lambda - 3+\lambda+1-2+\lambda$$

$$= (9+\lambda^2-6\lambda)(3-\lambda) - 7+3\lambda$$

$$= 27-9\lambda+3\lambda^2-\lambda^3-18\lambda+6\lambda^2-7+3\lambda$$

$$= -\lambda^3+9\lambda^2-24\lambda+20$$

$$= (\lambda-5)(\lambda-2)^2$$

$\lambda = 5$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

$$x_1 = t, \quad x_2 = t, \quad x_3 = t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : s \in \mathbb{R} \right\}$$

eigenpair: $\left(5, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$

$\lambda = 2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

$$x_1 = -s-t, \quad x_2 = s, \quad x_3 = t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t : s, t \in \mathbb{R} \right\}$$

eigenpairs: $\left(2, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$ & $\left(2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$

7.5 Eigenspaces

- linear combination of eigenvalues

Let $c, d \in \mathbb{F}$. $(\lambda_1, \vec{x})$ & $(\lambda_1, \vec{y})$ are eigenpairs of a matrix A over $\mathbb{F}$.

with the same eigenvalue λ_1.

$c\vec{x} + d\vec{y} \neq \vec{0}$. $\Rightarrow$ $(\lambda_1, c\vec{x}+d\vec{y})$ is also an eigenpair for A with eigenvalue λ_1.

ep. one eigenpair of $A = \begin{bmatrix} 3 & 12 \\ 1 & 2 \end{bmatrix}$ is $\left(6, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right)$

for any $c \in \mathbb{R} - \{0\}$. take $d = 0$.

$$A \begin{bmatrix} 4c \\ c \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4c \\ c \end{bmatrix} = \begin{bmatrix} 24c \\ 6c \end{bmatrix} = 6 \begin{bmatrix} 4c \\ c \end{bmatrix}$$

Any non-zero scalar multiple of the eigenvector $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. is also an eigenvector with eigenvalue $\lambda = 6$

ep. $\left(6, \begin{bmatrix} 8 \\ 2 \end{bmatrix}\right)$ $\left(6, \begin{bmatrix} 12 \\ 3 \end{bmatrix}\right)$ ...

proof:
$$\begin{aligned} A(c\vec{x}+d\vec{y}) &= cA\vec{x} + dA\vec{y} \\ &= c\lambda_1\vec{x} + d\lambda_1\vec{y} \\ &= \lambda_1(c\vec{x} + d\vec{y}) \end{aligned}$$

$\because c\vec{x}+d\vec{y} \neq 0$ $\therefore c\vec{x}+d\vec{y}$ is eigenvector of A with eigenvalue λ_1

$\therefore (\lambda_1, c\vec{x}+d\vec{y})$ is eigenpair for A.

- Eigenspace

Let $A \in M_{n \times n}(\mathbb{F})$ with eigenvalue λ.

The eigenspace of A associated with λ is $E_\lambda(A) = \mathrm{Null}(A - \lambda I)$

1) $\vec{0}$ is in eigenspace but $\vec{0}$ isn't eigenvector

2) λ is an eigenspace $\Leftrightarrow$ $E_\lambda = \{\vec{0}\}$

3) $E_0(A) = \mathrm{Null}(A - 0I) = \mathrm{Null}(A)$

if 0 is an eigenvalue

Q. How is E_λ connected to $S = \{$ eigenvectors with eigenvalue $\lambda\}$.

because $\vec{0}$ isn't eigenvector. $S = E_\lambda - \{0\}$.

$(E_0(A) = Null(A - 0I) = Null(A))$

Q. Find all eigenspaces for $A = \begin{bmatrix} 5 & -2 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & -3 \end{bmatrix}$

$C_A(\lambda) = \begin{vmatrix} 5-\lambda & -2 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & -3-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda)(-3-\lambda) = 0$

eigenvalues are $\lambda_1 = 5$. $\lambda_2 = 2$. $\lambda_3 = -3$

$\rightarrow \lambda_1 = 5$: $A - 5I = \begin{bmatrix} 0 & -2 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

eigenvectors : $\vec{w} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} s$ where $s \in \mathbb{R} - \{0\}$.

$\therefore E_5(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

$\begin{bmatrix} 5 & -2 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\rightarrow \lambda_2 = 2$ $A - 2I = \begin{bmatrix} 3 & -2 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

eigenvectors : $\vec{v} = \begin{bmatrix} \frac{2}{3}t \\ t \\ 0 \end{bmatrix}$ where $t \in \mathbb{R} - \{0\}$

$\therefore E_2(A) = \text{Span}\left\{\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}\right\}$

$\rightarrow \lambda_3 = -3$ $A + 3I = \begin{bmatrix} 8 & -2 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{4}{5} \\ 0 & 1 & \frac{6}{5} \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore E_3 = \text{Span}\left\{\begin{bmatrix} -4 \\ -6 \\ 5 \end{bmatrix}\right\}$.

7.6 Diagonalization

- Similar

Let $A, B \in M_{n\times n}(\mathbb{F})$.

If $\exists P \in M_{n\times n}(\mathbb{F})$, P is invertible s.t. $A = PBP^{-1}$

Then A is similar to B over $\mathbb{F}$.

ex. Let $B = \begin{bmatrix} 2 & -1 \\ 0 & 9 \end{bmatrix}$ $P = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & 0 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$.

$A = PBP^{-1}$

$= \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ -3 & 2 \end{bmatrix}$

A is similar to B over $\mathbb{R}$

$A = PBP^{-1}$

$P^{-1}(A)P = P^{-1}(PBP^{-1})P = IBI = B$

- Proposition 7.6.5

A & B are similar over $\mathbb{F}$ $\Rightarrow$ A & B have same char-poly & eigenvalues

proof: $A = PBP^{-1}$

$C_A(\lambda) = \det(A - \lambda I)$

$= \det(PBP^{-1} - \lambda PP^{-1})$ $\quad (\because I = PP^{-1})$

$= \det(P^{-1}[B - \lambda]P)$ $\quad (\because PP^{-1} = P^{-1}P = I)$

$= \det(P^{-1})\det(B - \lambda)\det(P)$ $\quad (\because \det(AB) = \det(A)\det(B))$

$= \det(B - \lambda)$ $\quad (\because \det(P^{-1})\det(P) = I)$

$= \det(B - I\lambda)$

$= C_B(\lambda)$

Corollary: Let $A, B \in M_{n\times n}(\mathbb{F})$. If A & B are similar over $\mathbb{F}$.

then 1) $\det(A) = \det(B)$

2) $tr(A) = tr(B)$

proof: 1) $A = PBP^{-1}$

$\therefore \det(A) = \det(PBP^{-1})$

$\det(PBP^{-1}) = \det(P)\det(B)\det(\frac{1}{P}) = \det(B)$

$\therefore \det(B) = \det(A)$

2) $tr(A) = (-1)^{n-1} c_{n-1} = tr(B)$

- diagonalization

对角形式 of matrix A. $\quad D = P^{-1}AP \quad\quad D \rightarrow \begin{bmatrix} \times & & 0 \\ & \times & \\ 0 & & \times \end{bmatrix}$

A is diagonalizable if $A = SDS^{-1}$

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

eigenspace

用法：① $A^n = SD\cancel{S^{-1}S}D\cancel{S^{-1}} \cdots \cancel{S}DS^{-1}$

$$= SD^nS^{-1}$$

$$= S \begin{bmatrix} \lambda_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^n \end{bmatrix} S^{-1}$$

② A is diagonalizable $\Leftrightarrow$ S存在. S^{-1}存在.

$\Leftrightarrow$ $\vec{v}_1 \cdots \vec{v}_n$ eigenvectors

$\Leftrightarrow$ $\vec{v}_1 \cdots \vec{v}_1$ linear independent

$\Leftrightarrow$ $\{\vec{v}_1, \cdots, \vec{v}_n\}$ form a basis for $\mathbb{F}^n$ ($\mathbb{R}^n$)

$\Leftrightarrow$ $\forall \vec{x} \in \mathbb{R}^n$.

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

$$A\vec{x} = A(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n)$$

$$= c_1A\vec{v}_1 + c_2A\vec{v}_2 + \cdots + c_nA\vec{v}_n.$$

$$= c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_n\lambda_n\vec{v}_n$$

$\therefore$ $(\vec{v}_1, \cdots, \vec{v}_n)$ is eigvec

- diagonalizable

$A \in M_{n\times n}(\mathbb{F}) \quad D \in M_{n\times n}(\mathbb{F})$. $\quad A$ similar to D

$\exists P \in M_{n\times n}(\mathbb{F})$ s.t $P^{-1}AP = D$

A diagonalizable over $\mathbb{F}$ $\quad\quad P$ diagonalizes A

- Diagonalizable $\Rightarrow$ n Eigenvalues

Let $A \in M_{n\times n}(\mathbb{F})$.

A is diagonalizable over $\mathbb{F}$. $\Rightarrow$ chara-polynomial has n roots in $\mathbb{F}$

P diagonalizes A $\Rightarrow$ diagonal entries $D = P^{-1}AP$ are eigenvalues of A.

proof: $\because$ A diagonalizable over $\mathbb{F}$.

$\therefore \exists D := P^{-1}AP$ and P similar to A.

$P^{-1}AP = D = \text{diag}(\lambda_1, \cdots \lambda_n)$

$\because$ A & D are similar

$$\therefore C_A(\lambda) = C_D(\lambda) = \det(D - \lambda I)$$
$$= \det(\text{diag}(\lambda_1 - \lambda, \cdots, \lambda_n - \lambda))$$
$$= (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$$

$\therefore C_A(\lambda)$ has n roots $\lambda_1 \cdots \lambda_n \in \mathbb{F}$.

$\lambda_1, \lambda_2 \cdots \lambda_n$ are eigenvalues of A.

- n distinct eigenvalues $\Rightarrow$ diagonalizable

Let $A \in M_{n\times n}(\mathbb{F})$ have n distinct eigenvalues $\lambda_1, \lambda_2 \cdots \lambda_n$ in $\mathbb{F}$.

$(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2) \cdots (\lambda_n, \vec{v}_n)$ be corresponding eigenpairs over $\mathbb{F}$.

let $P = [\vec{v}_1, \cdots \vec{v}_n]$. then 1) P is invertible

2) $P^{-1}AP = D = \text{diag}(\lambda_1, \lambda_2 \cdots \lambda_n)$

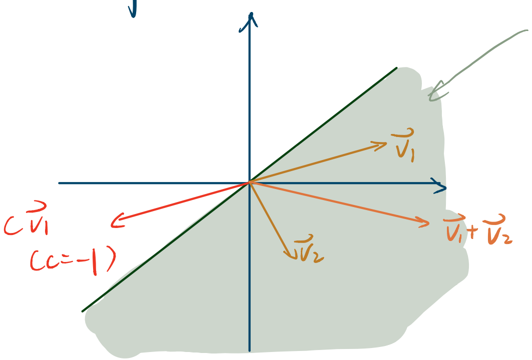
ex. Let $A = \begin{bmatrix} 1 & 0 & 18 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 0 & -3 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad P^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 18 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \text{diag}(1, 2, 9)$$

8.1 Subspace

- def.



V subset : $\{\begin{bmatrix} x \\ y \end{bmatrix} : x \geq y\}$

$\because c\vec{v}_1 \notin V$ when $c=-1$

$\therefore V$ 不合格 in subspace

(已知 $\vec{x} \in V$, $\vec{y} \in V$)

A subset V of $\mathbb{F}^n$ is called a subspace of $\mathbb{F}^n$ if :

1) $\vec{0} \in V$

2) $\vec{x}+\vec{y} \in V$ $\quad (\forall \vec{x}, \vec{y} \in V)$ closure under addition 加法

3) $c\vec{x} \in V$ $\quad (\forall \vec{x} \in V)$ closure under scalar multiplication. 乘系数

- Subspace Test

V is subset of $\mathbb{F}^n$.

V is subspace of $\mathbb{F}^n$ $\Leftrightarrow$ 同时满足
- a) V is non-empty.
- b) $c\vec{x}+\vec{y} \in V$ $\quad (\vec{x}, \vec{y} \in V, c \in \mathbb{F})$

proof. ($\Rightarrow$) Assume V is subspace of V.

$\vec{0} \in V$. $\therefore V \neq \emptyset$

Let $\vec{x}, \vec{y} \in V$. $c \in \mathbb{F}$. $\therefore c\vec{x} \in V$. (closure under scalar multi)

$\therefore c\vec{x}+\vec{y} \in V$. (closure under addition)

($\Leftarrow$) Assume (a) & (b) holds.

$\because V \neq \emptyset$. $\therefore \exists \vec{x} \in V$.

Take $\vec{y} = \vec{x}$, $c=-1$. $\quad c\vec{x}+\vec{y} = -\vec{x}+\vec{x} = \vec{0}$ $\quad \therefore \vec{0} \in V$

Take $\vec{x}, \vec{y} \in V$ $\quad$ Take $c=1$. $\quad \therefore \vec{x}+\vec{y} \in V$

Consider $\vec{y} = \vec{0}$. $\quad \vec{y} \in V$ $\quad \therefore c\vec{v} \in V$

- Examples of subspace

判断 V 是否是 subspace:
① $\{\vec{0}\} \in V$
② $\vec{v}+\vec{u}$
③ $c\vec{v}$.

a) $\{0\}$ is subspace of F^n. (smallest subspace)

proof: $c\vec{x}+\vec{y} = c\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} + \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}$

b) F is subspace of F^n. (largest subspace)

proof: $c\vec{x}+\vec{y} = c\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} + \begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix} = \begin{bmatrix}cx_1+y_1\\ \vdots\\ cx_n+y_n\end{bmatrix}$

c) $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_k\}$ is subset of F^n $\Rightarrow$ $\{\vec{v}_1 . \vec{v}_2 \cdots \vec{v}_k\}$ is subspace of F^n

proof: → let $\vec{v} \in \text{Span}\{\vec{v}_1 \cdots \vec{v}_p\}$. $c \in F$

$\vec{v} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$

$c\vec{v} = c(c_1\vec{v}_1 + \cdots + c_p\vec{v}_p)$

$= (cc_1)\vec{v}_1 + \cdots + (cc_p)\vec{v}_p$

$= \text{Span}\{\vec{v}_1 \cdots \vec{v}_p\}$

→ let $\vec{v} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$
$\vec{u} = d_1\vec{v}_1 + \cdots + d_p\vec{v}_p$.

$\vec{v}+\vec{u} = (c_1+d_1)\vec{v}_1 + \cdots + (c_p+d_p)\vec{v}_p \in \text{Span}\{\vec{v}_1 \cdots \vec{v}_p\}$

d) $\text{Null}(A)$ ($A\vec{x}=\vec{0}$) is subspace of F^n.

proof. ① $A\vec{0}=\vec{0}$. $\vec{0} \in \text{Null}(A)$

② Let $A\vec{x}_1=\vec{0}$ $A\vec{x}_2=\vec{0}$

$A(\vec{x}_1+\vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0}$

③ $A(c\vec{x}_1) = cA\vec{x}_1 = c\vec{0} = \vec{0}$

e) $Col(A)$ $(A \in M_{m\times n}(\mathbb{F}))$ is subspace of $\mathbb{F}^m$.

proof: $Col(A) = \{\vec{y} : \vec{y} = A\vec{x},\ \vec{x} \in \mathbb{R}^n\}$ $\quad A: m\times n$

① $\because \vec{0} = A\vec{0}.$ $\quad \therefore \vec{0} \in Col(A)$

② Let $\vec{y}_1 = A\vec{x}_1 \quad \vec{y}_2 = A\vec{x}_2 \quad \in Col(A)$

$\vec{y}_1 + \vec{y}_2 = A\vec{x}_1 + A\vec{x}_2 = A(\vec{x}_1 + \vec{x}_2) \in Col(A)$

③ $c\vec{y}_1 = cA\vec{x}_1 = A(c\vec{x}_1) \in Col(A)$

f) $Range(T)$ $(T: \mathbb{F}^n \to \mathbb{F}^m$ is linear transformation$)$

proof: $Range(T) = Col([T]_\varepsilon)$

by proved in (e). take $A = [T]_\varepsilon$.

g) $Null(T)$ $(T: \mathbb{F}^n \to \mathbb{F}^m$ is linear transformation$)$

proof: $[T]_\varepsilon \vec{x} = \vec{0}.$

by proved in (d). take $A = [T]_\varepsilon$.

h) E_λ $(A \in M_{n\times n}(\mathbb{F})\quad \lambda \in \mathbb{F})$

proof: $(A - \lambda I)\vec{x} = \vec{0}$

by proved in (d). take $A = A - \lambda I$

☆ $Span\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n\}$ is subspace 任何subspace都可写成span

① $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_n = \vec{0}$

② $\vec{w}_1 \in Span\{\vec{v}_1 \cdots \vec{v}_n\} \quad \vec{w}_2 \in Span\{\vec{v}_1 \cdots \vec{v}_n\}$

$\vec{w}_1 = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \quad \vec{w}_2 = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$

$\vec{w}_1 + \vec{w}_2 = (c_1+b_1)\vec{v}_1 + (c_2+b_2)\vec{v}_2 + \cdots + (c_n+b_n)\vec{v}_n$

$\therefore \vec{w}_1 + \vec{w}_2 \in Span\{\vec{v}_1, \cdots \vec{v}_n\}$

③ $c\vec{w}_1 = cc_1\vec{v}_1 + cc_2\vec{v}_2 + \cdots + cc_n\vec{v}_n$

$\therefore c\vec{w}_1 \in Span\{\vec{v}_1, \cdots \vec{v}_n\}$

Q. Are the following subsets - $\mathbb{R}^3$ subspaces of $\mathbb{R}^3$.

a) $S_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 = x_3 + 1 \right\}$

$\vec{0} \notin S_1$. $0+0 \neq 0+1$ $\therefore$ not a subspace

b) $S_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : |x_1| + |x_2| = |x_3| \right\}$

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \in S_2$ $\quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \notin S_2$ $\quad 0+0 \neq 4$

$\therefore$ not a subspace

c) $S_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 . \; 2x+5y+7z = 15$

for $S = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad 0+0+0 \neq 15$ $\quad \therefore$ not a subspace

d) $S_4 = \left\{ s \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + u \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \in \mathbb{R}^3 \; \text{s.t.} \; u \in \mathbb{R} \right\}$

可写作 $S_4 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \right\}$

$\because$ span is a subspan

$\therefore$ S_4 is subspace

- Efficient Spanning Set

1) $S_1 = \text{Span}\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}} \right\}$

S_1 is a subspace. $\{\vec{v}_1, \vec{v}_2\}$ is spanning set for S_1.

$\because \vec{v}_2 = 2\vec{v}_1$ $\quad \therefore \vec{v}_2$ is redundant.

If $\vec{u} \in S_1$. Then $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 = (c_1 + 2c_2)\vec{v}_1$

$\therefore \vec{u} \in \text{Span}\{\vec{v}_1\}$ $\quad \vec{u} \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

$\therefore S_1 = \text{Span}\{\vec{v}_1\}$.

2) $S_2 = \text{Span}\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}} \right\}$

S_2 is a subspace $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a spanning set for S_2

$\because \vec{v}_3 = \vec{v}_1 + 2\vec{v}_2$ $\quad \therefore \vec{v}_3$ is redundant.

$\because \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$.

$= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 + 2\vec{v}_2)$

$= (c_1 + c_3)\vec{v}_1 + (c_2 + 2c_3)\vec{v}_2$.

$\therefore S_2 \subseteq \text{Span}\{\vec{v}_1, \vec{v}_2\}$

$\because \vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 = b_1\vec{v}_1 + b_2\vec{v}_2 + 0\,\vec{v}_3$.

$\therefore S_2 = \text{Span}\{\vec{v}_1, \vec{v}_2\}$

So $\text{Span}\{\vec{v}_1, \vec{v}_2\} \subseteq S_2$.

8.2 Linear Dependence.

- def. linearly dependent

$\exists\, c_1, c_2 \cdots c_k \in \mathbb{F}$. not all 0. s.t $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$.

$\vec{v}_1 . \vec{v}_2 \cdots \vec{v}_k \in \mathbb{F}^n$ is linear dependence

↳ 说明其中一个 $\vec{v}$ 可 变换 成 $\vec{0}$

$\vec{v}_1 . \vec{v}_2 \cdots \vec{v}_k$ is linear independent if only sol is trivial sol $c_1 = c_2 = \cdots c_n = 0$

↳ 说明 $\vec{v}_1 \cdots \vec{v}_p$ 不可替代. 需满足 rank $(A) = \#$col.

ex. 1) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ linearly dependence $2\vec{v}_1 - \vec{v}_2 = \vec{0}$

2) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$ linearly dependence $\vec{v}_1 + 2\vec{v}_2 - \vec{v}_3 = \vec{0}$.

3) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ linearly independence $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 = \vec{0}$

$c_1 = c_2 = c_3 = 0$.

- Proposition 8.3.2

Let $S \subseteq \mathbb{F}^n$.

(a) $\vec{0} \in S$. $\Rightarrow$ S is linearly dep

proof: Let $S = \{\vec{v}_1, \cdots, \vec{v}_k, \vec{0}\}$.

$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{0} = \vec{0}$.

let $c_1 = c_2 = \cdots = c_k = 0$. $c_{k+1} = 1$. $\rightarrow$ non-trivial sol.

$\therefore$ linearly dependent.

(b) 若 $S = \{\vec{x}\}$, 则 S is lin-dep $\Leftrightarrow$ $\vec{x} = \vec{0}$

proof: ($\Leftarrow$) (a)

($\Rightarrow$) Assume $S = \{\vec{x}\}$. is linear dependence.

$\therefore c\vec{x} = \vec{0}$ has a sol where $c \neq 0$. $\therefore \vec{x} = \vec{0}$.

(c) 若 $S=\{\vec{x},\vec{y}\}$，则 S is lin-dep $\iff \vec{x}=c\vec{y}$. $(c\in\mathbb{R})$.

proof: $(\Rightarrow)$ $\{\vec{x},\vec{y}\}$ is lin dep.

Then $\exists a,b\in\mathbb{F}$. st $a\vec{x}+b\vec{y}=\vec{0}$ $(a\neq 0 \vee b\neq 0)$

Assume $a\neq 0$.

$a\vec{x}=-b\vec{y}$

$\vec{x}=-\frac{b}{a}\vec{y}$ $(a\neq 0)$

$\therefore \vec{y}$ is scalar multiple of $\vec{x}$.

$(\Leftarrow)$ Assume $\vec{y}=k\vec{x}$. $k\in\mathbb{F}$.

$k\vec{x}-\vec{y}=\vec{0}$. $\leftarrow$ non-trivial linear combination

$c_1\vec{x}+c_2\vec{y}=\vec{0}$. has sol $c_1=k$, $c_2=-1$.

Q. $S_1=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}2\\4\\7\end{bmatrix}\right\}$ not scalar multiple $\to$ linearly independence

$S_2=\left\{\begin{bmatrix}1+i\\1-i\end{bmatrix},\begin{bmatrix}2\\-2i\end{bmatrix}\right\}$ $\begin{bmatrix}1+i\\1-i\end{bmatrix}(1-i)-\begin{bmatrix}2\\-2i\end{bmatrix}=0$ linearly dependent

$S_3=\left\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}\to\vec{0}\in S$ linearly dependence

- linear dependence check.

$\vec{v_1}, \vec{v_2}\cdots\vec{v_k}$ $(k\geq 2)$. is linear dep

$\iff$ one of $\vec{v}$ can be written as linear comb of other vectors.

$(\Rightarrow)$ Assume $\{\vec{v_1}\cdots\vec{v_k}\}$ is linearly dependence

$\exists c_1,\cdots,c_k\in\mathbb{F}$. $c_1\vec{v_1}+\cdots+c_k\vec{v_k}=\vec{0}$. s.t. at least 1 of $c_i\neq 0$.
$c_1\neq 0$.

$\vec{v_1}=\frac{-c_2}{c_1}\vec{v_2}+\cdots+\frac{-c_k}{c_1}\vec{v_k}$ $(c_1\neq 0)$

$\therefore$ one of the vectors $(\vec{v_1})$ is a lin comb of the other.

($\Leftarrow$) Assume one of the vectors is lin comb of the other. (let it be $\vec{v}_k$)

$\exists b_1, \cdots, b_{k-1} \in \mathbb{F}$ s.t. $\vec{v}_k = b_1\vec{v}_1 + \cdots + b_{k-1}\vec{v}_{k-1}$

$\therefore b_1\vec{v}_1 + \cdots + b_{k-1}\vec{v}_{k-1} - \vec{v}_k = \vec{0}$. when $c_k = -1$

$\because c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = 0$ has a non-trivial sol.

$\therefore$ linear dependence

Q. Let S be a set of vectors of $\mathbb{F}^n$. Prove or disprove.

1. If S is linear independence. then $\forall T \subseteq S$. T is linear independence.

T

prove by contrapositive: If $\exists T \subseteq S$. T & S is linear depen.

Assume $T \subseteq S$ where T is linear-dependent.

$\therefore$ One of vectors in T can be written as linear combination of the other vectors in T.

But $T \subseteq S$. $\therefore$ One of vectors in S can be written as a linear comb. of the other in S. $\rightarrow$ use lin comb with 0 for the coefficients in front of the additional vectors

$\therefore S$ is linear independence

2. If S is linear dependence. then $\forall T \subseteq S$. T is linear dependent

F $\quad T = \emptyset$

3. If S is linear dependence. then $\forall T \subseteq S$. $T \neq \emptyset$. T is linear dependent

F $\quad S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ lin dep. $\qquad T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ lin ind

4. The union of 2 lin-ind set is lin-ind.

F $\quad S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $T = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$. $S \cup T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ lin-dep

5. The union of 2 lin-dep set is lin-dep.

T. $\quad S \subseteq S \cup T$. $\quad \therefore$ lin-dep.

8.3 Check Linear dep & indep 判断向量间的关系

- check.

linear dep. 其中 $\vec{v}$ 可被其它 vector 变换得到

linear indep 唯一解为 $c_1 = \cdots = c_n = 0$

$$[\vec{v_1}\ \vec{v_2}\ \cdots\ \vec{v_n}]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}.$$

↑全0

Q. Is the set $S = \left\{ \underset{\vec{v_1}}{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}, \underset{\vec{v_2}}{\begin{bmatrix} -2 \\ 5 \\ -6 \end{bmatrix}}, \underset{\vec{v_3}}{\begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix}} \right\}$ linear independence?

consider $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -2 & 5 & -2 & 0 \\ 3 & -6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & -9 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$c_1 = c_2 = c_3 = 0$

$\therefore$ S is linearly independence

Q. Let $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\} \subseteq \mathbb{F}^3$. prove that S is linearly dependent.

proof: consider $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} + c_4\vec{v_4} = \vec{0}$.

$[\vec{v_1}\ \vec{v_2}\ \vec{v_3}\ \vec{v_4} \mid \vec{0}]$ is the argumented matrix for this system of eqs

The coefficient matrix is 3×4. So by Rank Bounds rank $(A) \leq 3$.

no. col $= 4$.

$\therefore$ nullity $(A) \geq 1$.

So there is an infinite number of solutions.

There is more than just trivial sol

$\therefore$ S is linearly dependent.

- pivots and linear independence

可用于寻找 efficient spanning set

Let $S = \{\vec{v_1}, \vec{v_2} \cdots \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$.

Let $A = [\vec{v_1}\ \vec{v_2} \cdots \vec{v_k}]$ be $n \times k$ matrix

Suppose that rank $(A) = r$. A has pivots in cols $q_1\ q_2 \cdots q_r$.

$U = \{\vec{v_{q_1}}, \vec{v_{q_2}}, \cdots \vec{v_{q_r}}\}$

(a) S is linearly indep $\Leftrightarrow r = k$

consider $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} = \vec{0}$

$\{\vec{v_1}, \vec{v_2}, \cdots \vec{v_k}\}$ is linearly indep.

$\Leftrightarrow A\vec{c} = \vec{0}$ has a unique sol

$\Leftrightarrow$ rank $(A) = k$.

(b) U is linearly indep

consider $c_1\vec{v_{q_1}} + c_2\vec{v_{q_2}} + \cdots + c_k\vec{v_{q_k}} = \vec{0}$

coefficient matrix has rank r because it consists of only pivots columns of A.

To row reduce. this matrix RREF . $\rightarrow$ RREF.

So there is a unique solution

$\therefore U$ is linearly independence.

(c) $\vec{v} \in S \wedge \vec{v} \notin U \Rightarrow \{\vec{v_{q_1}}, \cdots \vec{v_{q_r}}, \vec{v}\}$ is linearly dep.

Suppose $r < k$.

$\therefore$ there's at least 1 non-pivot col. (let it be $\vec{v_j}$)

consider $c_1\vec{v_{q_1}} + c_2\vec{v_{q_2}} + \cdots + c_r\vec{v_{q_r}} + \alpha\vec{v_j} = 0$ Is

coefficient matrix will have rank r but there are $r+1$ co

So there will be a non-trivial sol.

$\therefore \{\vec{v_{q_1}}, \cdots \vec{v_{q_r}}, \vec{v}\}$ is linearly dep.

(d) $\text{Span}(U) = \text{Span}(S)$

$\because U \subseteq S \qquad \therefore \text{Span}(U) \subseteq \text{Span}(S)$

We need to prove that $\text{Span}(S) \subseteq \text{Span}(U)$

case 1: $U = S$ QED

case 2: $U \neq S$. $\exists \vec{v_j} \in S.\ \vec{v_j} \notin U$.

from (c) $c_1\vec{v_{q_1}} + c_2\vec{v_{q_2}} + \cdots + c_r\vec{v_{q_r}} + \alpha\vec{v_j} = 0$ has non-trivial sol

$\therefore \alpha \neq 0$. for this sol.

$\therefore \vec{v_j} = \frac{-d_1}{\alpha}\vec{v_{q_1}} + \cdots + \frac{-d_1}{\alpha}\vec{v_{q_r}}$

$\therefore$ any vectors $\in S\ \notin U$ is linear comb in U

$\therefore \text{Span}(S) \subseteq \text{Span}(U)$ QED

- Bound on number of linearly ind vectors

$\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}\right\} \quad 2<3.$

Let $S = \{\vec{v_1}, \vec{v_2} \cdots \vec{v_k}\} \subseteq F^n$. $n<k \Rightarrow S$ is linearly dep.

proof: $A = [\vec{v_1} \cdots \vec{v_k}]$ $n \times k$ matrix

$\text{rank}(A) \le n < k \qquad \therefore \text{rank}(A) \neq k$.

$\therefore c_1\vec{v_1} + \cdots + c_k\vec{v_k}$ has non-trivial solution

$\therefore S$ is lin-dep

如何提取 linear independent vector set

ep. $S = \left\{ \begin{array}{c}\vec{v_1}\\ \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\end{array}, \begin{array}{c}\vec{v_2}\\ \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\end{array}, \begin{array}{c}\vec{v_3}\\ \begin{bmatrix}3\\-5\\-5\\3\end{bmatrix}\end{array}, \begin{array}{c}\vec{v_4}\\ \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\end{array}, \begin{array}{c}\vec{v_5}\\ \begin{bmatrix}2\\1\\0\\2\end{bmatrix}\end{array} \right\}$

5 components in $\mathbb{R}^4$ $\therefore$ ind

提取有 pivot in col 对应 in vec

$A = [\vec{v_1}\ \vec{v_2}\ \vec{v_3}\ \vec{v_4}\ \vec{v_5}] \sim \begin{array}{c} \begin{bmatrix}1 & 0 & 3 & 0 & 2\\ 0 & 1 & -2 & 0 & 3\\ 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} \\ \begin{array}{ccccc}\uparrow & \uparrow & & \uparrow & \\ \vec{v_1} & \vec{v_2} & & \vec{v_4} & \end{array}\end{array}$

$\{\vec{v_1}, \vec{v_2}, \vec{v_4}\}$ is lin ind.

$\text{Span}(S) = \text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_4}\}$

8.4. Spanning Set

- Span of subset

Let V be a subspace of $\mathbb{F}^n$. $S = \{\vec{v}_1, \cdots \vec{v}_k\} \subseteq V$.

Then $\mathrm{Span}(S) \subseteq V$.

* 若想证 $\mathrm{Span}(S) = V$. We only need to prove $V \subseteq \mathrm{Span}(S)$

- Span $\mathbb{F}^n$ iff rank is n.

Let $S = \{\vec{v}_1 \cdots \vec{v}_k\}$ be a set of k vectors in $\mathbb{F}^n$.

let $A = \{\vec{v}_1 \cdots \vec{v}_k\}$. be the matrix whose cols are the vectors in S.

$$\mathrm{Span}(S) = \mathbb{F}^n \iff \mathrm{rank}(A) = n.$$

$\mathrm{Rank}(A) \le k$. So if S is a spanning set. 需满足 $k \ge n$.

proof: $\mathrm{Span}(S) = \mathbb{F}^n$

$\iff \forall \vec{v} \in \mathbb{F}^n .\ \exists c_1, \cdots c_k \in \mathbb{F}.\ c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{v}.$

$\iff A\vec{c} = \vec{v}$ is consistent $\forall \vec{v} \in \mathbb{F}^n$

$\iff \mathrm{rank}(A) = n$

- n vectors in $\mathbb{F}^n$ span iff linearly independent.

Let $S = \{\vec{v}_1 \cdots \vec{v}_n\}$. be a set of n vectors in $\mathbb{F}^n$.

S is linearly independent $\iff$ $\mathrm{Span}(S) = \mathbb{F}^n$.

proof: Let $A = [\vec{v}_1 \cdots \vec{v}_n]$

S Spans $\mathbb{F}^n \iff \mathrm{rank}(A) = n$.

by pivot linear independence. S is lin-indep $\iff \mathrm{rank}(A) = n$.

$\therefore$ 结合起来 S Spans $\mathbb{F}^n \iff S$ is lin-indep

8.5 Basis

- def. basis

最精简的形式

V is subspace of $\mathbb{F}^n$.

<u>basis</u> for V if 1) $\{V_1, V_2 \cdots V_m\}$ is linearly independent.

2) $V = \text{Span}\{V_1, V_2 \cdots V_m\}$

<u>dimention</u> of A : $m = \dim(V)$ (basis 的数量)

* $V = \{\vec{0}\}$ is a subspace. By convention a basis for V is $\emptyset$.

Span $\emptyset = \{\vec{0}\}$. linear combination of $\emptyset$ is $\vec{0}$.

如何找 basis

1. 找到 Span $\{\vec{v}_1 \cdots \vec{v}_n\} = \mathbb{R}^2$.
2. 再确认 linear independent

basis 不唯一. 但数量唯一. $= \dim(V)$

Q. Prove or disprove:

Let V be a subspace of $\mathbb{F}^n$. If $S = \{\vec{v}_1 \cdots \vec{v}_k\}$ is a spanning set for V.

Then $\forall \vec{v} \in \mathbb{F}^n$. $S_1 = S \cup \{\vec{v}\}$ is a spanning set for V.

False. Let $V = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ is a spanning set for V.

But $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ isn't spanning set for V.

改成 $\forall \vec{v} \in V$ True.

- Every subspace has a spanning set.

Let V be a subspace of $\mathbb{F}^n$. Then there exist vectors $\vec{v}_1 \cdots \vec{v}_k \in V$.

s.t $V = \text{Span}\{\vec{v}_1 \cdots \vec{v}_k\}$.

proof: If $V = \{\vec{0}\}$. then $\emptyset$ is by convention a spanning set for V.

consider $V \neq \{\vec{0}\}$.

$\because V$ is subspace . $V \neq \emptyset$.

$\therefore$ consider $\vec{v}_1 \in V$.

case 1 : $V = \text{Span}\{\vec{v}_1\}$ QED

case 2 : $\exists \vec{v}_2 \in V_1$ $\vec{v}_2 \notin \text{Span}\{\vec{v}_1\}$.

consider $\{\vec{v}_1, \vec{v}_2\}$. This set is linearly independent.

case 3 : $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ we are done.

otherwise. continue this process to get $V = \text{Span}\{\vec{v}_1 \cdots \vec{v}_k\}$

- Every subspace has a basis.

V be a subspace of $\mathbb{F}^n$. $\Rightarrow$ V has a basis.

- size of basis for $\mathbb{F}^n$.

$S = \{\vec{v}_1 \cdots \vec{v}_k\}$ be a set of k vectors in $\mathbb{F}^n$.

S is a basis for $\mathbb{F}^n$. $\Rightarrow$ $k = n$.

证明 $\text{Span}(S) = V$ 只需 证 $V \subseteq \text{Span}(S)$

Q. are the following set basis for $\mathbb{R}^3$?

1) $\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\0\end{bmatrix} \right\}$ # vector < 3 No.

2) $\left\{ \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\-1\\5\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$ # vector > 3 → lin-dep. No.

3) $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix} \right\}$ $A = \begin{bmatrix}1&2&3\\0&1&1\\0&0&0\end{bmatrix}$ rank(A) = 2 ≠ 3 No.

4) $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\1\\1\end{bmatrix} \right\}$ $A = \begin{bmatrix}1&2&3\\0&1&1\\0&0&1\end{bmatrix}$ rank(A) = 3 ∴ Yes.

判断 basis 1. # vector = # row → < # row not enough vectors; > # row lin-dep

2. rank(A) = # row

Q. Determine all values of $a \in \mathbb{R}$ s.t $\left\{ \begin{bmatrix}a\\2a\\2\end{bmatrix}, \begin{bmatrix}1\\2\\2a\end{bmatrix}, \begin{bmatrix}1\\2a\\2a+2\end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

$$\det(A) = \begin{bmatrix}a&1&1\\2a&2&2a\\2&2a&2a+2\end{bmatrix} = \det\begin{bmatrix}a&1&1\\0&0&2a-2\\2&2a&2a+2\end{bmatrix}$$

$$= -(2a-2)(2a^2-2)$$

$$= 4(a-1)^2(a+1) \neq 0.$$

So S will be a basis if $a \neq \pm 1$

8.6 Bases for Col(A) and Null(A)

- Basis for Col(A) (C)

Let $A = [\vec{a_1} \cdots \vec{a_n}] \in M_{m\times n}(\mathbb{F})$ and RREF(A) has pivots in cols $q_1 \cdots q_r$. $r = \text{rank}(A)$.

Then $\{\vec{a_{q_1}}, \cdots, \vec{a_{q_r}}\}$ is basis for Col(A)

e.g. Find a basis for col space

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ -1 & -1 & -1 & -4 & 2 \end{bmatrix} \qquad \text{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\uparrow \quad \uparrow \qquad\qquad \uparrow$$

basis for Col(A) is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ 所有带 pivot 的列

dim (Col(A)) = # of pivot

$$\boxed{\dim(\text{Col}(A)) + \dim(N(A)) = \#\ \text{Col of } A}$$

- Basis for Null(A)

Let $A \in M_{m\times n}(\mathbb{F})$. $\underline{A\vec{x} = \vec{0}}$.

sol to the system : $\text{Null}(A) = \{t_1\vec{x}_1 + \cdots + t_k\vec{x}_k : t_1 \cdots t_k \in \mathbb{F}\}$

$k = \text{nullity}(A) = n - \text{rank}(A)$ $\qquad t_i : 1 \le i \le k$.

e.g. Find a basis for null space

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 4 & 1 \\ -1 & -1 & -1 & -4 & 2 \end{bmatrix} \qquad \text{RREF}(A) = \left[\begin{array}{ccccc|c} 1 & 0 & 2 & 3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$-2t-3s \quad t-s \quad t \quad s \quad 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

dim (N(A)) = # of free variables $\quad$ span 的值. $\quad$ # null space basis = # of free

8.7 Dimension

- def. $\dim(V)$

of elements in a basis for a subspace V of $\mathbb{F}$.

V is subspace of $\mathbb{F}^n$.

$B = \{\vec{v}_1, \cdots \vec{v}_k\}$ $\quad$ $C = \{\vec{w}_1, \cdots \vec{w}_l\}$ are bases for V.

$\Rightarrow k = l$ $\quad$ 同一个 V 的 bases 中, vector 数相同

eg. P = a plane through origin.

$P = \{s\vec{a} + t\vec{b} : s, t \in \mathbb{R}\}$ $\quad$ $\vec{a} \neq k\vec{b}$ $\quad$ $k \in \mathbb{R}$.

$P = \text{Span}\{\vec{a}, \vec{b}\}$. $\quad$ a basis for P is $\{\vec{a}, \vec{b}\}$.

$\because \vec{a} \neq k\vec{b}$ $\quad$ $\therefore$ lin-indep.

$\therefore \dim(P) = 2$

- bound on dimension of subspace

V is subspace of $\mathbb{F}^n$. $\Leftrightarrow \dim(V) \leq n$.

Q. V & W are subspace of $\mathbb{F}^n$. $\quad W \subseteq V$

a) prove or disprove $\dim(W) \leq \dim(V)$

T.

$\because W \subseteq V$

$\therefore$ any indep subset in W is also lin indep subset in V.

$\therefore$ any basis for W can be "grown" into basis for V by adding more vectors if necessary.

$\therefore \dim(W) = \dim(V)$

b) prove or disprove $\dim(W) = \dim(V) \Leftrightarrow W = V$

T

($\Leftarrow$) obvious

($\Rightarrow$) Assume $\dim(W) = \dim(V) = k$.

Let $S = \{\vec{v}_1, \cdots, \vec{v}_k\}$. be a basis for W.

$W = \text{span}(S)$ $\quad$ S is lin-indep.

$\because S \subseteq W \subseteq V$.

$\therefore S$ is lin-indep set in V $\quad$ $\text{Span}(S) = V$.

$\therefore$ We can find a larger lin-indep set in V.

contradict that $\dim(V) = k$ $\quad$ Since $W = \text{Span}(S) = V$

- Rank & Nullity as dimensions.

Let $A \in M_{m \times n}(\mathbb{F})$. Then

a) $\text{rank}(A) = \dim(\text{Col}(A))$

b) $\text{nullity}(A) = \dim(\text{Null}(A))$

- Rank-Nullity theorem

$n = \text{rank}(A) + \text{nullity}(A) = \dim(\text{Col}(A)) + \dim(N(A))$

Q. Let $\vec{u}, \vec{v}, \vec{w}$ be 3 distinct basis of $\mathbb{R}^3$.

Prove $\{\vec{u}, \vec{v}, \vec{w}\}$ is basis of $\mathbb{R}^3 \Rightarrow \{\vec{u}+2\vec{v}+3\vec{w},\ 2\vec{v}+3\vec{w},\ 3\vec{w}\}$ is also basis of $\mathbb{R}^3$.

→ 证 linear-indep

$c_1(\vec{u}+2\vec{v}+3\vec{w}) + c_2(2\vec{v}+3\vec{w}) + c_3 \cdot 3\vec{w} = \vec{0}$

$c_1\vec{u} + (2c_1+2c_2)\vec{v} + (3c_1+3c_2+3c_3)\vec{w} = \vec{0}$

$\because \vec{u}, \vec{v}, \vec{w}$ is lin-indep

$\therefore a_1\vec{u} + a_2\vec{v} + a_3\vec{w}$ has only sol $\Rightarrow a_1 = a_2 = a_3 = 0$

$$\begin{cases} c_1 = 0 \\ 2c_1+2c_2 = 0 \\ 3c_1+3c_2+3c_3 = 0 \end{cases} \qquad \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

$A = [\vec{u}+2\vec{v}+3\vec{w} \quad 2\vec{v}+3\vec{w} \quad 3\vec{w}]$

→ 证 $\{\vec{u}+2\vec{v}+3\vec{w},\ 2\vec{v}+3\vec{w},\ 3\vec{w}\}$ span $\mathbb{R}^3$. $A\vec{x} = \vec{y}$ has sol $\forall \vec{y}$

$\because$ REF(A) has pivot in each row.

$\therefore A\vec{x} = \vec{y}$ always has sol.

$\therefore \{\vec{u}+2\vec{v}+3\vec{w},\ 2\vec{v}+3\vec{w},\ 3\vec{w}\}$ is basis of $\mathbb{R}^3$

Q.
$$A = \begin{bmatrix} 1 & 1 & 2 & -3 & 4 \\ -2 & -1 & -5 & 9 & -6 \\ 2 & 2 & 4 & -6 & 9 \end{bmatrix}$$

a) determine basis for Col(A). dim(Col(A))

b) determine dim(Null(A))

c) determine basis for Null(A)

a)
$$A = \begin{bmatrix} 1 & 1 & 2 & -3 & 4 \\ 0 & -1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
basis for Col(A): $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}$

dim(Col(A)) = 3

b) dim(Null(A)) = 5 − 3 = 2

c)
$$\begin{array}{ccccc} & & & & \\ \end{array}\left[\begin{array}{ccccc|c} 1 & 1 & 2 & -3 & 4 & 0 \\ 0 & -1 & -1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$
$\quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad 0$

$$\vec{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} s$$

basis for Null(A): $\{\ \}$

8.8 Coordinates

- Unique Representation Theorem

Let V be a subspace of $\mathbb{F}^n$. $B = \{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_k\}$ is a basis of V.

$\Rightarrow \forall \vec{v} \in V$. $\exists$ unique $c_1 \cdots c_k \in \mathbb{F}$. s.t. $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$

proof: ① proof there exist.

$\because \beta$ is a basis that it is a spanning set

$\therefore \forall \vec{v} \in \mathbb{F}^n$. $\exists c_1 \cdots c_n \in \mathbb{F}$ s.t $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$

② proof unique

Assume $\exists d_1 \cdots d_n \in \mathbb{F}$ s.t $d_1\vec{v}_1 + \cdots + d_n\vec{v}_n = \vec{v}$.

$\therefore c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = d_1\vec{v}_1 + \cdots + d_n\vec{v}_n$

$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n - d_1\vec{v}_1 - \cdots - d_n\vec{v}_n = \vec{0}$

$(c_1 - d_1)\vec{v}_1 + \cdots + (c_n - d_n)\vec{v}_n = \vec{0}$

$\because \beta$ is a basis. then $\vec{v}_1 \cdots \vec{v}_n$ are linear independent.

$\therefore c_1 - d_1 = c_2 - d_2 = \cdots = c_n - d_n = 0$

$\therefore c_1 = d_1 \quad \cdots \quad d_n = d_n$

- Coordinates with respect to B.

$B = \{\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n\}$ is a basis for $\mathbb{F}^n$. $v \in \mathbb{F}^n$.

$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \sum_{i=1}^{n} c_i\vec{v}_i \quad (c_i \in \mathbb{F}) \quad = \sum_{i=1}^{n} c_i\vec{v}_i \quad (c_i \in \mathbb{F})$

$c_1 \cdots c_n$: coordinates / components of $\vec{v}$ with respect to B.

* order of basis vectors matters.

- order basis.

ordered basis for $\mathbb{F}^n$ is basis $B = \{\vec{v}_1 \cdots \vec{v}_n\}$ with fixed ordering

$\{\vec{v}_1, \vec{v}_2\} \neq \{\vec{v}_2, \vec{v}_1\}$ standard ordering : $\varepsilon = \{\vec{e}_1 \cdots \vec{e}_n\}$

- coordinate vector.

coordinate vector of $\vec{v}$ with respect to B. $[\vec{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

ex. a) $\vec{v} = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$ $[\vec{v}]_\varepsilon = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$

b) $\vec{v} = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$ $\beta = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}$ $[\vec{v}]_\beta = ?$

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{v}.$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$ $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$ $[\vec{v}]_\beta = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

c) $[w]_\beta = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ $\beta = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \right\}$ $\vec{w} = ?$

$\vec{w} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -3 \end{bmatrix}$

- Linearity of Taking Coordinates

Let $B = \{\vec{v}_1 \cdots \vec{v}_n\}$ be an ordered basis for $\mathbb{F}^n$.

Then $[\ ]_B : \mathbb{F}^n \to \mathbb{F}^n$ defined by sending $\vec{v}$ to $[\vec{v}]_B$ is linear:

a) $\forall\, \vec{u}, \vec{v} \in \mathbb{F}^n$. $[\vec{u}+\vec{v}]_B = [\vec{u}]_B + [\vec{v}]_B$.

b) $\forall\, \vec{v} \in \mathbb{F}^n$ $[c\vec{v}]_B = c[\vec{v}]_B$.

Q. $\beta = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 3 \\ 8 \end{bmatrix}} \right\}$ an ordered basis for $\mathbb{R}^2$.

Find $\vec{w}$, where $[\vec{w}]_\beta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$\vec{w} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

M : matrix with $\vec{v}_1$ & $\vec{v}_2$ as its cols.

$M[\vec{x}]_\beta = \vec{x} = [\vec{x}]_\varepsilon$.

Change of coordinate matrix from basis β to basis ε. ${}_{\varepsilon}[I]_{\beta}$

Q. Find $[\vec{w}_i]_\beta$ given $\vec{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{w}_2 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

→ solve $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{w}_i$

$M \rightarrow [\vec{v}_1 \ \vec{v}_2]\vec{c} = \vec{w}_i$

$\vec{c} = [\vec{w}_i]_\beta$.

$\therefore M[\vec{w}_i]_\beta = \vec{w}_i$

Is M invertible? Yes. $\because$ Column are basis vectors.

$\therefore$ lin-indep. rank $(M) = 2$.

$[\vec{w}_i]_\beta = M^{-1}\vec{w}_i$ M^{-1}: the change of coordinate matrix from basis ε to basis β ${}_{\beta}[I]_{\varepsilon}$

$M^{-1} = \begin{bmatrix} -8 & 3 \\ 3 & -1 \end{bmatrix}$ $[\vec{w}_1]_\beta = \begin{bmatrix} -8 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$[\vec{w}_2]_\beta = \begin{bmatrix} -8 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 39 \\ -14 \end{bmatrix}$

- How to change between 2 non-standard bases for $\mathbb{F}^n$?

从 β 到 C. $\beta = \{\vec{v}_1, \cdots, \vec{v}_n\}$ $C = \{\vec{w}_1, \cdots, \vec{w}_n\}$

$\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ $[\vec{x}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$[\vec{x}]_C = [c_1\vec{v}_1 + \cdots + c_n\vec{v}_n]_C$

$= c_1[\vec{v}_1]_C + \cdots + c_n[\vec{v}_n]_C$

$= [[\vec{v}_1]_C \cdots [\vec{v}_n]_C]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ $[\vec{x}]_\beta$ ← change-of-coordinate matrix

– Changing a basis.

Let $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ and $\mathcal{C} = \{\vec{w_1}, \ldots, \vec{w_k}\}$ be ordered bases for a subspace V of $\mathbb{F}^n$.

The **change-of-basis** (or **change-of-coordinates**) matrix **from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates** is the $k \times k$ matrix

$$_{\mathcal{C}}[I]_{\mathcal{B}} = \left[\, [\vec{v_1}]_{\mathcal{C}}, \ldots, [\vec{v_k}]_{\mathcal{C}} \,\right]$$

whose columns are the $\mathcal{C}$-coordinates of the vectors $\vec{v_i}$ in $\mathcal{B}$.

Similarly, the **change-of-basis** (or **change-of-coordinates**) matrix **from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates** is the $k \times k$ matrix

$$\underset{\text{to}}{{}_{\mathcal{B}}}[I]\underset{\text{from}}{{}_{\mathcal{C}}} = \left[\, [\vec{w_1}]_{\mathcal{B}}, \ldots, [\vec{w_k}]_{\mathcal{B}} \,\right]$$

whose columns are the $\mathcal{B}$-coordinates of the vectors $\vec{w_i}$ in $\mathcal{C}$.

- $[\vec{x}]_C = {}_C[I]_B[\vec{x}]_B$.
- $[\vec{x}]_B = {}_B[I]_C[\vec{x}]_C$.
- $B = C$ $\quad {}_C[I]_C[\vec{x}]_{\mathcal{E}} = {}_C[I]_{\mathcal{E}}\vec{x}$.
- $C = \mathcal{E}$ $\quad {}_{\mathcal{E}}[I]_B[\vec{x}]_B$

– Inverse of change-of-basis Matrix

B. C are ordered basis

$${}_B[I]_C\,{}_C[I]_B = I_n.$$

$${}_B[I]_C = ({}_C[I]_B)^{-1}$$

proof: change coordinates twice

$${}_B[I]_C\,{}_C[I]_B[\vec{x}]_B = [\vec{x}]_B$$

By matrix equality $\quad {}_B[I]_C\,{}_C[I]_B = I_n.$

$$({}_B[I]_C)^{-1} = {}_C[I]_B$$

basis $\begin{cases} ① \text{ Span } V \\ ② \text{ lin-indep} \end{cases}$ # basis vec = dim(V) 不变

standard basis $\mathcal{E} = \{\vec{e}_1 \cdots \vec{e}_n\}$ $I = [\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n] = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

找 basis ① Col(A) 的 : REF(A) pivot 所在列
② Null(A) 的 : $A\vec{x} = \vec{0}$ 的 sol set # vec = # free variables
③ eigenspace 的 : 找 eigval. 代入 解入. 求 span
(和求 eig-space 一样)

coordinate $\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ coordinate vector $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = [\vec{v}]_B$ 坐标向量

$B\vec{c} = \vec{v}$. $\vec{c}$ 是 solution

→ 用 lin-trans. 看 coordinates vectors lin-indep ⟺ inverse

$T: V$ subspace $\to F^k$

$\dim(V) = k$ $\dim(F^k) = k$.

$\vec{v} \in V$ $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$

$T(\vec{v}) = \underset{\substack{\uparrow \\ k\times k}}{S} \cdot \vec{v}$. square matrix $S = [T(\vec{e}_1)\ T(\vec{e}_2) \cdots T(\vec{e}_k)]$

$S\vec{v}_1 = \vec{e}_1$
$S\vec{v}_2 = \vec{e}_2$ $[S\vec{v}_1\ S\vec{v}_2 \cdots S\vec{v}_k] = I$.
$\vdots$
$S\vec{v}_k = \vec{e}_k$ $S\underbrace{[\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_k]}_{V} = I$

$S\ V = I$. $S = V^{-1}$

→ change the basis

$B_1 = \{\vec{v}_1 \cdots \vec{v}_k\}$. $B_2 = \{\vec{a}_1 \cdots \vec{a}_k\}$ are basis of V

$\vec{v} \in V$

$[\vec{v}]_{B_1} = \vec{c}$ $\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ $[\vec{v}]_{B_2} = \vec{d}$ $\vec{v} = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k$.

Q: 已知 $\vec{c}$ 求 $\vec{d}$ 通过 matrix

$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \longrightarrow \vec{v} = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k$.

$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \xrightarrow{T} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$

$S = [T(\vec{e}_1)\ T(\vec{e}_2) \cdots T(\vec{e}_k)]$

$\vec{e}_1$: $\vec{e}_1 = [\vec{v}]_{B_2} \to \vec{v} = \vec{a}_1$ $T(\vec{e}_1) = [\vec{a}_1]_{B_1} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$

$S: C_{B_2 \to B_1} = [[a_1]_{B_1}\ [a_2]_{B_1} \cdots [a_k]_{B_1}]$

Q. Find coordinates of $\vec{w} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$ in the basis $B = \left\{ \begin{bmatrix} 1 \\ 8 \\ 10 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

① $B\vec{c} = \vec{w}$

$$\begin{bmatrix} 1 & 1 & 4 \\ 8 & 1 & 5 \\ 10 & 1 & 6 \end{bmatrix} \vec{c} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

解 $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 3 \\ 8 & 1 & 5 & 6 \\ 10 & 1 & 6 & 4 \end{array}\right]$

② $\begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$[\vec{w}]_{B_1} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$

$[\vec{w}]_B = C_{B_1 \to B} \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$

$C_{B_1 \to B} = \begin{bmatrix} [\vec{e_1}]_B & [\vec{e_2}]_B & [\vec{e_3}]_B \end{bmatrix}$

$C_{B \to B_1} = \begin{bmatrix} [\vec{b_1}]_{B_1} & [\vec{b_2}]_{B_1} & [\vec{b_3}]_{B_1} \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 4 \\ 8 & 1 & 5 \\ 10 & 1 & 6 \end{bmatrix}$

$C_{B_1 \to B} = [C_{B \to B_1}]^{-1} = \begin{bmatrix} 1 & 1 & 4 \\ 8 & 1 & 5 \\ 10 & 1 & 6 \end{bmatrix}^{-1}$

9.1 Matrix Representation – Linear Operator

- B Matrix of T

$T: \mathbb{F}^n \to \mathbb{F}^n$. $\quad B = \{\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n\}$ ordered basis

$$[T]_B = [\, [T(\vec{v}_1)]_B \;\; [T(\vec{v}_2)]_B \;\cdots\; [T(\vec{v}_n)]_B \,]$$

ep. $B = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} w_2 \\ w_1 \end{bmatrix} \right\}$ $\quad \text{proj}_{\vec{w}} \vec{w} = \vec{w} = 1 w_1 + 0 w_2$

$$[T]_B = [\text{proj}_{\vec{w}}]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- proposition 9.1.7

$$[T(\vec{v})]_B = [T]_B [\vec{v}]_B.$$

$$\begin{aligned} T(\vec{v}) &= T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \\ &= c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n) \end{aligned}$$

$$\begin{aligned} [T(\vec{v})]_B &= [c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n)]_B \\ &= [T(\vec{v}_1) \cdots T(\vec{v}_n)]_B \, \vec{c} \\ &= \underbrace{[\, [T(\vec{v}_1)]_B \cdots [T(\vec{v}_n)]_B \,]}_{\text{standard matrix on } B} [\vec{v}]_B \end{aligned}$$

$$\begin{tikzcd}
T: \mathbb{F}^n \; B \arrow[rr, "{[T(\vec{v})]_B = {}_{B_1}[I]_B [T]_{BB} [I]_{B_1} [\vec{v}]_B}"] \arrow[d] & & \mathbb{F}^n \; B \\
B_1 \arrow[rr, "\text{standard matrix of } T \text{ based } B_1"] & & B_1 \arrow[u]
\end{tikzcd}$$

proof: $\exists\, c_1 c_2 \cdots c_n \in \mathbb{F}$ s.t. $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$.

$\because T$ is linear

$$\begin{aligned} \therefore T(\vec{v}) &= T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) \\ &= c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \cdots + c_n T(\vec{v}_n) \end{aligned}$$

$$\begin{aligned} [T(\vec{v})]_B &= [c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n)]_B \\ &= c_1 [T(\vec{v}_1)]_B + \cdots + c_n [T(\vec{v}_n)]_B \\ &= [\, [T(\vec{v}_1)]_B \;\; [T(\vec{v}_2)]_B \cdots [T(\vec{v}_n)]_B \,] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\ &= [T]_B [\vec{v}]_B \end{aligned}$$

ep. $\beta = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix} \right\}$ $\quad \vec{x} = \begin{bmatrix} w_1 - w_2 \\ w_1 + w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix}$

$$[\vec{x}]_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[\mathrm{proj}_{\vec{w}}(\vec{x})]_\beta = [\mathrm{proj}_{\vec{w}}]_\beta [\vec{x}]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\therefore [\mathrm{proj}_{\vec{w}}(\vec{x})]_\beta = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

- Similarity of Matrix Representations

$T: \mathbb{F}^n \to \mathbb{F}^n$. $\quad \mathcal{B}$ & $\mathcal{C}$: ordered bases

$$[T]_C = {}_C[I]_B [T]_B \, {}_B[I]_C = ({}_B[I]_C)^{-1} [T]_B \, {}_B[I]_C$$

$[T]_B$ & $[T]_C$ are similar.

proof: $[T]_C = \big[[T(\vec{w}_1)]_C \quad [T(\vec{w}_2)]_C \quad \cdots \quad [T(\vec{w}_n)]_C \big]$

$= \big[{}_C[I]_B [T(\vec{w}_1)]_B \quad {}_C[I]_B [T(\vec{w}_2)]_B \cdots {}_C[I]_B [T(\vec{w}_n)]_B \big]$

$= {}_C[I]_B \big[[T]_B[\vec{w}_1]_B \quad [T]_B[\vec{w}_2]_B \quad \cdots \quad [T]_B[\vec{w}_n]_B \big]$ (9.1

$= {}_C[I]_B [T]_B \big[[\vec{w}_1]_B \quad [\vec{w}_2]_B \quad \cdots \quad [\vec{w}_n]_B \big]$ (def of ma- multi

$\because {}_B[I]_C = \big[[\vec{w}_1]_B \quad [\vec{w}_2]_B \quad \cdots \quad [\vec{w}_n]_B \big] \qquad {}_C[I]_B = ({}_B[I]_C)^{-1}$

$\therefore [T]_C = ({}_B[I]_C)^{-1} [T]_B \, {}_B[I]_C$

性质：① $[T]_B$ & $[T]_C$'s eig val 数与种类一致

$P_A(\lambda) = \det(A - \lambda I) = \det(CBC^{-1} - \lambda I)$
$P_B(\lambda) = \det(B - \lambda I)$

证明. $P_A(\lambda) = \det(CBC^{-1} - \lambda I)$
$= \det(CBC^{-1} - \lambda C I C^{-1})$
$= \det(CBC^{-1} - C(\lambda I)C^{-1})$
$= \det(C(BC^{-1} - \lambda I C^{-1}))$
$= \det(C(B - \lambda I)C^{-1})$
$= \det[C] \det(B - \lambda I) \det[C^{-1}]$
$= \det(B - \lambda I)$

- Finding Standard Matrix

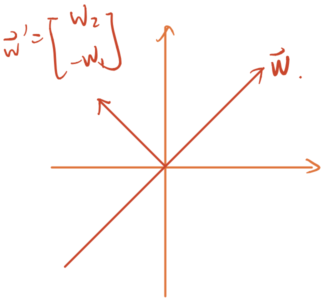
$T: \mathbb{F}^n \to \mathbb{F}^n$. $\quad B$: basis for $\mathbb{F}^n$. $\quad \mathcal{E}$: standard basis for $\mathbb{F}^n$.

$$[T]_\mathcal{E} = {}_\mathcal{E}[I]_B \, [T]_B \, {}_B[I]_\mathcal{E} = ({}_B[I]_\mathcal{E})^{-1} [T]_B \, {}_B[I]_\mathcal{E}$$

Q. $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2 \quad \vec{w} \neq \vec{0}$. $\quad \mathrm{proj}_{\vec{w}} : \mathbb{R}^2 \to \mathbb{R}^2$.

Find basis B. s.t $[\mathrm{proj}_{\vec{w}}]_B$ is diagonal

将B通过 $\mathrm{proj}_{\vec{w}}$ transform 后的 matrix 为 diagonal



① 找 standard matrix $\quad T(\vec{w}) = \vec{w} \quad T\left(\begin{bmatrix} w_2 \\ -w_1 \end{bmatrix}\right) = \vec{0}$.

$${}_{B_1}[I]_\mathcal{E} = [\,[\vec{w}]_\mathcal{E} \;\; [T(\vec{w})]_\mathcal{E}\,] = \begin{bmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{bmatrix} \quad \text{(inverse 公式)}$$

$$[T(\vec{v})]_{B_1} = [T_{B_1}]\,[\vec{v}]_{B_1}$$
$$[T(\vec{v})]_B = [T_B]\,[\vec{v}]_B$$

9.2 Diagonalization of Linear Operators

- Determine $[T]_B$.

$$[T]_\beta = \big[\, [T(\vec{v}_1)]_\beta \cdots [T(\vec{v}_n)]_\beta \,\big]$$

$T(\vec{v}_1) = \lambda\vec{v}_1 \quad \begin{bmatrix} \lambda \\ \vdots \\ 0 \end{bmatrix}$ $\qquad \vec{v}_i \in \beta. \qquad T(\vec{v}_i) = \lambda_i \vec{v}_i. \quad (\lambda_i \in \mathbb{F})$

- eig-pair of lin-op.

$T: \mathbb{F}^n \to \mathbb{F}^n$. be lin-op. $\qquad T(\underset{\substack{\uparrow \\ \text{eigenvec}}}{\vec{x}}) = \underset{\substack{\uparrow \\ \text{eigval}}}{\lambda}\vec{x}.$ of T.

- Eigenpair of T and $[T]_B$

$T: \mathbb{F}^n \to \mathbb{F}^n$

$(\lambda, \vec{x})$ is eig-pair of T $\iff$ $(\lambda, [\vec{x}]_B)$ is eig-pair of $[T]_B$

无论β的值是啥, T & $[T]_\beta$ share same "eigenstuff".

proof. $\quad T(\vec{x}) = \lambda\vec{x}$

$\iff [T(\vec{x})]_B = [\lambda\vec{x}]_B \qquad\qquad [\vec{v}]_\beta = [\vec{w}]_\beta \iff \vec{v} = \vec{w}$

$\iff [T]_\beta [\vec{x}]_\beta = \lambda [\vec{x}]_B$

- Diagonalizable

$T: \mathbb{F}^n \to \mathbb{F}^n$

$\exists B \in \mathbb{F}^n$. $[T]_B$ is diagonal matrix $\to$ T is diagonalizable over $\mathbb{F}$

eg. $T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T(\vec{x}) = \mathrm{proj}_{\vec{w}}\, \vec{x}$ diagonalizable?

Yes. choose basis: $\left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix} \right\}$

$T(\vec{w}) = \vec{w} = 1\,\vec{w} + 0\,\vec{w}_1$

$T(\vec{w}_1) = \vec{0} = 0\,\vec{w} + 0\,\vec{w}_1$

$[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \leftarrow$ diagonal

$\vec{w}$ & $\vec{w}_1$ are eigenvectors of T with eigenvalues 1 & 0.

- Eigenvector basis criterion for diagonalizability

  $T: \mathbb{F}^n \to \mathbb{F}^n$. be linear operator.

  $\exists$ ordered basis $B = \{\vec{v}_1 \cdots \vec{v}_n\}$ for $\mathbb{F}^n$ consisting of eigenvectors of T.

  $\Leftrightarrow$ T is diagonalizable over $\mathbb{F}$

proof ($\Rightarrow$) Assume T is diagonalizable

$\exists$ basis $B = \{\vec{v}_1, \cdots \vec{v}_n\}$ s.t. $[T]_B = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$

ith column of $[T]_B$: $[T(\vec{v}_i)]_B$ $= [T]_B [\vec{v}_i]_B$

$$= \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \vec{e}_i$$

$$= d_i \vec{e}_i$$

$$= d_i [\vec{v}_i]_B$$

$\therefore [\vec{v}_i]_B$ is an eigenvector of $[T]_B$

$\therefore \vec{v}_i$ is eigenvec of T

($\Leftarrow$) Assume $\exists$ a basis of eigenvectors $\beta = \{\vec{v}_1 \cdots \vec{v}_n\}$. $T(\vec{v}_i) = \lambda \vec{v}_i$

$[T(\vec{v}_i)]_\beta = [\lambda_i \vec{v}_i]_\beta = \lambda_i \vec{e}_i$

So $[T]_\beta = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is diagonalizable

- T diagonalizable $\Leftrightarrow$ $[T]_B$ diagonalizable

  $T: \mathbb{F}^n \to \mathbb{F}^n$. be linear operator. B is an ordered basis of $\mathbb{F}^n$.

  T is diagonalizable $\Leftrightarrow$ $[T]_B$ diagonalizable

proof:

($\Rightarrow$) Assume T diagonalizable

by eigenvec basis criterion for diagonalizability. $\exists$ C consist eigenvec of T.

$[T]_C = \operatorname{diag}(\lambda_1, \cdots \lambda_n)$. λ_i are eigenvals of T

$[T]_C = {}_C[I]_B [T]_{BB} [I]_C = ({}_B[I]_C)^{-1} [T]_{B\,B}[I]_C$

$\because [T]_C$ is diagonal

$\therefore [T]_B$ is diagonalizable

$\therefore P = {}_B[I]_C$ diag $[T]_B$

$(\Leftarrow)$ Assume $[T]_B$ is diag over $\mathbb{F}$.

$\therefore \exists P$. s.t $P^{-1}[T]_B P = D = \text{diag}(d_1, \cdots, d_n)$ $\quad D$ is diagonal

define $[\vec{v}_1 \cdots \vec{v}_n]$ s.t $[\vec{v}_i]_B = \vec{p}_i$. (i-th col of P)

$P = [\,[\vec{v}_1]_B \cdots [\vec{v}_n]_B\,]$.

Show $\{\vec{v}_1 \cdots \vec{v}_n\}$ is a basis & eigvec of T 即可

- Eigenvector basis criterion for diagonalizability - Matrix Version

A is diag $\Leftrightarrow \exists$ a basis of $\mathbb{F}$ 包含 eigenvectors of A.

proof. A is diag

$\Leftrightarrow [T_A]_\varepsilon$ is diag. (by T diag $\Leftrightarrow [T]_B$ diag)

$\Leftrightarrow T_A$ is diag (by eig bas criterion for diag)

$\Leftrightarrow \exists$ a basis B of $\mathbb{F}^n$ of eigvec of T_A (eig basis cri for diag)

9.3 Diagonalization and Power of Matrix

- The following are equivallent
  1. T is diagonalizable
  2. $\exists$ basis β s.t $[T]_\beta$ is diagonal
  3. $\exists$ basis β of eigenvectors of T $\quad (T(\vec{x}) = \lambda\vec{x})$
  4. $[T]_c$ is diag' $\forall c$
  5. $[T]_\varepsilon$ is diag' $\quad \beta = \{\vec{v}_1, \cdots \vec{v}_n\}$

$D = [T]_\beta = {}_\beta[I]_\varepsilon [T]_{\varepsilon\varepsilon}[I]_\beta = P^{-1}[T]_\varepsilon P$.

$P = {}_c[I]_\beta = [\vec{v}_1 \cdots \vec{v}_n]$

ep. Consider $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -2y - 3z \\ -2x - 3z \\ -2x + 2y + z \end{bmatrix}$

determine if T is diag' over $\mathbb{R}$.

$[T]_\varepsilon = \begin{bmatrix} 0 & -2 & -3 \\ -2 & 0 & -3 \\ -2 & 2 & 1 \end{bmatrix}$ $\qquad T$ is diag' $\Leftrightarrow [T]_\varepsilon$ is diag'

$[T]_\varepsilon$ was diag' with $P = \begin{bmatrix} -1 & -2 & 1 \\ -1 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$\therefore$ eig-pair of T: $\left(1, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)$ $\left(2, \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}\right)$ $\left(-2, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$

- eigenvec correspond to distinct eigval are linearly-indep.

  Let $A \in M_{n\times n}(\mathbb{F})$ have eig-pairs $(\lambda_1, \vec{v}_1)$ $(\lambda_2, \vec{v}_2)$ $\cdots$ $(\lambda_k, \vec{v}_k)$

  If $\lambda_1, \lambda_2 \cdots \lambda_k$ are all distinct, then $\{\vec{v}_1, \vec{v}_2 \cdots \vec{v}_k\}$ is linearly-indep

- algebraic multiplicity $(a\lambda_i)$

  largest pos int s.t. $(\lambda - \lambda_i)^{a\lambda_i}$ divides the char-poly $C_A\lambda_i$

  $a\lambda_i$ = # of time that λ_i repeat.

eg. $C_A(\lambda) = (\lambda-3)^4(\lambda+8)^2(\lambda-i)(\lambda+i)$

$\lambda=3. \quad \lambda=-8 \quad \lambda=i \quad \lambda=-i$

$a_3=4 \quad a_{-8}=2 \quad a_i=1 \quad a_{-i}=1$

- Geometric multiplicity (g_{λ_i})

Let λ_i be an eigval of $A \in M_{n\times n}(\mathbb{F})$

The geometric multiplicity of λ_i : $g_{\lambda_i} = \dim(E_{\lambda_i})$. $\leftarrow$ Null $(A-\lambda_i I)$

g_λ = largest num of lin-indep vectors we can get from E_{λ_i}

eg. $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & 3 \end{bmatrix} \qquad C_A(\lambda) = -(\lambda-3)^2(\lambda+3)$

$\lambda_1=3 \quad \lambda_2=-3$
$a_3=2 \quad a_{-3}=1$

$\lambda=3 \quad RREF(A-3I) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad$ basis for E_3 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\therefore$ nullity $(A-3I)=2 = g_3$.

$\lambda=-3 \quad RREF(A+3I) = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix} \qquad$ basis for E_3 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

$\therefore$ nullity $(A+3I)=1 = g_{-3}$

- Geometric & Algebraic multiplicities

$1 \le g_{\lambda_i} \le a_i$

- Diagonalizability test.

Let $A \in M_{n\times n}(\mathbb{F})$ with char-poly. $C_A(\lambda) = (\lambda-\lambda_1)^{a_{\lambda_1}} \cdots (\lambda-\lambda_k)^{a_{\lambda_k}} h(\lambda)$.

A is diagonalizable over $\mathbb{F}$ $\Leftrightarrow$ $h(\lambda)$ is constant polynomial
$a_{\lambda_i} = g_{\lambda_i}$

Q. $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & 3 \end{bmatrix}$

$C_A(\lambda) = -(\lambda-3)^2(\lambda+3)$

$\lambda = 3$ $\quad a_3 = 2$ $\quad E_3 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\quad g_3 = 2$

$\lambda = -3$ $\quad a_{-3} = 1$ $\quad E_{-3} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$ $\quad g_3 = 1$

↑ 系数

↑ span 中 vector 数

$g_i = a_i \ \forall i.$

3 eigval with repetition A is diagonal

$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ $\quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

Q. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad C_A(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \underline{\lambda^2+1} = (\lambda+i)(\lambda-i)$

↑ no real roots

$\therefore$ not diag'ble over $\mathbb{R}$.

diag'ble over $\mathbb{C}$

Q. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad C_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$

$\lambda = 1$ $\quad a_1 = 2$

$E_1 = \text{Null}(A-I)$ $\quad A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

nullity $(A-I) = 1 = g_1$

$a_1 \neq g_1 \quad \therefore A$ isn't dig'ble over $\mathbb{R}$. or $\mathbb{C}$

10.1

a non-empty set of objects V

a set of scalars (a field)

an operation which calculates 2 objects in V. (addition)

an operation which combines a object in V with a scalar from F (scalar multiplication)

$\mathbb{F}^n$ – vector

– vector space

A non-empty set of objects, $\mathbb{V}$, is a **vector space over a field, $\mathbb{F}$, under the operations of addition, $\oplus$, and scalar multiplication, $\odot$,** provided the following set of ten axioms are met.

C1. For all $\vec{x}, \vec{y} \in \mathbb{V}$, $\vec{x} \oplus \vec{y} \in \mathbb{V}$.
(Closure under Addition)

C2. For all $\vec{x} \in \mathbb{V}$ and all $c \in \mathbb{F}$, $c \odot \vec{x} \in \mathbb{V}$.
(Closure under Scalar Multiplication)

V1. For all $\vec{x}, \vec{y} \in \mathbb{V}$, $\vec{x} \oplus \vec{y} = \vec{y} \oplus \vec{x}$.
(Addition is Commutative)

V2. For all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{V}$, $(\vec{x} \oplus \vec{y}) \oplus \vec{z} = \vec{x} \oplus (\vec{y} \oplus \vec{z}) = \vec{x} \oplus \vec{y} \oplus \vec{z}$.
(Addition is Associative)

V3. There exists a vector $\vec{0} \in \mathbb{V}$ such that for all $\vec{x} \in \mathbb{V}$, $\vec{x} \oplus \vec{0} = \vec{0} \oplus \vec{x} = \vec{x}$.
(Additive Identity)

V4. For all $\vec{x} \in \mathbb{V}$, there exists a vector $-\vec{x} \in \mathbb{V}$ such that $\vec{x} \oplus (-\vec{x}) = (-\vec{x}) \oplus \vec{x} = \vec{0}$.
(Additive Inverse)

V5. For all $\vec{x}, \vec{y} \in \mathbb{V}$ and for all $c \in \mathbb{F}$, $c \odot (\vec{x} \oplus \vec{y}) = (c \odot \vec{x}) \oplus (c \odot \vec{y})$.
(Vector Addition Distributive Law)

sum of scalars

V6. For all $\vec{x} \in \mathbb{V}$ and for all $c, d \in \mathbb{F}$, $(c + d) \odot \vec{x} = (c \odot \vec{x}) \oplus (d \odot \vec{x})$.
(Scalar Addition Distributive Law)

product of scalars

V7. For all $\vec{x} \in \mathbb{V}$ and for all $c, d \in \mathbb{F}$, $(cd) \odot \vec{x} = c \odot (d \odot \vec{x})$.
(Scalar Multiplication is Associative)

V8. For all $\vec{x} \in \mathbb{V}$, $1 \odot \vec{x} = \vec{x}$.
(Multiplicative Identity)

- Vector

an element of vector space

eg. $\mathbb{R}^n$ - with usual addition and real mult

$\mathbb{C}^n$ - with usual addition and complex mult

$M_{m\times n}(\mathbb{F})$

$\oplus$ usual addition $\qquad -A = -1(A)$

$\odot$ usual scalar mult

eg. $L(\mathbb{F}^n, \mathbb{F}^m)$ $\qquad T: \mathbb{F}^n \to \mathbb{F}^m$

$(T_1 + T_2)(\vec{x}) = T_1(\vec{x}) + T_2(\vec{x})$

$(cT)(\vec{x}) = cT(\vec{x})$

$[T_1 + T_2]_\varepsilon = [T_1]_\varepsilon + [T_2]_\varepsilon$

$[cT]_\varepsilon = c[T]_\varepsilon$

additive identity is zero transformation

$T_0: \mathbb{F}^n \to \mathbb{F}^n$ $\qquad T_0(\vec{x}) = \vec{0}_{\mathbb{F}^m}$ $\qquad [T_0]_\varepsilon = O_{m\times n}$

eg. $P_n(\mathbb{F}) = \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{F}\}$

zero polynomial: $p(x) = 0$

- Create a new vec-space from an old one

$\mathbb{R}^2: \mathbb{F} = \mathbb{R}$ addition: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{bmatrix}$

$c \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 - c + 1 \\ cx_2 - c + 1 \end{bmatrix}$

$C_1 \cdot C_2$ hold $\to$ $V_1 \cdot V_2$ hold

$\vec{0}$? $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 + a - 1 \\ x_2 + b - 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad$ zero vector: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

additive inverse $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad \begin{bmatrix} x_1 + \alpha - 1 \\ x_2 + \beta - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$