Triangle inequality

$|x-y| \le |x-z| + |y-z|$

$|a+b| \le |a| + |b|$

解绝对值不等式

- 去绝对值
  - 讨论正负性 >0 / <0
  - 平方
    - $(|x|)^2 = x^2$
    - $(|a|-|b|)\times(|a|+|b|) = a^2-b^2$

算完要检查是否在区间中

$$|a|-|b| \le \left\{ \begin{matrix} |a+b| \\ |a-b| \end{matrix} \right\} \le |a|+|b|$$

解分式不等式

① 分母≠0

② 分式 ⇒ 整式（同乘分母）

Triangle inequality

- 证明 ☆ $|a| = |a+b-b|$
  $= |(a+b)+(-b)|$
  $\le |a+b| + |-b|$

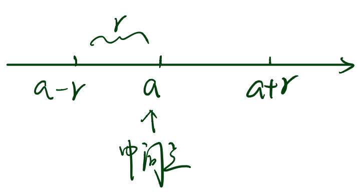
$|a+b| \le |a|+|b|$

$|x-y| \le |x-z|+|y-z|$

$2|xy| \le x^2+y^2$

$|x-y| \le |x-z| + |y-z|$

- 解不等式 $|x-a|<r$
  - 代数 $-r < x-a < r$
  - 几何



ex. $|x-1|-|x-3| \ge 5$ $(*)$

case 1. consider $x \in (-\infty, 1)$

$(*) \Rightarrow -(x-1)+(x-3) \ge 5$

$-2 \ge 5$ (impossible)

So, No solution on $(-\infty, 1)$

case 2. ……

Since $[1,3] \cap [\frac{9}{2}, \infty] = \emptyset$

No solution on $[1,3]$

……

Overall, no solution for the inequality

ex. $\dfrac{1+e^x}{1-e^x} < 2$ $(1-e^x \neq 0 \quad x \neq 0)$

case 1. $1-e^x>0$, $e^x<1$ $x<0$

$(*) \Rightarrow 1+e^x < 2(1-e^x)$

$e^x < \frac{1}{3}$

$x < \ln\frac{1}{3}$

case 2. $1-e^x<0$ …… $x > \ln\frac{1}{3}$

Overall, the solution to the inequality is $(-\infty, \ln\frac{1}{3}) \cup (0, \infty)$

ex. 证 $|a|-|b| \le |a+b|$

$|a| = |a+b-b|$

$= |(a+b)+(-b)|$

$\le |a+b| + |-b|$ (due to triangle inequality)

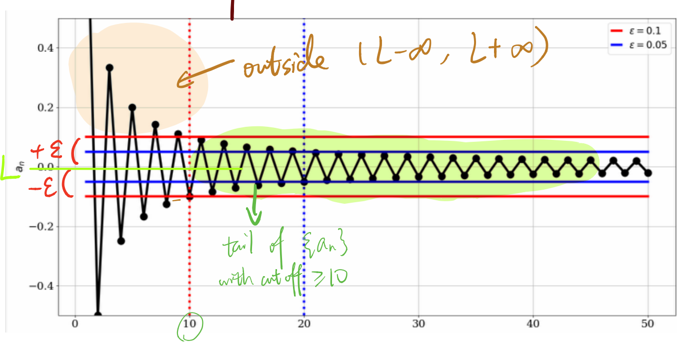
So, $|a|-|b| \le |a+b|$

ex. 解 $||x+1|<1$

法1（代数） $-1 < x+1 < 1$

法2（几何）

Sequence



- formal def. of lim I $\quad \lim_{x\to\infty} a_n = L$.

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$
s.t if $n \geq N$, then $|a_n - L| < \varepsilon$

- formal def. of lim II $\quad \lim_{x\to\infty} a_n = L$.

For $\lim_{\infty} a_n = L$. if $\forall \varepsilon > 0$.
then $(L-\varepsilon, L+\varepsilon)$ contains $\{a_n\}$

- MCT

if $\{a_n\}$
- bounded $\rightarrow$ converge to lub/glb
- not bounded $\rightarrow$ diverge to $\pm\infty$

bounded & mono $\rightarrow$ conv.

conv $\rightarrow$ bounded. (仅表示有上下限)

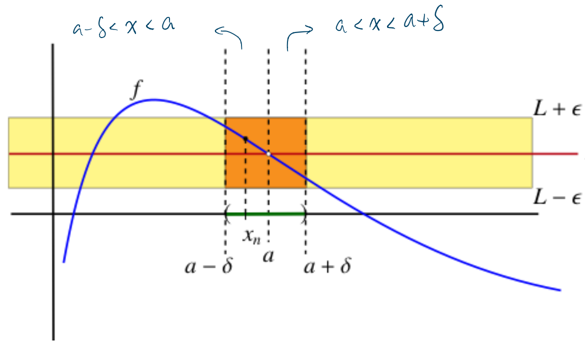
(sub seq 与原 seq 收敛性相同)

$\lim_{n\to\infty}$ 存在
- def — 求极限
- MCT — induction

$\lim_{n\to\infty}$ 不存在
- $\pm\infty$ diverge
- $\sin x$

所有不 converge 的都 diverge
$(-1)^n$ diverge. 但不 diverge to ∞

Function



$\varepsilon - \delta$ def of lim

- formal def. of lim I $\quad \lim_{x\to a} f(x) = L$

$\forall \varepsilon > 0$, $\exists \delta > 0$
s.t. if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$

- formal def of lim II. $\quad \lim_{x\to a} f(x) = \pm\infty$

$\forall M \in \mathbb{R}$. $\exists \delta > 0$
s.t. $0 < |x-a| < \delta$ $\quad f(x) > M$ $\quad$ 竖直渐近

三种竖直渐近
① 分母=0
② $\log x$ $\quad x=0$
③ $\tan x$ $\quad x=0$

证明 v.a 存在
$x \to 2$ $\quad f(x)$ ∄
$x \to 2^-$ $\quad f(x)$ ∄
$x \to 2^+$ $\quad f(x)$

- formal def of lim III. $\quad \lim_{x\to\pm\infty} f(x) = L$

$\forall \varepsilon > 0$ $\quad \exists N \in \mathbb{R}$.
s.t. $x > N$. $\quad |f(x) - L| < \varepsilon$ $\quad$ 水平渐近

- formal def of lim IV $\quad \lim_{x\to\pm\infty} f(x) = \pm\infty$

ep. $\lim_{x\to+\infty} f(x) = -\infty$
$\forall M \in \mathbb{R}$ $\quad \exists N \in \mathbb{R}$
s.t. $x > N$ $\quad f(x) < M$

- Sequential Char of lim

$\lim_{x\to a} f(x) = L \iff \lim_{n\to\infty} f(x_n) = L$.

for every $\{x_n\}$. where $\lim_{n\to\infty} x_n = a$
$x_n \neq a$.

计算

① ari R

1. For any $c \in \mathbb{R}$, $\lim\limits_{n\to\infty}(c \cdot a_n) = c \cdot \lim\limits_{n\to\infty} a_n = c \cdot L$.
2. $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = L \pm M$.
3. $\lim\limits_{n\to\infty} a_n b_n = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n = LM$.
4. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \dfrac{L}{M}$ if $M \neq 0$.
5. For any $k \in \mathbb{N}$, $\lim\limits_{n\to\infty} a_{n+k} = L$.

② Squeeze theorem

If $a_n \le b_n \le c_n$. $\lim\limits_{\infty} a_n = L = \lim\limits_{\infty} c_n$

then $\lim\limits_{n\to\infty} b_n = L$

③ alternative seq $(+,-,+,-\cdots)$ ep. $(-1)^n$

$$\frac{a_n x^n + \cdots + a_1 x + a_n}{b_m x^m + \cdots + b_1 x + b_0}$$

$$\lim_{x\to\pm\infty} a_n = \begin{cases} \text{若 } n=m & \dfrac{a_n}{b_m} \\ \text{若 } n<m & 0 \\ \text{若 } n>m & \pm\infty \end{cases}$$

$\frac{0}{0}$ undef $\quad \frac{c}{0} \to \pm\infty \quad \frac{\pm\infty}{0} \to \pm\infty$

$\frac{0}{c} \to 0 \quad \frac{c}{c} \quad \frac{\pm\infty}{0} \to \pm\infty$

$\frac{0}{\pm\infty} \to 0 \quad \frac{c}{\pm\infty} \to 0 \quad \frac{\pm\infty}{\pm\infty}$ undef

计算

① ari R

If $\lim\limits_{x\to a} f(x) = L$, $\lim\limits_{x\to a} g(x) = M$ **both exist**, then:

a). $\lim\limits_{x\to a}[c \cdot f(x)] = c \cdot \lim\limits_{x\to a} f(x) = c \cdot L$, where $c \in \mathbb{R}$

b). $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = L \pm M$

c). $\lim\limits_{x\to a}[f(x) \cdot g(x)] = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = L \cdot M$

d). $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{L}{M}$, if $M \neq 0$

② Squeeze theorem

If $h(x) \le f(x) \le g(x)$ $(x\to a)$ $\lim\limits_{x\to a} h(x) = L = \lim\limits_{x\to a} g(x)$

then $\lim\limits_{x\to a} f(x) = L$

③ Fundamental trigonometric & log limit

$\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1 \qquad \lim\limits_{x\to\infty} \dfrac{\ln x}{x} = 0$

$\lim\limits_{x\to\infty} \dfrac{\ln(x^p)}{x} = 0 \qquad \lim\limits_{x\to\infty} \dfrac{\ln x}{x^p} = 0 \quad p>0$

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_n}{b_m x^m + \cdots + b_1 x + b_0}$$

$$\lim_{x\to\pm\infty} f(x) = \begin{cases} \text{若 } n=m & \dfrac{a_n}{b_m} \\ \text{若 } n<m & 0 \\ \text{若 } n>m & \text{DNE} \end{cases}$$

$$\lim_{x\to a} f(x) = \frac{a_n x^n + \cdots + a_1 x + a_n}{b_m x^m + \cdots + b_1 x + b_0}$$

分解因式
上下约掉

for $p>0$,

$(\ln x)^p << x^p << p^x << x^x$

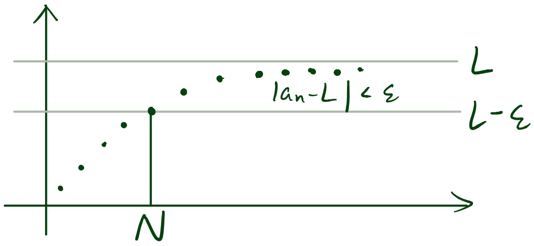
$(\text{as } x \to \infty)$

* 证 $\lim_{x\to\infty} a_n = L$ converge

① $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $n \geq N$ $\left|\frac{1}{n^2} - 0\right| < \varepsilon$

② 草稿 $|a_n - L| < \varepsilon$ $n > \cdots$ $N > \cdots$

③ Proof Let $\varepsilon > 0$
choose $N \in \mathbb{N}$ $N > \_\_\varepsilon$
assume $n \geq N$
$|a_n - L| = \cdots n \leq \cdots N < \cdots \varepsilon = \varepsilon$

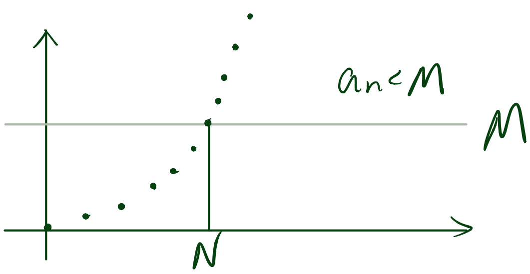


* 证 $\lim_{n\to\infty} a_n = \infty$ diverge

① $\forall M < 0$ $\exists N \in \mathbb{N}$ s.t $n \geq N$ $f(x) > M$

② 草稿 $a_n < M$ $N > \cdots M$

③ Proof. Let $M > 0$
choose $N \in \mathbb{N}$ $N > \_\_M$
assume $n \geq N$
$a_n < \cdots N < M$



* 证 $\{d_n\}$ converges & find lim

① 证 mono & bounded
b.c:
IH:
IS:

② by MCT, mono + bound = conv.

③ 求 limit
$\lim_{\infty} d_n = \lim_{\infty} d_{n+1} = L$

$L = \cdots$

* 证 $\lim_{x\to a} f(x) = L$ ε-δ def of lim

① $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $0 < |x - a| < \delta$ $|f(x) - L| < \varepsilon$

② 草稿 $|f(x) - L| < \varepsilon$ $\underbrace{|x-a|}_{\delta} < ?$

③ Proof. Let $\varepsilon > 0$.
choose $\delta = ?$
assume $0 < |x - a| < \delta$.
$|f(x) - L| = \cdots < \cdots \delta = \varepsilon$

* 证 $\lim_{x\to\pm\infty} f(x) = L$

* 证 $\lim_{x\to\pm\infty} f(x) = \pm\infty$

* 证 limit 不存在.

① Sequential Char of Lim

	1	-1
sin	$\frac{\pi}{2}$	$\frac{3\pi}{2}$
cos	π	2π

ep. 证 $\lim_{x\to 3} \frac{1}{x-3}$ DNE.

→ 找 seq $\{x_n\}$. $\lim_{n\to\infty} \{x_n\} = 3$.
↳ $3 + \frac{1}{n}$ / 有 sin/cos 选 π.

→ $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} n = \infty$
$\therefore \lim_{x\to 3} \frac{1}{x-3}$ DNE

② $\lim_{x\to a^+} \neq \lim_{x\to a^-}$

③ $\lim_{x\to a^+}$ / $\lim_{x\to a^-}$ / $\lim_{x\to a}$ DNE

even function $f(x) = f(-x)$
odd function $f(x) = -f(-x)$

Continuity

- Formal def of cont.

If $\lim_{x\to a} f(x) = f(a)$, then $f(x)$ is cont. at $x=a$

$\forall \varepsilon > 0 \quad \exists \delta > 0$ ← cut off.
s.t. $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

☆ $x=a \quad L=f(a)$

- ari R.

Let f and g be continuous at $x = a$, then

↳ Continuity of Sums: $f \pm g$ is continuous at $x = a$

↳ Continuity of Products: $f \cdot g$ is continuous at $x = a$

↳ Continuity of Quotients: $\frac{f}{g}$ is continuous at $x = a$ if $g(a) \neq 0$

- Cont composition.

若 $\lim_{x\to a} g(x) = L$. f cont at L.

$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(L)$

- lim & cont. of composition

从内到外

若 $\lim_{x\to a} f(x) = L$ $(f(x) \neq L)$. 且 $\lim_{x\to L} g(x) = M$

则 $\lim_{x\to a} g(f(x)) = M$

- IVT

If f cont at $[a,b]$ $\quad f(a) \lessgtr \alpha \lessgtr f(b)$

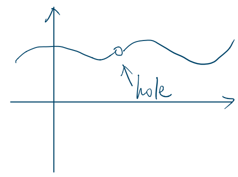
then $\exists c \in (a,b)$ s.t. $f(c) = \alpha$

- EVT

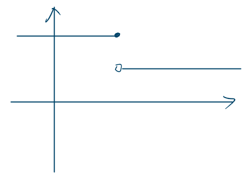
If f cont at $[a,b]$ $\quad \exists c_1 \,\&\, c_2 \in [a,b]$

then $f(c_1) \le f(x) \le f(c_2) \quad \forall x \in [a,b]$

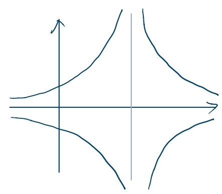
discontinuity



removable discontinuity

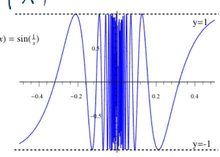


finite jump



vertical asymptote

sin(1/x)



oscillatory discontinuous

- 区间是否 continuity

① (a,b) open

$\forall x_0 \in (a,b) \quad \lim_{x\to x_0} f(x) = f(x_0)$

② $[a,b]$ close

$\forall x_0 \in (a,b) \quad \lim_{x\to x_0} f(x) = f(x_0)$

$\lim_{x\to a^+} f(x) = f(a)$

$\lim_{x\to b^-} f(x) = f(b)$

sin (a+b)
cos (a+b)

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

Derivative

limit def of derivatives

- def. $f'(a) = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h} = \lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

  tangent line $y = f(a) + f'(a)(x-a)$

- local extrema theorem.

  If ① c is local max/min ② $f'(c)$ exist.

  then $f'(c)=0$.

  critical point : $f'(x)$ DNE / $=0$

$\boxed{\begin{array}{l} \sec x = \dfrac{1}{\cos x} \\ \csc x = \dfrac{1}{\sin x} \\ \cot x = \dfrac{1}{\tan x} \end{array}}$

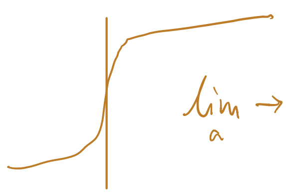
- MVT ☆

  If $f(x)$ satisfies

  ① $f(x)$ cont. for $x \in [a,b]$
  ② $f'(x)$ exists for $x \in (a,b)$

  Then there exists $c \in (a,b)$. s.t $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

连续
可导
斜率相同

* 

$\lim\limits_{a} \to \infty$ not differentiable

→ 计算

$$\log_a x = \frac{\ln x}{\ln a}$$

一找近似根

① linear approximation

联系

$L_a^f(x) = f(a) + f'(a)(x-a)$ 切线函数

$\text{error} = \left| f(x) - L_a^f(x) \right|$

upper bound of error: $\left| f(x) - L_a^f(x) \right| \le \frac{M}{2}(x-a)^2$

M: $f''(x)$ in 最大值

$(x-a)^2$: x 区间中最大值

② Newton's Method

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ x_1 需自己估算

一求导

① Chain Rule

② 三角函数 / exponential function

③ IFT ☆

$f(x)$ 满足 invertible on $[c,d]$. diff'ble on (c,d) $f'(a) \neq 0$

$f(a) = b$.

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

$$y = f^{-1}(b) + \frac{1}{f'(a)}(x-b) \qquad (L_a^f)^{-1}(x) = L_{f(a)}^{f^{-1}}(x)$$

↳ 求 $(\arcsin)'$ $(\arccos)'$ $(\arctan)'$

④ Implicit differentiation

两边加 $\frac{d}{dx}(\ldots)$ → 找 $\frac{dy}{dx}$ → x 代入 y.

* $y = x^{f(x)}$ → 同时加 ln. 再求导

⑤ L'HR. * 有时无法使用，需代入原式

若 $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0} = \frac{\infty}{\infty}$，则 $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

$= 0\cdot\infty \,/\, 0\cdot 0$ 变为除法

$= \infty - \infty$ 用 log / ln / 三角函数

$= 0^0 \,/\, \infty^0 \,/\, 1^\infty$ $e^{\ln f(x)}$

→ 证明

① arithmetic rule of diff' $(+ - \times \div)$ → derivative 定义

② $\sin x$ $\cos x$ e^x in f' → derivative 定义

→ 已知 $(\sin x)' = \cos x$. 用 ari rule

③ converge / diverge → Newton's method

④ Show 有且只有1个 root. → MVT + IVT

- assume there are 2 roots $x_1 . x_2$ $x_1 \neq x_2$

$f(x_1) = f(x_2) = 0$

$\because f'(x)$ cont & diff'ble

$\therefore$ By MVT $f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = 0$

- $f'(x) = \cdots > 0$ $\therefore$ 不存在2个根

- 找 $f(a) > 0$ $f(b) < 0$

$\because f(x)$ cont

$\therefore$ Due to IVT，存在一个根

函数图像特征

一阶导 → 斜率

- $f'(x)>0$ inc ↗
  $f'(x)<0$ dec ↘

条件: $f'(x)$ 存在 . $f'(x)=0$

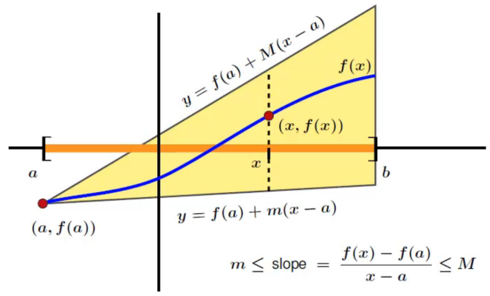
non-dec . $f'(x)\geq 0$

non-inc . $f'(x)\leq 0$

- BDT. 条件: ① cont. ② diff'ble

$$f(a)+m(x-a) \leq f(x) \leq f(a)+M(x-a)$$

m: $f'(x)$ min　　M: $f'(x)$ max



二阶导 → 开口朝上/下

- $f''>0$ concave up ∪
  $f''<0$ concave down ∩

点.

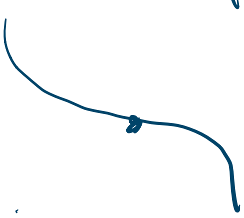
① critical point $f'(c)=0$ / DNE

② inflection point. $f''(c)=0$

def. cont & concavity change

③ global max/min $f'(c)=0$ 的点

或 $[a,b]$ 中的 a & b.



ex. curve sketching

① f 的取值范围

② 找 x & y 轴交点

③ 渐近线

H.A. $\lim_{x\to\pm\infty} f(x)$

V.A. 分母$=0$ / $\ln 0$

④ 求 $f'(x)$. → CP

⑤ 求 $f''(x)$ → POI

⑥ 以下所有的 x.

f''

f'

f

shape

Taylor's Theorem

几阶导　x 坐标

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$0 < k < n$ $\qquad T_{k,a}(a) = f^{(k)}(a)$

$$R_{n,a}(x) = f(x) - T_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$|f^{(n+1)}(c)| \le M$

$$\text{error} = |R_{n,a}(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$