MATH 237 Cheatsheet

Scalar functions

Level curves

Level curve $L = k$ of function $f(x, y)$ is the set of points x, y which satisfy $f(x, y) = k$. Can be generalized to level surfaces or level sets.

Cross sections

Cross section of a surface $z = f(x, y)$ is the intersection of $z = f(x, y)$ with a plane.

Limits

Neighborhood

r-neighborhood of point (a, b) is the set of points within distance r from (a, b):

$$N_r(a, b) = \{(x, y) \mid \|(x, y) - (a, b)\| < r\}$$

Limit

$\lim_{(x,y)\to(a,b)} f(x, y) = L$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(x, y) \in N_\delta(a, b)$,

$$|f(x) - L| < \varepsilon$$

- To prove a limit does not exist, show that there are two paths to (a, b) which results in different limits.

Squeeze theorem

If there exists a function $B(x, y)$ such that

$$|f(x, y) - L| \leq B(x, y) \quad \forall (x, y) \neq (a, b)$$

in some neighborhood of (a, b), and

$$\lim_{(x,y)\to(a,b)} B(x, y) = 0$$

Then, $\lim_{(x,y)\to(a,b)} f(x, y) = L$.

Linear approximation

Partial derivatives

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

- Variables other than the differentiated one is held constant.
- Clariaut's theorem: $f_{xy}(a, b) = f_{yx}(a, b)$ if they are continuous.

Hessian matrix

$$Hf(x, y) = \begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix}$$

Tangent plane/linearization

$$L_{a,b}(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Gradient

$$\nabla f(x, y) = \begin{bmatrix} f_x(x, y) & f_y(x, y) \end{bmatrix}$$

Second degree Taylor polynomial

$$P_{2,(a,b)}(x, y) = L_{a,b}(x, y) + \frac{1}{2}\big(f_{xx}(a, b)(x - a)^2$$
$$+ 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\big)$$

Taylor's theorem

$$f(x, y) = f(a, b) + L_{a,b}(x, y) + R_{1,(a,b)}(x, y)$$

where

$$R_{1,(a,b)}(x, y) = \frac{1}{2}\big(f_{xx}(x - a)^2$$
$$+ 2f_{xy}(x - a)(y - b) + f_{yy}(y - b)^2\big)$$

- If $f(x, y) \in C^2$ around (a, b), then there exists $M > 0$ such that $\left|R_{1,(a,b)}(x, y)\right| \leq M\|(x, y) - (a, b)\|^2$

Derivatives

Differentiable

$f(x, y)$ is differentiable at (a, b) if

$$\lim_{(x,y)\to(a,b)} \frac{\left|R_{1,(a,b)}(x, y)\right|}{\|(x, y) - (a, b)\|} = 0$$

where $R_{1,(a,b)}(x, y) = f(x, y) - L_{a,b}(x, y)$

- Linear approximation is only reasonable if f is differentiable at the point.

Mean value theorem

If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

Basic chain rule

Let $g(t) = f(x(t), y(t))$. Then,

$$g'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)$$
$$= \nabla f \cdot \frac{\partial(x, y)}{\partial t}$$

provided that f is differentiable at $(x(t), y(t))$ and $x(t)$ and $y(t)$ are differentiable at t.

Directional derivative

$$\mathrm{D}_{\vec{u}} f(a, b) = \frac{\mathrm{d}}{\mathrm{d}s} f(a + su_1, b + su_2)\Big|_{s=0}$$
$$= \nabla f(a, b) \cdot \hat{u}$$

- For gradient case f must be differentiable at (a, b).
- $\nabla f(a, b)$ is the direction of the greatest rate of change.

Optimization problems

Critical point

(a, b) is a critical point of $f(x, y)$ if $\frac{\partial f}{\partial x}(a, b) = 0$ or does not exist, and $\frac{\partial f}{\partial y}(a, b) = 0$ or does not exist.

- Find using first derivative.

Second derivative test

Let (a, b) be a critical point.

- If $Hf(a, b)$ is positive definite ($\det(Hf(a, b)) > 0$ and $a, b > 0$) then (a, b) is a local minimum.
- If $Hf(a, b)$ is negative definite ($\det(Hf(a, b)) > 0$ and $a, b < 0$) then (a, b) is a local maximum.
- If $Hf(a, b)$ is indefinite ($\det(Hf(a, b)) < 0$) then (a, b) is a saddle point.
- If $Hf(a, b)$ is semidefinite ($\det(Hf(a, b)) = 0$) then (a, b) is degenerate.

Algorithm for extreme values

To find max/min value on closed and bounded region S,

1. Find all critical points contained in S.
2. Find max/min points on the boundary of S.
3. Max/min point is on one of the points found in the previous steps.

Lagrange multiplier algorithm

To find max/min value given constraint $g(x, y) = k$, find all points satisfying

1. $\nabla f(a, b) = \lambda \nabla g(a, b)$ for some λ and $g(a, b) = k$.

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(a, b) = k$$

2. $\nabla g(a, b) = 0$ and $g(a, b) = k$.
3. Endpoint on curve $g(x, y) = k$.

Coordinate systems

Polar coordinates

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$

- $\frac{\partial(x,y)}{\partial(r,\theta)} = r$
- Cylinderical: add $z = z$

Spherical coordinates

$$x = \rho\sin(\varphi)\cos(\theta)$$
$$y = \rho\sin(\varphi)\sin(\theta)$$
$$z = \rho\cos(\varphi)$$
$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\cos\varphi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$
$$\tan\theta = \frac{y}{x}$$

- $\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \rho^2\sin(\varphi)$

Trig identities

- $\sin^2 x + \cos^2 x = 1$
- $\cos^2 x - \sin^2 x = \cos(2x)$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$

Mappings

Jacobian

$$\frac{\partial(u, v)}{\partial(x, y)} = \det(\mathrm{D}F(x, y)) = \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

- Inverse mapping theorem: if F has continuous partial derivatives on D_{xy} and is invertible with continuous partial derivatives on D_{uv}, then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(x,y)}{\partial(u,v)}^{-1}$.

Change of variable theorem

$$\iiint_{D_{xyz}} dV = \iiint_{D_{uvw}} \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dV$$