MATH 235 Cheatsheet

Abstract Vector Space

Vector space

A vector space over $\mathbb{F}$ has operations

- $+ : V \times V \to V$ (addition)
- $\cdot : \mathbb{F} \times V \to V$ (scalar multiplication)

and satisfies the axioms

- $\forall \vec{x}, \vec{y}, \vec{z}, \vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (associativity of addition)
- $\exists \vec{0}, \forall \vec{y}, \vec{0} + \vec{y} = \vec{y}$ (additive identity)
- $\forall \vec{x}, \exists -\vec{x}, \vec{x} + (-\vec{x}) = 0$ (additive inverse)
- $\forall \vec{x}, \vec{y}, \vec{x} + \vec{y} = \vec{y} + \vec{x}$ (commutativity of addition)
- $\forall \vec{x}, s, t, s(t\vec{x}) = (st)\vec{x}$ (associativity of scalar multiplication)
- $\forall \vec{x}, s, t, (s + t)\vec{x} = s\vec{x} + t\vec{x}$ (commutativity of scalar multiplication)
- $\forall \vec{x}, \vec{y}, s, s(\vec{x} + \vec{y}) = s\vec{x} + s\vec{y}$ (distributivity of scalar multiplication over vector addition)
- $\forall \vec{x}, 1\vec{x} = \vec{x}$ (multiplicative identity)

Properties of vector spaces:

- Zero vector $\vec{0}$ is unique
- Additive inverse is unique

Subspace

U is a subspace of V by **subspace test** if

- U is nonempty, e.g. $\vec{0} \in U$
- $\forall \vec{x}, \vec{y} \in U, \vec{x} + \vec{y} \in U$ (closed under addition)
- $\forall \vec{x} \in U, s \in \mathbb{F}, s\vec{x} \in U$ (closed under scalar multiplication)

Span

Given $S = \{\vec{v}_1, ..., \vec{v}_k\}$,

$$\mathrm{Span}(S) = \{t_1\vec{v}_1 + ... + t_k\vec{v}_k : t_1, ..., t_k \in \mathbb{F}\}$$

- S is a subspace of V.

Linear independence

Set of vectors S is LI iff

$$t_1\vec{v}_1 + ... + t_k\vec{v}_k = 0 \Rightarrow t_1, ..., t_k = 0$$

- Empty set is LI.
- S is LI iff no vector in S is a linear combination of other vectors in S.

Basis

A basis $\mathcal{B}$ for V is a LI set of vectors that spans V.

- $\forall \vec{v}, \exists! x_1, ..., x_n \in \mathbb{F}$,

$$\vec{v} = x_1\vec{b}_1 + ... + x_n\vec{b}_n$$

(unique representation theorem)

Dimension

- $\dim(V) = |\mathcal{B}|$ for any basis $\mathcal{B}$.
- $|S| > \dim(V) \Rightarrow S$ is LD.
- $|S| < \dim(V) \Rightarrow S$ cannot span V.
- $|S| = \dim(V) \Rightarrow S$ spans V iff S is LI.
- W is a subspace of $V \Rightarrow \dim(W) \le \dim(V)$.

Coordinate

Given $\mathcal{B} = \left\{\vec{b}_1, ..., \vec{b}_n\right\}$ and

$$\vec{x} = x_1\vec{b}_1 + ... + x_n\vec{b}_n$$

The coordinate vector w.r.t. $\mathcal{B}$ is

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- $[\vec{x}]_{\mathcal{B}}$ is linear over addition and scalar multiplication.

Linear Transformation

LT between abstract vectors

$L : V \to W$ is a linear transformation if

- $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$
- $L(t\vec{x}) = tL(\vec{x})$

Properties:

- $L(\vec{0}) = \vec{0}$

Rank and nullity

Given $L : V \to W$,

- $\mathrm{range}(L) = \{L(\vec{x}) : \vec{x} \in V\} \subseteq W$
- $\ker(L) = \left\{\vec{x} \in V : L(\vec{x}) = \vec{0}\right\} \subseteq V$
- $\mathrm{rank}(L) = \dim(\mathrm{range}(L))$
- $\mathrm{nullity}(L) = \dim(\ker(L))$
- $\mathrm{rank}(L) + \mathrm{nullity}(L) = \dim(V)$ (**rank-nullity theorem**)

LT matrix

Matrix representation of L:

$$[L(\vec{v})]_{\mathcal{C}} = A[\vec{v}]_{\mathcal{B}}$$

- $\vec{v} \in \mathrm{range}(L) \Rightarrow [\vec{v}]_{\mathcal{B}} \in \mathrm{Col}(L)$
- $\vec{w} \in \ker(L) \Rightarrow [\vec{v}]_{\mathcal{B}} \in \mathrm{Null}(L)$

Change of base matrix

$${}_{\mathcal{C}}[L]_{\mathcal{B}} = \left[\left[L\left(\vec{b}_1\right)\right]_{\mathcal{C}} \ \cdots \ \left[L\left(\vec{b}_n\right)\right]_{\mathcal{C}}\right]$$

- ${}_{\mathcal{C}}[\mathrm{id}]_{\mathcal{B}} = {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{C}}^{-1}$
- ${}_{\mathcal{C}}[L]_{\mathcal{B}}^{-1} = {}_{\mathcal{B}}[L^{-1}]_{\mathcal{C}}$ given L is invertible

Isomorphism

L is injective if

- $L(\vec{v}_1) = L(\vec{v}_2) \Rightarrow \vec{v}_1 = \vec{v}_2$.
- $\ker(L) = \left\{\vec{0}\right\}$.
- $\mathrm{nullity}(L) = 0$

L is surjective if

- $\mathrm{range}(L) = W$.

- $\text{rank}(L) = \dim(W)$

L is an isomorphism if

- L is injective and surjective.
- $V \cong W$ (V is isomorphic to W)
- If $\{\vec{v}_1, ..., \vec{v}_n\}$ is a basis for V, then $L(\vec{v}_1), ..., L(\vec{v}_n)$ is a basis for W.

Additional properties:

- $\dim(V) < \dim(W) \Rightarrow L$ cannot be surjective
- $\dim(V) > \dim(W) \Rightarrow L$ cannot be injective
- $\dim(V) = \dim(W) \Rightarrow L$ is injective iff L is surjective
- $V \cong W$ iff $\dim(V) = \dim(W)$
- L is an isomorphism iff there exists $L^{-1} : W \to V$ such that $LL^{-1} = L^{-1}L = \text{id}$

Diagonalization

Eigenvalue and eigenvector

$\lambda, \vec{v}$ such that

$$L(\vec{v}) = A\vec{v} = \lambda\vec{v}$$

Eigenspace

$$E_\lambda = \{\vec{v} \in V : A\vec{v} = \lambda\vec{v}\} = \text{Null}(A - \lambda I)$$

Diagonalization algorithm

1. Find λ as roots of characteristic polynomial
$$C_A = \det(A - \lambda I) = 0$$
2. For each λ, find eigenvectors $\vec{v}$ by solving
$$(A - \lambda I)\vec{v} = 0$$
3. Take $D = \text{diag}(\lambda_1, ..., \lambda_n)$, $P = [\vec{v}_1 \ \cdots \ \vec{v}_n]$ which satisfies $A = PDP^{-1}$

Similar

A and B are similar if there exists P such that $B = P^{-1}AP$. They also have the same:

- characteristic polynomial
- trace
- determinant
- eigenvalues
- rank and nullity

Diagonalizability test

A is diagonalizable iff forea each λ, algebraic multiplicity of λ = geometric multiplicity of λ.

Inner product spaces

Inner product

A function $\langle \vec{v}, \vec{w} \rangle$ such that

- $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}$
- $\langle \alpha\vec{v}, \vec{w} \rangle = \alpha\langle \vec{v}, \vec{w} \rangle$
- $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
- $\langle \vec{v}, \vec{v} \rangle \geq 0$
- $\langle \vec{v}, \vec{v} \rangle = 0 \Rightarrow \vec{v} = \vec{0}$

Additional properties:

- $\langle 0, \vec{v} \rangle = \langle \vec{v}, 0w_i \rangle = 0$
- $\langle \vec{v}, \alpha\vec{w} \rangle = \overline{\alpha}\langle \vec{v}, \vec{w} \rangle$
- Any finite-dimensional vector space has an inner product.

Examples of inner products

- Standard inner product on $\mathbb{C}^n$
$$\langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} v_i \overline{w_i}$$
- L^2 inner product on $\mathcal{C}[a, b]$
$$\langle p, q \rangle = \int_a^b p(x)\overline{q(x)}\,\mathrm{d}x$$
- Frobenius inner product on $\mathbb{M}^{m \times n}(\mathbb{C})$
  - Elementwise standard inner product
$$\langle A, B \rangle = \text{tr}(B^*A) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}\overline{B_{ij}}$$

Norm

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

- $\langle \vec{v}, \vec{w} \rangle = 0 \Rightarrow \vec{v}$ perp $\vec{w}$ (Orthogonal)
- $\vec{v} \perp \vec{w} \implies \|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$ (Pythagorean theorem)
- $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\|\|\vec{w}\|$ (Cauchy-Schwarz inequality)
- $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ (Triangle inequality)

Angle

$$\theta = \cos^{-1}\left(\frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\|\|\vec{w}\|}\right)$$

Distance

$$\text{dist}(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$$

- $\text{dist}(\vec{v}, \vec{w}) \geq 0$
- $\text{dist}(\vec{v}, \vec{w}) = 0$ iff $\vec{v} = \vec{w}$
- $\text{dist}(\vec{u}, \vec{w}) \leq \text{dist}(\vec{u}, \vec{v}) + \text{dist}(\vec{v}, \vec{w})$ (Triangle inequality)

Orthonormal basis

- $\{\vec{v}_1, ..., \vec{v}_k\}$ is orthogonal if $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for $i \neq j$
- It is orthornormal if additionally $\|\vec{v}_i\| = 1$ for all i
- It is an orthonormal basis if it is orthogonal and is a basis for V
- Every finite-dimensional inner product space has an orthonormal basis

Projection

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{w}\|^2}\vec{w}$$

$$\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{w}}(\vec{v})$$

$$\text{proj}_W(\vec{v}) = \text{proj}_{\vec{w}_1}(\vec{v}) + \cdots + \text{proj}_{\vec{w}_k}(\vec{v})$$

Gram-Schmidt orthogonalization

Given basis $\{\vec{v}_1, ..., \vec{v}_n\}$,

$$
\begin{aligned}
\vec{w}_1 &= \vec{v}_1 \\
\vec{w}_2 &= \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2} \vec{w}_1 \\
\vec{w}_3 &= \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{w}_1 \rangle}{\|\vec{w}_1\|^2} \vec{w}_1 - \frac{\langle \vec{v}_3, \vec{w}_2 \rangle}{\|\vec{w}_2\|^2} \vec{w}_2 \\
&\dots
\end{aligned}
$$

Orthogonal complement

Given subspace $W \subseteq V$,

$$W^{\perp} = \{\vec{v} \in V : \langle \vec{v}, \vec{w} \rangle = 0 \ \forall \vec{w} \in W\}$$

- $W^{\perp}$ is a subspace of V
- $\dim(W) + \dim(W^{\perp}) = \dim(V)$
- $W \cap W^{\perp} = \{\vec{0}\}$
- $(W^{\perp})^{\perp} = W$

Orthogonal projection

Given orthogonal basis $W = \{\vec{w}_1, \dots, \vec{w}_k\}$,

$$\vec{v} = \text{proj}_W(\vec{v}) + \text{perp}_W(\vec{v})$$

Least squares solution

Given system $A\vec{x} = \vec{b}$, $\vec{s}$ is a least squares solution to the system iff $A^{\top} A \vec{x} = A^{\top} \vec{b}$.

Unitary diagonalization

Adjoint

$$A^* = \overline{A^{\top}}$$

- $\langle \vec{v}, \vec{w} \rangle = \vec{w}^* \vec{v}$
- $\langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A^* \vec{w} \rangle$

Unitary/orthogonal matrix

$U \in \mathcal{M}_{n\times n}(\mathbb{F})$ is unitary if $U^* = U^{-1}$. If $\mathbb{F} = \mathbb{R}$ U is also orthogonal ($U^{\top} = U^{-1}$).

- $\langle U\vec{v}, U\vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle$
- $\|U\vec{v}\| = \|\vec{v}\|$
- Columns and rows of U form orthonormal bases.

Schur's triangularization theorem

For any matrix A there exists a unitary matrix U such that $T = U^* A U$ is upper triangular with eigenvalues of A on the diagonal.

- Any matrix over $\mathbb{C}$ is similar to an upper triangular matrix.

Cayley-Hamilton theorem

Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Then, $C_A(A) = 0$.

- $c_0 I + c_1 A + \dots + c_n A^n = 0$

Spectral theorem

- A is normal ($AA^* = A^*A$) iff A is unitarily diagonalizable.
- A is self-adjoint ($A = A^*$) iff A is unitarily diagonalizable with real eigenvalues.
- A over $\mathbb{R}$ is symmetric ($A = A^{\top}$) iff A is orthogonally diagonalizable.

Properties of normal matrices

Let A be normal.

- $\|A\vec{x}\| = \|A^*\vec{x}\|$
- If $(\lambda, \vec{x})$ is an eigenpair of A, then $(\overline{\lambda}, \vec{x})$ is an eigenpair of A^*.
- If $\vec{x}, \vec{y}$ are eigenvectors of A with different eigenvalues, then $\vec{x}$ and $\vec{y}$ are orthogonal.

Unitary diagonalization algorithm

1. Diagonalize $A = PDP^{-1}$ as usual.
2. Perform Gram-Schmidt procedure on the columns of P and obtain an orthogonal matrix U.
3. Result is $A = UDU^*$.

Quadratic form

Given $\vec{u} = (u_1, \dots, u_n)$, the quadratic form is a polynomial of form

$$Q(\vec{u}) = \sum_{i,j=1}^{n} a_{ij} u_i u_j$$

- Positive definite: $Q(\vec{u}) > 0 \ \forall \vec{u} \neq 0$.
- Positive semidefinite: $Q(\vec{u}) \geq 0 \ \forall \vec{u}$.
- Negative definite: $Q(\vec{u}) < 0 \ \forall \vec{u} \neq 0$.
- Negative semidefinite: $Q(\vec{u}) \leq 0 \ \forall \vec{u}$.
- Indefinite: $\exists \vec{u}, \vec{v}$, $Q(\vec{u}) > 0$, $Q(\vec{v}) < 0$

Singular value decomposition

Singular values and singular vectors

Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. The singular values of A are $\sigma_i = \sqrt{\lambda_i}$ for each eigenvalue λ_i of A^*A. The singular vectors are the eigenvectors of A^*A.

- A^*A is self-adjoint.
- $A^*A\vec{x} = \|A\vec{x}\|^2$.
- $\text{Null}(A^*A) = \text{Null}(A)$.
- $\text{Col}(A)^{\perp} = \text{Null}(A^*)$

SVD algorithm

Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ be a matrix with rank r.

1. Find eigenpairs $(\lambda_i, \vec{v}_i)$ for A^*A, where $\vec{v}_i$ is normalized.
2. Set $\sigma_i = \sqrt{\lambda_i}$ for $i \leq r$, generally ordered from largest to smallest.
3. Set $\vec{u}_i = A\frac{\vec{v}_i}{\sigma_i}$ for $i \leq r$.
4. If $r < m$, extend $\{\vec{u}_1, \dots, \vec{u}_r\}$ to an orthonormal basis of $\mathbb{F}^m$. Use Gram-Schmidt procedure.
5. Set $V = \{\vec{v}_1, \dots, \vec{v}_r\}$, $U = \{\vec{u}_1, \dots, \vec{u}_r\}$, $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r)$

This results in

$$A = U\Sigma V^*$$

- Compact SVD: remove zero rows/columns of Σ and columns above r of U and V.

Low rank approximation

Rank-k approximation of $A = U_r \Sigma_r V_r^*$ is

$$A_k = \sum_{i=1}^{k} \sigma_i \vec{u}_i \vec{v}_i^*$$

- $A = A_r$
- $\operatorname{rank}(A) = r$
- $\|A - A_k\| = \sqrt{\sum_{i=k+1}^{r} \sigma_i^2} \leq \sum_{i=k+1}^{r} \sigma_i$ where $\|X\|$ is the Frobenius norm.
- Eckert-Young theorem: $\|A - B\| \geq \|A - A_k\|$
  - A_k is the best rank-k approximation of A.

Pseudoinverse

Given $A = U \Sigma V^*$, the pseudoinverse of A is

$$A^\dagger = V \Sigma^\dagger U^*$$

where $\Sigma^\dagger$ is diagonal with $\Sigma_i^\dagger = \begin{cases} \sigma_i^{-1} & \Sigma_i \neq 0 \\ 0 & \Sigma_i = 0 \end{cases}$.

- Defined for matrices of any shape.
- If A is square and invertible, then $A^\dagger = A^{-1}$.
- Results in minimal norm solution

Minimal norm solution

Given system $A\vec{x} = \vec{b}$, the minimal norm solution is $\vec{x}_0 = A^\dagger \vec{b}$.

- If the system is consistent, then x_0 is the solution such that $\|\vec{x}\| \geq \|\vec{x}_0\|$
- If the system is inconsistent, then x_0 is the least squares solution.