1 FPNS

1.1 Representation

$$\pm 0.d_1 d_2 \dots d_t \times \beta^p; d_1 \geq 1; p \in [L, U]$$

1.2 Error

$$E_{\text{rel}} = \frac{|\hat{x} - x|}{|x|} \qquad E_{\text{machine}} = \left[\frac{1}{2}\right]^* \beta^{1-t}$$

$*$: rounding

$$\text{fl}(w) = w(1 + \delta_i) \qquad \delta_i \leq E_{\text{machine}}$$

1.3 Stability

$$\epsilon_n = (I_n)_A - (I_n)_E = \alpha((I_{n-1})_A - (I_{n-1})_E) = \alpha\epsilon_{n-1} = \alpha^n \epsilon_1$$

2 Polynomial interp

2.1 Vandermonde matrix

$$\underbrace{\begin{pmatrix} \mathbf{1} & \mathbf{x} & \mathbf{x}^2 & \cdots & \mathbf{x}^n \end{pmatrix}}_{V} \mathbf{c} = \mathbf{y} \qquad \det(V) = \prod_{i<j} (x_i - x_j)$$

2.2 Lagrange basis

$$L_i = \prod_{k=1, k\neq i}^{n} \frac{x - x_k}{x_i - x_k} \qquad p(x_i) = \sum_{i=1}^{n} y_i L_i(x)$$

2.3 Hermite

$$p_i(x) = a_i + b_i (x - x_i) + c_i (x - x_i)^2 + d_i (x - x_i)^3$$

$$\left(y_i, s_i, \frac{3y'_i - 2s_i - s_{i+1}}{\Delta x_i}, \frac{s_{i+1} + s_i - 2y'_i}{(\Delta x_i)^2} \right)$$

$$\Delta x_i = x_{i+1} - x_i, y'_i = \frac{\Delta y_i}{\Delta x_i}$$

2.4 Spline

$$T\mathbf{s} = \mathbf{r}$$

$$\begin{aligned} T_{i,i-1} &= \Delta x_i \\ T_{i,i} &= 2(\Delta x_{i-1} + \Delta x_i) \\ T_{i,i+1} &= \Delta x_{i-1} \\ r_i &= 3\left(\Delta x_i y'_{i-1} + \Delta x_{i-1} y'_i\right) \end{aligned}$$

- Natural: $S''(x_1) = 0$, $S''(x_n) = 0$

$$T_{1,1} = 1 \qquad T_{n,n} = 1$$

$$T_{1,2} = \frac{1}{2} \qquad T_{n,n-1} = \frac{1}{2}$$

$$r_1 = \frac{3}{2} y'_1 \qquad r_n = \frac{3}{2} y'_{n-1}$$

- Clamped: $S'(x_1) = s_1^*$, $S'(x_n) = s_n^*$

$$T_{1,1} = 1 \qquad T_{n,n} = 1$$

$$r_1 = s_1^* \qquad r_n = s_n^*$$

2.5 Arc-length parameterization

$$t_{i+1} = t_i + \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

3 Time stepping

3.1 Taylor series

$$y(a + h) = \sum_{k=0}^{\infty} \frac{y^{(k)}(a) h^k}{k!}$$

3.2 Schemes

- Forward Euler: $O(h^2)$

$$y_{n+1} = y_n + hf(t_n, y_n)$$

- *Backward/implicit Euler*: $O(h^2)$

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$$

- *Trapezoidal*: $O(h^3)$

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right)$$

- Modified Euler: $O(h^3)$

$$\begin{aligned} y^*_{n+1} &= y_n + hf(t_n, y_n) \\ y_{n+1} &= y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})\right) \end{aligned}$$

- RK2: $O(h^3)$

$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k_2 &= hf(t_n + h, y_n + k_1) \\ y_{n+1} &= y_n + \frac{k_1}{2} + \frac{k_2}{2} \end{aligned}$$

- RK4: $O(h^5)$

$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k_2 &= hf(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \\ k_3 &= hf(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \\ k_4 &= hf(t_n + h, y_n + k_3) \\ y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} \end{aligned}$$

- BDF2: $O(h^3)$

$$y_{n+1} = \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2}{3} hf(t_{n+1}, y_{n+1})$$

3.3 Truncation error

$$LTE = e_{n+1} = y(t_{n+1}) - y_{n+1}$$

3.4 Stability

Use

$$y'(t) = -\lambda y(t); y(0) = y_0$$

For most explicit schemes:

$$h < \frac{2}{\lambda}$$

3.5 Adaptive time stepping

$$e \approx \left|y^*_{n+1} - y_{n+1}\right|$$

4 Fourier transforms

4.1 Identities

$$\int_0^{2\pi} \cos kt \sin k't \; dt = 0, \;\; \forall k, k'$$
$$\int_0^{2\pi} \cos kt \cos k't \; dt = 0, \;\; k \neq k'$$
$$\int_0^{2\pi} \sin kt \sin k't \; dt = 0, \;\; k \neq k'$$
$$\int_0^{2\pi} \sin kt \; dt = 0 \qquad \int_0^{2\pi} \cos kt \; dt = 0$$

4.2 Roots of unity

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$
$$W = e^{\frac{2\pi i}{N}}$$
$$\sum_{j=0} W^{jk} W^{-jl} = N \delta_{k,l}$$
$$\delta_{k,l} = \langle k = l \rangle$$

4.3 DFT

$$f_n = \sum_{k=0}^{N-1} F_k W^{nk} = MF$$
$$F_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k W^{-nk} = \frac{1}{N} \overline{M^\top} F$$
$$M_{ij} = W^{ij} \qquad \overline{M^\top} M = \frac{1}{N} I$$

4.4 FFT

$$g_n = \frac{1}{2} \left(f_n + f_{n+\frac{N}{2}} \right)$$
$$h_n = \frac{1}{2} \left(f_n + f_{n-\frac{N}{2}} \right) W^{-n}$$
$$F_{\text{even}} = G, \quad F_{\text{odd}} = H$$

5 Linear algebra

5.1 FLOPS

- LU factor: $\frac{2n^3}{3} + O(n^2)$
- Forward+backward solve: $2n^2 + O(n)$
- Matrix multiply: $2n^3 + n^2$ (n^3 multiplys)

5.2 Norms

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$
$$\|x\|_1 = \sum_{i=1}^n |x_i|$$
$$\|x\|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$$
$$\|x\|_\infty = \max_i |x_i|$$
$$\|x\| = 0 \implies x = 0$$

$$\|A\| = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}$$
$$\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$$
$$\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$$
$$\|A\|_2 = \max_i \left| \lambda_i [M^\top M] \right|^{1/2}$$
$$\|AB\| \leq \|A\| + \|B\|$$
$$\|AB\| \leq \|A\| \, \|B\|$$

5.3 Conditioning

$$A(x + \Delta x) = b + \Delta b$$
$$\Delta x = A^{-1} \Delta b$$
$$\|\Delta x\| \leq \left\|A^{-1}\right\| \|\Delta b\|$$
$$\frac{\|\Delta x\|}{\|x + \Delta x\|} \leq \kappa(A) \frac{\|\Delta A\|}{\|A\|}$$

$$\kappa(A) = \|A\| \left\|A^{-1}\right\|$$
$$\kappa(A) \geq 1$$
$$\kappa(\alpha A) = \kappa(A)$$
$$\frac{\|\Delta x\|}{\|x + \Delta x\|} = \kappa(A) \frac{\|\Delta A\|}{\|A\|}$$

6 PageRank

6.1 PageRank matrix

$$P_{ij} = \frac{1}{\deg(j)} \langle G_{ij} = 1 \rangle$$
$$\mathbf{d}_i = \langle \deg(i) = 0 \rangle$$
$$P' = P + \frac{1}{R} \mathbf{e} \mathbf{d}^\top$$
$$M = \alpha P' + (1 - \alpha) \frac{1}{R} \mathbf{e} \mathbf{e}^\top$$
$$|\lambda_1| = 1 \qquad |\lambda_2| \approx \alpha \qquad |\lambda_2|^n \approx \alpha^n = \text{tol}$$

6.2 PageRank sparse

$$M\mathbf{p} = \alpha \left(P\mathbf{p} + \mathbf{e} \left(\frac{\mathbf{d}^\top \mathbf{p}}{R} \right) \right) + (1 - \alpha) \frac{\mathbf{e}}{R}$$