1 Linear problems

1.1 Outcomes

- Optimal: exists optimal solution $\bar{x}$ which maximizes objective function
  - Certificate: obj function $c^\top x + \bar{z}$ such that $c \leq 0$
- Unbounded: no upper/lower bound for value
  - Certificate: $\bar{x}$ and $d \geq \mathbf{0}$ such that $Ad = \mathbf{0}$, $c^\top d > 0$
- Infeasible: no feasible solutions
  - Certificate: y such that $y^\top A \geq \mathbf{0}$, $y^\top b < 0$

Certificates apply to max

Fundamental theorem of LP: exactly one of these outcomes is true

1.2 Canonical form

LP in SEF is in canonical form wrt basis B if

- $A_B = I$
- $c_B = \mathbf{0}$

Important identity for feasible solutions:

$$y^\top Ax = y^\top b$$

To convert to canonical form wrt B,

$$\begin{array}{ll} \max & (c^\top - y^\top A)x + \bar{z} + y^\top b \\ \text{s.t.} & A_B^{-1}Ax = A_B^{-1}b \\ & x \geq \mathbf{0} \end{array}$$

$$y = A_B^{-\top} c_B$$

or use tableau and pivot on augmented B

1.3 Simplex

Requires: LP and feasible basis B

1. Rewrite LP in canonical form wrt B, with $\bar{x}$ as basic feasible solution
2. If $C_N \leq \mathbf{0}$, $\bar{x}$ is **optimal**
3. Select k such that $c_k > 0$
4. If $A_k \leq \mathbf{0}$, the LP is **unbounded**
5. Choose r as the first index i corresponding to

$$\min \left\{ \frac{b_i}{A_{ik}} : A_{ik} > 0 \right\}$$

6. $B \leftarrow B + k - l$

1.4 Two-phase simplex

1. Use Simplex to determine optimal value with initial basis on I

$$\text{aux} : \min \{(0|1)x : (A|I)x = b, x \geq \mathbf{0}\}$$

2. If optimal value is positive, the LP is unbounded.

1.5 Convexity

- S is convex if for all $x_1, x_2 \in S$ and $\lambda \in [0,1]$, $\lambda x_1 + (1-\lambda)x_2 \in S$
- A convex hull of S is the smallest convex superset of S

1.6 Extreme points

- $\bar{x}$ is not an extreme point iff $\bar{x} = \lambda x_1 + (1-\lambda)x_2$ for distinct points x_1, x_2, and $\lambda \in (0,1)$
- $\bar{x}$ in polyhedron is an extreme point iff rank $A^= = n$

2 Duality examples

2.1 Graphs

$$\text{cut} : \delta(U) = \{uv \in E : u \in U, v \notin U\}$$

- An st-path intersects every st-cut

2.2 Shortest path problem

$$\begin{array}{ll} \min & \sum_{e \in E} c_e x_e \\ \text{s.t.} & \sum_{e \in \delta(U)} x_e \geq 1 \quad U \subseteq V, s \in U, t \notin U \\ & x_e \geq 0 \\ & x_e \in \mathbb{Z} \end{array}$$

2.3 Min-cost perfect matching

$$\begin{array}{ll} \min & \sum_{e \in E} c_e x_e \\ \text{s.t.} & \sum_{e \in \delta(v)} x_e = 1 \quad v \in V \\ & x_e \geq 0 \\ & x_e \in \mathbb{Z} \end{array}$$

Algorithm:

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3 Duality

3.1 Primal-dual pairs

Max $c^\top x$, $Ax = b$	Min $b^\top y$, $A^\top y = c$
$(\leq / = / \geq)$ constraint	$(\geq 0/\text{free}/\leq 0)$ variable
$(\geq 0/\text{free}/\leq 0)$ variable	$(\geq / = / \leq)$ constraint

3.2 Duality theorems

- $c^\top x \leq b^\top y$
- P is unbounded $\implies$ D is infeasible (weak)
- P is optimal $\implies$ D is optimal with same value (strong)

3.3 Complementary slackness

$$\max \{c^\top x : Ax \leq b\}$$
$$\min \{b^\top y : A^\top y = c, y \geq 0\}$$

- $\bar{x}$, $\bar{y}$ optimal iff for every row i, $A_i \bar{x} = b_i$ or $\bar{y}_i = 0$

3.4 Farka's lemma

- Exactly one is true:
  1. $Ax = b$, $x \geq 0$ has a solution
  2. There exists a vector y such that $A^\top y \geq 0$ and $b^\top y < 0$

4 Integer programs

4.1 Cutting planes

Inequality which satisfies

1. Valid for IP
2. Not valid for non-integral LP optimal

5 Nonlinear programs

5.1 Subgradients

s is a subgradient of f at $\bar{x}$ if

$$h(x) = f(\bar{x}) + s^\top (x - \bar{x}) \leq f(x)$$

5.2 KKT

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & g_i(x) \leq 0 \quad i \in [k] \end{array}$$

Suppose

1. g_i is convex
2. There exists a Slater point
3. $\bar{x}$ is a feasible solution
4. I is the set of indices i for which $g_i(\bar{x}) = 0$
5. For all $i \in I$, gradient $\nabla g_i(\bar{x})$ exists

Then $\bar{x}$ is optimal iff $-c \in \text{cone}\{\nabla g_i(\bar{x}) : i \in I\}$