The Hungarian Magic Cube

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THE Hungarian Magic Cube, commercially available* in this country since the autumn of 1979, has probably by now wreaked havoc in the mathematics departments of most universities, polytechnics, colleges and sixth forms—as well as other puzzle-minded communities. It was invented by Professor Ernö Rubik, a Hugarian sculptor, architect and designer and, I understand, first hit the mathematical world at the Helsinki Mathematics Congress in 1978.

The Cube (Fig. 1) has the appearance of a $3 \times 3 \times 3$ array of small cubes, but in the space which would be occupied by the small cube at the centre there is an ingenious mechanism which allows *complete faces* of the Cube to be turned either clockwise or anticlockwise,

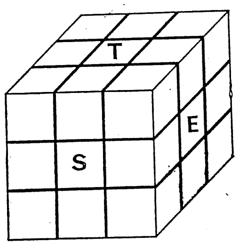


Fig. 1

but permits of no other kind of movement. When new the six faces of the Cube are each of a different "solid" colour. After as few as four random face-turns through multiples of 90°, the colours can become thoroughly mixed on all the faces—like a pretty patchwork quilt. The basic problem is, after *any* random mix, to restore the Cube to its "new" state, which I call the *home position*, with each face showing one solid colour.

The Cube is described by its Hungarian makers as "a toy, for children and adults, which develops logical thinking and spatial visualisation." Toy it may be, but it presents a serious and fascinating mathematical challenge. Experienced puzzle solvers may think they will master it in no time at all, only to discover many hours —or days—later that this is not as easy as they had supposed. For the benefit of those readers of the *Bulletin* who have put their mixed-up Cube aside in frustration to await another day, I give here in some detail a description of my own currently preferred method of solution (in about 80 to 90 face-turns at most) and outline how I arrived at it. I am grateful to Dr. Cyril Isenberg who gave me my first Magic Cube, to Dr. David Singmaster for sending me his "Notes on the Magic Cube" and for correspondence. It would be difficult now to be sure to what extent the sequences of turns given here are directly derived from the discoveries of others and which I discovered for myself. The process of improvement was gradual. One's personal objective may be to achieve a solution in the minimum number of face-turns (from *any* random mix), selecting a best possible sequence of turns at each stage from a number of choices; or to do so in least possible over-all time or with least possible "thought" even if this requires a few extra face-turns; or to do so with as few, easily memorised sequences as possible even if this requires many more than a minimum number of face-turns and, in consequence, significantly more time. Most people, not exploring the theoretical background, will probably settle for some compromise of their own—and stick to it when demonstrating their skill to others.

The Cube

The Cube may be assembled with the six coloured faces orientated in any one of 30 different ways. This is easily verified. The orientation of the colours is thus irrelevant to the discussion. With the Cube in its home position each of the small cubes forming its faces has its own unique location or "slot" and its uniquely correct orientation. Thus, for a solution, each small cube, if not already correctly *positioned*, *i.e.*, correctly located and orientated, has to be returned to its correct location and, either simultaneously or subsequently, brought into its correct orientation. As the faces are turned, centre small cubes turn about their own axes but do not change position relative to one another; the eight "corner cubes" always remain corner cubes, their "home positions" being defined by the colours of their *three* exposed faces; and the twelve "edge cubes" always remain edge cubes, their home positions being defined by the colours of their *two* exposed faces. When in its home location, if any one face of a corner cube is correctly orientated then so also will be the other two faces, and if either face of an edge cube is correctly orientated then so will be the other face.

The Cube and its constituent small cubes can be represented by a mapping as shown in Fig. 2, where the Top, Bottom, North, East, South, West faces are denoted by T, B, N, E, S, W, respectively.*

As the Cube's faces are defined by their centre small cubes and the relationship between these centre cubes is unchanged by any sequences of turns, the centre cube

* Magic Cubes can be obtained from Pentangle, Over Wallop, Hampshire, for approximately £5·50. They are also available from David Singmaster and Company, 12 Heath Grove, Buxton, Derbyshire SK17 9EH. Dr. Singmaster has also made available his "Notes on the Magic Cube" at £1·00, including postage.

* Dr. Singmaster uses Upper, Down, Back, Right, Front, Left (U, D, B, R, F, L). I had chosen my notation before reading his "Notes on the Magic Cube." The only confusion could be with the letter B for Bottom, Back (or Blue), but the context always makes the meaning clear.

of each face is permanently labelled on the mapping of the Cube, as held in the hand, by the appropriate letter as shown. I make the convention that the separate smaller square representing the Bottom face is looked at "from above," and the colours of the nine small-cube faces comprising the Bottom face of the Cube at any stage are indicated as they would be seen if the Cube were transparent and viewed from above. After any sequence

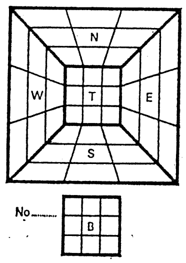


Fig. 2

of turns starting from the home position, some small cubes will be found to have taken up new locations and/or orientations. If the sequence is reversed these same small cubes will be returned to their home positions. I found the maps an essential ingredient while devising sequences of turns which would have particular desired effects. I also used self-stick spots of five different colours plus white to restore my Cube artificially and temporarily to the home position at any required moment, having, at the beginning, no knowledge of how to do this, but needing to in order to continue my self-instruction and search for improved techniques.

The sequences of turns used for the guide diagrams relate only to the Top "level" and, in two of them, the Middle level and are such that the nine small cubes of the Bottom level (and so of the Bottom face) are not affected. The separate smaller square of the mapping which represents the Bottom face is thus omitted as irrelevant in the guide diagrams in which only *changes* in location and/or orientation of small cubes resulting from a sequence of turns are recorded.

Notation

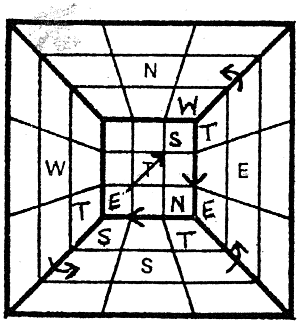
A clockwise turn of the face held at any particular stage at the Top level is denoted by T; an anticlockwise turn by T^{-1}. Thus T followed by T^{-1} restores the T-face to its original position. As in ordinary algebra T^2 indicates a repetition of the clockwise turn T. Since two clockwise turns have the same effect as two anticlockwise turns, $T^2 = (T^{-1})^2$, again as in ordinary algebra. In contrast to the conventions of group theory, a sequence of turns is written here in the order in which the turns are made reading from left to right. For example, $TSET^{-1}$ means "turn the Top face through 90° clockwise, then the South face through 90° clockwise, then the East face through 90° clockwise, and then the Top face through 90° anticlockwise. The "inverse" of the sequence $TSET^{-1}$ is $TE^{-1}S^{-1}T^{-1}$, *i.e.*, we read the sequence "backward" from right to left, and *reverse* each of the face turns in succession.

There is no way by which the location or orientation of a single small cube can be changed without also changing the location or orientation of some other "like" small cube. Moreover, we soon learn that the *locations* of two small cubes cannot be interchanged without a second pair of two small cubes also interchanging their locations—and mutual interchanges must necessarily be between "like" small cubes, *i.e.*, two corner cubes or two edge cubes. With this restriction any two pairs of like small cubes can be interchanged. It is also possible to "rotate" the locations of any three like small cubes. Thus, if A, B, C are any three like small cubes, then A can be moved to B's location, B to C's location and C to A's location. Moreover, when such a rotation is effected *specific faces* of these small cubes rotate, *i.e.*, if the face p of A moves to the precise position initially occupied by the face q of B, and the face q of B moves to the precise position occupied by face r of C, then face r of C moves to the precise position initially occupied by the face p of A. If this were not so then one small cube only could be left incorrectly orientated which, since the Cube is capable of being restored to its home position, cannot occur. Any three (or four) corner cubes, or any three (or four) edge cubes, can be brought to one common level of the Cube and the Cube can then be turned in the hand to bring this level to Top. The guide diagrams (see p. 89 for explanation) depict the effects of certain selected sequences of turns on the edge cubes and/or corner cubes at the Top level, the sequences being such that they leave the Middle and Bottom levels undisturbed.

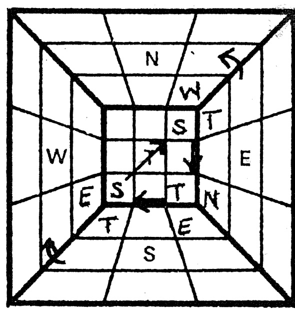
Consider (Fig. 3) the nine diagrams $C1$ to $C9$ which, the sequences being read forward or backward, give all possible rotations of the three corner cubes initially located at TSW, TNE, TES (the corner cube at TWN not being affected) such that the precise corner-cube faces rotate. We notice (and logic decrees) that, with the triangular rotations here described, either (*i*) all three corner cubes twist in the *same* clockwise ($C5$) or anticlockwise ($C1$) direction; or (*ii*) none of the three twists ($C9$); or (*iii*) two only of the three twist and in *opposite* directions ($C2$, 3, 4, 6, 7, 8). When a pair of corner cubes at the Top level interchange locations (along with some second pair of like small cubes) as in $C10$, 11, two and only two corner cubes twist—and they twist in opposite directions. Sequences can also be derived (Fig. 4, $C12$ to $C16$) which twist two, three or four of the corner cubes at the Top level (with or without disturbing the edge cubes) without altering their locations. If an even number of corner cubes requires twisting, then half of them will require a clockwise twist and half an anticlockwise twist. If three only require twisting, all must require a twist in the *same* direction.

Consider now the edge cubes (Fig. 5). The four guide diagrams $E1$, 4, 5, 6, read forward or backward, give all possible rotations of the three edge cubes initially located at TS, TN, TE (the edge cube at TW not being affected) such that the precise edge-cube faces rotate. We notice that either there is no flip ($E1$) or two edge cubes flip ($E4$, 5, 6 and $C10$). Sequences can also be derived ($E7$, 8, 9) which flip two or four edge cubes at the Top level without changing their locations. When a pair of edge cubes interchanges locations, along with some second pair of like small cubes, either (*i*) none flips ($E2$, 3); or (ii) by combining $E2$ or 3 with $E7$, 8 or 9 two or four

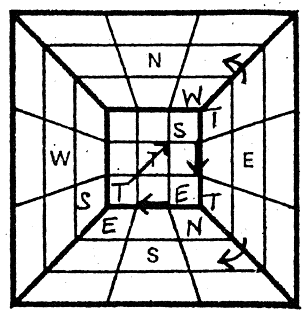
$C1: E^{-1}.TWT^{-1}.E.TW^{-1}T^{-1}$ (8)



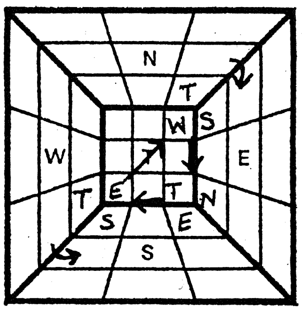
$C2: E^{2}T^{-1}.ETE^{-1}.B^{-1}.ET^{-1}E^{-1}.B.TE^{2}$ (12)



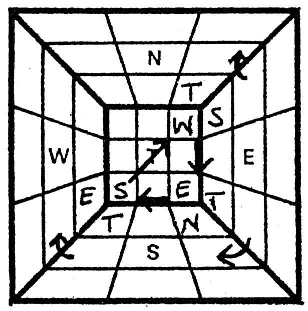
$C3: S^{2}T.SWS^{-1}.E^{-1}.SW^{-1}S^{-1}.E.T^{-1}S^{2}$ (12)



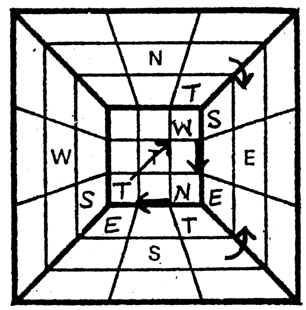
$C4: ENE^{-1}.S.EN^{-1}E^{-1}.S^{-1}$ (8)



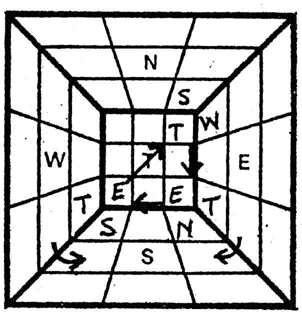
$C5: S.B^{2}.N^{2}BS^{2}.B^{-1}.N^{2}BS^{2}.B.S^{-1}$ (11)



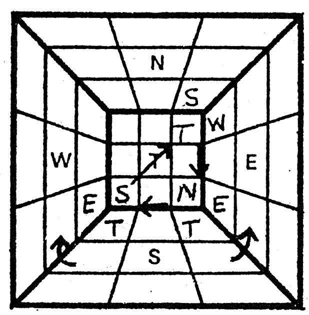
$C6: E^{-1}.S^{-1}W^{-1}S.E.S^{-1}WS$ (8)



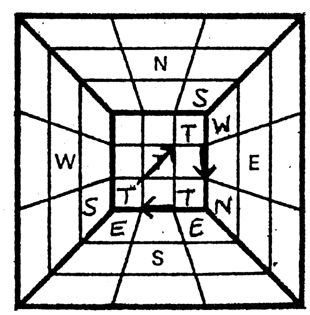
$C7: N^{2}.W^{2}NE^{2}.N^{-1}.W^{2}NE^{2}.N$ (9)



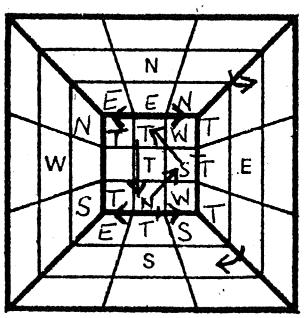
$C8: W.S^{2}WN^{2}.W^{-1}.S^{2}WN^{2}.W^{2}$ (9)



$C9: WB.ENE^{-1}.S^{-1}.EN^{-1}E^{-1}.S.B^{-1}W^{-1}$ (12)



$C10: S.ETE^{-1}T^{-1}.S^{-1}$ (6)



$C11: T.S.(ETE^{-1}T^{-1}).S^{-1}$ (7)

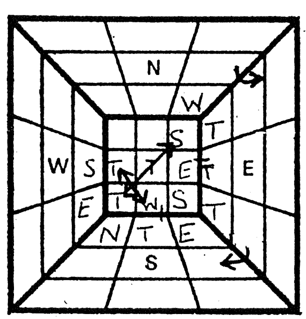


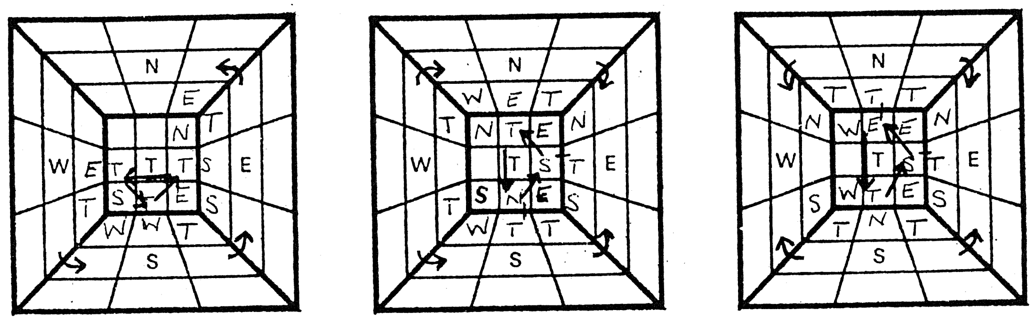
Fig. 3

If the turns which are entered above each diagram are read "forward," *i.e.*, from left to right, the rotation or interchanges is/are in the direction of the arrows, and the labelling shows the effects of the sequence operated as written. For example, in *C*1 the corner cube defined as *TSW* goes to the location initially occupied by the corner cube defined as *ETN* (*T*→*E*, *S*→*T*, *W*→*N*) that is the *T*-coloured face of *TSW* goes to the East face of the Cube as held, and so on. If the inverse of the sequence is used, then the corner cube initially occupying the "*TNE* corner" moves (in the opposite direction to that of the arrow) to the "*TSW* corner" with the *T*-coloured face of the "*TNE* corner cube" going to the South face of the Cube, the *N*-coloured face going to the West face of the Cube, and the *E*-coloured face going to the Top face of the Cube. Familiarity with this notation soon brings rapid facility in interpretation, making easy the required choices of sequences at any stage.

*C*12: $NT^2.N^{-1}T^{-1}.$
$N.T^{-1}N^{-1}T^2$ (8)

*C*13: $S(TET^{-1}E^{-1})^2.$
S^{-1} (10)

*C*14: $N^{-1}E^2(SES^{-1}E^{-1})^2$
E^2N (12)



*C*15: $E^{-1}BE.SBS^{-1}.T.$
$SB^{-1}S^{-1}.E^{-1}B^{-1}E.T^{-1}$ (14)

*C*16: $N^2.W^2NE^2.N^{-1}.$
$W^2NE^2N.S^{-1}W^{-1}S.$
$E^{-1}.S^1WS.E$ (16)

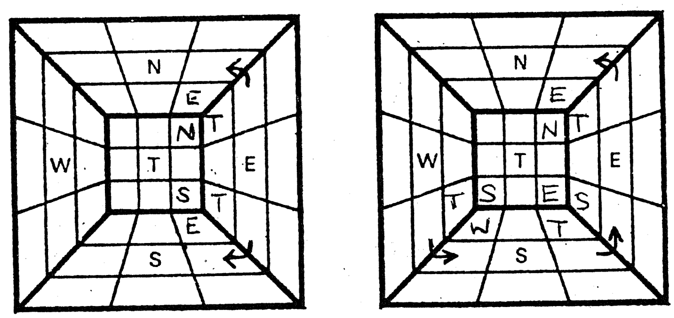


Fig. 4

flip; or (*iii*) two flip as with (*C*11). If edge-cube flips are required they must be even in number.

That closed triangular rotations of like small cubes give closed triangular rotations of precise faces of these small cubes is fundamental to the method of solution described here and in keeping to a minimum the total number of turns required. The sequences of turns which twist two corner cubes without changing their locations and those which flip two or four pairs of edge cubes without changing their locations are lengthy, necessitating a minimum of 14, 16 and 18 turns, respectively. There is thus considerable advantage, when choosing sequences of turns, if additional two-corner-cube-twists and edge flips can with foresight be avoided.

Working the method*

By this method all the small cubes of one face are first brought into their correct *positions* (*i.e.*, correct locations *and* orientations). This face is then turned to "Bottom" and, thereafter, the small cubes at the Bottom level are left undisturbed by any fully completed sequence of turns. The second stage is then to bring the corner cubes now at the Top level into their correct positions without regard to any changes in the positions of edge cubes at the Top and Middle levels. The third stage is to put any edge cubes which should be at the Middle level but are lying at the Top level (if there are such) into their precisely correct positions at the Middle level and/or if necessary ensure further that all four edge cubes at the Middle level are correctly positioned. The final fourth stage is then to make any corrections still necessary to the positions of the edge cubes at the Top level.

Stage One

This is simply accomplished without special technique and no guide diagrams are given. Put the edge cubes of a "first face"—I always use the White face as first face—into their correct positions by turning this face to and

* There are many different algorithms for restoring the Magic Cube to its home position and there are refinements, using the guide diagrams given here, of the method I now describe. I have, however, preferred not to complicate the description with small refinements merely to save relatively few turns. Establishing a method which gives a proven "minimax" number of turns for a solution is still an open challenge.

fro and bringing up the appropriate edge cubes. With an unfamiliar Cube it is reassuring to return this face to its "correct" position after each successive edge cube of the four belonging to this face has been put into position. The maximum number of turns for this operation should not be more than 8. To put all the corner cubes of this first face into their correct positions, bring each corner cube which belongs on the first face (if not already in exactly the correct location and orientation) "beneath" the location it is to occupy, holding the Cube so that the required location is at *TNE* and the recalcitrant small corner cube at *BNE*, the "first face" being held as Top. If the *T*-coloured face of the small cube at *BNE* is on the *E*-face, use EBE^{-1}; if the *T*-coloured face of the small cube at *BNE* is on the *N*-face, use $BEB^{-1}E^{-1}$; if the *T*-coloured face of the small cube at *BNE* is on the *B*-face, use $EB^2E^{-1}B^{-1}EBE^{-1}$. Time is saved by dealing first with wrongly positioned corner cubes with a *T*-coloured face which are not lying at the Top level, as the process of correcting these will often bring down to the Bottom level any corner cube with a *T*-coloured face which is at the *T* level but nonetheless wrongly positioned then to be dealt with as described above without unnecessary extra turns. Stage One is now complete in, say, about 25 turns. Hold the Cube from now on so that this first face is at Bottom. The check henceforth, at the end of every completed sequence of turns, is to ensure that the small cubes at this Bottom level have been left undisturbed.

Stage Two

To put all the corner cubes now at the new Top level into their correct positions without disturbing the Bottom level use the "*C*" guide diagrams. First, turn the Top face until at least one corner cube is correctly *located* (always possible). If none of the others is correctly located, a triangular rotation will make them so. If an adjacent corner cube (only) is also correctly located, then a 90° turn of the Top face will again ensure that one and only one of the four corner cubes is correctly located and a triangular rotation of the other three will bring them all into their correct locations. With the Cube held so that the single correctly-located corner cube is at *TNW*, the sequence depicted above diagram *C*1, used either forward or backward as required, is then sufficient to bring all four Top-level corners into their correct locations (without disturbing the Bottom level). However, as will be explained below, a judicious choice from among the 9 diagrams *C*1, 2, ..., 9 will usually significantly reduce the total number of turns required by bringing corner cubes into their correct (or most advantageous) orientation by the same turns which bring them into their correct locations.

If, when a first corner cube of the Top level is in its correct location, the corner cube diagonal to it (only) is also in its correct location, then the two other corner cubes must be required to interchange their locations. This interchange can be effected by the 7-turn sequence *C*11 (which also interchanges two edge cubes) or, after a 90° turn of the Top face, by the 6-turn sequence *C*10 which interchanges two pairs of corner cubes and causes three edge cubes to rotate (two of these "flipping"). Again the choice of which sequence is the more favourable depends on judgement. Here there is not much loss or saving either way.

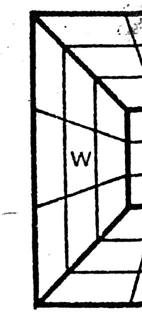
If, when a first corner cube of the Top level is in its correct location, two others are also, then all four are correctly located, and they can be brought into their correct orientations by the use of one or other sequences *C*12, 13, 14, 15. When a triangular rotation is required two possibilities occur. If the corner cube held at *TNW* has its *T*-coloured face upward, *i.e.*, if it is already correctly located *and* orientated, then one or other of the 9 sequences *C*1, 2, ..., 9, read either forward or backward, *must* correctly orientate as well as correctly locate all the other three corner cubes. This is because all 8 corner cubes *can* be restored to their correct positions; 5 are already correctly positioned; and the sequences given are the only permutations which fulfil the required conditions of re-locating the remaining 3 corner cubes. In contrast, if the corner cube held at *TNW* is *not* correctly orientated, then *none* of the sequences *C*1, 2, ..., 9 by itself will bring all the other three into their correct positions, otherwise the *TNW* corner cube alone would remain to be twisted which is not possible. However, by comparing the positions of the *T*-coloured faces of the three wrongly located corner cubes, it is always possible from among the sequences *C*1, 2, ..., 9 (read forward or backward) to select one to effect the correct direction of rotation, and we choose the shortest available, which will leave *one and only one* corner cube with its *T*-coloured face upward, *i.e.*, so that, whereas all the corner cubes will become correctly located, one and only one will be correctly orientated, leaving three requiring twists. These twists (as has been explained) must all three be in the same direction, clockwise or anticlockwise. They can be accomplished by the short 8-turn sequence *C*12 (used backward or forward)—which was the reason for ensuring that a three-corner-cube twisting was required. In the less frequent event of needing to use *C*10 or *C*11 to bring all the corner cubes into their correct locations there is not as much latitude and we may be left with two or four twists still required. We must then resort to the longer *C*13, *C*14 sequences which give two essentially different four-corner-cube twistings or the longer still *C*15 sequence which gives two-corner-cube twistings. We are not concerned at this Stage Two with the edge cube changes occurring in *C*12, 13, 14. The sequence *C*15 causes no incidental edge cube changes. *C*16 is added here merely for good measure. It gives a three-corner-cube twisting which causes no incidental edge-cube changes. It is not required in operating the algorithm described here.

Stage Three

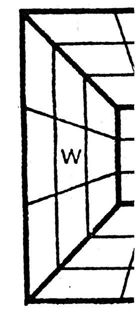
If, most unusually, all four of the edge cubes which should be at the Middle level are already at this level, they can be correctly located by means of the short sequences *M*1, *M*2, the latter if necessary used twice. If two or four are found, then or later, to require "flipping," this can be accomplished by means of *E*7, 8 or 9. If only one requires flipping, it will, as a last step of all, pair with a Top face edge cube also requiring to be flipped. At this Stage Three it is thus best left over.

More usually two or three edge cubes or one edge cube which should be at the Middle level will be lurking at the Top level. The technique to use while one is still inexperienced is to identify by coloured stick-on spots either (*i*) interchanges of precise faces of pairs of edge cubes which, by means of *E*2 or *E*3, will bring at least

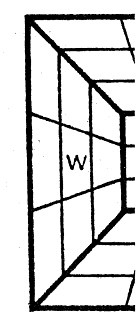
$E1: E^2T^{-1}$
$T^{-1}E^2$



$E4: N^{-1}T$
$T.E^{-1}S$



$E7: STB^{-1}.W$
$T.E^{-1}B^2T$



Fig

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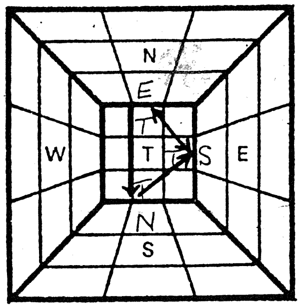
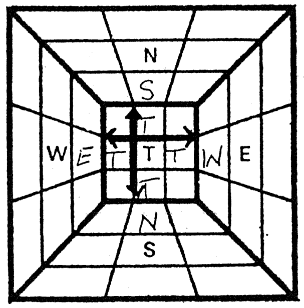
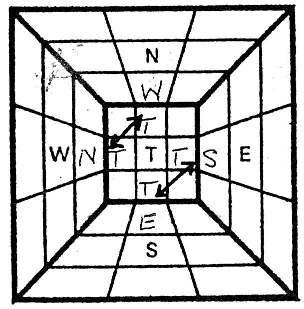
E1: $E^2T^{-1}.SN^{-1}.E^2.NS.$
$T^{-1}E^2$ (9)



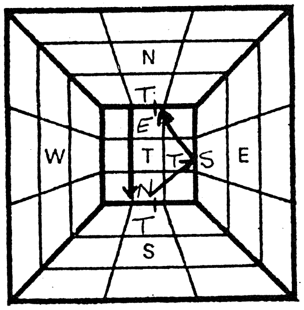
E2: $E^2W^2.B.E^2W^2.$
$T^2.E^2W^2.B.E^2W^2$ (11)



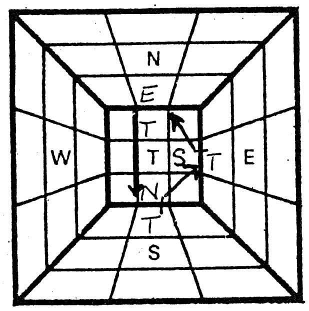
E3: $E^2W^2T^{-1}.$
$N^2S^2.E^2.N^2S^2.W^2.TW^2E^2$ (12)



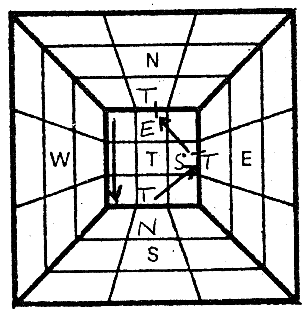
E4: $N^{-1}T^{-1}N.WSE.$
$T.E^{-1}S^{-1}W^{-1}$ (10)



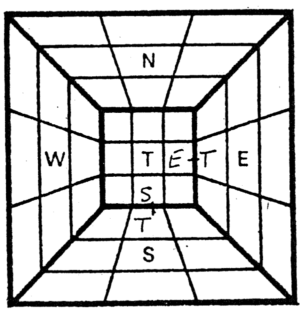
E5: $W^{-1}N^{-1}E^{-1}.T.$
$ENW.ST^{-1}S^{-1}$ (10)



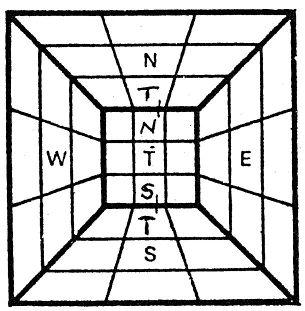
E6: $E^{-1}T^2E.NWS.$
$T^2.S^{-1}W^{-1}N^{-1}$ (10)



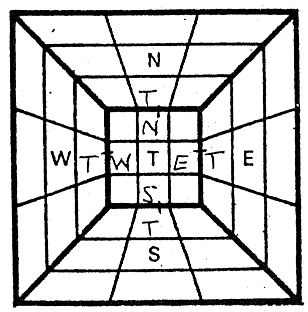
E7: $STB^{-1}.W^2T^2B^2E.$
$T.E^{-1}B^2T^2W^2.BT^{-1}S^{-1}.T^{-1}$
(16)



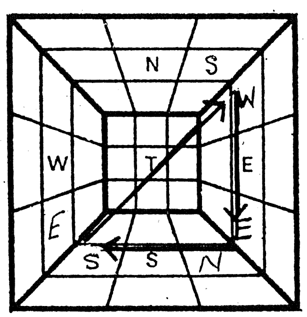
E8: $STB^{-1}.W^2T^2B^2E.$
$T^2.E^{-1}B^2T^2W^2.BT^{-1}S^{-1}.T^2$
(16)



E9: $W^2BN^2.EW^{-1}.N.EW^{-1}.B.$
$EW^{-1}.S.EW^{-1}.T.N^2B^{-1}W^2$
(18)



M1: $S^2.BT^{-1}.E^2.TB^{-1}$
(6)



M2: $T^2B^2.N^2.T^2B^2.S^2$
(6)

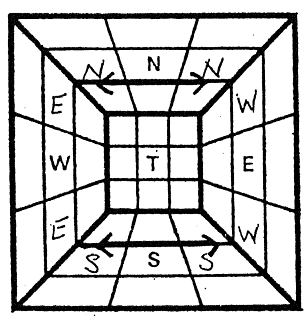


Fig. 5

two of four edge cubes into their correct locations *and* orientations or (*ii*) rotations of precise faces of trios of edge cubes which, again, by means of $E1$, 4, 5 or 6, will bring at least two of the three into their correct locations and orientations—priority at this Stage Three being given to correct positioning at the Middle level.

It is possible and somewhat easier (although not quicker) to use only triangular rotations, sticking a red spot (say) on the face at Top of an edge cube lying at Top level which belongs to the Middle level, a green spot (say) on the precise position (face) where we wish the red-spotted face to be, and a yellow spot (say) on the precise face where we wish the green-spotted face to be. The yellow-spotted face will then go to the position occupied by the red-spotted face to await if necessary a subsequent sequence of turns to correct its position.

With not more than two, or at most three, face turns the three spotted edge cubes can, by turning the Cube around in the hand, be brought to one common level temporarily held as Top. It helps to jot down as an *aide memoire* these "preliminary" face turns as they are made (clockwise or anticlockwise), denoting them by letters representing the *colours* of the Cube's faces for convenience. One or other of the sequences of turns $E1$, 4, 5, 6 (used forward or backward) will then always ensure that the red-, green-, yellow-, red-spotted faces rotate forward or backward as desired. The preliminary turns jotted down earlier are then executed in reverse, thus ensuring that the Bottom level of the Cube is restored to its correct position and that the red- and the green-spotted cubes are now, as intended, in precisely their correct home positions. The red-spotted cube, but not necessarily the green-spotted cube which may belong to the Top level, will be correctly at the Middle level. Three such triangular rotations will ensure that all four of the Middle level edge cubes are in their correct positions with benefit also having accrued toward the correct locating of edge cubes at the Top level. However, if two or more Middle level edge cubes are at the Top level the process is speeded by a judicious use first of two red and two green spots stuck on the *faces* of edge cubes to be interchanged, arranging to bring two edge cubes into their correct positions at the Middle level by means of $E2$ or $E3$. In this case it is easiest to bring all the spotted *faces* to the Top *face* (always possible by the same techniques as were used at Stage One to bring edge cubes of a "first face" into their correct positions), even if this requires several extra preliminary face turns. Provided these preliminary turns are carefully noted, there is no problem in reversing them, once the chosen sequence effecting the required interchanges of edge cubes has been executed, in order to restore the Bottom level of the Cube (and all other already correctly-positioned small cubes) to the correct home position(s).

With practice and some cunning Stage Three can be accomplished in at most 30 turns and often in many fewer. Moreover, with skill or good fortune, it is usually possible to avoid actual or potential turn-consuming final flips.

Stage Four

It remains to make any necessary corrections to the positioning of the edge cubes on the Top level. This can always be accomplished by either one or at most two of the sequences $E1$, 2, ..., 6, with the addition if necessary (not forgetting any leftover flip requirement at the Middle level) of one of the flip sequences $E7$, 8, 9.

By my calculation (and experience) this method gives a solution in an average of some 80 turns, never more than 90 unless there is some turn-wasting misjudgement and rarely significantly fewer than 80 unless chance eliminates the necessity for some intermediate step. The *time* taken depends on "the state of one's cube"—whether too stiff as when new, or too wobbly when much overused, or just right—and on personal skill in making turns to *exact* multiples of 90°—and with either hand. Dr. Singmaster purports to complete 80-turn solutions in under 4 minutes, but I need a little longer. To date I have not enjoyed the privilege of watching anyone operating the Cube and maybe others have skills I have not yet acquired—or perhaps my hands are smaller than Dr. Singmaster's? The fascination for the *observer* (as opposed to the probable obsessiveness of the addict experimenter) lies in the manner in which, during each sequence of turns, the Cube is jumbled into a continually changing kaleidoscope only to be restored at the very last turns of each sequence to an increasing state of order.

Arranging a

AVID SINGMA
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