PM 811 H^p Spaces **Assignment 3** **Due Thursday November 25.**

1. (a) Suppose A is a function algebra on X, $f \in A$, and E is a (clopen) subset of X so that $\operatorname{Re} f \ge 1$ on E and $\operatorname{Re} f \le -1$ on $X \setminus E$. Prove that χ_E belongs to A. **Hint:** use Runge's Theorem to find polynomials p_n that converge uniformly on $(\|f\|+1)\overline{\mathbb{D}} \cap \big\{|\operatorname{Re} z| \ge 1/2\big\}$ to a characteristic function.

   (b) Show that $H^\infty + \overline{H^\infty}$ is not dense in L^∞. **Hint:** First show this is equivalent to $\operatorname{Re} H^\infty$ being dense in $L^\infty_{\mathbb{R}}$. Let E be a measureable subset of $\mathbb{T}$ with $0 < |E| < 2\pi$, and set $f = 2\chi_E - 1$. Suppose that there is an $h \in H^\infty$ such that $\|f - \operatorname{Re} h\| < 1/2$. Apply part (a) with $X = \mathcal{M}(L^\infty)$.

2. Let A be a closed subalgebra of L^∞ containing H^∞.

   (a) Show that the restriction map from $\mathcal{M}(A)$ to $\mathcal{M}(H^\infty)$ is injective. Thus $\mathcal{M}(A)$ can be considered as a subset of $\mathcal{M}(H^\infty)$.

   (b) Show that if $H^\infty \subsetneq A \subset L^\infty$, then $A \supset H^\infty + C$. **Hint:** follow the strategy of Wermer's theorem, either evaluation at 0 is in $\mathcal{M}(A)$ or it isn't.

   (c) If g is invertible in A, show that there is an invertible h in H^∞ and an invertible unimodular function $u \in A$ so that $g = uh$. Hence show that A is generated by H^∞ and $\mathcal{U} = \{u \in A^{-1} : |u| = 1 \text{ a.e. on } \mathbb{T}\}$.

   (d) Show that if A is generated as a closed algebra by H^∞ and a *group* $\mathcal{U}$ of unimodular functions, then $\mathcal{M}(A) = \{\varphi \in \mathcal{M}(H^\infty) : |\varphi(u)| = 1 \text{ for all } u \in \mathcal{U}\}$.

   (e) What is $\mathcal{M}(H^\infty + C)$?

3. Use the maximal function to give a different proof of (part of) Fatou's theorem: if $f \in L^1$ and $u = P * f$ is its harmonic extension, show that $\lim_{\Gamma_\alpha(t) \ni z \to e^{it}} u(z) = f(t)$ a.e. as follows.

   (a) WLOG $f \in L^1_{\mathbb{R}}$. Show that $\Omega(t) = \limsup_{\Gamma_\alpha(t) \ni z \to e^{it}} u(z) - \liminf_{\Gamma_\alpha(t) \ni z \to e^{it}} u(z)$ is bounded by $C\, Mf(t)$, where C is a constant only depending on α.

   (b) Hence show that $|\{t : \Omega(t) > \varepsilon\}| < C/\varepsilon \|f\|_1$. Approximate f by trig polynomials to improve this estimate.

4. (a) Show that a conformal map of $\mathbb{D}$ onto $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$ induces an isometric map of H^∞ onto $H^\infty(\mathbb{H})$; and it takes interpolating sequences to interpolating sequences.

   (b) Consider the points $z_n = i + 10n$ in $\mathbb{H}$ for $n \in \mathbb{N}$. If $a = (a_n) \in \ell^\infty$ with $\|a\|_\infty \le 1$, define $f_1(z) = \sum_{n \ge 1} \dfrac{-4a_n}{(z - \bar{z}_n)^2}$. Prove that $f_1 \in H^\infty(\mathbb{H})$ with $\|f_1\|_\infty \le 10$ and $|f_1(z_n) - a_n| \le 1/6$ for $n \ge 1$.

   (c) Repeat this procedure with $a^{(1)} = \big(a_n - f_1(z_n)\big)$ to get a function f_2 with better estimates, etc. Hence construct an infinite series which converges to an interpolating function for a. Thus showing that $\{z_n : n \ge 1\}$ is an interpolating sequence.