PMATH 763, FALL 2010

Assignment #3 Due: February 26. Happy Reading Week!

1. Let $H = \mathrm{T}_3^0(\mathbb{R})$ and $\mathfrak{h} = \mathfrak{t}_3^0(\mathbb{R})$. These are known as the *Heisenberg* group and Lie algebra, respectively. For $x, y, z \in \mathbb{R}$ we let

$$g(x,y,z) = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Also, let $P = E_{12}, Q = E_{23}$ and $Z = E_{13}$, so that $\{P, Q, Z\}$ is a basis for $\mathfrak{h}$.

Now consider the vector space of functions

$$\mathcal{V} = \left\{ \varphi : \mathbb{R} \to \mathbb{C} \ : \ \varphi(s) = \sum_{j=1}^{k} p_j(s) e^{a_j s - s^2}, \text{ each } p_j(s) \text{ is a } \mathbb{C}\text{-polynomial}, a_j \in \mathbb{C}, k \in \mathbb{N} \right\}$$

Let us take for granted the fact that estimates $|\varphi(s)| \leq K e^{A|s| - s^2}$ always hold ($K, A \geq 0$), and hence the inner product

$$(\varphi, \psi) = \int_{-\infty}^{\infty} \varphi(s)\overline{\psi(s)}\, ds$$

("improper" Riemann integral) makes sense. Hence in linear operators on this inner product space, we may define

$$\mathrm{U}(\mathcal{V}) = \{U \in \mathcal{L}(\mathcal{V}) : (U\varphi, U\psi) = (\varphi, \psi) \text{ and } \mathfrak{u}(\mathcal{V}) = \{T \in \mathcal{L}(\mathcal{V}) : (T\varphi, \psi) = -(\varphi, T\psi)\}.$$

(a) Fix $h \in \mathbb{R} \setminus \{0\}$. For $g(x,y,z) \in H$, let

$$\rho_h(g(x,y,z))\varphi(s) = e^{i(hz+ys)}\varphi(s + hx).$$

Verify that $\rho_h(H) \subset \mathrm{U}(\mathcal{V})$ and that $\rho_h : H \to \mathrm{U}(\mathcal{V})$ is a group homomorphism.

[This is often called the *Schrödinger representation*.]

(b) Define for $X \in \mathfrak{h}$ and $\varphi \in \mathcal{V}$, $d\rho_h(X)\varphi(s) = \frac{d}{dt}\big|_{t=0} \rho_h(\exp(tX))\varphi(s)$. Show that $d\rho_h(\mathfrak{h}) \subset \mathfrak{u}(\mathcal{V})$ and, moreover, $d\rho_h : \mathfrak{h} \to \mathfrak{u}(\mathcal{V})$ defines a Lie algebra homomorphism.

[You may just have to mimic the proof from class that the differential is linear. However, the easy relations in $\mathfrak{h}$ will spare some grief in dealing with the Lie brackets.]

Remark. The notation is so chosen as P models momentum while Q models position. The relation $[d\rho_h(P), d\rho_h(Q)] = ihI$ is often called *Heisenberg's commutation relation.* The space $\mathcal{V}$ can be shown to be dense in $L^2(\mathbb{R})$, and is the minimal ρ_h-invariant subspace which contains the Gaussian $s \mapsto e^{-s^2}$.

(c) Show that, $\exp(d\rho_h(X)) = \rho_h(\exp(X))$, for each $X \in \mathfrak{h}$.

[Here, you may take for granted that all interchanges of summation of the "entire" analytic series for φ converge pointwise (even uniformly on compact sets) to the same function, and are thus sensible.]

(d) Determine if a version of Lie's theorem holds here: Is there $\varphi_0 \in \mathcal{V} \setminus \{0\}$ and a lineaer form λ on $\mathfrak{h}$ for which $d\rho_h(X)\varphi_0 = \lambda(X)\varphi_0$ for all $X \in \mathfrak{h}$?

2. Given a subset S of $\mathrm{GL}_n(\mathbb{R})$, let $\langle S \rangle$ denote the smallest closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ containing S. We let the commutator subgroup be given by $G' = \langle \{ghg^{-1}h^{-1} : g, h \in G\} \rangle$. Moreover we let $G^{(0)} = G$ and $G^{(k)} = (G^{(k-1)})'$ for $k \in \mathbb{N}$. We say that G is *solvable* if $G^{(m)} = \{I\}$ for some m.

   (a) Show that if G is a connected matrix Lie group, then its commutator subgroup has $\mathrm{Lie}(G') \supseteq [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{g} = \mathrm{Lie}(G)$. Deduce that if G is solvable then $\mathfrak{g}$ is solvable.

   (b) If $\mathfrak{g} \leq \mathfrak{gl}_n(\mathbb{R})$ is a Lie algebra, show that $\mathfrak{g}_\mathbb{C} = \mathfrak{g} + i\mathfrak{g} \leq \mathfrak{gl}_n(\mathbb{C})$ is a $\mathbb{C}$-Lie algebra. Moreover, any $\mathbb{R}$-linear Lie representation $\rho : \mathfrak{g} \to \mathfrak{gl}_d(\mathbb{C})$ extends to a $\mathbb{C}$-linear Lie representation $\rho_\mathbb{C} : \mathfrak{g}_\mathbb{C} \to \mathfrak{gl}_d(\mathbb{C})$, given by $\rho_\mathbb{C}(X + iY) = \rho(X) + i\rho(Y)$.

   (c) Suppose that G is a solvable connected matrix Lie group, and $\pi : G \to \mathrm{GL}(\mathcal{V})$ is a representation on an d-dimensional $\mathbb{C}$-vector space $\mathcal{V}$. Then show that there is a basis B for $\mathcal{V}$ with respect to which any matrix is upper triangular, i.e. $[\pi(g)]_B \in \mathrm{T}_d(\mathbb{C})$.

3. A matrix Lie algebra $\mathfrak{g}$ is called *reductive* if every abelian ideal $\mathfrak{a}$ lies in the centre $\mathfrak{z} = \mathrm{Z}(\mathfrak{g})$, and $\mathfrak{z} \cap \mathcal{D}(\mathfrak{g}) = \{0\}$.

   (a) Show that $\mathfrak{g}$ is reductive if and only if there is a semisimple ideal $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h}$, i.e. $\mathfrak{g} = \mathfrak{z} + \mathfrak{h}$ and $\mathfrak{z} \cap \mathfrak{h} = \{0\}$. Deduce that $\mathfrak{h} = \mathcal{D}(\mathfrak{g})$ and $\mathrm{rad}(\mathfrak{g}) = \mathfrak{z}$.

   [Read proof of Lie's theorem.]

   (b) Show that $\mathfrak{sl}_n(\mathbb{F})$ is a simple Lie algebra.

   [Start by showing that the smallest ideal generated by any of E_{ij} or $E_{ii} - E_{jj}$, $i \neq j$, must be all of $\mathfrak{sl}_n(\mathbb{F})$.]

   (c) Deduce that $\mathfrak{gl}_n(\mathbb{F})$ is a reductive Lie algebra.

4. Show that the Killing form on $\mathfrak{so}(n)$ is given by

$$B(X,Y) = (n-2)\mathrm{Tr}(XY)$$

and hence $\mathfrak{so}(n)$ is semisimple for $n \geq 3$.

5. **(a)** Show that if $\mathfrak{g}$ is a semisimple matrix Lie algebra, then $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is an isomorphism.

[Given $D \in \mathrm{Der}(\mathfrak{g})$ show that there is $X \in \mathfrak{g}$ such that $\mathrm{Tr}(D \circ \mathrm{ad}Y) = B(X,Y)$ for $Y \in G$. Then show that $D = \mathrm{ad}X$, i.e. $B((D - \mathrm{ad}X)Y, Z) = 0$ for all $Y, Z \in \mathfrak{g}$.]

(b) Show that if G is a connected matrix Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$ semisimple, then $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ has range $\mathrm{Aut}(\mathfrak{g})_0$ and kernel $\mathrm{Z}(G) = \{g \in G : ghg^{-1} = h \text{ for all } h \in G\}$. Deduce that $\mathrm{Z}(G)$ is discrete.

(c) Show that if G is a matrix Lie group for which the Killing form on $\mathfrak{g} = \mathrm{Lie}(G)$ is negative definite — i.e. $B(X,X) < 0$ for $X \in \mathfrak{g} \setminus \{0\}$ — then $\mathrm{Aut}(\mathfrak{g})$ is compact.

[Show that $\mathrm{Aut}(\mathfrak{g})$ is a subgroup of $\mathrm{O}(B)$.]

(Bonus) Show that if G is as in (c), above, then $\mathrm{Z}(G)$ is necessarily finite, hence G is compact.