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Les Houches Lectures on Fields, Strings and Duality*

Robbert Dijkgraaf

Department of Mathematics
University of Amsterdam
Plantage Muidergracht 24,
1018 TV Amsterdam
rhd@wins.uva.nl

Abstract

Notes of my 14 'lectures on everything' given at the 1995 Les Houches school. An introductory course in topological and conformal field theory, strings, gauge fields, supersymmetry and more. The presentation is more mathematical then usual and takes a modern point of view stressing moduli spaces, duality and the interconnectedness of the subject. An apocryphal lecture on BPS states and D-branes is added.

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Contents

1. Introduction

This is an almost literal write-up of the 14 lectures I gave at the 1995 Les Houches school *Quantum Symmetry*, plus an extra lecture on BPS states and D-branes that summarizes some important developments that have taken place after the school. It should be regarded as a broad-brushed sketch of modern views on quantum field theory and string theory, stressing moduli spaces, duality and the interconnectedness of the subject. I have taken perhaps a more mathematical point of view than is usual in these kind of introductory texts. I also tried to emphasize the common themes of various fields, perhaps at the expense of completeness and many details.

In the last decades, in a long series of abstractions and generalizations, string theory has emerged as the leading candidate for a fundamental theory of nature. Part of the motivation is (unfortunately, from a physical point of view) the intrinsic mathematical beauty of the theory. In fact, both directly and indirectly string theory has influenced various mathematical fields, and *vice versa* string theory has been influenced by recent mathematical development. In the last two years we have witnessed a marked change in pace in all this. In a confluence of a wide variety of ideas, many of them dating back to the 70s and 80s, the structure, internal consistency and beauty of string theory has greatly improved. We have now a much better and clearer picture of what string theory is about. Somehow the original concept of one-dimensional strings generalizing zero-dimensional point particles has become less central. The extended nature of the string and the infinite tower of oscillations that comes with it, serves more to naturally regularize the important field-theoretic massless excitations that include the gravitons and gauge bosons that mediate the unified forces. Indeed, it is not at all clear that the massive modes will form stable asymptotic states that can be (in principal) observed.

More than just a generalization of field theory, the crucial fact seems to be that string theory leads by itself naturally to the most important physical principles, such a quantum mechanics, general relativity, gauge theories and supersymmetry, without assuming these from the beginning. Indeed, it seems that any quantum theory that combines these four ingredients has to be a string theory. One therefore gets a the feeling that we are essentially dealing with a unique, very constrained object. The precise mathematical definition of this object still eludes us, but the remarkable internal consistency has been a constant source of new insights in the nature of quantum systems, the dynamics of gauge theories, the geometry of space-time, and last but not least a wonderful inspiration for beautiful mathematics.

Crucial in all the recent developments has been the concept of *duality.* In fact, duality has been a powerful idea in physics for a long time, both in statistical mechanics and field theory. In many respects duality in field theory and string theory can be used as an organizing principle, as I will try to do in these lectures.

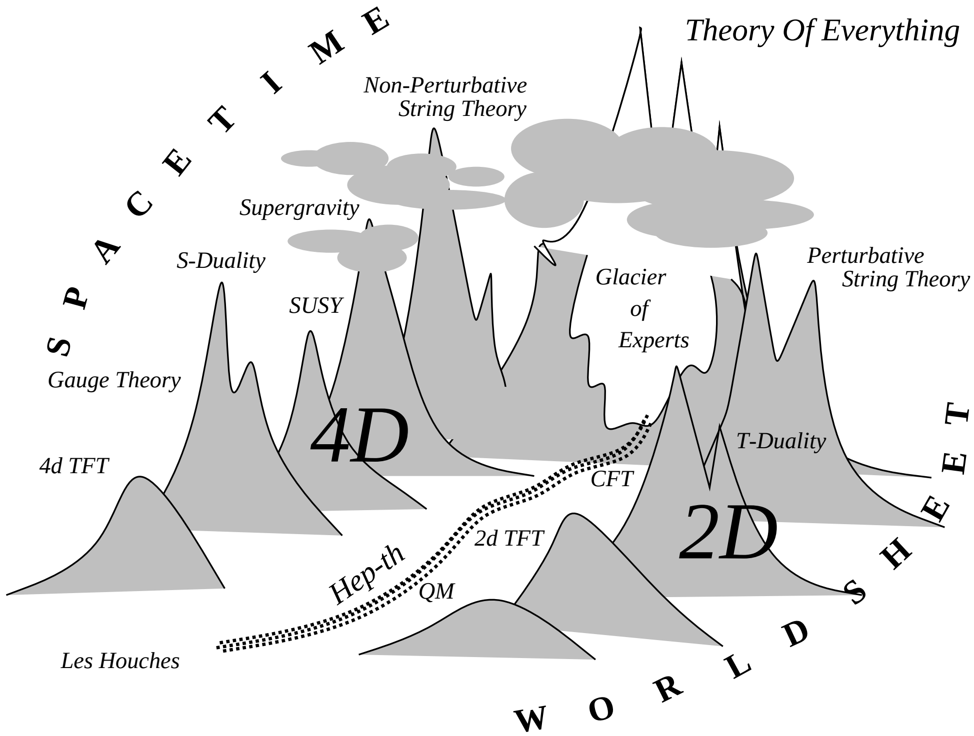


Fig. 1: *A road map for the lectures.*

The purpose of these notes is to give an introduction to many of the important elementary concepts from a point of view that is considerable more mathematical than usual, as was dictated by the particular audience of this Les Houches school. We will start at a (pedantically) low level and try to work our way up at a steady pace. In my lectures I used the analogy of a mountain walk (in a leisurely, Swiss style) and I gave the road map of *fig.* 1. as a summary of the course.

Let me describe it in some detail. The scenery consists of two mountain ranges that we can label as *2D* or *World-Sheet* and as *4D* or *Space-Time.* They both lead to the great summit of the *Theory of Everything.* Unfortunately, it is permanently hidden in a thick layer of clouds, so we have no guarantee that our trail will actually reach this summit. From the slopes we see the thick *Glacier of Experts.* Now and then we witness enormous avalanches on the glacier when big pieces of knowledge drop down and into the ferocious preprint flow called *Hep-th* that comes running down the glacier. This treacherous stream is very difficult to navigate, at least if you are a typical Les Houches student. We will walk two trails that explore the scenery, avoiding the glacier and the flow of preprints, that will each take half of the course of lectures.

The first one traverses the two-dimensional range. If the first few miles of this trail look familiar and uninteresting, I beg the reader patience, since the surroundings will change quickly. We will start with the small hills of *Quantum Mechanics,* after which we will move to the barren scenery of *2D Topological Field Theory,* a bare bones version of quantum field theory. In the next stage we will put some flesh to these bones, and explore *Conformal Field Theory.* Then we are ready to meet the first examples of *T-Duality* in the context of toroidal sigma models. After all this we make our first attempt to scale *String Theory.* However, we exclusively take a perturbative point of view using the language of Riemann surfaces. This will slow down our pace and, more important, our view. Slightly disappointed with the vista, we return to base camp and plan a second effort, avoiding the Riemann surfaces.

In our second walk we explore the four-dimensional world, keeping consistently a space-time perspective. Now we can use non-perturbative techniques to our advantage. We start by looking at *S-Duality* in abelian gauge theory. After a little detour to study four-dimensional *Topological Field Theory,* we then move on to *Supersymmetry*, in particular supersymmetric gauge theories, spending a lot of time on exploring the vacuum structure. Here the most prominent feature is the *Seiberg-Witten Mountain Pass* that allows us to leap forward into the non-perturbative world. The next mountain, *Supergravity*, we will mostly skip because the guide feels very uneasy at its slippery slopes. Instead we start immediately our second ascent of *String Theory*, now using the powerful tools of duality. After this strenuous trip, we spend some time at high altitude to look for stringy *BPS States*, that we can take home as a souvenir of our adventure and show our mathematical friends. Slightly dizzy of spending too much time at these great heights, we return back home, leaving the guide exhausted. (He had to recover for more than a full year, before he could even start recounting this experience!)

Needless to say, it is unthinkable to make this trip and cover all territory in fine detail, particularly if one walks in the high mountains. Sometimes we have to jump over a few crevasses, and we might not have always kept a careful balance. Whenever the reader gets lost, I hope the (always incomplete) references to the extensive literature can help to find the way back to the trail. The reader might even try a refreshing dip in the *Hep-th* river in the end. Enjoy our trip!

2. What is a quantum field theory?

Before we start the actual lectures, let us make a few very general philosophical remarks about quantum field theory — a hard to grasp subject, despite decades of physical and mathematical efforts.

2.1. Axioms vs. path-integrals

There are basically two ways in which one can approach quantum field theories: either in terms of axioms or in terms of quantization.

1. Axioms. Here we do not refer to any classical theory. One simply looks for a consistent set of amplitudes or correlation functions, satisfying certain well-defined sets of axioms. Here one can think of such diverse examples as the Wightman's formulation of axiomatic field theory [1] or Segal's geometric axioms of conformal field theory [2]. Although this approach is much preferred from a mathematical point of view, it is, unfortunately, highly non-trivial to define realistic four-dimensional QFTs along these lines. In fact, only for simpler systems such as topological field theories or two-dimensional theories can this powerful approach be used to its full extent and satisfaction.

2. Quantization. The textbook approach [3]. Here one starts with some classical action $S(\phi)$ for a classical field $\phi(x)$ and subsequently tries to make sense of the path-integral (over some appropriately defined big enough, infinite-dimensional space of field configurations)

$$A = \int \mathcal{D}\phi \, e^{iS(\phi)/\hbar} \, P(\phi), \tag{2.1}$$

with $P(\phi)$ some local expression representing the correlation function. One usually first tries to interpret this definition in perturbation theory in Planck's constant $\hbar$. In a certain regime, the amplitudes can have a meaningful, although most likely asymptotic, expansion of the form

$$A \sim \sum_n A_g \hbar^g, \tag{2.2}$$

where the perturbative coefficients A_g are computed as sums over all Feynman diagrams Γ with a fixed number of g loops [3]

$$A_g = \sum_\Gamma \frac{w_\Gamma}{\#\mathrm{Aut}\,(\Gamma)}. \tag{2.3}$$

Here the individual weight w_Γ of the diagram Γ is computed using the Feynman rules and we divide by the order of the symmetry group of the diagram (as one should do in all generating functions as a matter of principle). This relation with graphs explains why quantum field theories can describe point-particles and their interactions. The expansion of the path-integral in this combinatorical, diagrammatical fashion is of course a very general property of the asymptotics of integral of weights $e^{iS(\phi)}$, where $S(\phi)$ is a polynomial in ϕ. A famous finite-dimensional example of such an integral is the Airy function.

From this perturbative point of view there is no essential difference between a field theory and a string theory. String theories just give rise to amplitudes that allow expansions where the coefficients A_g are given as sums of *surfaces* with g handles. In general the weight of a particular surface is computed as an integral over the moduli space of inequivalent conformal metrics $\mathcal{M}_g$ on the surface [4]

$$A_g = \int_{\mathcal{M}_g} w_g. \tag{2.4}$$

The beauty of perturbative string theory, as contrasted with perturbative point-particle field theory, is that at fixed order g in the above expansion there is only one surface with

g handles to consider, whereas the number of connected graphs growths as $g!$. So string theory simplifies the combinatorics. In the usual low-energy limit reduction of string theory to the supergravity quantum field theory, these various diagrams can be recovered as particular limits of degenerated surfaces in the "corners" of the moduli space $\mathcal{M}_g$.

It is clear that this expansion in terms of surfaces will not be generated in any obvious fashion by a path-integral of the form (2.1). Indeed, string field theory, which tries to do exactly that, is a particularly complicated (and some would say, unnatural) effort to capture the surface expansion is terms of graphs. In order to do this successfully, one has to cut up the moduli space of Riemann surfaces $\mathcal{M}_g$ in cells, labeled by graphs, such that the Feynman sum over graphs reduces to an integral over $\mathcal{M}_g$. It is quite remarkable that this can be done in a completely well-defined and mathematically very beautiful way [5], and this formalism makes use of the most advanced concepts in the quantization of gauge systems, such as the BV-formalism [6]. But all in all, at this moment the conceptual advances in this approach are less striking.

However strong the intuition behind the semi-classical picture is, both in field theory and in string theory, the perturbative point of view does not adequately capture everything about a quantum field theory, as has become clear from our growing understanding of the phenomenon of duality.

2.2. Duality

Indeed, recently a particular powerful framework to think about quantum field theories has emerged, that is somehow complementary to the axiomatic or path-integral approach and that stresses the concept of a *moduli space* of quantum field theories. A central role is played by duality symmetries. (For a nice physical introduction see *e.g.* [8].) Through these dualities the semi-classical quantization point of view looses some of its uniqueness.

Just as is the case for many other fields of mathematics (in particular algebraic geometry) it is useful to consider *families* of the objects that one is studying. The space that parametrizes the objects is usually called a moduli space, and can be studied on its own. For example, in the case of Riemann surfaces we obtain the moduli space $\mathcal{M}_g$, first introduced by Riemann and one of the most intricated and beautiful spaces in algebraic geometry, see *e.g.* [7]. Other famous moduli spaces are the spaces of vector bundles, self-dual connections, holomorphic maps *etc.*

So, following this line of thought, it is natural to consider a family of quantum field theories. This gives us a parametrized set of amplitudes $A(\lambda)$, where the moduli $\lambda_1, \lambda_2, \ldots$ take value in a moduli space $\mathcal{M}$. Depending on the particular kind of quantum field theory that we are interested in, the space $\mathcal{M}$ will carry certain geometric structures. Indeed, one of the important issues in these lectures will be how properties of the QFT translate into properties of the moduli space.

The classical action underlying the QFT can be recovered in perturbation theory if there exists a point (or, more generally, a codimension one subspace) in $\mathcal{M}$ where one

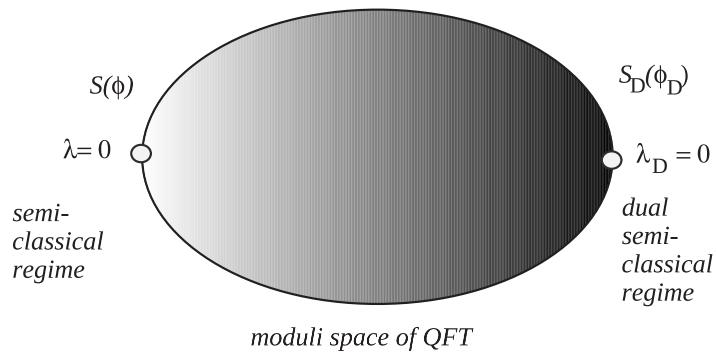


Fig. 2: *In the moduli space of a QFT we could have two possible semi-classical, perturbative regimes, described by fields ϕ with a classical action $S(\phi)$ and dual fields ϕ_D with a classical action $S_D(\phi_D)$. The transformation relating the variables in the two different patches is a duality transformation.*

of the moduli vanishes, say $\lambda = 0$, and where the coefficients A_g in the perturbative expansion of A in λ

$$A(\lambda) \sim \sum A_g \lambda^g \tag{2.5}$$

can be computed by Feynman diagrams. Out of the Feynman rules we can then "in weak coupling" reconstruct the action $S(\phi)$. After the fact we can then identify λ as Planck's constant. The classical action thus captures the description of the QFT in the neighbourhood of $\lambda = 0$. The zero-locus $\lambda = 0$ is then parametrized by the remaining moduli, that get now reinterpreted as a set of classical coupling constants that appear in the action $S(\phi)$.

And as we stressed above, this perturbative information is in general incomplete (*e.g.* has a zero radius of convergence) and has to be be supplemented by non-perturbative corrections. We can think of the usual semiclassical approach as giving some preferred set of local coordinates on $\mathcal{M}$ in the neighbourhood of $\lambda = 0$ (technically the normal bundle of the locus $\lambda = 0$).

However, it is very well possible that there is another point λ_0 in the moduli space with analogous properties, see *fig.* 2. That is, there can exist a second expansion parameter $\lambda_D = \lambda - \lambda_0$ and a second expansion

$$A(\lambda_0 + \lambda_D) \sim \sum B_g \lambda_D^g \tag{2.6}$$

with exactly the same properties as the original, semi-classical expansion (2.5). Out of the coefficients B_g we can now recover an *a priori* different classical action $S_D(\phi_D)$, possibly using a completely different set of fields ϕ_D. In that case we have two alternative "quantizations" that give rise to the same family of quantum field theories, and we speak of a dual description, with dual fields and a dual action. Note that we also have two candidates for Planck's constant, λ and the dual expansion parameter λ_D. This makes the concept of quantization ambiguous, to say the least.

There are various examples of such dual formulations, some known for a long time, some only discovered recently. A rather famous case of a duality transformation is the identification between free bosons and free fermions in two dimensions, or more generally the identification between the Sine-Gordon model and the massive Thirring model [9], see *e.g.* the textbook [10]. The two models describe respectively a scalar field $\varphi(x)$ with an interaction of the form

$$\lambda^{-1}\cos(\lambda^{\frac{1}{2}}\varphi)-1) \tag{2.7}$$

and a (massive) fermion field $\psi(x)$ with interaction

$$\lambda_D(\overline{\psi}\psi)^2. \tag{2.8}$$

The claim is that these models are physically equivalent, where the coupling constants λ, λ_D are related by

$$\lambda = \frac{1}{1+\lambda_D} \tag{2.9}$$

In particular weak coupling in the Sine-Gordon model corresponds to strong coupling in the Thirring model. In the solution of Coleman the fermion field ψ is written in terms of the boson φ as the non-polynomial vertex operator

$$\psi(x) =: e^{i\varphi(x)} : \tag{2.10}$$

This operator can be seen to create a soliton or kink in the Sine-Gordon model. *Vice versa,* the bosons are recovered in the fermionic model by "bosonizing" the charge current as

$$\partial\varphi(x) = \overline{\psi}\psi(x). \tag{2.11}$$

This two-dimensional example captures some important characteristics of the more general picture. Quite often the dual variables describe solitonic objects in the original weakly coupled system. These solitons become infinitely massive at weak coupling. At strong coupling however, the solitonic degrees of freedom can be come very light and take over the role as preferred classical fields.

For instance, the Montonen-Olive duality [11, 8] of four-dimensional $N = 4$ non-abelian super Yang-Mills theory interchanges the gauge bosons, that are typically thought of as fundamental fields, with the 't Hooft-Polyakov monopoles [12] that appear as solitonic solutions to the classical field equations, thereby also interchanging electric and magnetic fields. As such it is a quantum realization of the classical electric-magnetic duality of (abelian) Maxwell theory, that we will discuss at great length in §9. These kind of dualities have been generalized successfully to string theory recently, see *e.g.* [13, 14, 15, 16, 17] and the reviews [19, 20]. Indeed, string theory is at present our most fruitful source of these bizarre quantum equivalences. We will see examples of this later in the course.

An additional phenomenon that is nicely illustrated by the example of Montonen-Olive duality, is the concept of *self-duality.* Namely, it might be the case that the "new" dual theory we discovered at the second expansion point is actually identical to the old one, *i.e.*

$$S(\phi; \lambda') = S_D(\phi_D; \lambda'_D). \tag{2.12}$$

Here λ' denote the remaining moduli that parametrize the classical action and the subspace $\lambda = 0$. In the case that such a quantum identification exists, the moduli space $\mathcal{M}$ is clearly not parametrizing inequivalent quantum field theories, since the points $\lambda = 0$ and $\lambda_D = 0$ represent the same theory. We therefore have to divide by a further group G that identifies the dual descriptions. This group G is called the *duality group.* It can be regarded as the natural automorphism group of the QFT. The proper moduli space is then the quotient of $\mathcal{M}$ by G.

If the dual coupling constants λ and λ_D are related as

$$\lambda_D = 1/\lambda, \tag{2.13}$$

or more generally, if the limit $\lambda_D \to 0$ corresponds to $\lambda \to \infty$, we speak of an *S-duality.* This will then relate weak coupling $\lambda = 0$ to strong coupling $\lambda = \infty$. A typical example in statistical physics of such a duality is the Kramers-Wanier duality of the Ising model, another one is the Montonen-Olive duality mentioned above. In string theory these S-dualities are particularly powerful.

The main lesson that duality teaches us, is that "quantized fields" might not be the ultimate way to think about quantum field theories! It might be necessary to cover the moduli space $\mathcal{M}$ of quantum field theories (or strings for that matter) by local patches, in a way analogous to the coordinate patches by which we cover a manifold. In certain patches we have a formulation in terms of fields, actions, and path-integrals. The precise choice of fields and action might change from patch to patch. The transition functions are the duality transformations. There might also be parts of the moduli space where no such semi-classical description exist at all, and we are left with the "abstract" QFT.

This general description is of course terribly academic at this point. Only when we will meet various examples of duality groups during the lectures, it will become clear how powerful this notion of duality is. With these warnings out of the way, let us now start the proper lectures.

3. Quantum mechanics

We start on familiar grounds — quantum mechanics. In it simplest form quantum mechanics consists of a Hilbert space $\mathcal{H}$ and a unitary map

$$\Phi(t) : \mathcal{H} \to \mathcal{H} \tag{3.1}$$

describing the time evolution of the system during a time interval t. The composition law of time evolution tells us these transition amplitudes satisfy the relation

$$\Phi(t_1) \circ \Phi(t_2) = \Phi(t_1 + t_2), \tag{3.2}$$

which implies the existence of an hermitian Hamiltonian H with $\Phi(t) = e^{itH}$. We will often consider the Euclidean case, obtained by the analytic continuation $t \to it$, where $\Phi(t) = e^{-tH}$.

The Euclidean partition function is defined as

$$Z(t) = \operatorname{Tr} e^{-tH} \tag{3.3}$$

(if this makes mathematical sense) and will then be a good way to encode the spectrum and degeneracies of the system. Indeed, if the spectrum of H is discrete with eigenvalues ϵ_n and finite degeneracies $d(n) = \dim \mathcal{H}_n$ of the corresponding eigenspaces $\mathcal{H}_n$ we have

$$Z(t) = \sum_n d(n) e^{-\epsilon_n t}. \tag{3.4}$$

A particular useful quantum mechanical model is the point particle moving on a space X and its supersymmetric cousin that we will consider in a moment. In this case we have coordinates and momenta (q^μ, p_μ) that take value in the standard phase space T^*X, the total space of the tangent bundle to the manifold. In canonical quantization the Hilbert space is given as $\mathcal{H} = L^2(X)$ and we realize the operator p_μ as $-i\frac{\partial}{\partial q^\mu}$.

In the path-integral approach to quantization, we pick a Lagrangian of the form

$$L = p_\mu \dot{q}^\mu - H(p, q) \tag{3.5}$$

with $H(p, q)$ the Hamiltonian function. With this Lagrangian we compute the matrix elements

$$K_t(y, x) = \langle y | \Phi(t) | x \rangle \tag{3.6}$$

of the transition amplitude $\Phi(t)$ as the Feynman-Kac path-integral over all paths with $q(0) = x$, $q(t) = y$

$$K_t(y, x) = \int \mathcal{D}p\mathcal{D}q \, e^{-\int_0^t L}. \tag{3.7}$$

The additivity of time translations (3.2) gets reflected in the composition law of these kernels

$$K_{t_1+t_2}(z,x) = \int_X dy\ K_{t_1}(z,y) K_{t_2}(y,x). \tag{3.8}$$

The simplest case is actually the one with vanishing Hamiltonian, $H = 0$, which one could call topological quantum mechanics. This is an almost empty system. As we see by doing the integral over p, the path-integral localizes to $\dot{q} = 0$, *i.e.* to constant configurations $q \in X$. So the evolution is indeed trivial with

$$K_t(x,y) = \delta(x,y) \tag{3.9}$$

and $\Phi(t) = \mathbf{1}$. The only ingredient is this model is therefore the Hilbert space $\mathcal{H} = L^2(X)$.

For the usual point particle we pick a metric $g_{\mu\nu}$ on X and define $H = p^2$. The Hamiltonian is therefore represented as the Laplacian on X,

$$H = -\Delta. \tag{3.10}$$

In that case $K_t(x,y)$ reduces to the usual heat-kernel, satisfying

$$\Delta_x K_t(x,y) = \partial_t K_t(x,y), \qquad K_0(x,y) = \delta(x,y). \tag{3.11}$$

An important example is the case of a torus, $X = T^n$, represented as the quotient space $\mathbf{R}/2\pi\Lambda$, with Λ a lattice. Since the spectrum of p is now given by the dual lattice Λ^*, the partition function of this model becomes a theta-function (see §7.3)

$$Z(t) = \mathrm{Tr}_{\mathcal{H}} e^{-tH} = \sum_{p\in\Lambda^*} e^{-2\pi t p^2} \tag{3.12}$$

3.1. Supersymmetric quantum mechanics

$N = 1$ supersymmetric quantum mechanics leads to spinors on the manifold X and has proven to be of fundamental importance to understand for example index theorems. In topological applications, the $N = 2$ supersymmetric point particle is more useful (see *e.g.* [21], or for a recent analysis [22]). Here one introduces besides the bosonic variable q,p two more fermionic variables $\theta,\overline{\theta}$ satisfying

$$\theta^2 = \overline{\theta}^2 = 0, \qquad \theta\overline{\theta} = -\overline{\theta}\theta, \tag{3.13}$$

and a Lagrangian

$$L = p\dot{q} + i\overline{\theta}\dot{\theta} - H. \tag{3.14}$$

In canonical quantization we have $\{\theta, \overline{\theta}\} = 1$ with the reality condition $\theta^\dagger = \overline{\theta}$, so that we can consider θ as a coordinate and $\overline{\theta}$ as the conjugated variable

$$\overline{\theta} = \frac{\partial}{\partial \theta}, \tag{3.15}$$

which acts as an operator on wave functions

$$\psi(q, \theta) \in \mathcal{H} = L^2(\mathbf{R}^{1|1}) \cong \Omega^*(\mathbf{R}). \tag{3.16}$$

Supersymmetric wave functions can either be viewed as functions on the vector space $\mathbf{R}^{1|1}$, which is by definition the vector space with one even and one odd coordinate, or they can be viewed as differential forms on $\mathbf{R}$. In this latter point of view one considers θ as the one-form dq and expand the wave function as

$$\psi(q, \theta) = \psi^{(0)}(q) + \psi^{(1)}(q)\theta. \tag{3.17}$$

More generally, for a superparticle on a general n-manifold X, the Hilbert space is in this way interpreted as the total space of normalizable differential forms on X,

$$\mathcal{H} = L^2(\widehat{X}) \cong \Omega^*(X). \tag{3.18}$$

Here the $n|n$ dimensional supermanifold $\widehat{X}$ is defined as ΠTX, where the parity transformation Π declares the fibers of the tangent bundle TX to be anti-commuting. The second isomorphism is implied by the more general expansion of a super wave function as

$$\psi(q, \theta) = \sum_{k=0}^{n} \psi^{(k)}_{\mu_1,\ldots,\mu_k}(q)\theta^{\mu_1} \cdots \theta^{\mu_k}. \tag{3.19}$$

The coefficients $\psi^{(k)}_{\mu_1,\ldots,\mu_k}$ are completely antisymmetric and are naturally interpreted as differential k-forms on X.

Note that the Hilbert space has natural $\mathbf{Z}$ gradation. We can define fermion number F by given θ charge one and consequently $\overline{\theta}$ charge minus one. Fermion number will then correspond to the notion of degree for differential forms.

The supersymmetry transformations are given by

$$\delta q^\mu = \theta^\mu, \qquad \delta \overline{\theta}_\mu = i p_\mu, \tag{3.20}$$

so that $Q = i\theta^\mu p_\mu$. On wave functions Q equals the exterior differential d,

$$Q = \theta^\mu \frac{\partial}{\partial q^\mu} = d. \tag{3.21}$$

We thus find that $Q^2 = 0$ and its cohomology gives the de Rahm cohomology of X

$$H^*_Q(\mathcal{H}) = H^*_{dR}(X). \tag{3.22}$$

There is also the hermitian conjugated operator Q^*, defined by (here the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ enter)

$$Q^* = g^{\mu\nu}\overline{\theta}_\mu \frac{\partial}{\partial q^\nu} = d^* \tag{3.23}$$

These operators give the one-dimensional $N = 2$ supersymmetry algebra

$$\{Q, Q^*\} = H \quad \Longleftrightarrow \quad \{d, d^*\} = \Delta. \tag{3.24}$$

The supersymmetric ground states satisfy

$$Q|\psi\rangle = Q^*|\psi\rangle = 0, \tag{3.25}$$

and thus their wave functions correspond to harmonic differential forms,

$$d\psi = d^*\psi = 0. \tag{3.26}$$

The space $\mathcal{H}_0$ of supersymmetric ground states is thus canonically identified as

$$\mathcal{H}_0 = \mathrm{Harm}^*(X). \tag{3.27}$$

We can compute the superdimension of the Hilbert space (defined as the dimension of the even part minus the dimension of the odd part) in terms of the Witten index [23]

$$\mathrm{sdim}\,\mathcal{H} = \mathrm{Tr}\,(-1)^F = \mathrm{sdim}\,\mathcal{H}_0. \tag{3.28}$$

It equals the Euler number of the space X

$$\mathrm{sdim}\,\mathcal{H} = \chi(X) = \sum_k (-1)^k \dim H^k(X). \tag{3.29}$$

These relations between supersymmetry and algebraic topology were first stressed by Witten [24] and permeate the whole subject.

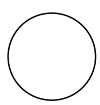
Perturbative quantum field theories that describe particles and their interactions can be obtained in "first quantized" form from quantum mechanics by making the metric on the world-line of the particle dynamical. We integrate the amplitudes over the world-line time $t \geq 0$.

For example, the usual free field massless propagator, associated to a line segment, takes the form

$$\text{———} = \int_0^\infty dt\, e^{-tH} = -\frac{1}{\Delta} = \frac{1}{p^2}. \tag{3.30}$$

To include interactions one considers "one-dimensional quantum gravity" where the space-time is an arbitrary graph Γ. The space of metrics modulo diffeomorphisms on such a graph is parametrized by an assignment of lengths $t_i \geq 0$ to all the edges of the graph. So, we sum over all graphs and integrate over the lengths with some particular weights. Of course, we can think of these lengths as the Schwinger parameters of the Feynman diagrams of quantum field theory. As such they are completely equivalent to the moduli of Riemann surfaces that appear in string theory and that we discuss at great length later on.

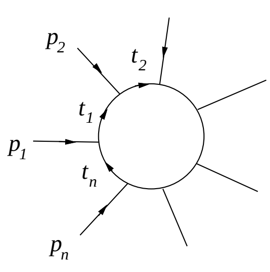
For the one-loop amplitude one finds in this way



$$= \tfrac{1}{2} \int_0^\infty \frac{dt}{t} \mathrm{Tr}\, e^{-tH} = -\tfrac{1}{2} \mathrm{Tr}\, \log \Delta = -\tfrac{1}{2} \log \det \Delta. \tag{3.31}$$

(The extra factor $1/t$ in the integrand has to do with the action of the circle group on the graph. We have to factor by the volume of this isometry group, which is given by t. The factor $\frac{1}{2}$ is a reflection of the $\mathbf{Z}_2$ symmetry of the circle graph, that flips the orientation.)

Simlarly, a more general one-loop scattering amplitude, with external momenta $p_1, \ldots, p_n$ satisfying $p_i^2 = 0$, can be represented as



$$= \int dt_1 \cdots \int dt_n \cdot \mathrm{Tr}\left(e^{ip_1 x} e^{-t_1 H} e^{ip_2 x} e^{-t_2 H} \cdots e^{ip_n x} e^{-t_n H} \right) \tag{3.32}$$

A more modern presentation of this first-quantized picture of QFT would use the notions of ghosts and BRST operators. We add to the QM system two anticommuting fields b and c with action

$$S_{gh} = \int dt\, b\dot{c}, \tag{3.33}$$

which gives canonical anticommutation relations $\{b, c\} = 1$ and a two-dimensional ghost Hilbert space $\mathcal{H}_{gh}$. It consists of the ghost vacuum of degree zero defined by $b|0\rangle = 0$ and the state $|1\rangle = c|0\rangle$ in degree one.

The full Hilbert space is now defined as the tensor product of the "matter" Hilbert space $\mathcal{H}_m = L^2(\mathbf{R}^n)$, for which we choose a momentum basis $|p\rangle$, $p \in \mathbf{R}$, and the ghost Hilbert space $\mathcal{H}_{gh}$

$$\mathcal{H} = \mathcal{H}_m \otimes \mathcal{H}_{gh} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}. \tag{3.34}$$

As indicated it has graded pieces in degree zero and one. The BRST operator is given by the degree one operator

$$Q = cH. \tag{3.35}$$

On a general state Q acts as

$$\begin{aligned} Q|p\rangle \otimes |0\rangle &= p^2 |p\rangle \otimes |1\rangle, \\ Q|p\rangle \otimes |1\rangle &= 0. \end{aligned} \tag{3.36}$$

The space V of "physical" states is now defined as the cohomology of Q,

$$V = H_Q^*(\mathcal{H}). \tag{3.37}$$

One easily verifies that these physical states are of the form

$$|phys\rangle = |p\rangle \otimes |1\rangle, \qquad p^2 = 0. \tag{3.38}$$

They are created out of the vacuum, $|vac\rangle = |0\rangle \otimes |0\rangle$, by the action of the vertex operators $\phi_p = c \cdot e^{ipx}$ with $p^2 = 0$,

$$|phys\rangle = \phi_p |vac\rangle. \tag{3.39}$$

Note that any expectation value of the form

$$\langle phys' | \{Q, \mathcal{O}\} | phys\rangle \tag{3.40}$$

vanishes, since Q annihilates both the bra and the ket. In particular we have the algebra

$$\{Q, b\} = H, \tag{3.41}$$

which tells us that within physical correlation functions the Hamiltonian vanishes. This vanishing of the Hamiltonian is the signature of a theory of quantum gravity. We can also introduce operators

$$\phi_p^{(1)} = \{b, \phi_p\} = e^{ipx} \tag{3.42}$$

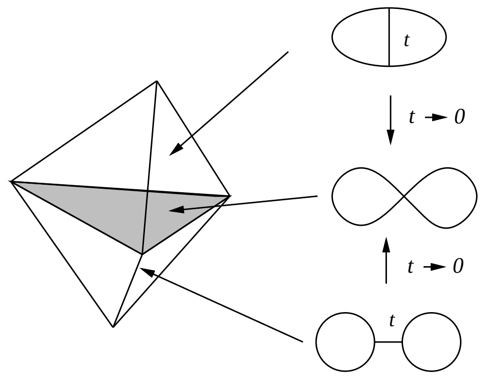


Fig. 3: *The Schwinger parameters of various Feyman graphs contributing to a particular amplitude can be glued together to form one connected moduli space.*

In terms of these operators the general one-loop amplitude (3.32) can then be written as

$$\int dt_1 \cdots \int dt_n \cdot \mathrm{Tr}\left(\phi_{p_1} e^{-t_1 H} \phi^{(1)}_{p_2} e^{-t_2 H} \cdots \phi^{(1)}_{p_n} e^{-t_n H}\right) \tag{3.43}$$

We will see later that this particular way of writing field theory amplitudes will be generalized in string theory.

As an aside we note that, for more complicated graphs, the Schwinger parameter spaces of the individual graphs can be glued together to form a connected moduli space that is quite analogous to the moduli space $\mathcal{M}_g$ of Riemann surfaces that will be a central point in the coming lectures. To every diagram Γ we can associate a cell whose points are parametrized by the length of the edges. These cells can be glued together in the following fashion. If one of the lengths becomes zero, the graph will change topology. We now glue the cell of this new graph to the side of the cell of the old graph, as is illustrated in *fig.* 3.. In this way we construct a cell complex that is a Feynman diagram analogue of $\mathcal{M}_g$ [26, 27].

4. Two-dimensional topological field theory

After our discussion of quantum mechanics and its relation to field theory, we now turn to to two-dimensional quantum field theories and their relation to string theory. The simplest type of quantum field theory is a topological field theory (TFT) which has the defining property that all amplitudes are independent of the local Riemannian structure. Two-dimensional TFTs will be the subject of this lecture. Here we will concentrate on the properties of one particular quantum field theory. Later in §8 we will generalize our analysis to include families of such theories.

4.1. Axioms of topological field theory

The axiomatic formulation of topological field theories has been given by Atiyah [28] following Segal [2] and uses the language of categories and functors. (A particularly thorough exposition of this approach is given in [29].)

Let us recall that a *category* contains a set of objects x and a set of arrows or morphisms $f : x \to y$, which should be regarded as abstract quantities satisfying an associative composition law. That is, given two arrows $f : x \to y$ and $g : y \to z$, we can form the composite arrow $g \circ f : x \to z$, such that $(h \circ g) \circ f = h \circ (g \circ f)$. A category further presumes that for each object x an identity arrow $\mathbf{1}_x : x \to x$ exists, satisfying $f \circ \mathbf{1}_x = \mathbf{1}_y \circ f = f$. A *functor* between two categories is a map that maps objects to objects, morphisms to morphisms, that respects all relations.

A good example is the category **Vect** of (say, complex, possibly graded) vector spaces, where the objects are obviously vector spaces and the morphisms correspond to linear maps between the vector spaces. This is actually an example of a so-called abelian tensor category, where the objects and morphisms can also be multiplied using the (associative and commutative) tensor product $\otimes$, and have an inverse, the linear dual V^*. If we have maps $\Phi_1 : V_1 \to W_1$ and $\Phi_2 : V_2 \to W_2$, we can form the tensor product map $\Phi_1 \otimes \Phi_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$. The unit is $\mathbf{C}$.

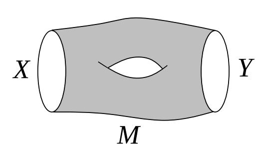
To define a topological quantum field theory, we start with the category $\mathbf{Man}(d)$ of d-dimensional manifolds, where the the objects are smooth, compact, oriented manifolds X (*not* defined up to isomorphism, and thus equipped with a given parametrization in local coordinates) and where the morphisms

$$M : X \to Y \tag{4.1}$$

are bordisms. That is, the morphism M is a smooth, oriented manifold of dimension $d+1$ with the property that it has two boundary components, isomorphic to X and Y, where the natural orientation coming from the orientation of M agrees with the orientation of Y but disagrees with the orientation of X,

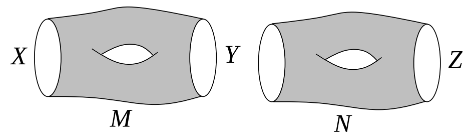
$$\partial M = (-X) \cup Y. \tag{4.2}$$

(So $-X$ indicates the same manifold X, now with the opposite orientation.) We will call the components X and Y "incoming" and "outgoing" respectively. In a figure we have



The properties of a category require that there exists a composition law of two bordisms

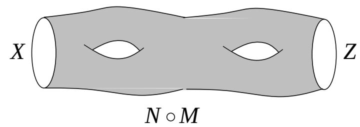
$M : X \to Y$ and $N : Y \to Z$,



producing a new bordism

$$N \circ M : X \to Z. \tag{4.3}$$

This composition law is obviously given by "gluing" the two boundaries,



that is, by identifying the two boundary components Y and $-Y$. We can do this in a unique way, since both copies of Y come with a parametrization. The identity morphism is given by the cylinder,

$$\mathbf{1}_X : X \times [0,1]. \tag{4.4}$$

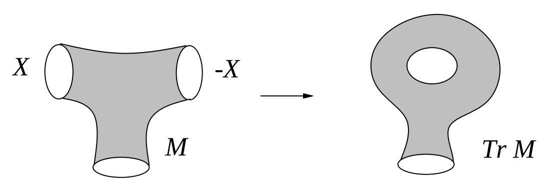
These definitions make $\mathbf{Man}(d)$ into a category.

We have one extra operation that is not standard in categories. We also want to be able to glue two boundary components of a single irreducible manifold: if the two boundary components of M contain a common factor X, we want to define the partial trace

$$\mathrm{Tr}_X : M \to \mathrm{Tr}_X(M). \tag{4.5}$$

This is best explained in a picture

Tr :



(4.6)

Note further that $\mathbf{Man}(d)$ is also a tensor category, with the product given by the disjoint union $\cup$ and unit the empty set $\emptyset$.

A $d+1$ dimensional *topological field theory* (TFT) can now be defined as a functor

$$\Phi : \mathbf{Man}(d) \to \mathbf{Vect} \tag{4.7}$$

from the category of d-manifolds to the category of vector spaces, satisfying certain extra properties. Concretely this means that to any space X we associate a vector space V_X,

$$\Phi : X \to V_X, \tag{4.8}$$

and to any bordism M a linear map Φ_M

$$(M : X \to Y) \quad \Rightarrow \quad (\Phi_M : V_X \to V_Y)\,. \tag{4.9}$$

The fact that Φ is a functor tells us that the amplitudes Φ_M should satisfy a factorization law

$$\Phi_{N\circ M} = \Phi_N \circ \Phi_M. \tag{4.10}$$

In case that the relevant operator is trace-class (or when all vector spaces V_X are finite-dimensional) we have an extra condition:

$$\Phi_{\mathrm{Tr}\,_X M} = \mathrm{Tr}\,_{V_X} \Phi_M. \tag{4.11}$$

Since both category are abelian tensor categories, we further demand that Φ respects the products $\otimes$ and $\cup$,

$$V_{X\cup Y} = V_X \otimes V_Y, \qquad V_{-X} = V_X^*, \qquad V_\emptyset = \mathbf{C}. \tag{4.12}$$

and

$$\Phi_{M\cup N} = \Phi_M \otimes \Phi_N, \qquad \Phi_{-M} = \Phi_M^*, \qquad \Phi_\emptyset = \mathbf{C}. \tag{4.13}$$

The existence of these amplitudes Φ_M can be physically motivated by the following path-integral argument. If $\phi(x)$ is a local set of fields in the theory, a state in the "Hilbert space" V_X will be a wave function $\Psi(\phi_X)$ on the space of field configurations $\phi_X(x)$ on the spacelike manifold X. The path-integral on M with fixed values ϕ_X, ϕ_Y at the boundaries X and Y then gives the kernel $K_M(\phi_Y, \phi_X)$ of the evolution operator Φ_M

$$K_M(\phi_Y, \phi_X) = \int\limits_{\phi|_X=\phi_X,\ \phi|_Y=\phi_Y} \mathcal{D}\phi\, e^{-S(\phi)} \tag{4.14}$$

The transition amplitude then relates an "incoming" wave function $\Psi_{in}(\phi_X)$ to an "outgoing" wave function $\Psi_{out}(\phi_Y)$ as

$$\Psi_{out}(\phi_Y) = \int \mathcal{D}\phi_X\, K_M(\phi_Y, \phi_X)\Psi_{in}(\phi_X) \tag{4.15}$$

in a huge generalization of the evolution operator e^{-tH} and its kernel $K_t(y, x)$ of quantum mechanics. The gluing law then corresponds to the composition law (3.8) in quantum mechanics

$$K_{N\circ M}(\phi_Z, \phi_X) = \int \mathcal{D}\phi_Y\, K_N(\phi_Z, \phi_Y) K_M(\phi_Y, \phi_X), \tag{4.16}$$

that tells us that evolving along M and then evolving along N is equivalent to evolving along $N \circ M$.

Some further scattered remarks:

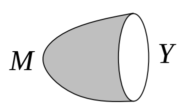
(1) We assume a natural action of the permutation group, possibly graded if we are dealing with a fermionic theory (the vector spaces V_X can be odd), if several boundary components are isomorphic.

(2) By definition the vector space associated with the empty set (a legitimate boundary!) is $\mathbf{C}$. Therefore, in case the $d+1$ dimensional manifold M is closed, $\partial M = \emptyset$, the corresponding morphism $\Phi_M : \mathbf{C} \to \mathbf{C}$ will be given by multiplication by a constant Z_M, the *partition function*. In path-integral language we have the identification

$$Z_M = \int \mathcal{D}\phi \, e^{-S(\phi)} \tag{4.17}$$

where one integrates over all field configurations on M.

(3) If M has a single outgoing boundary Y, so that $M : \emptyset \to Y$, the transition amplitude Φ_M is a map $\mathbf{C} \to V_Y$ and thus defines a special state



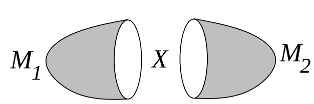
$$: \quad \Psi_M \in V_Y. \tag{4.18}$$

This can also be understood in path-integral language. The wave function is given by

$$\Psi_M(\phi_Y) = \int_{\phi|_Y = \phi_Y} \mathcal{D}\phi \, e^{-S(\phi)}. \tag{4.19}$$

Here we integrate over all possible extensions of the field ϕ_Y on Y to a field ϕ over all of M. In concrete situations, the field configurations on Y might be classified by certain topological indices that indicate whether or not this extension exists. (Here one can think about a spin structure for a fermion, or a vector bundle for a gauge field.) The vector space V_Y then splits into superselection sectors, and depending how the data can be extended over M, the vector Ψ_M will sit in one of these superselection sectors.

3. If a manifold M can be split into two parts M_1 and M_2 with common boundary $\partial M_1 = X = -\partial M_2$, we can compute the partition function as the inner product of the two states representing the two halves,



$$: \quad Z_M = \langle \Psi_{M_1}, \Psi_{M_2} \rangle. \tag{4.20}$$

This way of writing the partition function can be helpful if one wants to construct a new manifold M' by surgery on X. In that case one applies a diffeomorphism of X that is not in the identity component of $\pi_0\text{Diff}(X)$ and that is realized as an operator γ on the vector space V_X. The partition function of M' can then be written as

$$Z_{M'} = \langle \Psi_{M_1}, \gamma \Psi_{M_2} \rangle. \tag{4.21}$$

(4) If $M = [0,1] \times X$ and $N = S^1 \times X$, we can compute the dimension of the vector space V_X as the partition function of N (given the fact that this dimension is finite)

$$Z_N = \text{Tr}\, \Phi_M = \dim V_X. \tag{4.22}$$

4.2. Topological field theory in two dimensions

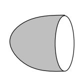
In this subsection we will restrict our investigations to two dimensions where our manifolds M are surfaces of genus g. This example has been explained quite often [25, 30, 31], but, with the risk of repeating myself once too often, I explain it here since it a necessary, though routine, part of our 2D route to string theory. For a very thorough review of these and other aspects of two-dimensional TFT see [32].

Since the only connected compact one-dimensional manifold is the circle S^1, we have only one vector space to consider

$$V = V_{S^1} \tag{4.23}$$

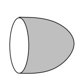
together with its dual. For convenience we will assume V to be finite dimensional. The data of a two-dimensional topological field theory are now obtained by considering respectively the sphere with one, two, and three holes. Let us briefly run through the argument.

(1) Following remark (2) in the previous section, the disk with an outgoing boundary gives rise to a particular state



$$\mathbf{1} \in V, \tag{4.24}$$

that we will denote as the identity, for reasons that become obvious in a moment. Similarly, the disk with one incoming boundary gives an element of the dual vector space,*i.e.* a linear functional

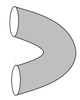


$$\langle \cdot \rangle_0 : V \to \mathbf{C}. \tag{4.25}$$

According to (4.20) the partition function of the two-sphere is then expressed as

$$Z(S^2) = \langle \mathbf{1} \rangle_0 \tag{4.26}$$

(2) The cylinder with one incoming and one outgoing boundary is just the identity map $\mathbf{1} : V \to V$. If we choose two incoming boundaries, it gives a bilinear map



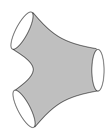
$$\eta : V \otimes V \to \mathbf{C}, \tag{4.27}$$

If we choose an explicit basis ϕ_i for V (with $\phi_0 = \mathbf{1}$) we obtain the graded symmetric matrix

$$\eta_{ij} = \eta(\phi_i, \phi_j) \tag{4.28}$$

By factorization, this inner product η will be non-degenerate (but not necessarily positive). It allows us to identify the incoming states in V with the outgoing states in the dual space V^*. The inverse is given by the cylinder with two outgoing boundaries. We will write $\eta^{-1}(\phi_i, \phi_j) = \eta^{ij}$. It is a simple exercise to check that $\eta \cdot \eta^{-1} = \eta \cdot^{-1} \eta = \mathbf{1}$. (Draw the corresponding pictures.)

(3) Finally the pair of pants, or the sphere with three holes, corresponds (again with the appropriate choice of orientations of the boundaries) to a map



$$c : V \otimes V \to V. \tag{4.29}$$

If we introduce the notation

$$\alpha \cdot \beta = c(\alpha, \beta), \tag{4.30}$$

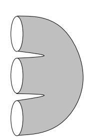
this makes V into an algebra, the operator product or Verlinde algebra of the topological field theory [33]. The multiplication c allows us to identify states with operators in a very simple way. It gives a map

$$c : V \to \mathrm{End}(V). \tag{4.31}$$

In a basis we will write the matrix corresponding to ϕ_i as C_i. We also introduce the structure coefficients ${c_{ij}}^k$ in

$$\phi_i \cdot \phi_j = \sum_k {c_{ij}}^k \phi_k. \tag{4.32}$$

Note that if we choose three incoming boundaries we get a map



$$c : \mathrm{Sym}^3 V \to \mathbf{C}, \tag{4.33}$$

which is characterized by the fully (graded) symmetric tensor

$$c_{ijk} = {c_{ij}}^l \eta_{lk}. \tag{4.34}$$

The inner product η is obtained by inserting the identity in this trilinear map, which gives in components

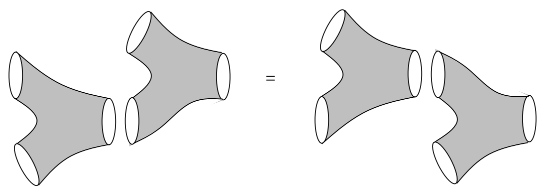
$$\eta_{ij} = c_{ij0}. \tag{4.35}$$

All these data suffice to calculate any partition or correlation function, since every surface can be reduced to a collection of three-holed spheres, cylinders and disks, by cutting it often enough. Of course, there are many inequivalent ways to factorize a particular surface. The final answer should however not depend on the particular choice of factorization. This gives further constraints on the data η and c.

For instance, a simple consequence of the symmetry of the 3-punctured sphere is the compatibility of the metric η with the multiplication

$$\eta(\alpha \cdot \beta, \gamma) = \eta(\alpha, \beta \cdot \gamma). \tag{4.36}$$

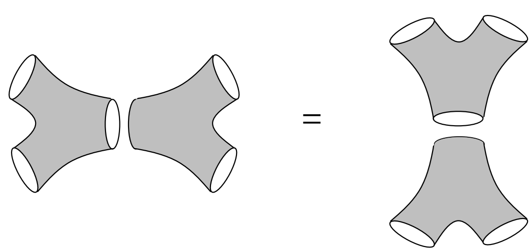
In a picture



Another relation expresses the inner product η in terms of the linear function $\langle \cdot \rangle_0$ as

$$\eta(\alpha, \beta) = \langle a \cdot \beta \rangle_0 \tag{4.37}$$

But most importantly, if we consider the sphere with four holes, there are two inequivalent ways of factorization



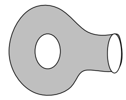
which translates in associativity of the algebra V

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma). \tag{4.38}$$

It can be easily checked that no further conditions will be found when we consider more complicated surfaces. An associative algebra V with such an invariant inner product

η is called a Frobenius algebra, and this simple object captures a 2d TFT completely. (For more on 2D TFTs see [32].)

With the concept of factorization, it is extremely easy to calculate higher genus partition and correlation functions. In fact, we can introduce an operator H that creates an handle. It is defined as the state associated to the torus with one puncture and has the representation



$$H = \sum_{i,j} c_i{}^{ij} \phi_j. \tag{4.39}$$

In this fashion a genus g partition function Z_g can be written as a genus zero correlation function

$$Z_g = \Big\langle \underbrace{H \cdots H}_{g} \Big\rangle_0 = \mathrm{Tr}\, H^{g-1}. \tag{4.40}$$

4.3. Example — quantum cohomology

Let us briefly discuss an important class of examples of Frobenius algebras: classical and quantum cohomology. We will see later how this is realized in (topological) sigma models.

We start with classical cohomology. Let X be a compact orientable manifold. We can take $V = H^*(X)$ with as multiplication the wedge product of differential forms $\alpha \cdot \beta = \alpha \wedge \beta$, which is graded commutative. On the states we now have an obvious inner product given by the intersection form

$$\eta(\alpha, \beta) = \int_X \alpha \wedge \beta. \tag{4.41}$$

Note that more generally the genus zero correlation functions are given as

$$\langle \alpha_1 \ldots \alpha_n \rangle_0 = \int_X \alpha_1 \wedge \cdots \wedge \alpha_n. \tag{4.42}$$

For example, if $X = \mathbf{P}^N$, the cohomology ring is simply generated by a generator x of degree two with the single relation

$$x^{N+1} = 0, \tag{4.43}$$

so $V = \mathbf{C}[x]/(x^{N+1})$, as an algebra.

It is clear that the cohomology ring $H^*(X)$ satisfies all the relations of a two-dimensional topological field theory. We note that we already represented $H^*(X)$ in supersymmetric quantum mechanics in §3.1. Here, however, we have a natural geometrical interpretation of the ring structure (although the multiplication can also be defined in quantum mechanics).

We really go beyond quantum mechanics if we consider quantum cohomology or Gromov-Witten invariants. For quantum cohomology we again take the vector space $V = H^*(X)$, but now define a new, improved "quantum" product [34, 35, 25, 36]. Let X now be a compact Kähler manifold and consider holomorphic maps[2] (sigma model instantons) from the Riemann sphere into X

$$x : \mathbf{P}^1 \to X, \qquad \overline{\partial} x = 0. \tag{4.44}$$

Let $\mathcal{N}$ denote the moduli space of such "stable" holomorphic maps (stability adds some special set of singular maps, making the moduli space compact). Let furthermore $\omega_1, \ldots, \omega_n$ be a integer basis of the Picard lattice $H^{1,1}(X) \cap H^2(X, \mathbf{Z})$. For any map we define a multi-degree $d = (d_1, \ldots, d_n)$ by

$$[x(\mathbf{P}^1)] = \sum d_i \omega_i. \tag{4.45}$$

The moduli space $\mathcal{N}$ will now decompose in components of different degree $\mathcal{N}_d$ (not necessarily irreducible)

$$\mathcal{N} = \bigcup_d \mathcal{N}_d, \tag{4.46}$$

with in particular the constant maps

$$\mathcal{N}_0 \cong X. \tag{4.47}$$

It is not easy to describe these instanton spaces for the generic case. However, there is a simple formula for the 'virtual dimension' that is defined as follows. (Here we consider maps $x : \Sigma \to X$, with Σ a general Riemann surface of genus g.) Consider the tangent space at a point $x \in \mathcal{N}_d$. This tangent space $T_x\mathcal{N}_d$ is by definition given by the infinitesimal maps δx satisfying $\overline{\partial} \delta x = 0$, *i.e.*

$$\delta x \in H^0(\Sigma, x^* T_X). \tag{4.48}$$

(T_X denotes the holomorphic $(1,0)$ tangent bundle of X.) We also have to consider the group $H^1(\Sigma, x^* T_X)$. The spaces H^0 and H^1 are the kernel and cokernel of the $\overline{\partial}$ operator on Σ twisted with the holomorphic vector bundle $x^* T_X$. The Hirzebruch-Riemann-Roch theorem gives an expression for the difference between the dimensions of these groups

$$\begin{aligned} \dim H^0 - \dim H^1 &= \int_\Sigma ch(x^* T_X) \cdot td(T_\Sigma) \\ &= n(1-g) - \int_\Sigma x^* c_1(X), \end{aligned} \tag{4.49}$$

[2] One can also take X to be a symplectic manifold and study pseudo-holomorphic maps, as introduced by Gromov to produce symplectic manifold invariants. This is how the subject started in mathematics [37].

with $n = \dim_{\mathbf{C}} X$. The RHS of this equation is known as the virtual dimension of the space $\mathcal{N}_d$. If H^1 vanishes, the virtual dimension equals the actual dimension. We see that the virtual dimension is independent of the homotopy of the map x in the Calabi-Yau case, with $c_1(X) = 0$,

$$\dim H^0 - \dim H^1 = n(1-g). \tag{4.50}$$

In the particular case of genus zero, $\mathcal{N}$ has virtual dimension n, the same as X.

We now choose parameters t^i and define the quantum cup product as

$$\langle \alpha_1 \ldots \alpha_n \rangle_q = \sum_d q^d \int_{\mathcal{N}_d} \varphi_i^* \alpha_1 \wedge \cdots \varphi_n^* \alpha_n, \tag{4.51}$$

where $q^d = \exp \sum d_i t^i$. Here we pull-back the cohomology classes α_i on X to the moduli space $\mathcal{N}$ of holomorphic maps by first considering the "universal instanton"

$$\Phi : \mathcal{N} \times \mathbf{P}^1 \to X, \qquad \Phi(x, z) = x(z), \tag{4.52}$$

and then choosing n sections $s_i : \mathcal{N} \to \mathbf{P}^1$. We then define $\varphi_i = \Phi \circ s_i$. One easily verifies that the answer does not depend on the various choices involved. This quantum product defines again an associative algebra. The associativity condition can be proved by a degeneration argument that we will sketch in more detail in §8.9. In the limit $q \to 0$ we recover the usual cohomology ring, which is the contribution of the identity component $\mathcal{N}_0$. For example, in the case of projective space, $X = \mathbf{P}^N$, the relation $x^{N+1} = 0$ gets deformed to [25]

$$x^{N+1} = q. \tag{4.53}$$

Many other cases have been worked out [38].

For later use we note here that for a Calabi-Yau three-fold the operator product coefficients c_{ijk} of the elements $\omega_i \in H^2(X)$ have an expansion [39, 40]

$$c_{ijk} = \int_X \omega_i \wedge \omega_j \wedge \omega_k + \sum_d N_d \frac{d_i d_j d_k q^d}{1 - q^d}, \tag{4.54}$$

where N_d counts the number of rational curves of degree d in X.

5. Riemann surfaces and moduli

We have seen that it is very simple to compute the partition function of a two-dimensional topological field theory for a genus g surface. It is simply given as

$$Z_g = \mathrm{Tr}\, H^{g-1}, \tag{5.1}$$

with H the handle operator. In string theory we also associate a number F_g, the g loop vacuum amplitude, to a topological surface of genus g, but in a much more involved way. Instead we consider the moduli space $\mathcal{M}_g$ of Riemann surfaces and define the genus g amplitude as

$$F_g = \int_{\mathcal{M}_g} Z_g, \tag{5.2}$$

in terms of a particular volume form

$$Z_g \in H^{top}(\overline{\mathcal{M}}_g), \tag{5.3}$$

that is produced by a two-dimensional conformal field theory. (Here the bar indicates the stable compactification, that we will discuss in §5.4.)

In terms of these higher-loop string amplitudes F_g the *space-time* partition function reads

$$Z_{spacetime} \sim \exp \sum_g \lambda^{2g-2} F_g \tag{5.4}$$

with λ the string coupling constant. (Unfortunately, these expansions do not converge, so as it stands $Z_{spacetime}$ is only a formal object.) Note that in the same notation we can say that a 2d TFT, which by definition does not depend on the complex structure on the surface, gives a partition function that is constant on $\mathcal{M}_g$ and thus can be regarded as an element of

$$Z_g \in H^0(\overline{\mathcal{M}}_g). \tag{5.5}$$

We now have to explain how quantum field theory leads to natural volume forms on $\mathcal{M}_g$. But before we do that, we spend the rest of this lecture to make some comments on Riemann surfaces and their moduli. This material is of course completely standard mathematics, see *e.g.* [41].

5.1. The moduli space of curves

There are basically three different ways to think about Riemann surfaces:

(1) As complex curves, *i.e.* one-dimensional complex varieties where the complex coordinates z and w in two patches are holomorphically related, $w = w(z)$. Equivalently, we are given a complex structure on the topological surface. Recall that a complex structure is a linear map J, defined in the tangent space at each point, that satisfies $J^2 = -1$ and the integrability condition $\nabla J = 0$. The complex structure allows us to split the complexified tangent space in holomorphic and anti-holomorphic vectors, with eigenvalues i and $-i$, and tells us what the (local) analytic coordinate is. Equivalently, we are given a $\overline{\partial}$ operator, and holomorphic functions are defined by $\overline{\partial} f = 0$.

(2) As algebraic curves, *i.e.* as the solutions of polynomial equations

$$f(x, y) = 0 \tag{5.6}$$

in two complex variables $x, y \in \mathbf{C}$. (Strictly speaking, solutions of homogeneous equations $f(x,y,z) = 0$ in complex projective space $\mathbf{P}^2$.)

(3) As surfaces with a conformal class of metrics. In two dimensions any metric g_{ab} defines a complex structure through the relation

$$J_a{}^b = \sqrt{g}\epsilon_{ac}g^{cb} \tag{5.7}$$

with ϵ_{ab} the Levi-Civita symbol and $g = \det g_{ab}$. Furthermore, all complex structures are obtained in this way. As we see, J is invariant under local rescaling (Weyl transformations) of the metric $g_{ab} \to \rho(x)g_{ab}$, and so is only determined by the conformal class of the metric. Locally we can choose coordinates x^a such that $g_{ab} = \rho(x)\delta_{ab}$, and in these coordinates the complex structure reduces to the usual identification of $\mathbf{R}^2 = \mathbf{C}$, with analytic coordinate $z = x^1 + ix^2$.

The moduli space $\mathcal{M}_g$ parametrizes all Riemann surfaces up to equivalence. It is an orbifold space (even a quasi-projective algebraic variety) of complex dimension

$$\dim_{\mathbf{C}} \mathcal{M}_g = \begin{cases} 0, & g = 0, \\ 1, & g = 1, \\ 3g-3, & g \geq 2. \end{cases} \tag{5.8}$$

This can be proved using deformation theory of the $\overline{\partial}$ operator on the surface Σ. We will assume $g \geq 2$ so that we are dealing with the generic situation. The deformed operator is written as $\overline{\partial}+\mu\partial$ with Beltrami differential μ, a section of $T_\Sigma \otimes \overline{T}^*_\Sigma$. There is no integrability condition for complex curves. (By dimensional reasons, an obstruction would lie in $H^2(T_\Sigma)$, which vanishes). Infinitesimal diffeomorphisms, which lead to equivalent complex structures, are generated by vector fields ξ and induce the equivalence $\mu \sim \mu + \overline{\partial}\xi$. Therefore, the tangent space at the point $\Sigma \in \mathcal{M}_g$ is given by

$$T_\Sigma \mathcal{M}_g \cong H^1(T_\Sigma) \tag{5.9}$$

Using Serre duality one finds that $H^1(T_\Sigma) \cong H^0(K^2)^*$, with $K = T^*_\Sigma$ the canonical line bundle on Σ. With Riemann-Roch one then computes that

$$\dim H^0(K^2) - \dim H^0(T) = 3g - 3. \tag{5.10}$$

For $g \geq 2$ the space $H^0(T)$ of holomorphic vector fields, *i.e.* infinitesimal automorphisms of the surface Σ, vanishes and the space of quadratic differentials has the dimension $\dim H^0(K^2) = 3g - 3$, which then equals the dimension of the moduli space. (For genus zero and one, we have $h^0(T) = 3,\ 1$ respectively.)

From the point of view of conformal metrics one starts with the space of all metrics modulo diffeomorphism, and chooses a unique representative in each conformal class. This can be conveniently done by requiring the curvature R to be constant. Taking into account the Gauss-Bonnet theorem

$$\int_\Sigma \frac{d^2z}{2\pi}\sqrt{g}R = 2-2g, \tag{5.11}$$

R can be normalized to be $1, 0, -1$ for genus $g = 0, 1, \geq 2$. (For genus one we also have to normalize $\int\sqrt{g} = 1$.) We still have to identify constant curvature metrics by the diffeomorphism group $\mathrm{Diff}(\Sigma)$, which does not act freely. Therefore the moduli space is not a smooth space; "symmetric" surfaces will be fixed points of certain transformations, which makes $\mathcal{M}_g$ into an orbifold. Actually, one can do this identification in two steps, by first taking the quotient by the identity component $\mathrm{Diff}(\Sigma)_0$ and subsequently the mapping class group Γ_g, which represents the global diffeomorphisms and is defined by the exact sequence

$$1 \to \mathrm{Diff}_0(\Sigma) \to \mathrm{Diff}(\Sigma) \to \Gamma_g \to 1. \tag{5.12}$$

The first step is a smooth operation that produces the so-called Teichmüller space $\mathcal{T}_g \cong \mathbf{C}^n$. However, the second step that expresses the moduli space as the quotient

$$\mathcal{M}_g = \mathcal{T}_g/\Gamma_g, \tag{5.13}$$

can have fixed points, that correspond to surfaces with extra automorphisms.

5.2. Example — genus one

A simple example is the case $g = 1$: the two-torus T^2 or elliptic curve. Let us consider this moduli space from the three equivalent points of view.

(1) First we use the language of complex curves. A complex curve of genus one, topologically a two-dimensional torus T^2, can be represented as $\mathbf{C}/\Lambda_\tau$ with the lattice

$$\Lambda_\tau = \mathbf{Z} \oplus \tau\mathbf{Z} \tag{5.14}$$

and τ an element of the upper half plane $\mathbf{H} = \mathcal{T}_1$ (defined by $\mathrm{Im}\,\tau > 0$). That is, we have identifications

$$z \sim z+1 \sim z+\tau. \tag{5.15}$$

which produces a topological two-torus, as illustrated in *fig.* 4. Note that we used here the automorphisms of the torus to put one of the two generators e_1, $e_2 \in \mathbf{C}$ of the rank two lattice Λ to 1. A more invariant expression for the modulus τ would be as the quotient e_2/e_1. Different basis choices for the lattice Λ_τ, that are related by elements in $SL(2,\mathbf{Z})$,

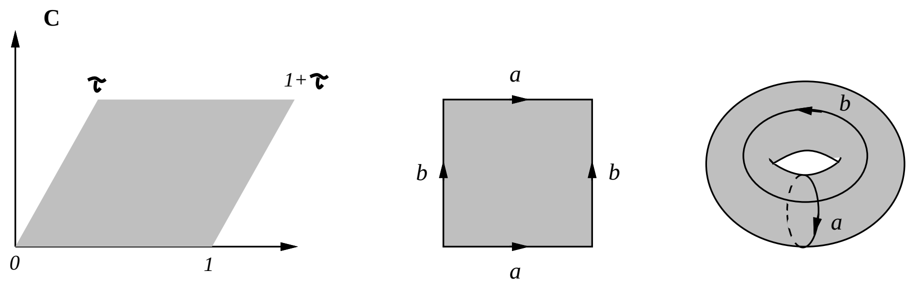


Fig. 4: *The torus or elliptic curve is obtained by quotienting the complex plane by a two-dimensional lattice.*

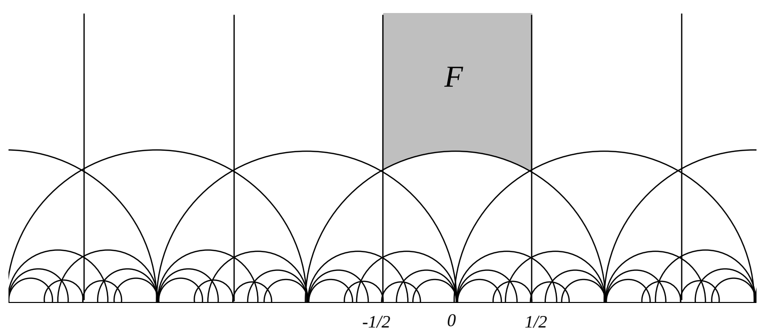


Fig. 5: *The action of the modular* $PSL(2, \mathbf{Z})$ *on the upper-half plane and a fundamental domain* $\mathcal{F}$*.*

give the same elliptic curve. This leads to a further identification of the modulus τ by the modular group $\Gamma_1 = PSL(2, \mathbf{Z})$. This group acts on τ by fractional linear transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbf{Z}). \tag{5.16}$$

It is generated by the transformations

$$\begin{aligned} T &: \tau \to \tau + 1, \\ S &: \tau \to -1/\tau, \end{aligned} \tag{5.17}$$

satisfying the relations $S^2 = (ST)^3 = 1$. The action on the upper half-plane is depicted in *fig.* 5.

The moduli space $\mathcal{M}_1$ equals the quotient of the upper half-plane $\mathbf{H}$ by the modular group $SL(2,\mathbf{Z})$ and can be represented by the well-known fundamental domain $\mathcal{F}$, defined by restricting $|\tau| \geq 1$, $|\mathrm{Re}\tau| \leq \frac{1}{2}$ and indicated in *fig.* 2. Topologically we have $\mathcal{F} \cong \mathbf{C}$. Note that, once we compactify the moduli space by adding the point $\tau = i\infty$, the resulting "stable " moduli space $\overline{\mathcal{M}}_1$ is topologically a (Riemann) sphere,

$$\overline{\mathcal{M}}_1 \cong \mathbf{P}^1 \tag{5.18}$$

It contains three orbifold singularities at $\tau = i$, $\tau = e^{2\pi i/3}$ and $\tau = i\infty$. These are respectively the fixed points of the transformations S, ST and T of order 2, 3 and ∞. The point at infinity corresponds to a singular elliptic curve, as we will see in a moment.

(2) From the point of algebraic curves, any genus one curve is represented by the cubic equation (elliptic curve)

$$y^2 = x^3 + ax + b, \tag{5.19}$$

for some constants $a, b \in \mathbf{C}$. However, this identification is not unique (the moduli space is one-dimensional and here we have two parameters) and in order to express the relation of the constants a, b with the modulus τ we first have to introduce some facts about modular forms that we will need at various places later in the lectures.

A modular form of weight k is a holomorphic function $f(\tau)$ on the upper half plane satisfying the condition

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \tag{5.20}$$

and which is well-behaved at infinity, *i.e.* has an expansion of the form

$$f(\tau) = \sum_{n\geq 0} a_n q^n, \qquad q = e^{2\pi i\tau}. \tag{5.21}$$

Since the product of two modular forms of weight k and l gives again a modular form, now of weight $k + l$, the space of modular forms is a ring. As a ring it is generated by the Eisenstein series $E_4(\tau)$ and $E_6(\tau)$ of weight 4 and 6 respectively. The normalized Eisenstein series $E_k(\tau)$, for $k \geq 4$ and even, are defined as

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n>0} \frac{n^{k-1} q^n}{1 - q^n} = \sum_{(m,n)=1} \frac{1}{(m + n\tau)^k} \tag{5.22}$$

There are two independent forms of weight 12, namely E_4^3 and E_6^2. Therefore, we can define a linear combination that vanishes at $q = 0$, the discriminant

$$\Delta = \frac{E_4^3 - E_6^2}{1728} = \eta^{24} = q - 24q^2 + \ldots \tag{5.23}$$

where η is Dedekind's eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n>0} (1 - q^n). \tag{5.24}$$

We need the discriminant Δ in our last definition, that of the modular j-function

$$j(\tau) = \frac{E_4^3}{\Delta}. \tag{5.25}$$

Since it is defined as a ratio of two modular forms both of weight 12, it transforms with weight zero. However, since the moduli space $\overline{\mathcal{M}}_1 \cong \mathbf{P}^1$ is compact, the only globally holomorphic function is constant. Therefore j has a pole at $q = 0$,

$$j = \frac{1}{q} + 744 + 196884q + \ldots \tag{5.26}$$

In fact, one can proof that the map $j : \overline{\mathcal{M}}_1 \to \mathbf{P}^1$ is a bijection, so the value of the j-function classifies the inequivalent elliptic curves. (One often speaks about $\mathcal{M}_1$ as the "j-line." Also, all meromorphic modular functions are necessarily rational expressions in the j-function, j generates the function field on $\mathcal{M}_1$.)

We can now express the constants a and b in terms of the modulus τ. One finds that $a = -\frac{\pi^2}{3} E_4(\tau)$ and $b = -\frac{2\pi^6}{27} E_6(\tau)$ so that the j-value of the curve (5.19) is given by

$$j(\tau) = 1728 \frac{4a^3}{4a^3 + 27b^2}. \tag{5.27}$$

This tells us which elliptic curves are equivalent and which are not.

(3) Finally, from the point of view of conformal structures, we have to pick a flat metric on T^2, which is of the general form

$$ds^2 = \lambda\, dx^2 + \mu\, dxdy + \nu\, dy^2, \tag{5.28}$$

with constants $\lambda, \mu, \nu \in \mathbf{R}$. Such a metric can always be written as

$$ds^2 = \rho |dx + \tau dy|^2, \qquad g_{ab} = \rho \begin{pmatrix} 1 & \mathrm{Re}\tau \\ \mathrm{Re}\tau & \tau\overline{\tau} \end{pmatrix} \tag{5.29}$$

We then immediately make contact with the complex structure point of view.

5.3. Surfaces with punctures

We can also include marked points on the Riemann surface. If we fix n different points $P_1, \ldots, P_n \in \Sigma_g$, the corresponding moduli space $\mathcal{M}_{g,n}$ has dimension

$$\dim_{\mathbf{C}} \mathcal{M}_{g,n} = 3g - 3 + n, \tag{5.30}$$

at least if $2g - 2 + n > 0$, one extra dimension for each puncture.

The simplest case is genus zero. We pick points $z_1, \ldots, z_n \in \mathbf{P}^1$ with $z_i \neq z_j$ if $i \neq j$. However, the moduli space $\mathcal{M}_{0,n}$ is not simply $(\mathbf{P}^1)^n$ minus diagonals, since $\mathbf{P}^1$ has a non-trivial group of automorphisms

$$\mathrm{Aut}\,(\mathbf{P}^1) = PGL(2, \mathbf{C}), \tag{5.31}$$

that we have to factor out. It acts by Möbius transformations on the coordinates z_i,

$$z \to \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbf{C},\ ad - bc \neq 0. \tag{5.32}$$

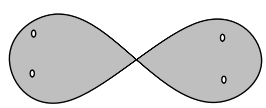
This quotient can be eliminated by fixing three points, say, $\{z_1, z_2, z_3\} = \{0, 1, \infty\}$. In this way we find for example that

$$\mathcal{M}_{0,3} \cong pt. \tag{5.33}$$

and

$$\mathcal{M}_{0,4} \cong \mathbf{P}^1 - \{0, 1, \infty\}. \tag{5.34}$$

Here we shouldn't confuse the Riemann surface and the moduli space! We can compactify this space to $\overline{\mathcal{M}}_{0,4} \cong \mathbf{P}^1$ by adding back the configurations $z = 0, 1, \infty$. However, it is better to think of adding singular curves with a node and two points on each component, that is, surfaces of the form

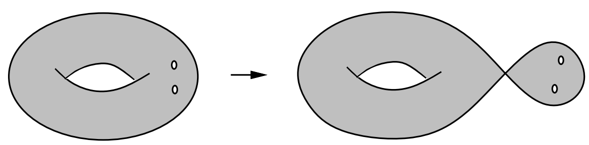


Let us explain this in some more detail.

5.4. The stable compactification

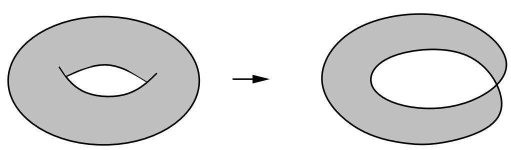
The moduli space $\mathcal{M}_{g,n}$ is non-compact because complex curves can become degenerate. There exists a direct, intuitive interpretation of the points at the boundary. There are basically two ways in which a surface can degenerate. If we think in terms of a conformal class of metrics, the surface can either form a node (equivalently, a long neck) or two marked points can collide. The boundary of $\mathcal{M}_{g,n}$ can be thought to lie at infinity. One would like to compactify the moduli space by adding points at infinity, not unlike how one compactifies the plane $\mathbf{R}^n$ to the n-dimensional sphere. In this case the points at infinity represent particular singular Riemann surfaces. The Knudsen-Deligne-Mumford or 'stable' compactification [42] tells us to add the following singular curves:

(1) The process in which two points z_1 and z_2 'collide', when the difference $q = z_1 - z_2$ tends to zero, can (after a coordinate transformation $z \to z/q$) alternatively be described as the process in which a sphere, that contains z_1 and z_2 at fixed distance, pinches off the surface by forming a neck of length $\log q$. These two descriptions are fully equivalent, but the latter is actually more in the spirit of conformal field theory, since we see in an obvious way the operator product expansion emerge. The natural final configuration is not simply the surface with $z_1 = z_2$, but it consists of a separate sphere containing the points z_1, z_2 and a third point where the infinite long tube was attached, together with the original surface with one marked point less.



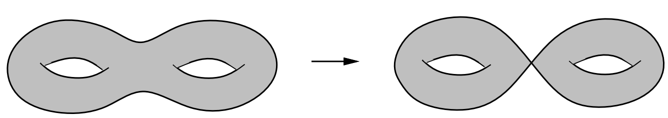
In the stable compactification we add this configuration as limit point. The crucial property of this compactification is that the points z_i never are allowed to come together.

(2) If a cycle of non-trivial homology pinches, we replace the surface by a surface with one handle less and two extra marked points, the attachment points of the infinitely thin handle.



So by this process we lower the genus by one.

(3) Finally, in case a dividing cycle pinches, the resulting surface consists of two disconnected surfaces of genus h and $g - h$, each having one extra puncture.



It can be shown that this prescription makes $\mathcal{M}_{g,n}$ into a compact orbifold space $\overline{\mathcal{M}}_{g,n}$ (and even a complex projective variety).

6. Conformal field theory

The next step in our hierarchy of two-dimensional field theories is conformal field theory (CFT). There are two mathematically sound ways to think about conformal field theories. Either in terms of representations of the Virasoro algebra and operator algebras, or in terms of Riemann surfaces. The algebraic and the geometric formalisms are equivalent, but we will mainly focus on the latter.

6.1. Algebraic approach

We can think about CFTs in terms of a Hilbert space $\mathcal{H}$ that carries a representation of the Virasoro algebra $Vir\oplus\overline{Vir}$ generated by the local holomorphic and anti-holomorphic vector fields

$$L_n = z^{n+1}\frac{\partial}{\partial z}, \qquad \overline{L}_n = \overline{z}^{n+1}\frac{\partial}{\partial \overline{z}} \tag{6.1}$$

on the cylinder $\mathbf{C}^*$. These generators can be assembled in the (anti)holomorphic stress tensor

$$\begin{aligned} T(z) &= \sum_n L_n z^{-n-2}, \\ \overline{T}(\overline{z}) &= \sum_n \overline{L}_n \overline{z}^{-n-2}. \end{aligned} \tag{6.2}$$

After quantization the Virasoro algebra Vir with central charge c reads

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{6.3}$$

To build a quantum field theory out of these representations, one has to introduce additional algebraic structure, in particular the operator algebra [43]. This produces in the end well-defined correlation functions

$$\langle \phi_1(P_1)\cdots\phi_n(P_n)\rangle_g, \tag{6.4}$$

where $\phi_i \in \mathcal{H}$ are states/operators and P_i are points on a genus g Riemann surface Σ. Ignoring the anomaly for a moment, these correlations functions are non-trivial functions on $\mathcal{M}_{g,n}$. Due to lack of time we will have to refer to the literature for a further discussion of this “algebraic” point of view. (For a more physical review see for example the excellent lectures by Ginsparg and Cardy at the 1988 Les Houches school [44, 45] and the reprint volume [48]. For a more mathematical exposition see *e.g.* [46, 47].)

6.2. Functorial approach

Alternatively, one can start with Segal’s functorial axioms [2] or what is known in the physics literature as the operator formalism [49]. Hereto one has to consider a much bigger, “dressed up” moduli space, denoted as $\mathcal{P}_{g,n}$. It consists of Riemann surfaces Σ of genus g and n marked points $P_i \in \Sigma$ together with a choice of local coordinates z_i

around the punctures. These local coordinates are chosen such that the point P_i is given by $z_i = 0$. Of course, there is a projection map

$$\mathcal{P}_{g,n} \xrightarrow{\pi} \mathcal{M}_{g,n} \tag{6.5}$$

(with infinite-dimensional fibers) that simply forgets the information about the local coordinate. The local coordinate not only allows one to "cut a hole" around the puncture by removing the disk $|z_i| < 1$, it also gives a parametrization of the resulting boundary S^1.

Generalizing the axioms of two-dimensional TFT, a CFT can now be regarded as a functor

$$\Phi : \mathbf{Riem} \to \mathbf{Hilb} \tag{6.6}$$

from the category **Riem** of these Riemann surfaces with parametrized boundaries to the category **Hilb** of Hilbert spaces. (Here we completely ignore the notion of the central charge c. Strictly speaking, everything here only holds for $c = 0$, which is actually the relevant case for string theory.)

That is, a CFT is map Φ that associates to each element in this extended moduli space $\mathcal{P}_{g,n}$ an element in the n-th tensor product of the Hilbert space

$$\Sigma \in \mathcal{P}_{g,n} \quad \Rightarrow \quad \Phi_\Sigma \in \mathcal{H}^{\otimes n}. \tag{6.7}$$

(Note that for a Hilbert space we can identify $\mathcal{H}$ with $\mathcal{H}^*$ so that we need not distinguish incoming and outgoing states.) If we compare this point of view with the axioms of a TFT we see two important differences:

(1) The morphisms now depend on the choice of complex structure;

(2) The vector space of states $\mathcal{H}$ associated to the circle is now infinite-dimensional and carries an hermitean inner product.

The action of the Virasoro algebra is recovered by considering the cylinder or annulus $\mathbf{C}^*$ which represents a map $\mathcal{H} \to \mathcal{H}$, but we will not go into this here, see [2].

The CFT correlation functions are obtained by choosing vectors $\phi_1, \ldots, \phi_n \in \mathcal{H}$ and inserting them into the linear forms $\Phi_\Sigma : \mathcal{H}^{\otimes n} \to \mathbf{C}$,

$$\langle \phi_1(P_1) \cdots \phi_n(P_n) \rangle_g = \Phi_\Sigma(\phi_1, \ldots, \phi_n) \tag{6.8}$$

The gluing property is more involved than in the case of a TFT, since it involves a local modulus q, comparable to e^{-t} in the case of quantum mechanics. We can glue two Riemann surfaces Σ_1 and Σ_2 at two punctures $P_1 \in \Sigma_1$ and $P_2 \in \Sigma_2$ by identifying the local coordinates z_1 and z_2 as

$$z_1 z_2 = q, \qquad q \in \mathbf{C}^*. \tag{6.9}$$

This is the inverse of the degeneration process we sketched in §5.4. The amplitudes of the new surface $\Sigma = \Sigma_1 \cup_q \Sigma_2$ are now given by

$$\Phi_\Sigma = \langle \Phi_{\Sigma_1}, q^{L_0}\overline{q}^{\overline{L}_0}\Phi_{\Sigma_2}\rangle. \tag{6.10}$$

This corresponds geometrically to cutting unit disks $|z_i| \leq 1$ out of the surfaces and inserting a cylinder of length $-\log|q|$ and rotating this cylinder ("twisting") around an angle $\arg q$. Note that the dimensions of the moduli spaces work out, since we introduce a new modulus q in the gluing process.

The Virasoro algebra can be recovered in the following general way: if we pick a non-single valued holomorphic vector field ξ on the surface, possibly with poles at the punctures, then we can associate to this a deformation of the complex structure $\mu = \overline{\partial}\xi$ (a quasi-conformal transformation). Thereby we obtain a vector field on the *moduli space* $\mathcal{P}_{g,n}$. We denote the Lie derivative of this vector field (as it acts on functions on the moduli space) as $\mathcal{L}_\xi$. On the other hand the Virasoro generators $L_n^{(i)}$ that act on the Hilbert space $\mathcal{H}$ at puncture $z_i = 0$, are given in terms of the stress-tensor as

$$L_n = \oint \frac{dz}{2\pi} z^{n+1} T(z). \tag{6.11}$$

More generally, we can define for an arbitrary local vector field ξ

$$L_\xi = \oint \frac{dz}{2\pi i} \xi(z) T(z). \tag{6.12}$$

The relation between the functorial approach and the Virasoro algebra is now given by the symbolic equation

$$\left(\mathcal{L}_\xi + \sum_i L_\xi^{(i)}\right)\Phi = 0, \tag{6.13}$$

that tells us that a deformation of the moduli in $\mathcal{P}_{g,n}$ can be translated into an action of *Vir* on the state spaces at the punctures.

6.3. Free bosons

Let us put our feet back on the ground and briefly review the ur-CFTs associated with free bosonic and fermionic fields, since we will need these results later. A free boson, with action

$$S = \frac{1}{2\pi}\int d^2z\, \partial x \overline{\partial} x \tag{6.14}$$

can be conveniently discussed in terms of the spin one currents ∂x and $\overline{\partial}x$. Because of the equation of motion, $\partial\overline{\partial}x = 0$, these currents have a meromorphic expansion

$$-i\partial x = \sum_{n\in\mathbf{Z}} \alpha_n z^{-n-1}, \qquad \alpha_n^\dagger = \alpha_{-n}. \tag{6.15}$$

The standard free field quantization procedure gives rise to a Fock space description of the Hilbert space. We have canonical commutation relations

$$[\alpha_n, \alpha_m] = n\delta_{n+m}, \tag{6.16}$$

and the Fock states are created out of the ground states $|p\rangle$, with $p \in \mathbf{R}$, defined by

$$\alpha_0|p\rangle = p|p\rangle, \qquad \alpha_n|p\rangle = 0, \quad n > 0. \tag{6.17}$$

The bosonic Fock space, that we will denote as $\mathcal{B}_p$, is spanned by states of the form

$$|\varphi\rangle = \alpha_{-n_1} \cdots \alpha_{-n_s}|p\rangle. \tag{6.18}$$

The operator-state correspondence relates these states to the vertex operators

$$\varphi(z) = \partial^{n_1}x \cdots \partial^{n_s}x e^{ipx(z)}. \tag{6.19}$$

The Fock space forms a $c = 1$ representations of the Virasoro algebra, generated by the stress-tensor

$$T = -\tfrac{1}{2}(\partial x)^2. \tag{6.20}$$

The space is graded by conformal dimension, the eigenvalues of the operator

$$L_0 = \oint zT(z) = \tfrac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n}\alpha_n \tag{6.21}$$

and $U(1)$ charge or (space-time) momentum

$$J_0 = -i \oint \partial x = \alpha_0. \tag{6.22}$$

The degeneracies for fixed eigenvalues can be read off from the character

$$\mathrm{Tr}_{\mathcal{B}_p}\left(y^{J_0} q^{L_0 + c/24}\right) = \frac{y^p q^{\frac{1}{2}p^2}}{\eta(q)}. \tag{6.23}$$

If we combine left-movers and right-movers, the full Hilbert space is of the form

$$\mathcal{H} = \int dp\, \mathcal{B}_p \otimes \overline{\mathcal{B}}_p. \tag{6.24}$$

Its partition function reads

$$
\begin{aligned}
Z &= \operatorname{Tr}_{\mathcal{H}} \left(q^{L_0+\frac{1}{24}} \overline{q}^{\overline{L}_0+\frac{1}{24}} \right) \\
&= \int dp \, \frac{q^{\frac{1}{2}p^2} \overline{q}^{\frac{1}{2}p^2}}{|\eta|^2} \\
&= \frac{1}{\sqrt{\operatorname{Im}\tau}|\eta|^2} = \frac{1}{\sqrt{\det' \Delta}}
\end{aligned}
\tag{6.25}
$$

with Δ the scalar Laplacian on T^2.

We can make a slight variation on this model by including a background charge Q. This modifies the stress-tensor to

$$
T = -\tfrac{1}{2}(\partial x)^2 - iQ\partial^2 x \tag{6.26}
$$

and gives $c = 1 - 3Q^2$. This shift produces

$$
L_0 \to L_0 + \tfrac{1}{2} Q J_0 \tag{6.27}
$$

and gives the vertex operator e^{ipx} a conformal dimension $\frac{1}{2}p(p+Q)$. In the character (6.23) this can be done by the "twist" $y \to yq^{Q/2}$.

6.4. Free fermions

Free two-dimensional chiral (Dirac) fermions b, c have an action

$$
S = \frac{1}{\pi} \int d^2 z \, b \overline{\partial} c \tag{6.28}
$$

where we can take the spins of b and c to be λ and $1-\lambda$, with λ half-integer for physical fermions (that obey spin-statistics) and λ integer for ghost fields (that violate spin-statistics). The equations of motion give $\overline{\partial} b = \overline{\partial} c = 0$, so we have again mode compositions of the form

$$
\begin{aligned}
b(z) &= \sum_{n \in \mathbf{Z}+\epsilon} b_n z^{-n-\lambda}, \\
c(z) &= \sum_{n \in \mathbf{Z}+\epsilon} c_n z^{-n-1+\lambda}.
\end{aligned}
\tag{6.29}
$$

Here $\epsilon = \frac{1}{2}, 0$ indicates the two possible spin structures on the circle, usually referred to as Neveu-Schwarz (NS) and Ramond (R) respectively.

Quantization starts from the canonical anti-commutation relations

$$\{b_n, c_m\} = \delta_{n+m,0}, \tag{6.30}$$

with

$$\{b_n, b_m\} = \{c_n, c_m\} = 0. \tag{6.31}$$

One defines a Fermi sea $|\mu\rangle$ satisfying

$$\begin{aligned} b_n|\mu\rangle &= 0, \qquad n > \mu - \lambda, \\ c_n|\mu\rangle &= 0, \qquad n \geq \lambda - \mu. \end{aligned} \tag{6.32}$$

and this gives rise to a fermionic Fock space, written as $\mathcal{F}^{\epsilon}_{\mu}$ and spanned by states of the form

$$b_{-m_1} \cdots b_{-m_r} c_{-n_1} \cdots c_{-n_s} |\mu\rangle. \tag{6.33}$$

This will form a representation of the Virasoro algebra with central charge

$$c = 1 - 3(1 - 2\lambda)^2, \tag{6.34}$$

and stress tensor

$$T = -\lambda b\partial c + (1 - \lambda)\partial bc. \tag{6.35}$$

We also have a $U(1)$ quantum number given by the eigenvalues of the operator

$$J_0 = -\oint b(z)c(z) \tag{6.36}$$

With these conventions the modes b_n carry charge -1 and c_n carry charge $+1$. The degeneracies for fixed eigenvalues of L_0 and J_0 follow from the expansion of the character

$$\mathrm{Tr}_{\mathcal{F}^{\epsilon}_{\mu}} \left(q^{L_0 + c/24} y^{J_0} \right) = y^{\mu} q^{\frac{1}{2}\mu(1+\mu-2\lambda)} \prod_{n \geq \lambda - \mu} (1 + yq^n) \prod_{m > \mu - \lambda} \left(1 + y^{-1} q^m \right) \tag{6.37}$$

7. Sigma models and T-duality

We now want to discuss some examples of less trivial CFTs describing strings moving on spaces with non-trivial topology. One of the interesting phenomena that we meet here is that two manifolds that are classically very different, turn out to be equivalent when considered as a background in string theory. This is one of the manifestations of duality. In its more sophisticated form this is known as mirror symmetry [50, 39, 51, 52], as discussed at great length in the lectures of Brian Greene [53]. Here we will mainly stick with the more tractable abelian dualities. These so-called T-dualities are reviewed in great detail in [54]. Actually, as we will see, abelian dualities are basically infinite-dimensional versions of the fact that under Fourier transformation the gaussian behaves as

$$e^{-ax^2} \to e^{-x^2/a}. \tag{7.1}$$

7.1. Two-dimensional sigma models

In this lecture we discuss two-dimensional sigma models. Here the fundamental fields are maps

$$x : \Sigma \to X, \tag{7.2}$$

where Σ is a closed Riemann surface (compact without boundary) and X is *a priori* an arbitrary compact n-dimensional Riemannian manifold. The corresponding quantum field theory is defined through the path-integral over all such smooth maps

$$Z = \int \mathcal{D}x \, e^{-S}, \tag{7.3}$$

where the action S is defined as follows. We choose a conformal class of metrics on the surface Σ, so that we obtain a Hodge star $*$, and pick coordinates x^μ and a metric $G_{\mu\nu}$ on X. The action is then given by

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma G_{\mu\nu}(x) dx^\mu \wedge *dx^\nu. \tag{7.4}$$

Here α' is the coupling constant of the sigma model that we usually simply set to one by absorbing it into the metric G. Note that weak coupling $\alpha' \to 0$ (in which sigma model perturbation theory makes sense and the classical action, in particular the target space metric can be recovered) corresponds to large volume of the target space X. It is well-known that the critical points of the action are given by the so-called *harmonic maps* — a beautiful subject in classical differential geometry (see *e.g.* [55]).

The action can be generalized by also picking a two-form $B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$ on X and adding the term

$$\frac{i}{4\pi} \int_\Sigma B_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{i}{2\pi} \int_\Sigma x^* B \tag{7.5}$$

to the action. Note that by Stokes' theorem this term is invariant under shifts $B \to B + d\Lambda$.

Actually, we have an even stronger quantum equivalence by which we shift $B \to B + 4\pi^2 C$ with C a closed two-form with quantized periods,

$$C \in H^2(X, \mathbf{Z}), \tag{7.6}$$

since the path-integral picks up a term

$$Z \to Z \cdot e^{2\pi i \int_\Sigma x^* C} = Z \tag{7.7}$$

When is such a sigma model conformal invariant? The action only uses the Hodge star, so classically the theory only depends on the conformal class of the metric on Σ. This conformal invariance is in general not preserved at the quantum level. It is ruined by the implicit but very important metric dependence of the measure $\mathcal{D}x$ in the path-integral. This fact is reflected by a non-vanishing beta-function, which for a sigma model is given by an expression in the target space Riemann tensor [56]

$$\beta_{\mu\nu} = R_{\mu\nu} + \alpha' \tfrac{1}{2} R_{\mu\kappa\lambda\rho} R_\nu{}^{\kappa\lambda\rho} + O((\alpha')^2) \tag{7.8}$$

So to first order in sigma model perturbation theory, conformal invariance implies that the target space should provide a solution to the vacuum Einstein equations

$$R_{\mu\nu} = 0. \tag{7.9}$$

One similarly derives an equation for the background $B_{\mu\nu}$ field, that reads to first order in α'

$$dB = 0. \tag{7.10}$$

Together with the gauge equivalence that we discussed above this gives us an element in the torus[3]

$$B \in H^2(X, \mathbf{R})/H^2(X, \mathbf{Z}). \tag{7.11}$$

Actually, for compact spaces, the full 'elliptic' equations $\beta_{\mu\nu} = 0$ only allows for flat space solutions,

$$R_{\mu\nu\lambda\rho} = 0 \tag{7.12}$$

leaving only tori (and singular quotients) as possible candidates for a conformal invariant sigma-model. This situation improves remarkably if we also include fermionic degrees of freedom, as we will indicate in §7.7. (But see Brian Greene's lectures [53] for more details.)

7.2. Toroidal models

The simplest cases for conformal invariant sigma-models are tori, so let us assume that X can be written as

$$X = T^n = \mathbf{R}^n / 2\pi\Lambda \tag{7.13}$$

[3]Here and everywhere we ignore torsion.

with Λ a rank n lattice. The action of these toroidal models is pure gaussian

$$S = \frac{1}{2\pi} \int_\Sigma (G_{\mu\nu} + B_{\mu\nu})\, \partial x^\mu \overline{\partial} x^\nu. \tag{7.14}$$

Duality is in this context the statement that the sigma-model model is insensitive (up to a change in normalization of the partition function) to the interchange of the torus with the dual torus

$$T^n \iff (T^n)^*. \tag{7.15}$$

This particular type duality transformation is usually referred to as *T-duality* [54]. It interchanges target spaces with large volumes and target spaces with small volumes. From the point of view of sigma model perturbation theory it relates strong and weak coupling

$$\alpha' \to 1/\alpha', \tag{7.16}$$

so we could have named it a *world-sheet* S-duality. But we prefer to reserve the term S-duality only for space-time strong-weak coupling transformation. Since a T-duality is an automorphism of the CFT and acts for every given Riemann surface, it is a perturbative symmetry from the point of string perturbation theory.

The simplest way to prove T-duality starts from the following path-integral [57] (we drop the B-field for convenience)

$$Z = \int \mathcal{D}A\mathcal{D}y \, \exp -\frac{1}{2\pi} \int \left(G_{\mu\nu} A^\mu \wedge *A^\nu + A^\mu \wedge dy_\mu \right) \tag{7.17}$$

with A^μ (for fixed $\mu = 1, \ldots, n = \dim X$) a vector-valued one-form on Σ and y_μ a set of scalars. On the one hand we can integrate out y_μ, which imposes the condition

$$dA^\mu = 0 \Rightarrow A^\mu = dx^\mu, \tag{7.18}$$

at least locally. This gives us back the sigma model on the original torus T^n. On the other hand we can integrate out the field A^μ which gives the integral

$$Z = \int \mathcal{D}y \, \exp -\frac{1}{2\pi} \int G^{\mu\nu} dy_\mu \wedge *dy_\nu \tag{7.19}$$

with $G^{\mu\nu}$ the dual metric on the dual torus $\widehat{T}^n$, $G^{\mu\nu} G_{\nu\lambda} = \delta^\mu{}_\lambda$. This produces the T-dual model.

Before we discuss the full T-duality group, let us first briefly review some general facts about lattices, that will also come in useful later, when we discuss four-dimensional geometry in §9.1.

7.3. Intermezzo — lattices

A lattice Γ is by definition a $\mathbf{Z}$ vector space,

$$\Gamma \cong \mathbf{Z}^n, \tag{7.20}$$

together with a non-degenerate, symmetric bilinear form (inner product)

$$Q : \Gamma \times \Gamma \to \mathbf{Z}. \tag{7.21}$$

This bilinear form will have a signature (p, q). The dual lattice Γ^* is defined as the set

$$\Gamma^* = \{x \in \mathbf{R}^n;\ Q(x, y) \in \mathbf{Z},\ \forall y \in \Gamma\}. \tag{7.22}$$

A lattice is self-dual, $\Gamma^* = \Gamma$, if and only if the inner product is unimodular

$$\det Q = \pm 1. \tag{7.23}$$

A lattice Γ is called even if

$$x^2 \in 2\mathbf{Z}, \qquad \forall x \in \Gamma, \tag{7.24}$$

and odd otherwise. For even lattices the signature $\sigma = p - q$ satisfies necessarily

$$\sigma \equiv 0 \pmod 8. \tag{7.25}$$

The classification of self-dual lattices now proceeds depending on whether the intersection form is even or odd and whether it is (positive or negative) definite or not [58]. The various possibilities are

	odd	even
indef	$p\mathbf{1} \oplus q(-\mathbf{1})$	$pH \oplus nE_8$
def	$n\mathbf{1}$, *exotic*	*exotic:* $E_8, D_{16}, \ldots$

Here H denotes the two-dimensional hyperbolic lattice

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7.26}$$

and E_8 the root lattice of the Lie algebra of the same name. The most remarkable fact is that there exists (for given rank and signature) a *unique* indefinite self-dual lattice —

a theorem by Hasse and Minkowski. In the odd case it carries simply the diagonal form, so the lattice equals the hypercube lattice. In the even case, where we will denote the lattice as $\Gamma^{p,q}$ with $p,q>0$, $p-q=0 \pmod 8$, it is given by a sum of hyperbolic and E_8 lattices,

$$\Gamma^{p+8n,p} = pH \oplus nE_8. \tag{7.27}$$

Unfortunately, the situation in the definite case is much more complicated, because there are many more possibilities apart from the obvious cubic lattice $n\mathbf{1}$. These are usually referred to as exotic lattices. Although the number of possibilities remains finite for given rank, it grows rather dramatically: for rank $8,16,24,32,\ldots$ we find $1,2,24,\sim 10^7,\ldots$ inequivalent even definite self-dual lattices. We will need these facts about lattices again if we consider four-manifolds in §9.1.

For any lattice Γ of positive signature $(n,0)$ we define the theta-function as

$$\theta_\Gamma(\tau) = \sum_{v\in\Gamma} q^{\frac{1}{2}v^2}, \qquad q = e^{2\pi i\tau},\ \operatorname{Im}\tau > 0. \tag{7.28}$$

Poisson resummation gives that

$$\theta_\Gamma(-1/\tau) = \left(\frac{i}{\tau}\right)^{n/2} \theta_{\Gamma^*}(\tau) \tag{7.29}$$

So for Γ even and self-dual, $\theta_\Gamma(\tau)$ is a modular form of weight $n/2$, *e.g.* in the simplest case

$$\theta_{E_8}(\tau) = E_4(\tau). \tag{7.30}$$

7.4. Spectrum and moduli of toroidal models

Returning to our discussion of toroidal sigma models, we want to show that the full duality symmetry group of the sigma model with target space T^n is given by

$$G = O(n,n,\mathbf{Z}), \tag{7.31}$$

by which we denote the group of automorphisms of the lattice

$$\Gamma^{n,n} = \Lambda^* \oplus \Lambda, \tag{7.32}$$

with inner product

$$p^2 = 2k\cdot m, \qquad p=(k,m),\ k\in\Lambda^*,\ m\in\Lambda. \tag{7.33}$$

This so-called Narain [59] lattice $\Gamma^{n,n}$ is an even self-dual lattice of signature (n,n). As we have seen, it is unique up to isomorphism,

$$\Gamma^{n,n} \cong \underbrace{H \oplus \cdots \oplus H}_{n\ times}. \tag{7.34}$$

The action of the duality symmetry group $O(n,n,\mathbf{Z})$ is particular clear if we consider the Hilbert space of the toroidal model. It is again given by a sum of tensor products of left-moving and right-moving Fock spaces as in §6.3, now however built on momentum states p_L and p_R that take value in the Narain lattice

$$\mathcal{H} = \bigoplus_{(p_L,p_R)\in\Gamma^{n,n}} \mathcal{B}_{p_L} \otimes \mathcal{B}_{p_R}, \tag{7.35}$$

with

$$p^\mu_{L,R} = k^\mu \pm m^\mu + B^{\mu\nu} m_\nu, \qquad k \in \Lambda^*,\ m \in \Lambda, \tag{7.36}$$

(Here indices are raised and lowered with the metric $G_{\mu\nu}$.) Note that the choice of metric and B-field determines the split of the momentum p,

$$p = p_L \oplus p_R, \qquad p^2 = p_L^2 - p_R^2. \tag{7.37}$$

The one-loop partition function reads

$$Z = \mathrm{Tr}_{\mathcal{H}} \left(q^{L_0 + \frac{n}{24}} \overline{q}^{\overline{L}_0 + \frac{n}{24}} \right) = \sum_{(p_L,p_R)\in\Gamma^{n,n}} \frac{q^{\frac{1}{2}p_L^2} \overline{q}^{\frac{1}{2}p_R^2}}{|\eta(q)|^{2n}} \tag{7.38}$$

One can prove quite straightforwardly that the partition function Z is invariant under modular transformations. Invariance under T and S transformations in $SL(2,\mathbf{Z})$ is implied by the fact that $\Gamma^{n,n}$ is even and self-dual, respectively. Furthermore, the expression is by inspection invariant under the action of $O(n,n,\mathbf{Z})$ on the lattice, since we sum over all elements.

The "classical" moduli space of this CFT is parametrized by the constant $n \times n$ matrix

$$G_{\mu\nu} + B_{\mu\nu}, \tag{7.39}$$

and is given by the Grassmannian $\mathrm{Gr}^{n,n}$ of maximal positive subspaces in $\mathbf{R}^{n,n}$

$$\begin{aligned} \mathrm{Gr}^{n,n} &= \{V \in \mathbf{R}^{n,n}, \dim V = n,\ (\cdot,\cdot)|_V > 0\} \\ &\cong O(n,n;\mathbf{R})/O(n;\mathbf{R}) \times O(n;\mathbf{R}). \end{aligned} \tag{7.40}$$

The "quantum" moduli space is defined as the left-quotient of this homogeneous space by the "arithmetic subgroup" $G = O(n,n,\mathbf{Z})$

$$\mathcal{M}_{T^n} = G\backslash \mathrm{Gr}^{n,n}. \tag{7.41}$$

To see how this Narain moduli space classifies toroidal sigma models, choose a $V \in \mathrm{Gr}^{n,n}$, so that we can write

$$\mathbf{R}^{n,n} = V \oplus V^{\perp}. \tag{7.42}$$

A vector $p \in \Gamma^{n,n}$ can in this decomposition be written as $p = (p_L, p_R)$ with $p_L \in V$, $p_R \in V^{\perp}$. The quadratic form p^2 will be the difference of two positive definite forms of rank n

$$p^2 = p_L^2 - p_R^2. \tag{7.43}$$

So we recover the spectrum described above. Note that inside the duality-group G we have a "geometrical" subgroup

$$SL(n,\mathbf{Z}) \subset O(n,n,\mathbf{Z}) \tag{7.44}$$

that represent the "large" diffeomorphisms of T^n.

7.5. The two-torus

A particular interesting case appears for $n = 2$, it is described in detail in [60]. Here we are dealing with $X = T^2$, a two-torus or elliptic curve, which is the simplest example of a Calabi-Yau space. We can represent the elliptic curve as

$$T^2 = \mathbf{C}/\mathbf{Z} \oplus \rho\mathbf{Z}, \tag{7.45}$$

with the modulus $\rho \in \mathbf{H}$, the upper half-plane, and naturally identified modulo the action of the modular group $PSL(2,\mathbf{Z})$. The Kähler form and the B-field (which has here only one non-zero component) combine conveniently in one complex (1,1) form

$$\omega = \tfrac{1}{2}(B + i g_{z\bar{z}}) dz \wedge d\bar{z}. \tag{7.46}$$

We can write this complexified Kähler form as

$$\omega = \sigma \left(\frac{\pi}{\mathrm{Im}\,\rho} dz \wedge d\bar{z} \right), \tag{7.47}$$

where we introduced a second complex modulus σ as

$$\int_{T^2} \omega = 2\pi i \sigma \tag{7.48}$$

Note that the area is given by $\text{Im}\,\sigma$ and should be positive, so σ also takes value in the upper half-plane. The local moduli are therefore

$$(\rho, \sigma) \in \mathbf{H} \times \mathbf{H} \tag{7.49}$$

which indeed represents the Grassmannian $\text{Gr}^{2,2}$. The sigma model action reads in this parametrization, with complex field $x = x^1 + \rho x^2$ that takes values on the torus $\mathbf{C}/\Lambda_\rho$ (see (5.14) for notation)

$$S = \int \frac{i\pi d^2 z}{\text{Im}\,\rho} \left(\overline{\sigma} \partial \overline{x} \overline{\partial} x - \sigma \partial x \overline{\partial} \overline{x} \right) \tag{7.50}$$

Now our claim is that there is also a modular $PSL(2, \mathbf{Z})$ group acting on the Kähler variable σ. In fact, one transformation, the usual shift of the B-field, is easily seen to be represented by

$$T : \sigma \to \sigma + 1. \tag{7.51}$$

The transformation $S : \sigma \to -1/\sigma$ relates small and volume, and equals the T-duality that we proved in §7.2.

One way to see the full duality symmetries is to notice that the Narain lattice can be written as

$$\Gamma^{2,2} = \bigoplus_{i=1}^{4} \mathbf{Z} e_i \tag{7.52}$$

with $e_i \in \mathbf{C}^2$ the vectors

$$e_1 = (1, 1), \quad e_2 = (\rho, \rho), \quad e_3 = (\sigma, \overline{\sigma}), \quad e_4 = (\rho\sigma, \rho\overline{\sigma}). \tag{7.53}$$

and where the signature $(2, 2)$ inner product of a vector (p_L, p_R) is given by

$$p^2 = \frac{p_L^2 - p_R^2}{\text{Im}\,\rho \cdot \text{Im}\,\sigma}. \tag{7.54}$$

From this explicit realization of the Narain lattice we can simply read off the full quantum symmetry group $O(2, 2, \mathbf{Z})$. In fact, we have the isomorphism

$$O(2, 2; \mathbf{Z}) = PSL(2, \mathbf{Z}) \times PSL(2, \mathbf{Z}) \times \mathbf{Z}_2. \tag{7.55}$$

The two copies of $\Gamma = PSL(2, \mathbf{Z})$ act in the obvious way on the two moduli ρ, σ and the $\mathbf{Z}_2$ interchanges them

$$(\rho, \sigma) \leftrightarrow (\sigma, \rho) \tag{7.56}$$

Note that this transformation interchanges the complex Kähler form ρ and the complex structure σ. It can be seen as the simplest realization of mirror symmetry. It can be

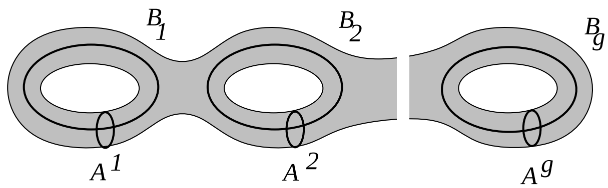


Fig. 6: *A Riemann surface with an homology basis.*

intuitively understood by picking a rectangular two-torus with radii R_1 and R_2. We then find

$$\rho = i\frac{R_1}{R_2}, \qquad \sigma = iR_1R_2. \tag{7.57}$$

A duality transformation on one S^1 will send $R_1 \to 1/R_1$ and thus interchange ρ and σ.

Concluding, the moduli space for a sigma model with target space T^2 is

$$\mathcal{M}_{T^2} \cong \left(\mathbf{P}^1_\rho \times \mathbf{P}^1_\sigma\right)/\mathbf{Z}_2 \tag{7.58}$$

7.6. Path-integral computation of the partition function

It might be instructive to also compute the partition function (7.38) of a toroidal model directly from the path-integral [61, 62]. We will do the general case of a field

$$x : \Sigma \to T^n \cong \mathbf{R}^n/2\pi\Lambda \tag{7.59}$$

on a genus g surface Σ.

We first choose an homology basis A^i, B_i, $i = 1, \ldots, g$, of $H_1(\Sigma, \mathbf{Z})$ as in *fig.* 6. In terms of these cycles the one-form dx has periods

$$\begin{aligned} \frac{1}{2\pi}\oint_{A^i} dx &= m^i \in \Lambda, \\ \frac{1}{2\pi}\oint_{B_i} dx &= n_i \in \Lambda. \end{aligned} \tag{7.60}$$

Now we can write

$$dx = \alpha + dy, \tag{7.61}$$

with y a proper function $\Sigma_g \to \mathbf{R}^n$ and α a harmonic vector-valued one-form, $d\alpha = d^*\alpha = 0$, with quantized periods

$$\frac{\alpha}{2\pi} \in H^1(\Sigma, \Lambda). \tag{7.62}$$

To compute the contribution to the action of this harmonic piece, the form α must be decomposed in terms of the basis $\omega_i \in H^0(K_\Sigma)$ $(i = 1, \ldots, g)$ of holomorphic (1,0) one-forms on Σ and their conjugates. These satisfy canonical normalizations of their periods

$$\begin{aligned}\oint_{A^i} \omega_j &= \delta^i{}_j, \\ \oint_{B_i} \omega_j &= \tau_{ij}.\end{aligned} \tag{7.63}$$

with τ_{ij} the period matrix, a complex symmetric matrix with $\operatorname{Im}\tau > 0$. Note that these abelian differentials satisfy

$$\int_\Sigma \omega_i \wedge \overline{\omega}_j = -2i \operatorname{Im}\tau_{ij}. \tag{7.64}$$

If we now write

$$\alpha = \lambda^i \omega_i + \overline{\lambda}^i \overline{\omega}_i, \tag{7.65}$$

we see that the periods are expressed as

$$\begin{aligned} m^i &= \lambda^i + \overline{\lambda}^i, \\ n_i &= \tau_{ij}\lambda^j + \overline{\tau}_{ij}\overline{\lambda}^j, \end{aligned} \tag{7.66}$$

so that the coefficients λ^i can be solved in terms of the winding numbers $m^i, n_i \in \Lambda$ as

$$\lambda = i\pi(\operatorname{Im}\tau)^{-1}(n - m\tau). \tag{7.67}$$

If we insert this expression for α in the action, we obtain the contribution

$$S_{m,n} = \tfrac{1}{2}\pi(n - m\overline{\tau})(\operatorname{Im}\tau)^{-1}(n - m\tau) + i\pi m \cdot B \cdot n. \tag{7.68}$$

So the partition function can be written as

$$Z = Z_{qu} \sum_{m,n\in\Lambda} e^{-S_{m,n}}. \tag{7.69}$$

Here Z_{qu} is the contribution of the single-valued fields y, a determinant that does not depend on the moduli of T^n (apart from the zero-mode).

In order to reach the canonical form (7.38) we have to perform a further Poisson resummation on the variables $n_i \in \Lambda$, exchanging it for the dual variables $k^i \in \Lambda^*$. The soliton sum then takes the familiar form

$$\sum_{m^i\in\Lambda,\ k^i\in\Lambda^*} \exp\left(i\pi(k+m)\tau(k+m) - i\pi(k-m)\tau(k-m)\right) \tag{7.70}$$

in which we recognize the left-moving and right-moving momenta $p_L = k + m$ and $p_R = k - m$. If we now restrict to $g = 1$ we obtain (7.38).

This computation is particular interesting for the case of a two-torus. If we compute the genus one partition function, we are dealing with maps

$$x : T^2_\tau \to T^2_{\rho,\sigma}, \tag{7.71}$$

where the suffices indicate the moduli. In the parametrization of §7.5 the soliton sum reads

$$\sum_{m,n\in\Lambda_\rho} \exp\left(-i\pi\sigma\frac{|n - m\overline{\tau}|^2}{\mathrm{Im}\,\tau\mathrm{Im}\,\rho} + i\pi\overline{\sigma}\frac{|n - m\tau|^2}{\mathrm{Im}\,\tau\mathrm{Im}\,\rho}\right) \tag{7.72}$$

were m and n are elements of the lattice $\Lambda_\rho = \mathbf{Z} \oplus \rho\mathbf{Z}$. After the Poisson resummation, we get the momentum sum

$$\sum_{k,m\in\Lambda_\tau} \exp\left(i\pi\tau\frac{|k - m\overline{\sigma}|^2}{\mathrm{Im}\,\rho\mathrm{Im}\,\sigma} - i\pi\overline{\tau}\frac{|k - m\sigma|^2}{\mathrm{Im}\,\rho\mathrm{Im}\,\sigma}\right) \tag{7.73}$$

Here we note a remarkable triality permuting the *three* moduli ρ, σ, τ [60].

7.7. Supersymmetric sigma models and Calabi-Yau spaces

If we add world-sheet supersymmetry, the set of conformal invariant sigma models becomes more interesting. For $N = 1$ supersymmetry not much happens: only tori and orbifolds survive. However, for $N = 2$ supersymmetry there is a much richer class, the Calabi-Yau spaces [63, 64].

Calabi-Yau manifolds, first introduced in the physics literature in [65], are Kähler spaces of complex dimension d for which the holonomy, which generically lies in $U(d)$ for a Kähler space, restricts to $SU(d)$. Calabi-Yau spaces can equivalently be defined by the property that for fixed Kähler class they have a unique solution to the vacuum Einstein equation

$$R_{i\overline{\jmath}} = 0, \tag{7.74}$$

i.e. they allow for a Ricci-flat Kähler metric. Indeed, the usual second order term in sigma model perturbation theory in the beta function (7.8) disappears for $N = 2$ models, as does a third order term. So in the context of $N = 2$ supersymmetry Ricci flat models stand a change to define a CFT. It turns out that there is a nontrivial fourth order correction, but this correction can always be compensated by adding a piece to the metric, that keeps the cohomology of the Kähler class invariant.

As conjectured by Calabi [63] and proven by Yau [64], a necessary and sufficient condition for Ricci flatness is given by the condition that the canonical line bundle of X, defined as

$$K_X = \bigwedge^d T^*_X \tag{7.75}$$

is trivial

$$c_1(K_X) = c_1(X) = 0. \tag{7.76}$$

So we have in particular one global holomorphic section Ω of K_X (unique up to a scale factor). That is, a CY space comes with a unique holomorphic volume-form

$$\Omega = \Omega(x) dz^1 \wedge \ldots \wedge dz^d \tag{7.77}$$

Therefore the space $H^{d,0}(X)$ is one-dimensional,

$$h^{d,0} = 1. \tag{7.78}$$

If a manifold has precisely $SU(d)$ holonomy and not less (so that it is a Calabi-Yau space in the strict sense) we further have

$$h^{1,0} = h^{2,0} = \ldots = h^{d-1,0} = 0. \tag{7.79}$$

In dimension $d \leq 3$ we have the following list of possible CY spaces

$d = 1$	T^2
$d = 2$	$T^4,\ K3$
$d = 3$	$T^6,\ T^2 \times K3,\ CY_3$

Here CY_3 denotes a simply connected three-fold; all tori have trivial holonomy group and $K3$ is the unique compact two-dimensional CY with holonomy $SU(2)$. We will not discuss supersymmetric sigma models on CY spaces in much detail here, since they are covered in detail in Brian Greene's lectures [53]. To be self-contained in these notes, let us just make a few comments about their moduli spaces, that we need later.

For Calabi-Yau three-folds we only understand the local structure of the sigma-model moduli space in full detail. It is clear that we have a unique CFT on X given a complex structure and a Kähler class. Deformations of complex structures are infinitesimally given by the cohomology group $H^1(T_X)$. Using the existence of the holomorphic three-form one can show that these deformations are unobstructed [66]. Furthermore this cohomology group is isomorphic with $H^{1,2}(X)$. Thus variations $\delta\sigma$ of the complex structure are parametrized by

$$\delta\sigma \in H^{2,1}(X), \tag{7.80}$$

and there are $h^{2,1}$ of these complex structure deformation parameters. The inequivalent Kähler deformations $\delta\rho$, including the B field) are elements of the cohomology group

$$\delta\rho \in H^{1,1}(X) = H^1(T^*_X). \tag{7.81}$$

These give rise to another $h^{1,1}$ complex moduli. Summarizing, the moduli space of CY sigma models takes the local form

$$T_X\mathcal{M} \cong H^{2,1}(X) \times H^{1,1}(X) \tag{7.82}$$

In the case of the two-torus we already saw the important role these moduli-deformations $\delta\rho, \delta\sigma$ played, and that there were strong relations between the two. Actually, in that case we could simply interchange the two. Mirror symmetry is the generalization of that phenomenon to higher dimensional manifolds [52].

7.8. Calabi-Yau moduli space and special geometry

Let us make a few further remarks about the moduli space of complex structures of Calabi-Yau three-folds. Let $\mathcal{M}$ be the moduli space of inequivalent complex structures on the CY space X. If $h^{2,1} = g$, then is $\mathcal{M}$ a g dimensional complex space. On X we can pick a holomorphic volume-form Ω. Let $\mathcal{L}$ denote the moduli space of pairs (X, Ω). Since the choice of Ω is unique up to multiplication by complex numbers, $\mathcal{L}$ is a line bundle over $\mathcal{M}$,

$$\mathcal{L} \xrightarrow{\pi} \mathcal{M}. \tag{7.83}$$

We can now consider the so-called period map that is defined as follows. First we recall that $H^3(X, \mathbf{C}) = H^3(X, \mathbf{Z}) \otimes \mathbf{C}$, where the cohomology group $H^3(X, \mathbf{Z})$ is the dual to the space $H_3(X, \mathbf{Z})$ of three-cycles. By the intersection form

$$\eta(\alpha, \beta) = \int_X \alpha \wedge \beta, \qquad \alpha, \beta \in H^3(X, \mathbf{Z}), \tag{7.84}$$

it is naturally a $2g + 2$ integer dimensional symplectic vector space. It is a topological object whose definition does not depend on the choice of complex structure.

If we choose a particular complex structure we get a Hodge decomposition of $H^3(X, \mathbf{C})$ in terms of the Dolbeault groups

$$H^3(X, \mathbf{C}) \cong H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \tag{7.85}$$

with $\overline{H}^{p,q} = H^{q,p}$. The piece $H^{3,0}$ represents the space of holomorphic volume forms and is one-dimensional. The period map now associates to the pair (X, Ω) the point in $H^3(X, \mathbf{C})$ represented by Ω. An important mathematical result is that this map

$$\mathcal{L} \to H^3(X, \mathbf{C}) \tag{7.86}$$

is injective [66]. This gives a description of $\mathcal{L}$ as a cone in $H^3(X, \mathbf{C})$. (It is a cone, since we can scale the holomorphic three-form by multiplication by $\mathbf{C}$.)

Now the period map can be described in more detail if we choose once and for all a basis in $H^3(X, \mathbf{Z})$ by picking a canonical basis A^i, B_i $(i = 0, \ldots, g)$ of homology three-cycles. This gives a dual basis α_i, β^i of integer three-forms. On this basis we can decompose the holomorphic (3,0) form as

$$\Omega = \phi^i \alpha_i + \mathcal{F}_i \beta^i \tag{7.87}$$

in terms of the periods

$$\begin{aligned} \oint_{A^i} \Omega &= \phi^i, \\ \oint_{B_i} \Omega &= \mathcal{F}_i. \end{aligned} \tag{7.88}$$

Because of the properties of the period map, the components ϕ^i may be used as local coordinates on $\mathcal{L}$, or, equivalently, as homogeneous coordinates on $\mathcal{M}$. The components $\mathcal{F}_i$ become then functions of the ϕ^i. We can now differentiate Ω with respect to these moduli ϕ^i. The $g+1$ partial derivatives

$$\omega_i = \partial_i \Omega \tag{7.89}$$

will give a basis of the space[4]

$$W = H^{3,0} \oplus H^{2,1}. \tag{7.90}$$

Note that we have

$$H^3(X, \mathbf{C}) = W \oplus \overline{W} \tag{7.91}$$

and this is a complex polarization of the symplectic vector space, which means concretely that

$$\int_X \omega_i \wedge \omega_j = 0. \tag{7.92}$$

The $g+1$ three-forms ω_i are given in components as

$$\omega_i = \alpha_i + \frac{\partial \mathcal{F}_j}{\partial \phi^i} \beta^j. \tag{7.93}$$

We will use the notation

$$\tau_{ij} = \frac{\partial \mathcal{F}_j}{\partial \phi^i}. \tag{7.94}$$

Now we claim that

$$\tau_{ij} = \tau_{ji}. \tag{7.95}$$

[4] Here we need the technical assumption of Griffiths transversality: the first order variation of a $(3,0)$ form gives at most a $(2,1)$ form [67].

This follows directly from relation (7.92), by using the fact that for *any* two 3-form Φ and Ψ

$$\int_X \Phi \wedge \Psi = \sum_i \left(\oint_{A^i} \Phi \oint_{B_i} \Psi - \oint_{A^i} \Psi \oint_{B_i} \Phi \right). \tag{7.96}$$

The LHS can only contribute if it is a form of total degree (3,3). Various identities are obtained by inserting forms for which the LHS vanishes. For instance, if we plug in ω_i and ω_j this is the case, and we find the relation

$$\tau_{ij} = \oint_{B^j} \omega_i = \oint_{B^i} \omega_j = \tau_{ji} \tag{7.97}$$

Since $\tau_{ij} = \partial_i \mathcal{F}_j$, this tells us that locally the period $\mathcal{F}_i$ is the derivative of a function on $\mathcal{L}$, the prepotential $\mathcal{F}(\phi)$

$$\mathcal{F}_i = \partial_i \mathcal{F}. \tag{7.98}$$

Note that this function is homogeneous of degree two. This relation is obtained easily by using the above trick again for the forms Ω and ω_i, which gives

$$0 = \int_X \Omega \wedge \omega_i = \phi^j \partial_j \mathcal{F}_i - \mathcal{F}_j. \tag{7.99}$$

So, $\mathcal{F}_i$ is homogeneous of degree 1, and therefore $\mathcal{F}$ is homogeneous of degree 2 (and therefore a section of the bundle $\mathcal{L}^{\otimes 2}$. Note that we can write

$$\mathcal{F} = \tfrac{1}{2} \phi^i \mathcal{F}_i = \tfrac{1}{2} \oint_{A^i} \Omega \oint_{B_i} \Omega. \tag{7.100}$$

Another useful identity is that

$$\Omega = \phi^i \omega_i. \tag{7.101}$$

That is, in the local coordinates ϕ^i on W, the three-form Ω is given by the vector ϕ.

Finally, there is a beautiful formula for the natural Weil-Peterson Kähler metric on the moduli space $\mathcal{M}$ [68]. (Note that such a metric canonically exist, since the complex deformations can be written as special metric deformations. The space of Riemannian metrics always carries a natural metric itself.) In the case of a family of Calabi-Yau spaces the metric is Kähler and given in terms of the Kähler potential K, which can be written as

$$e^{-K} = \frac{i}{2} \int_X \Omega \wedge \overline{\Omega} \tag{7.102}$$

or in terms of the special coordinates ϕ^i,

$$e^{-K} = \mathrm{Im}\, (\overline{\phi}^i \mathcal{F}_i) = (\phi, \overline{\phi}). \tag{7.103}$$

Here we use the inner product $(\cdot,\cdot)$ on the space W defined by

$$(x,y) = x^i (\mathrm{Im}\,\tau)_{ij} x^j. \tag{7.104}$$

Note that this inner product has signature $(g,1)$. The corresponding hermitian form can be written as

$$(x,\overline{y}) = \frac{i}{2}\int x\wedge \overline{y}. \tag{7.105}$$

The corresponding Kähler (Weil-Peterson or Zamolodchikov) metric is given by

$$G_{i\overline{\jmath}} = \partial_i \partial_{\overline{\jmath}} K, \tag{7.106}$$

so that for general vectors $x,y\in W$, with $x=x^i\omega_i$, we have

$$G(x,\overline{y}) = G_{i\overline{\jmath}} x^i \overline{y}^{\overline{\jmath}} = -\frac{(x,\overline{y})}{(\phi,\overline{\phi})} + \frac{(x,\overline{\phi})(\phi,\overline{y})}{(\phi,\overline{\phi})^2}. \tag{7.107}$$

Here we recall that the three-form Ω is expressed as $\Omega=\phi^i\omega_i$. Since e^{-K} gives the metric on the line bundle $\mathcal{L}$, almost by definition the corresponding Kähler form $\omega = \partial\overline{\partial}K$ has the property

$$\omega = 2\pi c_1(\mathcal{L}). \tag{7.108}$$

Hence we have the quantization condition $\omega/2\pi \in H^2(\mathcal{M},\mathbf{Z})$. (The technical term for such a Kähler metric that arises as the Chern class of a line bundle is *restricted Kähler*.)

The Kähler metric is by inspection degenerate in the direction of the vector $\phi=\Omega$,

$$G(x,\overline{\phi}) = G(\phi,\overline{y}) = 0 \tag{7.109}$$

In fact, here it is useful to introduce a slightly differently normalized metric, the so-called tt^*-metric (see §8.12) defined as

$$g_{i\overline{\jmath}} = e^{-K} G_{i\overline{\jmath}} \tag{7.110}$$

In terms of this metric we have

$$g(x,\overline{y}) = -(x,\overline{y}) + \frac{(x,\overline{z})(\phi,\overline{\phi})}{(\phi,\overline{\phi})} = -\frac{i}{2}\int x_\perp \wedge y_\perp \tag{7.111}$$

where $x_\perp$ is the $(2,1)$ part, satisfying $(x,\overline{\phi})=0$, of the general 3-form $x\in W = H^{3,0}\oplus H^{2,1}$. So, when we restrict to $H^{2,1}$, and thereby descend from $\mathcal{L}$ down to $\mathcal{M}$, we have

$$g_{i\overline{\jmath}} = -\mathrm{Im}\,\tau_{ij}. \tag{7.112}$$

More precisely, with $P_\perp$ the orthogonal projection on $H^{2,1}$,

$$g = -P_\perp \cdot \operatorname{Im}\tau \cdot P_\perp \tag{7.113}$$

We now see that g (and G) give a good metric on the moduli space $\mathcal{M}$.

This description of the geometry of the moduli space of complex structures of a CY manifold in terms of the homogeneous function $\mathcal{F}$ is known as *special geometry* [69]. Special geometry was first discovered in the context of $N = 2$ $D = 4$ supergravity [70]. It was then shown that CY spaces give a concrete realization [68]. This is not a coincidence, since the CY manifold can be used as a compactification in string theory. We will see later in §8.12 how the same structure also emerges in CFT, or more precisely, families of $N = 2$ string vacua.

It is interesting to see what happens at a singularity in $\mathcal{M}_X$. If the Calabi-Yau space develops a node, a particular three-cycle, say A^0, will shrink to zero volume. So the period

$$\phi^0 = \int_{A^0} \Omega \to 0 \tag{7.114}$$

will go to zero. Now one can prove that if we make a monodromy at the singular locus $\phi^0 = 0$, *i.e.* transform $\phi^0 \to e^{2\pi i}\phi^0$, there will be an action on the other cycles given by the Picard-Lefshetz formula [71]

$$C \to C + \eta(C, A_0)C. \tag{7.115}$$

So the dual cycle B_0 will transform as

$$B_0 \to B_0 + A_0, \tag{7.116}$$

and therefore we have

$$\partial_0\mathcal{F} = \int_{B_0} \Omega \;\to\; \partial_0\mathcal{F} + \phi^0. \tag{7.117}$$

From this we read off immediately that the singularity of $\mathcal{F}$ around $\phi^0 = 0$ takes the form

$$\mathcal{F} \sim \frac{1}{4\pi}(\phi^0)^2 \log \phi^0. \tag{7.118}$$

We will make use of the result later.

8. Perturbative string theory

From our deliberations on conformal field theories we now move upwards to the world-sheet formulation of perturbative (bosonic) string theory. This a subject for a lecture series in its own, and fortunately there are excellent set of lectures in the literature [72, 73, 74, 75] (as there are excellent text books [4]). So we will mainly focus on the more abstract mathematical aspects, that are usually not stressed so much in an introductory course.

8.1. Axioms for string vacuum

From an axiomatic point of view a perturbative string vacuum consists of three ingredients: a representation $\mathcal{H}$ of the Virasoro algebra (actually the classical Witt algebra), a BRST differential Q, and an anti-ghost field G that can be used to make volume forms on the moduli space $\mathcal{M}_{g,n}$. More precisely, we need (for a mathematical exposition see *e.g.* [76])

(1) A (necessarily non-unitary) CFT of central charge $c = 0$. That is, we have Virasoro generators $L_n, \overline{L}_n$ acting on a Hilbert space $\mathcal{H}$. This Hilbert space is graded by ghost/fermion number F,

$$\mathcal{H} = \bigoplus_{n\in\mathbf{Z}} \mathcal{H}^{(n)}. \tag{8.1}$$

Quite often we can introduce a bi-grading by separately conserved left-moving and right-moving ghost charges F_L and F_R with $F = F_L + F_R$.

(2) A BRST operator

$$Q : \mathcal{H}^{(n)} \to \mathcal{H}^{(n+1)} \tag{8.2}$$

satisfying $Q^2 = 0$. This makes the Hilbert space $\mathcal{H}$ into a complex.

(3) Anti-ghosts $G, \overline{G}$ that are primary spin two fields with a mode expansion

$$G(z) = \sum_k G_k z^{-k-2} \tag{8.3}$$

and similarly for $\overline{G}(\overline{z})$. The generators $G_n, \overline{G}_n$ are odd and anticommuting, and satisfy the algebra

$$\{Q, G_n\} = L_n, \qquad \{Q, \overline{G}_n\} = \overline{L}_n, \tag{8.4}$$

$$[L_n, G_m] = (n-m)G_{n+m}. \tag{8.5}$$

Given these three basic ingredients one defines the Hilbert space of basic states as

$$\mathcal{H}_{basic} = \ker G_0^- \cap \ker L_0^-, \tag{8.6}$$

with $G_0^{\pm} = G_0 \pm \overline{G}_0$, *etc.* Here we note the important relation

$$\{Q, G_0^-\} = L_0^-. \tag{8.7}$$

The space V of physical states is finally defined as the "semi-relative" cohomology

$$V = H_Q^*(\mathcal{H}_{basic}). \tag{8.8}$$

This last definition needs some explaining. Suppose we have a space X that carries a circle action, say for convenience a free action, so that we have a S^1 principal fiber bundle $\pi : X \to B$ over some base space B. Let ξ be the vector field that generates the S^1 action. Now suppose we want to compute the cohomology of B in terms of the cohomology of X. There is a general procedure to do this for arbitrary quotient spaces called equivariant cohomology [77], but here there is a much more simple procedure. To this end we consider the following two operators that act on the space $\Omega^*(X)$ of differential forms on X: the inner product ι_ξ with the vector field ξ, and the Lie derivative $\mathcal{L}_\xi$. These operators satisfy Cartan's relation

$$\{d, \iota_\xi\} = \mathcal{L}_\xi. \tag{8.9}$$

The differential forms on B pulled-back to X can now be characterized by two properties: they are S^1 invariant, and thus satisfy $\mathcal{L}_\xi \alpha = 0$, and they have no components in the direction of the fiber, $\iota_\xi \alpha = 0$. This is the definition of a basic form,

$$\Omega^*_{basic}(X) = \ker \iota_\xi \cap \ker \mathcal{L}_\xi. \tag{8.10}$$

The cohomology of B can now be computed by working with these basic forms. We see that to make an analogy with string theory, we have the translation

$$\Omega^*(X), \Omega^*_{basic}(X), H^*(B), d, \iota_\xi, \mathcal{L}_\xi \iff \mathcal{H}^*, \mathcal{H}^*_{basic}, V, Q, G_0^-, L_0^-. \tag{8.11}$$

In string theory the relevant circle action is the rotation of the string as implemented by the world-sheet momentum operator L_0^-.

With all this machinery in place, string amplitudes are defined by choosing physical vectors $\phi_1, \ldots, \phi_n \in V$ and doing the following integral over $\mathcal{M}_{g,n}$

$$A(\phi_1, \ldots, \phi_n) = \int_{\mathcal{M}_{g,n}} \langle \phi_1 \cdots \phi_n \prod_{I=1}^{3g-3+n} \int_\Sigma \mu_I G \int_\Sigma \overline{\mu}_I \overline{G} \rangle. \tag{8.12}$$

Here μ_I is a basis of Beltrami differentials spanning $T_\Sigma \mathcal{M}_{g,n}$. Since $G(z)$ is a section of K^2, the combination $\mu_I G$ is a $(1,1)$ form on the surface Σ and so the integral makes sense.

In particular we have the following form of the genus g vacuum amplitude

$$F_g = \int\limits_{\mathcal{M}_{g,n}} \langle \prod_{I=1}^{3g-3} \int_\Sigma \mu_I G \int_\Sigma \overline{\mu}_I \overline{G} \rangle. \tag{8.13}$$

By some formal manipulations this expression can be simplified for the case of the one-loop amplitude, giving [78]

$$F_1 = \frac{1}{2} \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\,\tau} \mathrm{Tr} \left(F_L F_R (-1)^F q^{L_0} \overline{q}^{\overline{L}_0} \right). \tag{8.14}$$

(The factor $\frac{1}{2}$ is because of the $\mathbf{Z}_2$ symmetry $z \to -z$ that any elliptic curve allows.)

8.2. Intermezzo — twisting and supersymmetry

Before we continue our discussion of string theory it is useful to review briefly the concept of twisted supersymmetries and non-local operators, since these concepts play an important role in the following and are quite generally applicable, also in the four-dimensional context in later lectures.

Supersymmetry transformations Q are fermionic operations that square into the translation operator. The usual supersymmetry algebra has the form

$$\{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta} P_\mu. \tag{8.15}$$

Here the supercharges Q_α are spinors on the space-time manifold, and P_μ is the momentum operator that generates translations. Now generically there are no covariant constant spinors on a curved manifold, so it is difficult to find manifolds that allow global supersymmetries. (Just as it is generally speaking not true that a manifold allows isometries.) In fact, to find a global supersymmetry that always exists, independent of the space-time topology, one needs a *scalar* supersymmetry. (There is always a covariant constant function, the constant function.) That is, we are looking for a slightly different supersymmetry algebra of the form

$$\{Q, G_\mu\} = P_\mu, \tag{8.16}$$

with Q a scalar and G_μ a vector, both odd and satisfying

$$Q^2 = 0, \qquad \{G_\mu, G_\nu\} = 0. \tag{8.17}$$

Since the "BRST operator" Q squares to zero, for any QFT with such an operator we can define in general the cohomology group

$$V = H_Q(\mathcal{H}) \tag{8.18}$$

of "physical" states annihilated by Q, modulo Q exact states. If Q symmetry is unbroken, *i.e.* if the vacuum satisfies $Q|vac\rangle = 0$, all expectation values of Q commutators vanish, since

$$\langle [Q, \mathcal{O}] \rangle = 0. \tag{8.19}$$

(We write $\langle \cdots \rangle = \langle vac| \cdots |vac\rangle$.) This implies that if a set of local operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$ all satisfy

$$[Q, \mathcal{O}_i(x)] = 0, \tag{8.20}$$

their correlation function

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle \tag{8.21}$$

is constant. This is proven algebraically by observing that

$$\frac{\partial \mathcal{O}}{\partial x^\mu} = [P_\mu, \mathcal{O}] = [Q, \mathcal{O}^{(1)}_\mu], \tag{8.22}$$

where we define the one-form

$$\mathcal{O}^{(1)}_\mu = [G_\mu, \mathcal{O}]. \tag{8.23}$$

We thus find that[5]

$$\frac{\partial}{\partial x^\mu} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \langle [Q, \mathcal{O}_{1,\mu}(x_1) \cdots \mathcal{O}_n(x_n)] \rangle = 0. \tag{8.24}$$

All this can be done in a more systematic way as follows [80]. In fact, regarding the Q symmetry as a spin zero supersymmetry is a very fruitful analogy. It is convenient to go to a "superspace" formulation of the theory where, in addition to the space-time coordinates x^μ we have additional Grassmannian coordinates θ^μ, just as when we considered supersymmetric quantum mechanics. Starting from a physical field $\mathcal{O}(x)$, that we will now denote as $\mathcal{O}^{(0)}(x)$ to indicate that it is a zero-form, the superfield $\mathcal{O}(x, \theta)$ is defined as

$$\begin{aligned} \mathcal{O}(x,\theta) &= e^{\theta^\mu G_\mu} \mathcal{O}(x) \\ &= \sum_k \mathcal{O}^{(k)}_{\mu_1 \ldots \mu_k}(x) \theta^{\mu_1} \cdots \theta^{\mu_k}. \end{aligned} \tag{8.25}$$

Here the fields $\mathcal{O}^{(k)}$ are generated from $\mathcal{O}^{(0)}$ by repeated application of G_μ

$$\mathcal{O}^{(k)}_{\mu_1 \ldots \mu_k}(x) = [G_{\mu_1}, [G_{\mu_2}, \ldots, [G_{\mu_k}, \mathcal{O}^{(0)}(x)] \ldots]]. \tag{8.26}$$

Since $\mathcal{O}^{(k)}_{\mu_1 \ldots \mu_k}$ is antisymmetric in all its indices, it represents a k-form, and we can write this relation as

$$[G, \mathcal{O}^{(k)}] = \mathcal{O}^{(k+1)}, \qquad G = G_\mu dx^\mu. \tag{8.27}$$

[5]We write in the following $[\cdot,\cdot]$ for the graded (anti)commutator in the following.

Since we have $[Q, G] = d$, these differential forms satisfy the important *descent equation* [80]

$$d\mathcal{O}^{(k)} = [Q, \mathcal{O}^{(k+1)}]. \tag{8.28}$$

We can draw two conclusions from this equation. First, it suggests a new class of *non-local* physical observables. If C is a k-dimensional closed submanifold, the descent equation shows that

$$\mathcal{O}_C = \int_C \mathcal{O}^{(k)}(x) \tag{8.29}$$

is a physical observable, since

$$[Q, \mathcal{O}_C] = \int_C d\mathcal{O}^{(k-1)} = \int_{\partial C} \mathcal{O}^{(k-1)} = 0. \tag{8.30}$$

Secondly, if C and C' represent the same class in $H_k(M)$, we have $C - C' = \partial B$ and

$$\mathcal{O}_C - \mathcal{O}_{C'} = \int_B d\mathcal{O}^{(k)} = [Q, \int_S \mathcal{O}^{(k+1)}]. \tag{8.31}$$

So the physical observable depends only on the homology class of C. That is, for each class in $H_k(M)$ and each element in $V = H^*_Q(\mathcal{H})$ we can construct a non-local operator

$$C \in H_k(M) \Rightarrow \mathcal{O}_C = \int_C \mathcal{O}^{(k)}(x) \tag{8.32}$$

Why is all this relevant for string theory? As we have stressed, string amplitudes are obtained by integrating over the moduli space. In particular we have to integrate over the positions of the vertex operators that create a certain string state in the push-forward $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$. These vertex operators have to be $(1,1)$ forms on the Riemann surface. These volume forms are obtained through the above procedure.

More precisely, we first observe that the translation operators are given by

$$\frac{\partial}{\partial z} = L_{-1}, \qquad \frac{\partial}{\partial \overline{z}} = \overline{L}_{-1}. \tag{8.33}$$

Part of the algebra of the anti-ghosts tells us that

$$\{Q, G_{-1}\} = L_{-1}, \qquad \{Q, \overline{G}_{-1}\} = \overline{L}_{-1}. \tag{8.34}$$

Therefore our scalar supersymmetry (8.16) is realized on the world-sheet.

Suppose we now pick a physical state, that is a cohomology class of Q. Such a state can be assumed to have conformal weight zero. If not, we use the relation

$$\{Q, G_0\} = L_0 \tag{8.35}$$

to prove that it is Q exact. Out of such a operator ϕ of weight zero we can now make the associated two-form as

$$\phi^{(2)} = [G_{-1}, [\overline{G}_{-1}, \phi]]. \tag{8.36}$$

The integrated vertex operators are now given by the non-local Q closed expressions

$$\phi_\Sigma = \int_\Sigma \phi^{(2)}. \tag{8.37}$$

8.3. Example — The critical bosonic string

We now return to string theory by giving some simple examples.

In the bosonic string (see *e.g.* [46]) the axioms are satisfied by starting with 26 bosonic free fields x^μ. These fields have stress-tensor

$$T = -\tfrac{1}{2}(\partial x^\mu)^2, \tag{8.38}$$

which generates a $c = 26$ representation of the Virasoro algebra. To this we add the ghosts of the bosonic string, a b, c system as described in §6.4 with $\lambda = 2$. So $b(z)$ has spin two and $c(z)$ has spin -1 and the central charge equals $c = -26$. The full Hilbert space is now given by

$$\mathcal{H} = \mathcal{H}^{26\ bosons} \otimes \mathcal{H}^{ghosts}. \tag{8.39}$$

It is graded by the ghost charge $F = -\oint bc$ and carries the action of the BRST charge

$$Q = \oint \left(-\tfrac{1}{2} c(\partial x)^2 + c\partial c b \right) \tag{8.40}$$

The anti-ghost is given by the b field (which gives its name to G)

$$G(z) = b(z), \tag{8.41}$$

which indeed has the defining property

$$\{Q, b(z)\} = T(z). \tag{8.42}$$

Physical fields typically take the form

$$\phi = c\overline{c}V, \tag{8.43}$$

with V a primary field in the matter sector of conformal dimension $(1,1)$. The most important fields are the states of the form

$$V_p^{\mu\nu} = \partial x^\mu \overline{\partial} x^\nu e^{ipx} \iff \alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}|p\rangle, \tag{8.44}$$

which represent the Fourier modes of the space-time fields $G_{\mu\nu}(x)$ and $B_{\mu\nu}(x)$.

8.4. Example — Twisted $N=2$ SCFT

Another important example is the twisting of the $N=2$ superconformal algebra in two dimensions, say with central charge $c=3d$. In that case we have the following currents: the stress-energy tensor T of spin 2, two supercurrents $G^\pm$ of spin 3/2, and a $U(1)$ current of spin 1. In these algebra states and fields are characterized by their conformal dimension h and charges q

$$L_0|\phi\rangle = h|\phi\rangle, \qquad J_0|\phi\rangle = q|\phi\rangle. \tag{8.45}$$

where L_0 and J_0 are the zero-modes of T and J. Twisting amounts to redefining the stress tensor as

$$T \to T + \tfrac{1}{2}\partial J, \tag{8.46}$$

which gives in particular $L_0 \to L_0 + \frac{1}{2}J_0$ and accordingly adds the charge to the conformal dimensions

$$h \to h + \tfrac{1}{2}q. \tag{8.47}$$

Since the supercurrent $G^\pm$ has $h = 3/2$ and $q = \pm 1$, this gives field $G = Q^+$ and $Q = Q^-$ of spin 2 and 1. The modes L_n, G_n, Q_n, J_n of the four currents form a closed algebra, which is simple the $N=2$ superconformal algebra written in a different basis

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}, & [J_m, J_n] &= d\cdot m\,\delta_{n+m,0},\\
[L_m, G_n] &= (m-n)G_{m+n}, & [J_m, G_n] &= -G_{m+n},\\
[L_m, Q_n] &= -n\,Q_{m+n}, & [J_m, Q_n] &= Q_{m+n},
\end{aligned}$$

$$\begin{aligned}
\{G_m, Q_n\} &= L_{m+n} + nJ_{m+n} + \tfrac{1}{2}d\cdot m(m+1)\delta_{n+m,0},\\
[L_m, J_n] &= -nJ_{m+n} - \tfrac{1}{2}d\cdot m(m+1)\delta_{m+n,0}.
\end{aligned} \tag{8.48}$$

So all the conditions of a string vacuum (and more) are satisfied.

We see in particular that out of this twisted $N=2$ we can produce a two-dimensional topological field theory with Hilbert space $V = H_Q(\mathcal{H})$. The ring V is in that case known as the chiral ring [85].

8.5. Example — twisted minimal model

As the simplest example of the twisting procedure we consider one free boson x with a background charge $Q = -1/\sqrt{3}$ so that the central charge vanishes $c = 0$. This is the twisted $k = 1$ $N = 2$ minimal model. Now the vertex operators

$$V_p = e^{ipy/\sqrt{3}} \tag{8.49}$$

have conformal dimension h and "ghost charge" q given by

$$h = \tfrac{1}{2}p(p-1), \qquad q = \tfrac{1}{3}p. \tag{8.50}$$

The BRST charge and the anti-ghost are defined as

$$Q = \oint V_3(z), \qquad G(z) = V_{-3}(z). \tag{8.51}$$

This string theory has two physical states

$$\phi_0 = \mathbf{1}, \qquad \phi_1 = V_1, \tag{8.52}$$

with $(h, q) = (0, 0)$ and $(0, \frac{1}{3})$. The algebra of these fields is trivial

$$\phi_1 \cdot \phi_1 = 0. \tag{8.53}$$

It is a nice exercise to compute the corresponding two-forms

$$\phi_0^{(2)} = 0, \qquad \phi_1^{(2)} = V_{-2}. \tag{8.54}$$

The only non-zero amplitude in this toy model is the four-point function

$$A(\phi_1, \phi_1, \phi_1, \phi_1) = \int d^2z\, |(z-1)^{\frac{2}{3}} z^{\frac{1}{3}}|^2 \tag{8.55}$$

which is some finite ratio of Γ-functions.

8.6. Example — topological string

In its simplest form the topological string is obtained by twisting a $N = 2$ sigma model on a Calabi-Yau X. (See [31] for an extensive review of topological strings and their role in non-critical string theory and two-dimensional quantum gravity, and see [82] for their importance for intersection theory on the moduli space of Riemann surfaces.)

For the simple case $X = \mathbf{C}$ (or T^2), the topological string is given by two free scalar fields x^1, x^2 that can be combined in complex fields $x = x_1 + ix_2$ and $\overline{x} = x^1 - ix^2$. The central charge $c = 2$ of this system is matched by a $\lambda = 1$ (b, c) system with central charge $c = -2$. The total stress-tensor is given as

$$T = -\partial x \partial \overline{x} - b \partial c \tag{8.56}$$

The BRST transformation rules are then taken as

$$\begin{aligned} Qx &= c \\ Q\overline{x} &= 0 \\ Qc &= 0 \\ Qb &= \partial x \end{aligned} \tag{8.57}$$

The anti-ghost is $G = b\partial x$. Typical physical fields are $e^{ip\overline{x}}$, $ce^{ip\overline{x}}$.

8.7. Functorial definition

After all these concrete examples, we return into the clouds and our abstract discussion of perturbative string theory. In order to give a more functorial definition of a string vacuum, using the language of categories, we have to make two generalizations beyond CFT [83, 84]

(1) The Hilbert space $\mathcal{H}$ is a complex, with differential Q. Its cohomology we will denote again as V.

(2) The amplitudes Φ_Σ are *differential forms* on the extended moduli space. So, we have maps

$$\Phi_\Sigma : \mathcal{H}^{\otimes n} \to \Omega^*(\mathcal{P}_{g,n}) \tag{8.58}$$

This structure is sometimes also called a *cohomological field theory* [81, 36].

If one wishes, one can introduce two new categories, the category **Comp** of Hilbert space complexes and equivariant maps, and the category **TRiem** of so-called topological Riemann surfaces [31]. The definition of these objects is such that their moduli space is given by the $3g - 3|3g - 3$ dimensional superspace

$$\widehat{\mathcal{M}}_{g,n} = \Pi T \mathcal{M}_{g,n} \tag{8.59}$$

so that functions on the moduli space become differential forms on $\mathcal{M}_{g,n}$. With all this in place, a perturbative string vacuum is a functor

$$\Phi : \mathbf{TRiem} \to \mathbf{Comp}. \tag{8.60}$$

We see that we have made the progression

$$TFT \Rightarrow CFT \Rightarrow Strings$$

and at the same time our amplitudes or morphisms became objects in bigger and bigger spaces

$$H^0(\mathcal{P}_{g,n}) \Rightarrow \Omega^0(\mathcal{P}_{g,n}) \Rightarrow \Omega^*(\mathcal{P}_{g,n}).$$

In string perturbation theory we have two differentials: the supercharge or BRST operator Q that acts on the Hilbert space $\mathcal{H}$ and the exterior differential d on the moduli space. Perturbative string theory is now abstractly defined by the extra relation

$$(d+Q)\,\Phi_\Sigma = 0. \tag{8.61}$$

The relation with the anti-ghost field

$$G(z) = \sum_n G_n z^{-n-2} \tag{8.62}$$

is as follows. The amplitudes are differential forms on moduli space with a degree determined by the total ghost charge. If we have a form of degree p, we can pick Beltrami differentials $\mu_1, \ldots, \mu_p \in T_\Sigma \mathcal{P}_{g,n} \cong H^1(T_\Sigma)$ and find the equivalence

$$\begin{aligned}\langle \prod_i \phi_1(P_i) \prod_{I=1}^{p} \Big(\int_\Sigma \mu_I G \int_\Sigma \overline{\mu}_I \overline{G} \Big) \rangle \\ = \langle \Phi_\Sigma(\phi_1, \ldots, \phi_n), \bigwedge_I (\mu_I \wedge \overline{\mu}_I) \rangle.\end{aligned} \tag{8.63}$$

If we now choose $\phi_i \in V = H^*_Q(\mathcal{H}_0)$ then the condition

$$d\Phi_\Sigma(\phi_1, \ldots, \phi_n) = 0 \tag{8.64}$$

tells us that Φ_Σ actually is a map

$$\Phi_\Sigma : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}). \tag{8.65}$$

Here we use that the fibers in the projection $\mathcal{P}_{g,n} \to \mathcal{M}_{g,n}$ do not contribute to the Q cohomology [84]. This is guaranteed by the restriction to basic states, as in (8.6). Indeed, the only non-trivial part in the infinite-dimensional fiber of the projection is the circle action obtained by rotating the local coordinate $z \to e^{i\theta} z$. This rotation is implemented by the operator L_0^-. This explains the analogy with computing the cohomology of the base manifold of a circle bundle that we mentioned in §8.1.

The final g loop string amplitudes, that computes scattering of string states, are then expressed by a further integral over $\mathcal{M}_{g,n}$ that picks out the top component

$$A(\phi_1,\ldots,\phi_n)=\int_{\mathcal{M}_{g,n}}\Phi_\Sigma(\phi_1,\ldots,\phi_n). \tag{8.66}$$

8.8. Tree-level amplitudes

This structure becomes more manageable in genus zero. As a simple example consider the sphere with three holes. Since we have seen that $\mathcal{M}_{0,3}=pt$, the three point function is simply given by a complex number

$$\langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle_0=\Phi_{0,3}(\phi_1,\phi_2,\phi_2)\in\mathbf{C}. \tag{8.67}$$

If we pick a basis $\phi_i\in V$ this gives us the structure coefficients of a TFT

$$c_{ijk}=\Phi_{0,3}(\phi_i,\phi_j,\phi_k). \tag{8.68}$$

So we see that we actually went through a little loop,

$$TFT\Rightarrow CFT\Rightarrow String\Rightarrow TFT. \tag{8.69}$$

For the more general case of an n-point function in genus zero, we obtain a cohomology class on $\overline{\mathcal{M}}_{0,n}$. In string theory we are particularly interested in the forms of top degree. We will define the n-point genus zero string amplitude as

$$\begin{aligned}c_{i_1\cdots i_n} &= \int_{\mathcal{M}_{0,n}}\Phi_{0,n}(\phi_{i_1},\ldots,\phi_{i_n})\\ &= \langle\phi_{i_1}(0)\phi_{i_2}(1)\phi_{i_3}(\infty)\prod_{k=4}^{n}\int\phi^{(2)}_{i_k}\rangle\end{aligned} \tag{8.70}$$

with $\phi^{(2)}=[G_{-1},[\overline{G}_{-1},\phi]]$, a $(1,1)$ form on the surface.

As written above this amplitude has not the manifest permutation symmetry of the indices $i_1,\ldots,i_n$. That this symmetry is still present we can show by considering again the automorphisms of $\mathbf{P}^1$. To this end we recall that the non-local operators can be combined into one superfield (8.25) on the topological Riemann sphere $\Pi T\mathbf{P}^1$,

$$\phi(z,\overline{z},\theta,\overline{\theta})=\phi^{(0)}+\phi^{(1,0)}\theta+\phi^{(0,1)}\overline{\theta}+\phi^{(2)}\theta\,\overline{\theta}. \tag{8.71}$$

Here we suppressed the $(z, \overline{z})$ dependence on the RHS. Let us now consider a correlation function on the sphere of the form

$$\Big\langle \prod_{i=1}^{n} \int \phi_i(z, \overline{z}, \theta, \overline{\theta}) \Big\rangle_0 . \tag{8.72}$$

This expression is not well-defined, since it is invariant under a fermionic extension of the $PGL(2, \mathbf{C})$ symmetry, generated by operators L_0, L_1, L_{-1} and G_0, G_1, G_{-1} and their complex conjugates. We have to factor out the infinite volume of this group in order to obtain a finite answer. These symmetries correspond to the super-Möbius transformations

$$z \to \frac{az+b}{cz+d}, \qquad \theta \to \frac{\theta + \alpha z^2 + \beta z + \gamma}{(cz+d)^2}. \tag{8.73}$$

This extended $PGL(2, C)$ symmetry can be used to fix three of the z-coordinates, say z_1, z_2, z_3, at 0, 1, and ∞, and put three of the θ-coordinates to zero. We choose these anti-commuting coordinates to be $\theta_1, \theta_2, \theta_3$. If we recall that

$$\int d^2\theta \cdot \phi = \phi^{(2)}, \qquad \phi\big|_{\theta=0} = \phi^{(0)}, \tag{8.74}$$

then we see that after gauge fixing we are left with a correlation function of the form

$$\Big\langle \phi_{i_1}^{(0)} \phi_{i_2}^{(0)} \phi_{i_3}^{(0)} \int \phi_{i_4}^{(2)} \cdots \int \phi_{i_s}^{(2)} \Big\rangle_0 . \tag{8.75}$$

Since we started from an expression that was explicitly symmetric in *all* indices $i_1, \ldots, i_s$, this correlator also has this permutational symmetry. That is, it does not matter which three operators we represent as zero-forms. The generalized $PGL(2, \mathbf{C})$ invariance tells us that we can interchange a zero and a two-form.

8.9. Families of string vacua

Out of the generating function of n-point functions we can construct a family of TFTs labeled by coordinates $t \in V^*$ by the definition

$$c_{ijk}(t) = \langle \phi_i \phi_j \phi_k \exp \int t^k \phi_k^{(2)} \rangle . \tag{8.76}$$

Physically one can think of deforming the two-dimensional action by

$$\delta S = \int t^n \phi_n^{(2)} \tag{8.77}$$

We claim that this defines a family of multiplications on the vector space V. Note that in this way we derive the higher n-point functions by taking derivatives at $t = 0$. In terms of the coefficients $c_{ijk}(t)$ the permutation symmetry of (8.75) gives the important integrability condition

$$\partial_i c_{jkl} = c_{ijkl} = \partial_j c_{ikl}. \tag{8.78}$$

We stress that this is an additional condition imposed on the family of algebras $c_{ijk}(t)$, and a consequence of the topological invariance. We can integrate this relation three times, at least locally, to find that the three-point functions are actually the third derivatives of a function $\mathcal{F}(t)$, the so-called free energy or *prepotential*

$$c_{ijk}(t) = \partial_i \partial_j \partial_k \mathcal{F}(t). \tag{8.79}$$

Symbolically, $\mathcal{F}(t)$ is defined as

$$\mathcal{F}(t) = \langle \exp \int t_i \phi_i^{(2)} \rangle. \tag{8.80}$$

For example, comparing with our discussion of quantum cohomology, formula (4.54) in §4.3, we see that in the case of a sigma model on a Calabi-Yau three-fold we have the following expression for $\mathcal{F}$

$$\mathcal{F}(t) = \int_X \frac{t^3}{3!} + \frac{\chi}{2}\zeta(3) + \sum_{rational\ C} Li_3\left(e^{2\pi t \cdot C}\right). \tag{8.81}$$

with $t \in H^{1,1}$, $Li_3(x) = \sum_{k>0} x^k/k^3$, $\zeta(3) = Li_3(1)$, and χ the Euler number of X. The sum is over all rational curves $C \cong \mathbf{P}^1$ in X. (The constant term is determined by mirror symmetry, or by a degeneration argument.)

As a corollary to the result (8.78), consider the special identity operator $\phi_0 = \mathbf{1}$. There is no corresponding two-form, since G commutes with the identity. So the coupling coefficient t_0 does not exist, there is no modulus associated to $\mathbf{1} \in V$. This fact, combined with integrability relation (8.78), shows that

$$0 = \partial_0 c_{ijk} = \partial_i c_{0jk} = \partial_i \eta_{jk}. \tag{8.82}$$

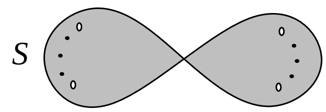
So we have shown that the metric or two-point function η is independent of the deformation parameters t_i.

We have claimed that also for $t \neq 0$ the coefficients $c_{ijk}(t)$ define a Frobenius algebra. That is, the generating function $\mathcal{F}(t)$ satisfies the so-called WDVV equation [25, 86, 31, 32, 36]

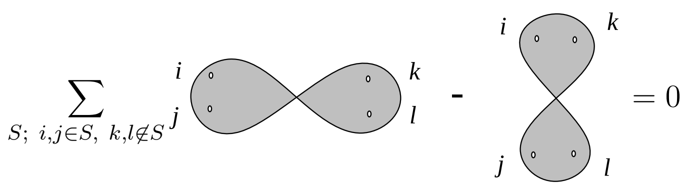
$$\partial_i \partial_j \partial_m \mathcal{F} \eta^{mn} \partial_n \partial_k \partial_l \mathcal{F} = \partial_i \partial_k \partial_m \mathcal{F} \eta^{mn} \partial_n \partial_j \partial_l \mathcal{F}. \tag{8.83}$$

This gives an infinite set of relations on the expansion coefficients $c_{i_1 \ldots i_k}$ of $\mathcal{F}$.

There is a nice mathematical way to prove this last equation starting from Segal's axioms [36]. Keel [87] has shown that the cohomology of $\overline{\mathcal{M}}_{0,n}$ is generated by divisors (codimension one subvarieties) D_S defined as follows. Pick a subset $S \subset \{1, \ldots, n\}$ with $2 \leq \#S \leq n-2$. D_S is now defined as the set of all singular rational curves with a single node, such that the punctures P_i, $i \in S$ lie all on one component, whereas the punctures P_j, $j \notin S$, all lie on the second component:



Clearly these divisors lie in the compactification divisor, the "boundary" of $\mathcal{M}_{0,n}$, since they represented singular curves. Now Keel has shown that these divisors generate all cohomology (as a ring) with a single constraint that is best expressed by a picture



(8.84)

This directly implies the associativity condition on c_{ijk}.

8.10. The Gauss-Manin connection

Let us quickly summarize the most important ingredients of the world-sheet discussion of string vacua and the associated topological field theory. It consists of a moduli space $\mathcal{M}$ of string vacua with over it a vector bundle V carrying a bilinear form η together with an associative multiplication on each fiber. In a concrete basis $\phi_i \in V$ this multiplication is given in terms of the structure coefficients

$$c_{ijk}(t) = \langle \phi_i \phi_j \phi_k \rangle_0 \tag{8.85}$$

Here t^i are the (special) coordinates on the moduli space $\mathcal{M}$ of TFTs.

More precisely, there are five important conditions that we can distinguish.

(1) The parameters t^i are coordinates on $\mathcal{M}$, *i.e.* the vector fields $D_i = \partial/\partial t_i$ commute

$$[D_i, D_j] = 0. \tag{8.86}$$

Moreover, the vector field D_0, corresponding to deformations by the identity operator vanishes.

(2) The coefficients c_{ijk} satisfy the integrability condition

$$D_i c_{jkl} = D_j c_{ikl}, \tag{8.87}$$

(symmetry of the 4-point function) that allows us to write them locally as

$$c_{ijk} = D_i D_j D_k \mathcal{F}, \tag{8.88}$$

in terms of the prepotential $\mathcal{F}(t)$.

(3) The algebra is associative for all values of the moduli,

$$c_{ij}{}^{n} c_{nkl} = c_{ik}{}^{n} c_{njl}. \tag{8.89}$$

(4) The bilinear form $\eta_{ij} = c_{0ij}$ is conserved,

$$D_i \eta_{jk} = 0, \tag{8.90}$$

which is equivalent to the statement that $D_0 c_{ijk} = 0$ using (2).

(5) The algebra is compatible with η

$$c_{ijk} \eta^{jk} = c_{ikj} \eta^{kj}, \tag{8.91}$$

which is equivalent to the complete symmetry of c_{ijk}.

A more succinct description of these conditions uses the matrices C_i defined by the natural map $c : V \to \mathrm{End}(V)$, with matrix elements

$$(C_i)_j{}^{k} = c_{ij}{}^{k} = c_{ijl} \eta^{lk}. \tag{8.92}$$

With these matrices the integrability condition reads

$$[C_i, D_j] = [C_j, D_i], \tag{8.93}$$

whereas the associativity reads

$$[C_i, C_j] = 0. \tag{8.94}$$

In fact, if we define the "Gauss-Manin connection" [32]

$$\nabla_i = D_i - \lambda C_i, \tag{8.95}$$

with spectral parameter λ, then the relations (1),(2) and (3) can be summarized in the statement that

$$[\nabla_i, \nabla_j] = 0, \tag{8.96}$$

for all values of λ. The compatibility of ∇ with η

$$\eta(\nabla\alpha, \beta) = \eta(\alpha, \nabla\beta) \tag{8.97}$$

gives the relations (4) $D_0\eta = 0$, and (5) $\eta(C_i\alpha, \beta)i = \eta(\alpha, C_i\beta)$. Why such a connection should exist will become clear in the later lectures when we consider the integral structure behind all this.

8.11. Anti-holomorphic dependence and special geometry

Up to now we only discussed the holomorphic coordinates t^i, which are analytical coordinates on the moduli space $\mathcal{M}$ of string vacua. In order to discuss the full geometry of $\mathcal{M}$ we need also the dependence on the complex-conjugate variables $t^{\bar{\imath}}$. This geometrical structure is called special geometry, and we have seen an example when we discussed the moduli space of Calabi-Yau three-folds in §7.7. The structure has been worked in the general context of twisted $N = 2$ models by Bershadsky, Cecotti, Ooguri and Vafa. These matters are explained in full detail in the beautiful papers [88, 78, 79], so we only summarize the most important results without proofs. This special geometry of moduli space of string vacua is only present for string vacua that give rise to space-time theories with $N = 2$ supersymmetry.

We have seen that the vector space V had a natural bilinear form (over $\mathbf{C}$)

$$\eta_{ij} = \eta(\phi_i, \phi_j). \tag{8.98}$$

However, in these models there is also an hermitian form

$$g_{i\bar{\jmath}} = g(\phi_i, \phi_{\bar{\jmath}}). \tag{8.99}$$

Here the complex conjugate fields $\phi_{\bar{\imath}}$ are defined in terms of some anti-linear map

$$\phi_{\bar{\imath}} = M_{\bar{\imath}}{}^i \phi_i, \tag{8.100}$$

satisfying $MM^* = -1$.

In our deformation problem we now have to allow for anti-holomorphic dependence on the variables $t^{\bar{\imath}}$. The corresponding deformations are described by local operators

$$\delta S = \int_\Sigma t^{\bar{\imath}} \phi^{(2)}_{\bar{\imath}}, \tag{8.101}$$

with

$$\phi^{(2)}_{\bar{\imath}} = [Q_{-1}, \{\overline{Q}_{-1}, \phi_{\bar{\imath}}\}]. \tag{8.102}$$

Since this field it explicitly Q-exact, its insertions will lead to total derivatives on $\mathcal{M}_{g,n}$, which can only give contributions at the compactification divisor. Indeed, naively one would have ignored these boundary terms and would have concluded that all amplitudes are holomorphic. If one is less naively, one has to investigate in a precise way the behavior on singular curves, and in this way one discovers that certain objects do obtain an antiholomorphic dependence. This has been done in [88, 78, 79], and we now give the results.

The hermitian form g leads to a Kähler metric on the moduli space $\mathcal{M}$. In fact, the canonical metric is the the so-called Zamolodchikov [89] or Weil-Peterson metric

$$G_{i\bar{\jmath}} = e^K g_{i\bar{\jmath}} \tag{8.103}$$

with

$$e^{-K} = g(\phi_0, \phi_{\bar{0}}) = \langle 0|0\rangle. \tag{8.104}$$

(Here ϕ_0 denotes the identity operator.) It is normalized so that $G_{0\bar{0}} = 1$. This metric appears in the two-point functions

$$\langle \phi_i(z) \phi_{\bar{\jmath}}(w)\rangle = \frac{G_{i\bar{\jmath}}}{|z-w|^2}. \tag{8.105}$$

It is a Kähler metric

$$G_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K, \tag{8.106}$$

with Kähler form K given in (8.104). Apart from the derivative D_i and the operator product matrices C_i, we now have conjugate objects $D_{\bar{\imath}}$ and $C_{\bar{\imath}}$. These satisfy the relations

$$[D_{\bar{\imath}}, C_j] = 0, \qquad [D_i, D_{\bar{\jmath}} = 0]. \tag{8.107}$$

These relations express the fact that the three-point function c_{ijk} is a holomorphic object

$$D_{\bar{\imath}} c_{ijk} = 0 \tag{8.108}$$

However there is an additional important relation that tells us how the curvature of the Kähler metric is related to the associative algebra, the so-called tt^* relation [88],

$$[D_i, D_{\bar{\imath}}] + [C_i, C_{\bar{\imath}}] = 0 \tag{8.109}$$

In terms of the Zamolodchikov metric $G_{i\bar{\jmath}}$ this gives the so-called special geometry or special Kähler relation

$$R_{i\bar{\jmath}k}{}^{l} = G_{k\bar{\jmath}}\delta_i{}^l + G_{i\bar{\jmath}}\delta_k{}^l - e^{2K} c_{ikn} G^{n\bar{n}} c_{\overline{\jmath m n}} G^{l\overline{m}} \tag{8.110}$$

Let us note that relations on $D_i, D_{\bar{\imath}}, C_i, C_{\bar{\imath}}$ can be summarized in terms of the Gauss-Manin connection as

$$[\nabla_i, \nabla_j] = [\nabla_{\bar{\imath}}, \nabla_{\bar{\jmath}}] = [\nabla_i, \nabla_{\bar{\jmath}}] = 0 \tag{8.111}$$

So apparently there is a natural flat connection on the moduli space of string vacua, that preserves the bilinear form η. The reasons why this connection exist will become apparent when we take a space-time point of view. There we will uncover an integer structure on V, related to electric and magnetic charges in the four-dimensional space-time.

The metric G satisfies a second important quantization condition: the associated Kähler form of type $(1,1)$,

$$\omega = \frac{i}{2} G_{i\bar{\jmath}} dz^i \wedge dz^{\bar{\jmath}}, \tag{8.112}$$

(which is by definition closed, $d\omega = 0$) has integer periods

$$\omega/2\pi \in H^2(\mathcal{M}, \mathbf{Z}) \tag{8.113}$$

In the language of geometric quantization, this means that the phase space $\mathcal{M}$ can be quantized. There exists a line bundle $\mathcal{L}$ over $\mathcal{M}$ with first Chern class

$$c_1(\mathcal{L}) = \omega/2\pi. \tag{8.114}$$

The Kähler form can thus be realized as the curvature of the $U(1)$ connection on $\mathcal{L}$.

8.12. Local special geometry

The above structure we also met in §7.8, when we discussed the moduli of Calabi-Yau manifolds, and is called *local special geometry* or *special Kähler geometry* [69]. The formal definition is as follows:

Definition: a *local special geometry* $(\mathcal{M}, V, \Omega)$ is defined by the following ingredients:

(1) A complex, Kähler manifold $\mathcal{M}$, the moduli space, of complex dimension

$$\dim \mathcal{M} = g. \tag{8.115}$$

(2) A holomorphic, flat $Sp(2g+2, \mathbf{Z})$ vector bundle $V \to \mathcal{M}$. We denote the symplectic form as η. In the case of the Calabi-Yau moduli space, this was the canonical bundle with fiber $H_3(X, \mathbf{Z})$ and η was given by the intersection form on these three-cycles. The flat connection is written as $\nabla, \overline{\nabla}$. It satisfies $d\eta(\alpha, \beta) = \eta(\nabla\alpha, \beta) + \eta(\alpha, \nabla\beta)$.

(3) A holomorphic section Ω of V, satisfying the conditions

$$\eta(\Omega, \nabla\Omega) = 0, \qquad \eta(\Omega, \overline{\Omega}) > 0 \tag{8.116}$$

(4) the Kähler form on the moduli space is given by

$$e^{-K} = \eta(\Omega, \overline{\Omega}) \tag{8.117}$$

To establish the correspondence between the description of the Calabi-Yau moduli space in §7.8 and the deformations of $N = 2$ string vacua in the previous subsection, note that the holomorphic section Ω determines a line bundle $\mathcal{L} \to \mathcal{M}$. In fact, in the CY case we can choose a polarization

$$V = V^{3,0} \oplus V^{2,1} \oplus V^{1,2} \oplus V^{0,3} \tag{8.118}$$

where $\overline{V}^{p,q} = V^{q,p}$ and $V^{3,0} \cong \mathcal{L}$ is the line bundle generated by Ω and $W = V^{3,0} \oplus V^{2,1}$ is the subspace generated by $\nabla\Omega$. Note that if $\alpha \in V^{p,q}$ and $\beta \in V^{p',q'}$ then the polarization satisfies the condition that

$$\eta(\alpha, \beta) \neq 0 \Rightarrow p + p' = q + q' = 3 \tag{8.119}$$

The description in terms of a local coordinate ϕ^i follows exactly the discussion in §7.8. First we choose a (local) canonical integer basis α_i, β^i of V using the flat connection ∇. We then expand the section Ω in terms of this basis in the familiar form[6]

$$\Omega = \phi^i \alpha_i + \mathcal{F}_i \beta^i. \tag{8.120}$$

Condition (3) now leads to the integrability relation

$$\mathcal{F}_i = \frac{\partial \mathcal{F}}{\partial \phi^i}, \tag{8.121}$$

[6]Here, in order to make contact with §7.8, we indicate the special local coordinates as ϕ^i instead of t^i.

which defines quite generally the prepotential $\mathcal{F}$ as a section of $\mathcal{L}^{\otimes 2}$. Finally, the all important three-point functions c_{ijk} are derived as

$$c_{ijk} = -\eta(\Omega, \nabla_i \nabla_j \nabla_k \Omega) \tag{8.122}$$

or in local coordinates,

$$c_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial \phi^i \partial \phi^j \partial \phi^k} \tag{8.123}$$

In the case of a Calabi-Yau model, the local parameters correspond to the deformations of the complex structure in $H^{2,1}(X)$. This then gives the so-called Type B chiral ring [35]

$$c_{ijk} = -\int_X \Omega \wedge \partial_i \partial_j \partial_k \Omega \tag{8.124}$$

This ring can be motivated as follows. We can define the cohomology groups

$$H^{-p,q}(X) = H^q_{\bar\partial}(Y, \bigwedge^p (T^{(1,0)}X)) \tag{8.125}$$

That is, we consider objects which are of the form

$$\alpha(x) = \alpha^{i_1 \ldots i_p}{}_{\bar\jmath_1 \ldots \bar\jmath_q} \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_p}} dx^{\bar\jmath_1} \wedge \ldots \wedge dx^{\bar\jmath_q}. \tag{8.126}$$

Note that compared with the usual Dolbeault groups, we replaced antisymmetric products of the holomorphic *cotangent bundle* with antisymmetric products of the holomorphic *tangent bundle*, which is essentially the operation of mirror symmetry. The unique holomorphic 3-form Ω can be used to give an isomorphism

$$\Omega : \bigwedge^p (T^{(1,0)}X) \to \bigwedge^{3-p} (T^{(1,0)}X)^*, \tag{8.127}$$

which also gives an isomorphism

$$H^{-p,q}(X) \cong H^{3-p,q}(X). \tag{8.128}$$

The space

$$H^{-*,*}(X) = \bigoplus_{0 \le p,q \le 3} H^{-p,q}(X) \tag{8.129}$$

acquires an obvious ring structure using the wedge product, both on forms and vector fields. Furthermore, since

$$H^{-3,3}(X) \cong H^{0,3}(X) = \mathbf{C}, \tag{8.130}$$

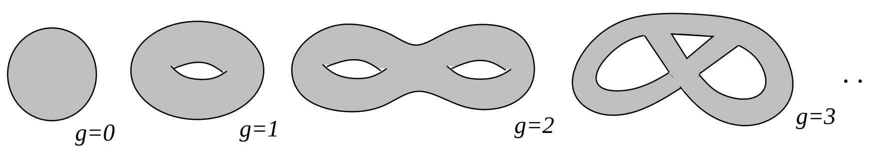
we have an integration formula, and the structure coefficients of the algebra can again be expressed in terms of intersection numbers. By definition this ring does depend crucially on the choice of complex structure — we use vector fields for the holomorphic indices and forms for the anti-holomorphic indices. There is however no dependence on the Kähler class. If we use the isomorphism $H^{-1,1} \cong H^{2,1}$ and choose an explicit basis for $H^{2,1}$ we reproduce the above formula.

9. Gauge theories and S-duality

We leave our discussion of the world-sheet formulation of string theory, and now turn to the space-time description, where we will try to reinterpret some of the mathematical structures that we found in the previous lectures. Hereto we first discuss in the following lecture S-duality in four-dimensional abelian gauge theories. However, before we do this, it might be useful to review some four-dimensional geometry. (The following sections closely follow [90].)

9.1. Introduction to four-dimensional geometry

Four-dimensional geometry is a very rich and difficult subject [91, 92, 93]. Since we have been living exclusively in two dimensions in the previous lectures, we might be tempted to think too simple about topology in other dimensions. For example, the classification problem of four-manifolds is very much different in nature from the classification of surfaces. Compact, orientable surfaces are topologically classified by an integer, their genus,



That the situation is not that simple in four dimensions we can already see by considering the most important invariant of any manifold, the fundamental group π_1.

It is a classical theorem in topology that any finitely representable group (*i.e.* a group that can be represented by a finite number of generators satisfying a finite number of relations — not a very severe restriction) can appear as the fundamental group of a four dimensional manifold. There is a surgery algorithm that constructs the required manifold starting from the generators and the relations. Markov's solution of the word problem shows that the question whether two finitely representable groups are actually isomorphic is undecidable. That is, there is no computing algorithm that can decide this question within a guaranteed finite time. This theorem has therefore rather dramatic consequences for the classification problem of manifolds in dimensions $d \geq 4$. There is simply not a conceivable 'list' of four-manifolds. This fundamental problem can however easily be circumvented by assuming that the four-manifold M is simply connected

$$\pi_1(M) = 0. \tag{9.1}$$

This we will often assume in the following.

After the fundamental group, the next important invariant is the second cohomology group $H^2(M)$. To understand better the role played by the second cohomology of a four-manifold, it might be instructive to consider first the analogous situation for two-dimensional surfaces, if only because it is so much more easily visualized.

For any two-dimensional surface Σ we can consider the first homology group $H_1(\Sigma)$ of homology cycles or equivalently the dual cohomology group $H^1(\Sigma)$ of cocycles. We can think of such a cocycle physically as a flat abelian gauge field A, $F = dA = 0$. The pairing between a cycle $C \in H_1$ and a cocycle $A \in H^1$ is the Wilson line

$$\oint_C A. \tag{9.2}$$

For a genus g surface H^1 has rank $b_1 = 2g$. It is naturally a symplectic space by the intersection form

$$Q(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta. \tag{9.3}$$

Q is an integer, unimodular *anti-symmetric* form. By a standard theorem in symplectic linear algebra there exists a canonical symplectic base $\alpha_1, \ldots, \alpha_g, \beta^1, \ldots, \beta^g \in H^2(\Sigma, \mathbf{Z})$, satisfying

$$Q(\alpha_i, \beta^j) = \delta_i{}^j, \tag{9.4}$$

with all other intersections vanishing. The dual homology basis A^i, B_i we described in §7.6 and looks like *fig.* 4. If we have a orientation-preserving diffeomorphism on the surface, it will act on the first cohomology by a symplectic transformation. That is, there is a map (actually a surjection)

$$\mathrm{Diff}^+(\Sigma) \to Sp(2g, \mathbf{Z}). \tag{9.5}$$

In four dimensions the analogue object to consider is the second cohomology group $H^2(M)$. A physical picture to keep in mind is that, instead of considering Wilson lines, we now look at magnetic fluxes F (satisfying the Bianchi identity $dF = 0$) through a two-cycle Σ (a linear combination of surfaces)

$$\int_\Sigma F. \tag{9.6}$$

In four dimensions we have an analogous intersection form

$$Q : H^2 \times H^2 \to \mathbf{R} \tag{9.7}$$

defined by

$$Q(\alpha,\beta) = \int_M \alpha \wedge \beta. \tag{9.8}$$

In the four-dimensional case Q is *symmetric* and non-degenerate. Over $\mathbf{R}$ such a form can be diagonalized and thus is easily classified by its rank $b_2 = \dim H^2$ and signature (b_2^+, b_2^-) (the number of positive respectively negative eigenvalues). The signature $\sigma(M)$ of the 4-manifold M is defined as the difference of positive and negative eigenvalues of Q

$$\sigma = b_2^+ - b_2^-. \tag{9.9}$$

It has a local expression due to Hirzebruch's signature theorem

$$\sigma(M) = \int_M \tfrac{1}{3}p_1 = -\frac{1}{24\pi^2}\int_M \operatorname{Tr} R \wedge R. \tag{9.10}$$

The signature σ and the Euler character χ are the two classical invariants of a four-manifold.

However, there is more sophisticated information hidden in the intersection matrix Q since it is actually defined on the lattice

$$\Gamma = H^2(M,\mathbf{Z}). \tag{9.11}$$

We can think of the elements of this lattice as *quantized* fluxes, in the sense that

$$F \in \Gamma \Rightarrow \int_\Sigma F \in \mathbf{Z} \tag{9.12}$$

for all surfaces Σ. By Poincaré duality the lattice Γ is self-dual, so we can use the classification results of §7.3.

It might be helpful to list some concrete examples of well-known 4-manifolds and their intersection forms

M	Q
$\mathbf{CP}^1$	$\mathbf{1}$
$\overline{\mathbf{CP}^1}$	$-\mathbf{1}$
$S^2 \times S^2$	H
T^4	$3\mathbf{1} \oplus 3(-\mathbf{1})$
$K3$	$3H \oplus 2(-E_8)$

Herea bar indicates a complex manifold with opposite orientation. (Every complex manifolds comes with a preferred orientation.)

Just as in the case of Riemann surfaces we can study the representation of the diffeomorphism group on Γ. Since this action should preserve the intersection form, there is a homomorphism of the group of orientation preserving diffeomorphisms $\mathrm{Diff}^+(M)$ into the arithmetic group $O(Q, \mathbf{Z}) = \mathrm{Aut}\,(\Gamma)$, that leaves the lattice fixed,

$$\mathrm{Diff}^+(M) \to O(Q, \mathbf{Z}). \tag{9.13}$$

The question is whether this map is actually onto (as was the case for the modular group for a Riemann surface). This is an important issue for Donaldson theory, where we construct differential invariants out of classes in Γ. Since a manifold invariant should by definition be invariant under diffeomorphisms, these Donaldson polynomials will necessarily be constructed out of tensors on H^2 that are invariant under the action of the image of Diff^+. In case that this image is the full group $O(Q, \mathbf{Z})$, standard invariant theory teaches us that the only invariants are tensors constructed out of the intersection form Q itself.

There is one more set of characteristic classes that we have not mentioned yet, the Stiefel-Whitney classes

$$w_i \in H^i(M, \mathbf{Z}_2). \tag{9.14}$$

For the case of a compact, simply-connected orientable manifold we have automatically $w_1 = w_3 = 0$. This leaves w_2 as the only new characteristic class. It is the obstruction to a spin structure. The class w_2 plays an important role if we consider the lattice reduced modulo 2, because of Wu's formula that states that for all $x \in \Gamma$

$$w_2 \cdot x = x^2 \ (\mathrm{mod}\ 2). \tag{9.15}$$

This implies that Q is necessarily even if $w_2 = 0$ or equivalently if M is spin.

9.2. The Lorentz group

It might be convenient to also review some standard facts concerning the Lorentz group $SO(3, 1)$ or its Euclidean version $SO(4)$ and its representations. Consider a 4-dimensional vector space $V \cong \mathbf{R}^4$ and the corresponding space of two-forms $\Lambda^2 \cong \mathbf{R}^6$. If we pick a volume form $e \in \Lambda^4$, then we can define the intersection form

$$q : \Lambda^2 \times \Lambda^2 \to \mathbf{R} \tag{9.16}$$

by

$$\alpha \wedge \beta = q(\alpha, \beta)e, \qquad \alpha, \beta \in \Lambda^2. \tag{9.17}$$

It is easily verified by explicit computation that the symmetric bilinear form q has signature $(3,3)$. The linear transformations $SL(4,\mathbf{R})$ preserve the volume form and induce linear maps on Λ^2 that preserve q. Thus we have a natural map

$$SL(4,\mathbf{R}) \to SO(3,3,\mathbf{R}). \tag{9.18}$$

This is actually a double cover of the identity component of $SO(3,3)$.

The intersection form q is not enough to decompose Λ^2 into maximal positive and negative subspaces

$$\Lambda^2 \cong \Lambda^2_+ \oplus \Lambda^2_-. \tag{9.19}$$

In fact, there is a moduli space of such decompositions parametrized by the grassmannian

$$\mathrm{Gr}^{3,3} = SO(3,3,\mathbf{R})/SO(3,\mathbf{R}) \times SO(3,\mathbf{R}). \tag{9.20}$$

To find such an explicit decomposition we can pick a metric (positive inner product) g on V. This induces a metric on Λ^2 (also denoted by g) which decomposes Λ^2 into orthogonal eigenspaces of the Hodge $*$-operator, defined by

$$g(\alpha,\beta) = q(\alpha, *\beta). \tag{9.21}$$

The $*$-operator squares to one

$$*^2 = 1, \tag{9.22}$$

and the spaces $\Lambda^2_\pm$ can now be defined as the eigenspaces with $* = \pm$: the self-dual (SD) respectively anti-self-dual (ASD) forms. The Hodge star only depends on the conformal class of the metric g, which can be identified with the grassmannian (9.20). The choice of a metric g gives thus rise to a natural homomorphism

$$SO(4,\mathbf{R}) \to SO(3) \times SO(3) = SO(3,3) \cap SO(6), \tag{9.23}$$

which is again a two-fold cover.

This local picture can be repeated globally by considering the tangent bundle TM. There is a splitting of the 2-forms into SD and ASD pieces

$$\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M). \tag{9.24}$$

Descending to de Rahm cohomology, we find a similar decomposition

$$H^2(M) = H^2_+(M) \oplus H^2_-(M) \tag{9.25}$$

with

$$b^2_\pm = \dim H^2_\pm. \tag{9.26}$$

We also notice that (anti)self-dual cocycles are necessarily harmonic

$$d\alpha = 0,\ *\alpha = \pm\alpha \ \Rightarrow\ d^*\alpha = 0. \tag{9.27}$$

Since we are also interested in fermions, we will in general consider representations of the universal cover group of $SO(4)$

$$Spin(4) \cong SU(2)_+ \times SU(2)_-. \tag{9.28}$$

Its irreducible representations are labeled by the dimensions $(\mathbf{n}_+, \mathbf{n}_-)$ with $\mathbf{n}_\pm = 1, 2, \ldots$. In particular we have

$$\begin{aligned}
&\textit{scalar}: && \psi && (\mathbf{1},\mathbf{1}),\\
&\textit{chiral spinor}: && \psi_\alpha && (\mathbf{2},\mathbf{1}),\\
&\textit{anti-chiral spinor}: && \psi_{\dot\alpha} && (\mathbf{1},\mathbf{2}),\\
&\textit{vector}: && \psi_{\alpha\dot\beta} \sim \psi_\mu && (\mathbf{2},\mathbf{2}),\\
&\textit{SD two-form}: && \psi_{\alpha\beta} = \psi_{\beta\alpha} \sim \psi^+_{\mu\nu} && (\mathbf{3},\mathbf{1}),\\
&\textit{ASD two-form}: && \psi_{\dot\alpha\dot\beta} = \psi_{\dot\beta\dot\alpha} \sim \psi^-_{\mu\nu} && (\mathbf{1},\mathbf{3}).
\end{aligned}$$

As we already mentioned in §9.2, the spinor bundles, with $\mathbf{n}_+ + \mathbf{n}_-$ odd, only exist if the 4-manifold is spin $w_2 = 0$.

9.3. Duality in Maxwell theory

Our first example of a space-time theory which has a non-trivial duality symmetry is Maxwell theory — $U(1)$ gauge theory on a four-manifold M. We pick a line bundle L on M and a connection A on L with curvature F. The curvature satisfies the Bianchi identity $dF = 0$ and has integer periods around two-cycles $\Sigma \subset M$

$$\frac{1}{2\pi}\int_\Sigma F \in \mathbf{Z}, \tag{9.29}$$

This gives of course the first Chern class

$$c_1(L) = \left[\frac{F}{2\pi}\right] \in H^2(M, \mathbf{Z}) \tag{9.30}$$

that classifies the line bundle L topologically.

To consider the Maxwell equations we further need a metric on M, or more precisely, a conformal structure or Hodge star $*$ (so the metric is only defined up to local rescalings.) The equations of motion now read $d^*F =$ or

$$d(*F) = 0. \tag{9.31}$$

Electric-magnetic duality or "abelian S-duality" is the observation that the equation $d^*F = 0$ together with the Bianchi identity

$$dF = 0 \tag{9.32}$$

are invariant under the transformation $F \leftrightarrow *F$ that interchanges the E and B field.

This duality is also present in the quantum theory, in fact it extends to the group $SL(2,\mathbf{Z})$. This can be made more precise by considering the path-integral. The Maxwell action can be concisely written if we introduce for any complex variable

$$\tau = \tau_1 + i\tau_2 \in \mathbf{H} \tag{9.33}$$

the operator $\widehat{\tau} : \Omega^2(M) \to \Omega^2(M)$ that acts on two-forms as

$$\widehat{\tau} = \begin{cases} \tau_1 + i\tau_2 *, & \text{Euclidean signature } (4,0),, \\ \tau_1 + \tau_2 *, & \text{Lorentzian signature } (3,1).. \end{cases} \tag{9.34}$$

(Note that in Lorentzian signature $*^2 = -1$.) In terms of the usual coupling constant g and the theta angle θ we have

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{9.35}$$

The Maxwell action, including the theta-angle, now reads

$$S = \frac{1}{4\pi} \int_M F \wedge \widehat{\tau} F. \tag{9.36}$$

The partition function is defined as the integral over the space of connections $\mathcal{A}$ (which includes a sum over all possible line bundles L) modulo the action of the gauge group $\mathcal{G}$

$$Z = \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A\, e^{iS} \tag{9.37}$$

We note immediately that under the theta-angle shift $\theta \to \theta + 2\pi$, or equivalently under the transformation

$$T:\ \tau \to \tau + 1, \tag{9.38}$$

the partition transforms by the phase factor

$$Z \to Z \cdot e^{2\pi i k}, \tag{9.39}$$

with

$$k = \frac{1}{8\pi^2}\int_M F \wedge F = \int_M \tfrac{1}{2}c_1^2 \in \tfrac{1}{2}\mathbf{Z}. \tag{9.40}$$

In case M is spin, so that the intersection form is even, k is an integer, and the T transformation is a quantum symmetry.

It is useful at this point to introduce the dual field strength, the two-form

$$F_D = 2\pi i \frac{\delta S}{\delta F} = \widehat{\tau} F. \tag{9.41}$$

In Maxwell theory (with a theta angle) we can then define electric and magnetic charges $e, m \in \mathbf{Z}$ as

$$\begin{aligned} e &= \frac{1}{2\pi}\int_{S^2} F_D, \\ m &= \frac{1}{2\pi}\int_{S^2} F. \end{aligned} \tag{9.42}$$

Here we choose a two-sphere S^2 surrounding a certain region in a three-dimensional spatial slice. Under the T-duality we have the Witten effect [94]: the field strength and dual field strength transform as

$$T: \begin{pmatrix} F_D \\ F \end{pmatrix} \to \begin{pmatrix} F_D + F \\ F \end{pmatrix}, \tag{9.43}$$

so that the charges transform as

$$T: \begin{pmatrix} e \\ m \end{pmatrix} \to \begin{pmatrix} e + m \\ m \end{pmatrix}. \tag{9.44}$$

In this way a monopole receives an electric charge equal to its magnetic charge and becomes a dyon (an object with both electric and magnetic charge).

We now claim that the full duality group of Maxwell theory is

$$G = SL(2, \mathbf{Z}), \tag{9.45}$$

and that it acts on the coupling constant by fractional linear transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{9.46}$$

In fact the vector (F_D, F) transforms as a doublet

$$\begin{pmatrix} F_D \\ F \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_D \\ F \end{pmatrix}. \tag{9.47}$$

To prove this duality we have to consider the second generator of $SL(2, \mathbf{Z})$, the well-known electric-magnetic S-duality

$$S: \ \tau \rightarrow -\frac{1}{\tau}, \tag{9.48}$$

that interchanges F and F_D:

$$S: \ \begin{pmatrix} F_D \\ F \end{pmatrix} \rightarrow \begin{pmatrix} -F \\ F_D \end{pmatrix}, \tag{9.49}$$

and consequently also the electric and magnetic charges

$$T: \ \begin{pmatrix} e \\ m \end{pmatrix} \rightarrow \begin{pmatrix} -m \\ e \end{pmatrix}. \tag{9.50}$$

Note that this transformation squares to minus the identity (the parity transform), $S^2 = -1$. Since the coupling constant g^2 transforms into $1/g^2$, S relates strong and weak coupling (even though this is a free theory, with no perturbation expansion!).

S-duality is derived in complete analogy with the proof of T-duality in the toroidal sigma model in §7.2. One starts with the path-integral

$$Z = \int \mathcal{D}F\mathcal{D}A_D \exp\left(iS(F) + \frac{i}{8\pi}\int_M F \wedge dA_D\right) \tag{9.51}$$

with F a two-form on M and A_D an abelian dual gauge field. The manipulation is familiar from §7.2. Integrating out A_D forces the Bianchi identity $dF = 0$, which allows us to write F locally as dA. Integrating out F produces the dual model, with $F_D = 2\pi i \delta S/\delta F = dA_D$, with dual coupling constant $-1/\tau$.

9.4. The partition function

We now consider the computation of the Euclidean partition function, following [95, 96]. The most interesting part is the contribution of the zero-modes. The zero-modes are here the first Chern classes

$$p = [F/2\pi] \in H^2(M, \mathbf{Z}). \tag{9.52}$$

For classical field configurations p will be an harmonic representative, $dp = d^*p = 0$. We can write a general two-form F that satisfies the Bianchi identity as

$$F = 2\pi p + d\alpha, \tag{9.53}$$

with α a proper one-form. Note that $\Gamma = H^2(M, \mathbf{Z})$ carries the intersection form

$$p^2 = \int_M p \wedge p, \tag{9.54}$$

which makes it into a self-dual lattice (even if M is spin). A choice of metric on M gives a decomposition of two-forms in their self-dual and anti-self-dual parts. We will write for the zero-mode piece

$$p = p_L + p_R, \tag{9.55}$$

with $*p_L = p_L$ and $*p_R = -p_R$, so that

$$p^2 = p_L^2 - p_R^2. \tag{9.56}$$

In this way the zero-mode contribution to the action reads

$$S = \pi(\tau p_L^2 - \overline{\tau} p_R^2) \tag{9.57}$$

The answer for the partition function now becomes a theta-function [95, 96]

$$Z = Z_{qu} \cdot \sum_{(p_L, p_R) \in \Gamma} q^{\frac{1}{2} p_L^2} \overline{q}^{\frac{1}{2} p_R^2}, \qquad q = e^{2\pi i \tau} \tag{9.58}$$

This generalized theta-function is identical to the zero-mode contribution of the toroidal sigma-model with Narain lattice Γ. The factor Z_{qu} represents the contribution from the oscillations and is a product of determinants. By the usual arguments of string theory, the theta function is a modular form (though not holomorphic) of weight $\frac{1}{2}(b_+, b_-)$ if the lattice Γ is even and self-dual, which is the case for spin manifolds.

One way to understand this appearance of theta functions has been suggested by Erik Verlinde [95]. Consider a model in six dimensions on the manifold $X^6 = M^4 \times T^2$, where the two-torus has modulus τ. The dynamical field is a two-form B with three-form field strength $H = dB$ and action

$$S = \int_X H \wedge *H. \tag{9.59}$$

Now demand that H is self-dual and has a decomposition as $H = \sum_i F_i \wedge C_i$ with F_i a two-form on M^4 and C_i a one-form on T^2. The self-duality on X^6 now tells us that the Hodge star on M^4 is correlated with the Hodge star on T^2. H has then a decomposition (with $dz = dx + \tau dy$)

$$H = F_+ dz + F_- d\overline{z} = F dx + F_D dy \tag{9.60}$$

(Note that $F = F_+ + F_-$ and $*F = F_+ - F_-$, so that $F_D = \tau_1 F + i\tau_2 * F = \tau F_+ + \overline{\tau} F_-$.) If (A, B) is a basis for $H_1(T^2)$ we can write

$$F = \oint_A H, \qquad F_D = \oint_B H \tag{9.61}$$

The partition function can now be computed either by first integrating over T^2, which reproduces the gauge theory computation, or by first integrating over M^4, which gives a sigma model with target space the torus $H^2(M)/\Gamma$. This Kaluza-Klein point of view will be substantially upgraded if we go to string theory.

9.5. Higher rank groups

We can make a simple generalization of Maxwell theory by considering an abelian gauge group of rank g. In that case we have one-form gauge fields A^i, $i = 1, \ldots, g$, with corresponding curvatures $F^i = dA^i$. We can now write an action with a matrix τ_{ij} of coupling constants and theta-angles

$$S = \frac{1}{4\pi} \int_M F^i \wedge \widehat{\tau}_{ij} F^j \tag{9.62}$$

Note that we can take τ_{ij} to be symmetric; the above action makes only sense if also $\operatorname{Im} \tau_{ij} > 0$. So we are dealing with a period matrix of an abelian variety of dimension g (see §11.6) for example the Jacobian of a genus g Riemann surface. The definitions of the dual field strengths are now

$$F_{D,i} = \widehat{\tau}_{ij} F^j, \tag{9.63}$$

and we have a $2g$ dimensional vector of electric and magnetic charges $e^i, m_i \in \mathbf{Z}$, $i = 1, \ldots, g$,

$$\begin{aligned} e^i &= \frac{1}{2\pi} \int F_{D,i}, \\ m_i &= \frac{1}{2\pi} \int F^i. \end{aligned} \tag{9.64}$$

The duality group is the rank $2g$ symplectic group

$$G = Sp(2g, \mathbf{Z}), \tag{9.65}$$

that acts on the coupling constant matrix as $\tau \to (A\tau + B)(C\tau + D)^{-1}$.

9.6. Dehn twists and monodromy

This model can be obtained by considering the six-dimensional model of §9.4, now on the manifold $M^4 \times \Sigma_g$. Note that from this point of view the group $Sp(2g, \mathbf{Z})$ is identified by (a quotient) of the mapping class group of the surface Σ_g. As such it is generated by Dehn twists. A Dehn twist D_C on a homology cycle C is defined by cutting the surface on C and gluing it back together after a 2π rotation. The action on the homology cycles is given by

$$D_C : C' \to C' + \eta(C, C')C \tag{9.66}$$

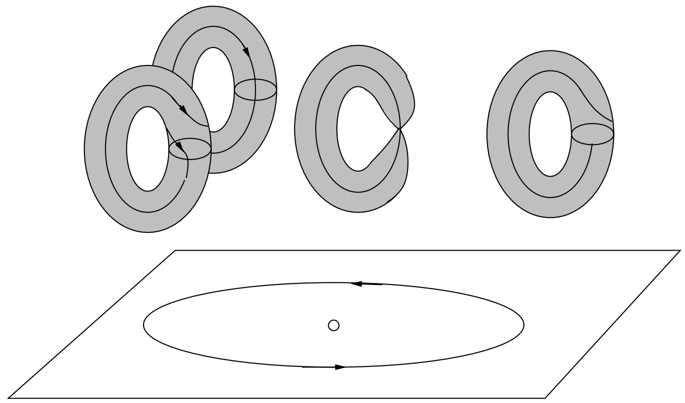


Fig. 7: *The action of a Dehn twist is implemented by a monodromy in the moduli space.*

where the symplectic form $\eta(\cdot,\cdot)$ denotes the intersection product on the first homology group $H_1(\Sigma_g, \mathbf{Z})$.

This Dehn twist can also be seen as a monodromy in the moduli space $\mathcal{M}_g$. If we pinch the cycle C we obtain a double point. The pinching process can be described by inserting first a tube connecting the two marked points with modulus q, as in our discussion of CFT. We write this modulus as $q = e^{-t+i\theta}$, so that the cycle C shrinks to zero in the limit $t \to \infty$. If we now fix t and all other moduli of the surface and follow the loop $\theta \to \theta + 2\pi$ in $\mathcal{M}_g$, the Dehn twist D_C is implemented, see *fig.* 7.

This is a special case of what is called in generality Picard-Lefschetz theory[7] [71]: modular/duality transformations are related to vanishing cycles in the "compactification" Σ_g. This fact we will meet again when we start studying Calabi-Yau compactifications in string theory.

10. Moduli spaces

We have stressed the importance of the concept of a moduli space in the study of quantum theories of fields and strings. There are basically three ways in which moduli spaces enter in field theory or string theory: (1) as special classical solutions, (2) as families of QFTs and (3) as families of inequivalent vacua. A priori the three ways are not related, however there are various deep relations between these points of view that

[7] Note that Picard-Lefschetz theory is the complex analogon of Morse theory [97]. In Morse theory we consider a real function $f : M \to \mathbf{R}$ and look at how the inverse image $M_x = f^{-1}(x)$ varies as a function of x. The essential behaviour is at the critical points $df = 0$ where the topology of M_x jumps, in a controlled way determined by the signature of the Hessian of f. In the complex case we have a complex function $f : M \to \mathbf{C}$ and one can go *around* a critical point, and study the corresponding transformation on the homology of $f^{-1}(x)$.

occur in string theory that we will use to our advantage.

10.1. Supersymmetric or BPS configurations

The first occurrence of moduli spaces is as spaces of special classical configurations of the (usually bosonic) fields ϕ, possibly modulo gauge transformations. A good example is given by the four-dimensional anti-self-duality (ASD) condition

$$F^+ = F + *F = 0, \tag{10.1}$$

that gives rise to the moduli space of instantons, defined as the set of solutions to this equation modulo gauge equivalence [92].

These configurations typically arise as supersymmetric or BPS configurations. If we denote the bosonic variables and fermionic variables generically as ϕ and ψ, then the supersymmetry variation $\delta\psi$ of the fermions reads (at $\psi = 0$)

$$\delta\psi = G(\phi)\epsilon, \tag{10.2}$$

BPS configurations are characterized by the property that for a particular subset of supersymmetry parameters ϵ the RHS vanishes.

For example, in $N = 2$ supersymmetric gauge theories that we will discuss in more detail in the next lectures, we have a relation

$$\delta\psi = F_{\mu\nu}\gamma^{\mu\nu}\epsilon, \tag{10.3}$$

so that for a chiral spinor ϵ that satisfies $\gamma_5\epsilon = \epsilon$, we find that ASD configurations with $F^+ = 0$ are invariant under half the number of supersymmetry variations.

These moduli spaces are often noncompact. In the case of ASD gauge fields this is a familiar consequence of the fact that the instanton equations allow solutions on flat space $\mathbf{R}^4$. Since the ASD equations depend on the choice of Hodge $*$, they are invariant under conformal transformations of the metric. So we can find a one-parameter family of rescaled solutions $x \to t \cdot x$. Since the instanton in the limit $t \to 0$ becomes a δ-function in the origin and therefore is no longer a smooth solution to the ASD equation, the moduli space on $\mathbf{R}^4$ is clearly non-compact. This argument can be repeated for general four-manifolds. In the limit $t \to 0$ we have almost point-like instantons on $\mathbf{R}^4$. We can now cut out a little disk containing most of the solution and "graft" this onto a point of X. Taubes has proven that this can made into a solution for the ASD equation on X [98]. We see that therefore also on a general manifold point-like instantons occur in the limit $t \to 0$.

The moduli space also can have singularities. For the ASD equations these singularities occur whenever the gauge group $\mathcal{G}$ does not act freely. This is the case if the holonomy group of the connection is $U(1)$ instead of $SU(2)$. We can then restrict the gauge group consistently to $U(1)$ and we are dealing with an abelian connection. So the only singularities are the abelian instantons. In that case the curvature is just a $\mathbf{R}$-valued 2-form $F \in H^2(X, \mathbf{R})$. It must however satisfy a quantization rule. The first Chern class of the $U(1)$ bundle should be an integer cohomology class. If we now also impose the ASD condition, we see that abelian instantons $*F = -F$ correspond one-to-one with elements of the lattice

$$H^2_- \cap H^2(X, \mathbf{Z}). \tag{10.4}$$

These singularities form in general not a serious problem. We should remember that we are free to pick a metric. If we choose a generic metric and thus a generic positioning of the subspace $H^2_- \subset H^2$, the intersection with the integer lattice $H^2(X, \mathbf{Z})$ will typically be zero, unless H^2_- has codimension zero (if $b^2_+ = 0$). (Also the case $b^2_+ = 1$ has to be treated carefully.) This is a rather general effect, quite often slight perturbations of our moduli problem can make the singularities disappear.

10.2. Localization in topological field theories

One of the applications of the moduli spaces of the type discussed above in QFT is that, under suitable circumstances, the path-integral can localize to these configurations,

$$Z = \int \mathcal{D}\phi \, e^{-S} \Rightarrow \int_{\mathcal{M}} \cdots \tag{10.5}$$

Localization is a crucial ingredient in topological field theories [80, 99, 100], see the reviews [21, 101, 102], that allows us to express partition and correlation functions as integrals over finite dimensional moduli spaces instead of over infinite field space. Of course, it is the unique structure of topological models that allows such a drastic reduction in degrees of freedom.

The mathematical idea of localization has rich applications. The most familiar one is the calculation of the Euler characteristic of a manifold X [103]. On the one hand it can be computed by picking a Riemannian metric and integrating over X the Euler density $\mathrm{Pf}\,(R)$ that is constructed out of the Riemann curvature tensor. On the other hand the Euler character can be computed by counting the number of zeroes of a generic vector field on the manifold.

In quantum mechanics the localization principle is well-known as the phenomenon of the Nicolai map [104]. This can be nicely illustrated by a zero-dimensional example. Consider a polynomial $s(x)$ and the "path-integral"

$$Z = \int_{-\infty}^{\infty} dx \, e^{-s^2/2}. \tag{10.6}$$

As it stands this integral has no interesting invariances and cannot be computed in any simplified way. If instead we consider the modified integral

$$Z = \int dx\, e^{-s^2/2}\, \frac{\partial s}{\partial x}, \tag{10.7}$$

we can compute it without much effort, since it can be rewritten as

$$Z = \int ds\, e^{-s^2/2} = \begin{cases} 0, & \text{if } s \text{ even,} \\ 1, & \text{if } s \text{ odd.} \end{cases} \tag{10.8}$$

So we see Z computes a simple invariant of s: the degree of the map $s : \mathbf{R} \to \mathbf{R}$, which is either zero or one depending on whether s is an even or odd polynomial. The partition function Z is thus invariant under all deformations of s that leave the boundary conditions at infinity invariant.

Now that we have established the "topological invariance" of Z we can make use of this by deforming the action s in such a way that the localization becomes evident. To do this, we rescale $s \to t \cdot s$ and take the limit $t \to \infty$. In this limit the gaussian factor will damp the integral for all values of x except for the zeroes of the function $s(x)$. Around these points we can perform a semi-classical approximation. In this way the computation localizes to a finite number of points. We compute Z as sum over the zeroes of s of a factor ± 1, very much in analogy with the Euler character,

$$Z = \sum_{s(x)=0} \operatorname{sgn} \det(ds). \tag{10.9}$$

One way to express this localization is that the semi-classical approximation gives an exact result.

A third way to compute the integral Z will be a metaphor for the actual computations in field theory. We start by rewriting the factor $\partial s/\partial x$ as a fermionic gaussian integral. We introduce two fermionic variables ψ, ρ and rewrite Z as

$$Z = \int dx d\psi d\rho\, e^{-S}, \tag{10.10}$$

with action

$$S = \tfrac{1}{2} s^2 + \rho \partial s \psi. \tag{10.11}$$

This action S has a BRST symmetry Q given by

$$Qx = \psi, \quad Q\psi = 0, \quad Q\rho = s(x), \tag{10.12}$$

As it stands this symmetry is not nilpotent. This property we obtain if we introduce a further bosonic auxiliary field H and write

$$Z = \int dx dH d\psi d\rho \, e^{-S}, \tag{10.13}$$

with

$$S = isH + \tfrac{1}{2}H^2 + \rho \partial s \psi. \tag{10.14}$$

The symmetry is now extended as

$$Q\rho = H, \qquad QH = 0. \tag{10.15}$$

With this extra auxiliary field H the BRST symmetry squares to zero "off- shell"

$$Q^2 = 0. \tag{10.16}$$

Now there is a simple localization argument for this BRST charge due to Witten [105]. Our integral Z is expressed as an integral over a $2|2$ dimensional superspace. On this space we have the action of the fermionic symmetry Q generated by a vector field ξ, that is, $Q = \mathcal{L}_\xi$. The odd vector field ξ squares to zero, $\xi^2 = \frac{1}{2}[\xi, \xi] = 0$, a non-trivial property for an odd vector field. The orbits of this group action are $0|1$ dimensional curves parametrized by an odd coordinate θ. Because of the fundamental identities of grassmannian calculus

$$\begin{aligned} \int d\theta \, 1 &= 0, \\ \int d\theta \, \theta &= 1, \end{aligned} \tag{10.17}$$

the integral of a constant function, such as the BRST-invariant action density e^{-S}, along the orbit will automatically give zero. Therefore the only non-vanishing contributions to the integral can come from the zero-dimensional orbits, that is, the fixed points of Q or equivalently the zeroes of the vector field ξ.

10.3. Quantization

A second application is that moduli spaces can be quantized. Here we consider a supersymmetric point particle moving on the moduli space $\mathcal{M}$. In this way there is a canonical way to associate a (graded) vector space V to the moduli problem as

$$V = H^*(\mathcal{M}). \tag{10.18}$$

The (super)dimension of this space can be computed using the Euler character, that often also allows a representation as a TFT partition function [106, 107, 108]

$$\mathrm{sdim}\, V = \chi(\mathcal{M}). \tag{10.19}$$

We will see an application of this in §13 when we discuss the quantization of D-branes.

10.4. Families of QFTs

The second way moduli spaces appear in our story is as families of quantum field theories. For example, in quantum mechanics we can consider a potential

$$V(x) = \sum t_n x^n \tag{10.20}$$

The coefficients t_n now label a family of systems. We have seen in detail how the moduli space of two-dimensional TFTs and CFTs is a rich object. In field theory such deformation will be of the general form

$$\delta S = \int \mathcal{O} \tag{10.21}$$

with $\mathcal{O}$ some set of local operators. The Hilbert space of the quantum field theory thus contains the tangent space to the moduli space $\mathcal{M}$.

10.5. Moduli spaces of vacua

The third way moduli spaces occur is as spaces of inequivalent vacuum states of a single QFT. Here we should be careful to distinguish the classical moduli space $\mathcal{M}_{cl}$ and the quantum moduli space $\mathcal{M}_{qu}$. For example, in the above case of a particle in a potential $V(x)$, the classical vacua correspond to the absolute minima of V, so that $\mathcal{M}_{cl}$ might be very complicated; quantum mechanically we know that tunneling will produce a unique vacuum state, so $\mathcal{M}_{qu} = pt$.

The phenomena of inequivalent vacua in a QFT is strongly related the appearance of scalar fields. Scalar fields have the unique property that they can have expectation values compatible with Poincaré invariance. In fact, when we consider a QFT with scalar fields ϕ on a noncompact spatial manifold, say $\mathbf{R}^n$, then we have to impose boundary conditions for the scalars

$$\phi(x) \stackrel{|x|\rightarrow\infty}{\rightarrow} \phi_0. \tag{10.22}$$

The values ϕ_0 will minimalize the energy and thus take value in some moduli space

$$\phi_0 \in \mathcal{M}_{cl}. \tag{10.23}$$

(Stated otherwise, in the path-integral we have to separate out the constant modes ϕ_0, since they are not L^2.)

If we consider the same QFT on a *compact* manifold, the constant mode ϕ_0 becomes L^2 and the partition function will be obtained as an integral over $\mathcal{M}$

$$Z = \int_{\mathcal{M}} d\phi_0 \cdots \tag{10.24}$$

11. Supersymmetric gauge theories

A good class of examples of field theories with scalar fields and non-trivial families of vacua are four-dimensional supersymmetric gauge theories, to which we will now turn.

11.1. Supersymmetric gauge theories

A general $D = 4$ supersymmetric gauge theory with $N \leq 4$ supersymmetries consists of the following field content: a gauge field A_μ together with N Majorana fermions $\lambda^I_\alpha, \overline{\lambda}_{\dot{\alpha},I}$, $I = 1, \ldots, N$, and $\frac{1}{2}N(N-1)$ real scalar fields, conveniently described by an $N \times N$ anti-symmetric matrix $\phi^{IJ} = -\phi^{JI}$. All these fields take action in the adjoint representation of the gauge group. The simplest supersymmetric action is of the form

$$S = \int \mathrm{Tr} \left(F \wedge *F + i\overline{\lambda}_I \not{D}\lambda^I + D\phi_{IJ} \wedge *D\phi^{IJ} + \sum_{I,K} [\phi^{IJ}, \phi_{JK}]^2 + \lambda_I [\phi^{IJ}, \lambda_J] \right) \tag{11.1}$$

The supersymmetry transformations satisfy

$$[Q^I_\alpha, \overline{Q}^J_{\dot{\alpha}}] = \delta^{IJ} \gamma^\mu_{\alpha\dot{\alpha}} P_\mu. \tag{11.2}$$

If we introduce a general linear combination

$$\delta = \eta_{\alpha,I} Q^I_\alpha + \overline{\eta}_{\dot{\alpha},I} \overline{Q}^I_{\dot{\alpha}}, \tag{11.3}$$

the transformations of the vector A_μ read in terms of the bispinor $A_{\alpha\dot{\alpha}} = A_\mu \gamma^\mu_{\alpha\dot{\alpha}}$

$$\delta A_{\alpha\dot{\alpha}} = i\lambda^I_\alpha \overline{\eta}_{\dot{\alpha},I} - i\eta_{\alpha,I} \overline{\lambda}^I_{\dot{\alpha}}. \tag{11.4}$$

The fermions transform in the two chirality components of the field strength

$$\begin{aligned} \delta\lambda^I_\alpha &= F_{\alpha\beta}\eta^{\beta,I} + i[\phi^{IJ}, \phi_{JK}]\eta^K_\alpha + D_{\alpha\dot{\alpha}}\phi^{IJ}\overline{\eta}^{\dot{\alpha}}_J, \\ \delta\overline{\lambda}^I_{\dot{\alpha}} &= F_{\dot{\alpha}\dot{\beta}}\overline{\eta}^{\dot{\beta},I} - i[\phi^{IJ}, \phi_{JK}]\overline{\eta}^K_\alpha + D_{\alpha\dot{\alpha}}\phi^{IJ}\eta^\alpha_J. \end{aligned} \tag{11.5}$$

Finally, the scalar fields transform as

$$\delta\phi^{IJ} = \lambda^I_\alpha \eta^{\alpha,J} + \overline{\lambda}^I_{\dot\alpha}\overline{\eta}^{\dot\alpha,J}. \tag{11.6}$$

This model has an internal $U(N)$ R-symmetry, under which the charge Q transforms as the fundamental representation $\mathbf{N}$ and $\overline{Q}$ transforms as the conjugate transformation $\overline{\mathbf{N}}$.

11.2. Twisting and Donaldson theory

We explained in section §8.2 how the twisting procedure can produce a TFT starting with a supersymmetric field theory. In four dimensions one needs a model with at least $N = 2$ supersymmetry [80]. This has a symmetry group

$$Spin(4) \times U(2)_R, \tag{11.7}$$

where $Spin(4)$ is the double cover of the Lorentz group $SO(4)$ and the internal R-symmetry $U(2)_R$ acts on the two supersymmetry charges Q^I which transform as a doublet. This group is (locally) isomorphic to

$$H = SU(2)_+ \times SU(2)_- \times SU(2)_R \times U(1), \tag{11.8}$$

Irreducible representations of this group are labeled by $(\mathbf{n}_+, \mathbf{n}_-, \mathbf{n}_R, q)$, where $\mathbf{n}$ indicates the dimension of the $SU(2)$ representation and q indicates the $U(1)$ representation $z \to e^{iq\theta}z$. The twisting procedure replaces the component $SU(2)_+$ by the diagonal subgroup of $SU(2)_+ \times SU(2)_R$. That is, the Lorentz spins are replaced as

$$(\mathbf{n}_+, \mathbf{n}_-) \Rightarrow (\mathbf{n}_+ \otimes \mathbf{n}_R, \mathbf{n}_-). \tag{11.9}$$

We see that the supercharges $Q^I_\alpha, \overline{Q}^J_{\dot\alpha}$ transform under H as $(\mathbf{2},\mathbf{1},\mathbf{2},1)$ and $(\mathbf{1},\mathbf{2},\mathbf{2},1)$. Consequently, under the twisting procedure they decompose as tensors of type $(\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1})$ and $(\mathbf{2},\mathbf{2})$, that is, as a scalar and a self-dual two-form, and as a vector

$$\begin{aligned} Q^I_\alpha &\Rightarrow Q, Q^+_{\mu\nu}, \\ \overline{Q}^J_{\dot\alpha} &\Rightarrow G_\mu. \end{aligned} \tag{11.10}$$

So we see that we obtain the required twisted supersymmetry algebra (8.16)

$$\{Q, G_\mu\} = P_\mu. \tag{11.11}$$

Therefore, we are dealing with a topological field theory.

Note that on a general curved four-manifold the holonomy group will be $SO(4) \cong SU(2)_+ \times SU(2)_-$, so the twisting procedure that changes the Lorentz properties will be a measurable effect. The twisted model is therefore physically speaking inequivalent to the untwisted model. This is similarly true on a complex manifold with holonomy group $U(2) = U(1)_+ \times SU(2)_-$. However, on a hyperkähler or Calabi-Yau manifold, where the structure group of the tangent bundle is simply $SU(2)_-$, the twisting is invisible. Twisting changes the $SU(2)_+$ representation, but that part of the frame bundle can be trivialized[8].

For an $N = 2$ gauge theory the resulting topological field theory was first introduced by Witten [80, 109]. It fits perfectly in the general framework we sketched. Starting point is $N = 2$ supersymmetric Yang-Mills theory. Its fundamental multiplet (A, λ^I, ϕ) consists of a connection A_μ, two spinors $(\lambda^I_\alpha, \lambda^I_{\dot{\alpha}})$ $(I = 1, 2)$ and a complex scalar field ϕ, all taking their values in the adjoint bundle. After twisting we recover the following fields

$$\begin{aligned} A_\mu &\Rightarrow A_\mu \\ \lambda^I_\alpha &\Rightarrow \psi_\mu \\ \lambda^I_{\dot{\alpha}} &\Rightarrow \rho^+_{\mu\nu}, \eta \\ \phi &\Rightarrow \phi \\ \overline{\phi} &\Rightarrow \overline{\phi} \end{aligned} \tag{11.12}$$

We recognize the fundamental topological multiplet (A, ψ, ρ) together with an equivariant multiplet $(\eta, \phi, \overline{\phi})$, that is due to the fact that we quotient by the gauge group. The BRST transformations can be derived from the $N = 2$ supersymmetry algebra that we gave before and read

$$\begin{aligned} \delta A_\mu &= \psi_\mu \\ \delta \psi_\mu &= D_\mu \phi \\ \delta \rho^+_{\mu\nu} &= F^+_{\mu\nu} \\ \delta \overline{\phi} &= \eta \end{aligned} \tag{11.13}$$

The "section" s in the general localization setup of §10.2 is thus identified as

$$s(A) = F^+ \tag{11.14}$$

and its zero locus is indeed the moduli space $\mathcal{M}$ of ASD connections.

11.3. Observables

In any TFT of cohomological type, the observables are the cohomology classes of the BRST operator. The most important classes of the twisted $N = 2$ SYM model are

[8]Stated otherwise, one does have (two) covariant constant spinors on a hyperkähler four-manifold. Unfortunately, in the compact case, the only hyperkähler spaces are T^4 and $K3$.

constructed out of the local operator

$$\mathcal{O}^{(0)} = \frac{1}{8\pi^2} \mathrm{Tr}\, \phi^2. \tag{11.15}$$

We construct its descendents by

$$d\mathcal{O}^{(i)} = \delta\mathcal{O}^{(i+1)}. \tag{11.16}$$

As we reviewed in section §8.2 any cycle gives rise to a physical operator via the map

$$C \in H_*(M) \;\Rightarrow\; \mathcal{O}_C = \int_C \mathcal{O}^{(i)}. \tag{11.17}$$

For twisted $N = 2$ Yang-Mills this gives the following operators: (We will assume that M is simply connected, so that only $H_0(M)$, $H_2(M)$ and $H_4(M)$ contribute.) Associated to two-cycles $\Sigma \subset M$ we find

$$\mathcal{O}_\Sigma = \int_\Sigma \mathcal{O}^{(2)} = \frac{1}{4\pi^2} \int_\Sigma \mathrm{Tr}\,(\phi F + \psi^2). \tag{11.18}$$

For the top form we recover the chern class or instanton number

$$\mathcal{O}_M = \int_M \mathcal{O}^{(4)} = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\, F \wedge F = ch_2. \tag{11.19}$$

Note that by Poincaré duality, the coupling constants to the operator $\int_C \mathcal{O}^{(i)}$ naturally takes value in $H^{4-i}(M)$. If we write

$$\mathcal{O} = \sum_i \mathcal{O}^{(i)} \in \Omega^*(M), \tag{11.20}$$

and $t \in H^*(M)$, then a general coupling to the action reads

$$\delta S = \int_M t \wedge \mathcal{O}. \tag{11.21}$$

(The integral naturally picks out the forms of total degree four.)

Since the path-integral localizes to the moduli space $\mathcal{M}$, the (expectation values of the) operators have a direct interpretation as cohomology classes on $\mathcal{M}$. In fact, we get Donaldson's map [110]

$$\mu : H_i(M) \cong H^{4-i}(M) \;\to\; H^{4-i}(\mathcal{M}). \tag{11.22}$$

The mathematical definition of this operation is roughly as follows. There is a natural bundle on the product space $M \times \mathcal{M}$. This bundle has a second Chern class $\widehat{c}_2$. We can now consider the differential form $\alpha \wedge \widehat{c}_2$, which is of degree $4+k$, and integrate it over the fiber X

$$\mu(\alpha) = \int_X a \wedge \widehat{c}_2. \tag{11.23}$$

We see that in the particular example of the identity $\mathbf{1}$ we have

$$\mu(\mathbf{1}) = \int_X c_2 = n \in \mathbf{Z}. \tag{11.24}$$

The famous Donaldson polynomials are now defined in terms of the generating functional

$$D(v,\lambda) = \left\langle \exp\left(\int_M t\mathcal{O}^{(4)} + v \wedge \mathcal{O}^{(2)} + \lambda \mathcal{O}^{(0)}\right)\right\rangle, \tag{11.25}$$

with coupling constants

$$t \in H^0(M),\ v \in H^2(M),\ \lambda \in H^4(M). \tag{11.26}$$

The generating function has an expansion

$$D(v,\lambda) = \sum_{n \geq 0} e^{-nt} D_n(v,\lambda), \tag{11.27}$$

where $D_n(v,\lambda)$ is a finite sum of intersection products on $\mathcal{M}_n$, the component of $\mathcal{M}$ of instanton charge n. However, it is a nontrivial matter to make rigorous sense of these intersections, since we have seen that the moduli space is non-compact. In fact, a tremendous amount of work goes in proving that the heuristic definitions below actually make sense in some precise way. It is a polynomial since only contributions of degree $d = \dim \mathcal{M}_n$ contribute (with $\deg(v)=2$, $\deg(\lambda)=4$.) The main theorem of Donaldson states that the polynomials D_n are diffeomorphism invariants if $b_2^+ > 1$.

11.4. Abelian models

After this detour to TFT, let us now consider $N=2$ (untwisted) Maxwell theory, with gauge group $G = U(1)^g$. We will write the gauge fields again as $A^i = A^i_\mu dx^\mu$, $i = 1,\ldots,g$. In that case it is convenient to combine the two real scalars into one (rank g vector-valued) complex scalar $\phi(x)$. The most general action, compatible with supersymmetry and at most second order in derivatives, is given by

$$S = \frac{1}{4\pi}\int \left(F^i \wedge \widehat{\tau}_{ij}(\phi) F^j + g_{i\bar{\jmath}}(\phi)\lambda^i_I \partial\!\!\!/ \lambda^{\bar{\jmath},I} + g_{i\bar{\jmath}}(\phi) d\phi^i \wedge * d\overline{\phi}^{\bar{\jmath}}\right), \tag{11.28}$$

where both the matrix of coupling constants τ_{ij} and the sigma-model metric $g_{i\bar{j}}$ are allowed to depend on ϕ. Supersymmetry restricts this dependence in the following way: The "period matrix" τ_{ij} should be the second derivative of an holomorphic function $\mathcal{F}(\phi)$ of ϕ, the prepotential,

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial\phi^i \partial\phi^j}, \qquad \frac{\partial \mathcal{F}}{\partial \overline{\phi}^i} = 0. \tag{11.29}$$

Furthermore, the kinetic terms for the fermions and scalars are directly related to this period matrix as

$$g_{i\bar{j}} = \mathrm{Im}\, \tau_{ij}. \tag{11.30}$$

Summarizing, the Lagrangian is essentially given by one complex "function" (it will turn out to be a section of a line bundle), the prepotential $\mathcal{F}(\phi)$. This fact is more or less immediately clear if one takes the superspace point of view.

In the case of a simple quadratic action, as we have been considering up to now, this prepotential is given by

$$\mathcal{F} = \tfrac{1}{2}\tau_{ij}\phi^i\phi^j. \tag{11.31}$$

Note that under S-duality $\tau \to -\tau^{-1}$, the prepotential transforms as under a Legendre transformation. This will be the general behaviour.

The important difference with the nonsupersymmetric case is that the coupling constant τ is now a function of the scalar fields, that take some constant asymptotic value $\phi \in \mathcal{M}$. So, instead of a space parametrizing a family of QFTs, we are dealing with one gauge theory (with given prepotential) that has a family of vacua with varying coupling constant $\tau(\phi)$.

In the quadratic case the abelian duality group $Sp(2g, \mathbf{Z})$ can be trivially extended to the fermions and scalars. We simply have to define dual scalar field as

$$\phi_{Di} = \tau_{ij}\phi^i, \tag{11.32}$$

similarly as we defined the dual field strength.

The question is now if and how these duality transformations get modified for general, not necessarily quadratic prepotential. For convenience we mainly restrict to the case of one $U(1)$ factor, so put $g = 1$. We claim that we still obtain a duality group $SL(2, \mathbf{Z})$ that acts by linear fraction transformations on the coupling constant τ, but that the transformation of the scalar fields is more involved. In fact, we have to define the general dual scalar fields as [111, 112]

$$\phi_{D,i} = \frac{\partial \mathcal{F}}{\partial \phi^i}. \tag{11.33}$$

Note that this implies that

$$\tau_{ij} = \partial_i \partial_j \mathcal{F} = \frac{\partial \phi_{D,i}}{\partial \phi^j}. \tag{11.34}$$

We now claim that the vector

$$\begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \tag{11.35}$$

transforms as a doublet under $SL(2, \mathbf{Z})$. Indeed, to show that this is consistent, remark that the transformation rule

$$\begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \tag{11.36}$$

implies that

$$\tau = \frac{\partial \phi_D}{\partial \phi} \to \frac{a\tau + b}{c\tau + d}. \tag{11.37}$$

Let us now see in detail how the T and S transformations that generate $SL(2, \mathbf{Z})$ act. For T we find

$$T : \begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \to \begin{pmatrix} \phi_D + \phi \\ \phi \end{pmatrix} \tag{11.38}$$

from which we recover the behaviour of the prepotential

$$T : \ \mathcal{F} \to \mathcal{F} + \tfrac{1}{2}\phi^2. \tag{11.39}$$

For the S-duality we find

$$S : \begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \to \begin{pmatrix} -\phi \\ \phi_D \end{pmatrix} \tag{11.40}$$

so that $\mathcal{F}(\phi) \to \mathcal{F}_D(\phi_D)$ with $\mathcal{F}'_D = \phi$. That is, $\mathcal{F}_D$ is the Legendre transform of $\mathcal{F}$

$$S : \ \mathcal{F} \to Legendre(\mathcal{F}). \tag{11.41}$$

11.5. Rigid special geometry

The question is now: on what space $\mathcal{M}$ take the scalar fields their values and what is the right definition of the prepotential $\mathcal{F}(\phi)$?

In fact, this is a good point to review the general structure of the moduli space of vacua of a theory with global (rigid) $N = 2$ supersymmetry — the so-called rigid special geometry [112]. We will then show that it reduces in local coordinates to the structure we have discussed above.

Local special geometry, as we discussed in §§7.8,8.11,8.12, arises in theories with local (gauged) supersymmetry, such as string theory and its low energy limit supergravity.

Rigid special geometry is a property of theories with global (non-gauged) supersymmetry, such as the four-dimensional gauge theories that we discuss here. The two structures are very closely related, as we will see from the following definition:

Definition: a *rigid special geometry* $(\mathcal{M}, V, \Omega)$ is defined by the following ingredients:

(1) A complex, Kähler manifold $\mathcal{M}$ (the moduli space) of dimension

$$\dim_{\mathbf{C}} \mathcal{M} = g. \tag{11.42}$$

(2) A holomorphic, flat $Sp(2g, \mathbf{Z})$ vector bundle $V \to \mathcal{M}$, *i.e.* a representation $\pi_1(\mathcal{M}) \to Sp(2g, \mathbf{Z})$ that produces a lattice bundle Γ such that $V = \Gamma \otimes \mathbf{C}$ can be given a holomorphic structure. We denote the symplectic form on the fiber as η. We further write $\nabla, \overline{\nabla}$ for the $(1,0)$ and $(0,1)$ pieces of the connection with

$$\nabla^2 = \overline{\nabla}^2 = [\nabla, \overline{\nabla}] = 0. \tag{11.43}$$

Compatibility with the symplectic form gives

$$d\eta(\alpha, \beta) = \eta(\nabla\alpha, \beta) + \eta(\alpha, \nabla\beta). \tag{11.44}$$

(3) A holomorphic section Ω of V,

$$\overline{\nabla}\Omega = 0, \tag{11.45}$$

satisfying the Lagrangian condition

$$\eta(\nabla\Omega, \nabla\Omega) = 0, \tag{11.46}$$

(so $\nabla\Omega$ spans a Lagrangian subspace in V) and the positivity constraint

$$-i\eta(\nabla_i\Omega, \overline{\nabla}_{\bar{\jmath}}\Omega) > 0. \tag{11.47}$$

(4) Finally, the Kähler form on the moduli space is given by

$$K = -\frac{i}{2}\eta(\Omega, \overline{\Omega}). \tag{11.48}$$

Note that the holomorphic section Ω gives a natural complex polarization

$$V = V^{(1,0)} \oplus V^{(0,1)}, \tag{11.49}$$

where $V^{(1,0)}$ is the image of $\nabla\Omega$.

To make contact with the previous formulas, notice that if we choose a (local) canonical **Z**-basis α_i, β^i $(i = 1, \ldots, g)$ of V with $\eta(\alpha_i, \beta^j) = \delta_i{}^j$, we can write the preferred section Ω as

$$\Omega = \phi^i \alpha_i + \mathcal{F}_i \beta^i. \tag{11.50}$$

Since Ω is a holomorphic section, we can locally use the components ϕ^i as analytic coordinates on $\mathcal{M}$ (special coordinates), so that the conjugate components $\mathcal{F}_i$ become (holomorphic) functions of ϕ^i. We then have

$$\nabla_i \Omega = \alpha_i + \partial_i \mathcal{F}_j \beta^j, \tag{11.51}$$

The condition $\eta(\nabla\Omega, \nabla\Omega)$ now gives

$$\partial_i \mathcal{F}_j = \partial_j \mathcal{F}_i, \tag{11.52}$$

which is an integrability condition that tells us that there exists a local holomorphic function $\mathcal{F}(\phi)$ with

$$\mathcal{F}_i = \frac{\partial \mathcal{F}}{\partial \phi^i}. \tag{11.53}$$

We now define the period matrix as

$$\tau_{ij} = \partial_i \partial_j \mathcal{F}. \tag{11.54}$$

Note that in these special coordinates the Kähler metric is given by

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K = \operatorname{Im} \tau_{ij} > 0, \tag{11.55}$$

also by definition. From this form one can derive that the Riemann tensor satisfies the following identity, that is often taken as the starting point of rigid special geometry:

$$R_{i\bar{\imath}j\bar{\jmath}} = -c_{ijk} \bar{c}_{\bar{\imath}\bar{\jmath}\bar{k}} g^{k\bar{k}}. \tag{11.56}$$

Here the "three-point functions" are defined as

$$c_{ijk} = \partial_i \partial_j \partial_k \mathcal{F}. \tag{11.57}$$

We note that these can be written more covariantly as

$$c_{ijk} = \eta(\nabla_i \nabla_j \Omega, \nabla_k \Omega) = -\eta(\nabla_i \Omega, \nabla_j \nabla_k \Omega). \tag{11.58}$$

Also note that if we have the further relation $\eta(\Omega, \nabla\Omega) = 0$, the prepotential is of the quadratic form

$$\mathcal{F} = \tfrac{1}{2}\tau_{ij}\phi^i\phi^j, \tag{11.59}$$

with constant matrix τ_{ij}.

11.6. Families of abelian varieties

There is a natural geometric realization of rigid special geometry using abelian varieties.

Consider an abelian variety X of complex dimension g (see *e.g.* [114]). That is, X is of the form W/Γ with $W \cong \mathbf{C}^g$ a complex vector space and $\Gamma \cong \mathbf{Z}^{2g}$ a rank $2g$ lattice. Topologically, X is a real $2g$ dimensional torus,

$$X \cong T^{2g}. \tag{11.60}$$

Such a complex torus is called an abelian variety if it can be embedded in projective space. There is a simple criterion that determines whether that is the case.

We have two natural coordinates to use on W and X. First we can consider the natural complex coordinates $z^1, \ldots, z^g$ coming from the isomorphism $V \cong \mathbf{C}^g$. Secondly, there are real coordinates $x^1, \ldots, x^{2g}$ from the isomorphism $W \cong \Gamma \otimes \mathbf{R}$. Here we have chosen a basis e_a in the lattice Γ.

Consider the space $H^1(X, \mathbf{C})$. This is a $2g$ dimensional complex vector space with a natural decomposition

$$H^1(X, \mathbf{C}) = H^{1,0} \oplus H^{0,1} \tag{11.61}$$

in terms of the one-form dz^i of type (1,0) and the conjugate forms $d\overline{z}^i$ of type (0,1). The dual space $H_1(X, \mathbf{Z})$ is canonically isomorphic with the lattice Γ. We use the same symbol to indicate a basis e_a of one-cycles in X. Given the natural pairing between homology and cohomology, we can define a $2g \times g$ period matrix

$$\oint_{e_a} dz^i = \pi_a{}^i \tag{11.62}$$

Since there is a natural dual basis of the space $H^1(X, \mathbf{Z})$ given by the $2g$ real one-forms dx^a, whose periods satisfy

$$\oint_{e_a} dx^b = \delta_a{}^b, \tag{11.63}$$

the period matrix $\pi_a{}^i$ can be seen to relate the two natural coordinates as

$$z^i = x^a \pi_a{}^i. \tag{11.64}$$

According to Kodaira's embedding theorem we now have to look for a two-form η (the Kähler form) of type $(1,1)$ that is closed, integer and positive. Now it is easy to write down an integer two-form in the real coordinates. Any form of the type

$$\eta = \tfrac{1}{2}\eta_{ab} dx^a \wedge dx^b \tag{11.65}$$

with integer antisymmetric matrix $\eta_{ab} = -\eta_{ba} \in \mathbf{Z}$ will do. We can rewrite this form in terms of the complex coordinates and now demand that the pieces of type (2,0) and (0,2) vanish, and the piece of type (1,1) is positive. These conditions translate immediately in terms of the period matrix π as the famous Riemann conditions. In fact, under these conditions we can always choose a basis (a_i, b^i) of Γ such that

$$\eta(a_i, b^j) = \delta_i{}^j, \tag{11.66}$$

and so that the periods are of the form

$$\begin{aligned} \oint_{a_i} dz^j &= \delta_i{}^j, \\ \oint_{b^i} dz^j &= \tau^{ij}. \end{aligned} \tag{11.67}$$

In terms of the period matrix τ_{ij} we then find that the Riemann conditions read

$$\tau_{ij} = \tau_{ji}, \qquad \operatorname{Im}\tau > 0. \tag{11.68}$$

In more invariant notation, the Riemann conditions tell us the decomposition of the $2g$ dimensional vector space $V = \Gamma \otimes \mathbf{C}$, given by

$$V \cong H^1(X, \mathbf{C}) = H^{1,0} \oplus H^{0,1}, \tag{11.69}$$

is a polarization for the Hodge form η. That is,

$$\eta(\alpha, \beta) = 0, \tag{11.70}$$

if α, β both in $V^{(1,0)}$ or $V^{(0,1)}$. We furthermore have the positivity condition

$$-i\eta(\alpha, \overline{\alpha}) > 0 \tag{11.71}$$

An important and famous class of examples of abelian varieties are the Jacobians of curves

$$X = H^1(\Sigma_g, \mathbf{C})/H^1(\Sigma, \mathbf{Z}) \tag{11.72}$$

Since abelian varieties are completely characterized by their period matrix τ, the moduli space $\mathcal{A}_g$ of (principally polarized) abelian varieties has a simple description as the quotient of a homogeneous space

$$\mathcal{A}_g \cong Sp(2g, \mathbf{Z}) \backslash Sp(2g, \mathbf{R}) / U(g) \tag{11.73}$$

Here $Sp(2g, \mathbf{Z})$ is the naturally symmetry group of the lattice Γ endowed with the symplectic form η. Over this moduli space we have a canonical flat, holomorphic $Sp(2g, \mathbf{Z})$ vector bundle V (the Hodge bundle) with fiber

$$V_X = H^1(X, \mathbf{C}) \tag{11.74}$$

How do we now get a solution of special geometry out of all this? We are clearly looking for a g dimensional family of abelian varieties parametrized by the moduli space $\mathcal{M}$. That is, we are looking for a map

$$\mathcal{M} \to \mathcal{A}_g \tag{11.75}$$

with the property that the holomorphic tangent bundle to $\mathcal{M}$ can be identified with the space $H^{1,0}(X, \mathbf{C})$. For more along this lines see *e.g.* [115].

11.7. BPS states

Any system with extended supersymmetry has a special subspace of the full Hilbert space $\mathcal{H}$, the so-called BPS space

$$\mathcal{H}_{BPS} \subset \mathcal{H}, \tag{11.76}$$

that consists of "small" supermultiplets. These BPS states play a fundamental role in understanding duality symmetries.

More precisely, suppose we are dealing with some supersymmetry algebra with a set of n supercharges Q^α (n will always be even.) The Hilbert space will decompose in irreducible representations of this algebra. Since the supersymmetry algebra will be of the general form

$$\{Q^\alpha, Q^\beta\} = \omega_i^{\alpha\beta} K^i, \tag{11.77}$$

with

$$[Q^\alpha, K^i] = 0, \qquad [K^i, K^j] = 0, \tag{11.78}$$

where the K^i are some set of bosonic charges, consisting of the translation operator P_μ and some extra set of central charges, usually denoted as Z. The symmetric bilinear forms

ω_i will always be non-degenerate. Therefore, when we consider a representation where the operators K^i have fixed generic eigenvalues k^i, so that the total bilinear form

$$\omega = \omega_i k^i \tag{11.79}$$

is non-degenerate, we are essentially dealing with a representation of a n-dimensional Clifford algebra. The dimension of the representation will therefore be $2^{n/2}$. However, for special values of the charges k^i, it might be the case that the bilinear form ω becomes accidentally degenerate. In that case there are certain linear combinations of supercharges that annihilate the representation. This representation is then called a BPS representation. If it satisfies the conditions

$$\epsilon_\alpha Q^\alpha |\mathrm{BPS}\rangle = 0, \tag{11.80}$$

for m independent spinors ϵ, the rank of the Clifford algebra will be $n - m$ and therefore the dimension of the representation will be $2^{(n-m)/2}$.

Let us now see what the role is of BPS states in models with global $N = 2$ supersymmetry. The four-dimensional $N = 2$ supersymmetry algebra is given as

$$\{Q^I_\alpha, Q_{\dot\beta,J}\} = \delta^I_J \gamma^\mu_{\alpha\dot\beta} P_\mu, \tag{11.81}$$

However, we also have to consider the other commutators, which can take the most general form

$$\begin{aligned} \{Q^I_\alpha, Q^J_\beta\} &= \epsilon^{IJ}\epsilon_{\alpha\beta} Z, \\ \{\overline{Q}_{\dot\alpha,I}, \overline{Q}_{\dot\beta,J}\} &= \epsilon_{IJ}\epsilon_{\dot\alpha\dot\beta}\overline{Z}, \end{aligned} \tag{11.82}$$

with complex central charge $Z \in \mathbf{C}$ satisfying

$$[Z, Q] = [Z, \overline{Q}] = [Z, P] = 0. \tag{11.83}$$

For a general state with eigenvalues $P_\mu = M\delta_{\mu,0}$ and Z, one easily derives an upper bound for the mass [113]

$$M^2 \geq |Z|^2. \tag{11.84}$$

In fact, states with $M^2 = |Z|^2$ are precisely the small BPS representation (of dimension 2 instead of 4).

The proof this statement follows a very general argument. For a given multiplet the BPS condition can be written as (we suppress the indices)

$$\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right) |\mathrm{BPS}\rangle = 0. \tag{11.85}$$

This condition holds with fixed $\epsilon, \overline{\epsilon}$ for all states in the BPS multiplet. If we take the commutator with the supercharges, we derive the conditions on the supersymmetry parameters of the form

$$\begin{aligned} \not{p}\epsilon + Z^{\dagger}\overline{\epsilon} &= 0, \\ Z\epsilon + \not{p}\overline{\epsilon} &= 0. \end{aligned} \tag{11.86}$$

Combining these equations with the mass shell condition $p^2 = M^2$, one deduces that M^2 coincides with the highest eigenvalue of $ZZ^{\dagger}$ and $Z^{\dagger}Z$, with ϵ and $\overline{\epsilon}$ being the corresponding eigenvectors,

$$\begin{aligned} (ZZ^{\dagger})\epsilon &= M^2\epsilon \\ (Z^{\dagger}Z)\overline{\epsilon} &= M^2\overline{\epsilon}. \end{aligned} \tag{11.87}$$

This determines the BPS masses completely in terms of the central charge Z. In the $N = 2$ context we find that $Z^{\dagger}Z = ZZ^{\dagger} = |Z|^2\mathbf{1}$, so that the above BPS mass formula is obtained immediately.

In the case of an $N = 2$ abelian gauge theory with coupling constant τ_{ij} the central charge Z takes a simple form. If a state carries charge $q = (e, m)$, with electric charges e_i and magnetic charges m^i, the central charge $Z(q)$ takes the form [113]

$$Z(q) = \phi^i(e_i - \tau_{ij}m^j) \tag{11.88}$$

This result can be generalized to the case where one has non-trivial moduli dependence $\tau_{ij}(\phi)$. Without proof, we mention that in that case the general mass formula takes the following form (see *e.g.* [8])

$$Z(q) = \eta(q, \Omega) = \phi^i e_i - \mathcal{F}_i m^i. \tag{11.89}$$

We note that this expression is explicitly $Sp(2g, \mathbf{Z})$ invariant.

11.8. Non-abelian $N = 2$ gauge theory

Our story has now come to the seminal work of Seiberg and Witten [111] on the non-abelian $N = 2$ SYM model. This work has been reviewed many times [116, 8], and we refer to these references (and the original papers!) for more details.

So we turn to $N = 2$ Yang-Mills theory with a non-abelian gauge group G and Lie algebra g. The fundamental multiplet is again of the form

$$(\mathcal{A}_{\mu}, \Lambda^I_{\alpha}, \overline{\Lambda}^I_{\dot{\alpha}}, \Phi), \tag{11.90}$$

where all fields take value in the adjoint bundle g. The important difference with the abelian case is that the action now contains a potential term of the scalars $\Phi \in g \otimes \mathbf{C}$ of the form

$$V(\Phi) = \int_M \mathrm{Tr}\,[\Phi, \Phi^\dagger]^2. \tag{11.91}$$

This potential vanishes if Φ is restricted to the complexified Cartan torus $t \otimes \mathbf{C}$. In the case $G = SU(2)$, to which we will restrict from now on, Φ is of the form

$$\Phi = \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}, \qquad \phi \in \mathbf{C}. \tag{11.92}$$

Global action of $SU(2)$ gives the further identification

$$\phi \leftrightarrow -\phi \tag{11.93}$$

(in the general case $\phi \in t \otimes \mathbf{C}/W$, with W the Weyl group), so that the *classical* moduli space of vacua is given by (here we added the point at infinity)

$$\mathcal{M}_{cl} \cong \mathbf{P}^1 \tag{11.94}$$

with good local coordinate $u = \phi^2 = \frac{1}{2}\mathrm{Tr}\,\Phi^2$. Note that we expect singularities at $u = 0$ and $u = \infty$.

What is quantum moduli space? This is the question Seiberg and Witten posed and answered. We first notice that a question about the vacuum structure refers to long distances, *i.e.* infrared physics. So we are only interested in the massless fields, whose correlation functions do not decay exponentially in distance. Since the "Higgs field" Φ will in general have a non-zero expectation value with $u = \frac{1}{2}\langle \mathrm{Tr}\,\Phi^2 \rangle \neq 0$, by the Higgs phenomenon all fields will acquire masses except for the component of the multiplet in the direction of Φ. That is, we can try to define a $U(1)$ gauge field

$$A = \mathrm{Tr}\,(\widehat{\Phi}\mathcal{A}), \qquad \widehat{\Phi} = \Phi/|\Phi|. \tag{11.95}$$

Of course, this definition does not make sense at the zeroes of Φ. Around such a zero the unit vector field $\widehat{\Phi}$ defines a possibly non-trivial element of $\pi_2(S^2)$ with winding number m, which represents an obstruction to deforming the zero away. These configurations are characterized by the property that for a two-sphere in $\mathbf{R}^3$ surrounding such a zero

$$\int_{S^2} dA = 2\pi m \tag{11.96}$$

and therefore the singularity carries possible magnetic charges. Indeed, these are the famous 't Hooft-Polyakov monopoles [12]. In the long-distance limit we can think of them as point-like objects. In order to correctly represent all $SU(2)$ gauge field configurations, we therefore have to add a gas of these monopoles. This translates into a second quantized monopole field. In fact, it is a long-standing conjecture of 't Hooft that we can understand the vacuum structure of a non-abelian gauge theory in terms of abelian gauge fields and magnetic monopoles [117]. This deep point of view has been brilliantly confirmed by the work of Seiberg and Witten. In general the monopoles will have positive masses, so in the infra-red limit (where we scale the metric on the volume of the four-manifold to infinity) it is not necessary to include the monopole fields in the action.

As we sketched above, after integrating out the massive fields, we are left with an effective abelian theory. Such a model is characterized (to quadratic order in derivatives) by a prepotential $\mathcal{F}(\phi)$. The coupling constant τ is given as its second derivative $\tau = \mathcal{F}''$. We further have a $Sp(2,\mathbf{Z}) \cong SL(2,\mathbf{Z})$ vector bundle $V = \Gamma \otimes \mathbf{C}$ with a holomorphic section

$$\Omega = \begin{pmatrix} \phi_D \\ \phi \end{pmatrix}, \qquad \phi_D = \frac{\partial \mathcal{F}}{\partial \phi}. \tag{11.97}$$

The electric and magnetic charges are, for given value of ϕ, given by

$$q = \begin{pmatrix} e \\ m \end{pmatrix} = \frac{1}{2\pi} \int_{S^2} \begin{pmatrix} F_D \\ F \end{pmatrix} \in \Gamma. \tag{11.98}$$

We now present the derivation of the prepotential $\mathcal{F}$.

11.9. The Seiberg-Witten solution

Starting point for the SW solution is the behaviour of the prepotential around the point $u = \infty$, where the leading contribution can be computed in perturbation theory. The coupling constant $\tau = \mathcal{F}''$ has an expansion

$$\tau = \tau_0 + \frac{i}{\pi} \log(\phi/\Lambda) + \sum_n a_n (\Lambda/\phi)^{4n} \tag{11.99}$$

where the first term τ_0 is the classical contribution, the second term represents the one-loop contribution (this is the only perturbative correction), whereas the other terms indicate instanton corrections, that are in general difficult to compute exactly. The logarithmic piece gets us started, since we derive the local monodromy of the $SL(2,\mathbf{Z})$ bundle around ∞. If $u \to e^{2\pi i} u$ we see that

$$\begin{pmatrix} \phi_D \\ \phi \end{pmatrix} = \begin{pmatrix} \mathcal{F}' \\ \phi \end{pmatrix} \to \begin{pmatrix} -\varphi_D + 2\varphi \\ -\phi \end{pmatrix} = \mathbf{M}_\infty \begin{pmatrix} \phi_D \\ \phi \end{pmatrix}. \tag{11.100}$$

With this monodromy

$$\mathbf{M}_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \tag{11.101}$$

we can try to extend the vector bundle to $\mathbf{P}^1 - \{0, \infty\}$ by having the opposite monodromy around the origin, where we expect our second singularity. However, we also have to satisfy the positivity constraint $\mathrm{Im}\,\tau > 0$. It turns out that this is impossible without introducing a third singularity. Indeed, the SW solution assumes the minimal amount of a total of three singularities, say at $u = \infty, 1$, and -1, with monodromy matrices

$$\begin{aligned} \mathbf{M}_1 &= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \\ \mathbf{M}_{-1} &= \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}. \end{aligned} \tag{11.102}$$

These matrices generate the $SL(2, \mathbf{Z})$ subgroup

$$\Gamma(2) = \{\gamma \in SL(2, \mathbf{Z});\ \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ (mod) } 2\}. \tag{11.103}$$

The quantum moduli space is given by the Ansatz

$$\mathcal{M}_{qu} = \mathbf{H}/\Gamma(2), \tag{11.104}$$

which is "almost" the moduli space of elliptic curves. In fact, $\mathcal{M}$ parametrizes elliptic curves Σ of the particular form

$$y^2 = (x^2 - 1)(x - u). \tag{11.105}$$

The bundle V is now the bundle with fiber over $\Sigma \in \mathcal{M}$ given by

$$V_\Sigma = H^1(\Sigma, \mathbf{Z}) \otimes \mathbf{C}. \tag{11.106}$$

So the charge lattice Γ is identified with the first homology group of the elliptic curve $\Gamma = H_1(\Sigma, \mathbf{Z})$.

To such a surface we can associate a natural modulus τ (of the elliptic curve) that always satisfies the positivity condition $\mathrm{Im}\,\tau > 0$. Note that the modulus can be expressed as a ratio of periods

$$\tau = \frac{\oint_B \omega}{\oint_A \omega}, \tag{11.107}$$

with (A, B) a canonical basis of $H_1(\Sigma)$ and $\omega = dz = dx + \tau dy$ the unique holomorphic one-form.

Now that we are given the bundle $V \to \mathcal{M}$, we should ask what the prepotential is. There is an obvious guess. Let's pick some one-form λ on Σ. We can then consider the following Ansatz

$$\begin{aligned} \phi &= \oint_A \lambda \\ \phi_D &= \oint_B \lambda. \end{aligned} \tag{11.108}$$

By definition this vector will transform correctly under $SL(2, \mathbf{Z})$. However, special geometry imposed the further constraint that

$$\tau = \frac{\partial \phi_D}{\partial \phi} = \frac{\oint_B \frac{\partial \lambda}{\partial u}}{\oint_A \frac{\partial \lambda}{\partial u}}. \tag{11.109}$$

If we compare with (11.107) we see that necessarily

$$\frac{\partial \lambda}{\partial u} = c\omega. \tag{11.110}$$

There is a (unique up to scalars) meromorphic one-form with this property. This then completes the solution.

11.10. Physical interpretation of the singularities

The two new singularities in $\mathcal{M}$ have a beautiful physical interpretation. Recall the formula for the BPS formula (11.89) for a dyon with charges (e, m) in a model with $N = 2$ supersymmetry:

$$M = |Z| = |\phi e - \phi_D m| \tag{11.111}$$

For fixed charges this mass varies with the moduli. Now, if at some point in the moduli space a linear combination $\phi e - \phi_D m$, with $e, m \in \mathbf{Z}$ vanishes, this has important physical consequences. It implies that the dyon with these particular charges will have zero mass. But the SW solution was based on the premises that the abelian (super) gauge field was the only massless field! Therefore, at these points in $\mathcal{M}$ we expect our formalism to break down. This breakdown will manifest itself in a singularity in the solution. There are three such points $u = 1, -1, \infty$.

At $u = \infty$ the situation is clear: here the non-abelian gauge fields with $(e, m) = (\pm 1, 0)$ become massless again, restoring the $SU(2)$ symmetry.

At $u = 1$ we have a very different situation. Here the monodromy $\mathbf{M}_1$ forces the component $\phi_D \to 0$. Thus magnetic monopoles with $(e, m) = (0, 1)$ get zero mass at this point. In the 't Hooft philosophy we had presumably taken into account the monopole fields, when we arrived at our effective action encoded by the prepotential $\mathcal{F}$. The fact that these fields actually become massless declares this procedure after the fact invalid. One should thus undo this integration procedure and reintroduce the monopole fields in the path-integral. This removes the singularity. It was simply the result of an overambitious simplification of the dynamics.

It is very important that it is not a physical singularity. There is no problem with having singularities in a moduli space that parametrizes a family of classical field configurations. Neither is there a problem if the moduli space parametrizes inequivalent quantum field theories. One simply decides not to consider such a model. (A good example of such a singularity is a sigma model on a singular target space.) However, one cannot allow singularities in a moduli space that parametrizes inequivalent vacua of a single theory. There is no way in which we can prevent the theory of exploring the region in moduli space that contains the singularity. (At least, if it is at finite distance, which is the case in all situations that we will discuss.) There must be a physical mechanism that "explains" the singularity, such as the massless monopole in the SW point $u = 1$.

One can reproduce the monodromy matrix $\mathbf{M}_1$ from this physical argument — an important consistency check on the solution. First we use S-duality to go to a dual description in which we use the expectation value of the dual scalar field ϕ_D as the local coordinate around $u = 1$ and have a corresponding dual gauge field A_D. The electric charges of this dual gauge field are the magnetic charges of the original gauge field A, that we recover in the perturbative expansion around $u = \infty$. In these new dual variables we have a dual prepotential $\mathcal{F}_D(\phi_D)$ in which we can express $\phi = \mathcal{F}_D'$. We can now make a *perturbative* calculation of the dual coupling constant $\tau_D = \mathcal{F}_D''$ by considering the one-loop contribution of the nearly massless monopoles. They give

$$\tau_D \sim -\frac{i}{\pi} \log(\phi_D/\Lambda) \tag{11.112}$$

from which we deduce that $\phi \sim -\frac{i}{\pi}\phi_D \log \phi_D$, so that the monodromy for $\phi_D \to e^{2\pi i}\phi_D$ is given by

$$\phi \to \phi - 2\phi_D. \tag{11.113}$$

This gives the required monodromy matrix

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{11.114}$$

In a similar fashion the massless dyon of charge $(1, 1)$ produces the monodromy matrix $\mathbf{M}_{-1}$ around the second singularity $u = -1$.

There is a nice geometric interpretation of this monodromy in terms of the elliptic curve. At the three singularities we have $\tau \to i\infty$, so that we are dealing with a singular elliptic curve of genus one. We can picture this singular curve by shrinking an homology cycle to zero. For example, in the case $u = 1$ the B-cycle shrinks to zero, so that also

$$\phi_D = \oint_B \lambda \to 0. \tag{11.115}$$

By doing a Dehn twist around this cycle, we recover the monodromy matrix, precisely as in our discussion in section §9.6.

11.11. Implications for four-manifold invariants

Somewhat as an aside, we can ask what the implications of the SW solution are for four-dimensional topology [118].

We have seen how a twisted version of $N = 2$ super Yang-Mills with gauge group $SU(2)$ led to Donaldson's manifold invariants by localization to the moduli space of non-abelian instantons. In this argument, we applied more or less semi-classical arguments. These arguments can be made valid if the gauge coupling constant, that controls quantum corrections, is assumed to be small. Thanks to the "running" of coupling constants in quantum gauge theories, that makes the coupling constant a non-trivial function of the scale, and thanks to asymptotic freedom, that tells us that the coupling will actually decrease if we make this scale very small in non-abelian $N = 2$ SYM theory, we can conclude that this semi-classical approximation is appropriate for small volumes of the four-manifold. But the Donaldson invariants are topological invariants, they do not depend on the choice of Riemannian metric (for $b^+ > 1$). We can therefore take this metric to be as small as we want, and can thus correctly apply semi-classical reasoning. This is the usual philosophy behind the application of non-abelian gauge theory to four-dimensional differential topology.

However, we can also take a completely contrary point of view and consider instead the infrared limit of large volume[9]. Then we believe that only the massless degrees of freedom are relevant, since all other fields have exponentially decaying interactions, which in the IR limit become point-like. Of course, these massless fields are described by an *effective* Lagrangian that takes correctly into account the quantum loops of all the massive fields that we have integrated out in the process. In this case, such an effective field theory will be an abelian gauge theory of the type we have been considering in §11.4.

If we now apply our localization formulas of §10.2 to the BRST fixed points of this effective abelian theory, we find that this model localizes to the solutions of the abelian instanton equation

$$F^+ = 0. \tag{11.116}$$

However, as we discussed already in section §10.1, for an abelian gauge field the curvature is necessarily quantized in integer fluxes, $[F/2\pi] \in H^2(M, \mathbf{Z})$. But F is also ASD, so

[9] This point of view is actually more intuitive. If we study topology we are typically interested in global, large scale features. At least, this is the colloquial use of the expression "topological" in physics.

$F = F^-$. For a generic metric and thus a generic positioning of the subspace $H^2_- \subset H^2$, the intersection with the integer lattice $H^2(M, \mathbf{Z})$ will only be non-zero if $H^2_+ = 0$ *i.e.* $b_2^+ = 0$, which one usually assumes not to be the case. We conclude therefore that for the generic four-manifold and generic metric the abelian BRST fixed point set is empty.

There is however one *caveat.* If we compute the partition function on a *compact* four-manifold M, as we will certainly do in the context of manifold invariants, there is no moduli space of disjunct vacua, as labeled by the quantum moduli space $\mathcal{M}_{qu}$. In fact, as we already have explained in §10.5, in this case one is forced to integrate over the constant values of the scalar fields too. We thus have to consider our model for all values of the modulus $u \in \mathbf{P}^1$ and integrate over u. Away from the singularities this is not a problem, but at $u = \pm 1$ we get a modified equation, because we also have to take into account the massless monopole and dyon fields respectively that we cannot ignore in the IR limit.

What is the form of their contribution? If we use an S-dual formulation, the massless fields are simply the field content of $N = 2$ QED, *i.e.* supersymmetric Maxwell theory with an additional massless fermion field (a neutrino). Independently of the work of Seiberg and Witten, one could have considered the twisted version of QED as a generator of four-manifold invariants after Witten's paper [80] in 1988 showed how $N = 2$ models can be twisted to produce topological field theories. However, somehow the believe was that only non-abelian Yang-Mills theories, that due to asymptotic freedom stand a much better change to exist mathematically, would be able to produce non-trivial differential geometry invariants! (Indeed, it is a beautiful fact that both in physics and in mathematics dimension four stands out as giving the richest phenomena.)

The extra $N = 2$ matter multiplet in QED takes the form $(M^I, \psi_\alpha, \overline{\psi}_{\dot\alpha})$, with ψ the usual fermion field and M^I a doublet of two complex scalar fields (sneutrinos). The twisting modifies the spin of these fields as

$$\begin{aligned} M^I &\Rightarrow M_\alpha \\ \psi_\alpha &\Rightarrow \psi_\alpha \\ \psi_{\dot\alpha} &\Rightarrow \rho_{\dot\alpha} \end{aligned} \tag{11.117}$$

We obtain a commuting bosonic variable M that transforms as a spinor! The BRST transformations are

$$\begin{aligned} \delta M_\alpha &= \psi_\alpha, \\ \delta \rho_{\dot\alpha} &= (\partial\!\!\!/ M)_{\dot\alpha}. \end{aligned} \tag{11.118}$$

Finally, because of the coupling of the matter multiplet to the gauge multiplet, the transformation for the field $\rho^+_{\mu\nu}$ in (11.13) gets modified to

$$\delta \rho^+_{\mu\nu} = F^+_{\mu\nu} - (\overline{M} M)^+_{\mu\nu}. \tag{11.119}$$

Looking at the transformation rules of the fields $(\rho^+_{\mu\nu}, \rho_{\dot\alpha})$ we see that in this case the section is given by

$$s = (F^+ - (\overline{\mathcal{M}} M)^+, DM) \tag{11.120}$$

Its zero locus is the Seiberg-Witten moduli space described in [118] and discussed in the lectures by Garcia. See also [119].

12. String vacua

This finishes our introduction!

We now arrive at the last stretch of our journey — the space-time physics of string theory. The most important new ingredient is of course the dynamical metric (together with its supersymmetric partners). String theory is a theory of quantum gravity and string perturbation theory can be used to compute weak-coupling quantum gravity effects. These perturbative computations follow the pattern described in the first half of the course. One picks a CFT describing the string vacuum around which we can do our perturbative expansion in terms of Riemann surfaces. The expansion coefficients of the n-point scattering amplitudes

$$A \sim \sum_g A_g \lambda^{2g-2} \tag{12.1}$$

are then computed as integrals over the moduli space $\mathcal{M}_{g,n}$. However, this is only part of the total picture.

First of all, the series expansion is only asymptotic, it does not converge[10], so additional nonperturbative information is needed to fix the final answer. These nonperturbative phenomena are not necessarily completely "stringy" in nature. They can sometimes be understood from a low-energy point of view, where we forget about the infinite set of massive states. In fact, the issues that we will discuss, such as dualities, moduli spaces and BPS states, can be even understood without studying much dynamical issues in quantum gravity.

12.1. Perturbative string theories

At present we know of five different perturbative string theories, *i.e.* theories that allow a consistent perturbation expansion in terms of Riemann surfaces, summarized in *table* 1. We apologize that we didn't clearly explained the precise world-sheet formulations of these theories. We did explain the bosonic string, but there are a few extra subtleties when we are dealing with super Riemann surfaces that we didn't want to get into.

Recently, two more string vacua have been discovered. These theories, called M-theory [16, 122] and F-theory [123] do not allow a perturbative formulation in terms of surfaces. In fact, the geometrical objects seem to involve $(2,1)$ dimensional super membranes and $(2,2)$ signature objects that are even less well understood. (See [124] for a conjectural definition of M-theory.) For completeness we add these two new models in *table* 2.

[10] See [120]. Indeed, one can estimate that string perturbation theory generates terms of order $2g!$ at g loops [121].

perturbative vacuum	*world-sheet geometry*	*space-time dimension*	*space-time susy*	*gauge group*
I	unoriented, open	(9,1)	$N = 1$	$Spin(32)/\mathbf{Z}_2$
IIA	super, closed	(9,1)	$N = (1, 1)$	$U(1)$
IIB	super, closed	(9,1)	$N = (2, 0)$	—
HO	heterotic, closed	(9,1)	$N = 1$	$Spin(32)/\mathbf{Z}_2$
HE	heterotic, closed	(9,1)	$N = 1$	$E_8 \times E_8$

Table 1: *The five known perturbative string theories.*

perturbative vacuum	*world-volume geometry*	*space-time dimension*	*space-time susy*	*gauge group*
M	super membranes?	(10,1)	$N = 1$	—
F	SD three-branes?	(10,2)	$N = 1$	—

Table 2: *The new string vacua M-theory and F-theory are less well-understood.*

The insight that all perturbative string theories are different expansion of one theory is now known as *string duality.* The precise way in which the different models are related is too involved to be explained in this set of lectures (but see *e.g.* [19, 20]. It is one of the amazing new insights following from string duality that these “theories” are all expansions of one and the same theory around different points in the moduli space of vacua. Indeed we have the beautiful picture of the moduli space *fig.* 8. given in [20] where we have various “cusps” around which the amplitudes allow systematic geometric expansions.

We will say briefly a few words about the world-sheet formulation of the perturbative string theories, although we do not have time to discuss this in full detail (see *e.g.* [4] or more recent lecture notes such as [75]). Of course, all models follow roughly the line discussed for the bosonic string in §8.3. We will not discuss the open string of Type I. According to the modern point of view it is by an S-duality related to the heterotic string. For the remaining strings we have the following left-moving and right-moving degrees of

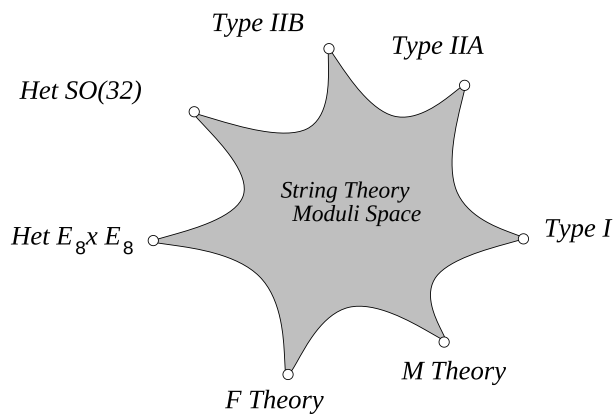


Fig. 8: *The moduli space of string vacua. Various perturbative theories are recovered in expansions around the cusps.*

freedom

string	*Left*	*Right*	
bosonic	x^μ	x^μ	$\mu = 1, \ldots, 26$
IIA & B	x^μ	x^μ	$\mu = 1, \ldots, 10$
	ψ^μ	ψ^μ	
heterotic	x^μ	x^μ	$\mu = 1, \ldots, 10$
	ψ^μ	x^I	$I = 1, \ldots, 16$

We add the bosonic string for good measure, although in the end it is inconsistent due to the existence of a tachyon. Here the 16 internal right-moving bosonic fields x^I of the heterotic string take their value in the torus defined by modding out the lattice $E_8 \oplus E_8$ or D_{16} giving the HE and HO theory respectively.

12.2. IIA or IIB

The distinction between the two type II theories has to do with space-time chirality. It will be important in the following, so we riefly review it. In the IIA theory we have supersymmetry generators $Q_\alpha, Q_{\dot\alpha}$ of both 10-dimensional chiralities, whereas in the type IIB theory the two supercharges Q^1_α, Q^2_α carry the same chirality. In terms of the world-sheet CFT description this distinction is made in the way the GSO projection is defined.

Recall that the 10 Majorana fermion fields ψ^μ are most conveniently quantized by combining them in pairs into 5 Dirac fermions

$$\psi = \psi^1 + i\psi^2, \qquad \psi^* = \psi^1 - i\psi^2. \tag{12.2}$$

These fields have an expansion of the form

$$\psi(z) = \sum_{n\in\mathbf{Z}+\epsilon} \psi_n z^{-n-\frac{1}{2}}, \tag{12.3}$$

where the spin structure $\epsilon = 0, \frac{1}{2}$ is determined by the NS or R boundary condition

$$\psi(e^{2\pi i}z) = \begin{cases} \psi(z), & \epsilon = \frac{1}{2} \ \ \text{(NS)}, \\ -\psi(z), & \epsilon = 0 \ \ \text{(R)}. \end{cases} \tag{12.4}$$

In the NS sector we have a well-defined vacuum state with

$$\psi_n|0\rangle = \psi^*_n|0\rangle = 0, \qquad n > 0. \tag{12.5}$$

The Fock space $\mathcal{F}_{NS}$ built on this vacuum has a natural $\mathbf{Z}_2$ grading by the fermion number $J_0 = 0, 1$ modulo 2,

$$\mathcal{F}_{NS} = \mathcal{F}^0_{NS} \oplus \mathcal{F}^1_{NS} \tag{12.6}$$

The GSO projection, that is an important ingredient in the definition of the superstring, tells us to take the odd piece $\mathcal{F}^1_{NS}$.

For the Ramond sector, the modes ψ_n are integer valued. This means that we have zero modes ψ_0, ψ^*_0 satisfying a Clifford algebra

$$\{\psi_0, \psi^*_0\} = 1 \tag{12.7}$$

Therefore the vacuum $|\sigma_\pm\rangle$ will be two-fold degenerate. It is a spinor of $SO(2)$, and it satisfies

$$\psi_n|\sigma_\pm\rangle = \psi^*_n|\sigma_\pm\rangle = 0, \qquad n > 0. \tag{12.8}$$

The states $|\sigma_\pm\rangle$ have conformal dimension $h = \frac{1}{8}$ and Fermi number $\pm\frac{1}{2}$. This can be seen by bosonization. Introduce a scalar field φ with $-i\partial\varphi = \psi\psi^*$ so that

$$\psi = e^{i\varphi}, \qquad \psi^* = e^{-i\varphi}. \tag{12.9}$$

The spin fields $\sigma_\pm$, that create the R ground states out of the NS vacuum now take the form

$$\sigma_\pm = e^{\pm\frac{i}{2}\varphi}. \tag{12.10}$$

The Ramond Fock space is again graded

$$\mathcal{F}_R = \mathcal{F}^+_R \oplus \mathcal{F}^-_R, \tag{12.11}$$

where the total Fermi number is $J_0 = \pm\frac{1}{2} \bmod 2$. The GSO projection instructs us to take either $\mathcal{F}_R^+$ or $\mathcal{F}_R^-$. It is hard to distinguish the two, since there is no intrinsic way to determine the absolute sign of the fermion charge J_0.

For the full multiplet of 10 fermion fields, the R ground states form a rank $2^5 = 32$ spinor of $Spin(10)$. In fact, after bosonization they take the form

$$\sigma_\alpha = e^{i\alpha\cdot\varphi}, \qquad \alpha = (\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}). \tag{12.12}$$

The spinor splits in two chiralities distinguished by

$$\sum_i \alpha_i = \pm\tfrac{1}{2} \bmod \mathbf{Z}. \tag{12.13}$$

The GSO projection will project on one chirality. Again, which chirality is a matter of taste, they will be permuted by a parity transformation. But there will be a marked difference when we combine left-movers and right-movers. The GSO projection is performed separately in both sectors, and we have a two-fold ambiguity in each sector. Of the total of four possibilities we can identify two by flipping the sign of the overall fermi number $F = J_0 + \overline{J}_0$.

The two remaining choices give physically distinct theories, the type IIA and type IIB string theories. As we mentioned already, they differ in the relative sign in the chiralities of the two supercharges. In a formula we obtain the following contribution to the Hilbert space

$$\begin{aligned} \text{IIA}: &\quad \left(\mathcal{F}^1_{NS} \oplus \mathcal{F}^+_R\right) \otimes \left(\overline{\mathcal{F}}^1_{NS} \oplus \overline{\mathcal{F}}^-_R\right), \\ \text{IIB}: &\quad \left(\mathcal{F}^1_{NS} \oplus \mathcal{F}^+_R\right) \otimes \left(\overline{\mathcal{F}}^1_{NS} \oplus \overline{\mathcal{F}}^+_R\right). \end{aligned} \tag{12.14}$$

This gives four different sectors in the Hilbert space

$$\begin{array}{cc} (\text{NS}, \text{NS}): & \textit{boson} \\ (\text{NS}, \text{R}): & \textit{fermion} \\ (\text{R}, \text{NS}): & \textit{fermion} \\ (\text{R}, \text{R}): & \textit{boson} \end{array} \tag{12.15}$$

The (NS,NS) sector contains the familiar massless bosonic fields: the metric $G_{\mu\nu}$, the two-form field $B_{\mu\nu}$ and the dilaton field Φ, all corresponding to states of the form

$$\psi^\mu_{-\frac{1}{2}} \psi^\nu_{-\frac{1}{2}} |p\rangle. \tag{12.16}$$

The massless bosonic fields in the (R,R) sector have long been neglected, but they play a crucial role in recent developments. By definition they are bispinors. However, we can further decompose them in terms of irreducible representations of $Spin(10)$. If $S^{\pm}$ denote the two chiral spinor representations, and Λ^k the representation of k-forms, we have

$$\begin{aligned} \text{IIA}: &\qquad S^+ \otimes S^- \cong \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4 \\ \text{IIB}: &\qquad S^+ \otimes S^+ \cong \Lambda^1 \oplus \Lambda^3 \oplus \Lambda^5_+ \end{aligned} \tag{12.17}$$

Here Λ^5_+ indicates the self-dual 5-forms. These relations are derived in the familiar way, we write a k-form as a bispinor using the Dirac matrices

$$\begin{aligned} F_{\alpha\dot\beta} &= \sum_{k\ even} F^{(k)}_{\mu_1\ldots\mu_k} (\gamma^{\mu_1}\cdots\gamma^{\mu_k})_{\alpha\dot\beta} \\ F_{\alpha\beta} &= \sum_{k\ odd} F^{(k)}_{\mu_1\ldots\mu_k} (\gamma^{\mu_1}\cdots\gamma^{\mu_k})_{\alpha\beta} \end{aligned} \tag{12.18}$$

Here the total differential form

$$F = \sum_{k\ odd/even} F^{(k)}, \qquad F^{(k)} \in \Omega^k(\mathbf{R}^{9,1}) \tag{12.19}$$

satisfies the self-duality condition $*F = F$.

The corresponding vertex operators couple to k-form field strengths. This we can see by looking at the equation of motion. The bispinor $F_{\alpha\beta}$ satisfies two Dirac equations, one on the left and one on the right. In terms of differential forms this gives the equations

$$dF = d^*F = 0, \tag{12.20}$$

which we recognize as the Bianchi identity and the Maxwell equation.

12.3. D-branes

Summarizing, in the type II superstring we have a collection of generalized gauge fields $A^{(k)}$, which are k-forms, with k odd and even in the IIA and IIB case respectively.

Now we are familiar with one such generalized gauge field, the two-form B field that we find in the (NS,NS) sector. The string carries a charge for this field, as can be seen from the sima-model coupling

$$\int d^2z\, \tfrac{1}{2} B_{\mu\nu}(x)\partial x^\mu \overline{\partial} x^\nu = \int_\Sigma B. \tag{12.21}$$

This minimal coupling is complete analogous to the way a particle with a one-dimensional world-line C couples with charge q to a one-form gauge field A

$$q \int d\tau\, A_\mu(x)\dot{x}^\mu = q \int_C A. \tag{12.22}$$

So we see that the string has charge $+1$ with respect to B. In a similar spirit the objects that would couple to a $p+1$ form gauge field $A^{(p+1)}$ are p-branes ("pea brains"). These p-dimensional extended objects sweep out a $p+1$ dimensional world-volume C_{p+1} if they propagate in time. The coupling takes the form

$$q \int_{C_{p+1}} A^{(p+1)}. \tag{12.23}$$

This coupling is invariant under gauge transformation which shift A by a p-form Λ

$$A \to A + d\Lambda, \tag{12.24}$$

at least if the wave function of the p-brane transforms at the same time by

$$\psi \to e^{iq \int_{brane} \Lambda} \psi, \tag{12.25}$$

which is the definition of charge q.

What are these mysterious extended objects that carry the charges of the RR fields? They cannot be present in the perturbative string spectrum, since we only find couplings to the field strength $F = dA$. (Since F is invariant under gauge transformations, the perturbative string states are neutral.) String theory has therefore to be complemented with non-perturbative states carrying the RR charges. These were found in a brilliant proposal by Polchinsky and called D-branes [125]. These D-branes are reviewed in great detail in a collection of excellent lecture notes [20], see also the lectures by Mike Douglas at this school [126]. So we will skip their world-sheet formulation. We cannot stress too much how important it is that for the first time we have an exact description of non-perturbative solitons in string theory.

12.4. Compactification

In string theory we will have to compactify the 10-dimensional space-time manifold on a 6-dimensional compact space X. This space has in general moduli ϕ that take value in a moduli space $\mathcal{M}_X$ that parametrizes the inequivalent geometrical structures we can choose on X. In perturbative string theory these moduli appear as the parameters describing the family of sigma-models, or more general CFTs that can appear as string

4D susy	*I/Het*	*IIA*	*IIB*	*M*	*F*
$N=8$	—	T^6	T^6	T^7	T^8
$N=4$	T^6	$K3\times T^2$	$K3\times T^2$	$K3\times T^3$	$K3\times T^4$
$N=2$	$K3\times T^2$	CY_3	$\widetilde{CY}_3$	$CY_3\times S^1$	$\widetilde{CY}_3\times T^2$
$N=1$	$CY_3\cong\widetilde{CY}_3$	*orient.*	*orient .*	G_2	CY_4

Table 3: *Compactifications in string theory.*

vacua; the proper moduli space is therefore the "quantum moduli space" that includes sigma-model quantum effects. It is not necessary that the compactification fiber X_x over the 4-dimensional space-time point $x\in\mathbf{R}^4$ is the same at each point; we can allow small local variations. Stated otherwise, the moduli $\phi\in\mathcal{M}_X$ become space-time scalar fields

$$\phi(x): M^4\rightarrow\mathcal{M}_X. \tag{12.26}$$

Therefore we have the all-important relation

$$\textit{families of CFTs}\subset\textit{vacua of string theory.} \tag{12.27}$$

Here we write $\subset$ instead of $\cong$ since we realize that string theory can have moduli such as the string coupling constant that are not present or difficult to understand in string perturbation theory.

The amount of supersymmetry we expect in four dimensions will dictate the combination of perturbative string plus compactification manifold. We have gathered these data in *table* 3. A few scattered remarks about this table: The amount of supersymmetry in $4D$ is simply given by the number of original supersymmetries in the uncompactified space-time times the number of covariantly constant spinors on the compactification space X. This latter number is determined by holonomy group of X. The holonomies are: trivial for T^n, $SU(2)$ for $K3$, $SU(3)$ for a Calabi-Yau three-fold CY_3, G_2 for a seven-dimensional manifold with this exceptional holonomy, and $SU(4)$ for a Calabi-Yau four-fold CY_4.

In this table all horizontal theories should give the same four dimensional physics. There should therefore be relations among the compactification data. In particular they should all have the same moduli space. The two equivalent Calabi-Yau three-folds CY_3 and $\widetilde{CY}_3$ are related by mirror symmetry.

In Type I or heterotic compactification we also have to pick a gauge bundle $V\rightarrow X$ over the compactification manifold X. This gauge bundle should satisfy the topological constraints

$$c_1(V)=0,\quad c_2(V)=c_2(X). \tag{12.28}$$

We have studied in section §§8.9–12 the abstract structure of string vacua. It corresponded to a family of 2d TFTs that satisfied certain integrability conditions. We now want to make contact with the space-time approach where they correspond with moduli spaces of supersymmetric vacua. Indeed, we know that upon compactification we obtain (in the low energy limit) a supergravity theory that contains a supersymmetric sigma model with target space the moduli space $\mathcal{M}_X$ of the compactification data X. This moduli space appears as the expectation values of the scalar fields ϕ^i and thus should be interpreted as a moduli space of vacua of a single theory (string theory respectively supergravity).

The scalar fields ϕ^i will have an action that is highly constrained by the requirements of supersymmetry. We have seen this in detail for the scalar components of the so-called vector multiplets in global $N = 2$ supersymmetry in §11.4 and this led directly to the rigid special geometry for the moduli space $\mathcal{M}$ as captured by the prepotential $\mathcal{F}$. The analogous discussion for local supersymmetry can be done along the same lines and leads directly to local special geometry that we met at various places. So we see that space-time considerations give a direct interpretation of the geometry of the moduli space of string vacua — one of the themes of this lecture series.

In this way we can reproduce the results from CFT obtained in §7 and §8. However, there is one thing we can do from a space-time perspective that is very difficult from the world-sheet point of view, namely to explain the integer structure and the corresponding canonical flat Gauss-Manin connection. Quite generally, the existence of the integer structure is related to charge quantization and charge lattices. Indeed, for a compactification on a Calabi-Yau X we have seen that the integer structure was related to the lattice $H_3(X, \mathbf{Z})$. This is the charge lattice for the the 4-form gauge field of the type IIB string theory. However, in perturbative string theory there are no objects that carry charge with respect to this field. The charged objects are 3-dimensional D-branes that can wrap around the 3-cycles of the Calabi-Yau, intrinsically non-perturbative objects that are invisible from the CFT perspective. So the D-branes finally complete our picture. In §13 their role in determining the BPS spectrum will be considered.

12.5. Singularities revisited

We have seen that the moduli spaces of string backgrounds might have singularities. These singularities have often a completely straightforward explanation. They simply parametrize the singular geometries of the target space X. There is no reason why a sigma model on such a singular space would have to make sense. However, once we realize that the same moduli space is now used to label vacua of string theory, we run into trouble. What will prevent the string of exploring these singular points? And what happens at these points, where CFT and thus string perturbation theory no longer makes sense?

Well, we have been at this point before. In the rigid special geometry solution of Seiberg and Witten we also met singularities. At these singularities the abelian gauge theory became infinite. But these infinities had a good explanation. They were the result of integrating out a nearly massless field, the monopole or dyon. If we had reinstated this degree of freedom, the model would make perfect sense at these points in moduli space.

It was the brilliant insight of Strominger [127] that precisely the same thing happens in string theory! For concreteness let's consider a compactification of the Type IIB theory on a Calabi-Yau space X. At a typical singularity the space X will develop a node. At this node a three-cycle C will shrink to zero, quite analogous to the shrinking of a one-cycle in the case of the SW solution. Is there an accompanying charged object that becomes massless? Yes, Type IIB string theory contains 3-branes that can wrap around the cycle C. There mass will be given (at least for BPS states) by the period of the holomorphic 3-form Ω,

$$M \sim \int_C \Omega. \tag{12.29}$$

So we see that in the singular limit $M \to 0$, and we have precisely the same situation. We have to introduce new degrees of freedom, describing the quantum fluctuations of this new light soliton.

12.6. String moduli spaces

What is the general structure for moduli spaces of inequivalent string vacua? In the low-energy limit string theory reduces to a particular supergravity model, and the vacuum manifolds for theories with local supersymmetry have been studied in great detail. In fact we always have the following ingredients:

(1) A moduli space $\mathcal{M}$ that carries certain geometric structures (special Kähler, hyperkähler, quaternionic) depending on the exact amount of supersymmetry.

(2) A charge lattice $\Gamma \cong \mathbf{Z}^r$ with rank r equal to the number of abelian gauge fields in the uncompactified dimensions[11].

(3) A duality group G that acts on Γ. In many cases this representation is actually irreducible.

(4) A homomorphism $\pi_1(\mathcal{M}) \to G$, that is, a (necessarily flat) Γ bundle over the moduli space $\mathcal{M}$.

(5) A BPS spectrum $\mathcal{H}_{BPS} \subset \mathcal{H}$.

Let us make some comments on this last ingredient. In supersymmetric theories the next step beyond the description of the vacua is the spectrum of BPS states. For extended supersymmetry there is a rich structure of such states. The BPS Hilbert space is graded

[11] Here the case $D = 4$ is special since there $*F$ is also a two form and thus an abelian gauge field gives rise to both electric and magnetic charges. Consequently, in four dimensions the rank of the lattice is twice the number of $U(1)$ gauge fields. Similarly, for $D = 5$ there is one extra generator.

by the charge lattice Γ (as is the full space $\mathcal{H}$)

$$\mathcal{H}_{BPS} = \bigoplus_{q\in\Gamma} \mathcal{H}^q. \tag{12.30}$$

The graded pieces $\mathcal{H}^q$ have typically finite dimensions

$$D(q) = \dim \mathcal{H}^q. \tag{12.31}$$

In fact, it is interesting to consider the corresponding entropy, defined as

$$S(q) = \log D(q). \tag{12.32}$$

Using relations with black hole physics, this entropy can be compared, for large charges q, to the so-called Bekenstein-Hawking [128] entropy S_{BH} that can be computed by classical (super)gravitational methods,

$$S(q) \stackrel{q\to\infty}{\longrightarrow} S_{BH}(q). \tag{12.33}$$

This gives an indication of the growth of the degeneracy of BPS states. In fact, in the interesting dimensions $d = 4, 5, 6$ one finds that

$$S(q) \sim \begin{cases} 2\pi|q|, & d = 6, \\ 2\pi|q|^{3/2}, & d = 5, \\ 2\pi|q|^4, & d = 4. \end{cases} \tag{12.34}$$

12.7. Example — Type II on T^6

As an example let us consider the compactification of the type IIA or IIB theory on the six-torus T^6. In that case one expects the following structure [15]:

The "classical" moduli space is given by the homogeneous space

$$\mathcal{M}_{cl} = E_{7(7)}/SU(8). \tag{12.35}$$

Here $E_{7(7)}$ indicates the maximally noncompact real version of the Lie group E_7, with 7 noncompact directions. The lattice Γ has rank 56 and it forms an irreducible representation of the U-duality group

$$G = E_{7(7)}(\mathbf{Z}) \tag{12.36}$$

a discrete group. The quantum moduli space is the quotient

$$\mathcal{M} = G\backslash\mathcal{M}_{cl} \tag{12.37}$$

The flat vector bundle is the obvious one.

The degeneracies $D(q)$ of BPS states are expected to grow with entropy [129]

$$S(q) = 2\pi\sqrt{Q_4(q)}, \tag{12.38}$$

where Q_4 is the unique quartic invariant of E_7 (that can be used to *define* E_7). We see that this is already an incredibly rich structure.

13. BPS states and D-branes

We have seen that the next step beyond describing the string vacua is the spectrum $\mathcal{H}_{BPS}$ of BPS states. How are these BPS vector spaces computed? This subject has seen a remarkable progress in the last year, after the introduction of Polchinsky's D-branes. Seminal papers in the field are, among others [125, 130, 131, 132, 133, 134]. There are two cases that are relatively well-understood, the perturbative states and the pure D-brane states, that we will now review.

13.1. Perturbative string states

Both in the type II and the heterotic string we can find BPS states in the perturbative string spectrum. These are by far the easiest states to understand. They are obtained by putting the right-handed bosonic and fermionic oscillators in their ground state. This is most easily done in the Green-Schwarz light-cone description of the physical Hilbert space (see [4]).

Recall that in this description the spectrum is generated by the following fields:

8 left-moving and 8 right-moving bosonic fields $x^i(z)$ and $x^i(\overline{z})$ that transform as a vector $\mathbf{8}_v$ under $SO(8)$;

8 integer-moded left-moving fermionic fields $S^\alpha(z)$ that transform as a spinor $\mathbf{8}_s$ of $SO(8)$;

8 integer-moded right-moving fermions $S^{\dot\alpha}(\overline{z})$ or $S^\alpha(\overline{z})$ that, depending on whether we are considering the type IIA or IIB theory, transform as a (conjugated) spinor $\mathbf{8}_c$ respectively $\mathbf{8}_s$.

Note that the three representations $\mathbf{8}_v$, $\mathbf{8}_s$, $\mathbf{8}_c$ of $SO(8)$ are related by triality (a necessary ingredient for space-time supersymmetry) and all carry an invariant real inner product.

These fields are quantized with the free action (written for the IIA case)

$$S = \frac{1}{\pi}\int d^2z \left(\tfrac{1}{2}\partial x^i\overline{\partial} x_i + S^\alpha\overline{\partial} S_\alpha + S^{\dot\alpha}\partial S_{\dot\alpha}\right). \tag{13.1}$$

For future use we introduce the explicit mode expansions

$$\begin{aligned} \partial x^i(z) &= \sum_k \alpha^i_k z^{-k-1}, \\ S^\alpha(z) &= \sum_k S^\alpha_k z^{-k}. \end{aligned} \tag{13.2}$$

Since the chiral fermions S^α are in the Ramond sector, their ground states form a representation of the Clifford algebra

$$\{S^\alpha_0, S^\beta_0\} = \delta^{\alpha\beta}. \tag{13.3}$$

If S^α would have transformed in $\mathbf{8}_v$, as is usually the case for a Clifford algebra, the spinor representation would have been $\mathbf{8}_s \oplus \mathbf{8}_c$. Since S^α transforms here as $\mathbf{8}_s$, one finds (after applying triality) that the R ground states give the representation $\mathbf{8}_v \oplus \mathbf{8}_c$. On the right-moving side, we have a similar picture, possibly with the interchange of chirality. In the GS formulation there is no GSO projection, therefore the ground states have the form

$$(\mathbf{8}_v \oplus \mathbf{8}_c) \oplus (\mathbf{8}_v \oplus \mathbf{8}_{c,s}) \tag{13.4}$$

The four terms in this product give the four sectors labeled (NS,NS) through (R,R) in the covariant approach. In particular $\mathbf{8}_v \otimes \mathbf{8}_v$ gives the light-cone modes of the $G_{\mu\nu}$ and $B_{\mu\nu}$ field.

The other, massive physical states are produced by acting with the creation operators α^i_{-n}, S^α_{-n} on these ground states. This gives a description of the Hilbert space tensor product of bosonic and fermionic Fock spaces both of rank 8. These left-moving and right-moving Fock spaces have a gradation by the number operators (N_L, N_R).

If we compactify on a torus T^n, we can fix momenta (p_L, p_R) in the Narain lattice $\Gamma^{n,n}$. The physical Hilbert space is then defined as the subspace satisfying the level-matching constraints

$$\tfrac{1}{2}p_L^2 + N_L = \tfrac{1}{2}p_R^2 + N_R. \tag{13.5}$$

Note that this can be written in the signature (n, n) inner product as

$$\tfrac{1}{2}p^2 = N_R - N_L. \tag{13.6}$$

The uncompactified momentum $k \in \mathbf{R}^{9-n,1}$ is fixed by the mass-shell condition

$$\tfrac{1}{2}k^2 + \tfrac{1}{2}p_L^2 + N_L = 0. \tag{13.7}$$

So for the type II string compactified on T^n the degeneracies $D(p)$ of physical states with momenta $p \in \Gamma^{n,n}$ are given by

$$D(p) = \sum_{n-m=p^2/2} d(n)d(m) \tag{13.8}$$

where the coefficients $d(k)$ are generated by

$$\sum_k d(k) q^k = 16 \prod_n \left(\frac{1+q^n}{1-q^n} \right)^8 \tag{13.9}$$

which is the character of 8 fermionic and 8 bosonic Fock spaces (the factor 16 counts the Ramond ground states).

For the heterotic string we tensor the left-moving supersymmetric spectrum-generating algebra we just described with 24 right-moving transverse bosonic oscillators $x_R^I(\overline{z})$. The compactified momenta (p_L, p_R) now take value in the extended Narain lattice

$$\Gamma^{n,n+16} \cong \Gamma^{n,n} \oplus (-E_8) \oplus (-E_8). \tag{13.10}$$

The right-moving degeneracies at oscillator level N_R are given by $c(N_R - 1)$ with

$$\sum c(k) q^k = \frac{1}{q \prod_n (1-q^n)^{24}} = \frac{1}{\eta(q)^{24}}, \tag{13.11}$$

the character of 24 bosonic Fock spaces. Because of the bosonic intercept (which produces massless states at level $N_R = 1$) the level-matching now gives

$$\tfrac{1}{2} p_L^2 + N_L = \tfrac{1}{2} p_R^2 + N_R - 1. \tag{13.12}$$

Note that this can be written as

$$\tfrac{1}{2} p^2 = N_R - N_L - 1. \tag{13.13}$$

So the full spectrum of perturbative heterotic string states with $p \in \Gamma^{n,n+16}$ takes the form

$$D(p) = \sum_{n-m=p^2/2} d(n) c(m). \tag{13.14}$$

13.2. Perturbative BPS states

For BPS states we simply put the left-moving supersymmetric oscillators in their ground states. That is we impose the conditions

$$\alpha_n^i |\mathrm{BPS}\rangle = S_n^\alpha |\mathrm{BPS}\rangle = 0, \qquad n > 0. \tag{13.15}$$

Why is this a BPS state? To understand this we have to consider the supersymmetry algebra. In the light-cone description of the physical states, the (left-moving) supersymmetry generators are given by (we put $p_+ = 1$)

$$\begin{aligned} Q^{\alpha} &= S_0^{\alpha}, \\ Q^{\dot\alpha} &= \oint \partial x^i \gamma_i^{\dot\alpha\beta} S_\beta. \end{aligned} \tag{13.16}$$

We can decompose $Q^{\dot\alpha}$ in zero-mode piece and a non-zero mode $Q_+^{\dot\alpha}$ piece as

$$Q^{\dot\alpha} = p^i \gamma_i^{\dot\alpha b} S_{0,\beta} + Q_+^{\dot\alpha}. \tag{13.17}$$

On ground states $Q_+^{\dot\alpha}$ will act trivially, by definition,

$$Q_+^{\dot\alpha}|\mathrm{BPS}\rangle = 0. \tag{13.18}$$

So, BPS states satisfy the condition

$$(\epsilon_\alpha Q^\alpha + \epsilon_{\dot\alpha} Q^{\dot\alpha})|\mathrm{BPS}\rangle = 0, \tag{13.19}$$

with

$$\epsilon^a = p^i \gamma_i^{\dot\alpha a} \epsilon_{\dot\alpha}. \tag{13.20}$$

Precisely the same argument holds for the heterotic string. So we see that 8 of the 16 left-moving supercharges annihilate these states. Consequently, for the type II and the heterotic strings these are 1/4 and 1/2 BPS states respectively.

The degeneracies of these BPS states is entirely given by the right-moving states, that can be arbitrary as long as level matching is satisfied. So we find that the number $D(p)$ of perturbative BPS states with given charge p are given simply by

$$D(p) = \begin{cases} d(\frac{1}{2}p^2), & \textit{type II}, \\ c(\frac{1}{2}p^2), & \textit{heterotic}. \end{cases} \tag{13.21}$$

13.3. D-brane states

A second class of BPS states that are now reasonable well-understood are the D-brane states, that can appear in the type II and type I string. We will concentrate on the type II states here.

In general these D-brane BPS states are believed to appear as follows (see *e.g.* [134]) In a compactification on a Calabi-Yau space X there exist special embedded subspaces

$C \subset X$ called supersymmetric cycles. To such a cycle C we can associate a charge q in the homology lattice

$$q = [C] \in H_*(X, \mathbf{Z}). \tag{13.22}$$

We can further associate to the cycle C a moduli space $\mathcal{M}_C$ that parametrizes the inequivalent supersymmetric embeddings. The BPS states should appear by "quantizing" this classical moduli space $\mathcal{M}_C$. In a naive way quantization means computing the ground states of the supersymmetric particle moving on the moduli space. These ground states are represented as harmonic differential forms. So roughly we have

$$\mathcal{H}_q \approx H^*(\mathcal{M}_C) \tag{13.23}$$

However, there are various subtleties in this reasoning that we will partly explain and partly sweep under the rug (see *e.g.* [135, 136] for a critical analysis).

What kind of cycles are appropriate? Here we have to distinguish between the type IIA and IIB theories. Recall that we had two sets of bosonic fields: NS fields and RR fields. Consequently we have a decomposition of the charge lattice as

$$\Gamma = \Gamma_{NS} \oplus \Gamma_{RR} \tag{13.24}$$

In the NS sector we find the graviton $G_{\mu\nu}$, 2-form $B_{\mu\nu}$ and the dilaton Φ. In the RR sector we have a series of generalized k-forms gauge fields $A^{(k)}$, where k can take all odd or even values in the IIA or IIB theory respectively. Moreover the total RR curvature

$$F = \sum dA^{(k)} \tag{13.25}$$

satisfied the self-duality condition $*F = F$.

If we compactify on X we can have abelian gauge fields out of both sectors. The NS sector only gives a contribution if X is the product with a torus T^n. Then we have the following charges: n momenta (from the metric G) and n winding numbers (from the 2-form field B) giving[12]

$$\Gamma_{NS} = \Gamma^{n,n}. \tag{13.26}$$

In the RR sector we produce after compactification an abelian gauge field out of a $(p+1)$ form A^{p+1} for every element $C_p \in H_p(X, \mathbf{Z})$. The "push-down"

$$A = \int_{C_p} A^{p+1} \tag{13.27}$$

[12] There are two exceptions to this rule: When we compactify down to five dimensions we can dualize the B field to give one more magnetic charge. In compactifications to four dimensions all gauge fields can be dualized, doubling the lattice.

gives a one-form in the uncompactified space-time. So the RR charge lattice is given by

$$\Gamma_{RR} = \begin{cases} H_{even}(X), & \text{type IIA}, \\ H_{odd}(X), & \text{type IIB}. \end{cases} \tag{13.28}$$

In the type IIA theory the most general D-brane state with charge

$$p \in H_{even}(X) \tag{13.29}$$

is realized as a sum of irreducible even-dimensional cycles C_i with multiplicity n_i

$$p = \sum_i n_i [C_i]. \tag{13.30}$$

In the Type IIA theory we demand that the cycles $C \subset X$ are holomorphically embedded for a complex structure on X compatible with the given Ricci flat metric [137]. We further need the additional date of a choice of holomorphic vector bundle $E_i \to C_i$ of rank n_i over each irreducible component. There is a concrete proposal for what the relevant moduli space $\mathcal{M}_C$ of such cycles should be: the moduli space of simple, semi-stable coherent sheaves $\mathcal{S}$ with Mukai vector [135, 136]

$$Ch(\mathcal{S}) \sqrt{\mathrm{td}\, X} = p. \tag{13.31}$$

In the type IIB theory we are dealing with odd dimensional cycles. Let us assume for a moment that we are dealing with a proper Calabi-Yau three-fold that is simply connected, so that $H_{odd}(X) = H_3(X)$. The correct notion of a supersymmetric three-cycle is now special Lagrangian [137, 138, 139].

A special Lagrangian L satisfies two conditions [140]: (1) it is a Lagrangian, *i.e.* the Kähler symplectic form vanishes when restricted to L, and (2) it minimalizes the volume in its homology class. More precisely, it has the property that (with an appropriate phase factor $e^{i\theta}$) the period of the holomorphic three-form Ω satisfies

$$\int_L e^{i\theta} \Omega = \mathrm{vol}(L). \tag{13.32}$$

For an irreducible BPS cycle we also have to pick a line bundle over L, as we did in the IIA case. The special Lagrangian calibration was introduced because its properties closely match those of holomorphic cycles that we met on the IIA side. In fact, Strominger, Yau and Zaslow [138] have been able to proof that the moduli space $\mathcal{M}_L$ has some very nice properties. It is always a Kähler manifold, even a Calabi-Yau space, and its complex dimension is given by

$$\dim_{\mathbf{C}} \mathcal{M}_L = b_1(L). \tag{13.33}$$

We will now consider a few interesting examples of compactifications where we know how to compute the BPS spectra.

13.4. Example — Type IIA on $K3$ = Heterotic on T^4

Our first example concerns the compactification of the Type IIA theory on $K3$ [132, 131]. By string duality this is equivalent to a compactification of the heterotic string on the four-torus. This compactification produces a supersymmetric theory in six dimensions with $N = 2$ supersymmetry. The charge lattice is of signature $(4, 20)$ and even, self-dual

$$\Gamma = \Gamma^{4,20}. \tag{13.34}$$

This lattice can be understood as the Narain lattice from the point of view of the heterotic string.

Viewed from the type IIA theory it is the (even) homology lattice of a $K3$ surface X

$$\Gamma = H_*(X, \mathbf{Z}), \tag{13.35}$$

which labels the RR charges. It indeed carries an intersection form that is even and self-dual. (There are no perturbative charged states in this compactification.) The duality group is

$$G = O(4, 20, \mathbf{Z}) = \mathrm{Aut}\,(\Gamma). \tag{13.36}$$

In the heterotic description these transformations are all perturbative T-duality symmetries. In the $K3$ compactification they correspond to quantum automorphisms of the $K3$ manifold [141].

For $K3$ surfaces the quantum moduli space is completely known [141]. The duality group is

$$G = O(4, 20, \mathbf{Z}), \tag{13.37}$$

which is the automorphism group of the unique even, self-dual lattice of signature $(4, 20)$

$$\Gamma^{4,20} = H \oplus H \oplus H \oplus (-E_8) \oplus (-E_8). \tag{13.38}$$

The quantum moduli space is given by the Narain space

$$\mathcal{M}_{K3} = G \backslash O(4, 20, \mathbf{R})/O(4, \mathbf{R}) \times O(20, \mathbf{R}). \tag{13.39}$$

One of the implication of string duality is that both compactifications should also have the same moduli space, which is indeed the case, since the above Narain space is well-known to classify the heterotic compactifications (see the discussion in §7.4).

The BPS spectrum has a straightforward perturbative description in terms of the heterotic string, it is simply a Fock space built on the lattice Γ. So the degeneracy $D(p)$ for a state with charge $p \in \Gamma$ is given by

$$D(p) = c(\tfrac{1}{2}p^2) \tag{13.40}$$

with generating function η^{-24}, see (13.11). According to the string duality hypothesis, these states should have an alternative description in terms of quantizing D-branes on $K3$.

This idea has been in fact (partly) confirmed. There are various cases where the moduli space of D-branes can be quantized and does give the required degeneracies. We have seen that in IIA theory a general D-brane configuration is a set of holomorphic[13] subvarieties $C_i \subset X$ with multiplicities n_i

$$p = \sum_i n_i [C_i], \tag{13.41}$$

and the moduli space $\mathcal{M}_C$ of such cycles is some set of appropriate sheaves. The strong version of string duality says that any two elements C, D of $H_*(X)$ with the same self-intersection number give rise to isomorphic moduli spaces

$$\mathcal{M}_C \cong \mathcal{M}_D \tag{13.42}$$

related by the duality group $O(4, 20, \mathbf{Z})$. Of course for the group $O(3, 19, \mathbf{Z})$, which is the image of $\mathrm{Diff}(X)$ in the second cohomology group, this is a trivial statement. As it stands, the strong duality statement is definitely not true with the above definition of $\mathcal{M}_C$ in terms of coherent sheaves, it most likely needs quantum corrections.

One can show that under certain conditions the moduli space $\mathcal{M}_C$ is isomorphic to the Hilbert scheme $X^{[N]}$ of subschemes of length N on an algebraic $K3$ surface X,

$$\mathcal{M}_C \cong X^{[N]}, \qquad N = \tfrac{1}{2}p^2. \tag{13.43}$$

Crucial in the correspondence with the heterotic description is the result of Göttsche [142], who computed the Hodge numbers of these Hilbert schemes. The Hilbert scheme can be understood as a resolution of the orbifold symmetric product spaces

$$X^{[N]} \xrightarrow{\pi} S^n X = X^N / S_N. \tag{13.44}$$

[13]Holomorphic for a complex structure on X that is compatible with the hyperkähler structure that is parametrized by the moduli space $\mathcal{M}_X$.

For algebraic surfaces, the Hilbert scheme is always smooth and its cohomology can be computed unambiguously. For the symmetric products we can do a similar computation, but we must be careful that we use the appropriate orbifold Euler character [143, 106]. Anyhow, the result is the generating function

$$\sum_{N\geq 0} q^N \chi(S^N X) = \prod_{n>0} \frac{1}{(1-q^n)^{24}} = \frac{q}{\eta(q)^{24}}. \tag{13.45}$$

There is a clear intuitive picture why we get a Fock space. Let α^μ_{-1} ($\mu = 1, \ldots, 24$) be a basis of $H_*(X)$. The suffix $_{-1}$ will be explained shortly. Now there are obvious cohomology classes that we can construct in the N-th symmetric product of X, one simply takes symmetric products of the classes on X

$$\alpha^{\mu_1}_{-1} \cdots \alpha^{\mu_N}_{-1}. \tag{13.46}$$

If this was everything the generating function (13.45) would read $1/(1-q)^{24}$. However, there are an infinity of mirror images of the classes α^μ_{-1}. In the diagonal X in the symmetric product S^2X we find a second copy of this class, denoted as α^μ_{-2}. Similarly, if a point on X^N is left invariant by a cycle of length n we get a class α^μ_{-n}. The (orbifold) cohomology is the symmetric algebra on all these generators (creation operators) α^μ_{-n}, *i.e.* a Fock space.

One case where the correspondence with Hilbert schemes is made rather easy, is the situation of N zero-branes and one four-brane. In that case we have $p^2 = 2N$, and the moduli space is indeed given by

$$\mathcal{M}_C = X^{[N-1]}, \tag{13.47}$$

so that we compute

$$D(p) = \chi(X^{[N-1]}) = c(N), \tag{13.48}$$

which checks with the heterotic prediction $c(p^2/2)$.

A particular other interesting case has been studied in [134] and by Yau and Zaslow in [144], namely the case that C is an irreducible curve of genus g with class $p = [C] \in H_2(X)$. In that case the adjunction formula expresses the genus as

$$p^2 = 2g - 2. \tag{13.49}$$

The moduli space $\mathcal{M}_C$, that parametrizes such holomorphic curves together with a choice of a holomorphic line bundle, is fibered over the local system $\mathbf{P}^g$ that parametrizes just the family of curves

$$\mathcal{M}_C \stackrel{\pi}{\rightarrow} \mathbf{P}^g. \tag{13.50}$$

The fiber of this map is the Jacobian Jac(C) of line bundles on C. This Jacobian is generically a torus, $\mathrm{Jac}(C) \cong T^{2g}$. Therefore the Euler character $\chi(\mathcal{M}_C)$ seems naively to vanish, since the Euler number of the fiber is zero. There is a catch in this argument, however, if the curve is singular and degenerates to a curve of lower genus with nodes. Over these exceptional subspaces the dimension of the fiber drops. Now in the case of a total degeneracy to a genus zero curve, the fiber is a point. Only these points, corresponding to nodal rational curves, can contribute to the Euler number. Therefore there is a beautiful prediction from string duality for the number of rational curves with g nodes on a $K3$ surface: it is given by the coefficient $c(g-1)$ of the generating function $1/\eta^{24}$. This checks with the explicit computations that have been done for low number of nodes [144].

13.5. Example — Type II on T^4

Our second example concerns the compactification of the Type IIA or IIB string on a four-torus. (T-duality will relate the two types.) This model has $N = 4$ supersymmetry in six dimensions. It has both NS and RR charged states. In fact, the duality group is [15]

$$G = O(5, 5, \mathbf{Z}), \tag{13.51}$$

that acts on the rank 16 charge lattice as a chiral **16** *spinor* representation. Of course, there is a perturbative T-duality subgroup

$$T = O(4, 4, \mathbf{Z}) \subset G, \tag{13.52}$$

under which the charge lattice decomposes as

$$\Gamma = \Gamma_{NS} \oplus \Gamma_{RR}. \tag{13.53}$$

Here both lattices are isomorphic to the even, self-dual Narain lattice $\Gamma^{4,4}$. For the NS lattice this is true by construction. Note that it transforms as a $\mathbf{8}_v$ under the T-duality group. The RR lattice we obtain (say in the Type IIA) theory) as

$$\Gamma_{RR} = H_{even}(T^4, \mathbf{Z}). \tag{13.54}$$

This has indeed an even, self-dual intersection product of signature $(4, 4)$, as we have seen in §9.1. So we also obtain $\Gamma_{RR} \cong \Gamma^{4,4}$. Note however that this lattice transforms as a spinor $\mathbf{8}_s$ under the T-duality group $O(4, 4, \mathbf{Z})$. (The Type IIB lattice $H_{odd}(T^4, \mathbf{Z})$ transforms as the conjugate spinor $\mathbf{8}_c$.)

Now we can make two computations. The perturbative states, with $p \in \Gamma_{NS}$, have a direct computable degeneracy in terms of a left-moving superstring partition function

$$D(p) = d(p^2/2), \tag{13.55}$$

with generating function (13.9). The full duality group G allows us to transform these perturbative states in pure D-brane states with $p \in \Gamma_R$. Here we have to make a similar computation as we did for the case of $K3$.

There is a conjecture with ample support that the relevant moduli space of torsion free coherent sheaves is indeed always a hyperkähler deformation of the Hilbert scheme or symmetric product $S^N T^4$ [145]. The relevant formula is now that the cohomology of these symmetric products produces the chiral superstring partition function. This is indeed the case, since

$$\sum_{N \geq 0} q^N \chi(S^N T^4) = \prod_{n>0} \left(\frac{1+q^n}{1-q^n} \right)^8 . \tag{13.56}$$

This is again a powerful check on string duality.

For a general state with charge p in the full charge lattice $\Gamma = \Gamma_{NS} \oplus \Gamma_R$ the degeneracy is given by the following formula [146]. There is natural bilinear map $\mathbf{16} \otimes \mathbf{16} \to \mathbf{10}$ in $O(5,5,\mathbf{Z})$ given by the Dirac matrices. So $\frac{1}{2}p^2$ is naturally a ten-dimensional vector in the Narain lattice

$$\tfrac{1}{2}p^2 \in \Gamma^{5,5}, \tag{13.57}$$

with components $\frac{1}{2}p^\alpha \gamma^\mu_{\alpha\beta} p^\beta$, $\mu = 1, \ldots, 10$. In general such a vector will be a multiple N of a primitive vector. The complete duality invariant formula for the degeneracy is now

$$D(p) = d(N). \tag{13.58}$$

13.6. Example — Type II on $K3 \times S^1$ = Heterotic on T^5

Our following example is a compactification down to five dimensions, best understood first from the heterotic perspective. In a compactification on T^5 we have clearly a Narain lattice $\Gamma^{5,21}$ of perturbative charges. However, the full charge lattice is one dimension higher,

$$\Gamma = \Gamma^{5,21} \oplus \mathbf{Z}, \tag{13.59}$$

because of the following reason. The B field, which has a three-form field strength, can be dualized in five dimensions to produce an extra gauge field A with $dA = *dB$. We will denote this extra charge as m. States with non-zero m charge are magnetic monopoles for the B field. Elementary string states can only carry electric B field charge. So, a generically charged BPS state with non-zero magnetic charge m cannot be described in terms of perturbative heterotic string states and we are stuck. (States with $m = 0$ can of course be described in CFT terms, but they are special. For example, their degeneracies do not grow as fast as the generic states with $m \neq 0$.)

On the other hand, there is a nice description of these states from the type IIA formulation in terms of strings and D-brane states on the manifold $S^1 \times K3$. Here we have a different decomposition of the charge lattice [147]. From the NS charges we first have the $\Gamma^{1,1}$ Narain lattice associated to the S^1 factor. The type IIA B-field gives an additional magnetic charge

$$\Gamma_{NS} = \Gamma^{1,1} \oplus \mathbf{Z} \tag{13.60}$$

In the type IIA formulation the RR charges come from the homology of the $K3$ factor,

$$\Gamma_{RR} = H_*(K3, \mathbf{Z}) \cong \Gamma^{4,20} \tag{13.61}$$

We first consider the moduli space associated to a D-brane state with R charge $p_R \in \Gamma_{RR}$. As we have seen, this moduli space is conjectured to be isomorphic to (a hyperkähler resolution) of the symmetric product $S^N K3$, with $N = \frac{1}{2}p_R^2 - 1$. In section §13.4 we computed the cohomology of this moduli space, or in physical terms we considered supersymmetric quantum mechanics on it. Now we are dealing with a fiber bundle $K3 \times S^1$. In the limit that the $K3$ fiber is much smaller then the circle[14], we might assume by an adiabatic argument that the D-branes are supersymmetric for each point of the circle. In this way we obtain a map

$$S^1 \to S^N K3, \tag{13.62}$$

i.e. an element of the loop space $\mathcal{L}(S^N K3)$. We now want to quantize this loop space. Note that it has a natural circle action, corresponding to rotations of the S^1, and the appropriate object to compute will be the S^1-equivariant Euler character of the loop space. In physics terms this means we have to consider a two-dimensional $N = 2$ supersymmetric sigma model with as target space this symmetric product [147].

BPS states will correspond to states that are in right-moving ground states. For any target space X, the spectrum of such states is computed by the elliptic genus [148]

$$\chi(X; q, y) = \mathrm{Tr}_{\mathcal{H}} \left((-1)^F y^{F_L} q^{L_0 - \frac{c}{24}} \right), \tag{13.63}$$

where $\mathcal{H}$ is the Hilbert space of the sigma model with RR boundary conditions on the fermions. If X is a Calabi-Yau space of complex dimension, $\chi(X; q, y)$ has beautiful modular properties. It is a weak Jacobi form of weight zero and index $d/2$ (with, as usual, $q = e^{2\pi i \tau}$, $y = e^{2\pi i z}$). This implies that the elliptic genus has an expansion in non-negative powers of q

$$\chi(X; q, y) = \sum_{n \geq 0,\ k} c(n, k) q^n y^k. \tag{13.64}$$

The coefficients $c(n, k)$ have a topological definition in terms of characteristic classes on X. One result we will need is the elliptic genus of $K3$. It has many representations. One

[14]Recall that we are computing an integer topological invariant, so we are allowed to pick a convenient metric.

that is easy to derive in the orbifold limit $K3 = T^4/\mathbf{Z}_2$ is the following expression in classical Jacobi theta functions

$$\chi(K3; q, y) = 8 \sum_{\alpha=2,3,4} \left(\frac{\vartheta_\alpha(\tau, z)}{\vartheta_\alpha(\tau, 0)} \right)^2 \tag{13.65}$$

Since $K3$ is a CY space, all its symmetric products will also be CYs. We can thus consider their elliptc genera. In fact, there is a nice formula for the generating function of the elliptic genera of the symmetric products, which generalizes Göttsche's result [149]

$$\sum_{N \geq 0} p^N \chi(S^N K3; q, y) = \prod_{n>0,m,k} (1 - p^n q^m y^k)^{-c(nm,k)}. \tag{13.66}$$

Here $c(m, k)$ are the expansion coefficients of the elliptic genus of $K3$. Let us expand the RHS in Fourier coefficients

$$\sum_{n,m,k} d(n, m, k) p^n q^m y^k. \tag{13.67}$$

According to the D-brane picture of the BPS states the final formula for the degeneracy of a BPS state with $p_R \in \Gamma_R$ and magnetic charge m and spin J is now given in terms of the elliptic genera of the symmetric products $S^N K3$ as

$$d(\tfrac{1}{2} p_R^2, m, J). \tag{13.68}$$

Using the above explicit formula, one can evaluate the asymptotic of these degeneracies and finds the expected entropy [150]

$$S(p_R, m, J) \sim 2\pi \sqrt{\tfrac{1}{2} m p_R^2 - J^2} \tag{13.69}$$

13.7. Example — Type IIA on X = Type IIB on Y

In our last example we compactify the IIA or IIB theory on a Calabi-Yau three-fold to four dimensions with $N = 2$ supersymmetry.

One of the most interesting applications of D-brane BPS states has been to the issue of mirror symmetry. The usual statement of mirror symmetry tells us that the Type IIA theory on a Calabi-Yau three-fold X is equivalent to the Type IIB theory compactified on a dual Calabi-Yau three-fold Y, with X and Y related by the mirror map. Indeed, we have seen how Type IIA and Type IIB were simply related by flipping the sign of the right-moving $U(1)$ charge. The two string theories are believed to be equivalent in the perturbative sector, which is the only thing CFT arguments will teach us anything about. It is no more than natural to conjecture that this equivalence extends to the nonperturbative sectors, and in particular to the D-brane BPS states.

Now we have also seen that the description of D-brane states was very different in the IIA and IIB theories. In the IIA setup we had to find a holomorphic subvariety $C \subset X$ with charge

$$[C] \in H_{even}(X). \tag{13.70}$$

The actual BPS states were obtained by quantizing the moduli space $\mathcal{M}_C$ associated to C. This moduli space is most likely the space of some appropriate sheaves, possibly with "quantum" corrections.

In the Type IIB theory the same BPS states should arise by quantizing the moduli space $\mathcal{M}_L$ that parametrizes the inequivalent special Lagrangians $L \subset Y$ with charge

$$[L] \in H_3(Y). \tag{13.71}$$

Now the "strong mirror conjecture" claims that not only do the BPS state spaces obtained from $\mathcal{M}_C$ and $\mathcal{M}_L$ agree, the actual moduli spaces should be isomorphic

$$\mathcal{M}_C \cong \mathcal{M}_L. \tag{13.72}$$

This conjecture has an immediate and powerful implication [138]: it allows us to directly construct the manifold X from the geometry of the mirror manifold Y. The argument is extremely simple and clear: There is one very special supersymmetric cycle on the IIA manifold X — a point $P \in H_0(X)$. The moduli space all possible "zero-branes" P is of course the original space X itself,

$$\mathcal{M}_P = X. \tag{13.73}$$

So if mirror symmetry holds this means that on the IIB manifold Y there must be one very special Lagrangian three-cycle L with the property that

$$\mathcal{M}_L \cong X. \tag{13.74}$$

The moduli space of L must therefore be a Calabi-Yau three-fold. As we mentioned the complex dimension of $\mathcal{M}_L$ is given by $b_1(L)$. So this condition forces the three-cycle L to be a three-torus

$$L \cong T^3. \tag{13.75}$$

So we find that, in order for a mirror pair (X, Y) to exist, the Calabi-Yau space Y must allow a fibration by three-tori over some (real) three-dimensional base manifold B,

$$Y \xrightarrow{\pi} B, \qquad \pi^{-1}(x) \cong T^3. \tag{13.76}$$

B parametrizes the different special Lagrangian embeddings of L. The full moduli space $\mathcal{M}_L$ also takes into account the choice of a line bundle on L. But the space of line bundles on a torus $T^3 L$ is nothing but the Jacobian or dual torus $Jac(T^3) = \widehat{T}^3$. So this gives $\mathcal{M}_L$ and thus also the IIA manifold X as a dual fibration

$$X = \mathcal{M}_L \xrightarrow{\widehat{\pi}} B, \qquad \widehat{\pi}^{-1}(x) \cong \widehat{T}^3. \tag{13.77}$$

Now going from T^3 to the dual torus $\widehat{T}^3$ is nothing but a T-duality, which explains the title of [138] "mirror symmetry is T-duality." This picture has been confirmed in examples [151].

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