A Relativistic Equation for Bound-State Problems

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The relativistic S-matrix formalism of Feynman is applied to the bound-state problem for two interacting Fermi-Dirac particles. The bound state is described by a wave function depending on separate times for each of the two particles. Two alternative integral equations for this wave function are derived with kernels in the form of an expansion in powers of g^2, the dimensionless coupling constant for the interaction. Each term in these expansions gives Lorentz-invariant equations. The validity and physical significance of these equations is discussed. In extreme nonrelativistic approximation and to lowest order in g^2 they reduce to the appropriate Schrödinger equation.

One of these integral equations is applied to the deuteron ground state using scalar mesons of mass μ with scalar coupling. For neutral mesons the Lorentz-invariant interaction is transformed into the sum of the instantaneous Yukawa interaction and a retarded correction term. The value obtained for g^2 differs only by a fraction proportional to $(\mu/M)^2$ from that obtained by using a phenomenological Yukawa potential. For a purely charged meson theory a correction term is obtained by a direct solution of the relativistic integral equation using only the first term in the expansion of the kernel. This correction is due to the fact that a nucleon can emit, or absorb, positive and negative mesons only alternately. The constant g^2 is increased by a fraction of $1.1(\mu/M)$ or 15 percent.

I. INTRODUCTION

VARIOUS mathematical formalisms, all equivalent physically, for a relativistic treatment of scattering problems for two or more particles have been developed in the last few years by Dyson, Feynman, Schwinger, and Tomonaga. It is the aim of this paper to present an extension of the Feynman formalism[1] to bound-state problems involving several particles. Throughout this paper we shall deal only with two Fermi-Dirac particles interacting with each other by means of an arbitrary (electrodynamic or mesonic) interaction in the absence of any external forces, particles, or quanta. The extension of the method to more complex bound states should, in principle, be straightforward, although the computational work might well become prohibitive.

For a scattering problem involving two such particles, the Feynman formalism consists essentially of giving a prescription for the amplitude function (or kernel) $K(3,4;1,2)$, representing the probability amplitude for the one particle to proceed from time and place $x_{\mu 1}$ to $x_{\mu 3}$, while the other particle proceeds from time and place $x_{\mu 2}$ to $x_{\mu 4}$. This meaning of the amplitude function suggests that we should describe a state of the system by a wave function $\psi(x_{\mu 1}, x_{\mu 2})$ with 16 spinor components, depending on a separate time for each of the particles as well as on the spatial coordinates of the two particles. Such an approach bears some similarity to the many-times formalism of Dirac, Fock, and Podolski[2] and of Bloch.[3] In a purely formal way $\psi(3,4)$ can be calculated, by the help of $K(3,4;1,2)$ and prescriptions given in FI and II, if $\psi(1,2)$ is known. $K(3,4;1,2)$ is defined in terms of a doubly infinite series, but a relatively simple integral equation can be derived for it. From this an integral equation can be derived for $\psi(3,4)$ with an inhomogeneous term depending on $\psi(1,2)$. $K(3,4;1,2)$ also obeys a differential-integral equation from which a homogeneous differential-integral equation can be derived for $\psi(3,4)$ which does not contain any boundary conditions for t_1 and t_2 explicitly.

These equations are derived in Sec. II and their validity discussed in Sec. III. These equations are applied to the deuteron ground state using a scalar meson theory with scalar interaction in Secs. IV and V.

II. DERIVATION OF THE EQUATIONS

Throughout this paper we shall use, wherever possible, the notation of FI and FII and shall put $\hbar=c=1$. We first give a formal derivation of two equations for the wave function $\psi(1,2)$ and the amplitude function $K(3,4;1,2)$ based on the expression given in FII for $K(3,4;1,2)$ and postpone a discussion of the validity of these equations till the following section.

Consider two Fermi-Dirac particles of masses m_a and m_b, respectively (electrons or nucleons), capable of interacting with each other through "the virtual emission and absorption of quanta" (photons or mesons). Let $G(1,2)$ be the interaction function corresponding to a simple exchange of one quantum written in its lorentz invariant form (proportional to a dimensionless coupling constant g^2). For the case of electrodynamics we have

$$G(1,2)=e^2\gamma_\mu{}^a\gamma_\mu{}^b\delta_+(s_{12}{}^2). \tag{1}$$

For scalar mesons with scalar coupling $G(1,2)$ is the relativistic generalization of the Yukawa potential, etc. (for further details see Sec. 10 of FII). We now define reducible and irreducible graphs in a manner similar to that used by Dyson;[4] i.e., we call a graph reducible if it can be split into two simpler graphs by drawing a line which cuts no quantum lines at all and each of the two particle lines only once. The remaining irreducible graphs can be ordered according to the power of the

[1] R. P. Feynman, Phys. Rev. **76**, 749 (1949), hereafter referred to as FI, and R. P. Feynman, Phys. Rev. **76**, 769 (1949), hereafter referred to as FII.
[2] Dirac, Fock, and Podolski, Physik. Z. d. Sowjetunion **2**, 468 (1932).
[3] F. Bloch, Physik. Z. d. Sowjetunion **5**, 301 (1934).
[4] F. J. Dyson, Phys. Rev. **75**, 486 and 1736 (1949).

coupling constant g^2 occurring in the expression we ascribe to the graph, this power being half the number of particle vertices in the graph. For the first power of g^2 we have only one graph, labeled 1 in Fig. 1, to which corresponds the expression $G(1,2)$. We define two more quantities G' and $G^{(1)}$

$$\begin{aligned} G(1,2)&=\Gamma_{a\tau}\Gamma_{b\tau}G'(1,2),\\ G^{(1)}(1,2;3,4)&=G(1,2)\delta^{(4)}(1,3)\delta^{(4)}(2,4), \end{aligned} \tag{2}$$

where $\delta^{(4)}(1,3)$ is the four-dimensional delta-function

$$\prod_{\mu=1}^{4}\delta(x_{\mu 1}-x_{\mu 3})$$

and Γ_a, a "vertex part," is an operator made up of Dirac matrices operating on particle a only, while Γ_b operates only on particle b. The index τ represents summation over different operator components (4 for electrodynamics; 1, 4 or 16 for simple meson theories, etc.); $G'(1,2)$, the "quantum propagator," is a function of $(x_{\mu 1}-x_{\mu 3})$ not containing any Dirac operators. To the other irreducible graphs we ascribe mathematical expressions $G^{(n)}(1,2;3,4)$. The expressions corresponding to the three graphs $2A$, $2B$, and $2C$, shown in Fig. 1, are

$$\begin{aligned} G^{(2A)}(1,2;3,4)=&-i\Gamma_{a\sigma}\Gamma_{b\tau}K_{+a}(3,1)K_{+b}(4,2)\\ &\times\Gamma_{a\tau}\Gamma_{b\sigma}G'(1,4)G'(2,3), \end{aligned} \tag{3a}$$

$$\begin{aligned} G^{(2B)}(1,2;3,4)=&-i\int d\tau_5\delta^{(4)}(2,4)\\ &\times\Gamma_{a\sigma}K_{+a}(3,5)\Gamma_{a\tau}K_{+a}(5,1)\\ &\times\Gamma_{a\sigma}\Gamma_{b\tau}G'(1,3)G'(2,5), \end{aligned} \tag{3b}$$

$$\begin{aligned} G^{(2C)}(1,2;3,4)=&-i\int d\tau_5\delta^{(4)}(2,4)\\ &\times\Gamma_{a\sigma}K_{+a}(3,5)\Gamma_{a\sigma}K_{+a}(5,1)\\ &\times\Gamma_{a\tau}\Gamma_{b\tau}G'(3,5)G'(1,2), \end{aligned} \tag{3c}$$

where $K_{+a}(1,3)$ is the amplitude function for the propagation of particle a as a free particle. The expressions corresponding to graphs $2C$ and $2D$, self-energy and mass-renormalization terms respectively, must be taken together with that for $2B$. As an example of more complicated graphs we choose $3A$:

$$\begin{aligned} G^{(3A)}(1,2;3,4)=&-\int\int d\tau_5 d\tau_6\Gamma_{a\sigma}K_{+a}(3,5)\\ &\times\Gamma_{a\tau}K_{+a}(5,1)\Gamma_{a\rho}\Gamma_{b\rho}K_{+b}(4,6)\\ &\times\Gamma_{b\sigma}K_{+b}(6,2)\Gamma_{b\tau}G'(3,6)\\ &\times G'(5,2)G'(1,4). \end{aligned} \tag{4}$$

The prescription for the general term $G^{(n)}(1,2;3,4)$ should be obvious from these few examples. It should be noted that, no matter how complicated the diagram (n), $G^{(n)}$ depends explicitly only on the four variables $x_{\mu 1}$, $x_{\mu 2}$, $x_{\mu 3}$, $x_{\mu 4}$.

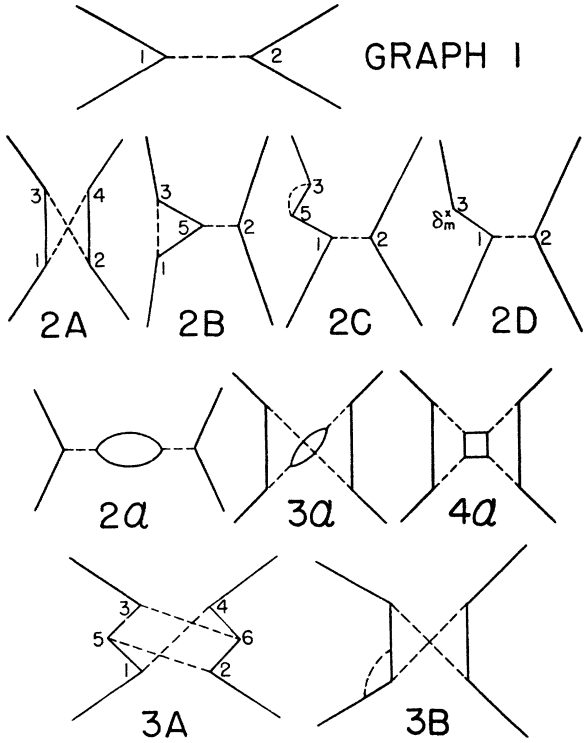

FIG. 1. Irreducible graphs of various orders. Solid lines denote Fermi-Dirac particles, broken lines denote quanta.

The expression given in FII for the amplitude function $K(3,4;1,2)$ consists of a doubly infinite sum of terms, one corresponding to each of the possible reducible or irreducible graphs, plus the term $K_{+a}(3,1)K_{+b}(4,2)$ corresponding to the propagation of both particles without the exchange of any quanta. For an irreducible graph (n) we have a term

$$\begin{aligned} K^{(n)}(3,4;1,2)=&-i\int\int\int\int d\tau_5 d\tau_6 d\tau_7 d\tau_8\\ &\times K_{+a}(3,5)K_{+b}(4,6)G^{(n)}(5,6;7,8)\\ &\times K_{+a}(7,1)K_{+b}(8,2). \end{aligned} \tag{5}$$

For a reducible graph which can be split up into two irreducible graphs, (n) and (m), respectively, we have a term

$$\begin{aligned} K^{(n,m)}(3,4;1,2)=&-i\int\int\int\int d\tau_5 d\tau_6 d\tau_7 d\tau_8\\ &\times K_{+a}(3,5)K_{+b}(4,6)G^{(n)}(5,6;7,8)\\ &\times K^{(m)}(7,8;1,2). \end{aligned} \tag{6}$$

For a graph reducible into 3 irreducible graphs we have

$$\begin{aligned} K^{(n,m,k)}(3,4;1,2)=&-i\int\int\int\int d\tau_5 d\tau_6 d\tau_7 d\tau_8\\ &\times K_{+a}(3,5)K_{+b}(4,6)G^{(n)}(5,6;7,8)\\ &\times K^{(m,k)}(7,8;1,2), \end{aligned} \tag{7}$$

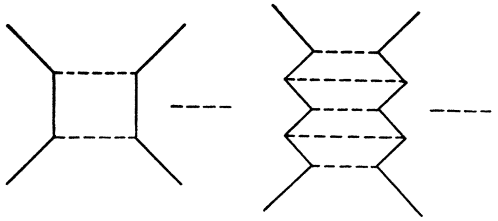

FIG. 2. Reducible graphs whose effect is included in the single term $G^{(1)}$.

and so on, for any reducible graph. Let $\bar{G}(1,2;3,4)$ be the sum of all expressions $G^{(n)}(1,2;3,4)$ for all *irreducible* graphs:

$$\bar{G}=\{G^{(1)}+G^{(2A)}+G^{(2B)}+G^{(2C)}+G^{(2D)}+G^{(2\alpha)}+G^{(3A)}+\cdots\}. \quad (8)$$

In Sec. 3 of FI an integral equation with a simple kernel is derived for the amplitude function for a single electron in an external field from an expression in the form of a singly infinite sum. Using Eq. (5) to (8) we obtain, by an analogous argument, an integral equation for $K(3,4;1,2)$ from the expression as a double sum, the kernel containing the singly infinite sum $\bar{G}(1,2;3,4)$:

$$K(3,4;1,2)-K_{+a}(3,1)K_{+b}(4,2)=i\int\int\int\int d\tau_5 d\tau_6 d\tau_7 d\tau_8 K_{+a}(3,5)K_{+b}(4,6)\times\bar{G}(5,6;7,8)K(7,8;1,2). \quad (9)$$

The expression (8) for $\bar{G}(1,2;3,4)$ consists of an expansion in powers of the coupling constant. Each term in this expansion can, in principle, be calculated, but no closed expression for $\bar{G}(1,2;3,4)$ has as yet been found; and hence the method described in this paper is not immediately applicable to bound-state problems for which the coupling constant is large. For problems for which the coupling constant is reasonably small, there is, however, considerable simplification to be gained from a treatment involving the function $\bar{G}$ over a method involving summation over all reducible graphs. If, for instance, in Eq. (9) $\bar{G}$ is replaced by merely the first term in its expansion $G^{(1)}$, this is equivalent to an inclusion of an infinite number of reducible "ladder-type" graphs, examples of which are given in Fig. 2, in addition to the irreducible graph 1 in the double-sum expression for $K(3,4;1,2)$. This means that some graphs of higher power in g^2 are included automatically in such an approximation, while others are not. The value of this for the case of small g^2 can perhaps be made plausible by means of the following physical arguments. In a bound state the particles interact with each other for an infinite (or at least very long) time. If g^2 is very small, the probability for "finding one virtual quantum in the field" is small and the probability for "finding two virtual quanta simultaneously" is much smaller still. Although the probability for the exchange of a quantum during a small time interval is fairly small, during the infinite time of existence of the bound state an indefinite number of quanta may be exchanged *successively*. It is just such processes that the ladder-type graphs deal with. On the other hand, all graphs omitted in such an approximate treatment, except for self energy and mass renormalization graphs, involve "crossed quantum lines" (for example, graph $2A$ or $3A$) or Lamb-shift-type terms (for example, $2B$). Such graphs refer to processes in which two or more quanta are "in the field simultaneously," which are indeed unimportant if the coupling constant is small. If $\bar{G}$ is thus replaced by $G^{(1)}$ in Eq. (9), this integral equation reduces, with the help of Eq. (2), to a simpler equation involving $G(1,2)$.

If one wants to include the higher terms in the expansion of $\bar{G}$, further simplification can be obtained by modifying the $\Gamma_{a\tau}$, G', etc., along lines outlined by Dyson.[4] Thus, by modifying G', graph 2α (vacuum polarization term) could be automatically included in the expression for graph 1. Graph 3α is then included in graph $2A$, etc. Similarly, a modification of $\Gamma_{a\tau}$ and $\Gamma_{b\sigma}$ would include graph $2B$ (Lamb shift type) in 1 and graph $3B$ in $2A$, etc. A modification in $G^{(2A)}$ would include graphs containing parts like graph 4α in graphs containing parts like graph $2A$ and so on.

We can, again in a purely formal way, write down an equation for the propagation of the wave function $\psi(1,2)$ in terms of $K(3,4;1,2)$ according to the prescriptions of FI and FII (see FI, Eq. (18)).

$$\psi(3,4)=\int\int K(3,4;1,2)N^a(1)N^b(2)\times\psi(1,2)d^3V_1 d^3V_2, \quad (10)$$

where, for any four-vector A_μ,

$$A^a\equiv A_\mu\gamma_{\mu a}.$$

In the $x_{\mu 1}$-space the integration is carried out over a closed 3-dimensional surface in space-time completely enclosing the point $x_{\mu 3}$. The essential parts of this surface are two spacelike surfaces, i.e., all of three-dimensional space at two times $t_1<t_3$ and $t_1'>t_3$. N_μ is the inward drawn normal to the surface. For the time t_1 only the components of $\psi(1,2)$ corresponding to positive energy of particle a occur, for t_1' only those for negative energy. The prescription for $x_{2\mu}$ and particle b is analogous.

Let $\phi_{1,2}(3,4)$ denote the expression obtained for $\psi(3,4)$ from Eq. (10) by replacing $K(3,4;1,2)$ by $K_{+a}(3,1)K_{+b}(4,2)$. Then $\phi_{1,2}(3,4)$ is a wave function which varies with t_3 and t_4 like the wave function for two free particles and is equal to $\psi(1,2)$ for $t_3=t_1$ and $t_4=t_2$. Substituting Eq. (9) into Eq. (10), we obtain an inhomogeneous integral equation for $\psi(3,4)$:

$$\psi(3,4)=\phi_{1,2}(3,4)-i\int\int\int\int d\tau_5 d\tau_6 d\tau_7 d\tau_8 \times K_{+a}(3,5)K_{+b}(4,6)\bar{G}(5,6;7,8)\psi(7,8). \quad (11)$$

It will be convenient to separate the motion of the center of mass from the relative motion. This is made slightly difficult by the fact that in relativity theory the coordinate of the center of mass cannot be defined. However, the total *momentum*, which is the momentum of the center-of-mass motion, can be defined. One can then select some coordinate, more or less arbitrarily, to represent the absolute position, in time and space, of the system, and use apart from this only relative coordinates. Let us denote the "absolute" coordinate by X_μ; it may be for instance the position of particle a itself, or a linear combination

$$X_\mu=\alpha x_{\mu 1}+(1-\alpha)x_{\mu 2} \tag{12}$$

with α any arbitrary constant. Let the relative coordinate be denoted by

$$x_\mu=x_{\mu 1}-x_{\mu 2}. \tag{12a}$$

Primed quantities, x_μ', X_μ', etc., denote the equivalent expressions with 1 and 2 replaced by 3 and 4. If the interaction function $G(1,2)$ does not depend on any external factors, it will be a function $G(x_\mu)$ of x_μ only. $\bar{G}(1,2;3,4)$ will then be a differential-operator function of the relative space-time coordinates x_μ and x_μ' and of the difference of the two "absolute" coordinates $(X_\mu-X_\mu')$, but not of X_μ itself. In this case we can look for solutions for our wave function which have the special form,

$$\psi(1,2)=\exp(iK_\mu X_\mu)\psi(x_\mu), \tag{13}$$

where K_μ is an arbitrary constant four-dimensional vector. Such a wave function represents an eigenstate of the operator P_μ of the total momentum and hence a stationary state with total energy K_4. The condition for this state being a bound state is then

$$K^2\equiv K_\mu K_\mu<(m_a+m_b)^2. \tag{14}$$

An arbitrary solution of Eq. (11) may be analyzed into fourier components of the form (13). If we select a fourier component satisfying the energy condition (14), then for this component the term $\phi_{1,2}(3,4)$ in Eq. (11) cannot contribute because it represents a free state of the two particles and hence the bound-state condition (14) cannot be true for it. For a bound state, we therefore obtain Eq. (11) without the inhomogeneous term,

$$\psi(3,4)=-i\int\int\int\int d\tau_5 d\tau_6 d\tau_7 d\tau_8 K_{+a}(3,5)\times K_{+b}(4,6)\bar{G}(5,6;7,8)\psi(7,8). \tag{11a}$$

Precisely this equation has been derived rigorously from quantum field theory by Gell-Mann and Low.[5] Its relevance also to states of "positive" energy can perhaps be made plausible from the fact that, in any stationary state, there will be an infinite number of scatterings, so that any free-particle function like $\phi_{1,2}(3,4)$, which may "initially" have been present, will be destroyed "in the course of time."

[5] M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).

In contrast to the rigorous derivations of Gell-Mann and Low, we have merely derived Eqs. (9), (11), and (11a) in a purely formal way from Eq. (10) and Feynman's expression for $K(3,4;1,2)$ in the form of a doubly infinite sum. Gell-Mann and Low[5] have shown that this expression for $K(3,4;1,2)$, and hence Eq. (9), can be derived rigorously from quantum field theory even for times for which the interaction is acting. The use of Eq. (10), however, is *not* correct unless the interaction can be neglected at and between times t_1 and t_2. If the interaction remains constant throughout all times, Eq. (11a), and not Eq. (11), is then correct.

III. FURTHER DEVELOPMENT OF THE EQUATION*

It is convenient to introduce momentum variables. If $\chi(p_3, p_4)$ is the fourier transform of $\psi(3,4)$, then Eq. (11a) transforms into

$$\chi(p_3,p_4)=i\int\int dp_7 dp_8[\not{p}_3{}^a-m_a]^{-1}[\not{p}_4{}^b-m_b]^{-1}\times\bar{G}(p_3,p_4;p_7,p_8)\chi(p_7,p_8), \tag{15}$$

where

$$\bar{G}(p_3,p_4;p_7,p_8)=(2\pi)^{-8}\int\int\int\int d\tau_5 d\tau_6 d\tau_7 d\tau_8\times\exp\{i[p_3x_5+p_4x_6-p_7x_7-p_8x_8]\}\times\bar{G}(5,6;7,8).$$

The symbols p_3, etc., denote four-vectors, p_3x_5 four-dimensional scalar products, and each of the integrals in Eq. (15) goes over four-dimensional momentum space. The superscript a in $\not{p}_3{}^a$ means that the Dirac γ-operators for particle a must be inserted.

Next, we may introduce total and relative momenta. Defining coordinates as in (12), (12a), we have for the conjugate momenta

$$\left.\begin{aligned} P_\mu&\equiv i\partial x_\mu=i\partial x_{\mu 1}+i\partial x_{\mu 2}\\ p_\mu&\equiv i\partial x_\mu=i(1-\alpha)\partial x_{\mu 1}-i\alpha\partial x_{\mu 2}.\end{aligned}\right\} \tag{12b}$$

Of these, P_μ is obviously the total momentum and is uniquely defined, whereas the definition of the relative momentum can be changed by changing α, which amounts to the addition of an arbitrary multiple of the total momentum. To facilitate the transition to the nonrelativistic limit, it is convenient to make the special choice

$$\alpha=m_a/(m_a+m_b) \tag{12c}$$

then in the nonrelativistic limit (12) goes over into the normal definition of the center of mass. All the calculations in this section, however, are independent of this choice.

With the change of variable in (12) to (12c), Eq. (15) reduces to

$$\psi(p_\mu)=i\int d^4p'\mathfrak{F}^{-1}\bar{G}(p,p';K)\psi(p_\mu'), \tag{16}$$

* These equations are summarized in the Appendix.

where

$$\bar{G}(p, p'; K)=(2\pi)^{-4}\int\int\int d\tau_x d\tau_{x'} d\tau_{X-X'} \times\exp\{i[K(X-X')+px-p'x']\} \times G(x, x'; X-X')$$

and

$$\mathfrak{F}=[(m_a/m_a+m_b)\boldsymbol{K}^a+\boldsymbol{p}^a-m_a] \times[(m_b/m_a+m_b)\boldsymbol{K}^b-\boldsymbol{p}^b-m_b]$$

Equation (16) is an integral equation for the momentum-space wave function $\psi(p_\mu)$ which depends on only four variables, namely, the components of the relative momentum of the two interacting particles. It differs from the ordinary Schrödinger momentum wave function by the appearance of the "relative energy" $\epsilon\equiv p_4$ as a fourth independent variable. $\bar{G}(p, p'; K)$ is the generalized interaction function in four-dimensional momentum space.

It is tempting to simplify Eq. (16) (or Eq. (15)) by multiplying it by the operator $\mathfrak{F}$, since it does not depend on the variables of integration p', P'. Equation (16) then leads to

$$\mathfrak{F}\psi(p_\mu)=i\int d^4p'\bar{G}(p, p'; K)\psi(p_\mu'). \tag{17}$$

From Eq. (17) we can derive the equivalent equation in coordinate space:

$$\mathfrak{F}\psi(x_\mu)=i\int d\tau_{x'}\bar{G}(x, x'; K)\psi(x_\mu'), \tag{18}$$

where

$$\bar{G}(x, x'; K)=\int d\tau_{X'}\exp[iK(X-X')]\bar{G}(x, x'; X-X')$$

and $\boldsymbol{p}^a$ is now considered as the differential operator $i\boldsymbol{\nabla}^a\equiv i\gamma_{\mu a}\partial_{x_\mu}$. As $\psi(p_\mu)$ depends on the "relative energy" p_4, so $\psi(x_\mu)$ in Eq. (18) depends not only on three relative distance variables $\mathbf{x}$ but also on $t\equiv x_4$, the "relative time" variable. Equation (18) is still manifestly lorentz-invariant and is to be considered as an eigenvalue equation in a four-dimensional space with K^2 as eigenvalue, the binding energy being $[m_a+m_b-|K|]$.

An integral-differential equation, similar to Eq. (18), can be obtained, without going to the center-of-mass system and without using Eq. (13), by applying to Eq. (11) *or* Eq. (11a) the operator $(i\boldsymbol{\nabla}_3{}^a-m_a)(i\boldsymbol{\nabla}_4{}^b-m_b)$. This yields

$$(i\boldsymbol{\nabla}_3{}^a-m_a)(i\boldsymbol{\nabla}_4{}^b-m_b)\psi(3,4) =i\int\int d\tau_5 d\tau_6\bar{G}(3,4; 5,6)\psi(5,6). \tag{19}$$

Equation (19) is the analog in our problem of the Dirac equation for a Fermi-Dirac particle in an external field. As discussed in FI, to obtain results equivalent to Dirac's hole theory both masses m_a and m_b in Eq. (19) (and in Eqs. (17), (18)) must be considered to contain an infinitesimal negative imaginary part.

If the coupling constant g^2 is small, we should obtain a reasonable approximation by replacing $\bar{G}(1,2; 3,4)$ by $G^{(1)}(1,2; 3,4)$. This substitution simplifies Eqs. (11), (11a), (15), (16), (17), (18), and (19) considerably, owing to the presence of the delta-functions in Eq. (2). In this approximation, Eq. (17) reduces to

$$\mathfrak{F}\psi(p_\mu)=(2\pi)^{-2}i\int d^4kG(k_\mu)\psi(p_\mu+k_\mu), \tag{17a}$$

where $G(k_\mu)$, the interaction function in momentum space, is the *four*-dimensional fourier transform of $G(1,2)$. Similarly, Eqs. (18) and (19) reduce to ordinary differential equations,

$$\mathfrak{F}\psi(x_\mu)=iG(x_\mu)\psi(x_\mu) \tag{18a}$$

and

$$(i\boldsymbol{\nabla}_1{}^a-m_a)(i\boldsymbol{\nabla}_2{}^b-m_b)\psi(1,2)=iG(1,2)\psi(1,2). \tag{19a}$$

Whereas any solution of Eq. (11a) also satisfies Eq. (19), the inverse is not necessarily true, since Eq. (19) is derived from Eq. (11) or Eq. (11a) by the application of two differential operators. The solutions of Eq. (11a) (or of the equivalent equations (15) and (16)) corresponding to a fixed value of the total energy determine uniquely stationary states of the system. In practice, however, it may be more convenient to solve Eq. (19) (or the equivalent (17) and (18)) than Eq. (11a). Since Eq. (19) is a necessary but not sufficient condition, only those solutions of Eq. (19) will be physically acceptable which can be shown to satisfy Eq. (11a). It would be more satisfactory from a practical point of view to have some criteria of "good behavior" of the wave function $\psi(1,2)$ which would determine directly which of the solutions of Eq. (19) correspond to physical states, without any reference to Eq. (11a). No completely satisfactory set of such criteria has as yet been found, but the following discussion of a purely fictitious adiabatic variation of the coupling constant g^2 may be useful in a search for such criteria and for an understanding of the physical significance of the wave-function $\psi(1,2)$.

Let T be the time coordinate X_4 of the center of mass in the coordinate system in which it is at rest. Let us assume that $G(1,2)$ is not absolutely invariant, but that the coupling constant g^2 is an extremely slowly (adiabatically) varying function of T, $g^2(T)$ such that, for very large positive and negative values of T, $g^2(T)$ is arbitrarily small, while for $-T_0<T<T_0$, $g^2(T)$ is practically constant and has its correct physical value g_0^2. Let us assume we have found a set of solutions of the eigenvalue equation (18), one solution for every value of g^2 between zero and g_0^2 such that both K^2 and $\psi(x_\mu)$ are "smooth continuous" functions of g^2. In the limit of extremely slow variation of $g^2(T)$ the wave

function is given by the adiabatic approximation

$$\psi(x_\mu;\ \mathbf{X},\ T)=\exp\left\{i\mathbf{K}\cdot\mathbf{X}+i\int_{-\infty}^{T}dT'K_4[g^2(T')]\right\}\times\psi[x_\mu;\ g^2(T)]. \quad (20)$$

For the solution $\psi(x_\mu;\ \mathbf{X},\ T)$ to be physically acceptable, $\psi[x_\mu;\ g^2(T)]$ for $g^2(\pm\infty)=0$ must then be a wave function describing two free Fermi-Dirac particles of masses m_a and m_b, respectively. If a particular set of solutions of Eq. (18) satisfies this "boundary condition," it automatically satisfies Eq. (11).†

The bound states we have discussed so far reduce, when the coupling constant is adiabatically reduced to zero, to a state in which only two particles and no other particle pairs or quanta are present. When the interaction is "on" $\psi(1,2)$ only represents the partial probability amplitude for the presence of only two particles. But it should then, in principle, be possible to express the partial probability amplitudes for the presence of any number of additional particle pairs and quanta uniquely in terms of $\psi(1,2)$ and $K^2(g^2)$. These expressions then contain expansions in positive powers of g^2 and reduce to zero in the limit of g^2 tending to zero. More complicated types of bound states may also exist. For instance there can be a bound state of two particles, plus an extra quantum. (For large enough values of g^2, the quantum itself may perhaps also be bound to the two particles.) The fundamental wave function for such a bound state would be of form $\psi(1,2;\ q_\mu)$, where q_μ denotes the coordinates of the quantum, and would satisfy a more complicated equation analogous to (11a). The probability amplitudes for the presence of two particles and no quanta, or of two quanta, etc., would then be expressible in terms of $\psi(1,2;\ q_\mu)$ and g^2 and would tend to zero as g^2 does.

In the nonrelativistic limit and for the case of small coupling constant, the formalism of this paper leads exactly to the ordinary nonrelativistic Schrödinger equation for a bound state for two particles. An example of this, and of the way Eq. (19) may be solved in practice, is given in the next section.

IV. SCALAR MESONS. EXTREME NONRELATIVISTIC LIMIT

We consider the problem of the deuteron ground state with scalar meson theory with scalar coupling. We shall solve only the approximate equation (19a), obtained from Eq. (19) by replacing $\tilde{G}$ by the first term in its expansion, $G^{(1)}$. It is most convenient to use the equivalent equation in relative-momentum space, Eq. (17a). This becomes

$$(\tfrac{1}{2}\boldsymbol{K}^a+\boldsymbol{p}^a-M)(\tfrac{1}{2}\boldsymbol{K}^b-\boldsymbol{p}^b-M)\psi(p_\mu)=-g^2(4\pi^3 i)^{-1}\int d^4k[k_\nu^2-\mu^2]^{-1}\psi(p_\mu+k_\mu), \quad (21)$$

where we have used the relativistic generalization of the Yukawa potential,

$$G(k_\mu)=g^2[\pi(k_\nu^2-\mu^2)]^{-1}. \quad (22)$$

M and μ are the nucleon and meson mass, respectively, each including an infinitesimal negative imaginary part, and g^2 is the dimensionless coupling constant equivalent to the fine structure constant in quantum electrodynamics. Without loss of generality, we can choose the coordinate system in which the center of mass is at rest. The first three components of K_μ are then zero and K_4 is the eigenvalue of Eq. (21).

The range of nuclear forces is large compared with the Compton wavelength of a nucleon, and hence the meson mass μ is small compared with the nucleon mass M. Furthermore, the binding energy of the deuteron is smaller than μ^2/M. Under these circumstances, a nonrelativistic calculation for the deuteron problem shows that the coupling constant g^2 is of the order of magnitude of μ/M and hence small compared with unity. It is further found that the momentum distribution in the deuteron falls off rapidly for momenta greater than μ; i.e., the "velocities" in the deuteron are of the order of magnitude of g^2 and hence small. Such a nonrelativistic solution should then be a good approximation and should be identical with the extreme nonrelativistic limit of our Eq. (21).

The wave function $\psi(p_\mu)$, or $\psi(x_\mu)$, is a 16-component spinor, the 16 components referring to the two possible spin directions and two signs of the energy for each of the particles. In analogy with the usual treatment of the nonrelativistic limit of the ordinary Dirac equation, we wish to carry out a reduction to the "large components" of Eq. (19a) or Eq. (21); i.e., we want an approximate equation involving only the spinor components referring to both particles in positive energy states. Various methods, analogous to various methods used for the Dirac equation,[6] are possible for such a reduction. One way is to transfer the term involving G to the left-hand side in the differential equation (19a) and to multiply this equation to the left by the operator $[(\boldsymbol{p}_1{}^a+m_a)(\boldsymbol{p}_2{}^b+m_b)-iG]$ which gives the equation,

$$\{(\boldsymbol{p}_1{}^2-m_a{}^2)(\boldsymbol{p}_2{}^2-m_b{}^2)-i[G,\ (\boldsymbol{p}_1{}^a\boldsymbol{p}_2{}^b+m_am_b)]_+ +i[G,\ (\boldsymbol{p}_1{}^am_b+\boldsymbol{p}_2{}^bm_a)]_--G^2\}\psi(1,2)=0, \quad (23)$$

where $\boldsymbol{p}_1{}^a=i\gamma_{\mu a}\partial_{x\mu 1}$ and Eq. (23) is considered as an ordinary differential equation in configuration space. A similar equation can be obtained from Eq. (21). We expand in powers of (p/M) and g^2 (which are of the same order of magnitude) and retain only the first two nonvanishing powers in this expansion (p^4 and p^5/M, respectively). The spatial components of γ_μ are of the order of (p/M). From Eq. (24), about to be derived,

† It would then seem that this criterion of good behavior would be sufficient; but it is not unambiguous, since the adiabatic variation of the wave function is not well-defined for the value of $g^2(T)$ which just gives a bound state of zero binding energy.

[6] H. A. Bethe, *Handbuch der Physik* (Verlag. Julius Springer, Berlin, 1933), Vol. 24/1, p. 304. H. A. Kramers, *Hand-u. Jahrbuch der chemischen Physik* (Akad. Verlag., Leipzig, 1938), Vol. 1, p. 295.

it follows that G is of the order of p^4/M^2. Hence,‡ the third and fourth terms in Eq. (23) contribute only terms of order p^6/M^2 or higher. Let $E\equiv(2M-K_4)$ be the binding energy, $\epsilon\equiv p_4$ the "relative energy" variable, and $\mathbf{p}\equiv\mathbf{p}_a=-\mathbf{p}_b$ the three-dimensional relative momentum. Equation (23) then reduces to the following approximate equation for the "large components," in the relative energy-momentum space, which does not contain any Dirac- (or spin-) operators at all:

$$(\tfrac{1}{2}E+\epsilon+p^2/2M)(\tfrac{1}{2}E-\epsilon+p^2/2M)\psi(\mathbf{p},\epsilon)$$
$$=-(g^2/2\pi i)\int\int d^3k d\omega[2\pi^2(k^2-\omega^2+\mu^2)]^{-1}\times\psi(\mathbf{p}+\mathbf{k},\epsilon+\omega). \quad (24)$$

Since in Eq. (21) M (but not p_4) contains an infinitesimal negative imaginary part, $\frac{1}{2}E$ in both factors of Eq. (24) also contains a negative imaginary part.

An iteration method for solving Eq. (24) will be described in the next section. However, if the relative-energy term ω^2 is omitted in the denominator on the right-hand side, we shall now show that this equation reduces exactly to the ordinary Schrödinger equation for two nonrelativistic particles. This omission of ω^2 is equivalent to replacing the lorentz-invariant, and hence retarded, Yukawa interaction by an interaction which is instantaneous in the center of mass system. We change the variable of integration from ω to $(\omega+\epsilon)$ and define

$$\phi(\mathbf{p})=\int_{-\infty}^{\infty}d\epsilon'\psi(\mathbf{p},\epsilon'). \quad (25)$$

The right-hand side of the modified Eq. (24) does not contain ϵ now, and on the left-hand side ψ is multiplied by a simple number (not operator). We can then write the modified Eq. (24) in the form,

$$\psi(\mathbf{p},\epsilon)=-g^2\{[(E/2)+(p^2/2M)+\epsilon]\times[(E/2)+(p^2/2M)-\epsilon]2\pi i\}^{-1}\times\int d^3k\phi(\mathbf{p}+\mathbf{k})[2\pi^2(k^2+\mu^2)]^{-1}. \quad (26)$$

The first two factors in Eq. (26) are now the only ones depending on ϵ, and they are easily integrated over ϵ, using the fact that E contains a small negative imaginary part. This yields a simple equation for $\phi(\mathbf{p})$:

$$(E+p^2/M)\phi(\mathbf{p})=-g^2\int d^3k\phi(\mathbf{p}+\mathbf{k})[2\pi^2(k^2+\mu^2)]^{-1}. \quad (27)$$

Equation (27) is identical with the Schrödinger equation in momentum space for two nonrelativistic nucleons interacting by means of a static central Yukawa potential.

The relative motion of the two nucleons could be described by a "mixed" wave function $\psi(\mathbf{p},t)$, instead of by $\psi(x_\mu)$ or $\psi(p_\mu)$, obtained by taking the fourier transform of $\psi(\mathbf{x},t)$ only for the three space coordinates but not for the relative time t. The function $\phi(\mathbf{p})$, which is the conventional momentum-space wave function, is then seen to be, except for a multiplicative constant, $\psi(\mathbf{p},t)$ for the special value $t=0$, i.e., for equal times for the two particles. The general expression for $\psi(\mathbf{p},t)$ can be obtained from Eqs. (26) and (27), and the total wave function is proportional to $\phi(\mathbf{p})$ times

$$\left.\begin{aligned}&\exp\{iET+i(\tfrac{1}{2}E+p^2/2M)t\}\\&\quad=\exp\{i(E+p^2/2M)t_1-i(p^2/2M)t_2\}\quad\text{for}\quad t<0,\\&\exp\{iET-i(\tfrac{1}{2}E+p^2/2M)t\}\\&\quad=\exp\{i(E+p^2/2M)t_2-i(p^2/2M)t_1\}\quad\text{for}\quad t>0.\end{aligned}\right\} \quad (28)$$

This wave function (28) can be said to correspond to propagation into the "future" as free waves (frequency $p^2/2M$) and into the "past" in a more complicated "bound" manner.

V. EXACT SOLUTION OF NONRELATIVISTIC EQUATION

The solutions of Eq. (24) differ from those of the approximation to it, Eq. (26), by terms of the relative order of magnitude of g^2. It was found that a straightforward application of a perturbation expansion in powers of g^2, starting with a solution of Eq. (26) as zero-order wave function, leads to wrong results even for a very small value of g^2. This is due to the fact that one of the integrands occurring in the solution of Eq. (24) has a pole for a value of ϵ for which the solution of Eq. (26) is a very poor approximation. However, in the preceding paper[7] a method was described for solving integral equations of a type similar to that of Eq. (24) by an iteration method which does not involve any expansion in powers of g^2.

Such an iteration method has been applied previously[8] to the solution of Eq. (27) for the ground state of the deuteron with an initial trial wave function of the form,

$$\phi_0(\mathbf{p})=[(E+p^2/M)(p^2a^{-2}+\mu^2)]^{-1}, \quad (29)$$

where a is a dimensionless parameter. By analogy with the prescriptions of the preceding paper[7] we assume an initial trial wave function for the ground-state solution of Eq. (24) in the form,

$$\psi_0(\mathbf{p},\epsilon)=[(\tfrac{1}{2}E+p^2/2M)^2-\epsilon^2]^{-1}\times(p^2a^{-2}-\epsilon^2b^{-2}+\mu^2)^{-1}, \quad (30)$$

where a and b are dimensionless coefficients which have to be determined and where E and μ are considered to have infinitesimal negative imaginary parts. For values of ϵ small compared with μ, the function ψ_0 differs little

‡ This demonstration involves some algebra.

[7] E. E. Salpeter, Phys. Rev. 84, 1226 (1951).
[8] J. S. Goldstein and E. E. Salpeter (to be published).

from the equivalent solution of Eq. (26). The first factor of Eq. (30) suggests that $\psi(\mathbf{p}, \epsilon)$ will be of interest mainly for values of $|\epsilon|$ of the order of magnitude of (μ^2/M), which is indeed much less than μ. The expression (30) for ψ_0 is then substituted for in ψ the integral on the right-hand side of Eq. (24), which yields a better approximation to $\psi(\mathbf{p}, \epsilon)$, i.e., the first iterated function $\psi_1(\mathbf{p}, \epsilon)$:

$$\psi_1(\mathbf{p}, \epsilon) = -g^2[(\tfrac{1}{2}E + p^2/2M)^2 - \epsilon^2]^{-1} \times \int\int d^3k d\omega \psi_0(\mathbf{p}+\mathbf{k}; \epsilon+\omega) \times [4\pi^3 i(k^2 - \omega^2 + \mu^2)]^{-1}. \quad (31)$$

In principle this iteration procedure could be repeated any number of times.

Since the binding energy of the deuteron is known, we shall consider E as given and g^2 as the eigenvalue of Eq. (24). We then find successive approximations g_1^2, g_2^2, etc., to g^2 by requiring that

$$\psi_{n+1}(0,0) = \psi_n(0,0). \quad (32)$$

For convenience in calculation it is useful to define two functions $a_n(p)$ and $b_n(p, \epsilon)$ by

$$\psi_n(\mathbf{p}, \epsilon) = [(\tfrac{1}{2}E + p^2/2M) - \epsilon^2]^{-1} \times \{p^2 a_n^{-2}(p) - \epsilon^2 b_n^{-2}(p, \epsilon) + \mu^2\}^{-1}. \quad (33)$$

The advantage of these two functions a_n and b_n is that they vary *slowly* with p and ϵ. We then determine "reasonable" values for a and b by requiring, for some conveniently chosen values of p and ϵ, that

$$\psi_1(p, 0)/\psi_1(0, 0) = \psi_0(p, 0)/\psi_0(0, 0), \text{ i.e., } a_1(p) = a; \quad (34)$$

$$\psi_1(p, \epsilon)/\psi_1(p, 0) = \psi_0(p, \epsilon)/\psi_0(p, 0), \text{ i.e., } b_1(p, \epsilon) = b. \quad (35)$$

For a more detailed discussion of the method used see the preceding paper.[7]

A meson mass μ of $275m_e$ agrees fairly well both with the mass of the charged π-meson[9] and with calculations[10] on the triplet effective range of the neutron-proton force. We therefore take a value of 0.150 for μ/M. For the ratio $(EM)^{\frac{1}{2}}/\mu$ we take a value of 0.32, which agrees fairly well with a deuteron binding energy E of 2.226 Mev and the above value for the meson mass. We determine the value to be used for a by applying condition (34) for $p=a$. For the extreme nonrelativistic equation (26) a similar choice[8] gives a value for a of approximately 1.85; and the three-dimensional equations equivalent to Eqs. (31) and (32) with $n=0$ give a value for g_1^2 of $2.39(\mu/M)$. This value of g_1^2 differs from the exact value of g^2 for Eq. (26) by only a fraction of one percent.[8]

Estimates for b for the four-dimensional equation (24) can be obtained easily from Eqs. (31) and (35) for the four cases of p and ϵ very small or very large compared with μ, estimates for a from Eqs. (31) and (34). These calculations for a and b could in principle be carried out for any value of (μ/M), but would be rather involved, since Eqs. (34) and (35) are coupled equations for the two unknowns a and b. In order to simplify the computations, an expansion in powers of (μ/M) was employed and only the first two terms in this expansion retained. Equation (35) involves, to this approximation, only the value 1.85 and not a itself and, for $p \ll \mu$ and $\epsilon \ll \mu$ gave $b=1.31$; for $p \ll \mu$ and $\epsilon \gg \mu$, $b=1.93$; for $p \gg \mu$ and any value for ϵ, $b=1.85$. For the following calculations a somewhat arbitrary value of $b=1.6$ was used. Equation (34) with $p=a$ was then solved for a, the solution involving a short numerical integration. With the values thus obtained for a and b, Eqs. (31) and (32), with $n=0$, give a value for g_1^2:

$$\left.\begin{aligned} a &= 1.85(1 - 0.5_6\mu/M), \quad b = 1.6; \\ g_1^2 &= 2.39(\mu/M)(1 + 0.9_8\mu/M). \end{aligned}\right\} \quad (36)$$

The integrals occurring in Eq. (31) cannot be carried out analytically for general p and ϵ, but the numerical work involved in evaluating $\psi_1(p, \epsilon)$ for any particular pair of values for p and ϵ is not very lengthy. The function $a_1(p)$ is real for all values of p; $b_1(p, \epsilon)$ is real for $\epsilon < \mu$ but is, in general, complex for $\epsilon > \mu$. The function $a_1(p)$ and the absolute value of $b_1(p, \epsilon)$ are slowly varying functions and, in the whole range $0 < p$, $\epsilon < M$, do not vary by more than about 50 percent. Hence, if required, $\psi_1(p, \epsilon)$, $a_1(p)$, and $b_1(p, \epsilon)$ could be evaluated for a few values each of p and ϵ and an interpolation method used on $a_1(p)$ and $b_1(p, \epsilon)$ for intermediate values of p and ϵ. A better approximation g_2^2 to the coupling constant for Eq. (24) could then be obtained after one more double integration, using Eq. (31) with $n=1$. However, g_1^2 is probably not in error by very much more than one or two percent. Since in the derivation of Eq. (24), terms of relative order of magnitude of $(\mu/M)^2 > 0.01$ have in any case been omitted, an evaluation of g_2^2 was not considered worth while.

The mixed wave function corresponding to $\psi_0(\mathbf{p}, \epsilon)$ is

$$\frac{\exp(iET)}{[b^2(p^2a^{-2} + \mu^2) - (E/2 + p^2/2M)^2]} \times \left\{ \frac{\exp[\pm i(\tfrac{1}{2}E + p^2/2M)t]}{(\tfrac{1}{2}E + p^2/2M)} - \frac{\exp[\pm ib(p^2a^{-2} + \mu^2)^{\frac{1}{2}}t]}{b(p^2a^{-2} + \mu^2)^{\frac{1}{2}}} \right\}, \quad (37)$$

where the + sign holds for $t<0$ and the − sign for $t>0$. The first term of (37) is of the same form as Eq. (28); the second term gives a contribution to the wave function which does *not* correspond to the propagation into the "future" as free waves and is connected with the fact that a retarded interaction is used. The two different denominators occurring in the first and second

[9] Smith, Barkas, Bishop, Bradner, and Gardner, Phys. Rev. **78**, 86(A) (1950).
[10] E. E. Salpeter, Phys. Rev. **82**, 60 (1951).

terms of (37) are of the order of magnitude of (μ^2/M) and μ, respectively. Thus, the absolute value of the second term is smaller than that of the first by a factor of the order of (μ/M).

Expression (36) for g_1^2 shows that appreciably different results are obtained if the relativistic retarded interaction function $(k^2-\omega^2+\mu^2)^{-1}$ is used instead of the instantaneous function $(k^2+\mu^2)^{-1}$. We can get a better insight into the significance of these differences by deriving an approximate equation involving a three-dimensional wave function from Eq. (24). We again define $\phi_n(\mathbf{p})$ by means of

$$\phi_n(\mathbf{p})=\int_{-\infty}^{\infty} d\epsilon'\psi_n(\mathbf{p},\epsilon').$$

In Eq. (24), unlike in Eq. (26), ϵ and ω cannot be eliminated unless the dependence of $\psi(\mathbf{p},\epsilon)$ on ϵ is known. But if the expression (30) is substituted into Eq. (31), then the integration over ω can be carried out. The ensuing expression for $\psi_1(\mathbf{p},\epsilon)$ can then be integrated over ϵ giving an expression for $\phi_1(\mathbf{p})$ not involving ϵ or ω. Finally, the integrand in this expression can be expressed in terms of $\phi_0(\mathbf{p})$. Neglecting terms of order of magnitude $(\mu/M)^2$ compared with unity, this expression becomes

$$(E+p^2/M)\phi_1(\mathbf{p})=-g^2\int d^3k[2\pi^2(k^2+\mu^2)]^{-1}\times\{1+F(\mathbf{p},\mathbf{k})\}\phi_0(\mathbf{p}+\mathbf{k}) \quad (38)$$

$$F(\mathbf{p},\mathbf{k})\equiv-\frac{E+(p^2+|\mathbf{p}+\mathbf{k}|^2)/2M}{(k^2+\mu^2)^{\frac{1}{2}}}+\frac{\frac{1}{2}E+|\mathbf{p}+\mathbf{k}|^2/2M}{b(p^2a^{-2}+\mu^2)^{\frac{1}{2}}+(k^2+\mu^2)^{\frac{1}{2}}}. \quad (39)$$

Equation (38) is identical with the equation for the iterated function $\phi_1(\mathbf{p})$ obtained from Eqs. (26) and (27), except for the presence of the additional term involving $F(\mathbf{p},\mathbf{k})$. Thus, the difference between the retarded and instantaneous interaction functions can, to first order in (μ/M), be included in an equation for the orthodox three-dimensional wave function $\phi(\mathbf{p})$ by adding an extra interaction function to $(k^2+\mu^2)^{-1}$. This additional term depends on $\mathbf{p}$ as well as on $\mathbf{k}$ and represents therefore a velocity-dependent potential. For $p,\ k,\ \lesssim\mu$, $F(\mathbf{p},\mathbf{k})$ is of the order of magnitude of (μ/M).

An approximation to g^2, somewhat more accurate than g_1^2, can also be obtained from Eq. (38). If we again consider (μ/M) small and expand in powers of it, $F(\mathbf{p},\mathbf{k})$ is small compared with unity throughout the whole range over which the integrand of Eq. (38) is appreciable. We can therefore use first-order perturbation theory, taking the ground-state solution of Eq. (27) as the unperturbed wave function and considering $(k^2+\mu^2)^{-1}F(\mathbf{p},\mathbf{k})$ as the perturbation potential. It should be noted that an attempt to use perturbation theory directly in Eq. (24) would be equivalent to an unjustified omission of the second half of expression (39) for $F(\mathbf{p},\mathbf{k})$. Using the expression (29) as an approximation to the unperturbed wave function the first order perturbation treatment for Eq. (38) was carried out. After some rather tedious numerical and analytical integrations we found, keeping the deuteron binding energy constant,

$$g^2=2.39(\mu/M)(1+1.0_7\mu/M). \quad (40)$$

The expression (40) represents only the first two terms in an expansion in powers of (μ/M). The coefficient 1.0_7 of the second term in Eq. (40) should be accurate to better than ten percent and agrees, as well as can be expected, with the equivalent coefficient in the less accurate expression (36) for g_1^2.

VI. DISCUSSION

We have found in Eq. (40) that the value of g^2 required to obtain the correct binding energy of the deuteron in relativistic theory differs from the non-relativistic value by an amount of relative order μ/M. This is of the same order as the coupling constant g^2, and as the "average" value of p/M in the deuteron ground state. Thus we get a correction of relative order v/c where v is the "average" nucleon velocity, in contrast to the well-known result in the theory of the hydrogen atom that relativistic effects (fine structure) are of relative order $(v/c)^2\approx(e^2/\hbar c)^2$.

This surprising result is not due to a mistake because, in the derivation of Eq. (24) from the Lorentz-invariant Eq. (21), only errors of the relative order (p/M^2) or smaller were made. Indeed, we shall prove that the result (40) is correct, neglecting terms of relative order $(\mu/M)^2$, if only *charged mesons can be emitted and absorbed* by nucleons. On the other hand, for *neutral* mesons, and also for electromagnetic interaction as in the hydrogen atom, our result in the last section must be corrected.

Indeed, Eq. (21), the basis of our calculations in the last section, differs from the complete Eq. (19) by the replacement of the full interaction $\bar{G}$ by the one-meson interaction $G^{(1)}$. But for a neutral field theory all other contributions $G^{(n)}$ also have to be included, and in particular the graphs $2A-2D$ of Fig. 1 will give contributions containing only one power of g^2 more than the "main" term $G^{(1)}$ which alone was included in (21). It must therefore be expected that these terms will give corrections to (40) of the order $g^2\approx\mu/M$, i.e., of the same order by which (40) differs from the nonrelativistic result. In charged meson theory, the graphs $2A-2D$ do *not* contribute as will be shown below.

From the example of the hydrogen fine structure, it should be expected that the contribution from the "two-meson graphs" $2A-2D$ not only is of the same order as the relativistic correction in (40) but actually cancels this correction exactly, leaving only corrections of the relative order $(\mu/M)^2$ to the nonrelativistic result.

Indeed, the *usual* derivation of Eq. (27) from field theory, by separation of a "static" interaction, seems to neglect only terms of relative order $(p/M)^2$. This has been confirmed by a detailed relativistic analysis of the problem by Dancoff.[11] We must therefore show that our theory also gives the nonrelativistic result to order $(\mu/M)^2$. This could of course be done by explicitly calculating the contributions of graphs $2A-2D$ and showing that they compensate exactly the relativistic correction term in (40), but this would be both tedious and unconvincing. Instead, we shall use a transformation of the theory which will give the result directly and will in addition be useful for the practical application of the theory to neutral fields.

Such a transformation is somewhat analogous to that described in Sec. 8 of FII for the elimination of the longitudinal field in quantum electrodynamics. Feynman showed that the formal incorporation of a certain interaction function, which depends explicitly on the energy change of the interacting particle, into the matrix elements for any arbitrary process undergone by a particle (initially and finally in a free state) does not affect the total amplitude function for any such physical process. For the case of electrodynamics the addition of these terms to the two longitudinal components of the retarded electrodynamic interaction results exactly in an instantaneous coulomb interaction. We shall now show that the formal incorporation of a slightly different interaction function does not affect the total amplitude function, for any physical processes involving neutral scalar mesons. The addition of this interaction to the retarded scalar meson interaction (22) results in the sum of an instantaneous Yukawa interaction and a more complicated, but smaller, retarded interaction.

Consider the probability amplitude function for any arbitrary graph for two Fermi-Dirac particles, initially and finally in a free state, interacting N times by means of arbitrary interactions with "vertex parts" Γ. For the special case of scalar meson interactions with scalar couplings Γ, defined in Eq. (2), is unity. Consider an additional interaction, involving an energy-momentum change of q_μ, with a velocity-dependent vertex-part Λ:

$$\Lambda=(\not{p}-M)\gamma_4-\gamma_4(\not{\pi}-M)=(\mathbf{p}+\boldsymbol{\pi})\cdot\boldsymbol{\gamma}\gamma_4-\omega \quad (41)$$

where p_μ and $\pi_\mu\equiv(p_\mu+q_\mu)$ are the energy-momentum vectors before and after the interaction respectively; ω is the energy change q_4 and $\mathbf{p}$, $\boldsymbol{\pi}$ the three-dimensional momenta. This interaction function is similar to the one used in FII except that γ_4 is replaced by unity.

We next modify the above mentioned arbitrary graph involving N Γ-interactions by including one additional interaction involving a vertex part Λ either before the first Γ-interaction of particle a or between the first and second, etc., the corresponding vertex parts for particle b being left arbitrary. We now sum the amplitude functions corresponding to these $(N+1)$ modified graphs, following the procedure explained in FII. The contribution to this sum of the first term in expression (41) for Λ acting between the nth and the $(n+1)$st Γ-interaction, plus that of the second term of Eq. (41) between the $(n-1)$th and nth Γ-interaction, contains the following factors:

$$\begin{aligned}&\cdots(\not{p}_{n-1}-M)^{-1}\Gamma_n(\not{p}_n-M)^{-1}\\&\qquad\times[(\not{p}_n-M)\gamma_4](\not{\pi}_n-M)^{-1}\cdots,\\&-\cdots(\not{p}_{n-1}-M)^{-1}[\gamma_4(\not{\pi}_{n-1}-M)]\\&\qquad\times(\not{\pi}_{n-1}-M)^{-1}\Gamma_n(\not{\pi}_n-M)^{-1}\cdots.\end{aligned} \quad (42)$$

The first term of Eq. (41) acting before the first Γ-interaction, and the second term acting after the last Γ-interaction, give no contribution, since particle a is initially and finally in a free state. The sum of the amplitude functions for these $(N+1)$ modified graphs is then equivalent to the sum for N modified graphs, in each of which one of the N Γ-interactions is replaced by a combined interaction with vertex part Λ'^a:

$$\Lambda'=\Gamma\gamma_4-\gamma_4\Gamma. \quad (43)$$

If each Γ is unity, then Λ' is zero and the introduction of the Λ-interactions does not affect the total amplitude function. The Λ used in FII (γ_4 replaced by unity) led to a Λ' equal to zero for any Γ.

In the center-of-mass system ($p_{a\mu}=-p_{b\mu}\equiv p_\mu$) we define an interaction between particles a and b involving an energy-momentum transfer $q_\mu=(\mathbf{k},\omega)$, with a velocity-dependent interaction function $L(p_\mu;\mathbf{k},\omega)$:

$$\begin{aligned}L=g^2[4\pi^2(k^2-\omega^2+\mu^2)(k^2+\mu^2)]^{-1}\\\times\{\Lambda^a(\Lambda^b-2\omega)+(\Lambda^a+2\omega)\Lambda^b\}.\end{aligned} \quad (44)$$

We can write the Lorentz-invariant retarded scalar meson interaction function $G(\mathbf{k},\omega)$, expression (22), as $\{[G-L]+L\}$. Using Eq. (44) and the second expression in Eq. (41), we find

$$G-L=\frac{g^2}{2\pi^2(k^2+\mu^2)}\times\left\{1-\frac{(\mathbf{p}+\boldsymbol{\pi})\cdot\boldsymbol{\gamma}^a\gamma_4{}^a(\mathbf{p}+\boldsymbol{\pi})\cdot\boldsymbol{\gamma}^b\gamma_4{}^b}{(k^2-\omega^2+\mu^2)}\right\}. \quad (45)$$

The first of the two terms of expression (45) is exactly the *instantaneous* Yukawa interaction function and, when substituted for G, leads to Eq. (27) and not to Eq. (24). The second term in Eq. (45) represents a velocity- (and spin-) dependent interaction and, since p, k, and $|\gamma M|$ are all of the order of magnitude of μ, is smaller than the first term by a factor of order $(\mu/M)^2$. Since Γ for the scalar meson interaction is unity, it follows from Eq. (43) that the additional interaction function $+L$ does not contribute to the

[11] S. M. Dancoff, Phys. Rev. **78**, 382 (1950).

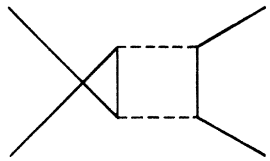

FIG. 3. Graph for a process involving virtual pair creation and annihilation, which is included in graph 2A.

total amplitude function for any interaction between the two particles.

The "crossed" graph 2A might be considered to represent the "presence of two mesons in flight simultaneously." The first part of Eq. (45) represents an instantaneous exchange of mesons, and one might expect that such an interaction should not contribute anything to the term $G^{(2A)}$ corresponding to graph 2A. In reality its contribution to $G^{(2A)}$ is not exactly zero, since processes involving intermediate states of negative energy (represented in Fig. 3) are included in graph 2A, but is smaller than the expression $G^{(2A)}$ for the retarded interaction function (22) by a factor of order $(\mu/M)^2$. The second term of Eq. (45) is itself small, and its contribution to $G^{(2A)}$ is still smaller by a factor of order (μ/M); the total sum of expressions $[G^{(2B)}+G^{(2C)}+G^{(2D)}]$ should be of the same order of magnitude as this contribution. We have therefore shown that replacing $\bar{G}$ by the instantaneous Yukawa interaction function $g^2/2\pi^2(k^2+\mu^2)$ only introduces errors of relative order of magnitude $(\mu/M)^2$, the leading correction term being given by the second term of Eq. (45).

However, for a purely *charged* scalar meson theory the position is radically different if we assume that a nucleon cannot change into a double charged or negatively charged proton by the emission or absorption of charged mesons and if we assume no coupling with neutral mesons at all. In this case a nucleon can emit (or absorb) positive and negative mesons only alternately, and the crossed graph 2A and Lamb shift graph 2B are completely forbidden by charge conservation. Since the nucleons in the deuteron are not free, the self-energy and mass-renormalization graphs 2C and 2D together may contribute nonvanishing terms; but these, as well as the terms due to more complicated graphs, are of a smaller order of magnitude than $G^{(2A)}$ with a retarded scalar interaction function. Therefore, Eq. (24) and the approximate solution derived for it in the last section give correctly the first two terms in an expansion in powers of (μ/M) for *charged* scalar mesons (in the absence of multiply charged nucleon states). It should be noted that the transformation carried out above for neutral mesons is based on the condition that a Λ-interaction can be introduced into any graph at *any* point. If we are restricted to charged mesons, and neutrons and protons only, this condition cannot be fulfilled (because of charge conservation) and the above transformation cannot be carried out.

For a purely charged scalar theory the coupling constant g^2 required for the interaction is then larger by approximately $1.1(\mu/M)$ or 15 percent, than for a purely neutral theory. For a Serber mixture of charged and neutral mesons (giving no force in p-states) the increase is only one quarter of this amount; for a Kemmer charge-symmetric mixture there is a decrease of very roughly four times this amount. It should be emphasized that calculations on a scalar meson theory were carried out in this paper merely to illustrate the methods developed and not because of any belief in this theory being correct physically.

The equations described in this paper are being applied to an investigation of the relativistic corrections to the fine structure and hyperfine structure of hydrogen. No way has as yet been found to apply this equation to problems in which the coupling constant is not small, e.g., the deuteron with pseudo-scalar mesons. Given any *phenomenological* interaction potential, however, one can always find a relativistically invariant generalization of it. This Lorentz-invariant interaction function could then be substituted in Eq. (14) or Eq. (19); and a relativistic, but purely phenomenological, equation for bound states for two particles is then obtained. An investigation of the theory of Fermi and Yang,[12] using such an equation, is planned.

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APPENDIX. SUMMARY OF EQUATIONS DERIVED IN SECS. II AND III

The amplitude function $K(3,4;1,2)$ is defined uniquely by the inhomogeneous integral equation (9). This definition is exactly equivalent to that given by Feynman in the form of a double infinite series. Equation (9) leads to Eq. (11) or (11a) for the wave function in coordinate space, $\psi(1,2)$. Equation (11) is obtained if the interaction is "on" only for a finite time, Eq. (11a) if the interaction is constant throughout all time. Equation (11) contains a boundary condition in the form of an inhomogeneous term $\phi_{1,2}(3,4)$ and therefore has only one solution. Equation (11a) is a homogeneous integral-differential equation and has an infinite number of solutions, each one corresponding to a "physically acceptable state." The fourier transform of Eq. (11a) is Eq. (15), an equation for the momentum-space wave function $\chi(p_1, p_2)$.

For the case of an interaction constant throughout all time, we may look for special solutions for our wave function satisfying the condition (13). These solutions correspond to states with a definite total momentum and total energy. For these states, Eq. (11a) reduces to a more special equation (which is not given explicitly in the text) and Eq. (15) to the special equation (16).

By applying certain operators to Eqs. (11a) or (11) we obtain Eq. (19) and an equivalent equation (not given explicitly) can be derived from Eq. (15). By applying equivalent operators to Eq. (16) and its counterpart in coordinate space, we obtain Eqs. (17) and (18), respectively. These four derived equations are more convenient for practical solutions, but are only necessary and not sufficient conditions. If the expression $\bar{G}$ is replaced by the first term in its expansion, $G^{(1)}$, Eqs. (17), (18), and (19) reduce to the approximate Eqs. (17a), (18a), and (19a), respectively.

[12] E. Fermi and C. N. Yang, Phys. Rev. **76**, 1739 (1949).