Riemann-Hilbert Analysis for Orthogonal Polynomials

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Summary. This is an introduction to the asymptotic analysis of orthogonal polynomials based on the steepest descent method for Riemann-Hilbert problems of Deift and Zhou. We consider in detail the polynomials that are orthogonal with respect to the modified Jacobi weight $(1-x)^{\alpha}(1+x)^{\beta}h(x)$ on $[-1,1]$ where $\alpha, \beta > -1$ and h is real analytic and positive on $[-1,1]$. These notes are based on joint work with Kenneth McLaughlin, Walter Van Assche and Maarten Vanlessen.

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1 Introduction

These lecture notes give an introduction to a recently developed method to obtain asymptotics for orthogonal polynomials. The method is called “steepest descent method for Riemann-Hilbert problems”. It is based on a characterization of orthogonal polynomials due to Fokas, Its, and Kitaev [15] in terms of a Riemann-Hilbert problem combined with the steepest descent method introduced by Deift and Zhou in [12] and further developed in [13, 11]. The application to orthogonal polynomials was initiated in the seminal papers of Bleher and Its [4] and Deift, Kriecherbauer, McLaughlin, Venakides and Zhou [8, 9]. These works were motivated by the connection between asymptotics of orthogonal polynomials and universality questions in random matrix theory [7, 27]. An excellent overview can be found in the book of Percy Deift [7], see also [10, 26]. Later developments related to orthogonal polynomials include [2, 3, 5, 18, 20, 21, 22, 23, 34].

In this exposition we will focus on the paper [23] by Kuijlaars, McLaughlin, Van Assche, and Vanlessen. That paper applies the Riemann-Hilbert technique to orthogonal polynomials on the interval $[-1,1]$ with respect to a modified Jacobi weight $(1-x)^\alpha(1+x)^\beta h(x)$ where h is a non-zero real analytic function on $[-1,1]$. It should be noted that the earlier works [4, 8, 9] dealt with orthogonal polynomials on the full real line. The fact that one works on a finite interval has some technical advantages and disadvantages. The disadvantage is that one has to pay special attention to the endpoints. The main advantage is that no rescaling is needed on $[-1,1]$, and that we can work with orthogonal polynomials with respect to a fixed weight function, instead of orthogonal polynomials with respect to varying weights on $\mathbb{R}$. Another advantage is that the analysis simplifies considerably on the interval $[-1,1]$, if the parameters α and β in the modified Jacobi weight are $\pm\frac{1}{2}$. In that case, there is no need for special endpoint analysis. The case $\alpha=\beta=\frac{1}{2}$ will be worked out in detail in the first part of this paper (up to Section 12).

For general $\alpha,\beta>-1$, the analysis requires the construction of a local parametrix near the endpoints. These local parametrices are built out of modified Bessel functions of order α and β. For orthogonal polynomials on the real line, one typically encounters a parametrix built out of Airy functions [7, 8, 9]. The explicit construction of a local parametrix is a technical (but essential and beautiful) part of the steepest descent method. This is explained in Section 14. The asymptotic behavior of the orthogonal polynomials can then be obtained in any region in the complex plane, including the interval $(-1,1)$ where the zeros are, and the endpoints ±1. We will give here the main term in the asymptotic expansion. It is possible to obtain a full asymptotic expansion, but for this we refer to [23]. We also refer to [23] for the asymptotics of the recurrence coefficients.

We will not discuss here the relation with random matrices. For this we refer to the papers [8] and [24] where the universality for the distribution of eigenvalue spacings was obtained from the Riemann-Hilbert method.

To end this introduction we recall some basic facts from complex analysis that will be used frequently in what follows. First we recall **Cauchy's formula**, which is the basis of all complex analysis. It says that

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(s)}{s-z} ds$$

whenever f is analytic in a domain Ω, γ is a simple closed, positively oriented curve in Ω, encircling a domain Ω_0 which also belongs to Ω, and $z \in \Omega_0$.

A second basic fact is **Liouville's theorem**, which says that a bounded entire function is constant. An extension of Liouville's theorem is the following. If f is entire and $f(z) = \mathcal{O}(z^n)$ as $z \to \infty$, then f is a polynomial of degree at most n.

Exercise 1. If you have not seen this extension of Liouville's theorem before, (or if you forgot about it) try to prove it.

We also recall **Morera's theorem**, which says that if f is continuous in a domain Ω and satisfies $\oint_\gamma f(z)dz = 0$ for all closed contours γ in Ω, then f is analytic in Ω.

We will also use some basic facts about isolated singularities of analytic functions. In a basic course in complex analysis you learn that an isolated singularity is either removable, a pole, or an essential singularity. **Riemann's theorem on removable singularities** says that if an analytic function is bounded near an isolated singularity, then the singularity is removable. The following is an extension of this result.

Exercise 2. Let $a \in \Omega$. If f is analytic in $\Omega \setminus \{a\}$, and $\lim_{z \to a} (z-a)f(z) = 0$, then a is a removable singularity of f.

2 Boundary values of analytic functions

We will deal with boundary values of analytic functions on curves. Suppose γ is a curve, which could be an arc, or a closed contour, or a system of arcs and contours. We will always consider **oriented curves**. The orientation induces a $+$side and a $-$side on γ. By definition, the $+$side is on the left, while traversing γ according to its orientation, and the $-$side is on the right.

All curves we consider are smooth (C^1 or even analytic), but the curves may have points of self-intersection or endpoints. At such points the $+$ and $-$sides are not defined. We use γ^o to denote γ without points of self-intersection and endpoints.

Let f be an analytic function on $\mathbb{C} \setminus \gamma$. The boundary values of f in $s \in \gamma^o$ are defined by

$$f_+(s) = \lim_{\substack{z \to s \\ z \text{ on } +\text{side}}} f(z), \qquad f_-(s) = \lim_{\substack{z \to s \\ z \text{ on } -\text{side}}} f(z),$$

provided these limits exist. If these limits exist for every $s \in \gamma^o$, and f_+ and f_- are continuous functions on γ^o, then we say that f has **continuous boundary values** on γ^o. It is possible to study boundary values in other senses, like L^p-sense, see [7, 14], but here we will always consider boundary values in the sense of continuous boundary values.

If the boundary values f_+ and f_- of f exists, and if we put $v(s) = f_+(s) - f_-(s)$, we see that f satisfies the following **Riemann-Hilbert problem** (boundary value problem for analytic functions)

(RH1) f is analytic in $\mathbb{C} \setminus \gamma$.
(RH2) $f_+(s) = f_-(s) + v(s)$ for $s \in \gamma^o$.

We say that v is the jump for f over γ^o.

Suppose now that conversely, we are given $v(s)$ for $s \in \gamma^o$. Then we may ask ourselves whether the above Riemann-Hilbert problem has a solution f, and whether the solution is unique. It is easy to see that the solution cannot be unique, since we can add an entire function to f and obtain another solution. So we need to impose an extra condition to guarantee uniqueness. This is typically an asymptotic condition, such as

(RH3) $f(z) \to 0$ as $z \to \infty$.

In this way we have normalized the Riemann-Hilbert problem at infinity. It is also possible to normalize at other points, but we will only meet problems where the normalization is at infinity.

It turns out that there is a unique solution if v is Hölder continuous on γ and if γ is a simple closed contour, or a finite disjoint union of simple closed contours, see [17, 28]. Then there are no points of self-intersection or endpoints so that $\gamma = \gamma^o$. In the case of points of self-intersection or endpoints, we need extra conditions at those points.

If γ is a simple closed contour, oriented positively, and if v is Hölder continuous on γ, then it can be shown that

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{v(s)}{s - z} ds \tag{2.1}$$

is the unique solution of the Riemann-Hilbert problem (RH1), (RH2), (RH3). It is called the **Cauchy transform** of v and we denote it also by $C(v)$. We will not go into the general existence theory here (for general theory, see e.g. [17, 28]), but leave it as an (easy) exercise for the case of analytic v.

Exercise 3. Assume that v is analytic in a domain Ω and that γ is a simple closed contour in Ω. Prove that the Cauchy transform of v satisfies the Riemann-Hilbert problem (RH1), (RH2), (RH3).
[Hint: Use a deformation of γ and Cauchy's formula.]

Exercise 4. Give an explicit solution of the Riemann-Hilbert problem (RH1), (RH2), (RH3), for the case where the jump v is a rational function with no poles on the simple closed contour γ.
[Hint: Use partial fraction decomposition of v.]

To establish uniqueness, one assumes as usual that there are two solutions f_1 and f_2. Then the difference $g = f_1 - f_2$ will solve a homogeneous Riemann-Hilbert problem with trivial jump $g_+ = g_-$ on γ. Then it follows from Morera's theorem that g is analytic on γ. Hence g is an entire function. From the asymptotic condition (RH3) it follows that $g(z) \to 0$ as $z \to \infty$, and therefore, by Liouville's theorem, g is identically zero.

We will use the above argument, based on Morera's theorem, also in other situations. We leave it as an exercise.

Exercise 5. Suppose that γ is a curve, or a system of curves, and that f is analytic on $\mathbb{C} \setminus \gamma$. Let γ_0 be an open subarc of γ^o so that f has continuous boundary values f_+ and f_- on γ_0 that satisfy $f_+ = f_-$ on γ_0. Show that f is analytic across γ_0.

In the case that γ has points of self-intersection or endpoints, extra conditions are necessary at the points of $\gamma \setminus \gamma^o$. We consider this for the case of the interval $[-1, 1]$ in the following exercise.

Exercise 6.

(a) Suppose that γ is the interval $[-1, 1]$ and v is a continuous function on $(-1, 1)$. Also assume that the Riemann-Hilbert problem (RH1), (RH2), (RH3) has a solution. Show that the solution is not unique.
(b) Show that there is a unique solution if we impose, in addition to (RH1), (RH2), (RH3), the conditions that

$$\lim_{z \to 1} (z - 1) f(z) = 0, \qquad \lim_{z \to -1} (z + 1) f(z) = 0.$$

The next step is to go from an additive Riemann-Hilbert problem to a multiplicative one. This means that instead of (RH2), we have a jump condition

(RH4) $f_+(s) = f_-(s) v(s)$ for $s \in \gamma^o$.

In this case, the asymptotic condition is typically

(RH5) $f(z) \to 1$ as $z \to \infty$.

If γ is a simple closed contour, and if v is continuous and non-zero on γ, then we define the index (or winding number) of v by

$$\operatorname{ind} v = \frac{1}{2\pi} \Delta_\gamma v(s).$$

This is $\frac{1}{2\pi}$ times the change in the argument of $v(s)$ as we go along γ once in the positive direction. The index is an integer. If the index is zero, then we can take a continuous branch of the logarithm of v on γ, and obtain from (RH4) the additive jump condition

$$(\log f)_+(s) = (\log f)_-(s) + \log v(s), \qquad s \in \gamma.$$

This has a solution as a Cauchy transform

$$\log f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{\log v(s)}{s-z} ds,$$

provided $\log v$ is Hölder continuous on γ. Then

$$f(z) = \exp\left(\frac{1}{2\pi i} \oint_\gamma \frac{\log v(s)}{s-z} ds \right) \tag{2.2}$$

solves the additive Riemann-Hilbert problem (RH1), (RH4), (RH5).

Exercise 7. How would you solve a Riemann-Hilbert problem with jump condition

$$f_+(s) f_-(s) = v(s)$$

for s on a simple closed contour γ?

3 Matrix Riemann-Hilbert problems

The Riemann-Hilbert problems that are associated with orthogonal polynomials are stated in a matrix form for 2×2 matrix valued analytic functions.

A matrix valued function $R : \mathbb{C} \setminus \gamma \to \mathbb{C}^{2\times 2}$ is analytic if all four entries of R are analytic functions on $\mathbb{C} \setminus \gamma$. Then a typical Riemann-Hilbert problem for 2×2 matrices is the following

(mRH1) $R : \mathbb{C} \setminus \gamma \to \mathbb{C}^{2\times 2}$ is analytic.
(mRH2) $R_+(s) = R_-(s)V(s)$ for $s \in \gamma^o$, where $V : \gamma^o \to \mathbb{C}^{2\times 2}$ is a given matrix valued function on γ^o.
(mRH3) $R(z) \to I$ as $z \to \infty$, where I denotes the 2×2 identity matrix.

Because of (mRH3) the problem is normalized at infinity. The matrix valued function V in (mRH2) is called the **jump matrix**. If $\gamma^o \neq \gamma$, then additional conditions have to be imposed at the points of self-intersection and the endpoints.

The existence theory of the **matrix Riemann-Hilbert problem** given by (mRH1), (mRH2), (mRH3) is quite complicated, and we will not deal with it. In the problems that we will meet, we know from the beginning that there is a solution. We know the solution because it is built out of orthogonal polynomials. See [6] for a systematic treatment of the general theory of matrix Riemann-Hilbert problems.

Also, we will only meet Riemann-Hilbert problems where the jump matrix V satisfies

$$\det V(s) = 1, \qquad s \in \gamma^o. \tag{3.1}$$

Then we can establish uniqueness of the Riemann-Hilbert problem (mRH1), (mRH2), (mRH3) on a simple closed contour. The argument is as follows.

First we consider the scalar function $\det R : \mathbb{C} \setminus \gamma \to \mathbb{C}$. It is analytic on $\mathbb{C} \setminus \gamma$, and in view of (mRH2) and (3.1) we have $(\det R)_+ = (\det R)_-$ on γ. Thus $\det R$ is an entire function. From (mRH3) it follows that $\det R(z) \to 1$ as $z \to \infty$, so that by Liouville's theorem $\det R(z) = 1$ for every $z \in \mathbb{C} \setminus \gamma$. Now suppose that $\tilde{R}$ is another solution of (mRH1), (mRH2), (mRH3). Since $R(z)$ has determinant 1, we can take the inverse, and we consider $X(z) = \tilde{R}(z)[R(z)]^{-1}$. Then X is clearly analytic on $\mathbb{C} \setminus \gamma$, and for $s \in \gamma$,

$$\begin{aligned} X_+(s) = \tilde{R}_+(s)[R_+(s)]^{-1} &= \tilde{R}_-(s)V(s)[R_-(s)V(s)]^{-1} \\ &= \tilde{R}_-(s)R_-(s) = X_-(s). \end{aligned}$$

Thus X is entire. Finally, we have $X(z) \to I$ as $z \to \infty$, so that by Liouville's theorem, we get $X(z) = I$ for every $z \in \mathbb{C} \setminus \gamma$, which shows that $\tilde{R} = R$. Thus R is the unique solution of (mRH1), (mRH2), (mRH3).

While existence of the solution is not an issue for us, we do need a result on the behavior of the solution R. We need to know that when V is close to the identity matrix on γ, then R is close to the identity matrix in the complex plane. We will need this result only for simple closed contours γ and it will be enough for us to deal with jump matrices that are analytic in a neighborhood of γ.

In order to specify the notion of closeness to the identity matrix, we need a norm on matrices. We can take any matrix norm, but for definiteness we will take the matrix infinity norm (maximum row sum) defined for 2×2 matrices R by

$$\|R\| = \max\left(|R_{11}| + |R_{12}|, |R_{21}| + |R_{22}|\right).$$

If $R(z)$ is a matrix-valued function defined on a set Ω, we define

$$\|R\|_\Omega = \sup_{z \in \Omega} \|R(z)\|,$$

where for $\|R(z)\|$ we use the infinity norm. If $R(z)$ is analytic on a domain Ω, then one may show that $\|R(z)\|$ is subharmonic as a function of z. If $R(z)$ is also continuous on $\bar{\Omega}$, then by the the maximum principle for subharmonic functions, it assumes its maximum value on the boundary of Ω.

With these preliminaries we can establish the following result. The following elementary complex analysis proof is due to A.I. Aptekarev [2].

Theorem 3.1. *Suppose γ is a positively oriented simple closed contour and Ω is an open neighborhood of γ. Then there exist constants C and $\delta > 0$ such that a solution R of the matrix Riemann-Hilbert problem (mRH1), (mRH2), (mRH3) with a jump matrix V that is analytic on Ω with*

$$\|V - I\|_\Omega < \delta,$$

satisfies

$$\|R(z) - I\| < C\|V - I\|_\Omega \tag{3.2}$$

for every $z \in \mathbb{C} \setminus \gamma$.

Proof. In the proof we use $\mathrm{ext}(\gamma)$ and $\mathrm{int}(\gamma)$ to denote the exterior and interior of γ, respectively. So, $\mathrm{ext}(\gamma)$ is the unbounded component of $\mathbb{C} \setminus \gamma$, and $\mathrm{int}(\gamma)$ is the bounded component. Together with γ, we also consider two simple closed curves γ_e and γ_i, both homotopic to γ in Ω, so that $\gamma_e \subset \Omega \cap \mathrm{ext}(\gamma)$ and $\gamma_i \subset \Omega \cap \mathrm{int}(\gamma)$, see Figure 1.

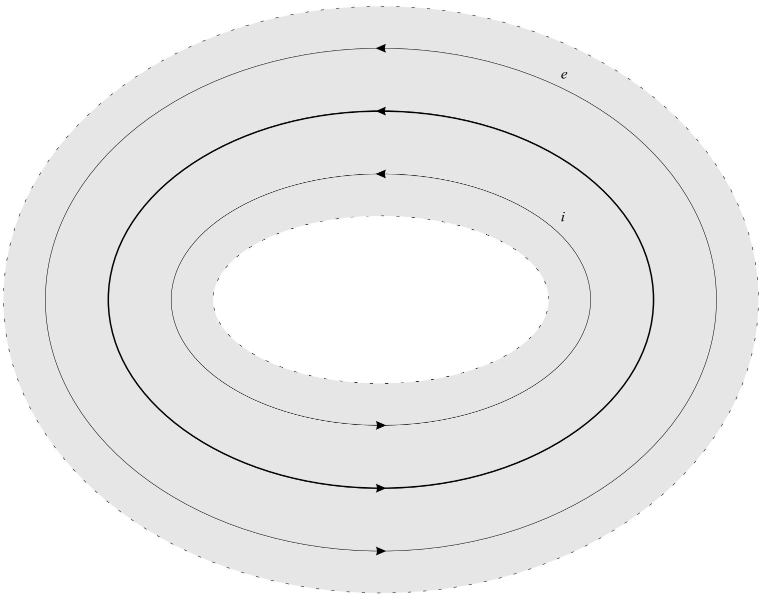

Fig. 1. Illustration for the proof of Theorem 3.1. The shaded region is the domain Ω, which contains the simple closed curves γ, γ_e, and γ_i.

We choose $r > 0$ so that

$$\min\left(\mathrm{dist}(z, \gamma_e), \mathrm{dist}(z, \gamma_i)\right) > r \qquad \text{for every } z \in \gamma, \tag{3.3}$$

where $\mathrm{dist}(z, \gamma_e)$ and $\mathrm{dist}(z, \gamma_i)$ denote the distances from z to the respective curves.

We write $\Delta = V - I$. Since $R_+ = R_- + R_-\Delta$, we may view $R_-\Delta$ as an additive jump for R on γ. By (2.1) and the asymptotic condition (mRH3) we thus have

$$R(z) = I + \frac{1}{2\pi i} \oint_\gamma \frac{R_-(s)\Delta(s)}{s - z} ds \tag{3.4}$$

for $z \in \mathbb{C} \setminus \gamma$. The integral in (3.4) is taken entrywise.

The idea of the proof is to show that for $\|\Delta\|_\Omega$ small enough, we have $\|R_-(s)\| \leq 4$ for every $s \in \gamma$. (Any other positive number than 4 would also do.) If we can prove this, then it follows by straightforward estimation on (3.4) that

$$\begin{aligned} \|R(z) - I\| &= \left\| \frac{1}{2\pi i} \oint_\gamma \frac{R_-(s)\Delta(s)}{s-z} ds \right\| \\ &\leq \frac{4l(\gamma)}{2\pi \mathrm{dist}(z,\gamma)} \|\Delta\|_\gamma \\ &\leq \frac{4l(\gamma)}{2\pi \mathrm{dist}(z,\gamma)} \|V - I\|_\Omega \end{aligned} \tag{3.5}$$

where $l(\gamma)$ is the length of γ. This then proves (3.2) for $\mathrm{dist}(z,\gamma) > r$ with constant

$$C = \frac{4l(\gamma)}{2\pi r}.$$

To handle the case when z is close to γ, we apply the same arguments to the curves γ_e and γ_i. Suppose for example that $z \in \mathrm{ext}(\gamma)$. Then we define

$$\tilde{R} = \begin{cases} R & \text{in } \mathrm{ext}(\gamma) \cup \mathrm{int}(\gamma_i) \\ RV^{-1} & \text{in } \mathrm{int}(\gamma) \cap \mathrm{ext}(\gamma_i) \end{cases}$$

Then $\tilde{R}_+ = \tilde{R}_-$ on γ so that $\tilde{R}$ is analytic across γ. On γ_i we have the jump $\tilde{R}_+ = \tilde{R}_- V$. The same arguments we will give below that lead to $\|R_-(s)\| \leq 4$ for $s \in \gamma$ will also show that $\|\tilde{R}_-(s)\| \leq 4$ for $s \in \gamma_i$ (provided $\|\Delta\|_\Omega$ is sufficiently small). Then an estimate similar to (3.5) shows that for every z,

$$\|\tilde{R}(z) - I\| \leq \frac{4l(\gamma_i)}{2\pi \mathrm{dist}(z,\gamma_i)} \|V - I\|_\Omega.$$

For $z \in \mathrm{ext}(\gamma)$, we have $\tilde{R}(z) = R(z)$ and $\mathrm{dist}(z,\gamma_i) > r$ by (3.3), so that we get (3.2) with a maybe different constant C. The same arguments apply for $z \in \mathrm{int}(\gamma)$. In that case we define $\tilde{R}$ so that it has a jump on γ_e.

So it remains to prove that $\|R_-(z)\| \leq 4$ for every $z \in \gamma$. In order to do this we put

$$M = \max_{z \in \gamma} \|R_-(z)\|.$$

Since $R_-(z)$ are the continuous boundary values for R taken from $\mathrm{ext}(\gamma)$, and R is analytic in $\mathrm{ext}(\gamma)$, including the point at infinity, we have by the maximum principle for subharmonic functions, that

$$\|R(z)\| \leq M, \qquad z \in \mathrm{ext}(\gamma).$$

We deform γ to γ_e lying in $\Omega \cap \mathrm{ext}(\gamma)$. Then $\mathrm{dist}(z,\gamma_e) > r$ for every $z \in \gamma$ by (3.3). For $z \in \mathrm{int}(\gamma)$, we then have

$$R(z) = I + \frac{1}{2\pi i} \oint_{\gamma_e} \frac{R(s)\Delta(s)}{s-z} ds.$$

Letting z go to γ from within $\mathrm{int}(\gamma)$, we then find

$$R_+(z) = I + \frac{1}{2\pi i} \oint_{\gamma_e} \frac{R(s)\Delta(s)}{s-z} ds, \qquad z \in \gamma,$$

and so, since $R_+ = R_-(I + \Delta)$,

$$R_-(z) = \left(I + \frac{1}{2\pi i} \oint_{\gamma_e} \frac{R(s)\Delta(s)}{s-z} ds \right) (I + \Delta(z))^{-1}, \qquad z \in \gamma.$$

We take norms, and estimate, where we use that $\|R(s)\| \leq M$ for $s \in \gamma_e$,

$$\|R_-(z)\| \leq \left(1 + \frac{l(\gamma_e)}{2\pi r} M \|\Delta\|_\Omega \right) \left\| (I + \Delta(z))^{-1} \right\|, \qquad z \in \gamma,$$

with $l(\gamma_e)$ the length of γ_e. If $\|\Delta(z)\| \leq \frac{1}{2}$ then $\left\| (I + \Delta(z))^{-1} \right\| \leq 1 + 2\|\Delta(z)\|$, which follows easily from estimating the Neumann series

$$(I + \Delta(z))^{-1} = \sum_{k=0}^{\infty} (-\Delta(z))^k .$$

So we assume $\delta > 0$ is small enough so that

$$\delta < \frac{1}{2} \qquad \text{and} \qquad \frac{l(\gamma_e)}{2\pi r} \delta (1 + 2\delta) < \frac{1}{2}.$$

Then, if $\|\Delta\|_\Omega < \delta$, we find for $z \in \gamma$,

$$\begin{aligned} \|R_-(z)\| &\leq \left(1 + \frac{l(\gamma_e)}{2\pi r} M\delta \right) (1 + 2\delta) \\ &= (1 + 2\delta) + \frac{l(\gamma_e)}{2\pi r} M\delta(1 + 2\delta) \\ &\leq 2 + \frac{1}{2} M. \end{aligned}$$

Taking the supremum for $z \in \gamma$, we get $M \leq 2 + \frac{1}{2}M$, which means that $M \leq 4$. So we have proved our claim that $\|R_-(z)\| \leq 4$ for every $z \in \gamma$, which completes the proof of the theorem.

Exercise 8. Analyze the proof of Theorem 3.1 and show that we can strengthen (3.2) to

$$\|R(z) - I\| \leq \frac{C}{1 + |z|} \|V - I\|_\Omega$$

for every $z \in \mathbb{C} \setminus \gamma$.

4 Riemann–Hilbert problem for orthogonal polynomials on the real line

Fokas, Its, and Kitaev [15] found a characterization of orthogonal polynomials in terms of a matrix Riemann-Hilbert problem.

We consider a weight function w on $\mathbb{R}$, which is smooth and has sufficient decay at $\pm\infty$, so that all moments $\int x^k w(x)\,dx$ exist. The weight induces a scalar product $\int f(x)g(x)w(x)dx$, and the Gram-Schmidt orthogonalization process applied to the sequence of monomials $1, x, x^2, \ldots,$ yields a sequence of **orthogonal polynomials** $\pi_0, \pi_1, \pi_2, \ldots$, that satisfy

$$\int \pi_n(x)\pi_m(x)w(x)\,dx = h_n\delta_{n,m}, \qquad h_n > 0.$$

We will choose the polynomials to be monic $\pi_n(x) = x^n + \cdots$. If we put

$$\gamma_n = h_n^{-1/2}, \qquad p_n(x) = \gamma_n \pi_n(x)$$

then the polynomials p_n are the orthonormal polynomials, i.e.,

$$\int p_n(x)p_m(x)w(x)dx = \delta_{n,m}.$$

The orthonormal polynomials satisfy a **three-term recurrence**

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)$$

with certain recurrence coefficients a_n and b_n. The monic form of the recurrence is

$$x\pi_n(x) = \pi_{n+1}(x) + b_n\pi_n(x) + a_n^2\pi_{n-1}(x).$$

Fokas, Its, Kitaev [15] formulated the following Riemann-Hilbert problem for a 2×2 matrix valued function $Y : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2}$.

(Y-RH1) Y is analytic in $\mathbb{C}\setminus\mathbb{R}$.

(Y-RH2) $Y_+(x) = Y_-(x)\begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$.

(Y-RH3) $Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

The asymptotic condition (Y-RH3) does not say that $Y(z)$ tends to the identity matrix as z tends to infinity (unless $n = 0$), so the problem is not normalized at infinity in the sense of (mRH3).

Theorem 4.1 (Fokas, Its, Kitaev). *The Riemann-Hilbert problem (Y-RH1)–(Y-RH3) for Y has a unique solution given by*

$$Y(z) = \begin{pmatrix} \pi_n(z) & C(\pi_n w)(z) \\ c_n\pi_{n-1}(z) & c_n C(\pi_{n-1}w)(z) \end{pmatrix} \tag{4.1}$$

where π_n *and* π_{n-1} *are the* ***monic*** *orthogonal polynomials of degrees* n *and* $n-1$*, respectively,* $C(\pi_j w)$ *is the Cauchy transform of* $\pi_j w$*,*

$$C(\pi_j w)(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\pi_j(x) w(x)}{x - z} dx,$$

and c_n *is the constant*

$$c_n = -2\pi i \gamma_{n-1}^2.$$

Proof. Consider the first row of Y. The condition (Y-RH2) gives for the first entry Y_{11}

$$(Y_{11})_+(x) = (Y_{11})_-(x), \qquad x \in \mathbb{R}.$$

Thus Y_{11} is an entire function. The asymptotic condition (Y-RH3) gives

$$Y_{11}(z) = z^n + \mathcal{O}(z^{n-1}) \qquad \text{as } z \to \infty.$$

By the extension of Liouville's theorem, this implies that Y_{11} is a monic polynomial of degree n. We call it P_n.

Now we look at Y_{12}. The jump condition (Y-RH2) gives

$$(Y_{12})_+(x) = (Y_{12})_-(x) + (Y_{11})_-(x) w(x).$$

We know already that $Y_{11} = P_n$, so that

$$(Y_{12})_+(x) = (Y_{12})_-(x) + P_n(x) w(x). \tag{4.2}$$

The asymptotic condition (Y-RH3) implies

$$Y_{12}(z) = \mathcal{O}\left(z^{-n-1}\right) \qquad \text{as } z \to \infty. \tag{4.3}$$

The conditions (4.2)–(4.3) constitute an additive scalar Riemann-Hilbert problem for Y_{12}. Its solution is given by the Cauchy transform

$$Y_{12}(z) = C(P_n w)(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{P_n(x) w(x)}{x - z} dx.$$

Now in general the Cauchy transform tends to zero like z^{-1} as $z \to \infty$, and not like z^{-n-1} as required in (4.3). We need extra conditions on the polynomial P_n to ensure that (4.3) is satisfied. We write

$$\frac{1}{x - z} = -\sum_{k=0}^{n-1} \frac{x^k}{z^{k+1}} + \frac{x^n}{z^n (x - z)}.$$

Then

$$\begin{aligned} Y_{12}(z) &= \frac{1}{2\pi i} \int P_n(x) w(x) \left[-\sum_{k=0}^{n-1} \frac{x^k}{z^{k+1}} + \frac{x^n}{z^n(x - z)} \right] dx \\ &= -\sum_{k=0}^{n-1} \frac{1}{2\pi i} \left[\int P_n(x) x^k w(x)\, dx \right] \frac{1}{z^{k+1}} + \mathcal{O}\left(z^{-n-1}\right). \end{aligned}$$

In order to have (4.3) we need that the coefficient of z^{-k-1} vanishes for $k = 0, \ldots, n-1$. Thus

$$\int P_n(x) x^k w(x)\, dx = 0, \qquad k = 0, \ldots, n-1.$$

This means that P_n is the orthogonal polynomial, and since P_n is monic, we have $P_n = \pi_n$. Thus we have shown that the first row of Y is equal to the expressions given in the equality (4.1). The equality for the second row is shown in a similar way. The details are left as an exercise.

Exercise 9. Show that the second row of Y is equal to the expressions given in (4.1).

Remark concerning the proof of Theorem 4.1 The above proof of Theorem 4.1 is not fully rigorous in two respects. First, we did not check that the jump condition (Y-RH2) is valid in the sense of continuous boundary values, and second, we did not check that the asymptotic condition (Y-RH3) holds uniformly as $z \to \infty$ in $\mathbb{C} \setminus \mathbb{R}$. This is not immediate since $\mathbb{R}$ is an unbounded contour.

Both of these questions are technical issues whose treatment falls outside the scope of this introduction. Suitable smoothness and decay properties have to be imposed on w. The reader is referred to [9, Appendix A] for a discussion of these matters. There it is shown that it is enough that $x^n w(x)$ belongs to the Sobolev space H^1 for every n.

5 Riemann–Hilbert problem for orthogonal polynomials on $[-1, 1]$

We will study polynomials that are orthogonal with respect to weights on the finite interval $[-1, 1]$. In particular we will consider modified Jacobi weight

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta} h(x), \qquad x \in (-1, 1) \tag{5.1}$$

where $\alpha, \beta > -1$ and h is positive on $[-1, 1]$ and analytic in a neighborhood of $[-1, 1]$. The weights (5.1) are a generalization of the Jacobi weights which have $h(x) \equiv 1$. In analogy with the case of the whole real line, the Riemann-Hilbert problem that characterizes the orthogonal polynomials has the following ingredients.

We look for a matrix valued function $Y : \mathbb{C} \setminus [-1, 1] \to \mathbb{C}^{2\times 2}$ that satisfies

(Y-RH1) Y is analytic in $\mathbb{C} \setminus [-1, 1]$.

(Y-RH2) $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}$ for $x \in (-1, 1)$

(Y-RH3) $Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$

Note that we restrict ourselves in the jump condition (Y-RH2) to the open interval $(-1,1)$. The jump is not defined at the endpoints ± 1, since the boundary values $Y_\pm$ are not defined there. If α or β (or both) is negative, there is also a problem with the definition of w at the endpoints.

We can show, as for the case of orthogonal polynomials on the real line, that

$$Y(z) = \begin{pmatrix} \pi_n(z) & C(\pi_n w)(z) \\ c_n \pi_{n-1}(z) & c_n C(\pi_{n-1} w)(z) \end{pmatrix} \tag{5.2}$$

is a solution of the Riemann-Hilbert problem, where now C denotes the Cauchy transform on $[-1,1]$, that is,

$$C(\pi_n w)(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\pi_n(x) w(x)}{x-z}\, dx.$$

However, this will not be the only solution. In order to ensure uniqueness we need extra conditions at the endpoints ± 1. The endpoint conditions are

(Y-RH4) As $z \to 1$, we have

$$Y(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z-1|^\alpha \\ 1 & |z-1|^\alpha \end{pmatrix} & \text{if } \alpha < 0, \\[2ex] \mathcal{O}\begin{pmatrix} 1 & \log|z-1| \\ 1 & \log|z-1| \end{pmatrix} & \text{if } \alpha = 0, \\[2ex] \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } \alpha > 0. \end{cases}$$

(Y-RH5) As $z \to -1$, we have

$$Y(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & |z+1|^\beta \\ 1 & |z+1|^\beta \end{pmatrix} & \text{if } \beta < 0, \\[2ex] \mathcal{O}\begin{pmatrix} 1 & \log|z+1| \\ 1 & \log|z+1| \end{pmatrix} & \text{if } \beta = 0, \\[2ex] \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } \beta > 0. \end{cases}$$

In (Y-RH4)–(Y-RH5) the $\mathcal{O}$ conditions are to be taken entrywise, so the condition (Y-RH4) in the case $\alpha < 0$ means that

$$\begin{aligned} Y_{11}(z) &= \mathcal{O}(1) & \qquad Y_{12}(z) &= \mathcal{O}(|z-1|^\alpha) \\ Y_{21}(z) &= \mathcal{O}(1) & \qquad Y_{22}(z) &= \mathcal{O}(|z-1|^\alpha) \end{aligned}$$

as $z \to 1$. So Y_{11} and Y_{21} should remain bounded at $z = 1$, while Y_{12} and Y_{22} are allowed to grow as $z \to 1$, but not faster than $\mathcal{O}(|z-1|^\alpha)$.

Now we can prove that Y given by (5.2) satisfies the boundary conditions (Y-RH4)–(Y-RH5), and that it is in fact the only solution to the Riemann-Hilbert problem (Y-RH1)–(Y-RH5). This is left to the reader as an exercise (see also [23]).

Exercise 10. Show that (5.2) satisfies the conditions (Y-RH4)–(Y-RH5).

Exercise 11. Show that the Riemann-Hilbert problem (Y-RH1)-(Y-RH5) for Y has (5.2) as its unique solution.

6 Basic idea of steepest descent method

The steepest descent method for Riemann-Hilbert problems consists of a sequence of explicit transformations, which in our case have the form

$$Y \mapsto T \mapsto S \mapsto R.$$

The ultimate goal is to arrive at a Riemann-Hilbert problem for R on a system of contours γ,

(R-RH1) R is analytic on $\mathbb{C} \setminus \gamma$,
(R-RH2) $R_+(s) = R_-(s)V(s)$ for $s \in \gamma$,
(R-RH3) $R(z) \to I$ as $z \to \infty$,

in which the jump matrix V is close to the identity.

Note that Y depends on n through the asymptotic condition

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix},$$

and so, to indicate the n-dependence, we may write $Y = Y^{(n)}$. Also the transformed functions T, S, and R depend on n, say $T = T^{(n)}$, $S = S^{(n)}$, and $R = R^{(n)}$. The jump matrix $V = V^{(n)}$ in (R-RH2) also depends on n. The contour γ, however, does not depend on n. The jump matrices that we will find have analytic continuations to a neighborhood of γ, which is also independent of n, and we will have

$$V^{(n)}(s) = I + \mathcal{O}\left(\frac{1}{n}\right) \qquad \text{as } n \to \infty,$$

uniformly for s in a neighborhood of γ. Then, from Theorem 3.1, we can conclude that

$$R^{(n)}(z) = I + \mathcal{O}\left(\frac{1}{n}\right) \qquad \text{as } n \to \infty,$$

uniformly for $z \in \mathbb{C} \setminus \gamma$. Tracing back the steps $Y^{(n)} \mapsto T^{(n)} \mapsto S^{(n)} \mapsto R^{(n)}$, we find asymptotics for $Y^{(n)}$, valid uniformly in the complex plane. So, in particular, since π_n is the $(1,1)$ entry of Y_{11}, we find asymptotic formulas for the orthogonal polynomials that are uniformly valid in every region of the complex plane.

The steepest descent method for Riemann-Hilbert methods is an alternative to more classical asymptotic methods that have been developed for differential equations or integral representations. The Jacobi polynomials $P_n^{(\alpha,\beta)}$

that are orthogonal with respect to $(1-x)^\alpha(1+x)^\beta$ have an integral representation and they satisfy a second order differential equation. As a result their asymptotic behavior as $n \to \infty$ is very well-known, see [31]. The orthogonal polynomials associated with the weights (5.1) do not have an integral representation or a differential equation, and so asymptotic methods that are based on these cannot be applied. The steepest descent method for Riemann-Hilbert problems is the first method that is able to give full asymptotic expansions for orthogonal polynomials in a number of cases where integral representations and differential equations are not available.

It must be noted that other methods, based on potential theory and approximation theory, have also been used for asymptotics of orthogonal polynomials [25, 30, 32]. These methods apply to weights with less smoothness, but the results are not as strong as the ones we will present here.

7 First transformation $Y \mapsto T$

The first transformation uses the mapping

$$\varphi(z) = z + (z^2-1)^{1/2}, \qquad z \in \mathbb{C} \setminus [-1,1].$$

That branch of the square root is chosen which is analytic in $\mathbb{C} \setminus [-1,1]$ and which is positive for $z > 1$. Thus $(z^2-1)^{1/2}$ is negative for real $z < -1$.

Exercise 12. Show the following

(a) φ is a one-to-one map from $\mathbb{C} \setminus [-1,1]$ onto the exterior of the unit disk.
(b) $\varphi(z) = 2z + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.
(c) $\varphi_+(x)\varphi_-(x) = 1$ for $x \in (-1,1)$.

The first transformation is

$$T(z) = \begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix} Y(z) \begin{pmatrix} \varphi(z)^{-n} & 0 \\ 0 & \varphi(z)^n \end{pmatrix}. \tag{7.1}$$

Then straightforward calculations show that T satisfies the Riemann-Hilbert problem

(T-RH1) T is analytic in $\mathbb{C} \setminus [-1,1]$.
(T-RH2) $T_+(x) = T_-(x) \begin{pmatrix} \varphi_+(x)^{-2n} & w(x) \\ 0 & \varphi_-(x)^{-2n} \end{pmatrix}$ for $x \in (-1,1)$.
(T-RH3) $T(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.
(T-RH4)–(T-RH5) T has the same behavior as Y near ± 1.

Exercise 13. Verify that the jump condition (T-RH2) and the asymptotic condition (T-RH3) hold.

The effect of the transformation $Y \mapsto T$ is that the problem is normalized at infinity, since $T(z) \to I$ as $z \to \infty$. This is good. What is not so good, is that the jump matrix for T is more complicated. The entries on the diagonal have absolute value one, and so for large n, they are rapidly oscillating as x varies over the interval $(-1,1)$. The effect of the next transformation will be to transform these oscillations into exponentially small terms.

Why did we choose to perform the transformation (7.1)? An easy answer would be: because we will see later that it works. A second answer could be based on a list of desirable properties that the function φ should have. The honest answer is that already a lot is known about orthogonal polynomials and their asymptotics, see, e.g., [16, 25, 29, 31, 33]. For example it is known that

$$\lim_{n\to\infty} (\pi_n(z))^{1/n} = \frac{\varphi(z)}{2}, \qquad z \in \mathbb{C} \setminus [-1,1] \tag{7.2}$$

where that branch of the nth root is chosen which behaves like z at infinity. This is the **nth root asymptotics** of the polynomials π_n. It is intimately connected with the weak convergence of zeros. The nth root asymptotics (7.2) holds for a very large class of weights w on $(-1,1)$. It is for example enough that $w > 0$ almost everywhere on $(-1,1)$.

A stronger kind of asymptotics is

$$\lim_{n\to\infty} \frac{2^n \pi_n(z)}{\varphi(z)^n} = \frac{\tilde{D}(\infty)}{\tilde{D}(z)}, \tag{7.3}$$

which is valid uniformly for $z \in \mathbb{C} \setminus [-1,1]$. The **strong asymptotics** (7.3) is valid for weights w satisfying the **Szegő condition**, that is,

$$\int_{-1}^{1} \frac{\log w(t)}{\sqrt{1-t^2}} dt > -\infty.$$

The function $\tilde{D}$ appearing in the right-hand side of (7.3) is known as the **Szegő function**. It is analytic and non-zero on $\mathbb{C} \setminus [-1,1]$, and there it is a finite limit

$$\lim_{z\to\infty} \tilde{D}(z) = \tilde{D}(\infty) \in (0,\infty).$$

Note that in what follows, we will use a different definition for the Szegő function, and we will call it D, instead of $\tilde{D}$.

Since we want to recover the asymptotics (7.2)–(7.3) (and more) for the modified Jacobi weights, we cannot avoid using the functions that appear there. This explains why we perform the transformation (7.1). The $(1,1)$ entry of T is

$$T_{11}(z) = \frac{2^n \pi_n(z)}{\varphi(z)^n}$$

and this is a quantity which we like. We have peeled off the main part of the asymptotics of π_n. By (7.3) we know that the limit of $T_{11}(z)$ exists as $n \to \infty$, and the limit is expressed in terms of the Szegő function associated with w. This indicates that the transformation $Y \mapsto T$ makes sense. It also indicates that we will have to use the Szegő function in one of our future transformations.

Exercise 14. Another idea would be to define

$$\tilde{T}(z) = Y(z) \begin{pmatrix} \varphi(z)^{-n} & 0 \\ 0 & \varphi(z)^n \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix}.$$

This would also lead to the $(1,1)$ entry being $\frac{2^n \pi_n(z)}{\varphi(z)}$. Work out the Riemann-Hilbert problem for $\tilde{T}$. What is the advantage of T over $\tilde{T}$?

Exercise 15. The transformation

$$\widehat{T}(z) = \begin{pmatrix} \varphi(z)^{-n} & 0 \\ 0 & \varphi(z)^n \end{pmatrix} Y(z) \begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix}$$

would lead to the same $(1,1)$ entry, but this transformation is a very bad idea. Why?

8 Second transformation $T \mapsto S$

The second transformation $T \mapsto S$ is based on a factorization of the jump matrix in (T-RH2)

$$\begin{pmatrix} \varphi_+^{-2n} & w \\ 0 & \varphi_-^{-2n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi_-^{-2n} & 1 \end{pmatrix} \begin{pmatrix} 0 & w \\ -\frac{1}{w} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi_+^{-2n} & 1 \end{pmatrix},$$

which can be verified by direct calculation.

Instead of making one jump across the interval $(-1,1)$, we can now think that we are making three jumps according to the above factorization. That is, if we cross the interval $(-1,1)$ from the upper half-plane into the lower half-plane, we will first make the jump $\begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi_+^{-2n} & 1 \end{pmatrix}$, then the jump $\begin{pmatrix} 0 & w \\ -\frac{1}{w} & 0 \end{pmatrix}$, and finally the jump $\begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi_-^{-2n} & 1 \end{pmatrix}$.

Now recall that $w(x) = (1-x)^\alpha (1+x)^\beta h(x)$ is the modified Jacobi weight. The extra factor h is positive on $[-1,1]$ and analytic in a neighborhood of $[-1,1]$. Then there is a neighborhood U of $[-1,1]$ so that h is analytic on U with positive real part, see Figure 2. All our future deformations will be contained in U.

We will consider $(1-z)^\alpha$ as an analytic function on $\mathbb{C} \setminus [1,\infty)$ where we take the branch which is positive for real $z < 1$. Similarly, we will view $(1+z)^\beta$ as an analytic function on $\mathbb{C} \setminus (-\infty,-1]$. Then

$$w(z) = (1-z)^\alpha (1+z)^\beta h(z)$$

is non-zero and analytic on $U \setminus ((-\infty,-1] \cup [1,\infty))$, and it is an analytic continuation of our weight $w(x)$.

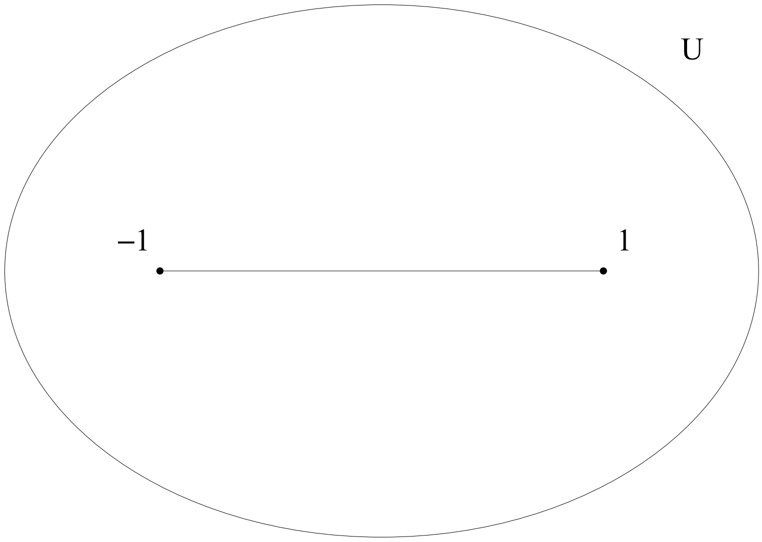

Fig. 2. Neighborhood U of $[-1,1]$ so that h is analytic with positive real part in U.

The two jump matrices $\begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi_+^{-2n} & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi_-^{-2n} & 1 \end{pmatrix}$ then have natural extensions into the upper and lower half-planes, respectively, both given by $\begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi^{-2n} & 1 \end{pmatrix}$. Note that for z away from the interval $[-1,1]$, we have $|\varphi(z)| > 1$, so that $\begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi(z)^{-2n} & 1 \end{pmatrix}$ is close to the identity matrix if n is large.

We open a lens-shaped region around $(-1,1)$ as shown in Figure 3 of the paper. The lens is assumed to be contained in the domain U. The upper and lower lips of the lens are denoted by Σ_1 and Σ_3 respectively. The interval $[-1,1]$ is denoted here by Σ_2.

Then we define the second transformation $T \mapsto S$ by

$$S = \begin{cases} T & \text{outside the lens} \\ T\begin{pmatrix} 1 & 0 \\ -\frac{1}{w}\varphi^{-2n} & 1 \end{pmatrix} & \text{in the upper part of the lens} \\ T\begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi^{-2n} & 1 \end{pmatrix} & \text{in the lower part of the lens.} \end{cases} \tag{8.1}$$

The transformation results in jumps for S on the interior of the three curves Σ_1, $\Sigma_2 = [-1,1]$ and Σ_3. It follows that S satisfies the following Riemann-Hilbert problem

(S-RH1) S is analytic in $\mathbb{C} \setminus (\Sigma_1 \cup [-1,1] \cup \Sigma_3)$.

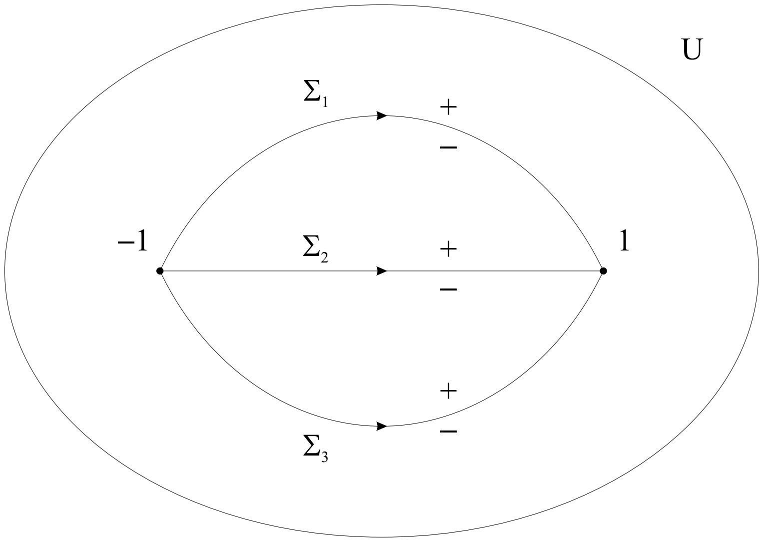

Fig. 3. Opening of lens in domain U.

(S-RH2) $S_+ = S_- \begin{pmatrix} 0 & w \\ -\frac{1}{w} & 0 \end{pmatrix}$ on $(-1,1)$.

$S_+ = S_- \begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi^{-2n} & 1 \end{pmatrix}$ on Σ_1^o and Σ_3^o.

(S-RH3) $S(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

(S-RH4) Conditions as $z \to 1$:

- For $\alpha < 0$:
$$S(z) = \mathcal{O}\begin{pmatrix} 1 & |z-1|^\alpha \\ 1 & |z-1|^\alpha \end{pmatrix}.$$
- For $\alpha = 0$:
$$S(z) = \mathcal{O}\begin{pmatrix} \log|z-1| & \log|z-1| \\ \log|z-1| & \log|z-1| \end{pmatrix}.$$
- For $\alpha > 0$:
$$S(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{as } z \to 1 \text{ outside the lens}, \\ \mathcal{O}\begin{pmatrix} |z-1|^{-\alpha} & 1 \\ |z-1|^{-\alpha} & 1 \end{pmatrix} & \text{as } z \to 1 \text{ inside the lens}. \end{cases}$$

(S-RH5) Similar conditions as $z \to -1$.

The endpoint condition (S-RH4) is rather awkward now, especially if $\alpha > 0$, where we distinguish between $z \to 1$ from within the lens, or from outside the lens. It turns out that they are necessary if we want a unique solution.

Exercise 16. Show that the Riemann-Hilbert problem (S-RH1)–(S-RH5) for S has a unique solution.
[Note: we already know that there is a solution, namely the one that we find after transformations $Y \mapsto T \mapsto S$. One way to prove that there is no other solution, is to show that these transformations are invertible. Another way is to assume that there is another solution $\tilde{S}$ and show that it must be equal to the S we already have.]

The opening of the lens in the transformation $T \mapsto S$ is also a crucial step in the papers [8, 9] by Deift et al., which deal with orthogonal polynomials on the real line, see also [7]. It transforms the oscillatory diagonal terms in the jump matrix for T into exponentially small off-diagonal terms in the jump matrix for S. Indeed, in (S-RH2) we have a jump matrix $\begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi^{-2n} & 1 \end{pmatrix}$ on Σ_1^o and Σ_3^o. Since $|\varphi(z)| > 1$ for z on Σ_1^o and Σ_3^o, the entry $\frac{1}{w}\varphi^{-2n}$ in the jump matrix tends to 0 exponentially fast. The convergence is uniform on compact subsets of Σ_1^o and Σ_3^o, but it is not uniform near the endpoints ± 1.

9 Special case $\alpha = \beta = -\frac{1}{2}$

For special values of α and β, the subsequent analysis simplifies considerably. These are the cases $\alpha = \pm\frac{1}{2}$, $\beta = \pm\frac{1}{2}$. We will treat the case $\alpha = \beta = -\frac{1}{2}$, so that

$$w(z) = (1 - z^2)^{-\frac{1}{2}} h(z).$$

In this case, we open up the lens further so that Σ_1 and Σ_3 coincide along two intervals $[-1-\delta, -1]$ and $[1, 1+\delta]$, where $\delta > 0$ is some positive number.

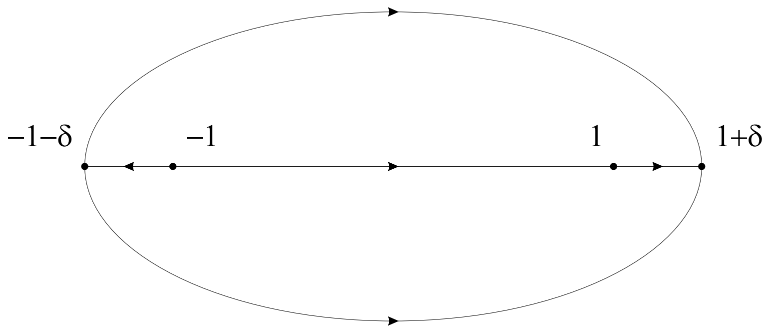

Fig. 4. Opening of lens in case $\alpha = \beta = -\frac{1}{2}$. The upper and lower lips of the lens coincide on the intervals $[-1 - \delta, -1]$ and $[1, 1 + \delta]$.

On the intervals $(-1 - \delta, -1)$ and $(1, 1 + \delta)$ two jumps are combined. If we calculate the total jump there, we have to be careful, since w has a jump on these intervals too. In fact, we have

$$w_+(x) = -w_-(x), \qquad \text{for } x > 1 \text{ or } x < -1 \text{ (with } x \in U),$$

which follows from the fact that $\alpha = \beta = -\frac{1}{2}$. Then we calculate on $(-1-\delta, -1)$ or $(1, 1+\delta)$,

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{w_-}\varphi^{-2n} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{w_+}\varphi^{-2n} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \left(\frac{1}{w_-} + \frac{1}{w_+}\right)\varphi^{-2n} & 1 \end{pmatrix} = I.$$

This means that S is analytic across $(-1-\delta, -1)$ and $(1, 1+\delta)$. The only remaining jumps are on $[-1, 1]$ and on a simple closed contour that we call γ. We choose to orient γ in the positive direction (counterclockwise). It means that in the upper half-plane we have to reverse the orientation as shown in Figure 5.

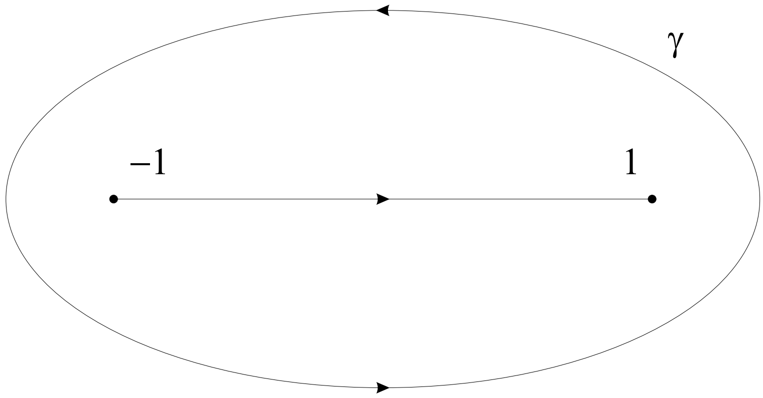

Fig. 5. Closed contour γ that encircles $[-1, 1]$. S has jumps only on γ and $[-1, 1]$.

It follows that in this special case S satisfies the following Riemann-Hilbert problem.

(S-RH1) S is analytic in $\mathbb{C} \setminus ([-1, 1] \cup \gamma)$.

(S-RH2) $S_+ = S_- \begin{pmatrix} 0 & w \\ -\frac{1}{w} & 0 \end{pmatrix}$ on $(-1, 1)$

$S_+ = S_- \begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi^{-2n} & 1 \end{pmatrix}$ on $\gamma \cap \{\mathrm{Im} z < 0\}$ and

$S_+ = S_- \begin{pmatrix} 1 & 0 \\ -\frac{1}{w}\varphi^{-2n} & 1 \end{pmatrix}$ on $\gamma \cap \{\mathrm{Im} z > 0\}$.

(S-RH3) $S(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ as $z \to \infty$.

(S-RH4) $S(z) = \mathcal{O}\begin{pmatrix} 1 & |z-1|^{-\frac{1}{2}} \\ 1 & |z-1|^{-\frac{1}{2}} \end{pmatrix}$ as $z \to 1$.

(S-RH5) $S(z) = \mathcal{O}\begin{pmatrix} 1 & |z+1|^{-\frac{1}{2}} \\ 1 & |z+1|^{-\frac{1}{2}} \end{pmatrix}$ as $z \to -1$.

Exercise 17. If you do the analysis in this section for the case $\alpha = \beta = +\frac{1}{2}$ then everything will be the same except for the endpoint conditions (S-RH4) and (S-RH5). Show that they change to $S(z) = \mathcal{O}\begin{pmatrix} |z-1|^{-\frac{1}{2}} & 1 \\ |z-1|^{-\frac{1}{2}} & 1 \end{pmatrix}$ as $z \to 1$, and $S(z) = \mathcal{O}\begin{pmatrix} |z+1|^{-\frac{1}{2}} & 1 \\ |z+1|^{-\frac{1}{2}} & 1 \end{pmatrix}$ as $z \to -1$, respectively.

10 Model Riemann Hilbert problem

The jump matrix for S is uniformly close to the identity matrix on the simple closed contour γ. Only the jump on the interval $[-1,1]$ is not close to the identity. This suggests to look at the following model Riemann-Hilbert problem, where we ignore the jump on γ. We look for $N : \mathbb{C} \setminus [-1,1] \to \mathbb{C}^{2\times 2}$ satisfying

(N-RH1) N is analytic in $\mathbb{C} \setminus [-1,1]$.

(N-RH2) $N_+(x) = N_-(x) \begin{pmatrix} 0 & w(x) \\ -\frac{1}{w(x)} & 0 \end{pmatrix}$ for $x \in (-1,1)$.

(N-RH3) $N(z) \to I$ as $z \to \infty$.

(N-RH4) $N(z) = \mathcal{O}\begin{pmatrix} 1 & |z-1|^{-\frac{1}{2}} \\ 1 & |z-1|^{-\frac{1}{2}} \end{pmatrix}$ as $z \to 1$.

(N-RH5) $N(z) = \mathcal{O}\begin{pmatrix} 1 & |z+1|^{-\frac{1}{2}} \\ 1 & |z+1|^{-\frac{1}{2}} \end{pmatrix}$ as $z \to -1$.

The conditions (N-RH4) and (N-RH5) are specific for the weights under consideration (i.e., modified Jacobi weights with $\alpha = \beta = -\frac{1}{2}$). For more general weights on $[-1,1]$, the corresponding problem for N would include the parts (N-RH1), (N-RH2) and (N-RH3), but (N-RH4) and (N-RH5) have to be modified.

Exercise 18. Let N be given by

$$N(z) = \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ \frac{a(z)-a^{-1}(z)}{-2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix},$$

where

$$a(z) = \frac{(z-1)^{1/4}}{(z+1)^{1/4}}.$$

Show that N satisfies parts (N-RH1), (N-RH2), and (N-RH3) with $w(x) \equiv 1$ (Legendre case). What would the conditions (N-RH4) and (N-RH5) be for this case?

The solution to the Riemann-Hilbert problem (N-RH1)–(N-RH5) is constructed with the use of the **Szegő function**. The Szegő function associated with a weight w on $[-1,1]$ is a scalar function $D : \mathbb{C} \setminus [-1,1] \to \mathbb{C}$ such that

(D-RH1) D is analytic and non-zero in $\mathbb{C} \setminus [-1,1]$.
(D-RH2) $D_+(x)D_-(x) = w(x)$ for $x \in (-1,1)$.
(D-RH3) the limit $\lim_{z\to\infty} D(z) = D_\infty$ exists and is a positive real number.

Note that (D-RH1)–(D-RH3) is a multiplicative scalar Riemann-Hilbert problem. We have not specified any endpoint conditions, so we cannot expect a unique solution. In general we want that $|D|$ behaves like $|w|^{1/2}$ also near the endpoints. So for a modified Jacobi weight we would add the endpoint conditions

(D-RH4) $D(z) = \mathcal{O}(|z-1|^{\alpha/2})$ as $z \to 1$,
(D-RH5) $D(z) = \mathcal{O}(|z+1|^{\beta/2})$ as $z \to -1$.

If the weight satisfies the Szegő condition

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}} dx > -\infty$$

then the Szegő function exists and is given by

$$D(z) = \exp\left(\frac{(z^2-1)^{1/2}}{2\pi} \int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}} \frac{dx}{x-z}\right). \tag{10.1}$$

Exercise 19. Show that $D(z)$ as given by (10.1) does indeed satisfy the jump condition $D_+D_- = w$.

Exercise 20. Show that the Szegő function for the pure Jacobi weight $w(x) = (1-x)^\alpha(1+x)^\beta$ is given by

$$D(z) = \left(\frac{(z-1)^\alpha(z+1)^\beta}{\varphi(z)^{\alpha+\beta}}\right)^{1/2},$$

with an appropriate branch of the square root.

Having D we can present the solution to the Riemann-Hilbert problem for N as follows.

$$N(z) = \begin{pmatrix} D_\infty & 0 \\ 0 & \frac{1}{D_\infty} \end{pmatrix} \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ \frac{a(z)-a^{-1}(z)}{-2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{D(z)} & 0 \\ 0 & D(z) \end{pmatrix}, \tag{10.2}$$

where

$$a(z) = \frac{(z-1)^{1/4}}{(z+1)^{1/4}}. \tag{10.3}$$

Exercise 21. Check that the jump condition (N-RH2) and endpoint conditions (N-RH4)–(N-RH5) are satisfied.
[Hint: The middle factor in the right-hand side of (10.2) appears as the solution for the Riemann-Hilbert problem for N in case $w \equiv 1$, see Exercise 18.]

Exercise 22. Show that $\det N(z) = 1$ for $z \in \mathbb{C} \setminus [-1,1]$.

11 Third transformation $S \mapsto R$

Now we can perform the final transformation $S \mapsto R$ in the case $\alpha = \beta = -\frac{1}{2}$. We define

$$R(z) = S(z)N^{-1}(z). \tag{11.1}$$

Since S and N have the same jump across $(-1,1)$ it is easy to see that $R_+(x) = R_-(x)$ for $x \in (-1,1)$, so that R is analytic across $(-1,1)$. Then R is analytic in $\mathbb{C} \setminus \gamma$ with possible singularities at ± 1. Since $\det N = 1$, we have from (N-RH4)

$$N^{-1}(z) = \mathcal{O}\begin{pmatrix} |z-1|^{-\frac{1}{2}} & |z-1|^{-\frac{1}{2}} \\ 1 & 1 \end{pmatrix}$$

as $z \to 1$. Thus

$$\begin{aligned} R(z) &= \mathcal{O}\begin{pmatrix} 1 & |z-1|^{-\frac{1}{2}} \\ 1 & |z-1|^{-\frac{1}{2}} \end{pmatrix} \mathcal{O}\begin{pmatrix} |z-1|^{-\frac{1}{2}} & |z-1|^{-\frac{1}{2}} \\ 1 & 1 \end{pmatrix} \\ &= \mathcal{O}\begin{pmatrix} |z-1|^{-\frac{1}{2}} & |z-1|^{-\frac{1}{2}} \\ |z-1|^{-\frac{1}{2}} & |z-1|^{-\frac{1}{2}} \end{pmatrix} \end{aligned}$$

as $z \to 1$. So all entries of R have an isolated singularity at $z = 1$ such that $R_{ij}(z) = \mathcal{O}\left(|z-1|^{-\frac{1}{2}}\right)$ as $z \to 1$. This implies that $z = 1$ is a removable singularity. Similarly it follows that $z = -1$ is a removable singularity.

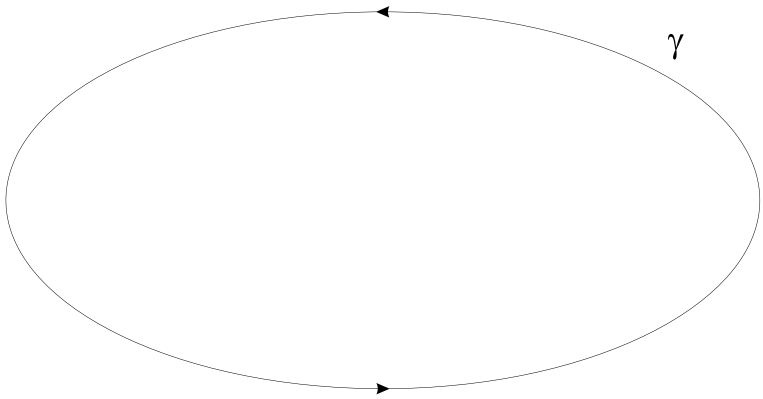

Fig. 6. Closed contour γ. R has a jump on γ only.

So R is analytic across the full interval $[-1,1]$, and so only has a jump on γ, as shown in Figure 6. We have the following Riemann-Hilbert problem for R.

(R-RH1) R is analytic on $\mathbb{C} \setminus \gamma$.
(R-RH2) $R_+(s) = R_-(s)V(s)$ where

$$V(s) = \begin{cases} N(s) \begin{pmatrix} 1 & 0 \\ \frac{1}{w(s)}\varphi(s)^{-2n} & 1 \end{pmatrix} N^{-1}(s) & \text{for } s \in \gamma \cap \{\operatorname{Im} z < 0\}, \\ N(s) \begin{pmatrix} 1 & 0 \\ -\frac{1}{w(s)}\varphi(s)^{-2n} & 1 \end{pmatrix} N^{-1}(s) & \text{for } s \in \gamma \cap \{\operatorname{Im} z > 0\}. \end{cases}$$

(R-RH3) $R(z) \to I$ as $z \to \infty$.

Observe that the jump matrix $V(s)$ is close to the identity matrix if n is large.

Exercise 23. Prove that V is analytic in a neighborhood Ω of γ, and that

$$\|V - I\|_\Omega = \mathcal{O}\left(e^{-cn}\right) \qquad \text{as } n \to \infty.$$

for some constant $c > 0$.

The Riemann-Hilbert problem for R is of the type discussed in Theorem 3.1. The problem is posed on a fixed contour γ (independent of n) and the jump matrix V is analytic in a neighborhood of γ where it is close to the identity. It follows from Theorem 3.1 that

$$R(z) = I + \mathcal{O}\left(e^{-cn}\right) \tag{11.2}$$

uniformly for z in $\mathbb{C} \setminus \gamma$. Tracing back the steps $Y \mapsto T \mapsto S \mapsto R$ we are then able to find asymptotics for Y as $n \to \infty$, and in particular for its $(1,1)$ entry, which is the orthogonal polynomial π_n.

Exercise 24. The analysis of Sections 9–11 goes through for all cases where the parameters α and β satisfy $\{\alpha, \beta\} \subset \{-\frac{1}{2}, \frac{1}{2}\}$. Work out the details for $\alpha = \beta = \frac{1}{2}$.
What goes wrong if $\alpha = \frac{3}{2}$?

12 Asymptotics for orthogonal polynomials (case $\alpha = \beta = -\frac{1}{2}$)

We repeat that the above analysis is valid for $\alpha = \beta = -\frac{1}{2}$, and according to the last exercise, can be extended to the cases $\alpha, \beta = \pm\frac{1}{2}$. Now we show how to get asymptotics from (11.2) for the orthogonal polynomials and for related quantities.

The easiest to obtain is asymptotics for $z \in \mathbb{C} \setminus [-1, 1]$. For a given $z \in \mathbb{C} \setminus [-1, 1]$, we can open the lens around $[-1, 1]$ so that z is in the exterior of γ. Then by (7.1), (8.1), (11.1), and (11.2),

$$\begin{aligned} Y(z) &= \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix} T(z) \begin{pmatrix} \varphi(z)^n & 0 \\ 0 & \varphi(z)^{-n} \end{pmatrix} \\ &= \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix} S(z) \begin{pmatrix} \varphi(z)^n & 0 \\ 0 & \varphi(z)^{-n} \end{pmatrix} \\ &= \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix} R(z) N(z) \begin{pmatrix} \varphi(z)^n & 0 \\ 0 & \varphi(z)^{-n} \end{pmatrix} \\ &= \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix} \left(I + \mathcal{O}\left(e^{-cn}\right)\right) N(z) \begin{pmatrix} \varphi(z)^n & 0 \\ 0 & \varphi(z)^{-n} \end{pmatrix} \end{aligned}$$

as $n \to \infty$. For the orthogonal polynomial $\pi_n(z) = Y_{11}(z)$ we get

$$\pi_n(z) = \left(\frac{\varphi(z)}{2}\right)^n \left[N_{11}(z)\left(1 + \mathcal{O}\left(e^{-cn}\right)\right) + N_{21}(z)\mathcal{O}\left(e^{-cn}\right)\right].$$

Since N_{11} does not become zero in $\mathbb{C} \setminus [-1,1]$, we get the strong asymptotic formula

$$\pi_n(z) = \left(\frac{\varphi(z)}{2}\right)^n N_{11}(z)\left(1 + \mathcal{O}\left(e^{-cn}\right)\right) \tag{12.1}$$

as $n \to \infty$. For $N_{11}(z)$ we have from (10.2) the explicit expression

$$N_{11}(z) = \frac{D_\infty}{D(z)} \frac{a(z) + a(z)^{-1}}{2} \tag{12.2}$$

in terms of the Szegő function D associated with w and the function $a(z) = \frac{(z-1)^{1/4}}{(z+1)^{1/4}}$. The formula (12.1) is valid uniformly for z in compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$.

For asymptotics on the interval $[-1,1]$ we have to work somewhat harder, the basic difference being that the transformation from T to S is non-trivial now. We take z in the upper part of the lens. Then the transformations (7.1), (8.1), and (11.1) yield

$$Y(z) = \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix} T(z) \begin{pmatrix} \varphi(z)^n & 0 \\ 0 & \varphi(z)^{-n} \end{pmatrix}$$

$$= \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix} S(z) \begin{pmatrix} 1 & 0 \\ \frac{1}{w(z)}\varphi(z)^{-2n} & 1 \end{pmatrix} \begin{pmatrix} \varphi(z)^n & 0 \\ 0 & \varphi(z)^{-n} \end{pmatrix}$$

$$= \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix} R(z)N(z) \begin{pmatrix} \varphi(z)^n & 0 \\ \frac{1}{w(z)}\varphi(z)^{-n} & \varphi(z)^{-n} \end{pmatrix}$$

$$= \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix} R(z) \begin{pmatrix} D_\infty & 0 \\ 0 & D_\infty^{-1} \end{pmatrix} \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ \frac{a(z)-a^{-1}(z)}{-2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix}$$

$$\times \begin{pmatrix} D(z)^{-1} & 0 \\ 0 & D(z) \end{pmatrix} \begin{pmatrix} \varphi(z)^n & 0 \\ \frac{1}{w(z)}\varphi(z)^{-n} & \varphi(z)^{-n} \end{pmatrix}.$$

So for the first column of Y we have

$$\begin{pmatrix} 2^n Y_{11}(z) \\ 2^{-n} Y_{21}(z) \end{pmatrix} = R(z) \begin{pmatrix} D_\infty & 0 \\ 0 & D_\infty^{-1} \end{pmatrix} \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ \frac{a(z)-a^{-1}(z)}{-2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix} \begin{pmatrix} \frac{\varphi(z)^n}{D(z)} \\ \frac{D(z)}{w(z)\varphi(z)^n} \end{pmatrix}.$$

Now we take $x \in [-1,1]$, and we let z tend to x from the upper part of the lens. So we have to take the $+$boundary values of all quantities involved. It is tedious, but straightforward, to check that for $x \in (-1,1)$,

$$\varphi_+(x) = x + \sqrt{1-x^2} = \exp\left(i \arccos x\right),$$

$$\frac{a_+(x) + a_+^{-1}(x)}{2} = \frac{1}{\sqrt{2}(1-x^2)^{\frac{1}{4}}} \exp\left(\frac{1}{2} i \arccos x - i\frac{\pi}{4}\right),$$

$$\frac{a_+(x) - a_+^{-1}(x)}{2i} = \frac{1}{\sqrt{2}(1-x^2)^{\frac{1}{4}}} \exp\left(-\frac{1}{2} i \arccos x + i\frac{\pi}{4}\right),$$

$$D_+(x) = \sqrt{w(x)}\, \exp\left(-i\psi(x)\right),$$

where $\psi(x)$ is a real-valued function, which is given by

$$\psi(x) = \frac{\sqrt{1-x^2}}{2\pi} \int_{-1}^{1} \frac{\log w(t)}{\sqrt{1-t^2}} \frac{dt}{t-x},$$

in which the integral is a principal value integral. Putting this all together we find for the orthogonal polynomial $\pi_n(x)$ with $x \in [-1,1]$,

$$\pi_n(x) = \frac{\sqrt{2}D_\infty}{2^n\sqrt{w(x)}(1-x^2)^{\frac{1}{4}}}\left[R_{11}(x)\cos\left(\left(n+\frac{1}{2}\right)\arccos x + \psi(x) - \frac{\pi}{4}\right)\right.$$
$$\left. - \frac{i}{D_\infty^2}R_{12}(x)\cos\left(\left(n-\frac{1}{2}\right)\arccos x + \psi(x) - \frac{\pi}{4}\right)\right], \tag{12.3}$$

where

$$R_{11}(x) = 1 + \mathcal{O}\left(e^{-cn}\right), \qquad R_{12}(x) = \mathcal{O}\left(e^{-cn}\right). \tag{12.4}$$

The asymptotic formula (12.3)–(12.4) is valid uniformly for $x \in [-1,1]$. The fact that this includes the endpoints ± 1 is special to the case $\alpha = \beta = -\frac{1}{2}$. For more general α and β, the formula (12.3) continues to hold on compact subsets of the open interval $(-1,1)$, but with error terms $R_{11}(x) = 1 + \mathcal{O}(\frac{1}{n})$ and $R_{12}(x) = \mathcal{O}(\frac{1}{n})$. Near the endpoints ± 1, there is a different asymptotic formula.

The formula (12.3) clearly displays the oscillatory behavior of $\pi_n(x)$ on the interval $[-1,1]$. The amplitude of the oscillations is $\frac{\sqrt{2}D_\infty}{2^n\sqrt{w(x)}(1-x^2)^{\frac{1}{4}}}$ and it is easy to check that this remains bounded as $x \to \pm 1$. The main oscillating term is $\cos\left(\left(n+\frac{1}{2}\right) + \psi(x) - \frac{\pi}{4}\right)$ with corrections that are exponentially small as $n \to \infty$.

Exercise 25. The orthogonal polynomials for the weight $(1-x^2)^{-\frac{1}{2}}$ are the Chebyshev polynomials of the first kind $T_n(x)$ with the property

$$T_n(x) = \cos(n \arccos x), \qquad x \in [-1,1].$$

The monic Chebyshev polynomials are

$$\pi_n(x) = \frac{1}{2^{n-1}}T_n(x) \qquad \text{if } n \geq 1.$$

Compare this with the asymptotic formula (12.3). What are R_{11} and R_{12} in this case?
[Hint: It may be shown that for a Jacobi weight $(1-x)^\alpha(1+x)^\beta$ one has $D_\infty = 2^{-\frac{\alpha+\beta}{2}}$ and $\psi(x) = \frac{\alpha+\beta}{2}\arccos x - \frac{\alpha\pi}{2}$.]

Exercise 26. The formula (12.3)–(12.4) is also valid for the case $\alpha = \beta = \frac{1}{2}$. This may seem strange at first since then the amplitude of the oscillations $\frac{\sqrt{2}D_\infty}{2^n\sqrt{w(x)}(1-x^2)^{\frac{1}{4}}}$ is unbounded as $x \to \pm 1$. Still the formula (12.3) is valid uniformly on the closed interval $[-1,1]$. How can this be explained?

Exercise 27. Deduce from (12.1)–(12.4) that the coefficients a_n and b_n in the recurrence relation

$$x\pi_n(x) = \pi_{n+1}(x) + b_n\pi_n(x) + a_n^2\pi_{n-1}(x)$$

satisfy

$$a_n = \frac{1}{2} + \mathcal{O}\left(e^{-cn}\right), \qquad b_n = \mathcal{O}\left(e^{-cn}\right). \tag{12.5}$$

Remark related to exercise 27: Geronimo [19] made a thorough study of orthogonal polynomials with recurrence coefficients that approach their limits at an exponential rate. He showed that (12.5) holds, if and only if the underlying orthogonality measure is a modified Jacobi weight $(1-x)^{\pm\frac{1}{2}}(1+x)^{\pm\frac{1}{2}}h(x)$, plus at most a finite number of discrete masspoints outside $[-1,1]$. I thank Jeff Geronimo for this remark.

13 Case of general α and β

For the case of a modified Jacobi weight $(1-x)^\alpha(1+x)^\beta h(x)$ with general exponents $\alpha, \beta > -1$, we cannot do the transformation $T \mapsto S$ as described in Section 9. In general we have to stay with the transformation $T \mapsto S$ as in Section 8. So we are left with a Riemann-Hilbert problem on a system of contours as shown in Figure 3.

We continue to use the Szegő function $D(z)$ characterized by (D-RH1)–(D-RH5), and the solution of the model Riemann-Hilbert problem

$$N(z) = \begin{pmatrix} D_\infty & 0 \\ 0 & \frac{1}{D_\infty} \end{pmatrix} \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ \frac{a(z)-a^{-1}(z)}{-2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{D(z)} & 0 \\ 0 & D(z) \end{pmatrix},$$

with $a(z) = \frac{(z-1)^{\frac{1}{4}}}{(z+1)^{\frac{1}{4}}}$. Note that N satisfies

(N-RH1) N is analytic in $\mathbb{C} \setminus [-1,1]$.

(N-RH2) $N_+(x) = N_-(x) \begin{pmatrix} 0 & w(x) \\ -\frac{1}{w(x)} & 0 \end{pmatrix}$ for $x \in (-1,1)$.

(N-RH3) $N(z) \to I$ as $z \to \infty$.

The aim is again to prove that S is close to N if n is large. However, the attempt to define $R = SN^{-1}$ and prove that $R \sim I$ does not work. The problem lies near the endpoints ± 1, as SN^{-1} is not bounded near ± 1.

The way out of this is a **local analysis** near the endpoints ± 1. We are going to construct a so-called **local parametrix** P in a disk $\{|z-1| < \delta\}$ centered at 1, where δ is a small, but fixed, positive number. The parametrix should satisfy the following local Riemann-Hilbert problem

(P-RH1) P is analytic in $\{|z-1| < \delta\} \setminus \Sigma$ and continuous in $\{|z-1| \leq \delta\} \setminus \Sigma$.

(P-RH2) P has the same jumps as S on $\Sigma \cap \{|z-1| < \delta\}$.
(P-RH3) $P = \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) N$ as $n \to \infty$, uniformly on $|z-1| = \delta$.
(P-RH4) P has the same behavior as S near 1.

Instead of an asymptotic condition, we now have in (P-RH3) a **matching condition**.

Similarly, we need a parametrix $\tilde{P}$ near -1 which should satisfy

($\tilde{\mathbf{P}}$-RH1) $\tilde{P}$ is analytic in $\{|z+1| < \delta\} \setminus \Sigma$ and continuous in $\{|z+1| \leq \delta\} \setminus \Sigma$.
($\tilde{\mathbf{P}}$-RH2) $\tilde{P}$ has the same jumps as S on $\Sigma \cap \{|z+1| < \delta\}$.
($\tilde{\mathbf{P}}$-RH3) $\tilde{P} = \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) N$ as $n \to \infty$, uniformly on $|z+1| = \delta$.
($\tilde{\mathbf{P}}$-RH4) $\tilde{P}$ has the same behavior as S near -1.

The construction of a local parametrix is done in [7, 9] with the help of Airy functions. Here we will need Bessel functions of order α. In the next section, we will outline the construction of P. In the remaining part of this section we will discuss how the transformation $S \mapsto R$ will be, assuming that we can find P and $\tilde{P}$.

We define R by

$$R(z) = \begin{cases} S(z)N(z)^{-1} & \text{if } |z-1| > \delta \text{ and } |z+1| > \delta, \\ S(z)P(z)^{-1} & \text{if } |z-1| < \delta, \\ S(z)\tilde{P}(z)^{-1} & \text{if } |z+1| < \delta. \end{cases} \tag{13.1}$$

Then R is analytic outside the system of contours γ shown in Figure 7.

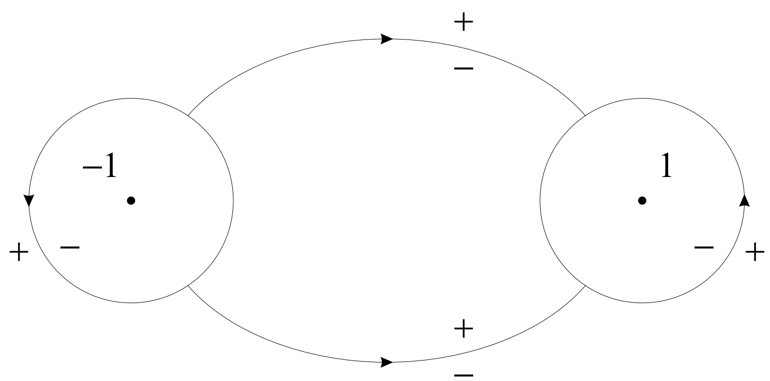

Fig. 7. System of contours γ so that R is analytic in $\mathbb{C} \setminus \gamma$. The system of contours γ consists of two circles of radius δ centered at ± 1, and two arcs joining these two circles.

R satisfies

(R-RH1) R is analytic on $\mathbb{C} \setminus \gamma$.
(R-RH2) $R_+ = R_- V$ on γ where

$$V = \begin{cases} PN^{-1} = I + \mathcal{O}\left(\frac{1}{n}\right) & \text{for } |z-1| = \delta, \\ \tilde{P}N^{-1} = I + \mathcal{O}\left(\frac{1}{n}\right) & \text{for } |z+1| = \delta, \\ N \begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi^{-2n} & 1 \end{pmatrix} N^{-1} = I + \mathcal{O}\left(e^{-cn}\right) & \\ \qquad\qquad \text{on } (\Sigma_1 \cup \Sigma_2) \cap \{|z-1| > \delta, |z+1| > \delta\}. \end{cases}$$

(R-RH3) $R(z) = I + \mathcal{O}\left(\dfrac{1}{z}\right)$ as $z \to \infty$.
(R-RH4) R remains bounded at the four points of self-intersection of γ.

Now the jump matrices are $I + \mathcal{O}\left(\frac{1}{n}\right)$ uniformly on γ. The contour γ is not a simple closed contour as in Theorem 3.1, so we cannot use that theorem directly. However, we can use the ideas in its proof to establish that we have $R(z) = I + \mathcal{O}\left(\frac{1}{n}\right)$.

Exercise 28. Prove that

$$R(z) = I + \mathcal{O}\left(\frac{1}{n}\right) \tag{13.2}$$

as $n \to \infty$, uniformly for $z \in \mathbb{C} \setminus \gamma$.

14 Construction of the local parametrix

The construction of the local parametrix P follows along a number of steps. More details can be found in [23].

Step 1: Reduction to constant jumps

We put for $z \in U \setminus (-\infty, 1]$,

$$W(z) = \left((z-1)^{\alpha}(z+1)^{\beta}h(z)\right)^{1/2},$$

where the branch of the square root is taken which is positive for $z > 1$. We seek P in the form

$$P = P^{(1)} \begin{pmatrix} W^{-1}\varphi^{-n} & 0 \\ 0 & W\varphi^{n} \end{pmatrix}.$$

In order to have (P-RH1), (P-RH2), and (P-RH4), we then get that $P^{(1)}$ should satisfy

(P1-RH1) $P^{(1)}$ is analytic in $\{|z-1| < \delta\} \setminus \Sigma$ and continuous in $\{|z-1| \leq \delta\} \setminus \Sigma$.

(P1-RH2) $P_+^{(1)} = P_-^{(1)} \begin{pmatrix} 1 & 0 \\ e^{\alpha\pi i} & 1 \end{pmatrix}$ on $\Sigma_1^o \cap \{|z-1| < \delta\}$,

$P_+^{(1)} = P_-^{(1)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $(1-\delta, 1)$,

$P_+^{(1)} = P_-^{(1)} \begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i} & 1 \end{pmatrix}$ on $\Sigma_3^o \cap \{|z-1| < \delta\}$.

(P1-RH4) Conditions as $z \to 1$:

- If $\alpha < 0$, then $P^{(1)}(z) = \mathcal{O}\begin{pmatrix} |z-1|^{\alpha/2} & |z-1|^{\alpha/2} \\ |z-1|^{\alpha/2} & |z-1|^{\alpha/2} \end{pmatrix}$.
- If $\alpha = 0$, then $P^{(1)}(z) = \mathcal{O}\begin{pmatrix} \log|z-1| & \log|z-1| \\ \log|z-1| & \log|z-1| \end{pmatrix}$.
- If $\alpha > 0$, then

$$P^{(1)}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z-1|^{\alpha/2} & |z-1|^{-\alpha/2} \\ |z-1|^{\alpha/2} & |z-1|^{-\alpha/2} \end{pmatrix} & \text{as } z \to 1 \text{ outside the lens,} \\ \\ \mathcal{O}\begin{pmatrix} |z-1|^{-\alpha/2} & |z-1|^{-\alpha/2} \\ |z-1|^{-\alpha/2} & |z-1|^{-\alpha/2} \end{pmatrix} & \text{as } z \to 1 \text{ inside the lens.} \end{cases}$$

For the moment we ignore the matching condition.

Step 2: Model Riemann-Hilbert problem

The constant jump problem we have for $P^{(1)}$ leads to a model problem for Ψ, defined in an auxiliary ζ-plane. The problem is posed on three semi-infinite rays γ_1, γ_2, and γ_3, where γ_2 is the negative real axis, $\gamma_1 = \{\arg\zeta = \sigma\}$, and $\gamma_3 = \{\arg\zeta = -\sigma\}$. Here σ is some angle in $(0, \pi)$, see Figure 8.

The Riemann-Hilbert problem for Ψ is:

(Ψ-RH1) Ψ is analytic in $\mathbb{C} \setminus (\gamma_1 \cup \gamma_2 \cup \gamma_3)$,

(Ψ-RH2) $\Psi_+ = \Psi_- \begin{pmatrix} 1 & 0 \\ e^{\alpha\pi i} & 1 \end{pmatrix}$ on γ_1^o,

$\Psi_+ = \Psi_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on γ_2^o,

$\Psi_+ = \Psi_- \begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i} & 1 \end{pmatrix}$ on γ_3^o.

(Ψ-RH4) Conditions as $\zeta \to 0$:

- If $\alpha < 0$, then $\Psi(\zeta) = \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \end{pmatrix}$.
- If $\alpha = 0$, then $\Psi(\zeta) = \mathcal{O}\begin{pmatrix} \log|\zeta| & \log|\zeta| \\ \log|\zeta| & \log|\zeta| \end{pmatrix}$.
- If $\alpha > 0$, then

$$\Psi(\zeta) = \begin{cases} \mathcal{O}\begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \end{pmatrix} \text{as } \zeta \to 0 \text{ with } |\arg\zeta| < \sigma, \\ \mathcal{O}\begin{pmatrix} |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \end{pmatrix} \text{as } \zeta \to 0 \text{ with } \sigma < |\arg\zeta| < \pi. \end{cases}$$

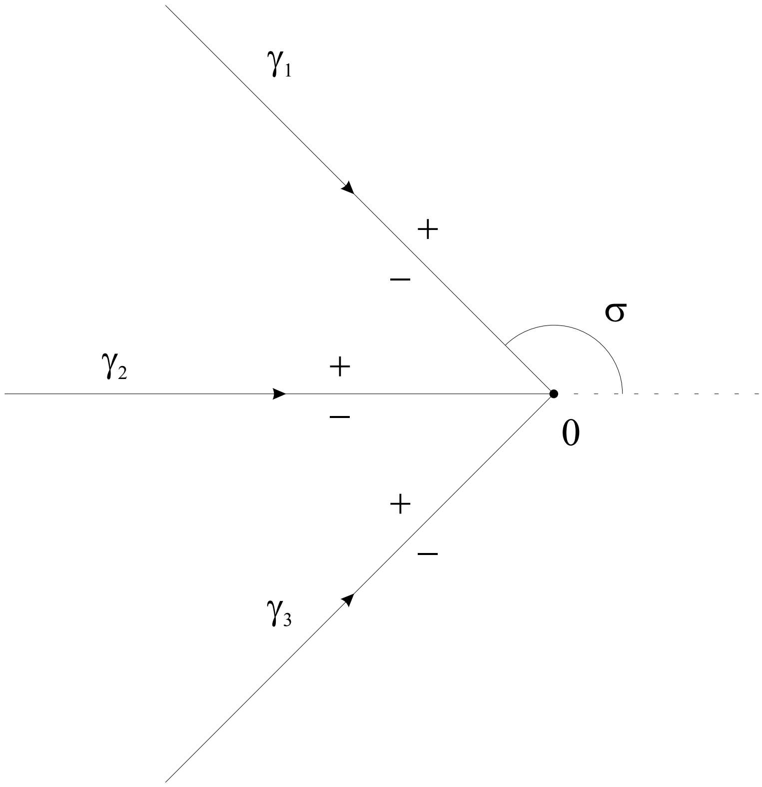

Fig. 8. Contours for the Riemann-Hilbert problem for Ψ.

There is no asymptotic condition (Ψ-RH3) for Ψ, so we cannot expect to have a unique solution. Indeed, there are in fact many solutions. In the next step we will construct one solution out of modified Bessel functions.

Step 3: Solution of model Riemann-Hilbert problem

The solution of the Riemann-Hilbert problem for Ψ will be built out of modified Bessel functions of order α, namely I_α and K_α. These are solutions of the modified Bessel differential equation

$$y'' + \frac{1}{\zeta} y' - \left(1 - \frac{\alpha^2}{\zeta^2}\right) y = 0.$$

The two functions $I_\alpha(2\zeta^{1/2})$ and $K_\alpha(2\zeta^{1/2})$ satisfy

$$y'' - \frac{1}{\zeta}\left(1 + \frac{\alpha^2}{4\zeta}\right) y = 0.$$

We consider these functions for $|\arg \zeta| < \pi$. On the negative real axis there is a jump. In fact we have the connection formulas, see [1, 9.6.30-31],

$$\begin{aligned}
I_\alpha(2\zeta^{1/2})_+ &= e^{\alpha\pi i} I_\alpha(2\zeta^{1/2})_- \\
K_\alpha(2\zeta^{1/2})_+ &= e^{-\alpha\pi i} K_\alpha(2\zeta^{1/2})_- - \pi i I_\alpha(2\zeta^{1/2})_-
\end{aligned}$$

for ζ on the negative real axis, oriented from left to right. We can put this in matrix-vector form

$$\left(I_\alpha(2\zeta^{1/2})\ \tfrac{i}{\pi}K_\alpha(2\zeta^{1/2})\right)_+ = \left(I_\alpha(2\zeta^{1/2})\ \tfrac{i}{\pi}K_\alpha(2\zeta^{1/2})\right)_- \begin{pmatrix} e^{\alpha\pi i} & 1 \\ 0 & e^{-\alpha\pi i} \end{pmatrix}.$$

Since the jump matrix is constant, it follows that the vector of derivatives satisfies the same jumps, and also if we multiply this vector by $2\pi i\zeta$. This has the effect of creating a matrix with determinant 1, due to the Wronskian relation [1, 9.6.15]). Thus

$$\begin{aligned}
&\begin{pmatrix} I_\alpha(2\zeta^{1/2}) & \frac{i}{\pi}K_\alpha(2\zeta^{1/2}) \\ 2\pi i\zeta^{1/2} I_\alpha'(2\zeta^{1/2}) & -2\zeta^{1/2}K_\alpha'(2\zeta^{1/2}) \end{pmatrix}_+ \\
&\quad = \begin{pmatrix} I_\alpha(2\zeta^{1/2}) & \frac{i}{\pi}K_\alpha(2\zeta^{1/2}) \\ 2\pi i\zeta^{1/2} I_\alpha'(2\zeta^{1/2}) & -2\zeta^{1/2}K_\alpha'(2\zeta^{1/2}) \end{pmatrix}_- \begin{pmatrix} e^{\alpha\pi i} & 1 \\ 0 & e^{-\alpha\pi i} \end{pmatrix}.
\end{aligned}$$

Now we have, as is easy to check,

$$\begin{pmatrix} e^{\alpha\pi i} & 1 \\ 0 & e^{-\alpha\pi i} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{\alpha\pi i} & 1 \end{pmatrix}.$$

This last product consists exactly of the three jump matrices in the Riemann-Hilbert problem for Ψ. It follows that if we define Ψ by

$$\Psi(\zeta) = \begin{cases} \Psi_0(\zeta) & \text{for } |\arg\zeta| < \sigma, \\[1ex] \Psi_0(\zeta)\begin{pmatrix} 1 & 0 \\ -e^{\alpha\pi i} & 1 \end{pmatrix} & \text{for } \sigma < \arg\zeta < \pi, \\[1ex] \Psi_0(\zeta)\begin{pmatrix} 1 & 0 \\ e^{-\alpha\pi i} & 1 \end{pmatrix} & \text{for } -\pi < \arg\zeta < -\sigma, \end{cases} \tag{14.1}$$

where

$$\Psi_0(\zeta) = \begin{pmatrix} I_\alpha(2\zeta^{1/2}) & \frac{i}{\pi}K_\alpha(2\zeta^{1/2}) \\ 2\pi i\zeta^{1/2} I_\alpha'(2\zeta^{1/2}) & -2\zeta^{1/2}K_\alpha'(2\zeta^{1/2}) \end{pmatrix}, \tag{14.2}$$

then Ψ satisfies the jump condition (Ψ-RH2). Clearly, (Ψ-RH1) is also satisfied. Because of the known behavior of the modified Bessel functions near 0, see [1, 9.6.7–9], Ψ also has the behavior (Ψ-RH4) near 0.

Step 4: Construction of $P^{(1)}$

Define for $z \in \mathbb{C} \setminus (-\infty, 1]$,

$$f(z) = \frac{1}{4}\left[\log \varphi(z)\right]^2, \tag{14.3}$$

where we choose the principal branch of the logarithm. Since $\varphi_+(x)\varphi_-(x) = 1$ for $x \in (-1,1)$, we easily get that $f_+(x) = f_-(x)$. So f is analytic in $\mathbb{C} \setminus (-\infty, -1]$. The behavior near $z = 1$ is

$$f(z) = \frac{1}{2}(z-1) - \frac{1}{12}(z-1)^2 + \mathcal{O}((z-1)^3) \qquad \text{as } z \to 1.$$

So f is a conformal mapping of a neighborhood of 1 onto a neighborhood of 0. We choose $\delta > 0$ sufficiently small so that $\zeta = f(z)$ maps the disk $\{|z-1| < \delta\}$ conformally onto a convex neighborhood of 0 in the ζ-plane. We still have some freedom in the precise location of the contours Σ_1 and Σ_3. Here we use this freedom to specify that $\Sigma_1 \cap \{|z-1| < \delta\}$ should be mapped by f to a part of the ray $\arg\zeta = \sigma$ (we choose any $\sigma \in (0, \pi)$), and $\Sigma_3 \cap \{|z-1| < \delta\}$ to a part of the ray $\arg\zeta = -\sigma$.

Then $\Psi(n^2 f(z))$ satisfies the properties (P1-RH1), (P1-RH2), and (P1-RH4) of the Riemann-Hilbert problem for $P^{(1)}$. This would actually be the case for any choice of conformal map $\zeta = f(z)$, mapping $z = 1$ to $\zeta = 0$, and which is real and positive for $z > 1$. The specific choice of f is dictated by the matching condition for P, which we will look at in a minute. This will also explain the factor n^2. But this will not be enough to be able to do the matching. There is an additional freedom we have in multiplying $\Psi(n^2 f(z))$ **on the left** by an analytic factor. So we put

$$P^{(1)}(z) = E(z)\Psi(n^2 f(z)), \tag{14.4}$$

where E is an analytic 2×2 matrix valued function in $\{|z-1| < \delta\}$. It will depend on n. The precise form of E will be given in the next subsection.

Exercise 29. Show that for any analytic factor E the definition (14.4) gives a matrix valued function $P^{(1)}$ that satisfies the jump condition (P1-RH2) and the condition (P1-RH4) near 1.

Step 5: The matching condition

The parametrix P we now have is

$$P(z) = E(z)\Psi(n^2 f(z)) \begin{pmatrix} W(z)^{-1}\varphi(z)^{-n} & 0 \\ 0 & W(z)\varphi(z)^n \end{pmatrix} \tag{14.5}$$

where we have not specified E yet. The conditions (P-RH1), (P-RH2), and (P-RH4) are satisfied. We also have to take care of the matching condition

$$P(z) = \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) N(z) \qquad \text{for } |z-1| = \delta.$$

To achieve the matching, $E(z)$ should be close to

$$N(z)\begin{pmatrix} W(z)\varphi(z)^n & 0 \\ 0 & W(z)^{-1}\varphi(z)^{-n}\end{pmatrix}\left[\Psi(n^2 f(z))\right]^{-1}.$$

The idea is to replace Ψ here with an approximation Ψ^a. For fixed z with $|z-1|=\delta$, the function $\Psi(\zeta)$ is evaluated at $\zeta = n^2 f(z)$, which grows as $n\to\infty$. So to figure out what approximation Ψ^a to use, we need large ζ asymptotics of the modified Bessel functions and their derivatives. These functions have a known asymptotic expansion, see [1, 9.7.1–4]. From this it follows that

$$\Psi(\zeta) = \begin{pmatrix} \frac{1}{\sqrt{2\pi}}\zeta^{-1/4} & 0 \\ 0 & \sqrt{2\pi}\zeta^{1/4}\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1+\mathcal{O}(\zeta^{-\frac{1}{2}}) & i+\mathcal{O}(\zeta^{-\frac{1}{2}}) \\ i+\mathcal{O}(\zeta^{-\frac{1}{2}}) & 1+\mathcal{O}(\zeta^{-\frac{1}{2}})\end{pmatrix} \begin{pmatrix} e^{2\zeta^{1/2}} & 0 \\ 0 & e^{-2\zeta^{1/2}}\end{pmatrix}.$$

Now we ignore the $\mathcal{O}(\zeta^{-\frac{1}{2}})$ terms, and we put

$$\Psi^a(\zeta) = \begin{pmatrix} \frac{1}{\sqrt{2\pi}}\zeta^{-1/4} & 0 \\ 0 & \sqrt{2\pi}\zeta^{1/4}\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1\end{pmatrix} \begin{pmatrix} e^{2\zeta^{1/2}} & 0 \\ 0 & e^{-2\zeta^{1/2}}\end{pmatrix},$$

and then define

$$E(z) = N(z)\begin{pmatrix} W(z)\varphi(z)^n & 0 \\ 0 & W(z)^{-1}\varphi(z)^{-n}\end{pmatrix}\left[\Psi^a(n^2 f(z))\right]^{-1}.$$

Note that $e^{-2\zeta^{1/2}} = \varphi(z)^n$ for $\zeta = n^2 f(z)$. Thus

$$E(z) = N(z)\begin{pmatrix} W(z) & 0 \\ 0 & W(z)^{-1}\end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1\end{pmatrix} \begin{pmatrix} \sqrt{2\pi n} f(z)^{1/4} & 0 \\ 0 & \frac{1}{\sqrt{2\pi n}} f(z)^{-1/4}\end{pmatrix}. \quad (14.6)$$

The fact that the exponential factor $\varphi(z)^n$ gets cancelled is the reason for the choice of the mapping f and the factor n^2 in $\Psi(n^2 f(z))$. With this choice for E, it is easy to check that P satisfies the matching condition (P-RH3). We leave it as an exercise to show that E is analytic in a full neighborhood of 1. This completes the construction of the parametrix P in the neighborhood of 1.

A similar construction with modified Bessel functions of order β yields a parametrix $\tilde{P}$ in the neighborhood of -1.

Exercise 30.

(a) Show that $E_+(x) = E_-(x)$ for $x \in (1-\delta, 1)$, so that E is analytic across $(1-\delta, 1)$.
[Hint: On $(1-\delta, 1)$ we have $(f^{1/4})_+ = i(f^{1/4})_-$, $W_+W_- = w$, and $N_+ = N_- \begin{pmatrix} 0 & w \\ -\frac{1}{w} & 0 \end{pmatrix}$.]

(b) Show that the isolated singularity of E at 1 is removable.
[Hint: Use that $W(z)/D(z)$ is bounded and bounded away from zero near $z = 1$.]

15 Asymptotics for orthogonal polynomials (general case)

Knowing that we can construct the local parametrices P and $\tilde{P}$ we can go back to Section 13 and conclude that $R(z) = I + \mathcal{O}(\frac{1}{n})$ uniformly for $z \in \mathbb{C} \setminus \gamma$, where γ is the system of contours shown in Figure 7.

Then we can go back to our transformations $Y \mapsto T \mapsto S \mapsto R$, to obtain asymptotics for Y, and in particular for the orthogonal polynomial $\pi_n(z) = Y_{11}(z)$. We summarize here the results. For $z \in \mathbb{C} \setminus [-1, 1]$, we obtain

$$\pi_n(z) = \frac{\varphi(z)^n}{2^n} \frac{D_\infty}{D(z)} \frac{a(z) + a(z)^{-1}}{2} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right). \tag{15.1}$$

The $\mathcal{O}(\frac{1}{n})$ term is uniform for z in compact subsets of $\mathbb{C} \setminus [-1, 1]$. The formula is the same as the one (15.1) we found for the case $\alpha = \beta = -\frac{1}{2}$, except for the error term.

For $x \in (-1 + \delta, 1 - \delta)$, we obtain

$$\pi_n(x) = \frac{\sqrt{2} D_\infty}{2^n \sqrt{w(x)} (1 - x^2)^{1/4}} \left(\cos\left(\left(n + \frac{1}{2}\right) \arccos x + \psi(x) - \frac{\pi}{4}\right) + \mathcal{O}\left(\frac{1}{n}\right)\right), \tag{15.2}$$

where $\psi(x) = -\arg D_+(x)$, compare also with (12.1).

Exercise 31. Check that we obtain (15.2) from taking the sum of $+$ and $-$ boundary values for the asymptotics (15.1) valid in $\mathbb{C} \setminus [-1, 1]$.

Near the endpoints ± 1 the asymptotic formula for $\pi_n(x)$ involves Bessel functions. For z in the upper part of the lens, inside the disk $\{|z - 1| < \delta\}$, the expression for $Y(z)$ involves a product of no less than thirteen matrices (even after some simplifications). To summarize we have by (7.1), (8.1), and (13.1),

$$Y(z) = \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix} R(z)P(z) \begin{pmatrix} 1 & 0 \\ \frac{1}{w}\varphi(z)^{-2n} & 1 \end{pmatrix} \begin{pmatrix} \varphi(z)^n & 0 \\ 0 & \varphi(z)^{-n} \end{pmatrix}, \tag{15.3}$$

with (due to (14.5), (14.6), (10.2), and (14.1)),

$$P(z) = E(z)\Psi(n^2 f(z)) \begin{pmatrix} W(z)^{-1}\varphi(z)^{-n} & 0 \\ 0 & W(z)\varphi(z)^n \end{pmatrix},$$

$$E(z) = N(z) \begin{pmatrix} W(z) & 0 \\ 0 & W(z)^{-1} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2\pi n} f(z)^{1/4} & 0 \\ 0 & \frac{1}{\sqrt{2\pi n}} f(z)^{-1/4} \end{pmatrix},$$

$$N(z) = \begin{pmatrix} D_\infty & 0 \\ 0 & D_\infty^{-1} \end{pmatrix} \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ \frac{a(z)-a^{-1}(z)}{-2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix} \begin{pmatrix} D(z)^{-1} & 0 \\ 0 & D(z) \end{pmatrix},$$

and

$$\Psi(\zeta) = \begin{pmatrix} I_\alpha(2\zeta^{1/2}) & \frac{i}{\pi} K_\alpha(2\zeta^{1/2}) \\ 2\pi i \zeta^{1/2} I'_\alpha(2\zeta^{1/2}) & -2\zeta^{1/2} K'_\alpha(2\zeta^{1/2}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{\alpha\pi i} & 1 \end{pmatrix}.$$

We start to evaluate the product (15.3) at the right. Plugging in the formula for $P(z)$, we get

$$Y(z) = \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix} R(z)E(z)\Psi(n^2 f(z)) \begin{pmatrix} W(z)^{-1} & 0 \\ \frac{W(z)}{w(z)} & W(z) \end{pmatrix}.$$

Since $W(z) = w(z)^{1/2} e^{\frac{1}{2}\alpha\pi i}$ in the region under consideration, we have

$$Y(z) = \begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix} R(z)E(z)\Psi(n^2 f(z)) \begin{pmatrix} e^{-\frac{1}{2}\alpha\pi i} & 0 \\ e^{\frac{1}{2}\alpha\pi i} & e^{\frac{1}{2}\alpha\pi i} \end{pmatrix} \begin{pmatrix} w(z)^{-1/2} & 0 \\ 0 & w(z)^{1/2} \end{pmatrix}. \tag{15.4}$$

Using the expression for $\Psi(\zeta)$, we get with $\zeta = n^2 f(z)$,

$$\begin{aligned} &\begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix} Y(z) \begin{pmatrix} w(z)^{1/2} & 0 \\ 0 & w(z)^{-1/2} \end{pmatrix} \\ &= R(z)E(z) \begin{pmatrix} I_\alpha(2\zeta^{1/2}) & \frac{i}{\pi} K_\alpha(2\zeta^{1/2}) \\ 2\pi i \zeta^{1/2} I'_\alpha(2\zeta^{1/2}) & -2\zeta^{1/2} K'_\alpha(2\zeta^{1/2}) \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}\alpha\pi i} & 0 \\ 0 & e^{\frac{1}{2}\alpha\pi i} \end{pmatrix}. \end{aligned}$$

At this point we see that the first column of $Y(z)$ can be expressed in terms of I_α and I'_α only. It will not involve K_α and K'_α. Continuing now only with the first column and focusing on the $(1,1)$ entry, we get

$$\begin{pmatrix} \pi_n(z) \\ * \end{pmatrix} = \frac{1}{2^n w(z)^{1/2} e^{\frac{1}{2}\alpha\pi i}} R(z) E(z) \begin{pmatrix} 1 & 0 \\ 0 & 2\pi n f(z)^{1/2} \end{pmatrix} \begin{pmatrix} I_\alpha(2\zeta^{1/2}) \\ i I'_\alpha(2\zeta^{1/2}) \end{pmatrix}, \tag{15.5}$$

where $*$ denotes an unspecified entry. Looking now at the formula for $E(z)$, we see that we pick up an overall factor $\sqrt{2\pi n} f(z)^{1/4}$. We get from (15.5)

$$\begin{pmatrix} \pi_n(z) \\ * \end{pmatrix} = \frac{\sqrt{2\pi n} f(z)^{1/4}}{2^n w(z)^{1/2} e^{\frac{1}{2}\alpha\pi i}} R(z) N(z) \begin{pmatrix} W(z) & 0 \\ 0 & W(z)^{-1} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} I_\alpha(2\zeta^{1/2}) \\ i I'_\alpha(2\zeta^{1/2}) \end{pmatrix}.$$

Next, we plug in the formula for N to obtain

$$\begin{pmatrix} \pi_n(z) \\ * \end{pmatrix} = \frac{\sqrt{2\pi n} f(z)^{1/4}}{2^n w(z)^{1/2} e^{\frac{1}{2}\alpha\pi i}} R(z) \begin{pmatrix} D_\infty & 0 \\ 0 & D_\infty^{-1} \end{pmatrix} \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ \frac{a(z)-a^{-1}(z)}{-2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix} \begin{pmatrix} \frac{W(z)}{D(z)} & 0 \\ 0 & \frac{D(z)}{W(z)} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} I_\alpha(2\zeta^{1/2}) \\ i I'_\alpha(2\zeta^{1/2}) \end{pmatrix}, \tag{15.6}$$

where we still have $\zeta = n^2 f(z)$.

Now we choose $x \in (1-\delta, 1]$ and let $z \to x$ from within the upper part of the lens. The asymptotics for $\pi_n(x)$ will then involve the $+$boundary values of all functions appearing in (15.6). First we note that $f(x) = -\frac{1}{4}(\arccos x)^2$, so that

$$f(z)^{1/4} \to e^{\frac{1}{4}\pi i} \frac{\sqrt{\arccos x}}{\sqrt{2}}.$$

We also get that

$$\begin{pmatrix} I_\alpha(2\zeta^{1/2}) \\ i I'_\alpha(2\zeta^{1/2}) \end{pmatrix} \to e^{\frac{1}{2}\alpha\pi i} \begin{pmatrix} J_\alpha(n \arccos x) \\ J'_\alpha(n \arccos x) \end{pmatrix},$$

where J_α is the usual Bessel function. Note also that

$$\frac{a_+(x) + a_+^{-1}(x)}{2} = \frac{\exp\left(\frac{1}{2} i \arccos x - i\frac{\pi}{4}\right)}{\sqrt{2}(1-x^2)^{\frac{1}{4}}}$$

and

$$\frac{a_+(x) - a_+^{-1}(x)}{2i} = \frac{\exp\left(-\frac{1}{2} i \arccos x + i\frac{\pi}{4}\right)}{\sqrt{2}(1-x^2)^{\frac{1}{4}}}$$

so that

$$\begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ \frac{a(z)-a^{-1}(z)}{-2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix} \to \frac{1}{\sqrt{2}(1-x^2)^{\frac{1}{4}}} \begin{pmatrix} e^{\frac{1}{2} i \arccos x - i\frac{\pi}{4}} & e^{-\frac{1}{2} i \arccos x + i\frac{\pi}{4}} \\ -e^{-\frac{1}{2} i \arccos x + i\frac{\pi}{4}} & e^{\frac{1}{2} i \arccos x - i\frac{\pi}{4}} \end{pmatrix}.$$

Finally we have that $W_+(x) = \sqrt{w(x)}e^{\frac{1}{2}\alpha\pi i}$ and $D_+(x) = \sqrt{w(x)}e^{-i\psi(x)}$, so that

$$\frac{W(z)}{D(z)} \to e^{\frac{1}{2}\alpha\pi i + \psi(x)i}.$$

We get from (15.6)

$$\begin{aligned}
\begin{pmatrix} \pi_n(x) \\ * \end{pmatrix} &= \frac{\sqrt{\pi n \arccos x}\, e^{\frac{1}{4}\pi i}}{2^n \sqrt{w(x)}\sqrt{2}(1-x^2)^{1/4}} R(x) \begin{pmatrix} D_\infty & 0 \\ 0 & D_\infty^{-1} \end{pmatrix} \\
&\quad \times \begin{pmatrix} e^{\frac{1}{2}i \arccos x - i\frac{\pi}{4}} & e^{-\frac{1}{2}i\arccos x + i\frac{\pi}{4}} \\ -e^{-\frac{1}{2}i\arccos x + i\frac{\pi}{4}} & e^{\frac{1}{2}i\arccos x - i\frac{\pi}{4}} \end{pmatrix} \\
&\quad \times \begin{pmatrix} e^{\frac{1}{2}\alpha\pi i + \psi(x)i} & 0 \\ 0 & e^{-\frac{1}{2}\alpha\pi i - \psi(x)i} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} J_\alpha(n\arccos x) \\ J_\alpha'(n\arccos x) \end{pmatrix} \qquad (15.7) \\
&= \frac{\sqrt{\pi n \arccos x}}{2^n \sqrt{w(x)}(1-x^2)^{1/4}} R(x) \begin{pmatrix} D_\infty & 0 \\ 0 & -iD_\infty^{-1} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \cos(\zeta_1(x)) & \sin(\zeta_1(x)) \\ \cos(\zeta_2(x)) & \sin(\zeta_2(x)) \end{pmatrix} \begin{pmatrix} J_\alpha(n\arccos x) \\ J_\alpha'(n\arccos x) \end{pmatrix}, \qquad (15.8)
\end{aligned}$$

where

$$\zeta_1(x) = \frac{1}{2}\arccos x + \frac{1}{2}\alpha\pi + \psi(x), \qquad \zeta_2(x) = -\frac{1}{2}\arccos x + \frac{1}{2}\alpha\pi + \psi(x).$$

Exercise 32. Check that the formula for $\pi_n(x)$ remains bounded as $x \to 1$. [Hint: First note that $\frac{\sqrt{\arccos x}}{(1-x^2)^{1/4}}$ has a limit for $x \to 1$. Next, we should combine $\frac{1}{\sqrt{w(x)}}$ with the Bessel functions $J_\alpha(n\arccos x)$ and $J_\alpha'(n\arccos x)$. Since $J_\alpha(z) \sim \frac{1}{\Gamma(\alpha+1)}\left(\frac{z}{2}\right)^\alpha$ as $z \to 0$, we get that $\frac{J_\alpha(n\arccos x)}{\sqrt{w(x)}}$ has a limit as $x \to 1$. Finally, we should control $\frac{J_\alpha'(n\arccos x)}{\sqrt{w(x)}}$, which is unbounded as $x \to 1$ (unless $\alpha = 0$). However it gets multiplied by $\sin\zeta_1(x)$ and $\sin\zeta_2(x)$. It may be shown that $\zeta_j(x) = \mathcal{O}(\sqrt{1-x})$ as $x \to 1$ for $j = 1, 2$, and this is enough to show that $\sin\zeta_j(x)\frac{J_\alpha'(n\arccos x)}{\sqrt{w(x)}}$ remains bounded as well.]

Exercise 33. Show that, uniformly for θ in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{1}{n^\alpha 2^n}\binom{2n+\alpha+\beta}{n}\pi_n\left(\cos\frac{\theta}{n}\right) = C(h)\left(\frac{2}{\theta}\right)^\alpha J_\alpha(\theta), \qquad (15.9)$$

where the constant $C(h)$ is given by

$$C(h) = \exp\left(\frac{1}{2\pi}\int_{-1}^{1} \frac{\log h(x) - \log h(1)}{\sqrt{1-x^2}}dx\right).$$

The limit (15.9) is the so-called Mehler-Heine formula, which is well-known for Jacobi polynomials, that is, for $h \equiv 1$, see, e.g., [1, 22.15.1] or [31]. From (15.9) one obtains the asymptotics of the largest zeros of π_n. Indeed, if $1 > x_1^{(n)} > x_2^{(n)} > \cdots$ denote the zeros of π_n, numbered in decreasing order, then (15.9) and Hurwitz's theorem imply that, for every $\nu \in \mathbb{N}$,

$$\lim_{n\to\infty} 2n^2 \left(1 - x_\nu^{(n)}\right) = j_{\alpha,\nu}^2,$$

where $0 < j_{\alpha,1} < j_{\alpha,2} < \cdots < j_{\alpha,\nu} < \cdots$ are the positive zeros of the Bessel function J_α. This property is well-known for Jacobi polynomials [1, 22.16.1].

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