PM 453/653 Assignment 1 Due Wednesday September 30.

1. The *Moore plane* is $\Gamma = \{(x, y) : x \in \mathbb{R},\ y \geq 0\}$ where for each (x, y) with $y > 0$, the usual open balls $B_r((x, y))$ of radius $r \leq y$ are open; and for $(x, 0) \in R = \{(x, 0) : x \in \mathbb{R}\}$, the sets $\{(x, 0)\} \cup B_r((x, r))$ are open.
   (a) Show that this determines a Hausdorff topology.
   (b) Show that Γ is separable, but that R with the induced topology is not.
   (c) Hence show that Γ is separable and first countable, but not second countable.
   (d) Show that $Q = \{(x, 0) : x \in \mathbb{Q}\}$ and $I = \{(x, 0) : x \notin \mathbb{Q}\}$ are disjoint closed sets; but if $I \subset U$ and $Q \subset V$, where U and V are open, then $U \cap V \neq \emptyset$. (Γ is not normal.)
   (e) Show that if $C \subset \Gamma$ is closed and $p \notin C$, there is a continuous function $f : \Gamma \to [0, 1]$ so that $f|_C = 0$ and $f(p) = 1$. (Γ is completely regular.)

2. Let $\mathbb{R}^{\mathbb{R}}$, the set of real functions on $\mathbb{R}$, have the product topology.
   (a) Show that a net $(f_\lambda)_{\lambda \in \Lambda}$ converges if and only if $\lim f_\lambda(x)$ exists for each $x \in \mathbb{R}$.
   (b) Let $E = \{f \in \mathbb{R}^{\mathbb{R}} : f(x) = 2$ except finitely often$\}$. Show that the zero function z is in the closure of E.
   (c) Construct a net in E with limit z.
   (d) Show that no sequence of points in E can converge to z.

3. Let $h : X \to Y$ be a continuous bijection of X onto Y. Suppose that X is compact and Y is Hausdorff. Prove that X and Y are homeomorphic.
   Hint: show that compact subsets of Hausdorff spaces are closed.

4. In a topological space X, suppose there is a net $(x_\lambda)_{\lambda \in \Lambda}$ converging to x; and for each λ, there is a net $(y_\mu^\lambda)_{\mu \in M_\lambda}$ converging to x_λ. Let $N = \{(\lambda, \mu) : \lambda \in \Lambda,\ \mu \in M_\lambda\}$ with $(\lambda, \mu) \leq (\lambda', \mu')$ iff $\lambda < \lambda'$ or $\lambda = \lambda'$ and $\mu \leq \mu'$ (*the lexicographic order*). Show that $(y_\mu^\lambda)_{(\lambda,\mu) \in N}$ has a subnet converging to x. Be explicit about using the Axiom of Choice.

5. Show that Tychonoff's Theorem implies the Axiom of Choice.
   Hint: Given non-empty sets X_α, set $Y_\alpha = \{p_\alpha\} \cup X_\alpha$. Let $\tau_\alpha = \{\emptyset, \{p_\alpha\}, X_\alpha, Y_\alpha\}$ be the topology on Y_α. Show that $\prod X_\alpha$ is a non-empty subset of $\prod Y_\alpha$.

6. (a) Show that there is an uncountable well-ordered set Ω_0 such that the initial segments $I(a) = \{x \in \Omega_0 : x < a\}$ for $a \in \Omega_0$ are all countable. Endow Ω_0 with the order topology. **Hint:** well-order $\mathbb{R}$ and select an appropriate initial segment.
   (b) Show that every increasing sequence in Ω_0 converges.
   (c) Show that Ω_0 is not compact, but that every sequence has a convergent subsequence (i.e. Ω_0 is *sequentially compact*) and every countable open cover has a finite subcover (i.e. Ω_0 is *countably compact*).
   (d) Show that every $A \subset \mathbb{R}$ is *Lindelof*: every open cover has a countable subcover.
   (e) Show that every continuous real-valued function on Ω_0 is eventually constant.
   Hint: Show $f(\Omega_0)$ must be compact. Then look at $\bigcap_{a \in \Omega_0} f(\Omega_0 \setminus I(a))$.