

Lecture 7

In lecture 6 we have found that neither MFT nor perturbation theory at small fluctuations around MFT work in $d < 4$.

The way to deal with this problem is RG which is a systematic procedure for eliminating short-wavelength degrees of freedom from LFW functional, since what's relevant for critical phenomena are long-wavelength fluctuations.

$$S[\varphi] = \int d^d x \left[\frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{c}{2} \varphi^2 + \frac{y}{4!} \varphi^4 \right]$$

$$\varphi(\vec{x}) = \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \varphi(\vec{k}) e^{i\vec{k} \cdot \vec{x}}$$

$\Lambda \sim \frac{\pi}{a}$ is the high-momentum cutoff.

$$\varphi(\vec{x}) = \varphi_L(\vec{x}) + \varphi_B(\vec{x})$$

$$\varphi_L(\vec{x}) = \int_0^{\Lambda/6} \frac{d^d k}{(2\pi)^d} \varphi(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \quad -\text{slow modes}$$

$$\varphi_B(\vec{x}) = \int_{\Lambda/6}^\Lambda \frac{d^d k}{(2\pi)^d} \varphi(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \quad -\text{fast modes}$$

B is slightly greater than one.

$$S[\varphi_L, \varphi_S] = S_0[\varphi_L] + S_0[\varphi_S] + S_{\text{int}}[\varphi_L, \varphi_S]$$

$$S_0[\varphi_L] = \frac{1}{2} \int_0^{\Lambda/\epsilon} \frac{d^d k}{(2\pi)^d} (k^2 + r) |\varphi_L(\vec{k})|^2$$

$$S_0[\varphi_S] = \frac{1}{2} \int_{\Lambda/\epsilon}^{\Lambda} \frac{d^d k}{(2\pi)^d} (k^2 + r) |\varphi_S(\vec{k})|^2$$

Now consider the φ^u term:

$$S_{\text{int}}[\varphi] = \frac{u}{4!} \int d^d x \varphi^u(x) = \\ = \frac{u}{4!} \int d^d x \int_0^{\Lambda} \frac{d^d k_1}{(2\pi)^d} \dots \frac{d^d k_4}{(2\pi)^d} \varphi(\vec{k}_1) \varphi(\vec{k}_2) \varphi(\vec{k}_3).$$

$$\cdot \varphi(\vec{k}_4) e^{i(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \cdot \vec{x}} = *$$

$$\int d^d x e^{i(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \cdot \vec{x}} = (2\pi)^d \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)$$

~~Define~~ Define:

$$\varphi_L(\vec{k}) = \begin{cases} \varphi(\vec{k}), & 0 < k < \Lambda/\epsilon \\ 0, & \Lambda/\epsilon < k < \Lambda \end{cases}$$

$$\varphi_S(\vec{k}) = \begin{cases} 0, & 0 < k < \Lambda/\epsilon \\ \varphi(\vec{k}), & \Lambda/\epsilon < k < \Lambda \end{cases}$$

$$\text{Then } \varphi(\vec{k}) = \varphi_L(\vec{k}) + \varphi_S(\vec{k})$$

Then we have:

$$\text{Smt}[\varphi_L, \varphi_S] = \frac{n}{q!} \int_0^{\infty} \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_q}{(2\pi)^d} \cdot$$

$$\cdot [\varphi_L(k_1) + \varphi_S(k_1)] \cdot [\varphi_L(k_2) + \varphi_S(k_2)] \cdot$$

$$\cdot [\varphi_L(k_3) + \varphi_S(k_3)] \cdot [\varphi_L(k_4) + \varphi_S(k_4)] \cdot$$

$$\cdot (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4)$$

Clearly, unlike in the case of the gaussian part of $S[\varphi]$, there will be cross terms here, coupling slow and fast modes, this is why we write $\text{Smt}[\varphi]$ as $\text{Smt}[\varphi_L, \varphi_S]$.

$$Z = \int D\varphi_L D\varphi_S e^{-S[\varphi_L, \varphi_S]}$$

Now imagine that we can do the integral over the fast modes φ_S . After the integration we obtain:

$$Z = \int D\varphi_L e^{-S[\varphi_L]}$$

$$e^{-S^1[\varphi_L]} = e^{-S_0[\varphi_L]} \int D\varphi_S e^{-S_0[\varphi_S]}.$$

$$\cdot e^{-S_{\text{int}}[\varphi_L, \varphi_S]} = \\ = e^{-S_0[\varphi_L]} \frac{\int D\varphi_S e^{-S_0[\varphi_S] - S_{\text{int}}[\varphi_L, \varphi_S]}}{\int D\varphi_S e^{-S_0[\varphi_S]}}.$$

$$\cdot \int D\varphi_S e^{-S_0[\varphi_S]}$$

$$\text{let } Z_{0S} = \int D\varphi_S e^{-S_0[\varphi_S]}$$

Then we obtain :

$$e^{-S^1[\varphi_L]} = e^{-S_0[\varphi_L]} \frac{1}{Z_{0S}} \int D\varphi_S e^{-S_{\text{int}}[\varphi_L, \varphi_S]}.$$

$$\cdot e^{-S_0[\varphi_S]} \cdot Z_{0S} =$$

$$= e^{-S_0[\varphi_L]} \left\langle e^{-S_{\text{int}}[\varphi_L, \varphi_S]} \right\rangle_{0S} Z_{0S}$$

Then we obtain:

$$S'[\varphi_c] = S_0[\varphi_c] - \ln \left\langle e^{-S_{\text{int}}[\varphi_c, \psi_b]} \right\rangle_{0>} - \ln Z_{0>}$$

Drop the last term - just a constant.

$$S'[\varphi_c] = S_0[\varphi_c] - \ln \left\langle e^{-S_{\text{int}}[\varphi_c, \psi_b]} \right\rangle_{0>}$$

Assumption: $S'[\varphi_c]$ still has term only up to 4th order in φ_c - will justify this later.

$$S'[\varphi_c] = \frac{1}{2} \int_0^{1/\bar{v}} \frac{d^d k}{(\bar{m})^d} \left(\tilde{r}_0 + \tilde{r}_2 k^2 \right) |\varphi_c(\vec{k})|^2 +$$

$$+ \frac{\tilde{v}}{4!} \int_0^{1/\bar{v}} \cancel{\int_0^{1/\bar{v}}} \frac{d^d k_1}{(\bar{m})^d} \cdots \frac{d^d k_4}{(\bar{m})^d} \cdot$$

$$\cdot \varphi_c(k_1^+) \varphi_c(k_2^+) \varphi_c(k_3^+) \varphi_c(k_4^+) (\bar{m})^4 \delta(k_1^+ + k_2^+ + k_3^+ + k_4^+)$$

Redefine momentum variables to make cutoff the same as before:

$$\vec{k}' = \vec{k} b$$

Also redefine φ_c variables to make the coefficient

if k^2 equal $\frac{t}{2}$, as before.

$$\varphi^i(\vec{k}^i) = z^{-1} \varphi_c(\vec{k})$$

z is called wavefunction renormalization.

$$S'[\varphi^i] = \frac{1}{2} \int_0^\infty \frac{d^d k^i}{(2\pi)^d} b^{-d} \left[\tilde{r}_0 + \tilde{r}_2 k^{i2} b^{-2} \right].$$

$$z^2 |\varphi^i(\vec{k}^i)|^2 + \frac{\tilde{m}}{4!} \int_0^\infty \frac{d^d k_1}{(2\pi)^d} \dots \frac{d^d k_4}{(2\pi)^d} \dots$$

$$b^{-4d} z^4 \varphi^i(\vec{k}_1^i) \varphi^i(\vec{k}_2^i) \varphi^i(\vec{k}_3^i) \varphi^i(\vec{k}_4^i)$$

$$(2\pi)^d \delta(\vec{k}_1^i + \vec{k}_2^i + \vec{k}_3^i + \vec{k}_4^i) b^d$$

From the k^{i2} term, z has to have the form:

$$z = b^{\frac{d+2-4}{2}}$$

(is called anomalous dimension (another critical exponent) and takes care of the ~~problematic~~).

$$\tilde{r}_2 \text{ coefficient: } b^{-\eta} = \tilde{r}_2^{-2}$$

After these transformations we obtain the same LFW functional as before, but with different coefficients r and m' :

$$S'[\varphi] = \int d^d x \left[\frac{1}{2} (\nabla \varphi)^2 + \frac{r'}{2} \varphi^2 + \frac{u'}{4!} \varphi^4 \right]$$

To summarize, there are 3 basic steps in the RF procedure:

1. Separate modes into fast and slow.
2. Integrate over fast modes.
3. Rescale momenta and φ_c to bring the LFW functional to the same form as before the transformation.

We can represent each LFW functional by a pair:

(r, u) . RF transformation maps:

$(r, u) \rightarrow (r', u')$ - both describe the same system.

What we are interested in are fixed points of RF:

$(r^*, u^*) \rightarrow (r^*, u^*)$

Why are we interested in fixed points?

Imagine the system had correlation length $\xi(r, u)$ before the RF transformation. After a single RF iteration it becomes:

$$\xi(r', u') = \frac{\xi(r, u)}{B}$$

At a fixed point, ξ must remain the same!

$$\zeta(r^*, u^*) = \frac{\zeta(r^*, u^*)}{\delta}$$

This can be true if $\zeta(r^*, u^*)$ is either zero or infinite.

The case when $\zeta(r^*, u^*)$ corresponds to critical points.

We will be able to find critical exponents from the behavior of r and u near the fixed point.

Now let us actually do the calculation explicitly.

$$S[\varphi] = \int d^d x \left[\frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} \varphi^2 + \frac{u}{4!} \varphi^4 \right]$$

~~Integrate over φ_B~~

$$S[\varphi] = S_0[\varphi_L] + S_0[\varphi_B] + S_{\text{int}}[\varphi_L, \varphi_B]$$

~~Integrate over φ_B~~ Integrate over φ_B :

$$e^{-S'[\varphi_L]} = e^{-S_0[\varphi_L]} \langle e^{-S_{\text{int}}[\varphi_L, \varphi_B]} \rangle_{0>}$$

$$\langle \dots \rangle_{0>} = \frac{\int D\varphi_B \dots e^{-S_0[\varphi_B]}}{\int D\varphi_B e^{-S_0[\varphi_B]}}$$

To evaluate the average convenient to use cumulant expansion?

Let r be any random variable.

Then the following result is true:

$$\langle e^r \rangle = e^{\langle r \rangle + \frac{1}{2} [\langle r^2 \rangle - \langle r \rangle^2] + \dots}$$

~~Proof~~ Proof :

Consider $g(t) = \ln \langle e^{tr} \rangle$ - generating function.

$g(t)$ has a Taylor expansion:

$$g(t) = \sum_{n=1}^{\infty} \lambda_n \frac{t^n}{n!}$$

λ_n is called n -th cumulant.

Clearly $\lambda_n = g^{(n)}(0)$

$$\lambda_1 = g^{(1)}(0) = \left. \frac{\langle re^{tr} \rangle}{\langle e^{tr} \rangle} \right|_{t=0} = \langle r \rangle$$

$$\begin{aligned} \lambda_2 &= g^{(2)}(0) = \left[\frac{\langle r^2 e^{tr} \rangle}{\langle e^{tr} \rangle} - \frac{\langle re^{tr} \rangle^2}{\langle e^{tr} \rangle^2} \right]_{t=0} = \\ &= \langle r^2 \rangle - \langle r \rangle^2 \end{aligned}$$

Then we obtain:

$$S'[\varphi_c] = S_0[\varphi_c] + \langle S_{\text{int}} \rangle_{00} - \\ - \frac{1}{2} \left[\langle S^2_{\text{int}} \rangle_{00} - \langle S_{\text{int}} \rangle_{00}^2 \right] + \dots$$

We want need to go beyond ~~second~~ second order in the cumulant expansion.

$$S_{\text{int}}[\varphi_c, \varphi_b] = \frac{u}{4!} \int_0^\infty \frac{d^d k_1}{(2\pi)^d} \dots \frac{d^d k_4}{(2\pi)^d} \cdot$$

$$\cdot [\varphi_c(k_1^+) + \varphi_b(k_1^-)] \cdot [\varphi_c(k_1^+) + \varphi_b(k_1^+)] \cdot \\ \cdot [\varphi_c(k_3^+) + \varphi_b(k_3^-)] \cdot [\varphi_c(k_3^+) + \varphi_b(k_3^+)] \cdot \\ \cdot (2\pi)^d \delta(k_1^+ + k_2^+ + k_3^+ + k_4^+)$$

Thus we will need to calculate averages of products of fast variables of the type:

$$\langle \varphi_b(k_1^+) \varphi_b(k_2^-) \dots \varphi_b(k_n^-) \rangle_0 = \\ = \frac{1}{Z_0} \int D\varphi_b \varphi_b(k_1^+) \dots \varphi_b(k_n^-) e^{-S_0[\varphi_b]}$$

To calculate these, we need to develop a bit of formalism.

Introduce a generating function:

$$G(y^*, y) = \frac{\int D\Phi e^{-\sum_{ij} y_i^* M_{ij} y_j + \sum_i (y_i^* y_i + y_i y_i^*)}}{\int D\Phi e^{-\sum_{ij} y_i^* M_{ij} y_j}}$$

Here y_i are complex variables, M_{ij} is a Hermitian positive definite matrix and:

$$D\Phi \equiv \prod_{i=1}^N \frac{d\Phi_i^R d\Phi_i^I}{\pi}$$

A simple generalization of the Hubbard-Stratonovich identity for real variables to complex variables gives:

$$\begin{aligned} \int D\Phi e^{-\sum_{ij} y_i^* M_{ij} y_j + \sum_i (y_i^* y_i + y_i y_i^*)} &= \\ = \frac{1}{\det M} e^{\sum_j y_j^* M_j^{-1} y_j} \end{aligned}$$

Thus we obtain:

$$G(y^*, y) = e^{\sum_j y_j^* M_j^{-1} y_j}$$

Averages of products of Φ_i variables can be calculated as derivatives of the generating function.

For example:

~~Integration of $\int \psi_i^* \psi_j d\Omega = \int \psi_i^* \psi_j e^{-\sum_j y_j^* M_{ij} \psi_j} d\Omega$~~

$$\langle \psi_i \psi_j^* \rangle = \frac{\int D\gamma \psi_i \psi_j^* e^{-\sum_j y_j^* M_{ij} \psi_j}}{\int D\gamma e^{-\sum_j y_j^* M_{ij} \psi_j}} \Big|_{y=0}$$

$$= \frac{\partial F(y^*, y)}{\partial y_i^* \partial y_j} \Big|_{y=0} = M_{ij}^{-1}$$

$$\langle \psi_i \psi_j^* \rangle = M_{ij}^{-1}$$