

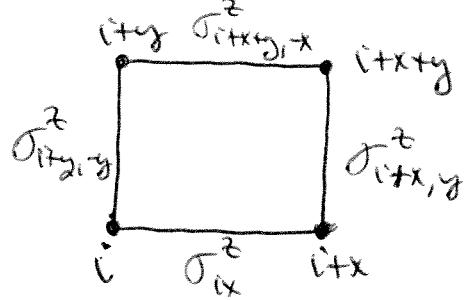
## Lecture 25

Considering Ising gauge Theory on a square lattice.

$$\mathcal{H} = -h \sum_{\langle i,j \rangle} \sigma_{i,j}^x - K \sum_D \prod_{i,j} \sigma_{i,j}^z$$

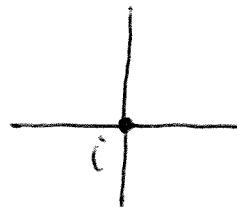
$\sigma^{x,z}$  are defined on the links of the square lattice.

$$\prod_D \sigma_{i,j}^z \equiv \sigma_{ix}^z \sigma_{i+x,y}^z \sigma_{i+x+y,-x}^z \sigma_{iy,y}^z$$



$\mathcal{H}$  is invariant under a gauge transformation:

$$G_i = \prod_j \sigma_{i,j}^x$$



$G_i$  flips  $\sigma^z \rightarrow -\sigma^z$  on all lines adjacent to site  $i$ . This is a local symmetry or gauge invariance.

Compare with global symmetry of the Ising model:

$$\mathcal{H} = -h \sum_i \sigma_i^x - J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - \text{this is only invariant under simultaneous change of sign of all } \sigma^z \text{'s.}$$

It's obvious that  $\langle \sigma_{ix}^z \rangle$  is always zero - Elitzur's theorem - local symmetry can not be spontaneously broken.  
 Nevertheless, there are two distinct phases.  
 The existence of two distinct phases can be shown by duality mapping.

To define the mapping it's convenient to fix a gauge.

Note that  $\sigma_i^2 = 1 \Rightarrow$  the eigenvalues of  $\sigma_i$  can be either +1 or -1.

~~Consider~~ Restrict to subset of states with eigenvalue +1.  
 For any state in this subset  $\sigma_i |+\rangle = |+\rangle$ . physical states are gauge invariant.  
 So  $\sigma_i = 1$

This means that  $\prod_x \sigma_{ix}^x = 1$

$$\sigma_{ix}^x \sigma_{i,-x}^x \sigma_{iy}^x \sigma_{i,-y}^x = 1$$

Solve this for  $\sigma_{iy}^x$ :

$$\sigma_{iy}^x = \sigma_{ix}^x \sigma_{i,-x}^x \sigma_{iy}^x$$

Analogously:

$$\sigma_{i-y}^x = \sigma_{i-y, y}^x = \sigma_{i-y,x}^x \sigma_{i-y,-x}^x \sigma_{i-y,-y}^y$$

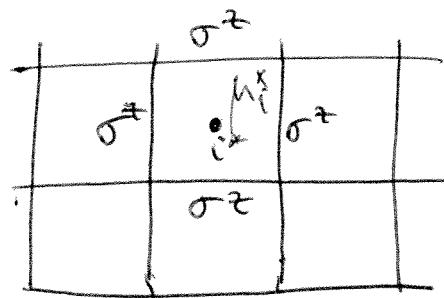
Then:

$$\sigma_{i,y}^x = \sigma_{i,x}^x \sigma_{i,-x}^x \sigma_{i-y,x}^x \sigma_{i-y,-x}^x \sigma_{i-y,x}^x \sigma_{i-y,-x}^x \dots$$

i.e.  $\sigma_{i,y}^x$  can always be expressed in terms of  $\sigma_{i,x}^x$ .

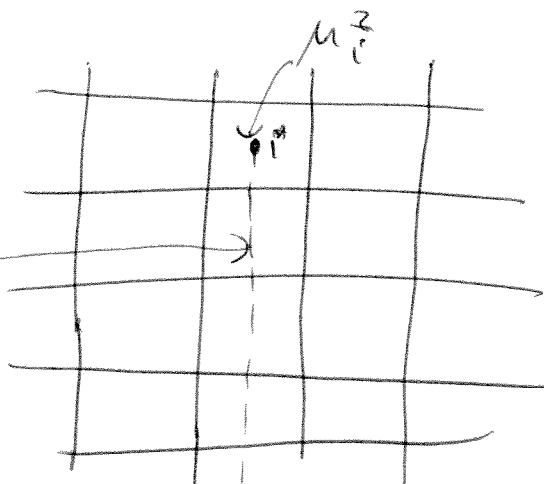
Define dual Ising variables as:

$$\mu_{i*}^x = \prod \sigma^z$$



$$\mu_{i*}^x = \prod \sigma_{i,x}^z$$

product along the cut



Clearly  $\mu_{i*}^x \mu_{i*}^z = \sigma_{i,x}^x$

$$\mu_{i*}^x \mu_{i*-x}^z = \sigma_{i,y}^x \rightarrow \text{thus follows from:}$$

$$\sigma_{i,y}^x = \sigma_{i,x}^x \sigma_{i,-x}^y \sigma_{i-y,x}^x \sigma_{i-y,-x}^y \dots$$

$$\text{Also } (\mu_{i*}^x)^2 = (\mu_{i*}^z)^2 = 1$$

$$\mu_{i*}^x \mu_{i*}^{z*} = -\mu_{i*}^z \mu_{i*}^x$$

Thus  $\mu_{i*}^x$  are also Pauli matrices.

Ising gauge theory Hamiltonian, rewritten in terms of dual variables, becomes:

$$\mathcal{H} = -h \sum_{\langle i,j \rangle} \mu_i^z \mu_j^z + K \sum_i \mu_i^x$$

This is simply a 2D transverse-field Ising model, which we know has two distinct phases.

Large  $\frac{K}{h}$  - all spins polarized in the x-direction - disordered phase.

Large  $\frac{h}{K}$  -  $\langle \mu_i^z \rangle \neq 0$  - ordered phase.

The large  $\frac{K}{h}$  phase is called deconfined,  
the large  $\frac{h}{K}$  - confined.

Deconfined phase is topologically ordered.

Consider the deconfined, large  $\frac{K}{h}$  phase.

Clearly one possible ground state corresponds to  $\sigma^z = 1$  on every link. ~~What are the excitations in this phase?~~ In the dual model this phase corresponds to spins  $\mu_i$  polarized in the x-direction. Excitations are ~~one~~ spins flipped to -x-direction.

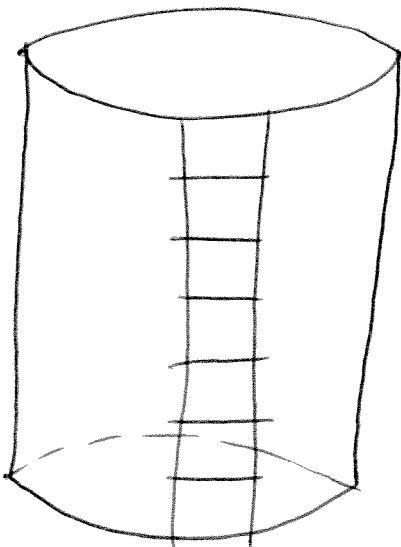
The operator, that flips dual spin from x to -x is  $\mu_i^z$ . In the original variables,  $\mu_i^z$  corresponds to a string of  $\sigma^x$  operators.

Such a string creates an "Ising vortex" or viron;

$\prod_C \sigma^z = -1$ , where  $C$  is any loop on  
the direct lattice, enclosing the dual lattice site  $i$ .

VIsions are gapped, i.e. cost a finite energy in the deconfined phase.

A distinct ground state from all  $\sigma^z \neq 1$  can be obtained ~~as follows~~ as follows:



Flip  $\sigma^z = -1$  on all horizontal lines as shown in figure.

Then if we take any loop  $C$  ~~around~~ around the cylinder:

$\prod_C \sigma^z = -1$  - there is a vision threading the holes of the cylinder.

This in the deconfined phase the GS is doubly degenerate.

On a torus, the ~~deconfined~~ deconfined GS would be 4-fold degenerate (two holes).

This is topological order - GS has degeneracy that depends on topology.

Compare with Ising model:

$$H = -h \sum_i \sigma_i^x - J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z$$

Paramagnetic GS at large  $\frac{h}{J}$  is unique.

Ferromagnetic GS at large  $\frac{J}{h}$  is doubly degenerate. The degeneracy is due to the global  $\sigma^z \rightarrow -\sigma^z$  symmetry of the Ising model.

Ising ~~is~~ gauge theory:

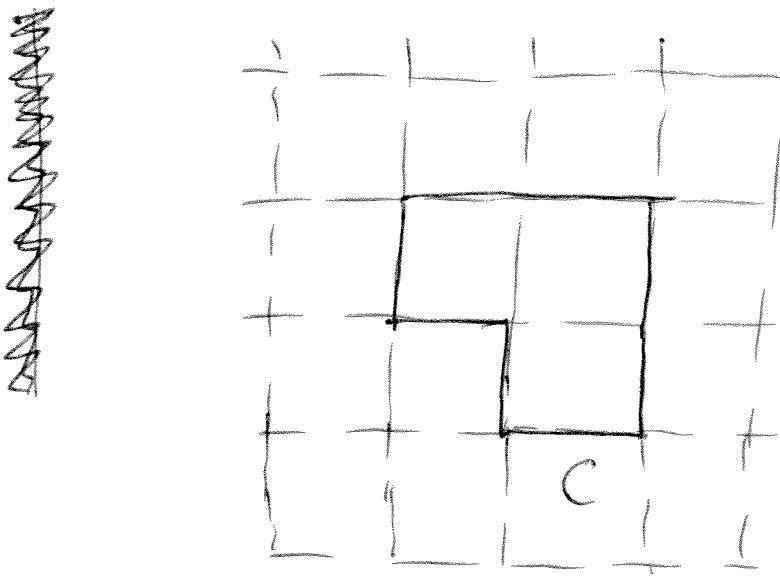
$$H = -h \sum_i \sigma_{ip}^x - k \sum_{\square} \prod_{\square} \sigma^z$$

Confined GS (no order) at large  $\frac{h}{k}$  - unique GS.

Deconfined GS at ~~large~~ large  $\frac{k}{h}$  - ~~topologically~~ ordered state - ~~topologically~~ topology-dependent degeneracy.

Another way to distinguish the two phases is in terms of gauge-invariant correlation functions.

Gauge-invariant quantities are products of  $\sigma^z$ 's along closed loops.



$$\langle \prod_c \sigma_{ip}^z \rangle = W(c)$$

It turns out that in the deconfined phase:

$$\langle \prod_c \sigma_{ip}^z \rangle \sim e^{-CP}, \text{ where } P \text{ is the perimeter of the loop.}$$

while in the confined phase:

$$\langle \prod_c \sigma_{ip}^z \rangle \sim e^{-CA}, A - \text{area enclosed.}$$

Indeed, consider the deconfined phase  $K \gg h$ .

Ground state  $|\Psi_0\rangle = \prod_i |\sigma_{ip}^z = 1\rangle$  for  $\frac{h}{K} \approx 0$ .

At order  $h^0$ :

~~$$\langle \Psi_0 | \prod_c \sigma_{ip}^z | \Psi_0 \rangle = \langle \Psi_0 | W(c) | \Psi_0 \rangle = 1$$~~

~~Orbital degeneracy~~

~~Rebreak orbital order~~

GS wavefunction with first order correction included is:

$$|4S\rangle = \prod_{i\mu} \frac{|\sigma_{i\mu}^z = 1\rangle + \frac{\hbar}{K} |\sigma_{i\mu}^z = -1\rangle}{\sqrt{1 + \frac{\hbar^2}{K^2}}}$$

$$\langle 4 | w(c) | 4 \rangle = \left( \frac{1 - \frac{\hbar^2}{K^2}}{1 + \frac{\hbar^2}{K^2}} \right)^P P(c) = \\ = e^{-h \left( \frac{1 + \frac{\hbar^2}{K^2}}{1 - \frac{\hbar^2}{K^2}} \right)} P(c)$$

- perimeter law.

Now consider the ~~non~~ confined phase,  $\hbar \gg K$ .

$$\text{At } K=0 \text{ the GS is } |4_0\rangle = \prod_{i\mu} |\sigma_{i\mu}^x = 1\rangle$$

$$\langle 4_0 | w(c) | 4_0 \rangle = \prod_{i\in c} \langle \sigma_{i\mu}^x | \sigma_{i\mu}^z | \sigma_{i\mu}^x \rangle = 0.$$

First order - corrected  $|4S\rangle$ :

$$|4S\rangle = |4_0\rangle + \frac{K}{\hbar} \sum_{i\in D} |\sigma_{i\mu}^x = 1\rangle$$

• GS wavefunction to first order in  $\frac{K}{h}$ :

$$|\Psi\rangle = \prod_{i\mu \in \alpha} |\sigma_{i\mu}^x = 1\rangle + \frac{K}{h} \prod_{i\mu \in \beta} |\sigma_{i\mu}^x = -1\rangle$$

$$\langle \Psi | w(c) | \Psi \rangle = \langle \Psi | \prod_{i\mu \in \alpha} \sigma_{i\mu}^z | \Psi \rangle =$$

$$= \langle \Psi | \prod_{i\mu \in \alpha} \sigma_{i\mu}^z | \Psi \rangle = \left(\frac{K}{h}\right)^{A(c)} =$$

$$= e^{-A(c) \ln\left(\frac{K}{h}\right)}$$

Now couple bosons to 2<sub>r</sub> gauge field.

$$\mathcal{H} = -\frac{eJ}{2} \sum_{i\mu} \sigma_{i\mu}^z (e^{i\phi_i} e^{-i\phi_{i\mu}} + h.c.) + \frac{1}{2C} \sum_i n_i^2 -$$

$$- h \sum_{i\mu} \sigma_{i\mu}^x - K \sum_{i\mu \in \alpha} \prod_{i\mu \in \beta} \sigma_{i\mu}^z$$

① Gauge invariance:

$$\sigma_{i\mu}^z \rightarrow \epsilon_i \epsilon_{i\mu} \sigma_{i\mu}^z$$

$$e^{i\phi_i} \rightarrow \epsilon_i e^{i\phi_i}, \quad \epsilon_i = \pm 1$$

~~cancel phase ambiguity~~

Confined phase - all particles are bound into ~~entangled~~ pairs -  
The energy cost of ~~keeping each pair separate~~  
~~separate~~ the two particles to a distance  $r$  is  $\sim h r$ .

Decoupled phase:  $\Gamma_{ij\mu}^+ = 1$  - particles are free to propagate.