

Lecture 22

In lecture 21 we derived continuum imaginary-time action for the transverse-field Ising model:

$$S[\varphi] = \int_0^\beta dt \int d^d x \left[\frac{1}{2} (\partial_t \varphi)^2 + \frac{1}{2} (\vec{\nabla} \varphi)^2 + \frac{g}{2} \varphi^2 + \frac{u}{4!} \varphi^4 \right]$$

Critical exponents can be calculated in $d = 3 - \epsilon$ dimensions

Correlation length ξ - ~~characterized~~ characterized spatial fluctuations of φ .

Need to also introduce correlation time τ .

In general $\tau \sim \xi^z$, z - dynamical critical exponent.

In the case of the transverse field Ising model $z=1$ since time and ~~space~~ space are on equal footing (action has relativistic form). In general $z \neq 1$.

A nontrivial problem is how to translate correlation functions of the classical $d+1$ -dimensional Ising model into time-dependent correlation functions of the d -dimensional transverse field Ising model.

Consider imaginary-time correlation function:

$$G_{ij}(\tau) = T_\tau \langle \sigma_i^z(\tau) \sigma_j^z(0) \rangle$$

Here $\sigma_i^z(\tau) = e^{H\tau} \sigma_i^z e^{-H\tau}$ - imaginary-time Heisenberg representation of spin operators.

T_τ is the time-ordering operator, puts operators with later times to the left.

We have periodic b.c. in imaginary time:

$$G_{ij}(\tau + \beta) = G_{ij}(\tau)$$

Then we have: $G_{ij}(\tau) = \frac{1}{\beta} \sum_{\omega_n} G_{ij}(\omega_n) e^{-i\omega_n \tau}$

$$\omega_n = \frac{2\pi n}{\beta} \quad \text{- Matsubara frequencies.}$$

Calculate $G_{ij}(\tau)$ ~~assuming~~ assuming we know exact eigenstates of the Hamiltonian.

$$G_{ij}(\tau) = \frac{1}{Z} \text{Tr} [e^{-\beta H} \sigma_i^z(\tau) \sigma_j^z(0)]$$

$$\text{let } H |n\rangle = E_n |n\rangle$$

$|n\rangle$ - complete set of eigenstates of H .

Inserting resolution of identity $1 = \sum_n |n\rangle \langle n|$

between the factors in $G_{ij}(\tau)$, we obtain:

$$G_{ij}(t) = \frac{1}{Z} \sum_{nm} \langle n | e^{-\beta H} | n \rangle \langle n | \sigma_i^z(t) | m \rangle.$$

$$\begin{aligned} \langle m | \sigma_j^z(0) | n \rangle &= \\ &= \frac{1}{Z} \sum_{nm} e^{-\beta E_n} e^{t(E_n - E_m)} \langle n | \sigma_i^z | m \rangle \langle m | \sigma_j^z | n \rangle \end{aligned}$$

$$\text{let } E_{mn} = E_m - E_n$$

$$G_{ij}(i\omega_n) = \int_0^\beta dt G_{ij}(t) e^{i\omega_n t}$$

$$G_{ij}(i\omega_n) = \frac{1}{Z} \sum_{mm'} e^{-\beta E_m} \langle m | \sigma_i^z | m' \rangle.$$

$$\langle m' | \sigma_j^z | m \rangle \int_0^\beta dt e^{(i\omega_n - E_{m'm})t}$$

$$\int_0^\beta dt e^{(i\omega_n - E_{m'm})t} = \frac{1}{i\omega_n - E_{m'm}} \left[e^{(i\omega_n - E_{m'm})\beta} - 1 \right]$$

$$= \frac{1}{i\omega_n - E_{m'm}} \left(e^{-E_{m'm}\beta} - 1 \right), \text{ since } e^{i\beta\omega_n} = 1.$$

Then we obtain:

$$G_{ij}(i\omega_n) = \frac{1}{Z} \sum_{m, m'} e^{-\beta E_m} (e^{-\beta E_{m'm}} - 1).$$

$$\frac{\langle m | \sigma_i^z | m' \rangle \langle m' | \sigma_j^z | m \rangle}{i\omega_n - E_{m'm}}$$

Introduce spectral density:

$$\rho_{ij}(\omega) = \frac{2\pi}{Z} \sum_{m, m'} e^{-\beta E_m} (1 - e^{-\beta E_{m'm}}).$$

$$\langle m | \sigma_i^z | m' \rangle \langle m' | \sigma_j^z | m \rangle \delta(\omega - E_{m'm})$$

Then we finally obtain:

$$G_{ij}(i\omega_n) = - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\rho_{ij}(\omega)}{i\omega_n - \omega}$$

$G_{ij}(i\omega_n)$ is what we can calculate from the classical $d+1$ -dimensional model.

Now let us connect it to measurable real-time properties.

Consider real-time response function:

$$\text{let } H = H_0 - \sum_i h_i(t) \sigma_i^z = H_0 + V(t)$$

$h_i(t)$ is a space and time-dependent magnetic field.

We want to see how system responds to $h_i(t)$ when h_i is small.

$$i \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle$$

Introduce "interaction representation" of operators:

$$\sigma_i^z(t) = e^{iH_0 t} \sigma_i^z e^{-iH_0 t}$$

$$\text{Then } i \frac{\partial}{\partial t} |\psi\rangle = V(t) |\psi\rangle$$

$$\text{Assume } h_i(-\infty) = 0$$

The general solution of the SE is:

$$\begin{aligned} |\psi(t)\rangle &= |\psi_0\rangle + \int_{-\infty}^t dt' \frac{\partial |\psi(t')\rangle}{\partial t'} = \\ &= |\psi_0\rangle - i \int_{-\infty}^t dt' V(t') |\psi(t')\rangle \end{aligned}$$

$$\text{Here } |\psi_0\rangle \equiv |\psi(t=-\infty)\rangle$$

This equation can be solved perturbatively in V :

At first order in V we obtain:

$$|\psi(t)\rangle = |\psi_0\rangle - i \int_{-\infty}^t dt' V(t') |\psi_0\rangle$$

Calculate $\delta \langle \sigma_i^z(t) \rangle = \langle \Psi(t) | \sigma_i^z | \Psi(t) \rangle - \langle \Psi_0 | \sigma_i^z | \Psi_0 \rangle$ - change in the expectation value of $\sigma_i^z(t)$ due to applied perturbation.

$$\delta \langle \sigma_i^z(t) \rangle = \left[\langle \Psi_0 | + \langle \Psi_0 | i \int_{-\infty}^t dt' V(t') \right]$$

$$\cdot \sigma_i^z(t) \cdot \left[| \Psi_0 \rangle - i \int_{-\infty}^t dt' V(t') | \Psi_0 \rangle \right] =$$

$$= i \int_{-\infty}^t dt' \langle \Psi_0 | V(t') \sigma_i^z(t) - \sigma_i^z(t) V(t') | \Psi_0 \rangle =$$

$$= i \int_{-\infty}^t dt' \sum_j \langle [\sigma_i^z(t), \sigma_j^z(t')] \rangle_0 h_j(t')$$

Introduce retarded response function:

$$\chi_{ij}(t-t') = i \theta(t-t') \langle [\sigma_i^z(t), \sigma_j^z(t')] \rangle$$

$$\delta \langle \sigma_i^z(t) \rangle = \int_{-\infty}^t dt' \sum_j \chi_{ij}(t-t') h_j(t')$$

Here $\langle \cdot \rangle$ means $\langle \Psi_0 | \cdot | \Psi_0 \rangle$ at $T=0$
and $\frac{1}{Z} \sum_n e^{-\beta E_n} \langle n | \cdot | n \rangle$ at $T \neq 0$.

Here $|n\rangle$ are eigenstates of H_0 .

let us relate χ to G .

$$\begin{aligned}
 \chi_{ij}(t) &= i\theta(t) \frac{1}{Z} \sum_n e^{-\beta E_n} \langle n | [\sigma_i^z(t), \sigma_j^z(0)] | n \rangle \\
 &= i\theta(t) \frac{1}{Z} \sum_{nm} e^{-\beta E_n} \left[\langle n | \sigma_i^z(t) | m \rangle \langle m | \sigma_j^z(0) | n \rangle \right. \\
 &\quad \left. - \langle n | \sigma_j^z(0) | m \rangle \langle m | \sigma_i^z(t) | n \rangle \right] = \\
 &= i\theta(t) \frac{1}{Z} \sum_{nm} e^{-\beta E_n} \left[e^{i(E_n - E_m)t} \langle n | \sigma_i^z | m \rangle \right. \\
 &\quad \left. \langle m | \sigma_j^z | n \rangle - e^{i(E_m - E_n)t} \langle n | \sigma_j^z | m \rangle \langle m | \sigma_i^z | n \rangle \right]
 \end{aligned}$$

let $E_{nm} = E_m - E_n$

Calculate Fourier transform of χ :

$$\chi_{ij}(\omega) = \int_{-\infty}^{\infty} dt \chi_{ij}(t) e^{i\omega t}$$

$$\begin{aligned}
 &\int_{-\infty}^{\infty} dt \theta(t) e^{iE_{nm}t} e^{i\omega t} = \\
 &= \int_0^{\infty} dt e^{i(\omega + E_{nm})t} =
 \end{aligned}$$

$$= \lim_{\eta \rightarrow 0^+} \int_0^{\infty} dt e^{i(\omega + E_{nm} + i\eta)t} = \frac{i}{\omega + E_{nm} + i\eta}$$

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$\eta = 0^+$

Then we obtain:

$$\chi_{ij}(\omega) = - \frac{1}{Z} \sum_{nm} e^{-\beta E_{nm}} \left[\frac{\langle n | \sigma_i^z | m \rangle \langle m | \sigma_j^z | n \rangle}{\omega + E_{nm} + i\eta} - \frac{\langle n | \sigma_j^z | m \rangle \langle m | \sigma_i^z | n \rangle}{\omega - E_{nm} + i\eta} \right]$$

Define dynamic structure factor:

$$S_{ij}(\omega) = \frac{2\pi}{Z} \sum_{nm} e^{-\beta E_n} \langle n | \sigma_i^z | m \rangle \langle m | \sigma_j^z | n \rangle \cdot$$

$$\cdot \delta(\omega - E_{nm})$$

Comparing with the definition of $\rho_{ij}(\omega)$, we have:

$$S_{ij}(\omega) = \frac{\rho_{ij}(\omega)}{1 - e^{-\beta\omega}}$$

In terms of $S_{ij}(\omega)$ we have:

$$\begin{aligned}
 X_{ij}(\omega) &= - \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left[\frac{S_{ij}(\omega')}{\omega - \omega' + iy} - \frac{S_{ji}(\omega')}{\omega + \omega' + iy} \right] \\
 &= \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \left[S_{ij}(\omega') - S_{ji}(-\omega') \right] \frac{1}{\omega - \omega' + iy}
 \end{aligned}$$

It is easy to show that $S_{ji}(-\omega) = e^{-\beta\hbar\omega} S_{ij}(\omega)$

Then we obtain

$$X_{ij}(\omega) = - \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} S_{ij}(\omega') (1 - e^{-\beta\hbar\omega'})$$

$$\frac{1}{\omega - \omega' + iy} = - \int_{-\infty}^{\infty} \frac{d\omega''}{2\pi} \frac{\rho_{ij}(\omega'')}{\omega - \omega' + iy}$$

Comparing with the imaginary time correlation function:

$$G_{ij}(\tau\omega_n) = - \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\rho_{ij}(\omega')}{i\omega_n - \omega'}$$

we obtain:

$$G_{ij}(\tau\omega_n \rightarrow \omega' + iy) = X_{ij}(\omega)$$

Real-time response function is obtained from imaginary-time correlation function by analytic continuation.

Now let's see how this works explicitly for the transverse-field Ising model.

$$G_{ij}(\tau) = \langle \sigma_i^z(\tau) \sigma_j^z(0) \rangle$$

Assume $0 \leq \tau \leq \beta$

$$\begin{aligned} G_{ij}(\tau) &= \frac{1}{Z} \text{Tr} \left[e^{-\beta H} \sigma_i^z(\tau) \sigma_j^z(0) \right] = \\ &= \frac{1}{Z} \text{Tr} \left[e^{-\beta H} e^{H\tau} \sigma_i^z e^{-H\tau} \sigma_j^z \right] = \\ &= \frac{1}{Z} \text{Tr} \left[e^{-(\beta-\tau)H} \sigma_i^z e^{-H\tau} \sigma_j^z \right] = \\ &= \frac{1}{Z} \sum_{\{\sigma^z(\tau_n)\}} \dots \sum_{\{\sigma^z(\tau_1)\}} \sigma_i^z(\tau) \sigma_j^z(0) e^{-S[\sigma^z]} \end{aligned}$$

Thus the ordered ~~imaginary-time~~ imaginary-time correlation function directly maps onto ~~classical~~ correlation function of the classical d+1-dimensional model.

Can calculate it from the continuum imaginary-time action:

$$\begin{aligned} S[\varphi] &= \int_0^\beta d\tau \int d^d x \left[\frac{1}{2} (\partial_\tau \varphi)^2 + \frac{1}{2} (\nabla \varphi)^2 + \right. \\ &\left. + \frac{c}{2} \varphi^2 + \frac{g}{4!} \varphi^4 \right] \end{aligned}$$

$$G_{ij}(\tau) = \langle \psi_i(\tau) \psi_j(0) \rangle$$

First consider $G_{ij}(\tau)$ above the transition in the disordered phase ($r > 0$).

$$S[\psi] \approx \int_0^\beta d\tau \int d^d x \left[\frac{1}{2} (\partial_\tau \psi)^2 + \frac{1}{2} (\nabla^2 \psi)^2 + \frac{r}{2} \psi^2 \right]$$

$$= \frac{1}{V\beta} \sum_{\vec{k}, \omega_n} \frac{1}{2} (k^2 + \omega_n^2 + r) |\psi(\vec{k}, \omega_n)|^2$$

$$G(\vec{k}, i\omega_n) = \frac{1}{V\beta} \langle |\psi(\vec{k}, \omega_n)|^2 \rangle = \frac{1}{k^2 + \omega_n^2 + r}$$

$$\chi(\vec{k}, \omega) = G(\vec{k}, i\omega_n \rightarrow \omega + iy) = \frac{1}{r + k^2 - (\omega + iy)^2} =$$

$$= \frac{1}{(\sqrt{r+k^2} - \omega + iy)(\sqrt{r+k^2} + \omega + iy)}$$

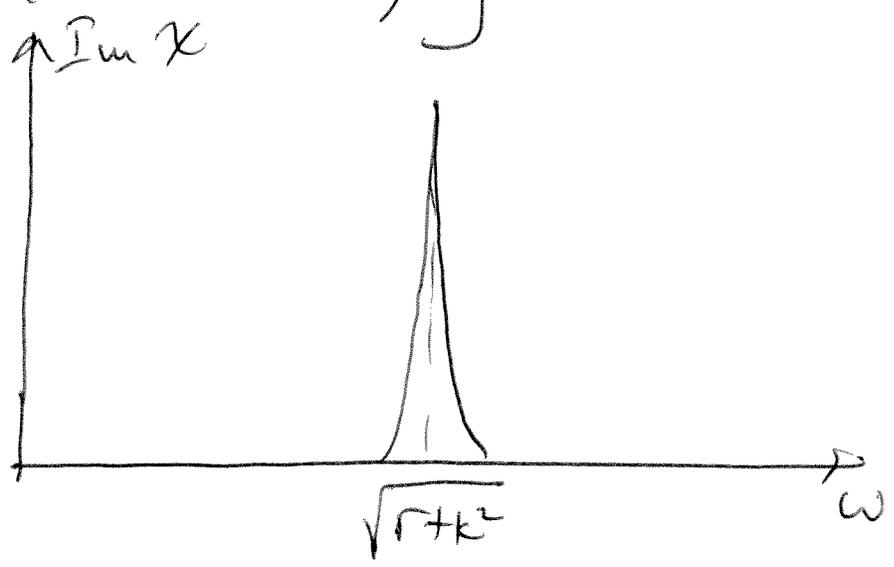
$$\frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$$

$$\lim_{y \rightarrow 0} \frac{1}{\pi} \frac{y}{x^2 + y^2} = \delta(x)$$

Thus $\text{Im} \frac{1}{x+iy} = -\pi \delta(x)$

Then we obtain:

$$\text{Im} \chi(\vec{k}, \omega) = \frac{\pi}{2\sqrt{\Gamma+k^2}} \left[\delta(\omega - \sqrt{\Gamma+k^2}) - \delta(\omega + \sqrt{\Gamma+k^2}) \right]$$



$\omega = \sqrt{\Gamma+k^2}$ - dispersion of quasiparticle excitations in the paramagnetic phase - has exactly the same form as dispersion of relativistic bosonic spinless particles!



Quasiparticle excitations correspond to flipped spins:



~~Spin~~ Spin flips propagate as relativistic particles.

Now see what happens at the critical point:

$$G(r) = \langle \psi(r) \psi(0) \rangle \sim \frac{1}{r^{D-2+\gamma}}$$

$$D = d+1$$

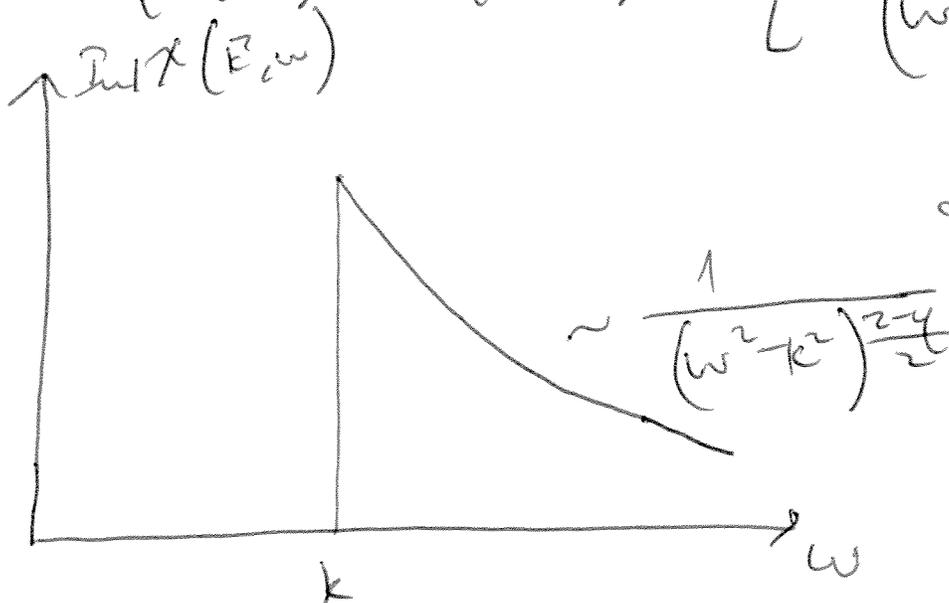
$$G(\vec{k}) = \int d^D r e^{-i\vec{k}\cdot\vec{r}} G(r) \sim \frac{1}{k^{2-\gamma}}$$

Restoring Matsubara frequency notation we have:

$$G(\vec{k}, i\omega) \sim \frac{1}{(k^2 + \omega^2)^{\frac{2-\gamma}{2}}}$$

$$\chi(\vec{k}, \omega) \sim \frac{1}{(k^2 - \omega^2)^{\frac{2-\gamma}{2}}}$$

$$\text{Im} \chi(\vec{k}, \omega) = \theta(\omega - k) \text{Im} \left[\frac{e^{-i\pi \left(\frac{2-\gamma}{2}\right)}}{(\omega^2 - k^2)^{\frac{2-\gamma}{2}}} \right]$$



critical point
does not have quasiparticle
excitations.