

## Lecture 18

~~Stability of superconducting state~~

Ginzburg - Landau ~~entanglement~~ free energy of a superconductor:

$$F = \int d^d x \left[ \frac{1}{2m^*} |(-i\hbar \vec{\nabla} + e^* \vec{A}) \Phi|^2 + \right. \\ \left. + \alpha |\Phi|^2 + \frac{B}{2} |\Phi|^4 + \frac{\vec{B}^2}{8\pi} - \frac{\vec{B} \cdot \vec{H}}{4\pi} \right]$$

Here  $m^* = 2m$ ,  $e^* = 2e$ ,  $\vec{H}$  is the external magnetic field,  $\vec{B}$  is the ~~total~~ total field inside the superconductor,  $\vec{B} = \vec{\nabla} \times \vec{A}$ .

$\Phi(\vec{x})$  is a macroscopic condensate wavefunction of the electron pairs.

Superconductivity is ~~not~~ superfluidity of electron pairs. However, unlike He atoms electron pairs are charged. This is why there is coupling to electromagnetic field.

~~the~~ thermodynamic equilibrium state of a superconductor is determined by minimizing  $F$ .

Vary  $F$  with respect to  $\Phi^*$ .

$$\delta_{\Phi^*} F = \int d^d x \left[ \alpha \Phi \delta \Phi^* + \beta \Phi |\Phi|^2 \delta \Phi^* + \right. \\ \left. + \frac{1}{2m^*} \left( i\hbar \vec{\nabla} \delta \Phi^* + e^* \vec{A} \delta \Phi^* \right) \cdot \left( -i\hbar \vec{\nabla} \Phi + e^* \vec{A} \Phi \right) \right]$$

$$\begin{aligned}
 & \int d^d x \vec{\nabla} \delta \Phi^* \cdot \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right) \Phi = \\
 &= \int d^d x \vec{\nabla} \cdot \left[ \delta \Phi^* \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right)^0 \Phi \right] - \\
 & - \int d^d x \delta \Phi^* \vec{\nabla} \cdot \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right) \Phi = \\
 &= - \int d^d x \delta \Phi^* \vec{\nabla} \cdot \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right) \Phi
 \end{aligned}$$

Then we obtain :

$$\begin{aligned}
 \delta \Phi^* F = & \int d^d x \left[ a \Phi + b |\Phi|^2 \Phi + \right. \\
 & \left. + \frac{1}{m^*} \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right)^2 \Phi \right] \delta \Phi^* = 0
 \end{aligned}$$

Thus we get the first GL equation:

$$\frac{1}{m^*} \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right)^2 \Phi + a \Phi + b |\Phi|^2 \Phi = 0$$

Look like a Schrödinger equation for a particle in a magnetic field, but nonlinear.

The magnetic field inside the sample is also a thermodynamic variable  $\Rightarrow$  also have to minimize  $F$  with respect to  $\vec{A}$ .

$$\delta \vec{A} F = \int d^d x \left[ \frac{1}{m^*} \frac{e^*}{c} \delta \vec{A} \Phi^* \cdot \left( -i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right) \Phi + \right. \\ \left. + \frac{1}{m^*} \left( i\hbar \vec{\nabla} + \frac{e^*}{c} \vec{A} \right) \Phi^* \cdot \frac{e^*}{c} \delta \vec{A} \Phi + \right. \\ \left. + \frac{1}{q_0} \left( \vec{\nabla} \times \vec{A} \right) \cdot \left( \vec{\nabla} \times \delta \vec{A} \right) - \frac{1}{q_0} \vec{\mu} \cdot \left( \vec{\nabla} \times \delta \vec{A} \right) \right]$$

Consider the last two terms:

$$\frac{1}{q_0} \int d^d x \left[ (\vec{\nabla} \times \vec{A}) - \vec{\mu} \right] \cdot (\vec{\nabla} \times \delta \vec{A}) = *$$

Use identity:

$$\vec{a} \cdot (\vec{\nabla} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{\nabla} \cdot (\vec{a} \times \vec{b})$$

$$* = \frac{1}{q_0} \int d^d x \delta \vec{A} \cdot \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \cancel{- \frac{1}{q_0} \oint d\vec{s} \cdot \left[ \delta \vec{A} \times (\vec{\nabla} \times \vec{A} - \vec{\mu}) \right]} =$$

$$= \frac{1}{q_0} \int d^d x \delta \vec{A} \cdot \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$

Then we obtain :

$$\delta_{\vec{A}} F = \int d^4x = \left[ -\frac{i\hbar e^*}{mc^*} (\vec{\Phi}^* \nabla \vec{\Phi} - \vec{\Phi} \nabla \vec{\Phi}^*) + \right. \\ \left. + \frac{e^{*2}}{m^* c^2} |\vec{\Phi}|^2 \vec{A} + \frac{1}{4\pi} \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \right] \cdot \delta \vec{A} = 0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

Then we obtain :

$$\vec{j} = \frac{i\hbar e^*}{mc^*} (\vec{\Phi}^* \nabla \vec{\Phi} - \vec{\Phi} \nabla \vec{\Phi}^*) - \frac{e^{*2}}{mc^*} |\vec{\Phi}|^2 \vec{A} - \text{superconducting current.}$$

Analyze some of the consequences of these equations.

Assume we are below  $T_c$  in the superconducting state :

$$\alpha \sim \frac{T - T_c}{T_c} \Rightarrow \alpha < 0.$$

First assume  $\vec{\Phi}$  is uniform.

Then the first FL equation becomes:

$$\alpha \vec{\Phi} + b |\vec{\Phi}|^2 \vec{\Phi} = 0$$

$$\text{The nontrivial solution is } |\vec{\Phi}| = \sqrt{\frac{10t}{b}}$$

$|\Phi|^2$  has the meaning of the density of electron pairs:

$$|\Phi|^2 = n_s$$

$$\text{Let } \Phi(\vec{x}) = \sqrt{n_s} e^{i\theta(\vec{x})}$$

In a uniform superconducting sample well below  $T_c$  we can expect  $n_s$  to be uniform, then only the phase depends on  $\vec{x}$ . Substituting  $\Phi(\vec{x})$  into the expression for the supercurrent, we obtain:

$$\vec{j} = -\frac{\hbar e^* n_s}{m^*} \left( \vec{\nabla} \theta + \frac{e^*}{\hbar c} \vec{A} \right)$$

thus the superconducting current is directly related to the gradient of the phase of the macroscopic wavefunction. This explains both the zero resistance and the Meissner effect - two defining properties of a superconductor.

Meissner effect is in fact most easily understood just by examining the free energy.

$$F = \int d^3x \left[ \frac{\hbar^2 n_s^2}{2m^*} \left( \vec{\nabla} \theta + \frac{e^*}{\hbar c} \vec{A} \right)^2 + \right. \\ \left. + \frac{1}{8\pi} \left( \vec{\nabla} \times \vec{A} - \vec{H} \right)^2 \right]$$

The physics behind superconductivity is phase rigidity, which is expressed by the first term in the free energy. Consider a simply-connected sample (no holes).

In this case we can always do the following gauge transformation:

$$\vec{A} \rightarrow \vec{A} - \frac{ie}{c} \vec{\nabla} \theta, \text{ since } \vec{\nabla} \times (\vec{\nabla} \theta) = 0.$$

Then we obtain:

$$F = \int d^4x \left[ \frac{e^* n_s}{2m^* c^*} \vec{A}^2 + \frac{1}{8\pi} (\vec{\nabla} \times \vec{A} - \vec{H})^2 \right]$$

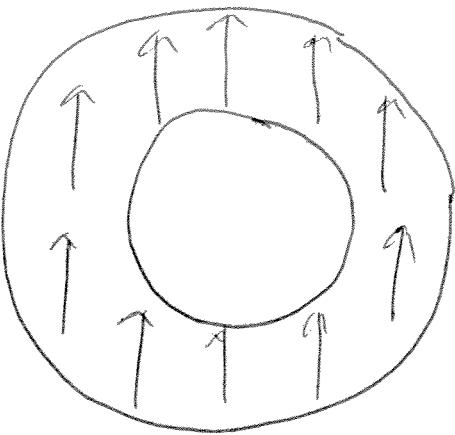
If  $\vec{B} \neq 0$  inside the sample, then  $\vec{A}$  must depend at least linearly on coordinates, since  $\vec{B} = \vec{\nabla} \times \vec{A}$ .

This means that  $\vec{A}^2$  contribution to the free energy will grow faster than the volume of infinite free energy density, which is impossible  $\Rightarrow B=0$  in the bulk of a superconducting sample — Meissner effect.

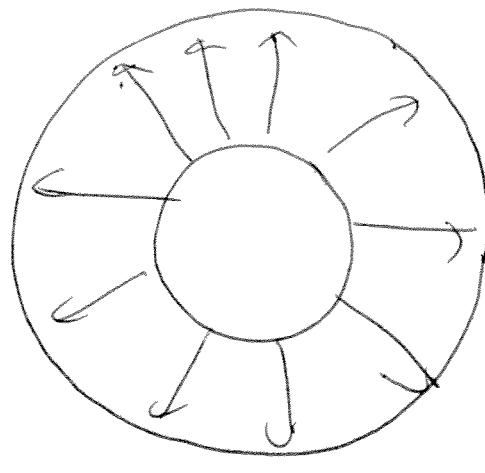
Now consider the persistent current effect.

$$\vec{j} = - \frac{ie^* n_s}{m^*} \vec{\nabla} \theta, \text{ can take } \vec{A}=0 \text{ in the bulk.}$$

Consider a sample in the form ~~of~~ a ring and consider two states: with and without supercurrent around the ring.



$$j=0$$



$$j \neq 0$$

The state with current has phase windings:

$$\oint \vec{\nabla}\theta \cdot d\vec{l} = 2\pi n, \quad n \text{ is integer.}$$

By the same arguments as in the case of vortices in XY model, the current-carrying state is topologically stable - need to cross an infinite energy barrier  $\Rightarrow$  current will ~~not~~ flow forever.

Another, more standard way to describe Meissner effect,

$$\vec{j} = - \frac{e^* n_s}{m^* c} \left( \vec{\nabla}\theta + \frac{e^*}{hc} \vec{A} \right)$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

Take curl of both sides of this equation:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \frac{4\pi}{c} \vec{\nabla} \times \vec{j}$$

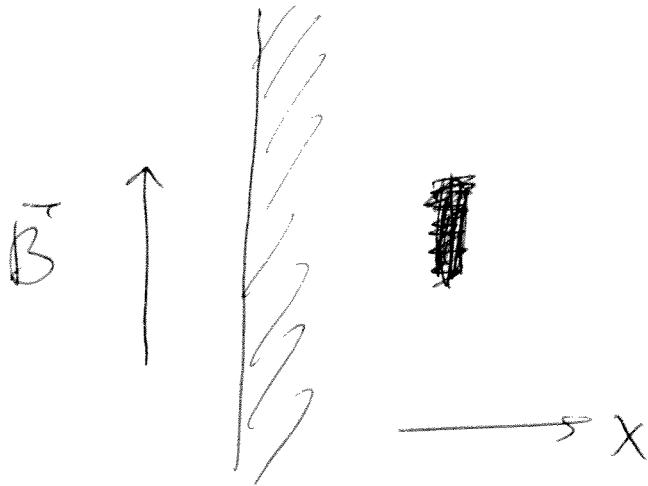
$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{B}) - \vec{\nabla}^2 \vec{B} = -\vec{\nabla}^2 \vec{B}$$

On the other hand:

$$\vec{\nabla} \times \vec{j} = - \frac{e^{*2} n_s}{m^* c} \vec{\nabla} \times \vec{A} = - \frac{e^{*2} n_s}{m^* c} \vec{B}$$

Thus we obtain:

$$\vec{\nabla}^2 \vec{B} = \frac{1}{d^2} \vec{B}, \quad d = \sqrt{\frac{m^* c^2}{4\pi e^* n_s}} \text{ - penetration depth.}$$



$$\frac{d^2 B}{dx^2} = \frac{1}{\lambda^2} B$$

$B(x) = B(0) e^{-\frac{x}{\lambda}}$  — field only penetrates a distance  $\lambda$  into the superconductor.

$\lambda \sim 100 \text{ \AA}$  in standard metallic superconductors,  
much larger in high- $T_c$  materials.

Second important length scale in superconductors — coherence length.

$$-\frac{\hbar^2}{m^*} \vec{\nabla}^2 \Phi + a\Phi + b|\Phi|^2\Phi = 0 \quad (\text{crossed out})$$

Assume for simplicity  $\Phi$  depends only on one coordinate,

$$-\frac{\hbar^2}{m^*} \frac{d^2 \Phi}{dx^2} + a\Phi + b|\Phi|^2\Phi = 0. \quad \text{say } x.$$

$$\text{let } \Phi(x \rightarrow \infty) = \sqrt{\frac{1 + \alpha}{b}} - \text{uniform bulk value.}$$

Define dimensionless  $\Phi$  as :

$$\tilde{\Phi}(x) = \frac{\Phi(x)}{\Phi(x-\infty)} \equiv \frac{\Phi(x)}{\Phi_0}$$

Rewriting the equation in terms of  $\tilde{\Phi}$ , we obtain:

$$-\frac{t^2}{m^*|\alpha|} \frac{d^2\tilde{\Phi}}{dx^2} - \tilde{\Phi} + |\tilde{\Phi}|^2 \tilde{\Phi} = 0$$

$$\xi^2 = \frac{t^2}{m^*|\alpha|} - \text{coherence length}.$$

$$\lambda = \frac{\lambda}{\xi}$$

$\lambda < \frac{1}{\sqrt{2}}$  - type I superconductors.

$\lambda > \frac{1}{\sqrt{2}}$  - type II superconductors.

Now let us consider fluctuations in superconductors.

$$S[\Phi] = \frac{1}{T} \int d^d x \left[ \frac{1}{m^*} \left| (-i\hbar \vec{v} + \frac{e^*}{c} \vec{A}) \Phi \right|^2 + \alpha |\Phi|^2 + \frac{b}{2} |\Phi|^4 + \frac{1}{8\pi} (\vec{v} \times \vec{A} - \vec{H})^2 \right]$$

~~$Z = \int D\Phi e^{-S[\Phi]}$~~

$$F = -T \ln Z$$

$$\text{let } \Phi_0 = \sqrt{\frac{|\alpha|}{b}} \text{ - uniform mean-field solution.}$$

Phase of the mean-field solution is arbitrary-spontaneous symmetry breaking. Can expect  $\Phi_0$  [then modes when we consider small fluctuations around mean-field solution.

$$\text{Write } \Phi(\vec{x}) = e^{i\theta(\vec{x})} (\Phi_0 + \psi(\vec{x}))$$

Assume  $\psi(\vec{x})$  is small.

~~Pick  $t = c = 1$  and redefine variables as~~

~~$\Phi$~~

~~$A$~~

Redefine variables as: take  $t = c = 1$

$$\sqrt{\frac{1}{2m}} \Phi \rightarrow \Phi, \quad \frac{\vec{A}}{\sqrt{4\pi}} \rightarrow \vec{A}, \quad \sqrt{4\pi} e^* \rightarrow e$$

Then  $S[\Phi]$  becomes,

$$S[\Phi] = \frac{1}{4} \int d^d x \left[ |(\vec{\nabla} - ie\vec{A})\Phi|^2 + a|\Phi|^2 + \frac{b}{2} |\Phi|^4 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2 \right], \text{ assuming } \vec{k} \approx 0.$$

Let's first set  $e=0$  - neutral order parameter.  
Then  $S[\Phi]$  describes a superfluid, like liquid He.

$$|\vec{\nabla}\Phi|^2 \approx \Phi_0 (\vec{\nabla}\theta)^2 + (\vec{\nabla}\phi)^2$$

$$|\Phi|^2 = \Phi_0^2 + \psi^2 + 2\Phi_0\psi$$

$$|\Phi|^4 = (\Phi_0 + \psi)^2 \approx \Phi_0^4 + 4\Phi_0^3\psi + 6\Phi_0^2\psi^2$$

Linear terms in  $\psi$  cancel.

Then we obtain:

$$S[\theta, \psi, \vec{A}] = \frac{1}{T} \int d^d x \left[ \Phi_0 (\vec{\nabla} \theta)^2 + 2|\alpha| \psi^2 + \right. \\ \left. + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2 \right]$$

~~Phase~~ Phase fluctuations are the ~~gold~~ Goldstone modes - energy cost vanishes when  $k \rightarrow 0$ .

~~Amplitude~~ Amplitude fluctuations have finite energy cost in the  $k \rightarrow 0$  limit.

Faige field fluctuations are decoupled from phase fluctuations and also have vanishing energy cost in the  $k \rightarrow 0$  limit.

Now consider the case  $e \neq 0$ .

In this case we obtain:

$$S[\theta, \psi, \vec{A}] = \frac{1}{T} \int d^d x \left[ \Phi_0^2 (\vec{\nabla} \theta - e \vec{A})^2 + (\vec{\nabla} \psi)^2 + \right. \\ \left. + 2|\alpha| \psi^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2 \right]$$

As before, in a simply-connected sample ~~we can~~ we can remove  $\vec{\nabla} \theta$  term by a gauge transformation of  $\vec{A}$ .

$$\vec{A} \rightarrow \vec{A} + \frac{1}{e} \vec{\theta}$$

Then we obtain:

$$S[\theta, \psi, \vec{A}] = \frac{1}{T} \int d^d x \left[ \frac{Q^2}{2} \vec{A}^2 + (\bar{\psi} \psi)^2 + 2|a| \psi^2 + \frac{1}{2} (\vec{\nabla} \times \vec{A})^2 \right]$$

Goldstone modes have disappeared!

This is Anderson-Higgs phenomenon — in systems with a "charged" order parameter there are no Goldstone modes even when there ~~exists~~ is an appearance of spontaneously broken symmetry.