

## Lecture 15

Continue ~~on~~ discussion of vortices ...

Calculate the energy of a vortex with winding number  $n$ .

$$H = \frac{J}{2} \int d^2x (\vec{\nabla}\theta)^2 - \text{continuum } \cancel{\text{and}} \text{ description OK}$$

away from the vortex core.

Minimize  $H$  subject to the constraint:

$$\oint_C \vec{\nabla}\theta \cdot d\vec{l} = 2\pi n \quad \text{where } C \text{ is any loop enclosing}$$

the vortex core.

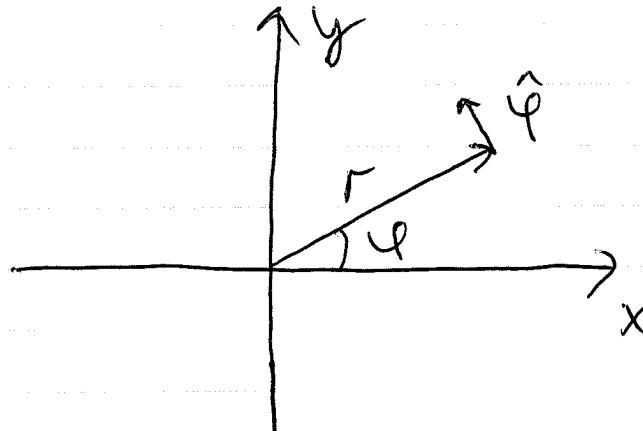
$$\delta H = \frac{J}{2} \int d^2x 2 \vec{\nabla}\delta\theta \cdot \vec{\nabla}\theta = -J \int d^2x \delta\theta \vec{\nabla}^2\theta = 0$$

Thus we have to solve:

$$\vec{\nabla}\theta = 0 \quad \text{subject to the constraint } \oint_C \vec{\nabla}\theta \cdot d\vec{l} = 2\pi n$$

The solution is:

$$\theta = n\varphi, \text{ where } \varphi \text{ is the polar angle:}$$



$$\vec{\nabla}\theta = n \frac{\hat{\vec{r}}}{r}$$

Then the energy of a vortex is given by:-

$$E_v = \frac{\gamma}{2} \int d^2x (\vec{\nabla}\theta)^2 = \frac{\gamma n^2}{2} \cdot 2\pi \int_a^L dr \cdot r \cdot \frac{1}{r^2} = \pi \gamma n^2 \ln\left(\frac{L}{a}\right)$$

Thus  $E_v$  diverges logarithmically with system size.

However, as we saw in lecture 14, removing a vortex costs even higher energy  $\sim \frac{\gamma L}{a}$ .

At  $T=0$  vortices won't appear since they cost infinite energy.

At finite  $T$  need to minimize free energy.

$$\text{Entropy of a vortex } S_v = \ln\left(\frac{L}{a}\right)^2$$

$$F_v = E_v - TS_v = (\pi\gamma - 2T) \ln\left(\frac{L}{a}\right)$$

We can expect vortices to start appearing when

$$T > T_{\text{KT}} = \frac{\pi\gamma}{2} \quad \text{- Kosterlitz-Thouless temperature}$$

Below  $T_{\text{KT}}$  correlation length is infinite:  $f(x) \sim \frac{1}{x^\eta}$

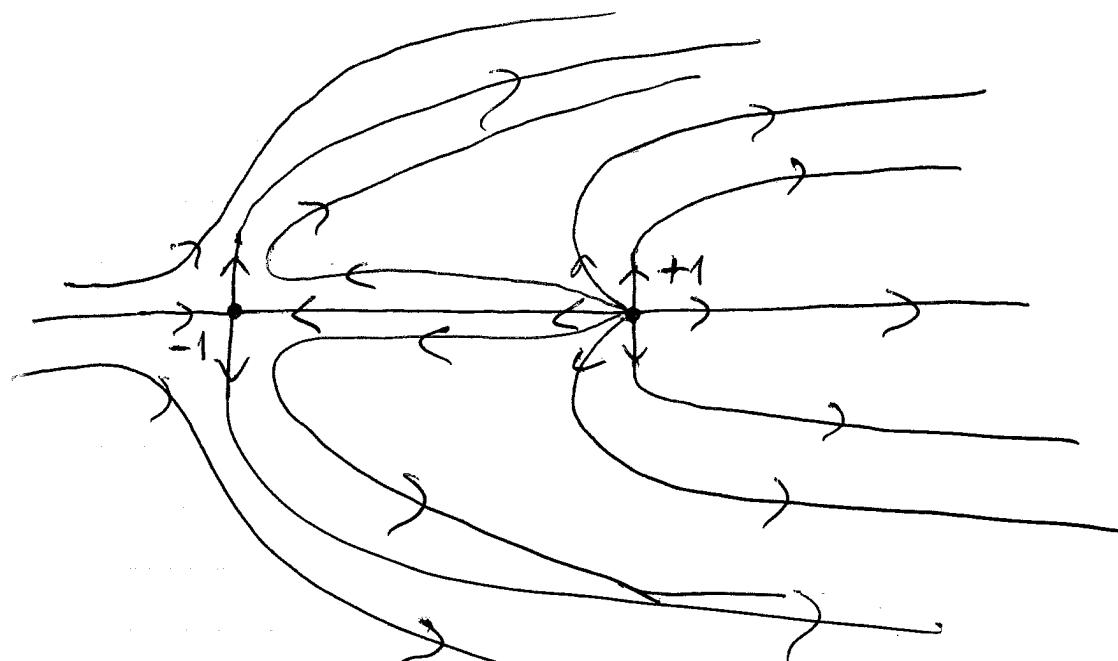
Above  $T_{\text{KT}}$  correlation length becomes finite:

$$f(x) \sim \frac{1}{x^\eta} e^{-\frac{x}{\xi}}$$

$\Sigma$  is related to the average spacing between the free vortices.

More physically correct way to think about the KT transition:

Below  $T_{KT}$  vortices exist, but are bound into vortex-antivortex pairs.



Far away from the pair there is no phase winding  $\Rightarrow$  the ~~total~~ energy is finite:

$$E_{\text{pair}} \approx 2\pi y \ln\left(\frac{R}{a}\right)$$

R - pair size.

Above  $T_{KT}$  the entropy wins and the vortex-antivortex pairs unbind  $\Rightarrow$  there are free vortices and antivortices.

KT transition is a vortex deconfinement transition.

Yet another way to think about KT transition is in terms of ~~rigidity~~ rigidity.

$$\mathcal{H} = \frac{\rho_s}{2} \int d\tilde{x} (\vec{\nabla}\theta)^2$$

$\rho_s$  is often called generalized rigidity, since  $\mathcal{H}$  looks like elastic energy of a solid where  $\theta$  is the analog of the displacement field in a solid.

$(\vec{\nabla}\theta)^2$  term penalized any ~~nonuniformity~~ nonuniformity in  $\theta$  just as elastic energy in a solid penalized any nonuniformity in the displacement field.

$\gamma$  is the ~~microscopic~~ microscopic rigidity, measured at the scale of the lattice constant.

~~At scales greater than the vortex-vortex distance~~

Continuum description is applicable at length scales, greater than the vortex-and-vortex pair size.

At such length scales  $\rho_s < \gamma$  - reduced by vortex-and-vortex pairs.

~~At TKT~~ At  $T_{KT}$   $\rho_s$  drops to zero discontinuously:

$$\frac{\rho_s(T_{KT})}{T_{KT}} = \frac{2}{\pi} - \text{"universal jump" of } \rho_s.$$

Thus another way to distinguish the low and high- $T$  phases in the  $d=2$  XY model is:

$$\rho_s(T < T_{KT}) > 0, \quad \rho_s(T > T_{KT}) = 0$$

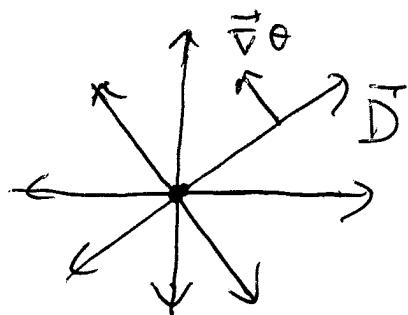
Describing KT transition seems like a difficult task since ~~order~~ can't describe it in terms of order parameter.

What makes it possible is the concept of duality, which is based on analogy to electrostatics.

Vortex core is a point, yet the presence of a vortex can be established by measuring phase winding arbitrarily far from the core. This suggests analogy to electric charge: charge is also pointlike, but its presence can be measured out an arbitrary distance by calculating ~~area~~ flux of the electric field through any surface, enclosing the charge.

Define "electric displacement field", associated with phase gradient, as:

$$\vec{D} = \vec{\nabla}\theta \times \hat{z}$$



$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\vec{\nabla}\theta \times \hat{z}) = \vec{\nabla} \times (\vec{\nabla}\theta) = 2\pi n \delta(\vec{x})$$

$n$  is the winding number. In the electrostatic analogy it plays the role of a charge.

$$\int \vec{\nabla} \times (\vec{\nabla} \theta) d^3x = \oint \vec{\nabla} \theta \cdot d\vec{l} = 2\pi n$$

$$\vec{\nabla} \theta = \frac{\hat{r} n}{r}$$

$$\vec{D} = \vec{\nabla} \theta \times \hat{z} = \frac{\hat{r} n}{r} \quad \text{- electric field of a point charge.}$$

Define "dielectric constant" as:

$$\epsilon = \frac{1}{4\pi \rho s}$$

$$\vec{D} = \epsilon \vec{E}$$

$$\text{Then } H = \frac{P_s}{2} \int d^3x (\vec{\nabla} \theta)^2 = \frac{1}{4\pi} \int d^3x \vec{E} \cdot \vec{D} - \text{electrostatic energy.}$$

Total electrostatic potential produced by a charge is:

$$U(r) = - \int_a^r \vec{E} \cdot d\vec{r}' = - \int_a^r n \frac{\hat{r}'}{\epsilon r'} \cdot d\vec{r}' = - \frac{n}{\epsilon} \ln \left( \frac{r}{a} \right)$$

Then the potential energy of a ~~one~~ system of interacting vortices and antivortices can be written as:

$$U = -\frac{1}{2\epsilon} \sum_{i \neq j} n_i n_j \ln \left| \frac{\vec{x}_i - \vec{x}_j}{a} \right|$$

~~REMOVED~~ This is a potential energy of a plasma.

At  $T < T_{\text{IG}}$  the vortex-antivortex plasma is "insulating" - all charges are bound into neutral pairs.

For  $T > T_{\text{IG}}$  the pairs dissociate and the plasma becomes "conducting".

We will make the above picture rigorous using duality transformation.

Start from XY model Hamiltonian on square lattice:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

Partition function:

$$Z = \int_0^{\pi} \prod_i \frac{d\theta_i}{2\pi} e^{\frac{J}{T} \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)}$$

Let  $\mu = \hat{x}, \hat{y}$  label unit vectors along the nearest-neighbor bonds of the lattice.

$$\text{Then } \langle ij \rangle = (i, i+\mu)$$

Convenient to introduce lattice gradient:

$$\Delta_\mu \theta_i = \theta_{i+\mu} - \theta_i$$

Then we have:

$$Z = \int_0^{\pi} \prod_i \frac{d\theta_i}{2\pi} e^{\frac{J}{T} \sum_{ij} \cos(\Delta_\mu \theta_i)}$$

$$e^{\frac{y}{T} \cos(\Delta_\mu \theta_i)}$$

$e^{\frac{y}{T} \cos(\Delta_\mu \theta_i)}$  is a periodic function of  $\Delta_\mu \theta_i$ .

Therefore it can be written as Fourier series,

$$e^{\frac{y}{T} \cos(\Delta_\mu \theta_i)} = \sum_{M_{1\mu}=-\infty}^{\infty} e^{i M_{1\mu} \Delta_\mu \theta_i} F(M_{1\mu})$$

Here  $M_{1\mu}$  are integers, defined on bands of the

$$F(M_{1\mu}) = \frac{1}{2\pi} \int_0^{2\pi} dx e^{-i M_{1\mu} x} e^{\frac{y}{T} \cos(x)} = \\ = I_{M_{1\mu}}\left(\frac{y}{T}\right)$$

$I$  is the modified Bessel function.

Use asymptotic form of  $I(x)$  at large  $x$ :

$$I_n(x) \approx \frac{1}{\sqrt{\pi x}} e^{x - \frac{n^2}{2x}}$$

This in fact gives a very good approximation to  $I$  even when  $x$  is not that large, i.e. of order 1.

Then we have:

$$e^{\frac{y}{T} \cos(\Delta_\mu \theta_i)} \approx \sqrt{\frac{T}{2\pi y}} e^{\frac{y}{T} \sum_{M_{1\mu}=-\infty}^{\infty} e^{i M_{1\mu} \Delta_\mu \theta_i} - \frac{T M_{1\mu}^2}{2y}}$$

The important point is that this approximation preserves the periodicity of  $e^{\frac{T}{2} \cos(\Delta_\mu \theta_i)}$  as a function of  $\Delta_\mu \theta_i$ .

Neglecting constant prefactors, we then obtain,

$$Z = \int_0^{2\pi} \prod_i \left[ \frac{d\theta_i}{2\pi} \right] e^{-\frac{T}{2J} \sum_i M_{ij\mu} + i \sum_{ij\mu} M_{ij\mu} \Delta_\mu \theta_i}$$

*{my}*

The advantage of this representation is that we can now integrate over  $\theta_i$  exactly.

~~$$\int d\theta_i \delta(\theta_i - \theta_i^*) = \int d\theta_i \delta(\theta_i)$$~~

$$\begin{aligned} \sum_{ij\mu} M_{ij\mu} \Delta_\mu \theta_i &\longleftrightarrow \int d^2x \vec{m} \cdot \vec{\nabla} \theta = \\ &= \int d^2x \vec{\nabla} \cdot (\vec{m} \theta) - \int d^2x \theta \vec{\nabla} \cdot \vec{m} = \\ &= - \int d^2x \theta \vec{\nabla} \cdot \vec{m} \end{aligned}$$

Thus  $\sum_{ij\mu} M_{ij\mu} \Delta_\mu \theta_i = - \sum_{ij\mu} \theta_i \Delta_\mu M_{ij\mu}$

~~$$\Delta_\mu M_{ij\mu} = M_{ij\mu} - M_{i-j,\mu}$$~~

$$\int_0^{2\pi} \frac{d\theta_i}{2\pi} e^{-i\theta_i \sum_\mu \Delta_\mu M_\mu} = \delta\left(\sum_\mu \Delta_\mu M_\mu\right) \quad 10$$

Thus integration over  $\theta$  produces the following constraint on  $M_\mu$  on every site:

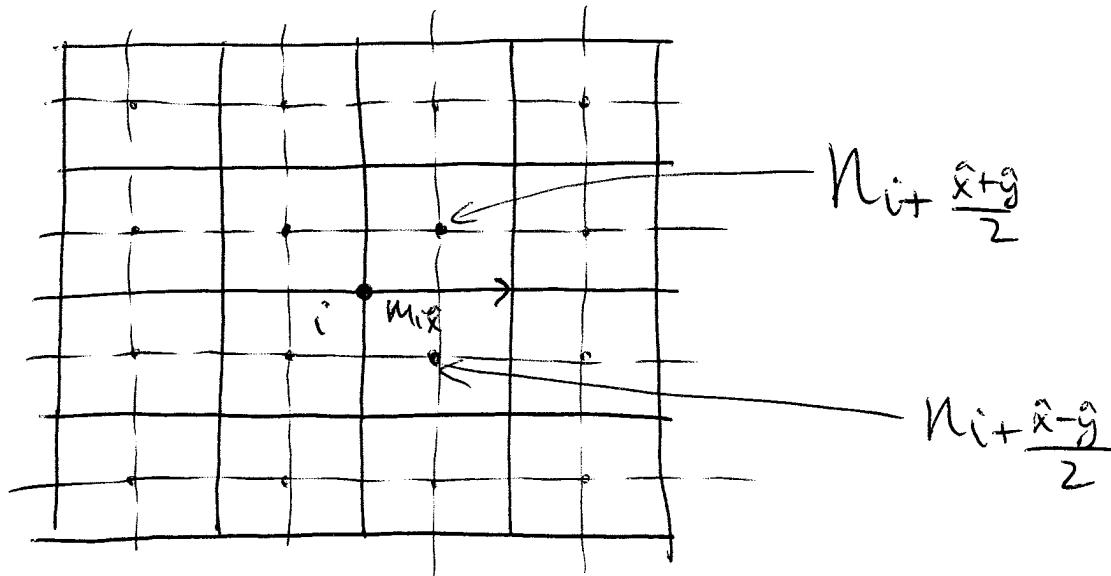
$$\vec{B} \cdot \vec{M}_i = \sum_\mu \Delta_\mu M_\mu = 0 \quad - \text{lattice divergence of } \vec{m} \text{ at every site is zero.}$$

The partition function becomes:

$$Z = \sum_{\{\vec{m}_i\}} \prod_i \delta\left(\vec{B} \cdot \vec{m}_i\right) e^{-\frac{I}{2} \sum_{i,\mu} m_i^2}$$

The constraint  $\vec{B} \cdot \vec{m} = 0$  can be solved as,

$\vec{m}_i = \vec{B} N_i \times \hat{z}$  where  $N_i$  are integer variables, defined on sites of the dual lattice.



$$U_{i\hat{x}} = U_i + \frac{\hat{x}+\hat{y}}{2} - U_{i+\frac{\hat{x}+\hat{y}}{2}}$$

$$U_{i\hat{y}} = U_i + \frac{\hat{y}-\hat{x}}{2} - U_{i+\frac{\hat{x}+\hat{y}}{2}}$$

$$U_{i-\hat{x},\hat{x}} = -U_i - \frac{\hat{x}+\hat{y}}{2} + U_i + \frac{\hat{y}-\hat{x}}{2}$$

$$U_{i-\hat{y},\hat{y}} = -U_i + \frac{\hat{x}-\hat{y}}{2} + U_i - \frac{\hat{x}+\hat{y}}{2}$$

Then we obtain:

$$\begin{aligned} \bar{J} \cdot \bar{U}_i &= \cancel{U_i + \frac{\hat{x}+\hat{y}}{2} + U_i + \frac{\hat{y}-\hat{x}}{2} + U_i - \frac{\hat{x}+\hat{y}}{2} + U_i - \frac{\hat{x}-\hat{y}}{2}} \quad U_{i\hat{x}} - U_{i-\hat{x},\hat{x}} + U_{i\hat{y}} - U_{i-\hat{y},\hat{y}} = \\ &= \cancel{U_i + \frac{\hat{x}+\hat{y}}{2}} - \cancel{U_i + \frac{\hat{x}-\hat{y}}{2}} - \left( \cancel{U_i + \frac{\hat{y}-\hat{x}}{2}} - \cancel{U_i - \frac{\hat{x}+\hat{y}}{2}} \right) + \\ &+ \cancel{U_i + \frac{\hat{y}-\hat{x}}{2}} - \cancel{U_i + \frac{\hat{x}+\hat{y}}{2}} - \left( \cancel{U_i - \frac{\hat{x}+\hat{y}}{2}} - \cancel{U_i + \frac{\hat{x}-\hat{y}}{2}} \right) = 0 \end{aligned}$$

Then the partition function can be rewritten as:

$$Z = \sum_{\{n^y\}} e^{-\frac{T}{2y} \sum_{i\mu} (A_\mu n_i)^2}$$

Here  $i$  and  $\mu$  label sites and nearest-neighbor directions of the dual lattice.

This is discrete gaussian model — it is an exact dual of the XY-model.

Note that the T-axis is inverted — high-T phase of the XY-model maps onto low-T phase of the discrete gaussian model and vice versa.

Dealing with integer variables is hard - e.g. can't take continuum limit. To make further progress, use Poisson resummation formula:

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\varphi \, g(\varphi) e^{-2\pi i m \varphi}$$

This follows from:

$$\sum_{m=-\infty}^{\infty} e^{-2\pi i m \varphi} = \sum_{n=-\infty}^{\infty} \delta(\varphi - n)$$

Using Poisson formula, we obtain:

$$Z = \int_{-\infty}^{\infty} \eta \, d\varphi_i \sum_{\{m_i\}} e^{-\frac{T}{2\beta} \sum_{i\mu} (\Delta_\mu \varphi_i)^2 - 2\pi i \sum_i \varphi_i m_i}.$$

To see the physical meaning of the variables  $m_i$ , integrate over  $\varphi_i$ :

$$\frac{T}{2\beta} \sum_{i\mu} (\Delta_\mu \varphi_i)^2 = \frac{1}{2} \sum_{ij} \varphi_i G_{ij}^{-1} \varphi_j$$

$$G_{ij}^{-1} = \frac{T}{j} \left[ 4\delta_{ij} - \sum_{\mu} (\delta_{j,i+\mu} + \delta_{j,i-\mu}) \right]$$

Diagonalize by Fourier transform.

$$G_{ij} = \frac{1}{N} \sum_{\vec{q}} F(\vec{q}) e^{i\vec{q} \cdot (\vec{x}_i - \vec{x}_j)}$$

$$F^{-1}(\vec{q}) = \frac{1}{N} \sum_{ij} f_{ij} e^{-i\vec{q} \cdot (\vec{x}_i - \vec{x}_j)} =$$

$$= \frac{T}{J} \left[ 4 - 2\cos(q_x a) - 2\cos(q_y a) \right] =$$

$$= \frac{2T}{J} \left[ 2 - \cos(q_x a) - \cos(q_y a) \right]$$

Then we have:

$$f_{ij} = \frac{1}{N} \sum_{\vec{q}} F(\vec{q}) e^{i\vec{q} \cdot (\vec{x}_i - \vec{x}_j)} =$$

$$= \frac{J}{2T} \frac{1}{N} \sum_{\vec{q}} \frac{e^{i\vec{q} \cdot (\vec{x}_i - \vec{x}_j)}}{2 - \cos(q_x a) - \cos(q_y a)}$$

If  $|x_i - x_j| \gg a$ , can expand the cosines:

$$G_{ij} = \frac{Ja^2}{2T} \int \frac{d^2 q}{(2\pi)^2} 2 \frac{e^{i\vec{q} \cdot (\vec{x}_i - \vec{x}_j)}}{q^2 a^2} =$$

$$= \frac{J}{T} \frac{1}{(2\pi)^2} \int_0^\infty d\varphi \int_0^{1/a} dq \cdot \frac{1}{q} e^{iq|x_i - x_j| \cos \varphi} =$$

$$= \frac{J}{2\pi T} \int_0^{1/a} dq \frac{\gamma_0(q|\vec{x}_i - \vec{x}_j|)}{q}$$

The integral is divergent at  $q \approx 0$ . This is a consequence of expanding the denominator at small  $q$ . We will fix this by,

$$G_{ij} = G_{ij} - G(0) + G(0)$$

~~the  $\delta$ -function~~

$$G_{ij} - G(0) = \frac{J}{2\pi T} \int_0^{1/a} dq \frac{\gamma_0(q|\vec{x}_i - \vec{x}_j|) - 1}{q} \approx -\frac{J}{2\pi T} \ln\left(\frac{|\vec{x}_i - \vec{x}_j|}{a}\right)$$

Then we obtain:

$$Z = \int_{-\infty}^{\infty} \prod_i d\varphi_i \sum_{\{m_i\}} e^{-\frac{1}{2} \varphi_i f_{ij}^{-1} \varphi_j - m_i \sum_i \varphi_i m_i} =$$

$$= \sum_{\{m_i\}} e^{-2\pi^2 \sum_{ij} m_i G_{ij} m_j} =$$

$$= \sum_{\{m_i\}} e^{\frac{iJ}{T} \sum_{ij} m_i m_j \ln\left(\frac{|\vec{x}_i - \vec{x}_j|}{a}\right)}$$

- partition function  
of vortex-antivortex plasma.