

Lecture 14

RG flow equation for NLSM:

$$\frac{d\tilde{T}}{dl} = -\varepsilon \tilde{T} + (n-2) K_d \tilde{T}^2$$

Fixed points:

$$1. \tilde{T}^* = 0$$

$$2. \tilde{T}^* = \frac{\varepsilon}{K_d (n-2)}$$

Analyze the stability.

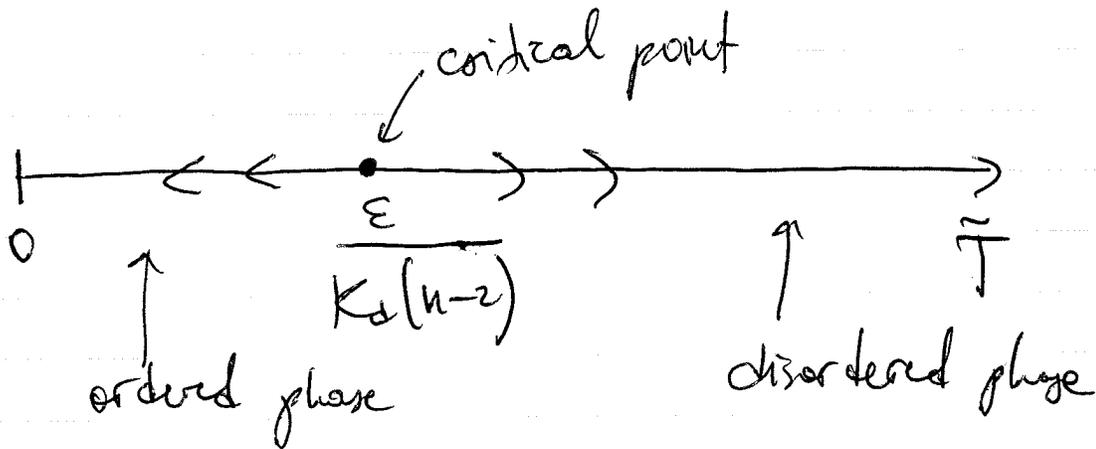
$$\beta(\tilde{T}) = -\varepsilon \tilde{T} + (n-2) K_d \tilde{T}^2$$

$$\frac{d\beta}{d\tilde{T}} = -\varepsilon + 2(n-2) K_d \tilde{T}$$

$$\left. \frac{d\beta}{d\tilde{T}} \right|_{\tilde{T}=0} = -\varepsilon$$

$$\left. \frac{d\beta}{d\tilde{T}} \right|_{\tilde{T} = \frac{\varepsilon}{K_d (n-2)}} = -\varepsilon + 2 K_d (n-2) \frac{\varepsilon}{K_d (n-2)} = \varepsilon$$

Thus $\tilde{T}^* = 0$ fixed point is stable, $\tilde{T}^* = \frac{\Sigma}{K_d(n-2)}$ fixed point is unstable.



$\tilde{T}_c \sim \epsilon$ - consistent with Mermin-Wagner theorem.

Behavior in $d=2$ is particularly interesting.

In this case we have:

$$\frac{d\tilde{T}}{dl} = (n-2)K_d\tilde{T}^2$$

$n > 2$ - always in disordered phase - MW theorem.

$n = 2$ - β -function vanishes (true to all orders in \tilde{T}).

This signals that $n=2$ (XY model) in $d=2$ is a special case - we will see that there is in fact a finite- T transition, even though there is no long-range order at any finite T .

Find T -dependence of the correlation length in $d=2$ for $n > 2$ - expect that correlation length diverges as $T \rightarrow 0$.

Consider correlation length at two consecutive steps of the RG, but before rescaling.

$$\Lambda, \Lambda' = \frac{\Lambda}{b}$$

$$\xi[\Lambda, \tilde{T}(\Lambda)] = \xi[\Lambda', \tilde{T}'(\Lambda')]$$

$$\beta(\tilde{T}) = \frac{d\tilde{T}}{d\ell}$$

$$\Lambda' = \frac{\Lambda}{b} = \Lambda e^{-\Delta\ell} \Rightarrow \Delta\ell = \ln\left(\frac{\Lambda}{\Lambda'}\right)$$

Then we have:

$$\beta(\tilde{T}) = \frac{d\tilde{T}}{d \ln\left(\frac{\Lambda}{\Lambda'}\right)} = - \frac{d\tilde{T}}{d \ln \Lambda'} \quad \text{assuming } \Lambda \text{ is fixed.}$$

Correlation length satisfies the equation (dropping the prime):

$$\frac{d\xi}{d \ln \Lambda} = \frac{\partial \xi}{\partial \ln \Lambda} + \frac{\partial \xi}{\partial \tilde{T}} \frac{d\tilde{T}}{d \ln \Lambda} = 0$$

Thus we have:

$$\frac{\partial \xi}{\partial \ln \Lambda} - \frac{\partial \xi}{\partial \tilde{T}} \beta(\tilde{T}) = 0$$

ξ has dimension of length and \tilde{T} is dimensionless.

$$\text{Therefore } \xi(\Lambda, \tilde{T}) = \Lambda^{-1} f(\tilde{T}).$$

Then we have:

$$\frac{\partial \xi}{\partial \ln \Lambda} = \frac{\partial \xi}{\partial \Lambda} \left(\frac{d \ln \Lambda}{d \Lambda} \right)^{-1} = \Lambda \frac{\partial \xi}{\partial \Lambda} =$$

$$= -\Lambda^{-1} f(\tilde{T}) = -\xi$$

Thus we have; now assuming Λ is fixed:

$$\xi + \frac{d\xi}{d\tilde{T}} \beta(\tilde{T}) = 0$$

$$\frac{d\xi}{\xi} = - \frac{d\tilde{T}}{\beta(\tilde{T})}$$

$$\int_{\xi_0}^{\xi} \frac{d\xi'}{\xi'} = - \int_{\tilde{T}_0}^{\tilde{T}} \frac{d\tilde{T}'}{\beta(\tilde{T}')}$$

$$\xi = \xi_0 e^{- \int_{\tilde{T}_0}^{\tilde{T}} \frac{d\tilde{T}'}{\beta(\tilde{T}')}}$$

$$\int_{\tilde{T}_0}^{\tilde{T}} \frac{d\tilde{T}'}{\beta(\tilde{T}')} = \int_{\tilde{T}_0}^{\tilde{T}} \frac{d\tilde{T}'}{(n-2)K_d \tilde{T}'^2} = *$$

Take $\tilde{T}_0 \rightarrow \infty$, then $\xi_0 = \xi(\tilde{T}_0) \sim a$

$$* = - \int_{\tilde{T}}^{\infty} \frac{d\tilde{T}'}{(n-2)K_d \tilde{T}'^2} = - \frac{1}{(n-2)K_d \tilde{T}}$$

$$K_d = K_2 = \frac{S_2}{(2\pi)^2} = \frac{1}{2\pi}$$

Thus we finally obtain:

$$\xi(\tilde{T}) = \xi_0 e^{\frac{2\pi}{(n-2)\tilde{T}}}$$

Correlation length diverges as $e^{\frac{1}{\tilde{T}}}$ as $\tilde{T} \rightarrow 0$.

Can also calculate correlation length critical exponent for $d > 2$.

For $d > 2$ $\beta(\tilde{T})$ vanishes at $\tilde{T}^* = \frac{\varepsilon}{K_d(n-2)}$

The dominant contribution to the integral:

$$\int_{\tilde{T}_0}^{\tilde{T}} \frac{d\tilde{T}'}{\beta(\tilde{T}')} \text{ comes from the neighborhood of } \tilde{T}^* \Rightarrow$$

\Rightarrow can expand $\beta(\tilde{T})$ near \tilde{T}^* :

$$\beta(\tilde{T}) \approx \left. \frac{d\beta}{d\tilde{T}} \right|_{\tilde{T}=\tilde{T}^*} (\tilde{T} - \tilde{T}^*) = \varepsilon (\tilde{T} - \tilde{T}^*)$$

$$\int_{\tilde{T}_0}^{\tilde{T}} \frac{d\tilde{T}'}{\beta(\tilde{T}')} = \int_{\tilde{T}_0}^{\tilde{T}} \frac{d\tilde{T}'}{\varepsilon(\tilde{T} - \tilde{T}^*)} =$$

$$= \frac{1}{\varepsilon} \ln \left| \frac{\tilde{T} - \tilde{T}^*}{\tilde{T}_0 - \tilde{T}^*} \right|$$

Thus we obtain:

$$\xi(\tilde{T}) \sim (\tilde{T} - \tilde{T}^*)^{-\frac{1}{\varepsilon}}, \quad \nu = \frac{1}{\varepsilon}$$

Compare with $4-\varepsilon$ expansion.

$$\nu = \frac{1}{2 - \frac{n+2}{n+8} \varepsilon} \approx \frac{1}{2} + \frac{n+2}{n+8} \varepsilon$$

Take $\varepsilon = 1$, i.e. $d=3$ in both cases,
 $n=3$ - Heisenberg model.

$$\nu_{2+\varepsilon} = 1, \quad \nu_{4-\varepsilon} = \text{[scribbled out]} \approx 0.6$$

The values don't agree - not surprising since $2+\varepsilon$ expansion neglects topological defects.

Correct value $\nu \approx 0.71$ - closer to the $4-\varepsilon$ expansion result.

+

We saw that $n=2, d=2$ is a special case, since ϵ -expansion fails even qualitatively.
The reason is new physics.
Go back to the $n=2$ NLSM.

$$S[\vec{n}] = \frac{\rho_s}{2T} \int d^2x \vec{\nabla} n^a \cdot \vec{\nabla} n^a$$

$$\vec{n} = n^x \hat{x} + n^y \hat{y}, \quad n^{x^2} + n^{y^2} = 1$$

Convenient to solve the constraint as:

$$n^x = \cos \theta, \quad n^y = \sin \theta$$

$$\vec{\nabla} n^a \cdot \vec{\nabla} n^a = \sin^2 \theta (\vec{\nabla} \theta)^2 + \cos^2 \theta (\vec{\nabla} \theta)^2 = (\vec{\nabla} \theta)^2$$

Then we obtain:

$$S[\theta] = \frac{\rho_s}{2T} \int d^2x (\vec{\nabla} \theta)^2$$

$\theta(\vec{x})$ is a real variable \Rightarrow this seems to be an exactly solvable model.

Calculate the correlation function:

$$G(\vec{x}) = \langle e^{i\theta(\vec{x})} e^{-i\theta(0)} \rangle$$

$$G(\vec{x}) = \frac{1}{Z} \int D\theta e^{i[\theta(\vec{x}) - \theta(0)]} e^{-S[\theta]} =$$

$$= e^{-\frac{1}{2} \langle [\theta(\vec{x}) - \theta(0)]^2 \rangle} \quad \text{--- true for gaussian } S[\theta]$$

$$\langle [\theta(\vec{x}) - \theta(0)]^2 \rangle = \langle \theta^2(\vec{x}) + \theta^2(0) - 2\theta(\vec{x})\theta(0) \rangle$$

$$= 2 \langle \theta^2(0) - \theta(\vec{x})\theta(0) \rangle =$$

$$= 2 \int_0^\Lambda \frac{d^2 k_1}{(2\pi)^2} \frac{d^2 k_2}{(2\pi)^2} \langle \theta(\vec{k}_1) \theta(\vec{k}_2) \rangle \cdot \left(1 - e^{i\vec{q} \cdot \vec{x}}\right)_{\vec{q} = \vec{k}_1 - \vec{k}_2}$$

$$\langle \theta(\vec{k}_1) \theta(\vec{k}_2) \rangle = \frac{T}{\rho_s k^2} (2\pi)^2 \delta(\vec{k}_1 + \vec{k}_2)$$

$$* = \frac{2T}{\rho_s} \int_0^\Lambda \frac{d^2 k}{(2\pi)^2} \frac{1 - e^{i\vec{k} \cdot \vec{x}}}{k^2} =$$

$$= \frac{2T}{\rho_s} \cdot \frac{1}{4\pi^2} \cdot \int_0^{2\pi} d\varphi \int_0^\Lambda dk \frac{1}{k} \left(1 - e^{ikx \cos \varphi}\right) =$$

$$= \frac{T}{\pi \rho_s} \int_0^\Lambda dk \frac{1 - J_0(kx)}{k} =$$

$$= \frac{T}{\pi \rho_s} \left[\int_0^{1/x} dk \frac{1 - J_0(kx)}{k} + \int_{1/x}^\Lambda dk \frac{1 - J_0(kx)}{k} \right]$$

$$\int_0^{1/x} dk \frac{1 - J_0(kx)}{k} \approx \int_0^{1/x} dk \frac{k^2 x^2}{4k} = \frac{1}{8}$$

Assume $x \gg \frac{1}{\Lambda} = a$

$$\int_{1/x}^{\Lambda} dk \frac{1 - J_0(kx)}{k} = \int_{1/x}^{1/a} dk \frac{1 - J_0(kx)}{k} \approx$$

$$\approx \int_{1/x}^{1/a} \frac{dk}{k} = \ln\left(\frac{x}{a}\right)$$

$$\text{Thus } \frac{1}{2} \langle [\theta(x^2) - \theta(0)]^2 \rangle = \frac{T}{2\pi\rho_s} \ln\left(\frac{x}{a}\right)$$

$$G(x^2) = e^{-\frac{T}{2\pi\rho_s} \ln\left(\frac{x}{a}\right)} = \left(\frac{x}{a}\right)^{-\gamma(T)}$$

$$\gamma(T) = \frac{T}{2\pi\rho_s}$$

Compare with $G(x) \sim \frac{1}{x^{d-2+\gamma}}$

$\gamma(T)$ - anomalous dimension, but it is nonuniversal here - depends on T and ρ_s .

Thus we have no long-range order, which agrees with MW theorem.

Correlation function is always powerlaw; like at a critical point but the critical exponent γ depends on T - looks like a line of critical points from $T=0$ to $T=\infty$.

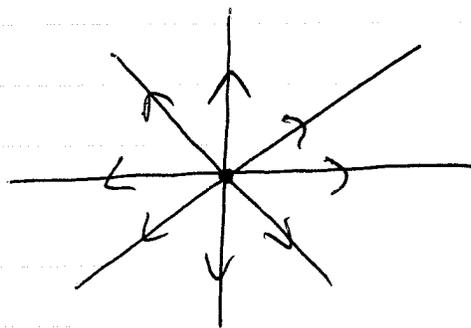
This result makes no sense physically: ~~arbitrary~~

for $T \gg \xi$ we certainly expect finite correlation length

$$G(x) \sim \frac{1}{x^\gamma} e^{-\frac{x}{\xi}}$$

What's wrong?

In our calculation we have implicitly neglected the possibility of topological defects of the order parameter, i.e. vortices.



Vortex is a configuration of the order parameter, where $\theta(\vec{r})$ winds by an integer multiple of 2π around a particular point in space. At this point θ is undefined and $\vec{\nabla}\theta$ ~~is~~ diverges \Rightarrow continuum limit is not ~~well~~ well-defined.

To include vortices we need to go back to the lattice description. Consider XY-model on square lattice.

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

$$\text{let } S_i^x = \cos \theta_i, \quad S_i^y = \sin \theta_i$$

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

Continuum limit - expand cosine to second order, assuming θ_i varies slowly on the scale of one lattice spacing:

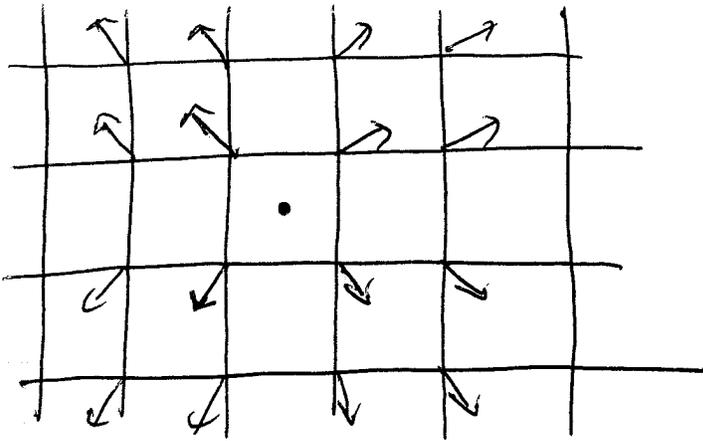
$$\cos(\theta_i - \theta_j) \approx 1 - \frac{1}{2} (\theta_i - \theta_j)^2 + \dots =$$

$$= 1 - \frac{1}{2} (\vec{x}_{ij} \cdot \vec{\nabla} \theta)^2 + \dots$$

$$\vec{x}_{ij} = \vec{x}_i - \vec{x}_j$$

$$H = \frac{J}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2 \rightarrow \frac{J}{2} \int d^2x (\vec{\nabla} \theta)^2$$

Continuum limit fails near vortex core since then the assumption that θ_i varies slowly on the scale of the lattice constant is ~~quite~~ wrong.



$$\oint_C d\vec{x} \cdot \vec{\nabla} \theta = 2\pi n$$

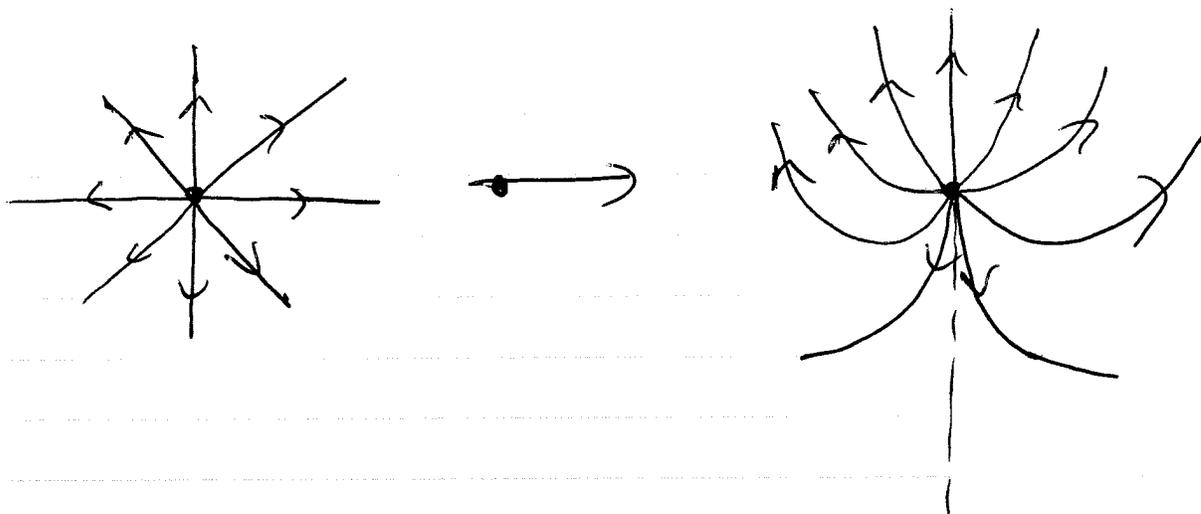
C is any closed loop enclosing vortex core.

n - winding number.

Since vortex is characterized by an integer number, it can not be made to disappear by any smooth deformation of the ~~spin~~ spin configuration.

~~The winding number of a vortex requires a non-trivial topology.~~

If we try to remove a vortex by continuously rotating spins, there will always be a line, starting from the core and going to infinity, along which the spins have to be rotated by an angle of order 2π .



The energy cost of such configuration is $\sim \gamma L$ higher than the energy cost of a vortex, where L is the size of the system. Thus in thermodynamic limit there will be an infinite energy barrier for unwinding a vortex \Rightarrow vortex is topologically stable.

Calculate the energy of a vortex with winding number n .

$$H = \frac{\gamma}{2} \int d^2x (\vec{\nabla} \theta)^2 \quad \text{— OK for away from the core.}$$

Minimize H subject to the constraint:

$$\oint \vec{\nabla} \theta \cdot d\vec{\ell} = 2\pi n$$

$$\delta H = \frac{\gamma}{2} \int d^2x \cdot 2 \vec{\nabla} \delta \theta \cdot \vec{\nabla} \theta =$$

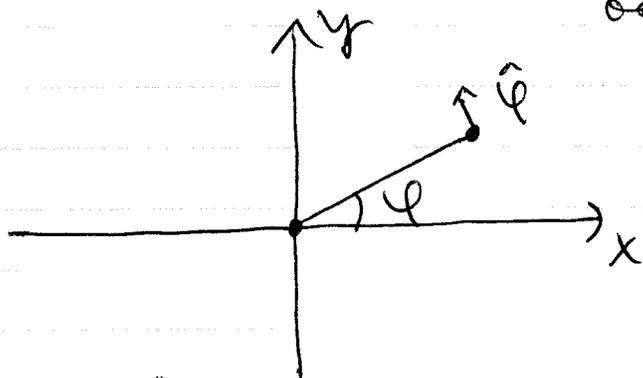
$$= -\gamma \int d^2x \delta \theta \nabla^2 \theta = 0$$

Thus we have to solve:

$$\nabla^2 \theta = 0 \quad \text{subject to constraint} \quad \oint \vec{\nabla} \theta \cdot d\vec{l} = 2\pi n$$

The solution is:

$\theta = n\varphi$, where φ is the polar angle, with origin at the vortex core.



$$\vec{\nabla} \theta = \frac{n\hat{\varphi}}{r}$$

Then the energy of a vortex is given by:

$$E_v = \frac{\gamma}{2} \int d^2x (\vec{\nabla} \theta)^2 = \frac{\gamma n^2}{2} \cdot 2\pi \int_a^L dr \cdot r \cdot \frac{1}{r^2} =$$

$$= \pi \gamma n^2 \ln \left(\frac{L}{a} \right)$$

Thus E_v diverges logarithmically with the system size L .

At finite T need to minimize free energy.

$$\text{Entropy of a vortex} \quad S_v = \ln \left(\frac{L}{a} \right)^2$$

$$F_v = E_v - TS_v = (\pi J - 2T) \ln\left(\frac{L}{a}\right)$$

Thus we can expect that vortices will start appearing above $T_{KT} = \frac{\pi J}{2}$ - Kosterlitz-Thouless temperature.

Below T_{KT} there are no free vortices and correlation length is infinite:

$$G(x) \sim \frac{1}{x^2}$$

above T_{KT} vortices appear and correlation length becomes finite:

$$G(x) \sim \frac{e^{-\frac{x}{\xi}}}{x^2}$$