

Lecture 12

As we saw in lecture 11, in systems with continuous symmetry, unlike in the Ising model case, there is no energy barrier to rotate the direction of $\vec{\varphi}$.

Thus we can expect the fluctuations of the direction \vec{n} to be very strong even far below the mean-field transition point, when the magnitude ρ is fixed.

(*)

Consider small fluctuations around the MF solution.

$$\text{let } \langle \vec{\varphi} \rangle = \rho \vec{n}.$$

$$\vec{n} = (1, 0, -\dots, 0)$$

$$\vec{\varphi}(x) = \rho \vec{n} + \delta \vec{\varphi}$$

$$\delta \vec{\varphi} = \delta \varphi_1 \vec{n} + \delta \vec{\varphi}_\perp$$

$$\delta \vec{\varphi}_\perp = (0, \delta \varphi_2, \dots, \delta \varphi_n)$$

Substitute into the bandon functional, leaving only terms of up to second order in fluctuations.

$$\begin{aligned} \vec{\varphi} \cdot \vec{\varphi} &= \rho^2 (\vec{n} + \delta \vec{\varphi}) \cdot (\vec{n} + \delta \vec{\varphi}) = \\ &= \rho^2 (1 + 2 \vec{n} \cdot \delta \vec{\varphi} + \delta \vec{\varphi} \cdot \delta \vec{\varphi}) = \\ &= \rho^2 (1 + 2 \delta \varphi_1 + \delta \vec{\varphi}_\perp \cdot \delta \vec{\varphi}_\perp) \end{aligned}$$

$$(\vec{\varphi} \cdot \vec{\varphi})^2 \approx \rho^u \left(1 + 6\delta\varphi_1^2 + 2\delta\vec{\varphi}_1 \cdot \delta\vec{\varphi}_1 + 4\delta\varphi_1 \right)$$

$$\vec{\nabla}\varphi^a \cdot \vec{\nabla}\varphi^a = \rho^2 \vec{\nabla}\delta\varphi_1 \cdot \vec{\nabla}\delta\varphi_1 + \rho^2 \vec{\nabla}\delta\varphi_1^a \cdot \vec{\nabla}\delta\varphi_1^a$$

Collecting everything, we obtain:

$$S[\vec{\varphi}] = \int d^d x \left[\frac{\rho^2}{2} \vec{\nabla}\delta\varphi_1 \cdot \vec{\nabla}\delta\varphi_1 + \frac{\rho^2}{2} \vec{\nabla}\delta\varphi_1^a \cdot \vec{\nabla}\delta\varphi_1^a \right]$$

$$+ \frac{r\rho^2}{2} \left(1 + 2\delta\varphi_1 + \delta\varphi_1^2 + \delta\vec{\varphi}_1 \cdot \delta\vec{\varphi}_1 \right) +$$

$$+ \frac{u\rho^4}{4} \left(1 + 4\delta\varphi_1 + 6\delta\varphi_1^2 + 2\delta\vec{\varphi}_1 \cdot \delta\vec{\varphi}_1 \right)$$

$$\rho = \sqrt{\frac{-r}{u}}$$

The coefficient of $\delta\varphi_1$:

$$r\rho^2 + u\rho^4 = -\frac{r^2}{u} + \frac{r^2}{u} = 0$$

The $\delta\vec{\varphi}_1 \cdot \delta\vec{\varphi}_1$ term has the same coefficient - also vanishes.

The coefficient of $\delta\varphi_1^2$:

$$\frac{r\rho^2}{2} + \frac{3}{2}u\rho^4 = -\frac{r^2}{2u} + \frac{3}{2}\frac{r^2}{u} = \frac{r^2}{u} = (\rho^2)r| -$$

doesn't vanish.

Then we obtain :

$$S[\delta\vec{\varphi}] = \int d^d x \frac{C^2}{2} \left[\vec{\nabla} \delta\varphi_1 \cdot \vec{\nabla} \delta\varphi_1 + \right. \\ \left. + \vec{\nabla} \delta\varphi_1^\alpha \cdot \vec{\nabla} \delta\varphi_1^\alpha + 2|\Gamma| \delta\varphi_1^2 \right]$$

Thus while fluctuations ~~of~~ of $\vec{\varphi}$ along the orthonormal direction \vec{n} cost finite energy, uniform deviation of ~~the~~ the direction of $\vec{\varphi}$ from \vec{n} doesn't cost any energy.

Rewrite in Fourier space :

$$\delta\varphi_1(\vec{k}) = \frac{1}{V} \sum_{\vec{k}} \delta\varphi_1(\vec{k}) e^{i\vec{k} \cdot \vec{x}}$$

$$\delta\varphi_1^\alpha(\vec{k}) = \frac{1}{V} \sum_{\vec{k}} \delta\varphi_1^\alpha(\vec{k}) e^{i\vec{k} \cdot \vec{x}}$$

$$S[\delta\vec{\varphi}] = \frac{C^2}{2V} \sum_{\vec{k}} \left[k^2 |\delta\varphi_1(\vec{k})|^2 + \right. \\ \left. + 2|\Gamma| |\delta\varphi_1(\vec{k})|^2 + k^2 \delta\varphi_1^\alpha(\vec{k}) \delta\varphi_1^\alpha(-\vec{k}) \right]$$

The energy cost for $\delta\varphi_1(\vec{k})$ is always nonzero.

The energy cost for $\delta\vec{\varphi}_1(\vec{k})$ vanishes as $k \rightarrow 0$.

$\delta\vec{\varphi}_1$ are called Goldstone modes.

Goldstone modes always arise as a consequence of spontaneous symmetry breaking in systems with continuous symmetry.

Examples of Goldstone modes: spin waves (magnons) in ferromagnets, phonons in crystals, π -mesons in nuclei.

Calculate correlation functions of $\delta\varphi_i$ and $\delta\varphi_{\perp}$.

$$G_1(\vec{x} - \vec{x}') = \langle \delta\varphi_i(\vec{x}) \delta\varphi_i(\vec{x}') \rangle = \\ = \frac{V}{\rho^2} \sum_{\vec{k}} \langle \delta\varphi_i(\vec{k}) \delta\varphi_i(-\vec{k}) \rangle e^{i\vec{k} \cdot (\vec{x} - \vec{x}')}}$$

$$\langle \delta\varphi_i(\vec{k}) \delta\varphi_i(-\vec{k}) \rangle = \frac{V}{\rho^2} \frac{1}{|\vec{k}|^2 + 2/\Gamma}$$

$$\langle \delta\varphi_{\perp}^a(\vec{k}) \delta\varphi_{\perp}^a(-\vec{k}) \rangle = \frac{V}{\rho^2} \frac{1}{|\vec{k}|^2}$$

It follows that:

$$G(\vec{x} - \vec{x}') \sim \frac{1}{|\vec{x} - \vec{x}'|^{d-2}} \ell^{-\frac{1}{\zeta}}$$

$$\zeta^2 = \frac{1}{2\Gamma}$$

$$G_{\perp}^a(\vec{x} - \vec{x}') \sim \frac{1}{|\vec{x} - \vec{x}'|^{d-2}}$$

The correlation length of foldstone modes is infinite.

Calculate the mean-square fluctuation of the order parameter due to foldstone modes:

$$\begin{aligned} \langle \delta\varphi_I^a(x) \delta\varphi_I^a(x') \rangle &= \sum_a f_I^a(0) = \\ &= \frac{n-1}{\rho^2} \sqrt{\sum_R \frac{1}{k^2}} = \frac{n-1}{\rho^2} \int_0^\infty \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} = \\ &= \frac{n-1}{\rho^2} \frac{S_d}{(2\pi)^d} \int_0^\infty \frac{k^{d-2} dk}{k^2} = \\ &= \frac{n-1}{\rho^2} \frac{S_d}{(2\pi)^d} \int_0^\infty dk k^{d-3} \end{aligned}$$

$d > 2$ - integral converges - fluctuations are finite.

$d \leq 2$ - integral diverges - no long-range order except at $T=0$.

No long range order at finite T in system with continuous symmetry in $d \leq 2$ (Mermin-Wagner theorem).

Foldstone modes are responsible for the loss of long-range order.

We want to describe the physics near the lower critical dimension $d=2$ in systems with continuous symmetry.

The basic observation is that the critical temperature vanishes as d approaches 2:

$$T_c(d) = T_c(2+\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0.$$

It turns out that, based on this, one can set up another ϵ -expansion - ~~starting~~ in small deviations from ~~dimensions~~ $d=2$.

We start again from the hamiltonian functional, written in terms of ρ and \vec{n} :

$$S[\rho, \vec{n}] = \int d^d x \left[\frac{1}{2} \rho^2 \vec{\nabla} n^a \cdot \vec{\nabla} n^a + \frac{1}{2} (\vec{\nabla} \rho)^2 + \right. \\ \left. + \frac{1}{2} \rho^2 + \frac{u}{4!} \rho^4 \right]$$

Near $d=2$ the real T_c will be far below the mean-field T_c . Then we can consider the magnitude of the order parameter ρ to be fixed and focus only on the fluctuations of \vec{n} . Then we obtain:

$$S[\vec{n}] = \int d^d x \cancel{\left[\frac{1}{2} \rho^2 \vec{\nabla} n^a \cdot \vec{\nabla} n^a \right]}$$

Have to remember that \vec{n} is a unit vector: $\vec{n} \cdot \vec{n} = 1$. This makes the problem highly nonlinear. ~~nonlinear~~

$S[\vec{u}]$ is often called a nonlinear sigma model (NLSM).

Since $Z = \int d\vec{u} e^{-S[\vec{u}]}$, the parameter ρ^2 is clearly proportional to $\frac{1}{T}$.

It is convenient to ~~redefine~~ redefine temperature and set $\rho^2 = \frac{1}{T}$.

$$S[\vec{u}] = \frac{1}{2T} \int d^d x (\vec{\nabla} u^a) \cdot (\vec{\nabla} u^a)$$

Let's single out a particular direction, say $(1, 0, -1, 0)$, which will be the ordering direction when $T < T_c$. Write \vec{u} as:

$$\vec{u} = (\sigma, \vec{\pi}^1, \dots, \vec{\pi}^{d-1})$$

$$\vec{\pi} = (\sigma, \vec{\pi}) , \sigma^2 + \vec{\pi}^2 = 1$$

$$S[\sigma, \vec{\pi}] = \frac{1}{2T} \int d^d x \left[(\vec{\nabla} \sigma)^2 + \vec{\nabla} \pi^a \cdot \vec{\nabla} \pi^a \right]$$

Hence the name "nonlinear sigma model".

Solve the constraint for σ :

$\sigma = \sqrt{1 - \vec{\pi}^2}$ — we will assume that $\sigma(\vec{x})$ has the same sign everywhere. This is an approximation which neglects topological defects of the order parameter: domain walls, vortices, hedgehogs. This misses important physics in some cases in $d=2$. For now let us proceed assuming this is OK.

$$(\vec{\sigma}_0)^2 = \vec{\sigma} \sqrt{1-\vec{n}^2} \cdot \vec{\sigma} \sqrt{1-\vec{n}^2}$$

$$Z = \int D\sigma D\vec{n} \delta(\sigma^2 + \vec{n}^2 - 1) e^{-S[\sigma, \vec{n}]}$$

~~Integrate over continuous space~~

To deal with the δ -function, ~~on~~ discrete space:
 assume variables are defined on a lattice with lattice constant a .
 Then we have:

$$\begin{aligned} & \int D\sigma D\vec{n} \delta(\sigma^2 + \vec{n}^2 - 1) \dots = \\ &= \int \prod_i d\sigma_i d\vec{n}_i \delta(\sigma_i^2 + \vec{n}_i^2 - 1) = \\ &= \int \prod_i \frac{d\vec{n}_i}{\sqrt{1-\vec{n}_i^2}} \end{aligned}$$

$$\begin{aligned} Z &= \int \prod_i \frac{d\vec{n}_i}{\sqrt{1-\vec{n}_i^2}} e^{-S[\vec{n}]} = \\ &= \int \prod_i d\vec{n} e^{-S[\vec{n}] - \frac{1}{2} \sum_i \ln(1-\vec{n}_i^2)} = \\ &= \int D\vec{n} e^{-S[\vec{n}] - \frac{1}{2a^d} \int d^d x \ln(1-\vec{n}^2)} \end{aligned}$$

Ans we can define a new hamon functional ~~functional~~ by:

$$S[\vec{\pi}] = \frac{1}{2T} \int d^d x \left[\vec{\nabla} \sqrt{1-\vec{\pi}^2} \cdot \vec{\nabla} \sqrt{1-\vec{\pi}^2} + \vec{\nabla} \vec{\pi}^a \cdot \vec{\nabla} \vec{\pi}^a \right] \\ + \frac{1}{2a^d} \int d^d x \ln(1-\vec{\pi}^2)$$

Expand to quartic order in $\vec{\pi}$:

$$\vec{\nabla} \sqrt{1-\vec{\pi}^2} \cdot \vec{\nabla} \sqrt{1-\vec{\pi}^2} \approx \frac{1}{4} (\vec{\nabla} \vec{\pi}^2)^2$$

$$\ln(1-\vec{\pi}^2) \approx -\vec{\pi}^2 - \frac{1}{2} \vec{\pi}^4$$

Then we obtain:

$$S[\vec{\pi}] = \frac{1}{2T} \int d^d x \left[\vec{\nabla} \vec{\pi}^a \cdot \vec{\nabla} \vec{\pi}^a + \frac{1}{4} (\vec{\nabla} \vec{\pi}^2)^2 - \right. \\ \left. - \frac{T}{a^d} \vec{\pi}^2 - \frac{T}{2a^d} \vec{\pi}^4 \right]$$

$$\dim[\vec{\pi}^2] - d + 2 - \dim[T] = 0.$$

$$\text{Thus } \dim[\vec{\pi}^2] = \dim[T] + d - 2 = \dim[T] + \epsilon$$

We will be looking for a fixed point T^* of the RG flow, corresponding to the critical point.

Expect $T^* \sim \epsilon$ near $d=2$.