Homework 5
Due Monday, April 6

1. Consider a one-dimensional (1D) classical Ising model:

$$H = -J \sum_{i=0}^{N} \sigma_i \sigma_{i+1},$$

where $\sigma_i = \pm 1$ and N is the total number of sites in the chain.

(a) Calculate the free energy of this system. *Hint: Start by writing the partition function as a product of N identical 2×2 matrices, each corresponding to a factor $e^{\frac{J}{T}\sigma_i \sigma_{i+1}}$.*

(b) Given the expression for the free energy argue that there is no phase transition in the 1D Ising model at any finite T.

(c) Using Hubbard-Stratonovich transformation derive the continuum ϕ^4 Landau functional for this system:

$$S[\phi] = \int dx \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{r}{2}\phi^2 + \frac{u}{4!}\phi^4 \right].$$

(d) Now reinterpret this Landau functional for $r < 0$ as an imaginary time action of a quantum particle, moving in a double-well potential:

$$S[\phi] = \int d\tau \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + \frac{r}{2}\phi^2 + \frac{u}{4!}\phi^4 \right].$$

What is the physical meaning of ϕ now?

(e) Quasiclassical (WKB) approximation to the quantum mechanics of this particle corresponds to the saddle-point (least action path) approximation for the imaginary-time evolution operator (we are using the units where $\hbar = 1$ as usual and keeping time imaginary for convenience):

$$Z = \int D\phi e^{-S[\phi]}.$$

Show that the saddle-point approximation gives the following imaginary-time differential equation for ϕ:

$$\frac{\partial^2 \phi}{\partial \tau^2} - r\phi - \frac{u}{6}\phi^3 = 0.$$

Find a solution of this equation satisfying

$$\phi(\tau = \pm\infty) \equiv \phi_\pm = \pm\sqrt{\frac{6|r|}{u}},$$

where $\phi_\pm$ correspond to the two minima of the double-well potential. This solution is called an *instanton* and represents quasiclassical description of the tunneling process, where the particle is in the left well at time $\tau = -\infty$ and eventually tunnels into the right well.

(f) What does the instanton correspond to in the language of the 1D Ising model?

(g) Recall the domain wall energy-entropy argument from lecture 11, which shows that $d = 1$ is the lower critical dimension for the Ising model. Reinterpret this argument in terms of instantons and the quantum mechanics of a particle in a double-well potential.

(h) Explain the absence of a phase transition in the 1D Ising model in terms of the mapping to a particle in a double-well potential. *Hint: Think what property of the ground state wavefunction of the particle translates into the absence of a phase transition in the 1D Ising model.*

2. Now consider a 2D classical Ising model. Reinterpret its Landau functional for $r < 0$ as the imaginary time action of a collection of interacting quantum particles, moving in the same one-dimensional double-well potential as above. Which term in the Landau functional does the interaction between the particles come from? What is its effect on the dynamics of the particles? What does the ordered state of the 2D Ising model correspond to in the quantum particle language? Argue qualitatively why in this case an ordered state is possible, unlike in the 1D Ising model case above. *Note: in this problem it's convenient to keep the "spatial" direction in the Landau functional discrete, taking continuum limit only in the "temporal" direction.*