Lecture 17

Continue semiclassical transport ...

In lecture 16 we derived a general expression for the linear-response conductivity of a metal:

$$\sigma_{ab} = e^2 \tau \int \frac{d\vec{k}}{4\pi^3} \delta(\varepsilon_k - \varepsilon_F) v_a v_b$$

This shows that conductivity (and most other transport coefficients) are determined by states at the Fermi surface.

Specializing to the case of a lightly-filled or an almost completely filled band, in which case effective mass approximation is valid:

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m^*}$$

we obtained a simpler expression for conductivity:

$$\sigma = \frac{ne^2\tau}{m^*}$$

This can be used for estimates of the impurity scattering time τ from experimentally measured conductivity.

Typical values of τ in metals are $10^{-14} - 10^{-15}$ sec.

Derive yet another expression for conductivity that's sometimes useful.

$$\sigma_{ab} = e^2\tau \int \frac{d\vec{k}}{4\pi^3}\, \delta(\varepsilon_k - \varepsilon_F)\, v_a v_b$$

let's rewrite this explicitly as an integral over the Fermi surface.

$d\vec{k} = dS_k\, dk_\perp$, where S is a constant energy surface

On the other hand $d\varepsilon_k = |\vec{\nabla}_k \varepsilon_k|\, dk_\perp$, since gradient of a function is always normal to the constant-value surface.

Therefore:

$$d\vec{k} = \frac{dS_k}{|\vec{\nabla}_k \varepsilon_k|}\, d\varepsilon_k = \frac{dS_k}{\hbar\, |v_k|}\, d\varepsilon_k$$

Then we can write the expression for conductivity as:

$$\sigma_{ab} = \frac{e^2\tau}{4\pi^3\hbar} \int dS_F\, \frac{v_{ka} v_{kb}}{|v_k|}$$

Specialize for the case of crystal with cubic symmetry:

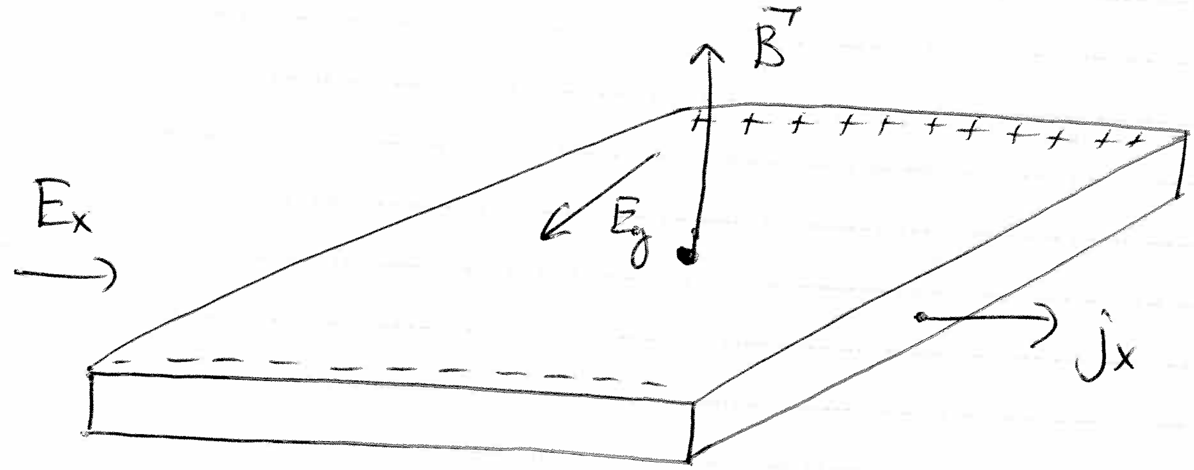
$$\sigma_{xx} = \frac{e^2\tau}{4\pi^3\hbar} \int dS_F\, \frac{v_{kx}^2}{|v_k|} = \frac{e^2\tau}{12\pi^3\hbar} \int dS_F\, |v_k|$$

In 2D:

$$\sigma_{xx} = \frac{e^2\tau}{2\pi^2\hbar} \int dl_F\, \frac{v_{kx}^2}{|v_k|} = \frac{e^2\tau}{4\pi^2\hbar} \int dl_F\, |v_k|$$

Start Hall effect.

Consider a metallic sample in crossed electric and magnetic fields.



In response to electric field in the x-direction, the current will flow in the x-direction. The electrons will experience a Lorentz force:

$\vec{F}_L = -\frac{e}{c}\,\vec{v}\times\vec{B}$, which is transverse to the direction of the applied electric field.

Electrons will accumulate on one side, leaving net positive charge on the other. Thus, there will be a transverse electric field E_y, proportional to both j_x and B.

This effect is characterized by Hall coefficient:

$$R_H = \frac{E_y}{j_x B}$$

Calculate R_H for a model.

~~Start from stationary Boltzmann equation in relaxation time approximation.~~

$$\cancel{\vec{\nabla} f \cdot \vec{v} + \frac{1}{\hbar} \vec{\nabla}_k f \cdot \vec{F} + \frac{f - f_0}{\tau} = 0}$$

Start from the Boltzmann equation in the relaxation time approximation, assuming uniform steady state:

$$\frac{1}{\hbar} \vec{\nabla}_k f \cdot \vec{F} + \frac{f - f_0}{\tau} = 0$$

$$\vec{F} = -e\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)$$

Need to be careful with linearization: Hall effect is linear in electric field but the Hall coefficient depends on a product of E and B ⇒ linearize only with respect to E.

$$f = f_0 + f_1$$

Assume $f_1 \sim E$ and leave only terms of first order in E.

$$-\frac{e}{\hbar}\left(\vec{\nabla}_k f_0 + \vec{\nabla}_k f_1\right) \cdot \left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) + \frac{f_1}{\tau} = 0.$$

Linearizing in E we obtain:

$$-\frac{e}{\hbar} \vec{\nabla}_k f_0 \cdot \left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) - \frac{e}{\hbar} \vec{\nabla}_k f_1 \cdot \frac{1}{c} \vec{v} \times \vec{B} + \frac{f_1}{\tau} = 0$$

$$\vec{\nabla}_k f_0(\varepsilon_k) = \frac{\partial f_0}{\partial \varepsilon_k} \vec{\nabla}_k \varepsilon_k = \hbar \frac{\partial f_0}{\partial \varepsilon_k} \vec{v}$$

$$-e\frac{\partial f_0}{\partial \varepsilon_k} \vec{v} \cdot \left(\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right) - \frac{e}{\hbar c} \vec{\nabla}_k f_1 \cdot (\vec{v} \times \vec{B}) + \frac{f_1}{\tau} =$$

$\vec{v} \cdot (\vec{v} \times \vec{B}) = 0 \Rightarrow$ second term drops out.

Thus we obtain:

$$-e\frac{\partial f_0}{\partial \varepsilon_k} \vec{v} \cdot \vec{E} - \frac{e}{\hbar c} \vec{\nabla}_k f_1 \cdot (\vec{v} \times \vec{B}) + \frac{f_1}{\tau} = 0.$$

~~[illegible]~~ The difference from the equation we had to solve before to find conductivity is the second term. This equation is not as easy to solve. One way is to guess the form of the solution. Look for solution in the following form:

$$f_1 = e\tau \frac{\partial f_0}{\partial \varepsilon_k} \vec{v} \cdot \vec{G}$$

Here $\vec{G}$ is an unknown vector, to be determined.

Motivation: this looks the same as the form of the solution without magnetic field with $\vec{G}$ replaced by $\vec{E}$.

Assume lightly filled band with $\varepsilon_k = \frac{\hbar^2 k^2}{2m^*}$.

$$\vec{v} = \frac{1}{\hbar} \vec{\nabla}_k \varepsilon_k = \frac{\hbar \vec{k}}{m^*}$$

$$\vec\nabla_k f_1 = \vec\nabla_k \left(e\tau \frac{\partial f_0}{\partial \varepsilon_k} \vec v \cdot \vec G \right) =$$

$$= e\tau \frac{\partial f_0}{\partial \varepsilon_k} \frac{\hbar}{m^*} \vec G + e\tau \frac{\partial^2 f_0}{\partial \varepsilon_k^2} \hbar \vec v \left(\vec v \cdot \vec G \right)$$

Plug this into the Boltzmann equation.

The second term above won't contribute, since the contribution will be proportional to:

$$\vec v \cdot \left(\vec v \times \vec B \right) = 0.$$

Then we obtain:

$$-e \frac{\partial f_0}{\partial \varepsilon_k} \vec v \cdot \vec E - \frac{e^2 \tau}{m^* c} \frac{\partial f_0}{\partial \varepsilon_k} \vec G \cdot \left(\vec v \times \vec B \right) +$$

$$+ e \frac{\partial f_0}{\partial \varepsilon_k} \vec v \cdot \vec G = 0$$

Cancelling the common factor $-e \frac{\partial f_0}{\partial \varepsilon_k}$, we obtain:

$$\vec v \cdot \vec E + \frac{e\tau}{m^* c} \vec G \cdot \left(\vec v \times \vec B \right) - \vec v \cdot \vec G = 0$$

Use $\vec A \cdot \left(\vec B \times \vec C \right) = \vec B \cdot \left(\vec C \times \vec A \right)$

$$\vec v \cdot \vec E + \frac{e\tau}{m^* c} \vec v \cdot \left(\vec B \times \vec G \right) - \vec v \cdot \vec G = 0$$

$$\vec{v}\cdot\left(\vec{E}+\frac{e\tau}{m^*c}\vec{B}\times\vec{G}-\vec{G}\right)=0$$

Thus we have a solution for all $\vec{v}$ if:

$$\vec{E}=\vec{G}-\frac{e\tau}{m^*c}\vec{B}\times\vec{G}$$

Now note that since the form of solution for f_1 is the same as in the conductivity calculation, only with $\vec{E}$ replaced by $\vec{G}$, if we calculated the current density using f_1, we would get:

$$\vec{j}=\sigma\vec{G}\ ,\ \sigma=\frac{ne^2\tau}{m^*}$$

Thus $\vec{G}=\dfrac{\vec{j}}{\sigma}$.

$$\vec{E}=\frac{\vec{j}}{\sigma}-\frac{e\tau}{m^*c}\,\frac{m^*}{ne^2\tau}\,\vec{B}\times\vec{j}=$$

$$=\frac{\vec{j}}{\sigma}-\frac{1}{nec}\vec{B}\times\vec{j}$$

Thus the Hall coefficient is given by:

$$R_H=-\frac{1}{nec}$$

$$\vec{E}=\frac{\vec{j}}{\sigma}+R_H\,\vec{B}\times\vec{j}$$

From here we can read off the resistivity tensor for electrons in magnetic field:

$$\vec{E} = \rho \vec{j}$$

$$\rho = \begin{pmatrix} 1/\sigma & -R_H B \\ R_H B & 1/\sigma \end{pmatrix}$$

Hall effect measurements are one of the ways to determine charge carrier density:

$$R_H = -\frac{1}{nec}$$ – only density and fundamental constants.

Note that R_H depends on the first power of the charge $\Rightarrow$ if we repeated the same calculation for an almost filled band, we would get an opposite sign for the Hall coefficient:

$$R_H = \frac{1}{nec}$$