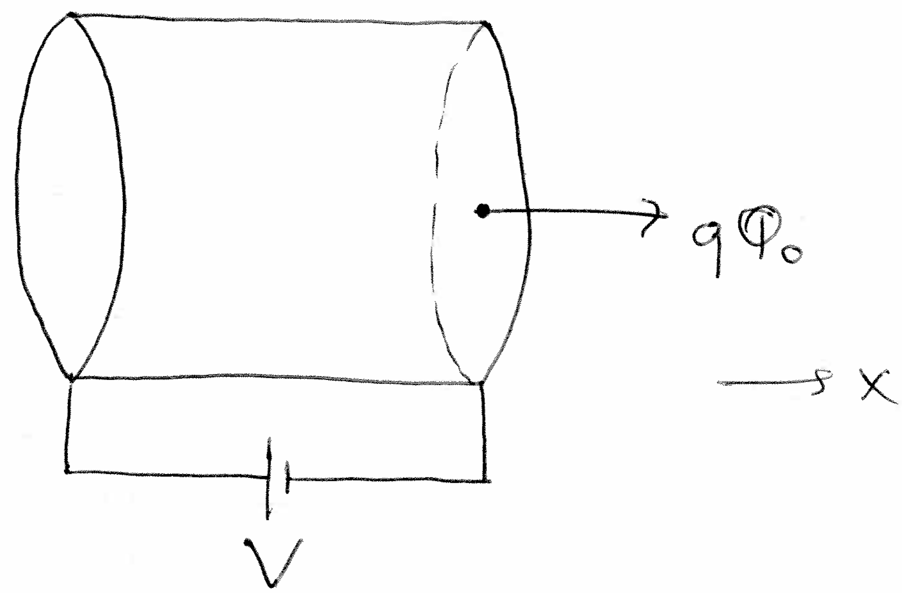
lecture 25

Recap laughlin argument.



$V = E L_x$.

The eigenstates in the presence of electric field E in the x-direction and ~~[illegible]~~ magnetic flux $q\Phi_0$ through the hole of the cylinder have the form:

$$E_{nk} = \hbar\omega_c\left(n+\frac{1}{2}\right) - \frac{\hbar E c}{B}\left(k - \frac{2\pi q}{L_y}\right) - \frac{m}{2}\left(\frac{Ec}{B}\right)^2$$

Eigenstates have their centers-of-mass shifted to:

$$X_0 = -k\ell^2 + \frac{2\pi q}{L_y}\ell^2 - \frac{m\ell^2}{\hbar}\left(\frac{Ec}{B}\right)$$

As we adiabatically increase q from 0 to 1 the eigenstates move to the right. When $q=1$, the set of eigenstates is exactly the same as at $q=0$ since $k = \frac{2\pi n_k}{L_y}$ with integer n_k.

. Since, by assumption, the Fermi level in the bulk of the sample is in the mobility gap and only the extended states are sensitive to the flux through the hole of the cylinder, adiabatic addition of a flux quantum ~~[illegible]~~ can not excite electrons to higher Landau levels. The only thing that happens is the transfer of exactly one electron per filled Landau level from left edge of the sample to right. The corresponding work, done on the system is:

$$\Delta E = eV\nu .$$

and the transferred charge is $\Delta Q = -e\nu$.

This immediately leads to quantization of ρ_{xy}:

$$\rho_{xy} = \frac{h}{\nu e^2} .$$

To summarize, validity of the Laughlin argument depends on the following points:

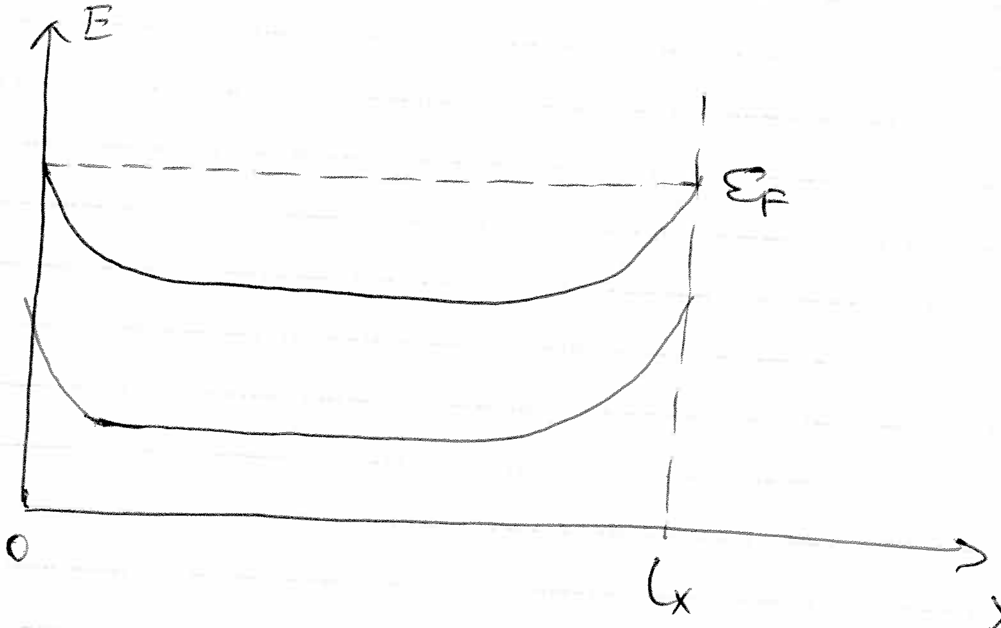
1. As we change the magnetic flux through the hole, only extended states near the Landau level centers are sensitive to this change. This is because locally the flux through the hole is invisible since $\nabla \times \delta \vec{A} = 0$.

2. There is a mobility gap between bands of extended states, which ensures $\rho_{xx} = 0$ at $T=0$.

3. Mobility gap also ensures that electrons are not excited to higher Landau levels during the flux injection,

and the only thing that happens is the transfer of exactly one electron per filled Landau level from the left edge of the sample to the right.

The last point implies that while there is a mobility gap in the bulk of the sample, the edges do not have a gap and the extended states exist at the Fermi level.

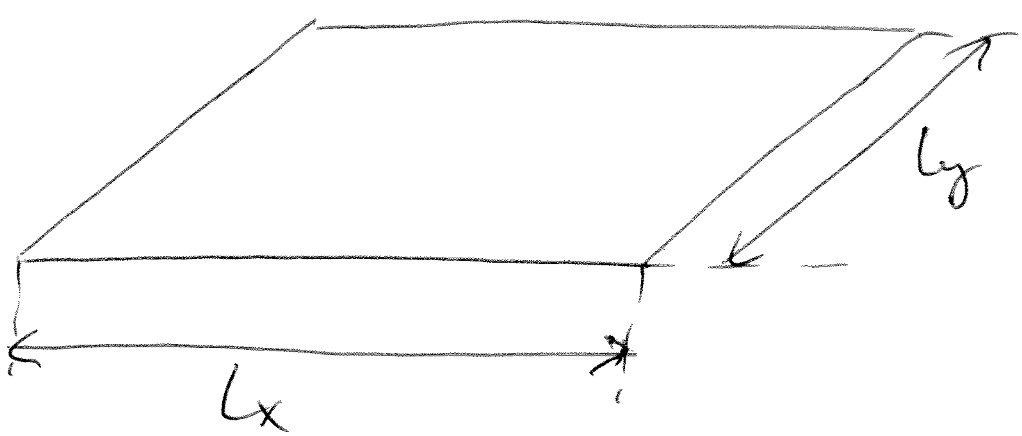


Laughlin argument also reveals a very interesting property of the integer quantum Hall insulator: it has extended states, sensitive to topological properties of the sample, like flux through the hole of the cylinder, if the sample is bent into a cylinder. Such insulators are called <u>topological insulators</u>.

Precise quantization of ρ_{xy} is in fact a topological property — this is why it's so robust and precise.

- Another important issue we need to address: experimentalists measure not the Hall resistivity ρ_{xy}, but the Hall resistance R_{xy}.

Normally $R = \rho \frac{L}{A}$, i.e. even if ρ_{xy} is precisely quantized, R_{xy} doesn't have to be, since it depends on the sample geometry. However, in 2D this turns out not to be the case.



$$V_y = R_{yx} I_x \;, \quad R_{yx} = \frac{V_y}{I_x} = \frac{L_y E_y}{L_y j_x} =$$

$$= \frac{E_y}{j_x} = \rho_{yx}$$, i.e. in 2D $R_{yx} = \rho_{yx}$ – this is why the precise quantization is directly observable experimentally.

Quantum Hall effect gives the most precise measurement of $\frac{h}{e^2}$ and, correspondingly, the fine structure constant $\alpha = \frac{e^2}{\hbar c}$.

- So far we have been looking at the integer quantum Hall effect, where $\rho_{xy} = \frac{h}{\nu e^2}$ with integer ν - Landau level filling factor.

In 1983, Tsui, Stormer and Gossard, also from Bell Labs, discovered fractional quantum Hall effect (FQHE): plateau in ρ_{xy} vs. B at $\nu = \frac{1}{3}$, i.e. when the lowest Landau level (LLL) is only 1/3-filled. Subsequently, plateaus at many other rational fractions were discovered.

As we have seen in the IQHE case, a gap in the spectrum is required to observe plateaus in ρ_{xy}.

At a fractional ν, gap can only arise due to electron-electron interactions, the single-electron spectrum doesn't have a gap.
Thus the FQHE problem is much more complicated, but also much more interesting and rich.

The $\nu = \frac{1}{3}$ problem was solved by Laughlin in 1983, which earned him Nobel prize in 1998.

Laughlin's solution was to write down, based on certain arguments, a many-body wavefunction for $\frac{1}{3}$-filled Landau level.

· The wavefunction has the form:

$$\Psi(z_1, \ldots, z_N) = \prod_{i<j} |z_i - z_j|^3 \, e^{-\frac{1}{4\ell^2} \sum_{i=1}^{N} |z_i|^2}$$

where $z = x + iy$ are electron coordinates.

The factor $\prod_{i<j} |z_i - z_j|^3$ efficiently minimizes Coulomb interaction energy between the electrons, since it vanishes when any two electrons approach each other.

Laughlin state has many interesting properties:

1. Topological order – when the system is put on a topologically nontrivial manifold, such as torus or cylinder the ground state develops nontrivial degeneracy, which depends on the topology: e.g. it is 3 on a torus and 1 on a sphere.

2. Excitations above the ground state carry fractions of the charge of the electron, e.g. $q = \pm\frac{e}{3}$ for $\nu = \frac{1}{3}$.

For this problem it's convenient to use symmetric gauge:

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}.$$

The single-electron eigenfunctions in the lowest Landau level have the form:

$$\varphi_m(z) = \frac{1}{\sqrt{2\pi l^2 2^m m!}}\left(z/l\right)^m e^{-\frac{1}{4l^2}|z|^2}$$

where $m = 0, 1, \dots, N_\varphi - 1$.

Let us first write down the many-electron ground state wavefunction for the case $\nu = 1$ — completely filled lowest Landau level. Since all single-particle states are filled, Pauli principle dictates that the ground state is simply a Slater determinant, with one electron in every single-particle state $\varphi_m(z)$.

It is easy to see that it is given by:

$$\Psi(z_1, \dots, z_{N_\varphi}) = \prod_{i<j}^{N_\varphi}(z_i - z_j)\, e^{-\frac{1}{4}\sum_{i=1}^{N_\varphi}|z_i|^2},$$

l was absorbed into z's for convenience.

Laughlin's insight is based on the so-called plasma analogy.

Consider probability density, corresponding to Ψ:

$$|\Psi(z_1, \dots, z_{N_\varphi})|^2 = \prod_{i<j}^{N_\varphi}|z_i - z_j|^2 e^{-\frac{1}{2}\sum_{i=1}^{N_\varphi}|z_i|^2}$$

Take the norm of this wavefunction:

$$Z = \int dz_1 \cdots dz_{N_\phi} \left| \Psi(z_1, \dots, z_{N_\phi}) \right|^2$$

and consider this to be partition function of a classical statistical mechanics system with a Boltzmann weight:

$$\left| \Psi(z_1, \dots, z_{N_\phi}) \right|^2 \equiv e^{-\beta E_{class}}$$

β would be equal to $\frac{1}{k_B T}$ in a real classical stat. mech. problem.

Here we take $\beta = \frac{2}{m}$. Then E_{class} is given by:

$$E_{class} = -m^2 \sum_{i<j} \ln |z_i - z_j| + \frac{m}{4} \sum_i |z_i|^2$$

$m = 1$ in the $\nu = 1$ case.

It turns out that E_{class} is the potential energy of a classical 2D plasma of particles with charge m in a uniform neutrality background.
First term is the interaction energy of these particles with each other.

To understand the second term, we note that:

$$\nabla^2 \frac{1}{4}|z|^2 = \nabla^2 \frac{1}{4\ell^2}(x^2+y^2) = \frac{1}{\ell^2} = 2\pi \frac{1}{2\pi\ell^2}$$

We can interpret this as 2D Poisson equation

$$\nabla^2 \varphi = -2\pi \rho \quad \text{, with } \varphi = \frac{1}{4}|z|^2 \text{ and } \rho = -\frac{1}{2\pi\ell^2}$$

Thus the second term in E_{class} corresponds to interaction of the particles with uniform background charge $\rho = -\frac{1}{2\pi\ell^2}$.

Since the plasma must be neutral and uniform, we have:

$nm + \rho = 0$, where n is the particle density = electron density in the original quantum Hall problem.

Thus $n = \frac{1}{2\pi\ell^2 m}$. For $m=1$ we have:

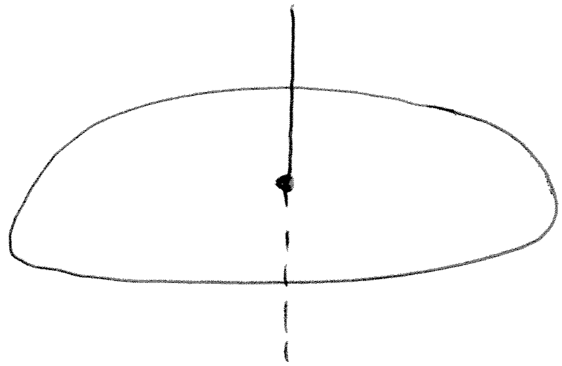
$n = \frac{1}{2\pi\ell^2}$ — this is indeed the density corresponding to $\nu = 1$: $\nu = \frac{N}{N_\varphi} = \frac{N \cdot 2\pi\ell^2}{L_x L_y} = n \cdot 2\pi\ell^2 \Rightarrow$

$$\Rightarrow n = \frac{\nu}{2\pi\ell^2}.$$

Thus for $\nu = \frac{1}{3}$ we can take $m=3$. This corresponds precisely to the Laughlin wavefunction.

It can be shown that excited states are separated from the Laughlin ground state by a finite gap.

To demonstrate fractionally charged excitations, consider the following thought experiment.



Take a sample at a rational filling factor $\nu = \frac{p}{q}$ in a gapped fractional quantum Hall ground state. Pierce it with an infinitely thin solenoid through the origin. Slowly change the flux through the solenoid from 0 to Φ_0. When the flux is Φ_0, it is invisible to the electrons, i.e. the system is mapped back onto itself. But, just as in the Laughlin argument for IQHE, charge has been transferred in the process, this time from infinity to the origin. Indeed, time-dependent flux gives rise to azimuthal electric field in the sample,

$$E_\varphi(r) = -\frac{1}{2\pi r}\frac{1}{c}\frac{\partial \Phi}{\partial t}$$

This field gives rise to radial current:

$$j_r = \sigma_{xy} E_\varphi = -\frac{\nu e^2}{h} E_\varphi = \frac{\nu e^2}{h}\frac{1}{2\pi r c}\frac{\partial \Phi}{\partial t}$$

Integrating this over time, we get the total charge transferred:

$$\Delta Q = 2\pi r \int_{-\infty}^{\infty} dt\, j_r = \frac{\nu e^2}{h}\frac{1}{c}\Phi_0 = \nu e$$

Thus we have generated an excited state with charge νe, localized near the origin.