Lecture 33

Ginzburg – Landau free energy of a superconductor:

$$F = \int d\vec{r} \left[\frac{1}{2m^*} \left| \left(-i\hbar\vec{\nabla} + \frac{e^*}{c}\vec{A}\right)\Phi(\vec{r})\right|^2 + \right.$$

$$\left. + a|\Phi|^2 + \frac{b}{2}|\Phi|^4 + \frac{\vec{B}^2}{8\pi} - \frac{\vec{B}\cdot\vec{H}}{4\pi} \right]$$

$m^* = 2m$, $e^* = 2e$, $\vec{H}$ is the external magnetic field.

Thermodynamic equilibrium state of a superconductor is determined by minimizing F.

Vary F with respect to $\Phi^*(\vec{r})$.

$$\delta_{\Phi^*} F = \int d\vec{r} \left[a\Phi\delta\Phi^* + b\Phi|\Phi|^2\delta\Phi^* + \right.$$

$$\left. + \frac{1}{2m^*}\left(i\hbar\vec{\nabla}\delta\Phi^* + \frac{e^*}{c}\vec{A}\delta\Phi^*\right)\cdot\left(-i\hbar\vec{\nabla}\Phi + \frac{e^*}{c}\vec{A}\Phi\right)\right]$$

$$\int d\vec{r}\, \vec{\nabla}\delta\Phi^* \cdot \left(-i\hbar\vec{\nabla} + \frac{e^*}{c}\vec{A}\right)\Phi =$$

$$= -\int d\vec{r}\, \delta\Phi^* \vec{\nabla}\cdot\left(-i\hbar\vec{\nabla} + \frac{e^*}{c}\vec{A}\right)\Phi +$$

$$+ \int d\vec{r}\, \vec{\nabla}\cdot\left[\delta\Phi^*\left(-i\hbar\vec{\nabla} + \frac{e^*}{c}\vec{A}\right)\Phi\right] =$$

$$= -\int d\vec{r}\, \delta\varphi^* \, \vec{\nabla}\cdot\left(-i\hbar\vec{\nabla} + \frac{e^*}{c}\vec{A}\right)\varphi +$$

$$+ \oint \delta\varphi^* \left(-i\hbar\vec{\nabla}\varphi + \frac{e^*}{c}\vec{A}\varphi\right)\cdot d\vec{S}$$

Here, S is the surface of the sample.
Neglect the surface term – this only determines boundary conditions for φ in a finite sample.
Then we obtain:

$$\delta_{\varphi^*} F = \int d\vec{r} \left[a\varphi + b\varphi|\varphi|^2 + \right.$$

$$\left. + \frac{1}{2m^*}\left(-i\hbar\vec{\nabla} + \frac{e^*}{c}\vec{A}\right)^2 \varphi \right] \delta\varphi^* = 0$$

$\delta_{\varphi^*} F$ can only be zero if the expression in the square brackets vanishes at every point $\vec{r}$.
Thus we obtain the first GL equation:

$$\frac{1}{2m^*}\left(-i\hbar\vec{\nabla} + \frac{e^*}{c}\vec{A}\right)^2\varphi + a\varphi + b|\varphi|^2\varphi = 0.$$

This looks like a Schrodinger equation ~~for~~ a particle in magnetic field, but it's nonlinear.

The magnetic field inside the sample $\vec{B} = \vec{\nabla}\times\vec{A}$ is also a thermodynamic variable $\Rightarrow$ we also have to vary with respect to $\vec{A}$.

$$\delta_{\vec{A}} F = \int d\vec{r} \left[\frac{1}{2m^*} \frac{e^*}{c} \delta\vec{A}\, \varphi^* \cdot \left(-i\hbar\vec{\nabla} + \frac{e^*}{c}\vec{A} \right) \varphi + \right.$$

$$+ \frac{1}{2m^*} \left(i\hbar\vec{\nabla} + \frac{e^*}{c}\vec{A} \right) \varphi^* \cdot \frac{e^*}{c} \delta\vec{A}\, \varphi \; +$$

$$\frac{1}{4\pi} \left(\vec{\nabla} \times \vec{A} \right) \cdot \left(\vec{\nabla} \times \delta\vec{A} \right) -$$

$$\left. - \frac{\vec{H}}{4\pi} \cdot \left(\vec{\nabla} \times \delta\vec{A} \right) \right]$$

Consider the last two terms:

$$\frac{1}{4\pi} \int d\vec{r} \left[\left(\vec{\nabla} \times \vec{A} \right) - \vec{H} \right] \left(\vec{\nabla} \times \delta\vec{A} \right) \overset{*}{=}$$

Use identity:

$$\vec{a} \cdot \left(\vec{\nabla} \times \vec{b} \right) = \vec{b} \cdot \left(\vec{\nabla} \times \vec{a} \right) - \vec{\nabla} \cdot \left(\vec{a} \times \vec{b} \right)$$

$$\overset{*}{=} \frac{1}{4\pi} \int d\vec{r}\, \delta\vec{A} \cdot \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) \; -$$

$$- \frac{1}{4\pi} \oint d\vec{S} \cdot \left[\delta\vec{A} \times \left(\vec{\nabla} \times \vec{A} - \vec{H} \right) \right]$$

The surface integral vanishes since magnetic field at the surface of the superconductor is fixed by boundary conditions. Thus $\delta \vec{A}|_S = 0$.

Then we obtain:

$$\delta_{\vec{A}} F = \int d\vec{r} \left[\frac{-i\hbar e^*}{2m^* c}\left(\varphi^* \vec{\nabla}\varphi - \varphi\vec{\nabla}\varphi^*\right) + \right.$$

$$\left. + \frac{e^{*2}}{m^* c^2}|\varphi|^2 \vec{A} + \frac{1}{4\pi}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)\right]\cdot\delta\vec{A} = 0.$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$$

Thus we obtain:

$$\vec{j} = \frac{i\hbar e^*}{2m^*}\left(\varphi^*\vec{\nabla}\varphi - \varphi\vec{\nabla}\varphi^*\right) - \frac{e^{*2}}{m^* c}|\varphi|^2\vec{A}$$

Thus we have two equations:

$$\frac{1}{2m^*}\left(-i\hbar\vec{\nabla} + \frac{e^*}{c}\vec{A}\right)^2\varphi + a\varphi + b|\varphi|^2\varphi = 0$$

$$\vec{j} = \frac{i\hbar e^*}{2m^*}\left(\varphi^*\vec{\nabla}\varphi - \varphi\vec{\nabla}\varphi^*\right) - \frac{e^{*2}}{m^* c}|\varphi|^2\vec{A}$$

Analyze some of the consequences of these equations.

Assume we are below T_c in the superconducting state.

First assume φ is uniform.

- Then the first equation is:

$$a\Psi + b|\Psi|^2\Psi = 0.$$

The nontrivial solution is $|\Psi| = \sqrt{\frac{|a|}{b}}$.

If $\Psi(\vec{r})$ is the wavefunction of the pair condensate, then $|\Psi|^2$ should have the meaning of the density of pairs:

$$|\Psi|^2 = n_s^*$$

$$\text{let } \Psi(\vec{r}) = \sqrt{n_s^*}\, e^{i\theta(\vec{r})}$$

In a uniform superconducting sample we can expect n_s^* to be uniform, then only the phase $\theta(\vec{r})$ depends on $\vec{r}$.

Calculating the current, we obtain:

$$\vec{j} = -\frac{\hbar e^* n_s^*}{m^*}\left(\vec{\nabla}\theta + \frac{e^*}{\hbar c}\vec{A}\right)$$

Thus the superconducting current is directly related to the gradient of the phase of the macroscopic condensate wavefunction. This explains both the Meissner effect and the zero resistance. First consider Meissner effect.

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$$

Take the curl of both sides of this equation.

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=\frac{4\pi}{c}\,\vec{\nabla}\times\vec{j}$$

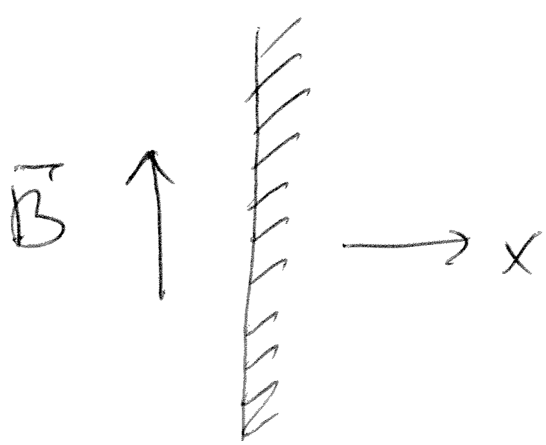
$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right)-\vec{\nabla}^2\vec{B}=-\nabla^2\vec{B}$$

$$\vec{\nabla}\times\vec{j}=-\frac{e^{*2}n_s^*}{m^*c}\,\vec{\nabla}\times\vec{A}=-\frac{e^{*2}n_s^*}{m^*c}\vec{B}$$ since $\vec{\nabla}\times\left(\vec{\nabla}\theta\right)\equiv 0$.

Thus we obtain:

$$\nabla^2\vec{B}=\frac{4\pi e^{*2}n_s^*}{m^*c^2}\vec{B}=\frac{1}{\lambda^2}\vec{B}$$

$$\lambda=\sqrt{\frac{m^*c^2}{4\pi e^{*2}n_s^*}}$$ - penetration depth.



$$\frac{d^2B}{dx^2}=\frac{1}{\lambda^2}B\Rightarrow B(x)=B(0)e^{-\frac{x}{\lambda}}$$ field only penetrates a distance λ into the superconductor.

$\lambda\sim 100\,\mathring{A}$ in simple metallic superconductors.

- Another way to understand Meissner effect:

Consider the free energy:

$$F = \int d\vec{r} \left[\frac{\hbar^2 n_s^{*2}}{2m^*} \left(\vec{\nabla}\theta + \frac{e^*}{\hbar c}\vec{A} \right)^2 + \right.$$

$$\left. + \frac{1}{8\pi}\left((\vec{\nabla}\times\vec{A}) - \vec{H} \right)^2 \right]$$

If the ~~[illegible]~~ sample is simply-connected, can always do the following gauge transformation:

$$\vec{A} \to \vec{A} - \frac{\hbar c}{e^*}\vec{\nabla}\theta \quad , \text{ since } \vec{\nabla}\times(\vec{\nabla}\theta) = 0.$$

Then we obtain:

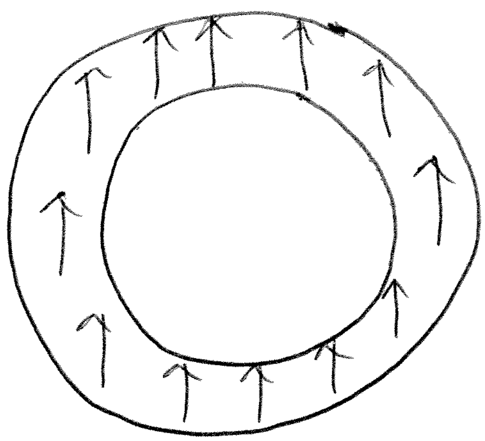
$$F = \int d\vec{r} \left[\frac{e^{*2} n_s^{*2}}{2m^* c^2}\vec{A}^2 + \frac{1}{8\pi}\left((\vec{\nabla}\times\vec{A}) - \vec{H} \right)^2 \right]$$

If $\vec{B} \neq 0$ inside the sample then $\vec{A}$ must depend at least linearly on the coordinates $\Rightarrow$ F will grow faster than the volume $\Rightarrow$ infinite free energy density, which is impossible $\Rightarrow B = 0$ in the bulk of the sample.

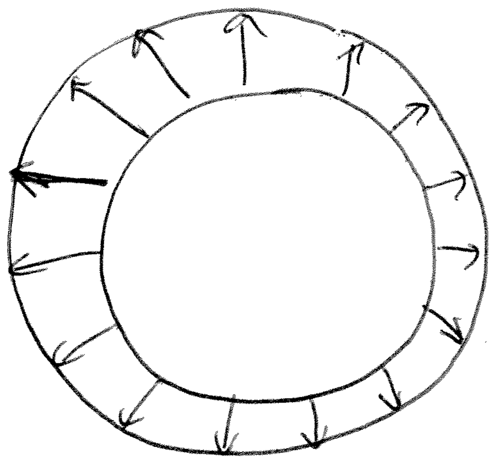
~~Consider~~ Now consider the question of persistent currents in a superconductor.

$$\vec{j} = -\frac{\hbar e^* n_s^*}{m^*}\vec{\nabla}\theta$$ — can take $\vec{A}=0$ in the bulk.

Consider a sample in the form of a ring.



— state without a current corresponds to $\vec{\nabla}\theta = 0$ or uniform phase, which is represented by arrows.



— current flowing around the ring.

$$\oint \vec{\nabla}\theta \cdot d\vec{l} = 2\pi n$$, where n is integer.

Going from $n=1$ to $n=0$ requires "unwinding" the phase in the whole ring — costs infinite energy in a macroscopic ring $\Rightarrow$ current will flow forever.