Lecture 34

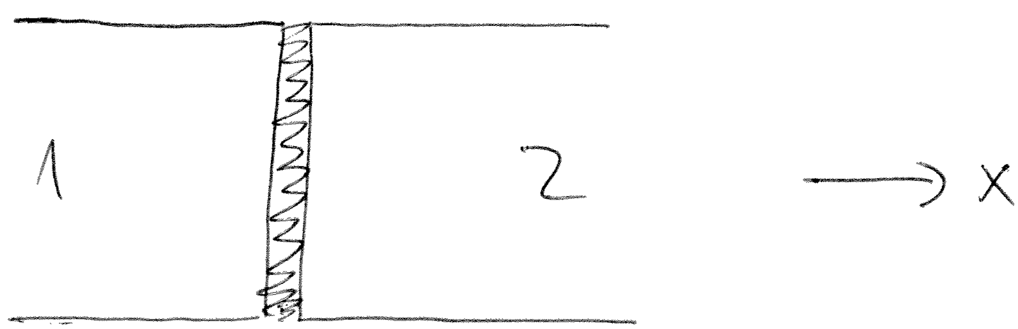
Josephson effects

Imagine we have two pieces of the same superconducting material at the same temperature, but completely isolated from each other.
Their macroscopic wavefunctions are $\varphi_1(\vec{r})$ and $\varphi_2(\vec{r})$.
Since the two pieces are made from the same material and are at the same temperature, $|\varphi_1| = |\varphi_2| = \sqrt{n_s^*}$.

The phases of the wavefunctions, however, are generally different:

$$\varphi_1(\vec{r}) = \sqrt{n_s^*}\, e^{i\theta_1(\vec{r})}, \quad \varphi_2(\vec{r}) = \sqrt{n_s^*}\, e^{i\theta_2(\vec{r})}.$$

Now let's establish a weak contact between the two pieces; e.g. through a thin insulating barrier.



If electron pairs are able to tunnel through the barrier a new macroscopic wavefunction will be established, which describes the whole system.

The total wavefunction can be found from the following Schrodinger equation:

$$\mathcal{H}\Phi = E\Phi$$

$$\mathcal{H} = \begin{pmatrix} 0 & -t \\ -t & 0 \end{pmatrix}$$, where t is the pair tunneling matrix element.

The lowest-energy eigenstate has the form:

$$\Phi(\vec{r}) = \frac{1}{\sqrt{2}}\left(\Phi_1(\vec{r}) + \Phi_2(\vec{r})\right)$$

Calculate the supercurrent density in this state:

$$\vec{j} = \frac{i\hbar e^*}{2m^*}\left(\Phi^*\vec{\nabla}\Phi - \Phi\vec{\nabla}\Phi^*\right)$$

$\vec{\nabla}\Phi$ is nonnegligible only in the junction area, where it can be estimated as:

$$\vec{\nabla}\Phi \approx \frac{\Phi_2 - \Phi_1}{L}\hat{x}$$, where L is the width of the junction.

$$\Phi^*\vec{\nabla}\Phi - \Phi\vec{\nabla}\Phi^* \sim (\Phi_1^* + \Phi_2^*)(\Phi_2 - \Phi_1) - (\Phi_1 + \Phi_2)(\Phi_2^* - \Phi_1^*)$$

$$= \Phi_1^*\Phi_2 - |\Phi_1|^2 + |\Phi_2|^2 - \Phi_2^*\Phi_1 - \Phi_2^*\Phi_1 + |\Phi_1|^2 -$$

$$- |\Phi_2|^2 + \Phi_1^*\Phi_2 = 2\left(\Phi_1^*\Phi_2 - \Phi_2^*\Phi_1\right) =$$

$$= -2n_s^*\left(e^{i\theta} - e^{-i\theta}\right) = -4in_s^*\sin\theta$$, where

$\theta = \theta_1 - \theta_2$ – phase difference between the two sides.

Thus the current across the junction is given by:

$$I = I_c \sin(\theta_1 - \theta_2) = I_c \sin\theta$$

Here I_c is the maximum supercurrent that can flow across the junction. Its value depends on the specific parameters of the junction.

This is DC Josephson effect – dissipationless current across a junction between two superconductors in the absence of any voltage.
Current is directly related to the phase difference across the junction – macroscopic quantum effect.

Now let's see what happens when we apply voltage across the junction.
In this case we need to solve time-dependent Schrodinger equation:

$$i\hbar \frac{\partial \varphi}{\partial t} = \mathcal{H}\varphi,$$

$$\text{where } \mathcal{H} = \begin{pmatrix} eV & -t \\ -t & -eV \end{pmatrix}$$

Pairs experience potential drop of $2eV$ across the junction.

Writing this in components, we have:

$$i\hbar \frac{\partial \varphi_1}{\partial t} = eV\varphi_1 - t\,\varphi_2$$

$$i\hbar \frac{\partial \varphi_2}{\partial t} = -t\,\varphi_1 - eV\varphi_2$$

Complex conjugate equations are:

$$-i\hbar \frac{\partial \varphi_1^*}{\partial t} = eV\varphi_1^* - t\,\varphi_2^*$$

$$-i\hbar \frac{\partial \varphi_2^*}{\partial t} = -t\,\varphi_1^* - eV\varphi_2^*$$

Now evaluate the following:

$$i\hbar \frac{\partial}{\partial t}\left(\varphi_2^*\varphi_1 - \varphi_1^*\varphi_2\right) =$$

$$= i\hbar\left(\varphi_2^* \frac{\partial \varphi_1}{\partial t} + \varphi_1 \frac{\partial \varphi_2^*}{\partial t} - \varphi_1^* \frac{\partial \varphi_2}{\partial t} - \varphi_2 \frac{\partial \varphi_1^*}{\partial t}\right) =$$

$$= eV\varphi_2^*\varphi_1 - \cancel{t|\varphi_2|^2} + \cancel{t|\varphi_1|^2} + eV\varphi_2^*\varphi_1 +$$

$$+ \cancel{t|\varphi_1|^2} + eV\varphi_1^*\varphi_2 - \cancel{t|\varphi_2|^2} + eV\varphi_1^*\varphi_2 =$$

$$= 2eV\left(\varphi_1^*\varphi_2 + \varphi_2^*\varphi_1\right) = 4eVn_s^*\cos\theta$$

On the other hand:

$$i\hbar \frac{\partial}{\partial t}\left(\varphi_2^*\varphi_1 - \varphi_1^*\varphi_2\right) = -2\hbar n_s^* \frac{\partial}{\partial t}\sin\theta =$$

$$= -2\hbar n_s^* \frac{\partial \theta}{\partial t}\cos\theta$$

Thus we obtain:

$$\hbar \frac{\partial \theta}{\partial t} = -2eV$$ — Josephson relation.

Assuming a DC voltage, we have:

$$\theta(t) = \theta(0) - \frac{2eV}{\hbar} t$$

Then the current across the junction is given by:

$$I(t) = I_c \sin\theta = I_c \sin\left[\theta(0) - \frac{2eV}{\hbar} t\right]$$

Thus a DC voltage across the junction leads to an AC current — AC Josephson effect.

The frequency of the current $\omega = \frac{2eV}{\hbar}$.

Since both the frequency and the voltage can be measured with a very high precision, AC Josephson effect gives an extremely precise measurement of $e/\hbar$.

In fact, AC Josephson effect and the quantum Hall effect, which gives a precise measurement of $\frac{e^2}{\hbar}$, in combination provide the most precise experimental values for e and $\hbar$ — the most fundamental physical constants.