Homework 6
Due Monday, November 16 in class

1. Consider electrons in a two-dimensional sample with dimensions L_x and L_y. Assume there are $N = nL_xL_y$ electrons and assume that the sample is so clean that the impurity potential can be neglected.

   (a) Calculate and plot the magnetic field dependence of the chemical potential μ for temperatures $k_BT \ll \hbar\omega_c$, where ω_c is the cyclotron frequency.

   (b) Real samples are of course never perfectly two-dimensional. Assume the sample thickness is $d = 100\AA$ and let the electron density be $n = 10^{18}\text{cm}^{-3}$. What is the typical magnetic field required to observe the integer quantum Hall effect? What temperatures are low enough for this field?

2. Show that the free energy of *classical* particles with no internal magnetic moment is always independent of magnetic field. This means that all phenomena, involving the dependence of thermodynamic properties of the electron gas on magnetic field, are intrinsically quantum mechanical. *Hint: write down the partition function Z for a collection of N classical particles. Let the particles interact with each other via an arbitrary potential U, which depends only on the positions of the interacting particles. Show that the magnetic field dependence can be eliminated from the partition function by a variable change.*

3. This problem illustrates why disorder is crucial for the existence of the quantum Hall effect. Consider an infinite disorder-free two-dimensional sample with electron density n in a constant uniform perpendicular magnetic field $\mathbf{B} = B\hat{z}$. Such a sample is clearly translationally invariant and there is no preferred frame of reference. Assume there is no electric field applied in the lab frame.

   (a) Transform to a reference frame moving with velocity $-\mathbf{v}$ with respect to the lab frame. What is the current density, carried by electrons in this frame?

   (b) To lowest order in v/c, what are the electric and magnetic fields in the moving frame?

Problem 1

(a) When $k_BT \ll \hbar\omega_c$, μ is approximately equal to the Fermi energy, $\mu \approx E_F$.

E_F will always coincide with the highest unfilled Landau level.

The degeneracy of each Landau level, neglecting spin, is $N_\varphi = \frac{\Phi}{\Phi_0} = \frac{B L_x L_y}{\Phi_0}$

Let N be the total number of electrons: $N = n L_x L_y$.

Assume first that $N < N_\varphi$ – magnetic field is large.

In this case all electrons are accomodated in the lowest Landau level and $E_F = \frac{\hbar\omega_c}{2}$. As B is reduced eventually $N = N_\varphi$. At this point E_F will jump to the second Landau level, $E_F = \frac{3\hbar\omega_c}{2}$ and the second Landau level will start to get populated, and so on.

Jumps occur when $N = \nu N_\varphi$, ~~[illegible]~~ where ν is the number of filled Landau levels.

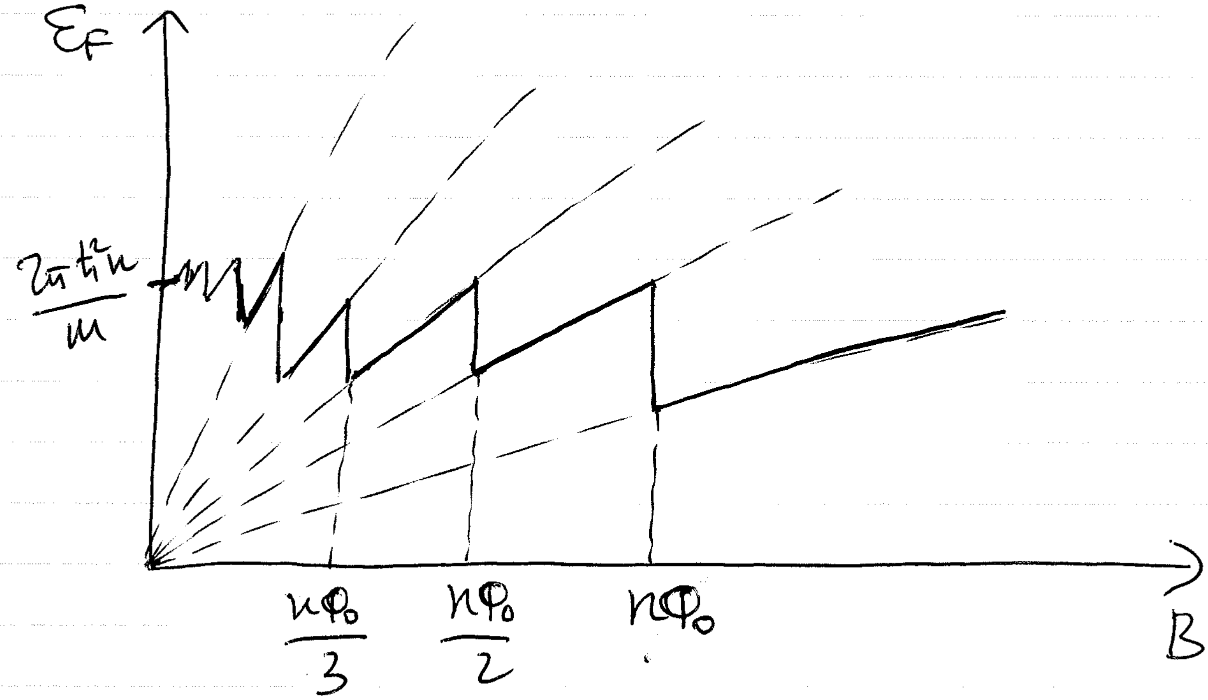
$$N = \nu N_\varphi \Rightarrow n L_x L_y = \nu \frac{B L_x L_y}{\Phi_0}$$

Thus $B_\nu = \frac{n \Phi_0}{\nu}$ – the values of magnetic field at which E_F jumps to a higher Landau level.

As $B \to 0$ $\mathcal{E}_F$ must converge to the Fermi energy of a free electron gas in 2D:

$$\frac{\pi k_F^2}{(2\pi)^2} = n$$

$$\mathcal{E}_F = \frac{2\pi\hbar^2 n}{m}$$ - spin degeneracy is neglected here.



(b) The 2D density is $n = 10^{18}\,\text{cm}^{-3} \cdot d = 10^{12}\,\text{cm}^{-2}$

Quantum Hall effect will be observed when the filling factor $\nu \sim 1$.

$$\nu = \frac{N}{N_\varphi} = \frac{n L_x L_y}{\frac{B L_x L_y}{\Phi_0}} = n\frac{\Phi_0}{B}$$

thus $B \approx n\Phi_0 = 10^{12} \times 2.07 \cdot 10^{-7}$ Gauss $= 2 \cdot 10^5$ Gauss $=$
$= 20$ Tesla

$$T \ll \frac{\hbar \omega_c}{k_B} = 27\,K.$$

Problem 2

Hamiltonian for classical particles of charge $-e$ in magnetic field is:

$$\mathcal{H} = \sum_{i=1}^{N} \frac{1}{2m}\left(\vec{p}_i + \frac{e}{c}\vec{A}_i\right)^2 + \sum_{i<j} V(\vec{r}_i - \vec{r}_j)$$

Here $\vec{A}_i \equiv \vec{A}(\vec{r}_i)$ and $V(\vec{r}_i - \vec{r}_j)$ is arbitrary interaction between the particles.

The partition function is given by:

$$Z = \int d\vec{p}_1 \ldots d\vec{p}_N\, d\vec{r}_1 \ldots d\vec{r}_N\, e^{-\frac{\mathcal{H}}{k_B T}}$$

The integral is over the whole phase space.
Since $\vec{A}$ only depends on the coordinates of the particles and not on their momenta we can do the following change of variables when doing the integrals over momenta:

$\vec{p}_i' = \vec{p}_i + \frac{e}{c}\vec{A}_i$ and integrate over $\vec{p}_i$ instead of $\vec{p}_i$. Then it's clear that Z and, therefore, the free energy $F = -k_B T \ln Z$ do not depend on the magnetic field.

Problem 3

(a) In the frame, moving with velocity $-\vec{v}$ relative to the lab frame, electrons appear to be moving with velocity $\vec{v}$! Thus the current density in the moving frame is:

$$\vec{j} = -ne\vec{v}$$

(b) In the lab frame we have:

$$\vec{E} = 0, \quad \vec{B} = B\hat{z}.$$

Transforming to the moving frame, we obtain, to lowest order in v/c:

$$\vec{E} = -\frac{1}{c}\vec{v} \times \vec{B}, \quad \vec{B} = B\hat{z}.$$

(c) Expressing $\vec{v}$ in terms of $\vec{j}$, we obtain:

$$\vec{E} = \frac{B}{nec}\vec{j} \times \hat{z}$$

From here we can read off the resistivity tensor:

$$\rho = \begin{pmatrix} 0 & \frac{B}{nec} \\ -\frac{B}{nec} & 0 \end{pmatrix}$$

Thus the resistivity is not quantized, the Hall effect is classical.

In the presence of disorder, lab frame is not equivalent to other frames since impurities are stationary in the lab frame.
Thus the above reasoning becomes invalid.
This demonstrates that disorder is necessary for the quantum Hall effect.