Lecture 23

Continue electrons in 2DEG in perpendicular magnetic field ...

$$\mathcal{H} = \frac{1}{2m}\left(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2$$

Landau gauge: $\vec{A} = xB\hat{y}$

$$\Psi(x,y) = e^{iky}\,\Phi(x)$$

Get the following equation for $\Phi(x)$:

$$-\frac{\hbar^2}{2m}\frac{d^2\Phi}{dx^2} + \frac{m\omega_c^2}{2}\left(x + k\ell^2\right)^2\Phi = E\Phi$$

This is the Hamiltonian of a harmonic oscillator with frequency ω_c, centered at $x = -k\ell^2$.

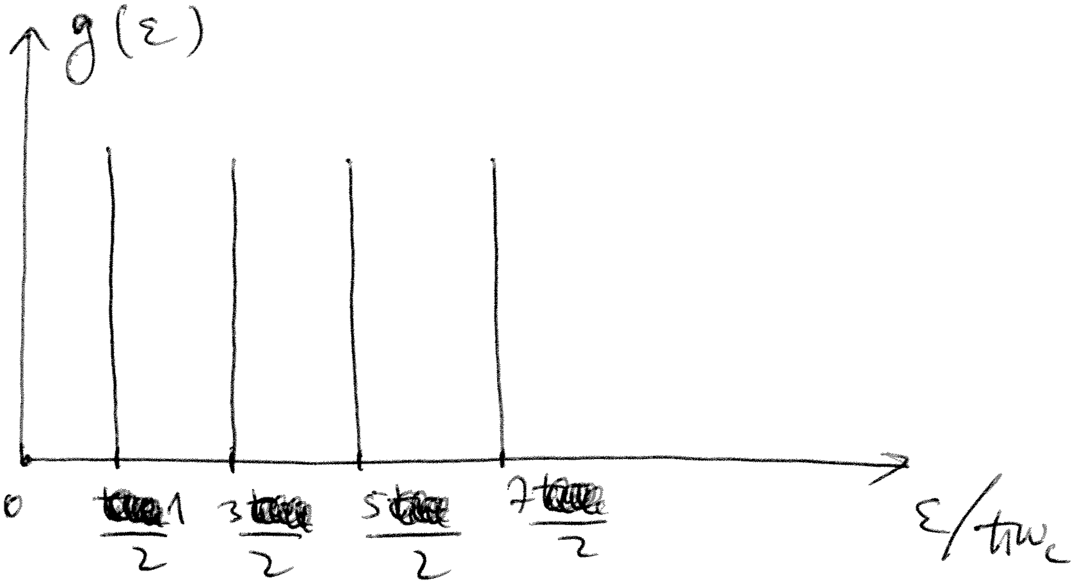
The energy spectrum and the eigenstates thus have the following form:

$$E_{nk} = \hbar\omega_c\left(n + \frac{1}{2}\right), \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\Phi_{nk}(x) = \frac{1}{\sqrt{2^n n!\,\ell\sqrt{\pi}}}\, e^{-\frac{1}{2\ell^2}(x+k\ell^2)^2} H_n\!\left(\frac{x+k\ell^2}{\ell}\right)$$

H_n is a Hermite polynomial.

Thus the spectrum of electrons is drastically changed. Instead of continuous spectrum $E_k = \frac{\hbar^2 k^2}{2m}$, we have discrete levels. These are called Landau levels.



Note that E_{nk} is independent of k $\Rightarrow$ each Landau level contains an infinite (in thermodynamic limit) number of degenerate states, corresponding to different values of k.

Find the allowed values of k.

Assume periodic boundary conditions in the y-direction.

$$\psi(x, y + L_y) = \psi(x, y)$$

This gives $k = \frac{2\pi n}{L_y}$, $n = 0, \pm 1, \pm 2, \ldots$

Note that k also determines the center-of-mass position of $\varphi_{nk}(x)$ in the x-direction.

This gives the following condition:

$$0 \le k\ell^2 \le L_x$$

Thus $0 \le k \le \frac{L_x}{\ell^2}$

Then the total number of states (not including spin) per Landau level is given by:

$$N_\varphi = \frac{L_x/\ell^2}{\frac{2\pi}{L_y}} = \frac{L_x L_y}{2\pi\ell^2}$$

Thus $2\pi\ell^2$ has the meaning of area per state. This can also be written as:

$$N_\varphi = \frac{L_x L_y}{2\pi\ell^2} = \frac{L_x L_y\, eB}{2\pi\hbar c} = \frac{\Phi}{\Phi_0}$$

$\Phi = L_x L_y B$ – total magnetic flux through the sample.

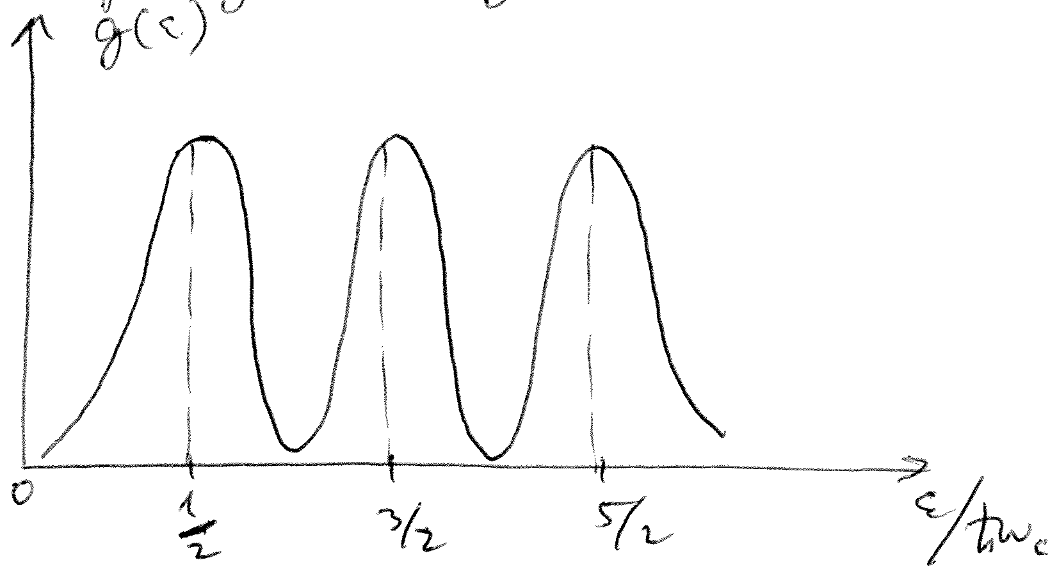
$\Phi_0 = \frac{2\pi\hbar c}{e} = \frac{hc}{e}$ – magnetic flux quantum.

$\Phi_0 = 2{,}07 \times 10^{-7}$ gauss $\cdot$ cm^2.

Introduce Landau level filling factor:

$\nu = \frac{N}{N_\varphi}$, N – total number of electrons.

Return to the density of states.
In a perfectly clean 2DEG density of states in magnetic field consists of a series of δ-function peaks at $\varepsilon_n = \hbar\omega_c\left(n+\frac{1}{2}\right)$.
In real 2DEG there is always some amount of impurity scattering.
Impurity scattering broadens the δ-function peaks.



Broadening can be understood as a consequence of the Heisenberg uncertainty principle: electron exists in a given quantum state for a time τ - mean time between collisions:

$$\Delta\varepsilon\,\tau \sim \hbar$$

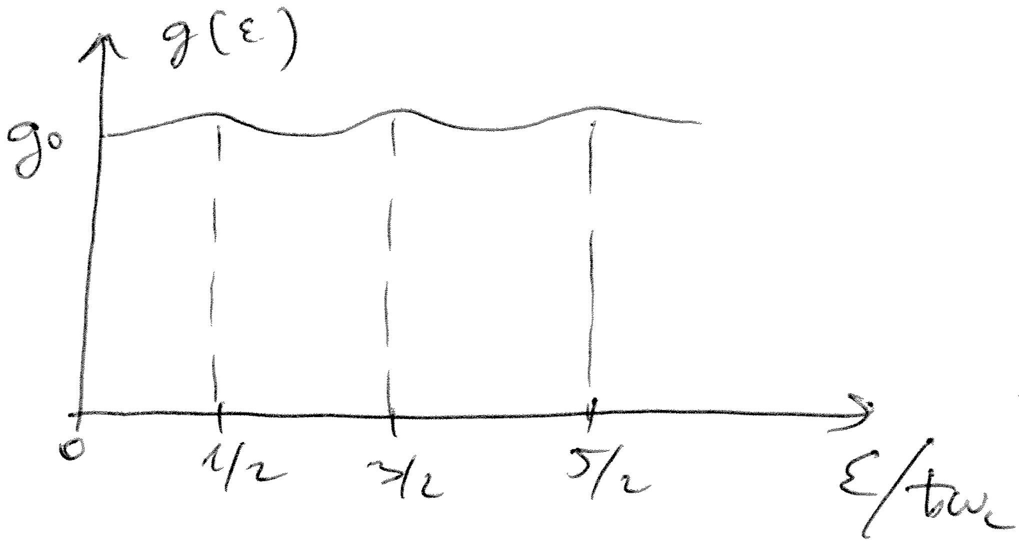
$$\Delta\varepsilon \sim \frac{\hbar}{\tau}$$ - width of Landau levels.

This gives another interpretation to the applicability of semiclassical transport theory conditions:

can be
$\omega_c\tau \ll 1$

written as:

~~[illegible]~~ $\hbar\omega_c \ll \frac{\hbar}{\tau}$ - distance between Landau levels is much less than broadening $\Rightarrow$ ~~[illegible]~~ cannot resolve individual Landau levels.

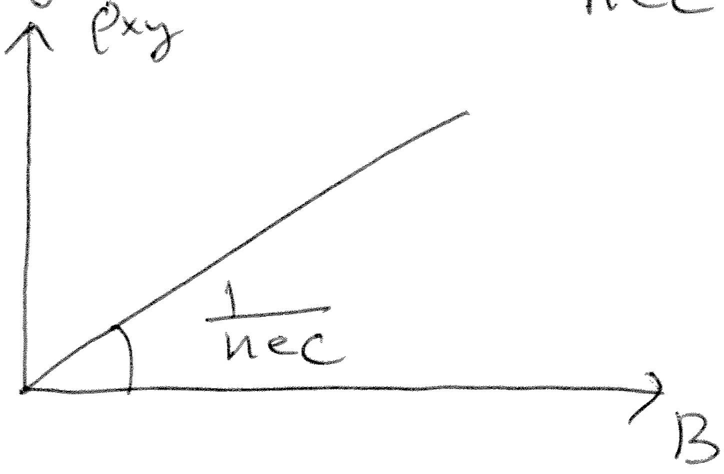


$g_0 = \frac{m}{\pi\hbar^2}$ - density of states in 2D without magnetic field.

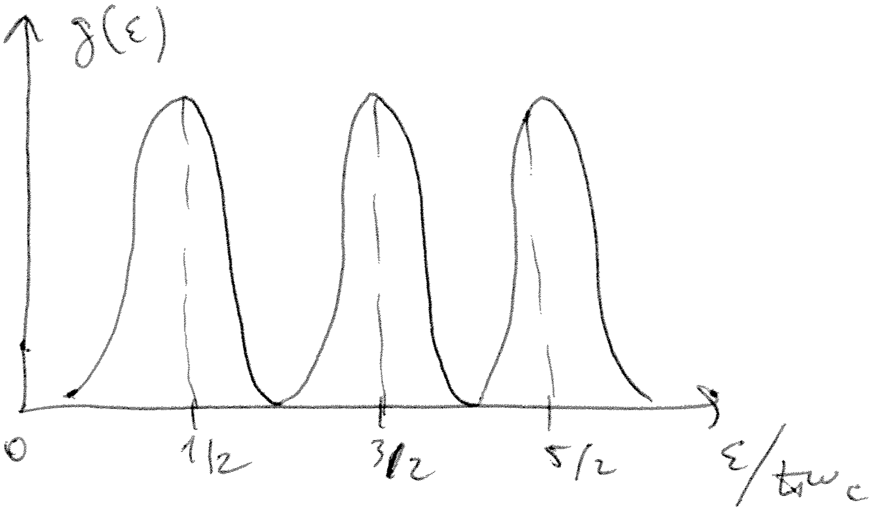
In this case semiclassical theory works.

Hall resistivity (lecture 17) is given by:

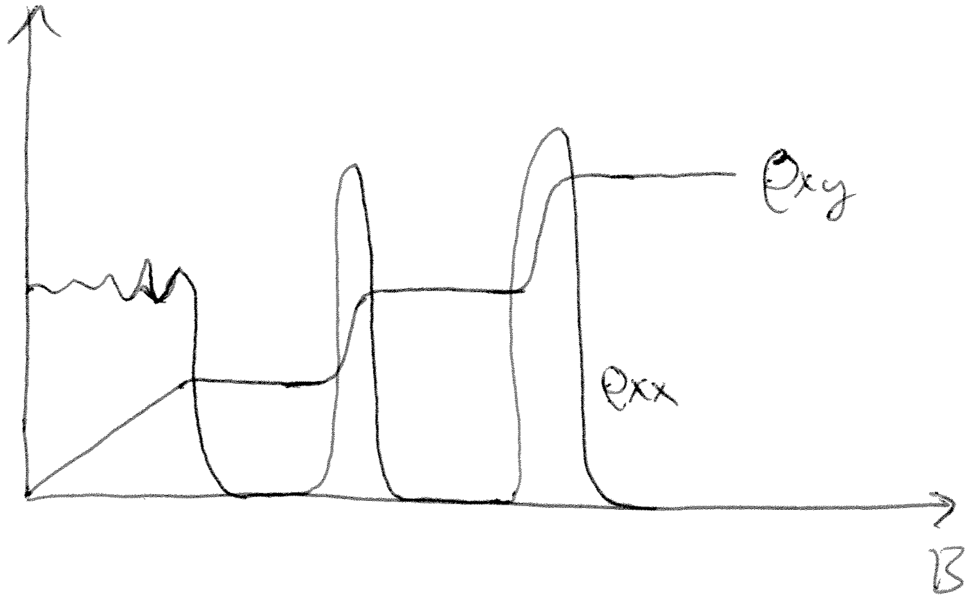
$$\rho_{xy} = -R_H B = \frac{B}{nec}$$



When $\omega_c \tau > 1$ $\hbar\omega_c > \frac{\hbar}{\tau}$ individual Landau levels can be resolved.



In this regime measurement of the Hall effect gives something completely different.



ρ_{xy} develops perfectly flat plateaus exactly at $\rho_{xy} = \frac{h}{e^2 \nu}$, where ν can be integer (integer quantum Hall effect) or some rational fraction (fractional quantum Hall effect).

At the same time as ρ_{xy} develops a plateau, ρ_{xx} goes exactly (in the $T \to 0$ limit) to zero.

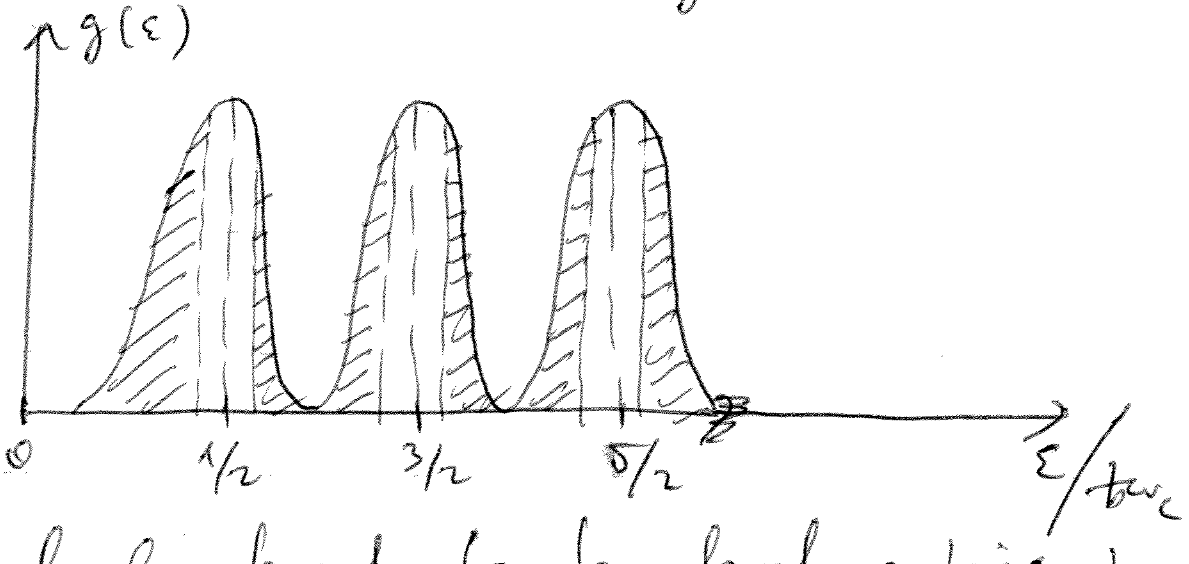
ν turns out to be equal to the Landau level filling factor $\nu = \frac{N}{N_\varphi}$.

$$\frac{h}{e^2} = 25813\ \Omega$$ - quantum of resistance.

The quantization of ρ_{xy} is precise to an accuracy of about 10^{-10} and is completely universal, i.e. independent of the sample shape, impurity composition and other details — it is the most precise measurement of h/e^2.

First consider IQHE. It occurs when ν is an integer, i.e. some number of Landau levels becomes completely filled.

To understand IQHE need to ~~[illegible]~~ first understand the structure of the density of states in more detail.



Each broadened Landau level contains two classes of states.

The states near the center of each Landau level are extended:

$$\psi_{nk}(\vec{r}) = e^{iky}\,\varphi_{nk}(x)$$

Extended states are weakly-modified eigenstates of a clean 2DEG in perpendicular field.

The states between the Landau level centers (shaded areas) are localized: these states are strongly modified by impurity scattering and, roughly, correspond to electrons localized in deeper minima of the impurity potential.

$$\psi(\vec{r}) \sim e^{-\frac{r}{\xi}}, \quad \xi \text{ - localization length.}$$

Such states can not participate in transport: even though there are states arbitrarily close in energy to a given localized state, they will typically be very far spatially - transport through such states is impossible.

Thus when the Fermi level is within the region of localized states, the 2DEG is an insulator. Such an insulator is called an Anderson insulator.

The gap between the bands of extended states near the centers of neighboring Landau levels is called mobility gap.

When we change magnetic field B, the peaks in the density of states will move through the Fermi energy ε_F.
When ε_F goes through bands of extended states near the centers of the peaks $\rho_{xx} \neq 0$

When ε_F is in the mobility gap, ρ_{xy} doesn't change because the states in the mobility gap are localized, and don't contribute to transport. At the same time $\rho_{xx} \to 0$.
In a usual insulator we would have $\sigma_{xx} \to 0$ and $\rho_{xx} \to \infty$.

Calculate σ_{xx}:

$$\sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2} = 0$$ – as it should be in an insulator.

$$\sigma_{xy} = -\frac{\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2} = -\frac{1}{\rho_{xy}}$$

$$\rho = \begin{pmatrix} 0 & \frac{h}{\nu e^2} \\ -\frac{h}{\nu e^2} & 0 \end{pmatrix} ; \quad \sigma = \begin{pmatrix} 0 & -\frac{\nu e^2}{h} \\ \frac{\nu e^2}{h} & 0 \end{pmatrix}$$

Thus in the integer quantum Hall insulator both $\sigma_{xx} = 0$ and $\rho_{xx} = 0$. – the system looks simultaneously like an insulator and like a perfect conductor.