Solutions

Homework 5
Due Friday, November 6 in class

1. In class we calculated the DC conductivity of a metal (response to a time-independent electric field) by solving a static Boltzmann equation. If the applied electric field is not static, time-dependent Boltzmann equation needs to be solved.

   (a) Assume the time dependence of the electric field is of the form $\mathbf{E}(t) = \mathbf{E}e^{-i\omega t}$. Solve the time-dependent linearized Boltzmann equation and find the frequency-dependent conductivity $\sigma(\omega)$. *Hint: assume that the time dependence of the distribution function is of the form* $f_1 \sim e^{-i\omega t}$. Plot $\mathrm{Re}[\sigma(\omega)]$ versus ω. The characteristic shape of this dependence is often referred to as the *Drude peak.*

   (b) Now assume that electric field is applied at time $t = 0$ and held constant afterwards. Find the current density $\mathbf{j}(t)$ for all $t > 0$.

   You can assume that the effective mass approximation is valid in both cases.

2. Assume conduction band dispersion in a semiconductor has the form

$$\epsilon_c(\mathbf{k}) = \epsilon_c + \frac{\hbar^2 k_x^2}{2m_{cx}} + \frac{\hbar^2 k_y^2}{2m_{cy}} + \frac{\hbar^2 k_z^2}{2m_{cz}}.$$

   Prove that the density of states is given by:

$$g(\epsilon) = \frac{m_c^{3/2}}{\pi^2 \hbar^3} \sqrt{2|\epsilon - \epsilon_c|},$$

   where $m_c = (m_{cx} m_{cy} m_{cz})^{1/3}$.

Problem 1

(a) Start from the time-dependent linearized Boltzmann equation:

$$\frac{\partial f_1}{\partial t} + \frac{1}{\hbar}\vec{\nabla}_k f_0 \cdot \vec{F} + \frac{f_1}{\tau} = 0$$

$$\vec{F}(t) = -e\vec{E}(t) = -e\vec{E}e^{-i\omega t}$$

look for solution in the form:

$$f_1(t) = f_1 e^{-i\omega t}$$

Plugging this into the Boltzmann equation, we obtain:

$$-i\omega f_1 - \frac{e}{\hbar}\vec{\nabla}_k f_0 \cdot \vec{E} + \frac{f_1}{\tau} = 0 .$$

The factor $e^{-i\omega t}$ can be cancelled since it appears in all terms.

The solution is:

$$f_1 = \frac{e\tau}{\hbar}\frac{1}{1-i\omega\tau}\vec{\nabla}_k f_0 \cdot \vec{E}$$

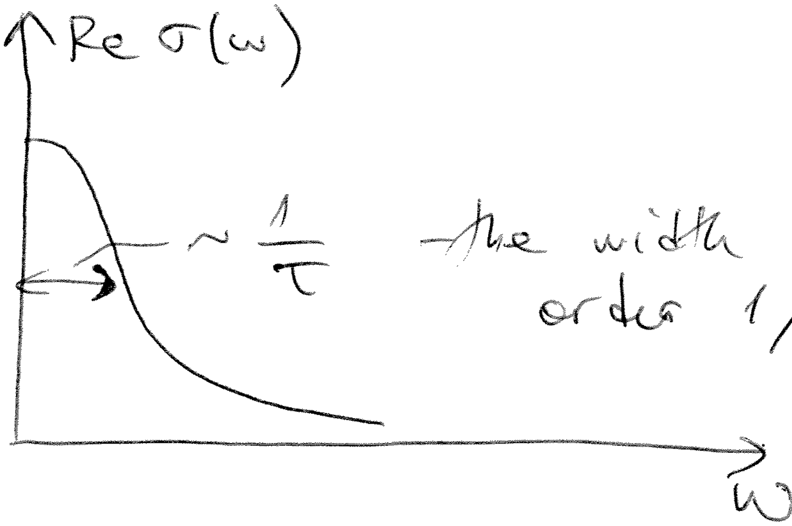
The only difference from the DC case is the factor of $\frac{1}{1-i\omega\tau}$.

Thus the AC conductivity is given by:

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \quad \text{where } \sigma_0 = \frac{ne^2\tau}{m^*} \text{ is}$$

the DC conductivity.

$$\operatorname{Re}\sigma(\omega) = \frac{\sigma_0}{1+\omega^2\tau^2}$$



the width of the peak is of order $1/\tau$.

(b) We have (as found in part a):

$$\vec{j}(\omega) = \sigma(\omega)\vec{E}(\omega)$$

This means:

$$\vec{j}(t) = \int_{-\infty}^{\infty} dt' \, \sigma(t-t')\vec{E}(t')$$

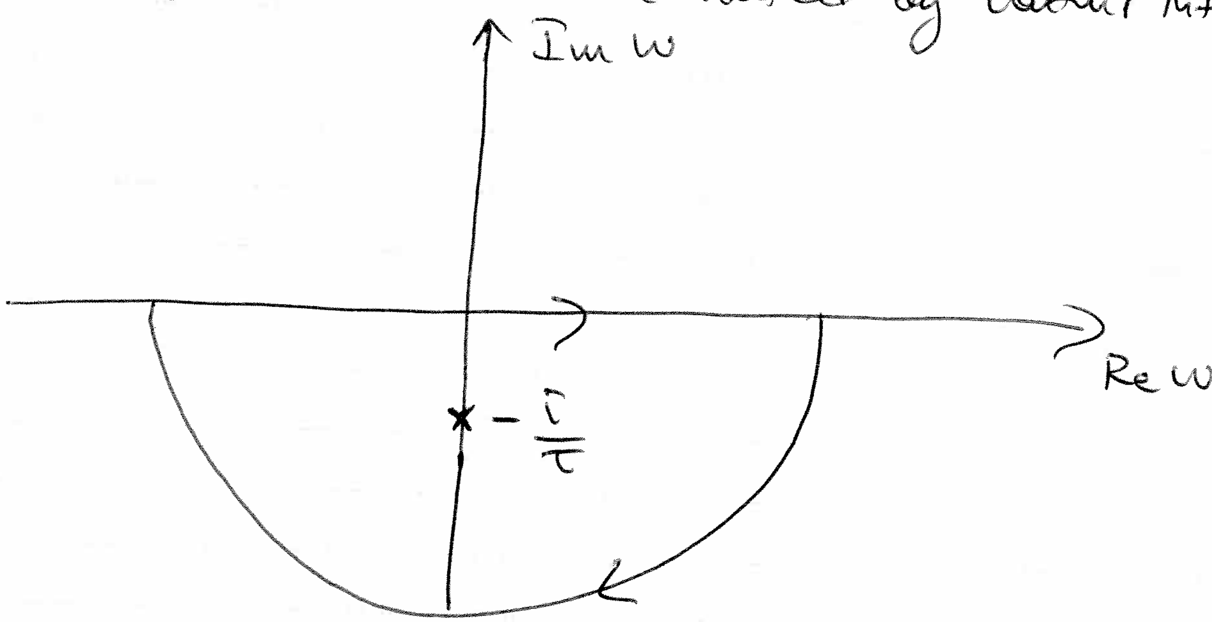
where $\vec{j}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \vec{j}(\omega) e^{-i\omega t}$

Find $\sigma(t)$.

$$\sigma(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\sigma_0}{1-i\omega\tau} e^{-i\omega t} =$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\sigma_0 \frac{i}{\tau}}{\omega + \frac{i}{\tau}} e^{-i\omega t}$$

This integral can be calculated by contour integration.



For $t>0$ the contour has to be closed in the lower half-plane for the integral to converge since $e^{\mathrm{Im}\,\omega\, t} \to 0$ as $|\omega| \to \infty$ on the contour.

Then we obtain:

$$\sigma(t) = -2\pi i \operatorname{Res}\left[\frac{1}{2\pi} \frac{\sigma_0 \frac{i}{\tau}}{\omega + \frac{i}{\tau}} e^{-i\omega t}\right]_{\omega = -\frac{i}{\tau}} =$$

$$= -2\pi i \, \frac{1}{2\pi} \sigma_0 \, \frac{i}{\tau} \, e^{-\frac{t}{\tau}} = \frac{\sigma_0}{\tau} e^{-\frac{t}{\tau}}$$

$$\sigma(t) = \frac{\sigma_0}{\tau} e^{-\frac{t}{\tau}} \quad \text{for } t > 0.$$

For $t < 0$ the contour has to be closed in the upper-half-plane and then the integral is zero, since there are no poles.

Thus
$$\sigma(t) = \begin{cases} \frac{\sigma_0}{\tau} e^{-\frac{t}{\tau}}, & t > 0 \\ 0, & t < 0 \end{cases}$$

The fact that $\sigma(t) = 0$ for $t < 0$ expresses causality: $\vec{j}(t)$ can only be determined by $\vec{E}(t')$ at $t' < t$.

$$\vec{j}(t) = \int_{-\infty}^{\infty} dt' \, \sigma(t - t') \, \vec{E}(t') =$$

$$= \int_{-\infty}^{t} dt' \, \sigma(t - t') \, \vec{E}(t') = \vec{E} \int_{0}^{t} dt' \, \sigma(t - t')$$

$$\int_0^t dt' \, \sigma(t - t') = \frac{\sigma_0}{\tau} e^{-\frac{t}{\tau}} \int_0^t e^{\frac{t'}{\tau}} \, dt' =$$

$$= \sigma_0 e^{-\frac{t}{\tau}} \left(e^{-\frac{t}{\tau}} - 1 \right) = \sigma_0 \left(1 - e^{-\frac{t}{\tau}} \right)$$

Thus we finally obtain:

$$\vec{j}(t) = \sigma_0 \vec{E}\left(1 - e^{-\frac{t}{\tau}}\right)$$

Problem 2

Define $N(\varepsilon)$ – number of states with energy less than ε.
Density of states can then be calculated as:

$$g(\varepsilon) = \frac{dN(\varepsilon)}{d\varepsilon}$$

On the other hand, $N(\varepsilon)$ is given by:

$$N(\varepsilon) = 2 \cdot \frac{1}{(2\pi)^3} V(\varepsilon)$$

, where $V(\varepsilon)$ is the volume in momentum space occupied by states with energy less than ε.

We have:

$$\varepsilon_c(\vec{k}) = \varepsilon_c + \frac{\hbar^2 k_x^2}{2m_{cx}} + \frac{\hbar^2 k_y^2}{2m_{cy}} + \frac{\hbar^2 k_z^2}{2m_{cz}}$$

Equation for the surface of the above volume is:

$$\varepsilon - \varepsilon_c = \frac{\hbar^2 k_x^2}{2m_{cx}} + \frac{\hbar^2 k_y^2}{2m_{cy}} + \frac{\hbar^2 k_z^2}{2m_{cz}}$$

This can be written in the canonical form of the equation for an ellipsoid:

$$\frac{\hbar^2 k_x^2}{2m_{cx}(\varepsilon-\varepsilon_c)} + \frac{\hbar^2 k_y^2}{2m_{cy}(\varepsilon-\varepsilon_c)} + \frac{\hbar^2 k_z^2}{2m_{cz}(\varepsilon-\varepsilon_c)} = 1$$

This implies $\varepsilon > \varepsilon_c$.

Axes of the ellipsoid are:

$$a = \sqrt{\frac{2m_{cx}(\varepsilon-\varepsilon_c)}{\hbar^2}}, \quad b = \sqrt{\frac{2m_{cy}(\varepsilon-\varepsilon_c)}{\hbar^2}},$$

$$c = \sqrt{\frac{2m_{cz}(\varepsilon-\varepsilon_c)}{\hbar^2}}.$$

The volume of an ellipsoid is given by:

$$V = \frac{4\pi}{3} abc = \frac{4\pi}{3}\sqrt{m_{cx} m_{cy} m_{cz}}\left[\frac{2(\varepsilon-\varepsilon_c)}{\hbar^2}\right]^{3/2}$$

$N(\varepsilon)$ is given by:

$$N(\varepsilon) = \frac{V}{4\pi^3} = \frac{2^{3/2} m_c^{3/2}}{3\pi^2 \hbar^3}(\varepsilon-\varepsilon_c)^{3/2}, \text{ where}$$

$$m_c = (m_{cx} m_{cy} m_{cz})^{1/3}.$$

$$g(\varepsilon) = \frac{dN(\varepsilon)}{d\varepsilon} = \frac{m_c^{3/2}}{\pi^2 \hbar^3}\sqrt{2(\varepsilon-\varepsilon_c)}$$