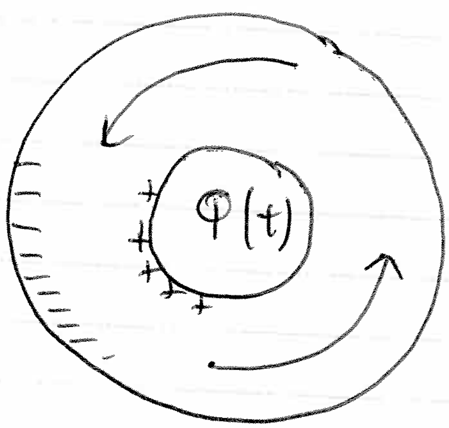
# Lecture 24

Continue the quantum Hall effect...

In lecture 23 we found that a quantum Hall system ~~[illegible]~~ appears to have both features of a perfect insulator and a perfect conductor (in the $T \to 0$ limit).

$$\rho = \begin{pmatrix} 0 & \frac{h}{\nu e^2} \\ -\frac{h}{\nu e^2} & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -\frac{\nu e^2}{h} \\ \frac{\nu e^2}{h} & 0 \end{pmatrix}$$

To understand this, consider the following thought experiment. Take a sample in the form of a Corbino ring:




In addition to a constant perpendicular magnetic field apply a time-dependent magnetic flux $\Phi(t)$, only through the hole in the ring (insert a solenoid). Time-dependent flux leads to azimuthal electric field in the sample:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{\partial \Phi}{\partial t}$$ – the integral is over some contour inside the sample, enclosing the hole.

$\sigma_{xx} = 0 \Rightarrow$ this field does not lead to azimuthal currents. But $\sigma_{xy} \neq 0 \Rightarrow$ it does lead to a radial current, which will in turn lead to radial polarization and radial electric field.

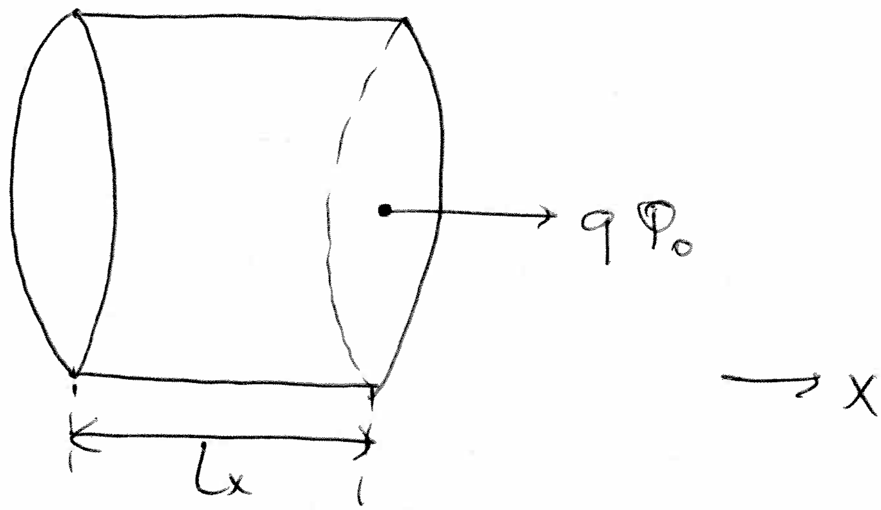
This radial electric field will lead to azimuthal current, which will flow without resistance since $\rho_{xx} = 0$.

The radial electric field is permanent: it can't decay through radial currents since $\sigma_{xx} = 0 \Rightarrow$ the sample behaves as a perfect insulator. But the azimuthal current also never decays $\Rightarrow$ the sample behaves as a perfect conductor at the same time.

Let us now derive quantization of the Hall resistivity. So far we have only explained the existence of the plateaus in $\rho_{xy}$ vs. $B$, but haven't shown why $\rho_{xy}$ has to be exactly quantized to $\frac{h}{\nu e^2}$.

We will use the beautiful argument due to R. B. Laughlin PRB 23, 5632 (1981).

Imagine that we take our 2DEG sample and bend it into a cylinder in the y-z plane:




Width is $L_x$, circumference is $L_y$.

The magnetic field $B$ is perpendicular to the surface of the cylinder at every point.

Apply electric field $\vec{E} = E_0 \hat{x}$ across the width of the cylinder.

In addition, add magnetic flux $\Phi = q \Phi_0$ through the hole of the cylinder, where $q$ is some number.

The additional vector potential, corresponding to this flux through the hole, is:

$$\delta \vec{A} = -q \frac{\Phi_0}{L_y} \hat{y}$$

$$\oint q \frac{\Phi_0}{L_y} \hat{y} \cdot d\vec{l} = q \Phi_0$$ – the integral is over a closed path on the cylinder, enclosing the hole.

The Hamiltonian becomes:

$$\mathcal{H}(q) = \frac{1}{2m}\left(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A} - q\frac{e}{c}\frac{\Phi_0}{L_y}\hat{y}\right)^2 + eE_0 x.$$

Note that since the curl of $\delta\vec{A}$ is zero, it doesn't change the perpendicular field $\vec{B}$.

$$\mathcal{H}(q) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} + \frac{eB}{c}x - q\frac{e}{c}\frac{\Phi_0}{L_y}\right)^2 + eE_0 x$$

look for solution of $\mathcal{H}\Psi = E\Psi$ in the same form as before:

$$\Psi(x,y) = e^{iky}\varphi(x)$$

We obtain:

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + \frac{m\omega_c^2}{2}\left(x + k\ell^2 - q\frac{\Phi_0}{BL_y}\right)^2\varphi + eE_0 x\,\varphi = E\varphi$$

$$q\frac{\Phi_0}{BL_y} = q\frac{2\pi\hbar c}{eBL_y} = \frac{2\pi q}{L_y}\ell^2$$

$$\frac{m\omega_c^2}{2}\left(x + k\ell^2 - \frac{2\pi q}{L_y}\ell^2\right)^2 + eE_0 x =$$

$$= \frac{m\omega_c^2}{2}\left[x^2 + 2x\left(k - \frac{2\pi q}{L_y}\right)\ell^2 + \left(k - \frac{2\pi q}{L_y}\right)^2\ell^4 + \right.$$

$$\left. + \frac{2eE_0}{m\omega_c^2}\,x\right] =$$

$$\cancel{\frac{eE_0}{m\omega_c^2} = \frac{eE_0}{m}\,\frac{m^2c^2}{e^2B^2} = \frac{mc^2E_x}{eB^2}}$$

$$= \frac{m\omega_c^2}{2}\left[x^2 + 2x\ell^2\left(k - \frac{2\pi q}{L_y} + \frac{eE}{m\omega_c^2\ell^2}\right) + \right.$$

$$\left. + \left(k - \frac{2\pi q}{L_y}\right)^2\ell^4\right] =^{*}$$

$$\frac{eE}{m\omega_c^2\ell^2} = \frac{eE}{m\,\frac{e^2B^2}{m^2c^2}\cdot\frac{\hbar c}{eB}} = \frac{mEc}{\hbar B} = \frac{m v_d}{\hbar}$$

$$\cancel{* = \frac{m\omega_c^2}{2}\left[x^2 + 2x\ell^2\left(k - \frac{2\pi q}{L_y}\right)\right.}$$

$v_d = \frac{cE}{B}$ – classical drift velocity in crossed electric and magnetic fields.

$$* = \frac{m\omega_c^2}{2}\left[x^2 + 2x\ell^2\left(k - \frac{2\pi q}{L_y} + \frac{m v_d}{\hbar}\right) + \right.$$

$$+ \left(k - \frac{2\pi q}{L_y} + \frac{m v_d}{\hbar}\right)^2\ell^4 - 2\frac{m v_d}{\hbar}\left(k - \frac{2\pi q}{L_y}\right)\ell^4 -$$

$$-\left(\frac{m v_d}{\hbar}\right)^2 \ell^4 \Big] =$$

$$= \frac{m\omega_c^2}{2}\left[X + \ell^2\left(k - \frac{2\pi q}{L_y} + \frac{m v_d}{\hbar}\right)\right]^2 \cancel{-}$$

$$\cancel{-}\ \frac{m\omega_c^2}{2}\cdot 2\,\frac{m v_d}{\hbar}\left(k - \frac{2\pi q}{L_y}\right)\ell^4 - \frac{m\omega_c^2}{2}\left(\frac{m v_d}{\hbar}\right)^2 \ell^4 =$$

$$= \frac{m\omega_c^2}{2}\left[X + \ell^2\left(k - \frac{2\pi q}{L_y} + \frac{m v_d}{\hbar}\right)\right]^2 -$$

$$- m^2\,\frac{e^2B^2}{m^2c^2\hbar}\,v_d\,\frac{\hbar^2c^2}{e^2B^2}\left(k - \frac{2\pi q}{L_y}\right) -$$

$$- \frac{m}{2\hbar^2}\,\frac{e^2B^2}{m^2c^2}\,m^2 v_d^2\,\frac{\hbar^2c^2}{e^2B^2} =$$

$$= \frac{m\omega_c^2}{2}\left[X + \ell^2\left(k - \frac{2\pi q}{L_y} + \frac{m v_d}{\hbar}\right)\right]^2 -$$

$$- \hbar v_d\left(k - \frac{2\pi q}{L_y}\right) - \frac{m v_d^2}{2}$$

Thus we finally obtain:

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + \frac{m\omega_c^2}{2}\left[X + \ell^2\left(k - \frac{2\pi q}{L_y} + \frac{m v_d}{\hbar}\right)\right]^2\varphi -$$

$$- \hbar v_d\left(k - \frac{2\pi q}{L_y}\right)\varphi - \frac{m v_d^2}{2}\varphi = E\varphi$$

Thus the eigenenergies are given by:

$$E_{nk} = \hbar\omega_c\left(n+\frac{1}{2}\right) - \hbar v_d\left(k-\frac{2\pi q}{L_y}\right) - \frac{m v_d^2}{2}$$

Eigenfunctions:

$$\varphi_{nk}(x) \sim e^{-\frac{1}{2\ell^2}\left[x+\ell^2\left(k-\frac{2\pi q}{L_y}+\frac{m v_d}{\hbar}\right)\right]^2} \cdot H_n\left[\frac{x+\ell^2\left(k-\frac{2\pi q}{L_y}+\frac{m v_d}{\hbar}\right)}{\ell}\right]$$

Thus the eigenstate energies now depend on $k$, and the center-of-mass of each eigenstate is shifted from $x=-\ell^2 k$ to $x=-\ell^2\left(k-\frac{2\pi q}{L_y}+\frac{m v_d}{\hbar}\right)$.

Note that $k=\frac{2\pi n_k}{L_y}$ due to periodic b.c.

Now imagine that we slowly (adiabatically) change $q$ from 0 to 1. As $q$ increases from zero, the center-of-mass positions of the eigenstates move to the right. When $q$ reaches 1, the term $\frac{2\pi q}{L_y}$ can be absorbed into $\frac{2\pi n_k}{L_y}$, i.e.

The system maps back onto itself.

$q=1$ correspond to flux $\Phi=\Phi_0$ – one flux quanta through the hole of the cylinder.
In general, any integer number of flux quanta through the hole will be "invisible" to the electrons.

But, due to the $\hbar v_d \left(k - \frac{2\pi q}{L_y}\right)$ term in $E_{nk}$, we have done work on the system in the process of changing $q$ from 0 to 1: the energy has increased by $\Delta E_{nk} = \frac{2\pi \hbar v_d}{L_y} = \frac{2\pi \hbar c E}{B L_y}$ per each eigenstate. This is due to the fact that we have moved electrons along the applied electric field E.

The total energy cost is $\Delta E = \Delta E_{nk} \cdot \nu \cdot N_\varphi =$

$$= \frac{2\pi \hbar c E}{B L_y} \cdot \nu \cdot \frac{L_x L_y}{2\pi l^2} = \frac{2\pi \hbar c E L_x}{B \cdot 2\pi} \, \frac{eB}{\hbar c} \nu =$$

$= e E L_x \nu = e V \nu$ – this is the energy cost of transferring one electron per each filled Landau level across the sample in the x-direction.

Now we can calculate the current in the x-direction if we connect the sample into a closed circuit.

We have:

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} = \oint \vec{E}_y \cdot d\vec{l} = \oint dl\, \rho_{yx} j_x$$

Here the integral is over a closed contour on the cylinder, enclosing the hole and the flux $\Phi(t)$, and we have used that $\rho_{xx} = 0$.

$$j_x = \frac{I_x}{l_y}$$ is the current density in the x-direction.

Integrate the above equation from $-\infty$ to $\infty$, assuming that the flux changes by one flux quantum:

$$\frac{1}{c}\Phi_0 = \oint dl\, \rho_{yx} \int_{-\infty}^{\infty} dt\, j_x = \rho_{yx} \int_{-\infty}^{\infty} dt\, I_x .$$

$\int_{-\infty}^{\infty} dt\, I_x = \Delta Q$ – the total charge, transferred from the left edge of the sample to the right in the process of changing the flux through the hole by $\Phi_0$.

As we have found, $\Delta Q = -\nu e$ – one electron per occupied Landau level.

Thus we obtain:

$$\rho_{yx} = -\frac{1}{c}\Phi_0 \frac{1}{\nu e} = -\frac{hc}{ec}\frac{1}{\nu e} = -\frac{h}{\nu e^2}$$

Thus we have derived the quantization of $\rho_{xy}$.

The above argument is very powerful, since it is very general and universal.

It's validity depends on the following three implicit assumptions that we have made:

1. Only extended states are sensitive to the flux through the hole of the cylinder: locally the flux is invisible since $\vec{\nabla} \times \delta\vec{A} = 0$ and it doesn't correspond to any physical magnetic field through the sample.

2. There is mobility gap, which ensures $\rho_{xx} = 0$.

3. After we adiabatically insert one flux quantum, the system after the insertion is exactly the same as before. Since localized states in the mobility gap do not feel the flux, the energy change can only be due to transferring electrons from one edge to the other.

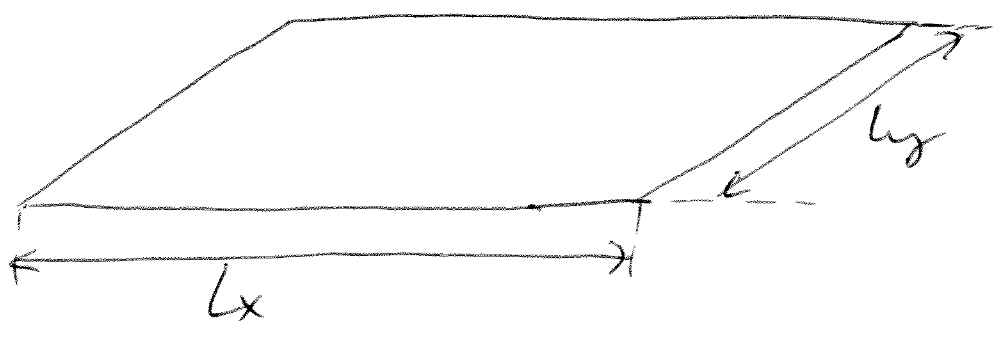
The last point implies that while there is a mobility gap in the bulk of the sample, the edges are gapless.

Laughlin argument demonstrates another very interesting property of the quantum Hall insulator: it has extended states, sensitive to the topology of the sample and such topology-related properties as flux through the hole. Such insulators are called topological insulators. Other topological insulators, not requiring magnetic field, exist in nature.

Precise quantization of $\rho_{xy}$ is a topological property – this is why it is so precise and insensitive to the details.

Another important issue that needs to be addressed: experimentalists directly measure not resistivity $\rho_{xy}$ but resistance $R_{xy}$.

Normally $R = \rho \frac{L}{A}$, i.e. even if $\rho_{xy}$ is precisely quantized, $R_{xy}$ doesn't have to be, since it depends on sample geometry. This is, however, not the case in 2D.




$$V_y = R_{yx} I_x \quad ; \quad R_{yx} = \frac{V_y}{I_x} = \frac{L_y E_y}{L_y j_x} =$$

$$= \frac{E_y}{j_x} = \rho_{yx} \quad ; \quad \text{i.e. in 2D } R_{yx} = \rho_{yx}.$$