Homework 8
Due Friday, December 4 in class

1. Consider a Heisenberg antiferromagnet on a 4-site linear chain. The Hamiltonian is:

$$H = J\left(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_3 \cdot \mathbf{S}_4 + \mathbf{S}_4 \cdot \mathbf{S}_1\right),$$

where $J > 0$ and $\mathbf{S}_i$ are spins of magnitude S.

   (a) Assuming the spins are classical vectors, sketch the arrangement of spins that minimizes the energy. Find the classical ground state energy $E_0^{c\ell}$.

   (b) Now assume the spins are quantum and show that the exact quantum mechanical ground state energy is given by:

$$E_0^{qm} = -4JS^2\left(1 + \frac{1}{2S}\right).$$

   Hint: start by showing that the Hamiltonian can be written as:

$$H = \frac{J}{2}\left[(\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4)^2 - (\mathbf{S}_1 + \mathbf{S}_3)^2 - (\mathbf{S}_2 + \mathbf{S}_4)^2\right].$$

   Now think how the spins in each of the three terms above should add up to minimize the energy.

   (c) Compare $E_0^{c\ell}$ and E_0^{qm} and comment on the result.

   (d) Now repeat the same calculation for a ferromagnetic chain with $J < 0$. Compare $E_0^{c\ell}$ and E_0^{qm} in this case. Comment on the difference from the antiferromagnetic chain case.

2. *Classical theory of spin waves.*
Consider Heisenberg model on a 3-dimensional cubic lattice:

$$H = -\frac{J}{2}\sum_{\langle ij\rangle} \mathbf{S}_i \cdot \mathbf{S}_j,$$

where $J > 0$, i and j are nearest-neighbor sites and $\mathbf{S}_i$ are classical vectors of length S, not quantum spin operators. Recall that the equation of motion for a spin can generally be written as:

$$\frac{d\mathbf{S}_i}{dt} = \mathbf{S}_i \times \mathbf{B}_i,$$

where $\mathbf{B}_i$ is the total magnetic field acting on the spin at site i. The field in this case is due to the nearest-neighbor spins, interacting with a given spin, and can be calculated as:

$$\mathbf{B}_i = -\frac{\partial H}{\partial \mathbf{S}_i} = J \sum_j \mathbf{S}_j,$$

where the sum over j is restricted to the nearest neighbors of i. Assume that $T = 0$ and the spins are polarized in the z-direction. Using the equation of motion above find the dispersion $\omega_{\mathbf{k}}$ of small fluctuations around the fully polarized state. Compare with the result for quantized spin fluctuations (magnons) we obtained in class.
Hint: linearize the equations of motion with respect to fluctuations, Fourier transform the linearized equations and find the eigenmodes.