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Asymptotics of Orthogonal Polynomials: Some Old, Some New, Some Identities

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Abstract. We briefly review some asymptotics of orthonormal polynomials. Then we derive the Bernstein–Szegő, the Riemann–Hilbert (or Fokas–Its–Kitaev), and Rakhmanov projection identities for orthogonal polynomials and attempt a comparison of their applications in asymptotics.

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1. An Overview of Results

Suppose that I is a finite or infinite interval and that $w\colon I \to [0, \infty)$ is a measurable function, positive on a set of positive measure, and with all moments

$$\int_I x^j w(x)\,\mathrm{d}x, \qquad j = 0, 1, 2, \ldots,$$

finite. Then we may define orthonormal polynomials

$$p_n(x) := p_n(w, x) = \gamma_n x^n + \cdots, \qquad \gamma_n > 0,$$

satisfying

$$\int_I p_n p_m w = \delta_{mn}. \tag{1}$$

One of their characteristic features is the three-term recurrence relation

$$xp_n(x) = A_{n+1} p_{n+1}(x) + B_n p_n(x) + A_n p_{n-1}(x), \tag{2}$$

where

$$A_n = \frac{\gamma_{n-1}}{\gamma_n} > 0; \qquad B_n = \int_I xp_n\,(x)\; p_{n-1}(x) w(x)\,\mathrm{d}x \in \mathbb{R}. \tag{3}$$

The number of applications of orthonormal polynomials seems to grow exponentially with time; they are useful in topics ranging from combinatorics to quantum mechanics; from graph theory to group representations; from numerical analysis to number theory; from statistical physics to signal processing. To review such a vast enterprise would require a proceedings even more extensive than that of the 1990 NATO conference [88].

Some of the applications involve orthogonal polynomials of fixed degree; others involve the behaviour of p_n as $n \to \infty$. In this paper, we shall focus on a small slice of the latter. Amongst the notable applications of these asymptotics are:

(I) analysis of linear predictors in the theory of stochastic processes [43];
(II) analysis of processes of approximation such as numerical integration, convergence of orthonormal expansions and polynomial and rational interpolation; [32, 79, 110];
(III) universality conjectures in random matrix theory [21, 26, 92];
(IV) investigation of numerical analysis algorithms for finding eigenvalues of matrices, for example the QD algorithm [52];
(V) Fisher–Hartwig conjectures related to Ising models with large numbers of particles [13, 14];
(VI) estimation of entropy in various contexts [4, 5].

In attempting to review the myriad of asymptotics that are currently available, one is forced to take account of two classifying features: the *region of validity*, that is the range of z for which the behavior of $p_n(z)$, $n \to \infty$ is being described; and the *strength of the asymptotic*.

Some insight into the former is provided by the Chebyshev polynomials

$$T_n(x) := \cos(n \arccos x) = 2^{n-1}x^n + \cdots, \quad n \geqslant 1,$$

with orthogonality relation

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{\pi}{2}\delta_{mn}, \quad m, n \geqslant 1.$$

The identity

$$T_n(\cos\theta) = \cos n\theta, \quad \theta \in [-\pi, \pi] \tag{4}$$

may be regarded as an exact asymptotic description of the behaviour of T_n throughout $[-1, 1]$. From this we may first deduce for $u = \mathrm{e}^{i\theta}$ and then for all $u \in \mathbb{C}\backslash\{0\}$, that

$$T_n\big(\tfrac{1}{2}(u + u^{-1})\big) = \tfrac{1}{2}(u^n + u^{-n}).$$

Solving the equation $z = \frac{1}{2}(u + u^{-1})$ leads to the familiar Joukowski or aerofoil map

$$u = \varphi(z) := z + \sqrt{z^2 - 1}, \quad z \in \mathbb{C}\backslash[-1, 1], \tag{5}$$

that maps the exterior of $[-1, 1]$ conformally onto the exterior of the unit ball. Since $|\varphi(z)| > 1$, we obtain

$$T_n(z) = \frac{1}{2}\big(\varphi(z)^n + \varphi(z)^{-n}\big) = \frac{\varphi(z)^n}{2}\big(1 + \mathrm{o}(1)\big), \quad n \to \infty,$$

uniformly in closed subsets of $\mathbb{C}\backslash[-1, 1]$. This exponential growth off the interval of orthogonality is fairly typical. Where the Chebyshev polynomials are atypical is the fact that the asymptotic formula (4) holds uniformly on the whole interval of orthogonality. In most cases, there is a different asymptotic behaviour inside I and close to its endpoints. Around the latter, the description of the asymptotic involves Bessel functions or Airy functions.

In summary, there are often three regions to distinguish when dealing with $I = [-1, 1]$, a finite interval:

(I) Asymptotics in the exterior $\mathbb{C}\backslash[-1, 1]$ of the interval of orthogonality;
(II) Asymptotics in suitable subintervals, for example compact subintervals, of $(-1, 1)$;
(III) Asymptotics close to the endpoints ± 1 of I.

In describing the strength of an asymptotic, one tends to focus on the exterior $\mathbb{C}\backslash[-1, 1]$, and also on the leading coefficient γ_n. It is a remarkable feature that in many cases knowledge of the behaviour of γ_n alone is enough to derive asymptotics for $p_n(z)$ for $z \in \mathbb{C}\backslash[-1, 1]$. Since p_n behaves at ∞ like $\gamma_n z^n$ and maximum modulus principles may be applied to $p_n(z)/\varphi(z)^n$ in $\overline{\mathbb{C}}\backslash[-1, 1]$, this is not all that surprising. The following table outlines the four main asymptotics of γ_n and p_n, $n \to \infty$, off the interval of orthogonality.

Name	γ_n	$p_n(z),\ z \notin [-1, 1]$
(I) nth root	$\gamma_n^{1/n} \to 2$	$p_n(z)^{1/n} \to \varphi(z)$
(II) ratio	$A_n = \frac{\gamma_{n-1}}{\gamma_n} \to \frac{1}{2};\ B_n \to 0$	$\frac{p_{n+1}(z)}{p_n(z)} \to \varphi(z)$
(III) Szegő/power	$\gamma_n/2^n \to c_0$	$p_n(z)/\varphi(z)^n \to g(z)$
(IV) strong Szegő	$\Big(\prod_{j=1}^n \gamma_j\Big)/\Big(2^{\frac{n(n+1)}{2}} c_0^n n^{c_1}\Big) \to c_2$	

Here c_j, $0 \leqslant j \leqslant 2$, are positive constants, and g is an explicitly given function. It is fairly clear that

$$(\text{IV}) \Rightarrow (\text{III}) \Rightarrow (\text{II}) \Rightarrow (\text{I})$$

and hardly surprising that none of the converse relations holds in general. The relationship between nth root and ratio asymptotics is similar to that between the nth root test and ratio test for convergence of power series. Let us expand a little on (I)–(IV).

In terms of asymptotics on $I = [-1, 1]$, the nth root asymptotic is associated with

$$\limsup_{n\to\infty} \big|p_n(x)\big|^{1/n} = 1, \quad x \in [-1, 1]\backslash\mathcal{E},$$

where $\mathcal{E}$ has linear Lebesgue measure 0 (and in fact logarithmic capacity 0).

One of the most attractive features of nth root asymptotics is their link to distribution of zeros of p_n. Let us write

$$p_n(x) = \gamma_n \prod_{j=1}^{n} (x - x_{jn}),$$

where the zeros $\{x_{jn}\}_{j=1}^{n}$ all lie in $[-1, 1]$. Then (I) above is equivalent to the zeros displaying arc sine distribution: $\forall -1 \leqslant a < b \leqslant 1$,

$$\lim_{n\to\infty} \frac{1}{n} \#\{j : 1 \leqslant j \leqslant n \text{ and } x_{jn} \in [a, b]\} = \int_a^b \frac{\mathrm{d}t}{\pi\sqrt{1-t^2}}.$$

It was Faber who in 1922 [29], first established nth root asymptotics, under fairly severe conditions on w. A major advance was made by Erdős and Turan, in the course of investigating convergence of Lagrange interpolation in the late 1930's [27, 28]. They assumed that $I = [-1, 1]$ and $w > 0$ a.e. on $[-1, 1]$. This was the beginning of intensive efforts to characterize nth root asymptotics in terms of the weight w, involving authors such as A. Ambroladze, P. Erdős, G. Freud, P. Korovkin, H. Stahl, V. Totik, J. Ullman, H. Widom and M. Wyneken, culminating in the recent monograph of Stahl and Totik [110], see also [2]. This monograph deals with the far more general case of orthogonal polynomials corresponding to a measure with compact support in the complex plane. It also presents applications to rational and Padé approximation.

Ratio asymptotics are historically probably the latest arrivals of (I)–(IV). A most penetrating study of these was presented in a memoir of P. Nevai [84]. There the equivalence of the asymptotic for $p_{n+1}(z)/p_n(z)$ and that for the recurrence coefficients, namely

$$\lim_{n\to\infty} A_n = \frac{1}{2}; \qquad \lim_{n\to\infty} B_n = 0 \tag{6}$$

was established. It is less obvious than the corresponding link between asymptotics for γ_n and p_n in (I). One of the main achievements in ratio asymptotics has been Rakhmanov's theorem, which establishes (6) under merely the Erdős–Turan criterion, namely that $w > 0$ a.e. on $[-1, 1]$ [102, 103].

The main focus of Nevai's memoir was to place hypotheses on $\{A_n\}$, $\{B_n\}$, such as (6), rather than on the weight w. This is a very important alternative starting point for analysis of orthonormal polynomials and continues to attract a lot of attention, especially when more general hypotheses than (6) are placed on the recurrence coefficients – for example when they are asymptotically periodic, or display some other definite asymptotic pattern. The most recent monograph on the topic seems to be Van Assche's lecture notes of 1987 [119]. There is certainly a need for an extensive survey article, or even monograph, to cover subsequent developments due to Van Assche, Aptekarev, Geronimo, Golinsky, Lopez, Magnus, Mate, Nevai, Totik, Peherstorfer and many others [6, 8, 34 – 36, 38 – 40, 42, 98, 100, 119, 120].

The next step up is the Szegő or power asymptotics (III). Historically, simultaneously with Faber's work, it was also the first general asymptotic: it appeared in 1920–21 in papers that did not even contain the phrase 'orthogonal polynomials' in the title. If a vote was conducted as to the crowning achievement of orthogonal polynomials in the 20th century, it would almost certainly be Szegő's theory (with more than a two-thirds majority). It has had ramifications that stretch from analytic function theory (Hardy spaces) to best approximation to ... zero distribution. Let $w\colon [-1,1] \to \mathbb{R}$ satisfy Szegő's condition

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}}\,\mathrm{d}x > -\infty \tag{7}$$

or, equivalently,

$$\int_{0}^{\pi} \log w(\cos\theta)\,\mathrm{d}\theta > -\infty.$$

Let

$$f(\theta) := w(\cos\theta)|\sin\theta|, \quad \theta \in [-\pi, \pi] \tag{8}$$

and define the associated *Szegő function*

$$D(f; z) := \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \log f(t)\frac{\mathrm{e}^{it}+z}{\mathrm{e}^{it}-z}\,\mathrm{d}t\right), \quad |z| < 1. \tag{9}$$

Many would recognize this as an *outer function* from the theory of Hardy spaces (apart from 4 replacing 2). Szegő proved that

$$\lim_{n\to\infty} p_n(z)/\varphi(z)^n = \frac{1}{\sqrt{2\pi}} D\left(f; \frac{1}{\varphi(z)}\right)^{-1}, \tag{10}$$

uniformly for z in closed subsets of $\mathbb{C}\backslash[-1,1]$ and, consequently, taking $z = \infty$,

$$\lim_{n\to\infty} \gamma_n/2^n = \frac{1}{\sqrt{2\pi}} D(f; 0)^{-1}. \tag{11}$$

There is also a remarkable converse: if $p_n(z)/\varphi(z)^n$ is bounded independently of n on some contour enclosing $[-1,1]$ in its interior, then (7) must be true. In fact a lot less is required than uniform boundedness on such a contour. In terms of asymptotics on $[-1,1]$, (10) or (11) are essentially equivalent to the mean asymptotic

$$\lim_{n\to\infty} \int_{0}^{\pi} \left| p_n(\cos\theta) f(\theta)^{1/2} - \sqrt{\frac{2}{\pi}} \cos\big(n\theta + \arg D(f; \mathrm{e}^{i\theta})\big)\right|^2 \mathrm{d}\theta = 0. \tag{12}$$

If instead of this asymptotic in the mean, one wants an asymptotic that holds uniformly for $x = \cos\theta$ in a compact subinterval of $(-1,1)$, one needs to assume more than just (7): a weak smoothness condition, such as a Dini–Lipschitz condition suffices [32, 112]. The treatment of asymptotics near ± 1 is far more delicate and generally has been established under more severe hypotheses. For special weights such as Jacobi weights, one may apply steepest descent to integral representations. Another promising approach, involving hypotheses on the recurrence coefficients, has been given by Aptekarev [3]. The Riemann–Hilbert techniques of Deift *et al*. [21 – 26, 49] seem to offer a new approach for this question. Again, the topic of pointwise asymptotics on $[-1, 1]$ is a whole subject on its own and could well do with a lengthy survey.

Szegő's theory has been extended by Widom [122] to weights with support on several intervals; at least one conjecture that Widom left unsolved was recently resolved by Peherstorfer [97]. Other extensions have been to the arcs of a circle [41], curves in the plane, ... While Szegő's original theory is presented at an accessible level in several monographs, it is a pity that its ramifications – especially Widom's theory – has not received an 'entry level' treatment.

Some 30 years after his seminal papers of the 1920's, Szegő published another celebrated work. He showed that it is possible to strengthen (III), giving (IV), the first example of the *strong Szegő limit theorem*. He presented his results within the framework of orthogonal polynomials on the unit circle, but they can be translated to the form above [43, p. 91]. Szegő assumed that f of (8) has a derivative f' that satisfies a Lipschitz condition of some positive order in $[-\pi, \pi]$. Subsequently this hypothesis was weakened by especially Widom, who gave an alternative proof; and by many others, including Tracy, Basor, Böttcher, Silbermann, Spitkovskii, Golinskii, etc.

Strong Szegő Limit Theorems have applications in statistical physics, in Ising models, where f has some sort of singularity, such as a jump discontinuity. There the Fisher–Hartwig conjectures were formulated, and as far I know, they have still not been totally resolved. The little book of Grenander and Szegő remains an excellent introduction; more modern developments are covered by Böttcher and Silbermann at a relatively introductory level in [14]. A deeper treatment is given in their earlier monograph [13].

Of course there are many important asymptotics for weights on compact sets that do not fit into the above classification. Amongst them are the comparative asymptotics studied by Mate, Nevai, Totik, Lopez, Golinskii, Rakhmanov, Peherstorfer [42, 62, 63, 75 – 77, 99, 105] and others; the asymptotics for varying weights that have so many applications, including to orthogonal rational functions [15, 63, 90, 91]; results on zero distribution due to Mhaskar, Saff [83] and others; orthogonal polynomials associated with measures with discrete support [9, 53]; and those associated with q-series; finer asymptotics for classical weights, orthogonal polynomials for Sobolev inner products, ...

There are still interesting problems regarding asymptotic behaviour of orthogonal polynomials associated with weights satisfying Szegő's condition on the unit circle: how do they behave *inside* the unit circle (not outside) as the degree approaches ∞? See [7].

While the developments associated with weights on a finite interval in the last three decades have been impressive, those associated with weights on infinite intervals have been spectacular. Although there is nothing as complete as Szegő's theory (and probably there never will be), there has been the development of asymptotics of all types, valid under very general conditions. A model is provided by the exponential weights

$$w(x) := \exp(-2|x|^{\alpha}), \quad x \in \mathbb{R},\ \alpha > 0.$$

This led to the idea of writing very general weights in exponential form:

$$w(x) = \exp\bigl(-2Q(x)\bigr), \quad x \in I,$$

where $I = (c, d)$ may be finite or infinite. Even more fundamental has been the systematic use of potential theory, in a form designed to deal with the exponent or *external field* Q.

One of the striking implications of the potential theory is that when working with weighted polynomials $P\exp(-Q)$ (P of degree $\leqslant n$), all the interesting features occur on an interval that depends on Q and on the degree n, but not on the particular P. This had been used in the 1970's by G. Freud and P. Nevai [87], but potential theory enables one to find the exact interval of interest. Suppose for example that Q is convex, and that $Q(x)$ and $|Q'(x)| \to \infty$ as $x \to c, d$. Then one may define for $r > 0$, the *Mhaskar–Rakhmanov–Saff numbers* $a_{-r} < a_r$ by the equations

$$\begin{aligned} r &= \frac{1}{\pi}\int_{a_{-r}}^{a_r} \frac{xQ'(x)}{\sqrt{(x-a_{-r})(a_r-x)}}\,\mathrm{d}x; \\ 0 &= \frac{1}{\pi}\int_{a_{-r}}^{a_r} \frac{Q'(x)}{\sqrt{(x-a_{-r})(a_r-x)}}\,\mathrm{d}x. \end{aligned} \tag{13}$$

It is of course not obvious that $a_{\pm r}$ exist or are unique, but this is the case. One may also show that $a_{-r} \to c$; $a_r \to d$ as $r \to \infty$.

One illustration of their importance is the Mhaskar–Saff identity [80–82]

$$\|P\mathrm{e}^{-Q}\|_{L_\infty(I)} = \|P\mathrm{e}^{-Q}\|_{L_\infty[a_{-n},a_n]}, \tag{14}$$

valid for all polynomials P of degree $\leqslant n$. Moreover, $[a_{-n}, a_n]$ is asymptotically the smallest interval for this to hold, and $P\mathrm{e}^{-Q}$ decays exponentially as we recede from $[a_{-n}, a_n]$. The connection with orthonormal polynomials becomes more obvious when one considers the L_2 analogue:

$$\begin{aligned} \left(\int_I P^2 w\right)^{1/2} &= \|P\mathrm{e}^{-Q}\|_{L_2(I)} \leqslant C\|P\mathrm{e}^{-Q}\|_{L_2[a_{-n},a_n]} \\ &= C\left(\int_{a_{-n}}^{a_n} P^2 w\right)^{1/2}, \end{aligned}$$

where C is independent of P and n. Moreover, if $\varepsilon > 0$ is fixed, the contribution of the integral over $I \backslash [a_{-n}(1+\varepsilon), a_n(1+\varepsilon)]$ decays exponentially as $n \to \infty$. This has the consequence that most of the interesting features of weighted polynomials $P\mathrm{e}^{-Q}$ occur in, or close to, $[a_{-n}, a_n]$. In particular, all the zeros of $p_n(x)$ lie in $(a_{-n-1/2}, a_{n+1/2})$ [59].

Consider the simplest example

$$Q(x) = |x|^\alpha, \quad x \in \mathbb{R}.$$

Here as Q is even, $a_{-r} = -a_r$ and (13) simplifies to

$$r = \frac{2}{\pi} \int_0^1 \frac{a_r x Q'(a_r x)}{\sqrt{1-x^2}} \, \mathrm{d}x.$$

Solving gives

$$a_r = \left(r \left[\frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha+1}{2})} \right] \right)^{1/\alpha}.$$

Another example is the non-even Q,

$$Q(x) := \begin{cases} x^\alpha, & x \in (0, \infty); \\ |x|^\beta, & x \in (-\infty, 0]. \end{cases}$$

Suppose for example that $\beta \geqslant \alpha > 1$. Then [16, 17] for some constants $c_1, c_2 > 0$

$$a_{-r} = -c_1 r^{\frac{1}{\alpha}\frac{2\alpha-1}{2\beta-1}}; \quad a_r = c_2 r^{\frac{1}{\alpha}}.$$

The asymptotics associated with exponential weights are inherently more complicated than those for $[-1, 1]$. In their description, it is useful to map $[-1, 1]$ linearly onto $[a_{-n}, a_n]$: let $\delta_n := \frac{1}{2}(a_n - a_{-n})$ and

$$\begin{aligned} x &= L_n(s) := (s+1)\delta_n + a_{-n}, \quad s \in [-1, 1] \\ &\iff s = L_n^{[-1]}(x) = \frac{1}{\delta_n}\left[x - \frac{a_n + a_{-n}}{2}\right], \quad x \in [a_{-n}, a_n]. \end{aligned}$$

One then may regard

$$w(x) = \exp\bigl(-2Q(x)\bigr) = \exp\bigl(-2Q\bigl(L_n(s)\bigr)\bigr),$$

as a sequence of weights in the variable $s \in [-1, 1]$. Moreover, all the zeros of the transformed polynomial $p_n(L_n(z))$ lie in, or very close to, $[-1, 1]$. Effectively this is a normalizing transformation – and normalization plays such an important role in so many different types of asymptotic behaviour. In the case of non-even Q, the linear map was first extensively used by Buyarov and Rakhmanov [16 – 18].

Following is a partial list of asymptotics as $n \to \infty$:

Name	γ_n	$p_n(L_n(z)),\ z \notin [-1,1]$
(I) nth root	$\gamma_n^{1/n}\delta_n \to c_0$	$p_n(L_n(z))^{1/n} \to c_1 \exp(U(\nu; z))$
(II) ratio	$\frac{\gamma_{n-1}}{\gamma_n \delta_n} \to \frac{1}{2};\ \frac{B_n}{\delta_n} \to 0$	$\frac{p_{n+1}(L_n(z))}{p_n(L_n(z))} \to g(z)$
(III) Szegő/power	$\gamma_n (\frac{\delta_n}{2})^{n+1/2} D(f_n; 0) \to c_2$	$\frac{\delta_n^{1/2} p_n(L_n(z))}{\varphi(z)^n} D\Big(f_n; \frac{1}{\varphi(z)}\Big)^{-1} \to h(z)$
(IV) strong Szegő	$c_2^n n^{c_4} \prod_{j=1}^{n} \Big[\gamma_j \Big(\frac{\delta_j}{2}\Big)^{j+\frac{1}{2}} D(f_j; 0)\Big] \to c_5$	

Here the constants c_j, $0 \leqslant j \leqslant 4$ and the functions f_n, g, h, U may be given an explicit representation under suitable hypotheses on w. For example, U is an exponential of a potential,

$$U(z) = \exp\left(\int_{-1}^{1} \log|z-t|\nu(t)\,\mathrm{d}t\right), \tag{15}$$

where ν is a non-negative density function of total mass 1: $\nu \geqslant 0$ and

$$\int_{-1}^{1} \nu = 1.$$

The function f_n appearing in the power asymptotic is

$$f_n(\theta) := w\big(L_n(\cos\theta)\big)|\sin\theta|, \quad \theta \in [-\pi, \pi].$$

Historically, remarkably precise results were obtained for the Hermite weight $\exp(-x^2)$ by Plancherel–Rotach starting in the 1920's with many other later contributions. Weights such as $\exp(-x^{2m})$, $m = 1, 2, 3, \ldots$ received a detailed and very precise treatment in the early 1980's [12, 46, 47, 78, 85, 109]. Moreover, starting from hypotheses on the recurrence coefficients $\{A_n\}$, $\{B_n\}$, Nevai and Dehesa [89] had obtained one form of zero distribution in the 1970's, while J. Ullman had started to use potential theory on the problem [116, 117].

However, the first very general asymptotics were obtained independently by E. A. Rakhmanov [104] and H. N. Mhaskar and E. B. Saff [80–82] in the early 1980's. E. A. Rakhmanov considered weights w, where

$$\lim_{|x|\to\infty} \left(\log\frac{1}{w(x)}\right)\Big/|x|^{\alpha} = c > 0.$$

(Mhaskar and Saff treated the underlying exponential weight $\exp(-2|x|^{\alpha})$ directly for all $\alpha > 0$.) Here the weight is asymptotically equivalent to the even weight $\exp(-2|x|^{\alpha})$, apart from a scaling on x, and consequently the linear transformation L_n above simplifies substantially, to

$$L_n(z) = a_n z = \mathrm{c}n^{1/\alpha} z.$$

Moreover, the density function ν, which in this case is called the Ullman or Nevai–Ullman density, admits the simple representation

$$\nu(x) = \frac{\alpha}{\pi}\int_{|x|}^{1} \frac{t^{\alpha-1}}{\sqrt{t^2-x^2}}\,\mathrm{d}t, \quad x \in (-1,1).$$

Rakhmanov also established logarithmic asymptotics such as

$$\lim_{n\to\infty} \log\left|p_n(z)\right|/n^{1-1/\alpha} = c|\operatorname{Im} z|,$$

and even log-log asymptotics

$$\lim_{n\to\infty} \frac{\log\log|p_n(z)|}{\log n} = 1 - \frac{1}{\alpha}.$$

The latter was established for $z \in \mathbb{C}\backslash\mathbb{R}$ when the weight w satisfies

$$\lim_{|x|\to\infty} \frac{\log\log\frac{1}{w(x)}}{\log|x|} = \alpha > 1.$$

This last condition is reminiscent of the formula defining the order of an entire function. Indeed, there are points of contact between the theory of entire functions and weights on the real line, not least in handling rates of growth and decay at ∞.

The papers [80 – 82, 104] led to a rapid series of developments. In the nth root line, extensions to very general exponential weights have been given by Buyarov, Gonchar, Mhaskar, Rakhmanov, Saff, Totik and others. See [79, 108]. Despite the great generality of the results, one would guess that a complete treatment is a long way off. After all, even for weights with compact support, advances are still being made. In many cases, one has to replace the asymptotic for p_n in (I) above by

$$p_n\big(L_n(z)\big)^{1/n} - c_n \exp\big(U(\nu_n; z)\big) \to 0, \quad n \to \infty,$$

where $c_n > 0$, $n \geqslant 1$, and the density function ν_n varies with n. In the case of convex Q,

$$\nu_n(x) = \frac{\delta_n\sqrt{1-x^2}}{n\pi^2}\int_{-1}^{1} \frac{Q'(L_n(s)) - Q'(L_n(x))}{s-x}\frac{\mathrm{d}s}{\sqrt{1-s^2}}, \quad x \in [-1, 1].$$

As is the case with weights with compact support, there is a close relationship between nth root asymptotics and zero distribution. Indeed the papers of Mhaskar and Saff and Rakhmanov, as well as the earlier work of Dehesa and Nevai and Ullmann concentrated on zero distribution. The scaling involving $L_n(z)$ is really a normalization, and leads to the most natural formulation of distribution of zeros. However, other scalings have been used, and these do lead to interesting and significant results, most notably for Q decaying very slowly, in which case the usual scaling does not yield much information [50, 118, 121].

Not long after the first nth root asymptotics, ratio asymptotics were established by the author and Mhaskar and Saff [70, 71], thereby resolving the Freud conjectures for the weights $w(x) = \exp(-2|x|^{\alpha})$, $\alpha > 0$. The latter asserted that the recurrence coefficients $\{A_n\}$ satisfy

$$\lim_{n\to\infty} A_n/n^{1/\alpha} = c_\alpha > 0,$$

or in terms of the Mhaskar–Rakhmanov–Saff numbers associated with this weight,

$$\lim_{n\to\infty} A_n/a_n = \frac{1}{2}.$$

Of course, because the weight is even, $B_n = 0$ and $a_{-n} = -a_n$. (The conjecture had earlier been established by Alphonse Magnus [73] for α a positive even integer and subsequently refined to an asymptotic expansion by Mate, Nevai and Zaslavsky [78].) Beautiful and important asymptotics involving hypotheses on the recurrence coefficients for weights with non-compact support have been established by Van Assche, Geronimus, Kuijlaars, and others [36, 50, 55]. Some of the latter are closer in spirit to Szegő, rather than ratio, asymptotics.

Somewhat surprisingly the techniques used to prove the ratio asymptotics turned out to be sufficient for the stronger Szegő asymptotics [72]. The methods involved the classical Bernstein–Szegő identities, which will be discussed in the next section, and weighted polynomial approximations. E. A. Rakhmanov announced asymptotics of this type in [65], but the proofs appeared later [106] – and contained fundamentally new ideas.

The most general conditions to date for establishing Szegő asymptotics have been given by Totik in his seminal lecture notes [113], which gave a new approach to constructing weighted polynomial approximations. These ideas have been used by the author and A. L. Levin in discussing exponential weights on a general (not necessarily infinite interval) I [61].

In terms of asymptotics for $p_n(x)$ for $x \in I$, there is a mean asymptotic involving values of $p_n(x)$ for $x \in [a_{-n}, a_n]$ that is similar to (12). Asymptotics that hold uniformly on suitable proper subintervals of I were established for general classes of weights by E. A. Rakhmanov [106] and the author [66] in the late 1980's. Recent work in this direction, for fairly general even and non-even exponential weights on a finite or infinite interval I, are announced in [60] and will appear in [61].

In terms of sharper asymptotics, that imply strong Szegő limit theorems, there are the exciting new results of the group of Deift, Kriecherbauer, McLaughlin *et al.*, that use Riemann–Hilbert techniques [21 – 26, 49]. These will be briefly discussed in Section 4. These methods are fundamentally new and different, and promise to revolutionise much of the asymptotic theory of orthogonal polynomials. The asymptotic (IV) in the table above follows from recent results of Kriecherbauer and McLaughlin for γ_n for the weights $\exp(-2|x|^\alpha)$, $\alpha > 0$ [49]. For the case of α a positive even integer, it is implied by the asymptotic expansions of Mate, Nevai and Zaslavsky for A_n [78]. What is interesting though, is that in both cases, the results come out of a finer asymptotic for γ_n or A_n, rather than out of an approach involving Toeplitz determinants, and so is quite different from the techniques used on the unit circle or $[-1, 1]$. Other powerful techniques that provide sharper asymptotics for the recurrence coefficients have been explored by Wong and his coworkers [11, 101] and by Chen, Ismail and Van Assche [20].

Not only do the Riemann–Hilbert methods imply strong Szegő limit theorems, but they also yield asymptotics for the orthonormal polynomials in all parts of the plane. In particular they yield asymptotics near the endpoints of $[a_{-n}, a_n]$, the first methods to do this in some degree of generality. A recent lecture notes of P. Deift [21] provides a clear introduction to the Riemann–Hilbert method.

2. Identities

Amongst the many tools that have been used in asymptotics of orthogonal polynomials, possibly the most important in the last twenty years has been potential theory. A detailed discussion of its application is available elsewhere [56, 79, 108, 110]. So in this section, we shall focus on another key ingredient of asymptotics: identities for special weights that are useful for general classes of weights.

The philosophy is very simple: suppose that we know for a special weight, v say, an explicit expression for its nth orthonormal polynomial, which we denote by $q_n(x)$. Suppose also that for a given weight w, we wish to compute the behaviour of $p_n(x) = p_n(w, x)$ as $n \to \infty$. Then if we can ensure that $w \simeq v$, that is w is close to v, then one expects that $p_n(x) \approx q_n(x)$. Of course, to justify this involves a lot of technical detail. One of the oldest and still most useful ways to proceed rigorously, is to use Korous' identity. It is based on the reproducing kernel

$$K_n(v, x, t) := \sum_{j=0}^{n-1} q_j(x) q_j(t).$$

Let $\gamma_n(v)$ and $\gamma_n(w)$ denote respectively the leading coefficients of $q_n(x)$ and $p_n(x)$. Then since the next left-hand side has degree $\leqslant n - 1$,

$$\begin{aligned} p_n(x) - \frac{\gamma_n(w)}{\gamma_n(v)} q_n(x) &= \int_I K_n(v, x, t) \left[p_n(t) - \frac{\gamma_n(w)}{\gamma_n(v)} q_n(t) \right] v(t)\, \mathrm{d}t \\ &= \int_I K_n(v, x, t) p_n(t) v(t)\, \mathrm{d}t \\ &= \int_I K_n(v, x, t) p_n(t) \left[v(t) - \frac{v(x)}{w(x)} w(t) \right] \mathrm{d}t, \end{aligned}$$

by orthogonality. We next need the Christoffel–Darboux formula

$$K_n(v, x, t) = \frac{\gamma_{n-1}(v)}{\gamma_n(v)} \frac{q_n(x) q_{n-1}(t) - q_{n-1}(x) q_n(t)}{x - t}.$$

Let us also define

$$R(t, x) := \frac{1 - \frac{v(x)}{w(x)} \frac{w(t)}{v(t)}}{x - t}.$$

We see that

$$p_n(x) - \frac{\gamma_n(w)}{\gamma_n(v)} q_n(x)$$

$$= \frac{\gamma_{n-1}(v)}{\gamma_n(v)} \left[q_n(x) \int_I q_{n-1}(t) p_n(t) R(t,x) v(t)\, dt - \right.$$
$$\left. - q_{n-1}(x) \int_I q_n(t) p_n(t) R(t,x) v(t)\, dt \right]. \tag{16}$$

Now if $w \approx v$, then $R(t, x)$ is small in some sense. If, moreover, we have for the special weight v, bounds on $q_n(x)$ and $(\gamma_{n-1}(v))/\gamma_n(v)$, we may then use Cauchy–Schwarz on the integrals in (16) and orthonormality of p_n with respect to w to show that the left-hand side of (16) is small. As sketched here, this requires global estimates for v/w, which are not always available. More sophisticated 'local' versions of this argument are often applied [76, 77, 99, 105].

How do we choose the special weights, for which an identity is available? In this section, we shall discuss three classes of identities that have yielded impressive results for exponential weights:

(I) Bernstein–Szegő;
(II) Fokas–Its–Kitaev (Riemann–Hilbert);
(III) Rakhmanov's projection identity.

The next section contains a detailed treatment of Bernstein–Szegő identities, especially as this is the technique most used by this author. In Section 2.2, we present an application to universality limit relations, in a very special case. In Section 2.3, we establish the Fokas–Its–Kitaev identity and briefly discuss its spectacular application to the weights $\exp(-|x|^\alpha)$, $\alpha > 0$. Finally, in Section 2.4, we discuss an identity of Rakhmanov. An attempt at comparing the applications of these three identities is given in Section 2.5.

2.1. BERNSTEIN–SZEGŐ IDENTITIES

A Bernstein–Szegő weight has the form

$$w(x) = \sqrt{1 - x^2}/S(x), \quad x \in (-1, 1),$$

where S is a polynomial that is positive on $[-1, 1]$, except possibly for simple zeros at ± 1. The identities associated with them have been very widely applied in deriving asymptotics for more general weights. They have analogues on arcs of the unit circle and have been extended, for example, by N. I. Achieser [1] and later by F. Peherstorfer to weights supported on finitely many disjoint intervals [93–95]. There are generalisations, due to P. Nevai [86], where the polynomial S is replaced by an expression involving a Hilbert transform. It is surprising that many of these extensions have been done so recently!

In this section, we state and prove some of the identities associated with classical Bernstein–Szegő weights on $[-1, 1]$, in several different formulations. Since many of these are stated and proved in most of the standard texts, the reader may well ask why? The reason is that some of the formulations are different; moreover, some, such as that for the Christoffel functions, are either inaccessible, or 'hidden away' in the classical texts. The explicit formulae for these special weights also illustrate asymptotics that hold more generally. We emphasise that our treatment is not complete because of our concentration on the classical form – but that is the one most used in asymptotics.

We begin by recalling the representation of a positive trigonometric polynomial: let $s(\theta)$ be a trigonometric polynomial of degree k that is positive in $[-\pi, \pi]$. We may write

$$s(\theta) = \left|h(\mathrm{e}^{i\theta})\right|^2, \quad \theta \in [-\pi, \pi], \tag{17}$$

where h is an algebraic polynomial of degree k, with $h(0) > 0$, and with all its zeros in $\{z : |z| > 1\}$. The proof is elementary [112, p. 4]: we may write

$$s(\theta) = \mathrm{e}^{-ik\theta} H(\mathrm{e}^{i\theta}),$$

where $H(z)$ is an algebraic polynomial of degree $2k$. One may choose for an appropriate c,

$$h(z) := c \prod_{a:\ H(a)=0 \text{ and } |a|<1} (z - a).$$

Now suppose that S is an algebraic polynomial of degree ℓ, positive in $[-1, 1]$ except possibly for simple zeros at ± 1. Then $S(\cos\theta)$ is a trigonometric polynomial of degree ℓ involving only cosine terms. If $S(x)$ has a zero at ± 1, we may factor out a term $\pm 1 - x$, and then apply (17) to deduce that

$$S(\cos\theta) = \left|h(\mathrm{e}^{i\theta})\right|^2, \quad \theta \in [-\pi, \pi], \tag{18}$$

where h is an algebraic polynomial of degree ℓ, with $h(0) > 0$, and with all zeros in $\{z : |z| > 1\}$, except possibly for simple zeros at ± 1 corresponding to zeros of S at ± 1. (For the factor $1 - x = 1 - \cos\theta$, we may write, with $z = \mathrm{e}^{i\theta}$,

$$1 - \cos\theta = \frac{-1}{2z}(z - 1)^2.)$$

We also note that since $S(\cos\theta)$ involves only cosine terms, the coefficients of h are real.

THEOREM 2.1. *Let S be a polynomial of degree ℓ, positive in* $[-1, 1]$, *except possibly for simple zeros at* ± 1, *and let*

$$w(x) := \frac{\sqrt{1 - x^2}}{S(x)}, \quad x \in (-1, 1). \tag{19}$$

Represent S in the form (18). *Then for $n > \ell/2$, $x = \cos\theta$ and $z = \mathrm{e}^{i\theta}$,*

$$p_n(x) = \sqrt{\frac{2}{\pi}} (\sin\theta)^{-1} \operatorname{Im}\left\{z^{n+1}\overline{h(z)}\right\}; \tag{20}$$

$$\gamma_n = \sqrt{\frac{2}{\pi}} h(0) 2^n. \tag{21}$$

Proof. Let us denote the right-hand side of (20) by $p(x)$. We first note that as h has real coefficients, say

$$h(z) = \sum_{j=0}^{\ell} h_j z^j,$$

then at least if $n \geqslant \ell$,

$$p(x) = p(\cos\theta) = \sqrt{\frac{2}{\pi}} \sum_{j=0}^{\ell} h_j \frac{\sin(n+1-j)\theta}{\sin\theta} = \sqrt{\frac{2}{\pi}} \sum_{j=0}^{\ell} h_j U_{n-j}(\cos\theta),$$

where U_k denotes the Chebyshev polynomial of the second kind of degree k, so that

$$U_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta}.$$

As U_n has leading coefficient 2^n, we see that p is a polynomial of degree n with leading coefficient given by (21). If $\ell/2 < n < \ell$, then we may express those terms in the sum with $n - j < 0$ as 0 for $n - j = -1$ and as $-h_j U_{j-n-2}(\cos\theta)$, for $n - j \leqslant -2$. Then we see that p is still of degree n.

We now establish the orthogonality relations. Write

$$\int_{-1}^{1} p U_k w = \frac{1}{2} \int_{-\pi}^{\pi} p(\cos\theta) U_k(\cos\theta) w(\cos\theta) |\sin\theta| \, \mathrm{d}\theta.$$

Here we have used evenness of the latter integrand in θ. Continue this as

$$\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \operatorname{Im}\left(\int_{-\pi}^{\pi} z^{n+1} \frac{\overline{h(z)}}{\sin\theta} \frac{\sin(k+1)\theta}{\sin\theta} \frac{\sin^2\theta}{S(\cos\theta)} \, \mathrm{d}\theta \right) \\
&= \frac{1}{\sqrt{2\pi}} \operatorname{Im}\left(\int_{-\pi}^{\pi} z^{n+1} \overline{h(z)} \frac{z^{k+1} - z^{-k-1}}{2i} \frac{1}{|h(z)|^2} \, \mathrm{d}\theta \right) \\
&= \frac{1}{\sqrt{2\pi}} \operatorname{Im}\left(\frac{-1}{2} \int_{|z|=1} z^{n-k-1} \frac{z^{2k+2} - 1}{h(z)} \, \mathrm{d}z \right).
\end{aligned}$$

In the second last step, we used (18). If $k \leqslant n - 1$, the integrand is analytic in the closed unit ball (the possible zeros of h at ± 1 are matched by those of $z^{2k+2} - 1$), so Cauchy's integral theorem shows the integral is 0. So p is an orthogonal polynomial. If $k = n$, the integrand has a simple pole at 0, and the residue calculus gives

$$\int_{-1}^{1} p U_n w = \sqrt{\frac{\pi}{2}} \frac{1}{h(0)}.$$

Finally we may write

$$U_n = \frac{2^n}{\gamma_n} p + \text{ polynomial of degree } \leqslant n-1,$$

whence

$$\int_{-1}^{1} p^2 w = \frac{\gamma_n}{2^n} \sqrt{\frac{\pi}{2}} \frac{1}{h(0)} = 1,$$

by (21). So, $p = p_n$. □

The one drawback of the above formula is the need to first find the polynomial h in the representation (18) of $S(\cos\theta)$. By expressing h in terms of a Szegő function, we obtain a more explicit representation for p_n. Recall that, corresponding to w, we may define a weight $f(\theta)$ on $[-\pi, \pi]$ (or equivalently on the unit circle),

$$f(\theta) := w(\cos\theta)|\sin\theta|, \quad \theta \in [-\pi, \pi] \tag{22}$$

and the corresponding Szegő function

$$D(f; z) := \exp\left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \log f(t) \frac{\mathrm{e}^{it} + z}{\mathrm{e}^{it} - z} \,\mathrm{d}t\right), \quad |z| < 1. \tag{23}$$

The latter is uniquely determined by the following three properties [112, Ch. 10]:

(I) $D(f; z)$ is analytic and non-zero in $|z| < 1$;

(II) $D(f; 0) > 0$;

(III) The radial limits

$$\lim_{r \to 1-} D(f; r\mathrm{e}^{i\theta}) =: D(f; \mathrm{e}^{i\theta})$$

exist for almost all $\theta \in [-\pi, \pi]$ and

$$\left|D(f; \mathrm{e}^{i\theta})\right|^2 = f(\theta) \quad \text{a.e. } \theta \in [-\pi, \pi]. \tag{24}$$

(For those with some H_p background, of course radial limits may be replaced by non-tangential ones.) When f is positive and continuous on $[-\pi, \pi]$, there is the stronger boundary behaviour

$$\lim_{r \to 1-} \max_{\theta \in [-\pi, \pi]} \left|\left|D(f; r\mathrm{e}^{i\theta})\right|^2 - f(\theta)\right| = 0. \tag{25}$$

Moreover, the identity (24) holds for all $\theta \in [-\pi, \pi]$.

In the rest of this section, we assume that w is as in Theorem 2.1, and that f is given by (22). Unless otherwise specified, we also assume that $x = \cos\theta \in (-1, 1)$, that $\theta \in (0, \pi)$ and $z = \mathrm{e}^{i\theta}$.

THEOREM 2.2. *For* $n > \ell/2$, $x = \cos\theta$ *and* $z = \mathrm{e}^{i\theta}$,

$$p_n(x) = \frac{1}{\sqrt{2\pi}} \left[z^n D(f; z^{-1})^{-1} + z^{-n} D(f; z)^{-1}\right] \tag{26}$$

$$= \sqrt{\frac{2}{\pi}} \operatorname{Re}\left[z^n D(f; z^{-1})^{-1}\right]. \tag{27}$$

Moreover,

$$\gamma_n = \frac{2^n}{\sqrt{2\pi}} D(f;0)^{-1}. \tag{28}$$

Proof. We first claim that

$$D\big(S(\cos\cdot);z\big) = h(z). \tag{29}$$

Let

$$g(z) := D\big(S(\cos\cdot);z\big)/h(z).$$

Then $g(0) > 0$, g and $1/g$ are analytic in the unit ball, and if we assume that S is positive on all of $[-1,1]$, (18) above shows that as $r \to 1-$, we have uniformly for $\theta \in [-\pi,\pi]$,

$$\left|g(re^{i\theta})\right|^2 = \frac{|D(S(\cos\cdot);re^{i\theta})|^2}{|h(re^{i\theta})|^2} \to \frac{S(\cos\theta)}{|h(e^{i\theta})|^2} = 1.$$

Since g and g^{-1} are analytic in the unit ball, the maximum-modulus principle yields in a straightforward manner that $g(z) \equiv g(0)$ and then also $g(0) = 1$, so $g(z) \equiv 1$. In the case where S has simple zeros at ± 1, this argument may be modified using, for example, the identity

$$1 - x^2 = \sin^2\theta = \left|\frac{1-z^2}{2}\right|^2.$$

This identity and the uniqueness of the Szegő function also implies that

$$D(\sin^2\cdot;z) = \frac{1-z^2}{2}, \quad |z| < 1 \tag{30}$$

and, hence, also

$$D(\sin^2\cdot;z) = \frac{1-z^2}{2} = -iz\sin\theta, \quad z = e^{i\theta}. \tag{31}$$

Then for $|z| = 1$, obvious multiplicativity properties of $D(\cdot;z)$ give

$$D(f;z) = D\left(\frac{\sin^2\cdot}{S(\cos\cdot)};z\right) = \frac{D(\sin^2\cdot;z)}{D(S(\cos\cdot);z)} = \frac{-iz\sin\theta}{h(z)}$$

so that from Theorem 2.1,

$$\begin{aligned} p_n(x) &= \sqrt{\frac{2}{\pi}}\,\mathrm{Im}\left\{z^{n+1}\overline{\frac{h(z)}{\sin\theta}}\right\} = \sqrt{\frac{2}{\pi}}\,\mathrm{Im}\left\{z^{n+1}\overline{\frac{-iz}{D(f;z)}}\right\} \\ &= \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left\{z^n\overline{D(f;z)}^{-1}\right\}. \end{aligned}$$

Next, the evenness of f implies that

$$\overline{D(f;z)} = D(f;\bar{z}) = D(f;z^{-1}).$$

Then (27) and (26) follow. Finally, as (31) shows that

$$h(0) = D\big(S(\cos\ \cdot);0\big) = \frac{1}{2}\,\frac{D(S(\cos\ \cdot);0)}{D(\sin^2\ \cdot\,;0)} = \frac{1}{2}D\,(f;0)^{-1},$$

we obtain (28) from (21). □

One immediate consequence is a formula for $p_n'(x)$:

COROLLARY 2.3. *For $n > \ell/2$, $x = \cos\theta$ and $z = \mathrm{e}^{i\theta}$,*

$$\begin{aligned} & p_n'(x)\sqrt{1-x^2}\sqrt{\frac{\pi}{2}} \\ &\quad = n\,\mathrm{Im}\big\{z^n D(f;z^{-1})^{-1}\big\} + \mathrm{Im}\left\{z^{n-1}\frac{D'(f;z^{-1})}{D(f;z^{-1})^2}\right\}. \end{aligned} \tag{32}$$

Proof. Note that if $x = \cos\theta$ and $z = \mathrm{e}^{i\theta}$, then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{iz}{\sqrt{1-x^2}}.$$

Differentiating (27) gives

$$\begin{aligned} p_n'(x) &= \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left\{\frac{\mathrm{d}}{\mathrm{d}z}\big[z^n D(f;z^{-1})^{-1}\big]\frac{\mathrm{d}z}{\mathrm{d}x}\right\} \\ &= \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{1-x^2}}\,\mathrm{Im}\left\{z\frac{\mathrm{d}}{\mathrm{d}z}\big[z^n D(f;z^{-1})^{-1}\big]\right\} \end{aligned}$$

and then (32) follows. □

The last identity is useful in deriving explicit representations for Christoffel functions. Recall that the nth Christoffel function for the weight w is

$$\lambda_n(w,x) := \inf_{\deg(P)\leqslant n-1} \frac{\int_{-1}^{1} P^2 w}{P^2(x)}. \tag{33}$$

It is well known and easy to prove that

$$\lambda_n(w,x) = 1\Big/\sum_{j=0}^{n-1} p_j^2(x).$$

An extensive, and still relevant, survey of the use of Christoffel functions in orthogonal polynomials and weighted approximation was given by Nevai [87]. From the Christoffel–Darboux formula

$$K_n(x, y) := \sum_{k=0}^{n-1} p_k(x) p_k(x) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x - y}, \tag{34}$$

and l'Hospital's rule, it is easily seen that

$$\lambda_n^{-1}(w, x) = K_n(x, x) = \frac{\gamma_{n-1}}{\gamma_n}\big[p_n'(x)p_{n-1}(x) - p_{n-1}'(x)p_n(x)\big]. \tag{35}$$

We can now prove an identity for $\lambda_n^{-1}(w, x)$. It has been proved in [112, p. 320] in one form in the course of a proof of equiconvergence of orthonormal expansions. It was stated with some misprints as an exercise on asymptotics in Freud [32, p. 269], and was used in [66].

THEOREM 2.4. *For $n > \ell/2 + 1$, $x = \cos\theta$ and $z = \mathrm{e}^{i\theta}$,*

$$\begin{aligned}
&\pi\lambda_n^{-1}(w, x)w(x)\sqrt{1-x^2} \\
&\quad = n - \frac{1}{2} + \operatorname{Re}\left\{\frac{zD'(f; z)}{D(f; z)}\right\} + \frac{1}{2\sqrt{1-x^2}} \operatorname{Im}\left\{z^{2n-1}\frac{D(f; z)}{D(f; z^{-1})}\right\}.
\end{aligned} \tag{36}$$

Proof. Let us set

$$F(z) := D(f; z),$$

and recall that since f is even in θ,

$$\overline{F(z)} = F(\overline{z}), \qquad z = \mathrm{e}^{i\theta}.$$

Then (35), (26) and (32) and the fact that $(\gamma_{n-1})/\gamma_n = \frac{1}{2}$ imply that

$$\begin{aligned}
&4\pi i\lambda_n^{-1}(w, x)\sqrt{1-x^2} \\
&\quad = 2i\left\{\left[\sqrt{\frac{\pi}{2}}\, p_n'(x)\sqrt{1-x^2}\right]\left[\sqrt{2\pi}\, p_{n-1}(x)\right] - \right. \\
&\qquad \left. - \left[\sqrt{\frac{\pi}{2}}\, p_{n-1}'(x)\sqrt{1-x^2}\right]\left[\sqrt{2\pi}\, p_n(x)\right]\right\} \\
&\quad = \left[n\big(z^n F(\overline{z})^{-1} - z^{-n}F(z)^{-1}\big) + \left(\frac{z^{n-1}F'(\overline{z})}{F(\overline{z})^2} - \frac{z^{-n+1}F'(z)}{F(z)^2}\right)\right] \times \\
&\qquad \times \left[z^{n-1}F(\overline{z})^{-1} + z^{-n+1}F(z)^{-1}\right] - \\
&\qquad - \left[(n-1)\big(z^{n-1}F(\overline{z})^{-1} - z^{-n+1}F(z)^{-1}\big) + \right. \\
&\qquad \left. + \left(\frac{z^{n-2}F'(\overline{z})}{F(\overline{z})^2} - \frac{z^{-n+2}F'(z)}{F(z)^2}\right)\right] \times \left[z^n F(\overline{z})^{-1} + z^{-n}F(z)^{-1}\right].
\end{aligned}$$

By collecting coefficients of like terms, we continue this as

$$\begin{aligned}
&= F(\overline{z})^{-2}z^{2n-1} + F(z)^{-1}F(\overline{z})^{-1}(2n-1)\{z - z^{-1}\} - F(z)^{-2}z^{-2n+1} + \\
&\quad + \frac{F'(\overline{z})}{F(\overline{z})^2F(z)}(1 - z^{-2}) + \frac{F'(z)}{F(z)^2F(\overline{z})}(-1 + z^2) \\
&= F(z)^{-1}F(\overline{z})^{-1}(2n-1)2i\sin\theta + \\
&\quad + 2i\,\mathrm{Im}(z^{2n-1}F(\overline{z})^{-2}) + 2i\,\mathrm{Im}\left(\frac{F'(\overline{z})}{F(\overline{z})^2F(z)}(1 - z^{-2})\right).
\end{aligned} \tag{37}$$

Using the evenness of f, we see that

$$F(z)F(\overline{z}) = \left|F(z)\right|^2 = f(\theta) = w(\cos\theta)|\sin\theta| = w(x)\sqrt{1-x^2}$$

so multiplying (37) by $w(x)/(4i)$ gives (36). □

We also record an identity for the reproducing kernel $K_n(x, y)$:

THEOREM 2.5. *Let* $x = \cos\theta$, $z = \mathrm{e}^{i\theta}$ and $y = \cos\phi$; $w = \mathrm{e}^{i\phi}$. *Then for* $n > \ell/2 + 1$,

(a) $$\begin{aligned}
&p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y) \\
&\quad = \frac{1}{\pi}\,\mathrm{Re}\Big[D(f; z^{-1})z^{-n}\big\{D(f; w)^{-1}w^{-n}(w - z) + \\
&\qquad\quad + D(f; w^{-1})^{-1}w^n(w^{-1} - z)\big\}\Big].
\end{aligned} \tag{38}$$

(b) $$\begin{aligned}
K_n(x, y) &= \sum_{k=0}^{n-1} p_k(x)p_k(y) \\
&= -\frac{1}{2\pi}\,\mathrm{Im}\left[D(f; z)^{-1}z^{\frac{1}{2}-n}\left\{\frac{D(f; w^{-1})^{-1}w^{n-\frac{1}{2}}}{\sin(\frac{\theta-\phi}{2})} + \right.\right. \\
&\qquad \left.\left. + \frac{D(f; w)^{-1}w^{\frac{1}{2}-n}}{\sin(\frac{\theta+\phi}{2})}\right\}\right].
\end{aligned} \tag{39}$$

Proof. (a) Let $F(z) := D(f; z)$ as before. By (26),

$$\begin{aligned}
&(2\pi)[p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)] \\
&= \left[z^nF(z^{-1})^{-1} + z^{-n}F(z)^{-1}\right]\left[w^{n-1}F(w^{-1})^{-1} + w^{-n+1}F(w)^{-1}\right] - \\
&\quad - \left[z^{n-1}F(z^{-1})^{-1} + z^{-n+1}F(z)^{-1}\right]\left[w^nF(w^{-1})^{-1} + w^{-n}F(w)^{-1}\right] \\
&= F(z^{-1})^{-1}F(w^{-1})^{-1}\{z^nw^{n-1} - z^{n-1}w^n\} + \\
&\quad + F(z)^{-1}F(w)^{-1}\{z^{-n}w^{-n+1} - z^{-n+1}w^{-n}\} + \\
&\quad + F(z)^{-1}F(w^{-1})^{-1}\{z^{-n}w^{n-1} - z^{-n+1}w^n\} + \\
&\quad + F(z^{-1})^{-1}F(w)^{-1}\{z^nw^{-n+1} - z^{n-1}w^{-n}\}
\end{aligned}$$

$$\begin{aligned}
&= 2\,\mathrm{Re}\big\{F(z)^{-1}F(w)^{-1}(zw)^{-n}\{w-z\}\big\} + \\
&\quad +2\,\mathrm{Re}\big\{F(z)^{-1}F(w^{-1})^{-1}(z^{-1}w)^{n}\{w^{-1}-z\}\big\} \\
&= 2\,\mathrm{Re}\big\{F(z)^{-1}z^{-n}\big[F(w)^{-1}\{w-z\}w^{-n} + F(w^{-1})^{-1}\{w^{-1}-z\}w^{n}\big]\big\}.
\end{aligned}$$

Then (38) follows.

(b) We use the Christoffel–Darboux formula (34) and (28) to deduce that

$$K_n(x,y) = \frac{1}{2}\frac{p_n(x)p_{n-1}(y) - p_n(y)p_n(x)}{x-y}.$$

Here

$$x - y = -2\sin\left(\frac{\theta-\phi}{2}\right)\sin\left(\frac{\theta+\phi}{2}\right)$$

and

$$\begin{aligned}
w - z &= -(wz)^{1/2}2i\,\sin\left(\frac{\theta-\phi}{2}\right); \\
w^{-1} - z &= -(w^{-1}z)^{1/2}2i\,\sin\left(\frac{\theta+\phi}{2}\right).
\end{aligned}$$

These last four identities and (a) give

$$\begin{aligned}
&2\pi K_n(x,y) \\
&\quad = -\,\mathrm{Im}\left\{F(z)^{-1}z^{\frac{1}{2}-n}\left[\frac{F(w)^{-1}w^{\frac{1}{2}-n}}{\sin(\frac{\theta+\phi}{2})} + \frac{F(w^{-1})^{-1}w^{n-\frac{1}{2}}}{\sin(\frac{\theta-\phi}{2})}\right]\right\}.
\end{aligned}$$ □

We next rewrite some of the above formulae in terms of the argument of $D(f;z)$. If we write

$$D(f;z) = \big|D(f;z)\big|\exp\big(i\arg D(f;z)\big),$$

then

$$\begin{aligned}
\arg D(f; re^{i\theta}) &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\log f(t)\,\mathrm{Im}\left(\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\right)\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\log f(t)\frac{r\sin(\theta-t)}{1+r^2-2r\cos(\theta-t)}\,\mathrm{d}t.
\end{aligned}$$

As $r \to 1-$,

$$\frac{r\sin(\theta-t)}{1+r^2-2r\cos(\theta-t)} \to \frac{\sin(\theta-t)}{2(1-\cos(\theta-t))} = \frac{1}{2}\cot\left(\frac{\theta-t}{2}\right),$$

so one expects that $\arg D(f; re^{i\theta})$ approaches

$$\frac{1}{4\pi}PV\int_{-\pi}^{\pi}\log f(t)\cot\left(\frac{\theta-t}{2}\right)\mathrm{d}t =: \Gamma(f;\theta). \tag{40}$$

Here because of the non-integrable singularity of $\cot((\theta - t)/2)$ at $t = \theta$, the integral must be taken in a *Cauchy principal value* sense:

$$PV \int_{-\pi}^{\pi} \log f(t) \cot\left(\frac{\theta - t}{2}\right) \mathrm{d}t$$
$$:= \lim_{\varepsilon \to 0+} \int_{[-\pi,\pi]\setminus[\theta-\varepsilon,\theta+\varepsilon]} \log f(t) \cot\left(\frac{\theta - t}{2}\right) \mathrm{d}t.$$

It is a well known fact in the theory of singular integrals, that the above limits (as $r \to 1-$ and $\varepsilon \to 0+$) exist a.e. assuming just $\log f \in L_1[-\pi, \pi]$. For our f of (22), which is differentiable in $(-1, 1)$, the limit exists everywhere in $(-1, 1)$. $\Gamma(f;\ \cdot)$ is often called the *conjugate function* of $\frac{1}{2} \log f$ and sometimes, its *Hilbert transform on the circle* [33, 44, 48, 111, 123].

Recalling (24), we see that

$$D(f; \mathrm{e}^{i\theta}) = f(\theta)^{1/2} \exp\bigl(i\Gamma(f;\theta)\bigr), \tag{41}$$

and differentiating formally with respect to θ gives

$$\frac{\mathrm{e}^{i\theta} D'(f; \mathrm{e}^{i\theta})}{D(f; \mathrm{e}^{i\theta})} = -\frac{i}{2} \frac{f'(\theta)}{f(\theta)} + \Gamma'(f;\theta). \tag{42}$$

This formula is meaningful when $\Gamma'(f;\theta)$ exists. For f given by (22) and $\theta \in [-\pi, \pi]\setminus\{0\}$, we shall effectively prove its existence in Lemma 2.7(b) below.

In some applications it is useful to express the formulae for p_n, p_n' and λ_n^{-1} in terms of Γ:

THEOREM 2.6. *For $n > \ell/2 + 1$, $x = \cos\theta$ and $z = \mathrm{e}^{i\theta}$,*

(a) $$p_n(x) w(x)^{1/2} (1 - x^2)^{1/4} \sqrt{\frac{\pi}{2}} = \cos\bigl(n\theta + \Gamma(f;\theta)\bigr). \tag{43}$$

(b) $$p_n'(x) w(x)^{1/2} (1 - x^2)^{3/4} \sqrt{\frac{\pi}{2}} = \bigl(n + \Gamma'(f;\theta)\bigr) \sin\bigl(n\theta + \Gamma(f;\theta)\bigr) + {} + \frac{1}{2} \frac{f'(\theta)}{f(\theta)} \cos\bigl(n\theta + \Gamma(f;\theta)\bigr). \tag{44}$$

(c) $$\pi \lambda_n^{-1}(w, x) w(x) \sqrt{1 - x^2} = n - \frac{1}{2} + \Gamma'(f;\theta) + {} + \frac{1}{2\sqrt{1 - x^2}} \sin\bigl((2n - 1)\theta + 2\Gamma(f;\theta)\bigr). \tag{45}$$

Proof. This follows from (27), (32), (36), (41), (42) and the fact that the evenness of $f(\cdot)$ and the oddness of cot imply that $\Gamma(f;\cdot)$ is odd – see (40). $\square$

We shall present one final set of formulae, couched in language that is appropriate in the context of exponential weights. For $g \in L_1[-1,1]$, define its Hilbert transform

$$H[g](x) := PV \int_{-1}^{1} \frac{g(t)}{t-x}\,\mathrm{d}t, \quad \text{a.e. } x \in (-1,1). \tag{46}$$

If also $g'(\cdot)\sqrt{1-\cdot^2} \in L_1[-1,1]$, we define for a.e. $x \in (-1,1)$,

$$L[g](x) := \frac{1}{\pi} H\big[g'(\cdot)\sqrt{1-\cdot^2}\big](x) = \frac{PV}{\pi} \int_{-1}^{1} \frac{g'(t)\sqrt{1-t^2}}{t-x}\,\mathrm{d}t. \tag{47}$$

Using the identity (see, for example, [108, p. 225])

$$\frac{PV}{\pi} \int_{-1}^{1} \frac{1}{t-x} \frac{\mathrm{d}t}{\sqrt{1-t^2}} = 0, \quad x \in (-1,1), \tag{48}$$

we may write for a.e. $x \in (-1,1)$,

$$L[g](x) = \frac{PV}{\pi} \int_{-1}^{1} \frac{g'(t)(1-t^2) - g'(x)(1-x^2)}{t-x} \frac{\mathrm{d}t}{\sqrt{1-t^2}}. \tag{49}$$

Since

$$\frac{\sqrt{1-t^2}}{t-x} = \frac{(1-x^2)}{\sqrt{1-t^2}(t-x)} - \frac{x}{\sqrt{1-t^2}} - \frac{t}{\sqrt{1-t^2}},$$

we see from (47) that

$$\begin{aligned} L[g](x) \;&=\; \pi\sqrt{1-x^2}\sigma[g](x) - \frac{x}{\pi}\int_{-1}^{1} \frac{g'(t)}{\sqrt{1-t^2}}\,\mathrm{d}t \;- \\ &\quad - \frac{1}{\pi}\int_{-1}^{1} \frac{tg'(t)}{\sqrt{1-t^2}}\,\mathrm{d}t, \end{aligned} \tag{50}$$

where

$$\sigma[g](x) = \frac{\sqrt{1-x^2}}{\pi^2} PV \int_{-1}^{1} \frac{g'(t)}{t-x} \frac{\mathrm{d}t}{\sqrt{1-t^2}}. \tag{51}$$

Of course, this will be meaningful only if all the integrals in (50) and (51) converge in a suitable sense.

If we write our weight w of (19) in the form

$$w(x) = \exp(-2Q(x)) \tag{52}$$

so that

$$Q(x) = -\frac{1}{4}\log(1-x^2) + \frac{1}{2}\log S(x), \tag{53}$$

and if we take $g := Q$, then the expression $\sigma[g] = \sigma[Q]$ is one of the most commonly used formulae for the density function of the equilibrium distribution for Q, provided the interval of support of Q is $[-1, 1]$. For further orientation on this, see [79] or [108]. Moreover, one frequently then has

$$\frac{1}{\pi}\int_{-1}^{1}\frac{Q'(t)}{\sqrt{1-t^2}}\,\mathrm{d}t = 0; \qquad \frac{1}{\pi}\int_{-1}^{1}\frac{tQ'(t)}{\sqrt{1-t^2}}\,\mathrm{d}t = n \tag{54}$$

so that

$$L[Q](x) = \pi\sqrt{1-x^2}\,\sigma[Q](x) - n. \tag{55}$$

Unfortunately for the Q of (53), the integrals in (50) and (51) diverge, due to a non-integrable singularity at ± 1. In contrast, $L[Q](x)$ is well defined for $x \in (-1, 1)$. Nevertheless, (55) provides insight into the relationship between $L[Q]$ and quantities more commonly used in potential theory.

Before stating our final formulae for p_n and λ_n^{-1} in terms of $L[Q]$, we need to establish the relationship between Γ and L:

LEMMA 2.7. *Let w, f be given by* (19) *and* (22) *respectively and write $x = \cos\theta, \theta \in (0, \pi)$. Let Γ and L be defined by* (40) *and* (47) *respectively.*

$$\text{(a)} \quad \Gamma(f;\theta) = \theta - \frac{\pi}{2} - \frac{\sqrt{1-x^2}}{2\pi}\int_{-1}^{1}\frac{\log S(s) - \log S(x)}{s-x}\,\frac{\mathrm{d}s}{\sqrt{1-s^2}}. \tag{56}$$

$$\text{(b)} \quad \Gamma'(f;\theta) = \frac{1}{2} + L[Q](x). \tag{57}$$

$$\text{(c)} \quad \Gamma(f;\theta) = \frac{\theta}{2} - \tau + \int_{x}^{1}\frac{L[Q](t)}{\sqrt{1-t^2}}\,\mathrm{d}t, \tag{58}$$

where

$$\tau := \begin{cases} \frac{\pi}{2}, & S(1) \neq 0; \\ 0, & S(1) = 0. \end{cases} \tag{59}$$

Proof. (a) Recall that

$$f(\theta) = \sin^2\theta / S(\cos\theta),$$

so that

$$\Gamma(f;\theta) = \Gamma(\sin^2\cdot;\theta) - \Gamma\big(S(\cos\cdot);\theta\big). \tag{60}$$

But by (31), for $z = e^{i\theta}$,

$$D(\sin^2 \cdot; z) = -iz \sin\theta$$
$$\Rightarrow \Gamma(\sin^2 \cdot; \theta) = \arg D(\sin^2 \cdot; e^{i\theta}) = \theta - \frac{\pi}{2}. \tag{61}$$

Next, we make the substitutions $x = \cos\theta$; $s = \cos t$ in the integral in the right-hand side of (56), namely in

$$H(x) := \frac{\sqrt{1-x^2}}{2\pi} \int_{-1}^{1} \frac{\log S(s) - \log S(x)}{s-x} \frac{ds}{\sqrt{1-s^2}}. \tag{62}$$

Using (48), we obtain

$$H(x) = \frac{\sin\theta}{4\pi} PV \int_{-\pi}^{\pi} \frac{\log S(\cos t)}{\cos t - \cos\theta} dt.$$

(The substitution in the principal value integral is easily justified.) Since

$$\frac{\sin\theta}{\cos t - \cos\theta} = \frac{\sin(\frac{\theta-t}{2})\cos(\frac{\theta+t}{2}) + \cos(\frac{\theta-t}{2})\sin(\frac{\theta+t}{2})}{2\sin(\frac{\theta-t}{2})\sin(\frac{\theta+t}{2})},$$

we see that

$$\begin{aligned} H(x) &= \frac{PV}{8\pi} \int_{-\pi}^{\pi} \log S(\cos t)\left[\cot\left(\frac{\theta+t}{2}\right) + \cot\left(\frac{\theta-t}{2}\right)\right] dt \\ &= \frac{PV}{4\pi} \int_{-\pi}^{\pi} \log S(\cos t) \cot\left(\frac{\theta-t}{2}\right) dt \\ &= \Gamma\big(S(\cos \cdot); \theta\big). \end{aligned} \tag{63}$$

Together (60)–(63) give (56).

(b) Now from (56) and (62),

$$\begin{aligned} \Gamma'(f;\theta) &= 1 + H'(x)\sin\theta \\ &= 1 - \frac{x}{2\pi} \int_{-1}^{1} \frac{\log S(s) - \log S(x)}{s-x} \frac{ds}{\sqrt{1-s^2}} + \\ &\quad + \frac{1-x^2}{2\pi} \int_{-1}^{1} \frac{d}{dx}\left(\frac{\log S(s) - \log S(x)}{s-x}\right) \frac{ds}{\sqrt{1-s^2}} \\ &= 1 - \frac{x}{2\pi} \int_{-1}^{1} \frac{\log S(s) - \log S(x)}{s-x} \frac{ds}{\sqrt{1-s^2}} + \\ &\quad + \frac{1-x^2}{2\pi} \int_{-1}^{1} \frac{\log S(s) - \log S(x) - \frac{S'(x)}{S(x)}(s-x)}{(s-x)^2} \frac{ds}{\sqrt{1-s^2}}. \end{aligned} \tag{64}$$

The interchange of series and derivative is justified by the uniform convergence in x in the last integral, provided x is restricted to a closed subinterval of $(-1, 1)$ (recall that $\log S$ is infinitely differentiable in $(-1, 1)$). Using the identity

$$\frac{1-x^2}{s-x} = \frac{1-s^2}{s-x} + s + x$$

in the second integral in (64), and using also (48) gives

$$\begin{aligned}\Gamma'(f;\theta) &= 1+\frac{PV}{2\pi}\int_{-1}^{1}\frac{\log S(s)-\log S(x)}{s-x}\times\\ &\quad\times\left\{-x+\frac{1-s^2}{s-x}+s+x\right\}\frac{\mathrm{d}s}{\sqrt{1-s^2}}\\ &= 1-\frac{PV}{2\pi}\int_{-1}^{1}\big[\log S(s)-\log S(x)\big]\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{\sqrt{1-s^2}}{s-x}\right\}\mathrm{d}s.\end{aligned}$$

Integrating by parts gives

$$\Gamma'(f;\theta)=1+\frac{PV}{2\pi}\int_{-1}^{1}\frac{S'(s)}{S(s)}\frac{\sqrt{1-s^2}}{s-x}\,\mathrm{d}s. \tag{65}$$

This may be justified by first integrating by parts over $(-1+\varepsilon, x-\varepsilon)$ and $(x+\varepsilon, 1-\varepsilon)$ and then letting $\varepsilon\to 0+$. The limits exist as $(S'(s)/S(s))\sqrt{1-s^2}$ is differentiable in $(-1,1)$ and is $O(1/\sqrt{1-s^2})$ as $s\to\pm 1$. Using again (48) gives

$$\begin{aligned}&\Gamma'(f;\theta)\\ &\quad=1+\frac{1}{2\pi}\int_{-1}^{1}\frac{(S'/S)(s)(1-s^2)-(S'/S)(x)(1-x^2)}{s-x}\,\frac{\mathrm{d}s}{\sqrt{1-s^2}}.\end{aligned} \tag{66}$$

Finally, since

$$Q(x)=-\frac{1}{2}\log w(x)=\frac{1}{2}\log S(x)-\frac{1}{4}\log(1-x^2),$$

we see from (49) that

$$\begin{aligned}L[Q](x) &= \frac{1}{\pi}\int_{-1}^{1}\frac{Q'(s)(1-s^2)-Q'(x)(1-x^2)}{s-x}\,\frac{\mathrm{d}s}{\sqrt{1-s^2}}\\ &= \frac{1}{2\pi}\int_{-1}^{1}\frac{(S'/S)(s)(1-s^2)-(S'/S)(x)(1-x^2)}{s-x}\,\frac{\mathrm{d}s}{\sqrt{1-s^2}}+\\ &\quad+\frac{1}{2\pi}\int_{-1}^{1}\frac{\mathrm{d}s}{\sqrt{1-s^2}}\\ &= \Gamma'(f;\theta)-1+\frac{1}{2},\end{aligned}$$

by (66). So we have (57).

(c) Integrating (57) for θ from 0 to some $\phi\in(0,\pi)$ gives

$$\Gamma(f;\phi)-\Gamma(f;0)=\frac{\phi}{2}+\int_0^{\phi}L[Q](\cos\theta)\,\mathrm{d}\theta.$$

Now as $x\to 1-$, or equivalently as $\theta\to 0+$, we see from (56) that if $S(1)\neq 0$,

$$\Gamma(f;\theta)\to -\frac{\pi}{2}.$$

Then (58) follows if we change ϕ to θ. If S has a simple zero at 1, then we write

$$f(\theta) := 1/S_1(\cos\theta),$$

where $S_1(1) \neq 0$. As in (a),

$$\begin{aligned}\Gamma(f;\theta) &= \Gamma(1;\theta) - \Gamma\big(S_1(\cos\,\cdot);\theta\big) = -\Gamma\big(S_1(\cos\,\cdot\,;\theta)\big)\\ &= -\frac{\sqrt{1-x^2}}{2\pi}\int_{-1}^{1}\frac{\log S_1(s) - \log S_1(x)}{s-x}\,\frac{\mathrm{d}s}{\sqrt{1-s^2}},\end{aligned}$$

just as in (a). Since $\log S_1(x)$ is differentiable at $x = 1$, we may show as above that $\Gamma(S_1(\cos\,\cdot);\theta) \to 0$ as $\theta \to 0+$, and then (58) follows as before. □

We may now reformulate Theorem 2.6 in terms of $L[Q]$:

THEOREM 2.8. *Let w be given by* (19) *and Q be given by* (53). *Let τ be given by* (59). *For $n > \ell/2 + 1$, $x = \cos\theta$ and $z = \mathrm{e}^{i\theta}$,*

(a) $$p_n(x)w(x)^{1/2}(1-x^2)^{1/4}\sqrt{\frac{\pi}{2}} = \cos\left(\left(n+\frac{1}{2}\right)\theta - \tau + \int_x^1 \frac{L[Q](t)}{\sqrt{1-t^2}}\,\mathrm{d}t\right). \tag{67}$$

(b) $$\begin{aligned}p_n'(x)w(x)^{1/2}(1-x^2)^{3/4}\sqrt{\frac{\pi}{2}} &= \left(n+\frac{1}{2}+L[Q](x)\right)\sin\left(\left(n+\frac{1}{2}\right)\theta - \tau + \int_x^1 \frac{L[Q](t)}{\sqrt{1-t^2}}\,\mathrm{d}t\right) +\\ &\quad + \frac{1}{2}\frac{f'(\theta)}{f(\theta)}\cos\left(\left(n+\frac{1}{2}\right)\theta - \tau + \int_x^1 \frac{L[Q](t)}{\sqrt{1-t^2}}\,\mathrm{d}t\right).\end{aligned} \tag{68}$$

(c) $$\pi\lambda_n^{-1}(w,x)w(x)\sqrt{1-x^2} = n + L[Q](x) + \frac{1}{2\sqrt{1-x^2}}\sin\left(2n\theta - 2\tau + 2\int_x^1 \frac{L[Q](t)}{\sqrt{1-t^2}}\,\mathrm{d}t\right). \tag{69}$$

Proof. This follows directly from Theorem 2.6 and Lemma 2.7(b), (c). □

2.2. BERNSTEIN–SZEGŐ IN UNIVERSALITY LIMITS

We have stressed that the main applications of the Bernstein–Szegő identity has been to asymptotics of orthogonal polynomials. Its cousin on the circle underlies the Szegő/power asymptotics for orthogonal polynomials both on the unit circle and $[-1,1]$, and the identity on $[-1,1]$ underlies one of the main approaches for proving asymptotics for orthogonal polynomials associated with exponential weights on the real line [61, 72, 113].

In this section, we shall illustrate a different application, to universality limit relations. In the theory of random matrices, the distribution of eigenvalues of matrices in small intervals reduces to a technical limit relation involving orthogonal polynomials. For a weight w, with nth reproducing kernel,

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x)p_k(y),$$

one form of the universality limit relation is

$$\lim_{n\to\infty} K_n^*\left(u + \frac{s}{K_n^*(u,u)}, u + \frac{t}{K_n^*(u,u)}\right)\Big/ K_n^*(u,u) = \frac{\sin \pi(s-t)}{\pi(s-t)}, \qquad (70)$$

where $s, t \in \mathbb{R}$, $u \in I$ and

$$K_n^*(x,y) := K_n(x,y)w(x)^{1/2}w(y)^{1/2}. \qquad (71)$$

Various forms and analogues of this have been explored in [21, 26, 92], etc. The most impressive rigorous approach has been given in [26].

In this section, we illustrate the limit (70) in the very simple case of Bernstein–Szegő weights. In addition to K_n^*, we use

$$K_n^{\#}(x,y) := K_n(x,y)\big[w(x)(1-x^2)^{1/2}\big]^{1/2}\big[w(y)(1-y^2)^{1/2}\big]^{1/2}. \qquad (72)$$

We also need the modulus of continuity of a function restricted to an interval $[a,b]$: if this interval is contained in the domain of definition of a real valued function g, then we define for $\varepsilon \geqslant 0$,

$$\omega_{[a,b]}(g;\varepsilon) := \sup\big\{\big|g(s)-g(t)\big| : |s-t| \leqslant \varepsilon \text{ and } s,\ t \in [a,b]\big\}.$$

We can now prove:

THEOREM 2.9. *Let w be given by* (19) *and $n > \ell/2 + 1$. Let $u \in (-1,1)$ and write $u = \cos\psi$, where $\psi \in (0,\pi)$. Let $s, t \in \mathbb{R}$ and assume that*

$$\delta := \frac{|s-t|}{K_n^{\#}(u,u)} \in [0,\pi]. \qquad (73)$$

Let $[a,b] \subset (0,\pi)$ and assume that it contains ψ, $\psi + s/(K_n^{\#}(u,u))$ and $\psi + t/(K_n^{\#}(u,u))$. Then

$$\begin{aligned}&\left|K_n^{\#}\left(\cos\left(\psi + \frac{s}{K_n^{\#}(u,u)}\right), \cos\left(\psi + \frac{t}{K_n^{\#}(u,u)}\right)\right) - \right.\\ &\qquad \left. - \frac{\sin(\pi(s-t))}{2\pi \sin(\frac{s-t}{2K_n^{\#}(u,u)})}\right| \\ &\quad \leqslant \frac{1}{2}\omega_{[a,b]}(\Gamma'(f;\ \cdot);\delta) + \left(\frac{1}{4} + \frac{1}{2\pi}\right)\frac{1}{\min\{\sin a, \sin b\}}. \end{aligned} \qquad (74)$$

Proof. Let us write

$$\begin{aligned}\theta &:= \psi + \frac{s}{K_n^{\#}(u,u)}; \quad x := \cos\theta;\\ \phi &:= \psi + \frac{t}{K_n^{\#}(u,u)}; \quad y := \cos\phi. \end{aligned}\tag{75}$$

By the Christoffel–Darboux formula, the fact that $(\gamma_{n-1})/\gamma_n = \frac{1}{2}$, and (43),

$$\pi(x-y)K_n^{\#}(x,y) = \cos\alpha\cos(\beta-\phi) - \cos\beta\cos(\alpha-\theta),$$

where

$$\alpha := n\theta + \Gamma(f;\theta); \qquad \beta := n\phi + \Gamma(f;\phi).$$

Then we deduce that

$$\begin{aligned}&-2\pi\sin\left(\frac{\theta-\phi}{2}\right)\sin\left(\frac{\theta+\phi}{2}\right)K_n^{\#}(x,y)\\ &\quad= \frac{1}{2}\big[\cos(\alpha+\beta-\phi)+\cos(\alpha-\beta+\phi)\big] -\\ &\qquad -\frac{1}{2}\big[\cos(\alpha+\beta-\theta)+\cos(\alpha-\beta-\theta)\big]\\ &\quad= -\sin\left(\alpha+\beta-\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) -\\ &\qquad -\sin\left(\alpha-\beta+\frac{\phi-\theta}{2}\right)\sin\left(\frac{\theta+\phi}{2}\right).\end{aligned}$$

Hence

$$\begin{aligned}&\left|K_n^{\#}(x,y) - \frac{\sin(\alpha-\beta+\frac{\phi-\theta}{2})}{2\pi\sin(\frac{\theta-\phi}{2})}\right|\\ &\quad\leqslant \frac{1}{2\pi|\sin(\frac{\theta+\phi}{2})|}\\ &\quad\leqslant \frac{1}{2\pi\min\{\sin a,\sin b\}}.\end{aligned}\tag{76}$$

Next,

$$\alpha-\beta+\frac{\phi-\theta}{2} = \left(n-\frac{1}{2}\right)(\theta-\phi) + \Gamma(f;\theta) - \Gamma(f;\phi).$$

Here by (45),

$$\begin{aligned}\pi K_n^{\#}(u,u) &= \pi\lambda_n^{-1}(w,u)w(u)\sqrt{1-u^2}\\ &= n - \frac{1}{2} + \Gamma'(f;\psi) + \eta,\end{aligned}$$

where

$$\eta := \frac{1}{2\sqrt{1-u^2}} \sin\big((2n-1)\psi + 2\Gamma(f;\psi)\big).$$

Thus, substituting for $n - 1/2$,

$$\begin{aligned} \alpha - \beta + \frac{\phi-\theta}{2} &= \pi K_n^{\#}(u,u)(\theta-\phi) + \Gamma(f;\theta) - \\ &\quad - \Gamma(f;\phi) - \big(\Gamma'(f;\psi) + \eta\big)(\theta-\phi) \\ &= \pi(s-t) + (\theta-\phi)(\Gamma'(f;\xi) - \Gamma'(f;\psi) - \eta) \\ &=: \pi(s-t) + \varepsilon, \end{aligned}$$

where ξ is between θ and ϕ and we have used our choice (75) of θ, ϕ. Then we see that by our choice (73) of δ,

$$\begin{aligned} &\left|\sin\left(\alpha - \beta + \frac{\phi-\theta}{2}\right) - \sin\big(\pi(s-t)\big)\right| \leqslant |\varepsilon| \\ &\quad \leqslant \delta\left(\omega_{[a,b]}\big(\Gamma'(f;\,\cdot);\delta\big) + \frac{1}{2\sqrt{1-u^2}}\right) \end{aligned}$$

and hence

$$\begin{aligned} &\left|\frac{\sin(\alpha-\beta+\frac{\phi-\theta}{2})}{2\pi\sin(\frac{\theta-\phi}{2})} - \frac{\sin(\pi(s-t))}{2\pi\sin(\frac{s-t}{2K_n^{\#}(u,u)})}\right| \\ &\quad \leqslant \frac{\delta}{2\pi\sin\frac{\delta}{2}}\left(\omega_{[a,b]}\big(\Gamma'(f;\,\cdot);\delta\big) + \frac{1}{2\sqrt{1-u^2}}\right) \\ &\quad \leqslant \frac{1}{2}\left(\omega_{[a,b]}\big(\Gamma'(f;\,\cdot);\delta\big) + \frac{1}{2\sqrt{1-u^2}}\right), \end{aligned}$$

by the inequality $\sin\delta/2 \geqslant \delta/\pi$. This, (76), and the fact that $\psi \in [a,b]$ yield the result. $\square$

Note that the estimate in (74) holds without a division by $K_n^*(u,u)$. We can now transform that estimate into:

THEOREM 2.10. *Let w be given by* (19). *Let J be a closed subinterval of* $(-1,1)$ *and $\mathcal{K} \subset \mathbb{R}$ be bounded. Then uniformly for $u \in J$, and $\sigma, \tau \in \mathcal{K}$, we have as $n \to \infty$,*

$$\begin{aligned} &K_n^*\left(u + \frac{\sigma}{K_n^*(u,u)}, u + \frac{\tau}{K_n^*(u,u)}\right)\Big/ K_n^*(u,u) \\ &\quad = \frac{\sin(\pi(\sigma-\tau))}{\pi(\sigma-\tau)} + \mathrm{O}\left(\frac{1}{n}\right). \end{aligned} \tag{77}$$

Proof. We first show that the right-hand side of (74) may be bounded independently of n. Now by (57) and (47),

$$\Gamma'(f;\theta) = L[Q](\cos\theta) + \frac{1}{2} = \frac{1}{\pi} H\big[Q'(\,\cdot\,)\sqrt{1-\,\cdot^{2}}\big](\cos\theta) + \frac{1}{2}.$$

Here, Q is given by (53). Since Q is infinitely differentiable in $(-1,1)$, Privalov's Theorem (see, for example, [44, p. 94]) shows that $\Gamma'(f;\,\cdot)$ satisfies a Lipschitz condition of order α in each compact subinterval of $(-1,1)$ and for any $0<\alpha<1$. Then it follows that

$$\omega_{[a,b]}\big(\Gamma'(f;\,\cdot);\delta\big) \leqslant C\delta^{a}, \quad \delta>0,$$

where C depends on $[a,b]\subset(0,\pi)$. Next, it follows from (69) that provided x is restricted to a compact subinterval of $(-1,1)$,

$$\pi K_n^{\#}(x,x) = n + \mathrm{O}(1),$$

so for s, t in a compact set, we see that δ of (73) satisfies, uniformly in u, s, t,

$$\delta = \mathrm{O}\left(\frac{1}{n}\right).$$

Thus we obtain uniformly for $u=\cos\psi\in J$, s, t in a compact set

$$\left| K_n^{\#}\left(\cos\left(\psi+\frac{s}{K_n^{\#}(u,u)}\right), \cos\left(\psi+\frac{t}{K_n^{\#}(u,u)}\right)\right) - \right.$$
$$\left. - \frac{\sin(\pi(s-t))}{2\pi\sin\left(\frac{s-t}{2K_n^{\#}(u,u)}\right)}\right| \leqslant C. \tag{78}$$

Next, for $\sigma,\tau\in\mathcal{K}$, $u\in J$ and large enough n, we may write for some $s=s(\sigma)$,

$$\cos\psi + \frac{\sigma}{K_n^{*}(u,u)} = \cos\psi + \frac{\sigma\sin\psi}{K_n^{\#}(u,u)} = \cos\left(\psi+\frac{s}{K_n^{\#}(u,u)}\right),$$

where s also varies in a compact subset of $\mathbb{R}$. This follows by a Taylor series expansion to second order, and since sin is bounded below by a positive constant in each compact subinterval of $(0,\pi)$. Similarly, for some $t=t(\tau)$,

$$\cos\psi + \frac{\tau}{K_n^{*}(u,u)} = \cos\psi + \frac{\tau\sin\psi}{K_n^{\#}(u,u)} = \cos\left(\psi+\frac{t}{K_n^{\#}(u,u)}\right).$$

Then

$$\begin{aligned}
\frac{\sigma-\tau}{s-t} &= \frac{\cos\left(\psi+\frac{s}{K_n^{\#}(u,u)}\right) - \cos\left(\psi+\frac{t}{K_n^{\#}(u,u)}\right)}{(s-t)/K_n^{*}(u,u)} \\
&= -\sin\left(\psi+\frac{\xi}{K_n^{\#}(u,u)}\right)\frac{K_n^{*}(u,u)}{K_n^{\#}(u,u)} \\
&= \left(-\sin\psi + \mathrm{O}\left(\frac{1}{n}\right)\right)\frac{1}{\sin\psi} \\
&= -1+\mathrm{O}\left(\frac{1}{n}\right),
\end{aligned}$$

uniformly for $\sigma, \tau \in \mathcal{K}$, $u \in J$ and with the convention that the left-hand side is taken as -1 if $\sigma = \tau$. Since the function $v \to \sin v/v$, with value 1 at $v = 0$, is continuously differentiable in $(-\pi, \pi)$, we deduce that uniformly in $\sigma, t \in \mathcal{K}$,

$$\frac{\sin(\pi(\sigma - \tau))}{\pi(\sigma - \tau)} = \frac{\sin(\pi(s - t))}{\pi(s - t)} + \mathrm{O}\left(\frac{1}{n}\right)$$

and also uniformly in $\sigma, \tau \in \mathcal{K}$, $u \in J$,

$$\sin\left(\frac{s - t}{2K_n^{\#}(u, u)}\right) \Big/ \left(\frac{s - t}{2K_n^{\#}(u, u)}\right) = 1 + \mathrm{O}\left(\frac{1}{n}\right)$$

so we may reformulate (78) as

$$\left| K_n^{\#}\left(u + \frac{\sigma}{K_n^*(u, u)}, u + \frac{\tau}{K_n^*(u, u)}\right) \Big/ K_n^{\#}(u, u) - \frac{\sin(\pi(\sigma - \tau))}{\pi(\sigma - \tau)} \right| \leqslant \frac{C}{n},$$

uniformly in u, σ, τ. Finally,

$$\left(1 - \left[u + \frac{\sigma}{K_n^*(u, u)}\right]^2\right)^{1/4} = (1 - u^2)^{1/4} + \mathrm{O}\left(\frac{1}{n}\right),$$

uniformly in u, σ, τ, with a similar relation when τ replaces σ, and then (77) follows. □

Obviously the very narrow class of weights treated in Theorem 2.10 limits its interest. However, via Korous' method – as outlined in the beginning of this section – one may extend Theorem 2.9 to more general weights, that admit suitable polynomial approximation. This would still be for weights on a fixed interval. The real question, which seems well worth exploring, is whether a Korous type approach can yield universality limits for varying weights and hence for exponential weights on the real line. Would this, for example, compete with the strength of results in [26, Lemma 6.1]?

2.3. THE FOKAS–ITS–KITAEV (RIEMANN–HILBERT) IDENTITY

While the Bernstein–Szegő identity is based ultimately on Cauchy's integral theorem and integral formula, the Fokas–Its–Kitaev identity is based on the Sokhotskii–Plemelj formulas. These may be viewed as the boundary behaviour form of Cauchy's integral formula: an excellent introduction appears in Henrici's ode to complex analysis [44]. Suppose for example, that we have a measurable function $h\colon \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{|h(t)|}{1 + |t|}\,\mathrm{d}t < \infty. \tag{79}$$

Then one may define its Cauchy transform

$$\mathcal{C}[h](z) := \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{h(t)}{t-z}\, \mathrm{d}t, \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

(Sometimes, this is called the Stieltjes transform, or Hilbert transform, ...) This is of course analytic in $\mathbb{C}\backslash\mathbb{R}$. Let us define the boundary values from the upper and lower half planes,

$$\begin{aligned} \mathcal{C}[h]_+(x) &:= \lim_{y\to 0+} \mathcal{C}[h](x+iy); \\ \mathcal{C}[h]_-(x) &:= \lim_{y\to 0+} \mathcal{C}[h](x-iy), \end{aligned}$$

whenever the limits exist. (For those familiar with boundary behaviour of analytic functions, of course the radial limits may be replaced by non-tangential ones.) The Sokhotskii–Plemelj formulas assert that whenever the limits exist,

$$\begin{aligned} \mathcal{C}[h]_+(x) - \mathcal{C}[h]_-(x) &= h(x); \\ \mathcal{C}[h]_+(x) + \mathcal{C}[h]_-(x) &= \frac{1}{\pi i} PV \int_{-\infty}^{\infty} \frac{h(t)}{t-x}\, \mathrm{d}t. \end{aligned} \tag{80}$$

Here, as earlier, PV denotes Cauchy principal value. In particular (79) ensures that the limits and hence (80) hold a.e. Moreover, if h satisfies a Lipschitz condition of some positive order in an interval, then (80) holds in the interior of that interval.

Riemann–Hilbert problems involve replacing a difference of boundary values by their ratio: for example, one looks for a function G analytic in $\mathbb{C}\backslash\mathbb{R}$ satisfying

$$G_+(x) = h(x)G_-(x), \quad x \in \mathbb{R},$$

for a given function h, and subject to some normalization condition on G. An applicable connection between Riemann–Hilbert problems and orthogonal polynomials was first drawn in the early 1990's [30, 31] by Fokas, Its and Kitaev for a weight w on $I = \mathbb{R}$. The formulation involves 2×2 complex valued matrix functions $Y(\cdot)$: we write $Y(z) \in \mathbb{C}^{2\times 2}$.

THEOREM 2.11. *Let w: $\mathbb{R} \to [0,\infty)$ have all moments $\int_{\mathbb{R}} x^j w(x)\,\mathrm{d}x$, $j = 0, 1, 2, \ldots$ finite and assume that for each $j \geqslant 0$, $w(s)s^j$ satisfies a Lipschitz condition of some positive order throughout $\mathbb{R}$. Let $n \geqslant 1$. Consider the following Riemann–Hilbert problem:*

(I) $Y\colon \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}^{2\times 2}$ *is analytic;*

(II) $$Y_+(x) = Y_-(x)\begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R}. \tag{81}$$

(III) $$Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathrm{O}\left(\frac{1}{|z|}\right), \quad |z| \to \infty. \tag{82}$$

The problem has a unique solution

$$Y(z)=\begin{pmatrix} p_n(z)/\gamma_n & \mathcal{C}[p_n w](z)/\gamma_n \\ -2\pi i\gamma_{n-1}p_{n-1}(z) & -2\pi i\gamma_{n-1}\mathcal{C}[p_{n-1}w](z)\end{pmatrix}. \tag{83}$$

Proof. We follow [26, 49]: first we establish

Uniqueness. Firstly, by taking determinants in (81), we see that

$$\det Y_+(x)=\det Y_-(x), \quad x\in\mathbb{R}.$$

The fact that w satisfies a Lipschitz condition of positive order in $\mathbb{R}$ ensures that each of the entries in the matrices Y_+, Y_- does the same, at least in finite intervals – by Privalov's theorem on singular integrals. In particular $\det Y_\pm$ are continuous there. (Alternatively, the Lipschitz condition implies uniform convergence in compact intervals to the boundary values and hence also continuity.) So $\det Y_\pm$ are continuous on $\mathbb{R}$, while $\det Y(z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$ and hence $\det Y(z)$ is actually an entire function. The order relation (82) forces

$$\det Y(z)\to 1, \quad |z|\to\infty,$$

and so by Liouville's Theorem, $Y(z)\equiv 1$ in $\mathbb{C}$.

If Z is another solution of (I)–(III), we let

$$R(z):=Z(z)Y(z)^{-1},$$

which is analytic in $\mathbb{C}\backslash\mathbb{R}$. Using (81) on Y and Z, we see that for $x\in\mathbb{R}$,

$$R_+(x)=\left[Z_-(x)\begin{pmatrix}1 & w(x)\\ 0 & 1\end{pmatrix}\right]\left[Y_-(x)\begin{pmatrix}1 & w(x)\\ 0 & 1\end{pmatrix}\right]^{-1}=R_-(x).$$

So as above, R is entire. Finally, (82) shows that

$$R(z)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+O\left(\frac{1}{|z|}\right), \quad |z|\to\infty,$$

and again Liouville's Theorem shows that each of the entries of R is constant. Thus R is the identity matrix, and we have established uniqueness.

Existence. We now show that Y of (83) solves (I)–(III). Firstly, (I) is immediate. To verify (II), we must satisfy four equations, two of which reduce to

$$p_{j+}(x)=p_{j-}(x), \quad x\in\mathbb{R},$$

namely continuity of p_j, $j=n-1,n$. The other two are

$$\begin{aligned}&C[p_n w]_+(x)/\gamma_n = p_{n-}(x)w(x)/\gamma_n + C[p_n w]_-(x)/\gamma_n;\\ &-2\pi i\gamma_{n-1}C[p_{n-1}w]_+(x)\\ &\quad = -2\pi i\gamma_{n-1}p_{n-1,-}(x)w(x) - 2\pi i\gamma_{n-1}C[p_{n-1}w]_-(x).\end{aligned}$$

These follow immediately from the Sokhotskii–Plemelj formula (80). The verification of (III) also reduces to four order relations. Two are immediate because $p_n(z)/\gamma_n = z^n + \cdots$ and because $p_{n-1}(z)$ has degree $n-1$. The non-trivial ones are

$$z^n \mathcal{C}[p_n w](z)/\gamma_n = \mathrm{O}\left(\frac{1}{z}\right); \quad |z| \to \infty, \tag{84}$$

$$-2\pi i \gamma_{n-1} z^n \mathcal{C}[p_{n-1} w](z) = 1 + \mathrm{O}\left(\frac{1}{z}\right); \quad |z| \to \infty. \tag{85}$$

These are more complex, because the integral defining the Cauchy transform extends over the whole real line, and z can be close to, or even on, the real line. So we take some care over them. Let us establish (85). We write

$$\frac{1}{s-z} = \frac{s^n}{z^n(s-z)} - \sum_{k=0}^{n-1} \frac{s^k}{z^{k+1}}$$

and use orthogonality to deduce that

$$\begin{aligned}&-2\pi i \gamma_{n-1} z^n \mathcal{C}[p_{n-1} w](z)\\ &\quad = -\gamma_{n-1} \int_{-\infty}^{\infty} \frac{(p_{n-1}w)(s)s^n}{s-z}\,\mathrm{d}s + \gamma_{n-1} \int_{-\infty}^{\infty} (p_{n-1}w)(s)s^{n-1}\,\mathrm{d}s.\end{aligned}$$

Because of orthonormality, the second term on the right-hand side equals

$$\gamma_{n-1} \int_{-\infty}^{\infty} (p_{n-1}w)(s)s^{n-1}\,\mathrm{d}s = \int_{-\infty}^{\infty} (p_{n-1}w)^2(s)\,\mathrm{d}s = 1.$$

We must show that

$$\int_{-\infty}^{\infty} \frac{(p_{n-1}w)(s)s^n}{s-z}\,\mathrm{d}s = \mathrm{O}\left(\frac{1}{z}\right), \quad |z| \to \infty \tag{86}$$

and then (85) follows. (If $z \in \mathbb{R}$, the integral must be taken in a Cauchy principal value sense.) We split the integral into three pieces. Firstly,

$$\left| \int_{\{s:|s-z| \geqslant \frac{|z|}{2}\}} \frac{(p_{n-1}w)(s)s^n}{s-z}\,\mathrm{d}s \right| \leqslant \frac{2}{|z|} \int_{-\infty}^{\infty} |p_{n-1}w|(s)|s|^n\,\mathrm{d}s.$$

Secondly,

$$\left| \int_{\{s:1 \leqslant |s-z| \leqslant \frac{|z|}{2}\}} \frac{(p_{n-1}w)(s)s^n}{s-z}\,\mathrm{d}s \right| \leqslant \frac{2}{|z|} \int_{-\infty}^{\infty} |p_{n-1}w|(s)|s|^{n+1}\,\mathrm{d}s.$$

Write $z = x + iy$. To deal with the remaining range, namely $\{s : |s - z| < 1\}$, we note that it is non-empty only when $|y| < 1$. We write

$$\begin{aligned}
&\int_{\{s:|s-z|<1\}} \frac{(p_{n-1}w)(s)s^n}{s-z}\,\mathrm{d}s \\
&= \int_{x-\sqrt{1-y^2}}^{x+\sqrt{1-y^2}} \frac{(p_{n-1}w)(s)s^{n+1} - (p_{n-1}w)(x)x^{n+1}}{s-z}\,\frac{\mathrm{d}s}{s} + \\
&\quad +(p_{n-1}w)(x)x^{n+1} \int_{x-\sqrt{1-y^2}}^{x+\sqrt{1-y^2}} \frac{\mathrm{d}s}{(s-z)s} \\
&=: I_1 + I_2.
\end{aligned}$$

Firstly, for large z, and equivalently for large x (recall $|y| < 1$),

$$\begin{aligned}
|I_1| &\leqslant \frac{2}{|z|} \int_{x-1}^{x+1} \left| \frac{(p_{n-1}w)(s)s^{n+1} - (p_{n-1}w)(x)x^{n+1}}{s-z} \right| \mathrm{d}s \\
&\leqslant \frac{2}{|z|} \int_{x-1}^{x+1} \left| \frac{(p_{n-1}w)(s)s^{n+1} - (p_{n-1}w)(x)x^{n+1}}{s-x} \right| \mathrm{d}s.
\end{aligned}$$

Our hypothesis that $w(s)s^j$, $j \geqslant 0$, satisfies a Lipschitz condition uniformly on the real line implies that this last integral is bounded independently of x. Next, because of symmetry of the interval of integration about x,

$$\begin{aligned}
|I_2| &= \frac{|(p_{n-1}w)(x)x^{n+1}|}{|z|} \left| \int_{x-\sqrt{1-y^2}}^{x+\sqrt{1-y^2}} \left(\frac{1}{s-z} - \frac{1}{s} \right) \mathrm{d}s \right| \\
&= \frac{|(p_{n-1}w)(x)x^{n+1}|}{|z|} \left| \int_{x-\sqrt{1-y^2}}^{x+\sqrt{1-y^2}} \frac{iy\,\mathrm{d}s}{(s-x)^2 + y^2} - \right. \\
&\quad \left. - \log\left(\frac{x + \sqrt{1-y^2}}{x - \sqrt{1-y^2}} \right) \right| \\
&\leqslant \frac{|(p_{n-1}w)(x)x^{n+2} - (p_{n-1}w)(0)0^{n+2}|}{|z||x|} \times \\
&\quad \times \left[\int_{-\infty}^{\infty} \frac{\mathrm{d}u}{u^2+1} + \log\left(\frac{|x|+1}{|x|-1} \right) \right] \\
&\leqslant \frac{C}{|z|},
\end{aligned}$$

where C is independent of z. We have again used the fact that for each k, $w(s)s^k$ satisfies a Lipschitz condition uniformly on the real line, and we can assume that the Lipschitz constant is < 1. So we have (86) and hence (85). The proof of (84) is similar, but easier. $\square$

Who would have guessed that the above identities could be so useful in establishing asymptotic properties of orthogonal polynomials? A one variable problem has been converted into a 2×2 matrix problem involving singular integrals and boundary values! Compare this with the Bernstein–Szegő formula, whose utility is almost immediate, especially when one recalls how much is known about polynomial approximation.

But in the hands of the group around P. Deift, to whom Riemann–Hilbert problems are bread and butter, the identities above (and some extensions of them) have yielded remarkably precise results. The above problem is transformed by a succession of maps/substitutions, some of which boil down to the mapping of the Mhaskar–Rakhmanov–Saff interval $[a_{-n}, a_n]$ onto $[-1, 1]$, and to use of potential theory. Then, the real line is deformed into a suitable contour, and a two-dimensional version of steepest descent is applied. The details are of course nontrivial. Historically, the first use of this circle of ideas was by Bleher and Its [10], though their method differs substantially from that of the group around Deift.

As an illustration of the power of the method, we quote a small part of the impressive recent results of Kriecherbauer and McLaughlin [49]:

THEOREM 2.12. *Let* $\alpha > 0$ *and*

$$w(x) := \exp\bigl(-2|x|^{\alpha}\bigr), \quad x \in \mathbb{R}.$$

Let

$$a_n = a_n(\alpha) := n^{1/\alpha}\left[\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha+1}{2})}\right]^{1/\alpha}, \quad n \geqslant 1.$$

Then

$$\gamma_n\left(\frac{a_n}{2}\right)^{n+\frac{1}{2}} \mathrm{e}^{-n/\alpha}\sqrt{2\pi} = 1 + \frac{\alpha - 4}{24\alpha n} + \frac{\varepsilon_n}{n}, \quad n \to \infty, \tag{87}$$

where

$$\varepsilon_n = \mathrm{O}\bigl((\log n)^{-2}\bigr), \quad n \to \infty. \tag{88}$$

Consequently, for some c_1, c_2 *independent of* n,

$$\prod_{j=1}^{n}\left[\gamma_j\left(\frac{a_j}{2}\right)^{j+\frac{1}{2}} \mathrm{e}^{-j/\alpha}\sqrt{2\pi}\right] = c_1 n^{c_2}\bigl(1 + \mathrm{o}(1)\bigr), \quad n \to \infty. \tag{89}$$

What is so impressive is the precision in the asymptotic in (87), leading to the strong Szegő limit (89) for all $\alpha > 0$. For α a positive even integer, the asymptotic expansion for the recurrence coefficients given by Mate, Nevai and Zaslavsky [78] implied both (87) and (89). We emphasise that more detailed information regarding ε_n is given in [49]. That, together with the asymptotics for p_n that are established in all parts of the plane – most notably near $\pm a_n$, illustrate the impressive power of the Riemann–Hilbert technique.

Undoubtedly investigations in the next few years will reveal the full potential of this exciting new method.

2.4. RAKHMANOV'S PROJECTION IDENTITY

Rakhmanov's projection identity first appeared in his 1992 paper [106], as part of his proof of asymptotics for orthonormal polynomials for the weights $\exp(-|x|^\alpha)$, $\alpha > 1$, on the real line. There the identity was applied on a growing sequence of intervals, namely the Mhaskar–Rakhmanov–Saff intervals $[-a_r, a_r]$ for appropriate choices of r.

Here we shall present the identity for a weight w on $[-1, 1]$ and for a fixed n. We are forced to use some potential theory but have attempted to keep it to a minimum. Those readers requiring further orientation can refer to [79] or [108]. We write

$$w(x) = \exp\bigl(-2Q(x)\bigr), \quad x \in (-1, 1) \tag{90}$$

and assume that there is a finite, absolutely continuous measure

$$\mathrm{d}\nu(x) = \nu'(x)\,\mathrm{d}x \quad \text{on } [-1, 1] \tag{91}$$

of total mass n, that is

$$\int_{-1}^{1} \mathrm{d}\nu = n, \tag{92}$$

satisfying the equilibrium condition

$$\int_{-1}^{1} \log|x - t|^{-1}\,\mathrm{d}\nu(t) + Q(x) = \alpha, \quad x \in (-1, 1). \tag{93}$$

Here α is a (real) constant. A positive measure $\mathrm{d}\nu$ satisfying this last relation is called the equilibrium measure of total mass n for the external field Q. It exists and is unique, if, for example, Q is convex on $(-1, 1)$ and satisfies some other conditions that guarantee that $\mathrm{d}\nu$ has mass n. (The argument that follows does not seem to require that $\mathrm{d}\nu$ is positive though!) Let us set

$$\mathrm{d}\mu(x) := \mathrm{d}\nu(x) + \frac{1}{2}\frac{\mathrm{d}x}{\pi\sqrt{1 - x^2}}, \quad x \in (-1, 1) \tag{94}$$

so that

$$\int_{-1}^{1} \mathrm{d}\mu = n + \frac{1}{2}. \tag{95}$$

Also let

$$c_n := \frac{1}{\sqrt{\pi}} \exp(\alpha); \tag{96}$$

$$\phi_n(x) := \pi \int_x^1 \mathrm{d}\mu(t) - \frac{\pi}{4}; \tag{97}$$

$$A(x) := \left[2\pi\sqrt{1-x^2}w(x)\right]^{-1/2}; \tag{98}$$

$$\begin{aligned} R_n(x) :&= 2A(x)\cos\phi_n(x) \\ &= \sqrt{\frac{2}{\pi}}\left[\sqrt{1-x^2}w(x)\right]^{-1/2}\cos\left(\pi\int_x^1 \mathrm{d}\mu(t) - \frac{\pi}{4}\right). \end{aligned} \tag{99}$$

We note that $R_n(x)$ is the expression expected to describe the behaviour of $p_n(x)$ as $n \to \infty$:

$$\begin{aligned} &(1-x^2)^{1/4}w(x)^{1/2}p_n(x) \\ &\quad = \sqrt{\frac{2}{\pi}}\cos\left(\pi\int_x^1 \mathrm{d}\mu(t) - \frac{\pi}{4}\right) + \mathrm{o}(1). \end{aligned}$$

Compare (67) and (55). Of course here we are fixing n and adjusting w so that the equilibrium measure $\mathrm{d}\nu$ for w has total mass n and is supported on $[-1, 1]$.

Define an inner product and norm with respect to the weight w by

$$(f, g) := \int_{-1}^1 fgw; \tag{100}$$

$$\|f\| := (f, f)^{1/2} = \left(\int_{-1}^1 f^2 w\right)^{1/2}.$$

Finally, let as usual, $K_n(x, t)$ denote the nth reproducing kernel for w, as at (34). We can now state:

THEOREM 2.13. *Let* $w = \exp(-2Q)$ *be a weight on* $[-1, 1]$ *and assume that there is a measure* $\mathrm{d}\nu$ *satisfying* (91)–(93) *and, moreover, that in each compact subinterval of* $(-1, 1)$, $\nu'(x)$ *satisfies a Lipschitz condition of positive order. Assume the notation* (94) *and* (96)–(99). *Let*

$$E_n := \inf_{\deg(P)\leqslant n} \|R_n - P\|. \tag{101}$$

Then the inf *is attained by* $(c_n/\gamma_n)p_n$ *and only this polynomial, so that*

$$E_n = \left\|R_n - \frac{c_n}{\gamma_n}p_n\right\|. \tag{102}$$

Moreover,

$$\frac{c_n}{\gamma_n} p_n(x) = \int_{-1}^{1} K_{n+1}(x,t) R_n(t) w(t)\, \mathrm{d}t, \tag{103}$$

and

$$E_n^2 = 1 - \left(\frac{\gamma_n}{c_n}\right)^2 + e_n, \tag{104}$$

where

$$e_n := \frac{1}{\pi} \int_{-1}^{1} \cos\bigl(2\phi_n(x)\bigr) \frac{\mathrm{d}x}{\sqrt{1-x^2}}. \tag{105}$$

What is remarkable is that the expression that we expect to describe the asymptotic behaviour of p_n, as $n \to \infty$, namely $R_n(x)$, appears in an extremal problem which is uniquely solved by p_n (suitably normalized). This really helps to motivate the asymptotic form. Furthermore, it seems immediately obvious that it should be useful in studying asymptotics. It was used by E. A. Rakhmanov to good effect in [106] in providing a compact proof of asymptotics for the orthonormal polynomials for exponential weights. There, though, this was not the only new identity: to estimate e_n, E_n, and hence to show that $\gamma_n/c_n \to 1-$, other strikingly original ideas and identities were developed and used.

We shall need three lemmas as a prelude to the proof of Theorem 2.13. The first involves the Cauchy transform of $\mathrm{d}\mu$. Recall that

$$\mathcal{C}[\mathrm{d}\mu](z) := \frac{1}{2\pi i} \int_{-1}^{1} \frac{\mathrm{d}\mu(t)}{t-z}, \quad z \in \mathbb{C}\backslash[-1,1].$$

The Sokhotskii–Plemelj formulas (80) may be recast in the form

$$\mathcal{C}[\mathrm{d}\mu]_{\pm}(x) = \frac{PV}{2\pi i} \int_{-1}^{1} \frac{\mathrm{d}\mu(t)}{t-x} \pm \frac{1}{2}\mu'(x), \quad x \in (-1,1). \tag{106}$$

Our hypothesis of the Lipschitz condition on ν' is enough to guarantee that this last relation holds pointwise, and that all functions involved also satisfy a Lipschitz condition. We shall apply this to the complex potential associated with $\mathrm{d}\mu$. Define

$$\begin{aligned} V^{\nu}(z) &:= \int_{-1}^{1} \log|z-t|^{-1}\, \mathrm{d}\nu(t); \\ V(z) &:= \int_{-1}^{1} \log|z-t|^{-1}\, \mathrm{d}\mu(t). \end{aligned}$$

Notice that we use a superscript to distinguish the potential V^{ν} for $\mathrm{d}\nu$ from that for $\mathrm{d}\mu$. Note too that by definition of μ,

$$\begin{aligned} V(x) &= V^{\nu}(x) + \frac{1}{2}\int_{-1}^{1} \log|x-t|^{-1} \frac{\mathrm{d}t}{\pi\sqrt{1-t^2}} \\ &= V^{\nu}(x) + \frac{1}{2}\log 2, \quad x \in [-1,1]. \end{aligned} \tag{107}$$

The last step involves an elementary identity of potential theory – see, for example, [72, p. 30]. Define the complex potential for $\mathrm{d}\mu$,

$$U(z) := \int_{-1}^{1} \log(z-t)^{-1}\,\mathrm{d}\mu(t), \quad z \in \mathbb{C}\backslash[-1,1] \tag{108}$$

and define the conjugate function for V by

$$U(z) =: V(z) + iV^*(z), \quad z \in \mathbb{C}\backslash[-1,1]. \tag{109}$$

Both V^* and U are multivalued in $\overline{\mathbb{C}}\backslash[-1,1]$, but we consider the singlevalued branch of V^* in $\mathbb{C}\backslash(-\infty,1]$ normalized by

$$V^*(z) = 0, \quad z \in [1,\infty). \tag{110}$$

Our first lemma deals with the boundary values $U_\pm$ of U from the upper and lower half planes:

LEMMA 2.14. *The complex potential U has a continuous extension to* $[-1,1]$. *In addition for $x \in (-1,1)$,*

$$U_\pm(x) = V(x) \mp i\pi \int_x^1 \mathrm{d}\mu. \tag{111}$$

Proof. Now for $z \in \mathbb{C}\backslash[-1,1]$,

$$U'(z) = \int_{-1}^{1} \frac{\mathrm{d}\mu(t)}{t-z} = 2\pi i\,\mathcal{C}[\mathrm{d}\mu](z).$$

Then the Sokhotskii–Plemelj formulas (106) give

$$V_\pm^{*\prime}(x) = \operatorname{Im} U'_\pm(x) = 2\pi \operatorname{Re} \mathcal{C}[\mathrm{d}\mu]_\pm(x) = \pm\pi\mu'(x), \quad x \in (-1,1). \tag{112}$$

Both the functions $V_\pm^*$ are continuous in $(-1,1)$ and the assertion concerning the continuous extension of U follows. Next, by our choice of branches, $V_\pm^*(1) = 0$, so for $x \in (-1,1)$,

$$\begin{aligned} U_\pm(x) &= V_\pm(x) + i0 + i\int_1^x V_\pm^{*'}(t)\,\mathrm{d}t \\ &= V(x) \pm i\pi \int_1^x \mathrm{d}\mu, \end{aligned}$$

by (112) and the absolute continuity of $\mathrm{d}\mu$ with respect to Lebesgue measure. □

Now define

$$W(z) := c_n(z^2-1)^{-1/4}\exp\bigl(-U(z)\bigr), \quad z \in \mathbb{C}\backslash[-1,1], \tag{113}$$

where the branch of $(z^2-1)^{-1/4}$ is chosen so that

$$(z^2-1)^{-1/4} > 0, \quad z \in (1,\infty).$$

Since μ has total mass $n+\frac{1}{2}$, we see that (with a multivalued log),

$$U(z) = -\left(n+\frac{1}{2}\right)\log z + \mathrm{o}(1), \quad |z| \to \infty$$

so

$$W(z)/z^n \to c_n, \quad |z| \to \infty. \tag{114}$$

Thus $W(z)$ has a single valued analytic continuation to $\mathbb{C}\backslash[-1,1]$ with pole at ∞. We need a technical lemma:

LEMMA 2.15. *For* $x \in (-1,1)$,

$$\text{(i)} \quad W_\pm(x) = A(x)\exp\bigl(\pm i\phi_n(x)\bigr); \tag{115}$$

$$\text{(ii)} \quad \operatorname{Re} W_\pm(x) = \frac{1}{2}R_n(x); \tag{116}$$

$$\text{(iii)} \quad \bigl|W_\pm(x)\bigr|^{-2} = 2\pi w(x)\sqrt{1-x^2}. \tag{117}$$

Proof. From (107) and then (93),

$$V(x) = V^\nu(x) + \tfrac{1}{2}\log 2 = -Q(x) + \alpha + \tfrac{1}{2}\log 2.$$

Also

$$\bigl((z^2-1)^{-1/4}\bigr)_\pm(x) = (1-x^2)^{-1/4}\exp(\mp i\pi/4),$$

so from (113) and then (111) and (96),

$$\begin{aligned} W_\pm(x) &= c_n(1-x^2)^{-1/4}\exp(\mp i\pi/4)\exp(-U_\pm(x)) \\ &= \frac{1}{\sqrt{2\pi}}(1-x^2)^{-1/4}\exp\bigl(Q(x)\bigr)\exp\left(\pm i\left[\pi\int_x^1 \mathrm{d}\mu - \frac{\pi}{4}\right]\right) \\ &= A(x)\exp\bigl(\pm i\phi_n(x)\bigr). \end{aligned}$$

Thus we have (i). Then (ii) follows from (99), and (iii) follows from the definition (98) of A. □

Our final lemma is:

LEMMA 2.16. *If q is a polynomial of degree n with real coefficients and leading coefficient k,*

$$(q, R_n) = \int_{-1}^{1} qR_n w = k/c_n. \tag{118}$$

Proof. Now

$$u(z) := \mathrm{Re}\left(\frac{q}{W}(z)\right)$$

is harmonic in $\mathbb{C}\backslash[-1, 1]$ and by (114),

$$u(z) \to k/c_n, \quad |z| \to \infty,$$

so it is also harmonic at ∞. Let

$$g(z) := \begin{cases} u\left(\frac{1}{2}(z + z^{-1})\right), & 0 < |z| < 1, \\ k/c_n, & z = 0. \end{cases}$$

Then g is harmonic in the open unit ball. If $\theta \in (0, \pi)$, then as $z \to \mathrm{e}^{\pm i\theta}$ from inside the unit ball,

$$\begin{aligned} g(z) &\to \mathrm{Re}\left(\frac{q}{W_\pm}(\cos\theta)\right) \\ &= \mathrm{Re}\left(\frac{(qW_\pm)(\cos\theta)}{|W_\pm(\cos\theta)|^2}\right) \\ &= \pi(qR_n w)(\cos\theta)|\sin\theta|, \end{aligned}$$

by (116) and (117). The mean value property for harmonic functions gives

$$\begin{aligned} k/c_n &= g(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\mathrm{e}^{i\theta})\,\mathrm{d}\theta \\ &= \frac{1}{2}\int_{-\pi}^{\pi}(qR_n w)(\cos\theta)|\sin\theta|\,\mathrm{d}\theta \\ &= \int_{-1}^{1} qR_n w. \end{aligned}$$

□

We turn to

The Proof of Theorem 2.13. First note that if q is a polynomial of degree $\leqslant n$, with real coefficients and leading coefficient k, orthonormality gives

$$\left(p_n\frac{\gamma_n}{c_n}, q\right) = \left(\frac{\gamma_n}{c_n}p_n, \frac{k}{\gamma_n}p_n\right) = \frac{k}{c_n} = (R_n, q)$$

by Lemma 2.16, so

$$\left(R_n - p_n\frac{\gamma_n}{c_n}, q\right) = 0. \tag{119}$$

Then

$$\begin{aligned}\|R_n - q\|^2 &= \left\| \left(R_n - p_n \frac{\gamma_n}{c_n} \right) + \left(p_n \frac{\gamma_n}{c_n} - q \right) \right\|^2 \\ &= \left\| R_n - p_n \frac{\gamma_n}{c_n} \right\|^2 + \left\| p_n \frac{\gamma_n}{c_n} - q \right\|^2 .\end{aligned}$$

It follows that

$$\begin{aligned}E_n^2 &= \min_{\deg(P)\leqslant n} \|R_n - P\|^2 \\ &= \left\| R_n - p_n \frac{\gamma_n}{c_n} \right\|^2 \\ &= \|R_n\|^2 - 2\left(p_n \frac{\gamma_n}{c_n}, R_n \right) + \left\| p_n \frac{\gamma_n}{c_n} \right\|^2 \\ &= \|R_n\|^2 - 2\left(p_n \frac{\gamma_n}{c_n}, p_n \frac{\gamma_n}{c_n} \right) + \left\| p_n \frac{\gamma_n}{c_n} \right\|^2 \\ &= \|R_n\|^2 - \left(\frac{\gamma_n}{c_n} \right)^2 . \end{aligned} \tag{120}$$

We have used (119). Next, using the identity

$$2(\operatorname{Re} z)^2 = \operatorname{Re}(z^2) + |z|^2, \quad z \in \mathbb{C},$$

and Lemma 2.15(ii), (iii), we see that

$$\begin{aligned}R_n(x)^2 &= 4(\operatorname{Re} W_\pm(x))^2 \\ &= 2\operatorname{Re}\big(W_\pm(x)^2\big) + 2\big|W_\pm(x)\big|^2 \\ &= 2A^2(x)\cos\big(2\phi_n(x)\big) + \frac{1}{\pi w(x)\sqrt{1-x^2}}\end{aligned}$$

so that

$$\begin{aligned}\|R_n\|^2 &= \int_{-1}^{1} R_n^2 w \\ &= \frac{1}{\pi}\int_{-1}^{1} \cos\big(2\phi_n(x)\big)\frac{\mathrm{d}x}{\sqrt{1-x^2}} + \frac{1}{\pi}\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^2}} \\ &= e_n + 1.\end{aligned}$$

Putting this in (120) gives (104). Finally, (103) follows directly from (102), since $\int_{-1}^{1} K_{n+1}(t,x)R_n(t)w(t)\,\mathrm{d}t$ is the $(n+1)$st partial sum of the orthonormal expansion of R_n and so is the unique polynomial giving the minimum in E_n. Thus by uniqueness, it equals $p_n(\gamma_n/c_n)$. □

2.5. A COMPARISON?

In attempting to compare the three identities presented above, and their applicability to asymptotics, I am forced to admit my own lack of expertise – my experience has been primarily with Bernstein–Szegő identities. Another criticism is that the three identities presented in the previous three sections vary in their distance to the actual asymptotic: I have not presented the full passage from identity to asymptotic. Nevertheless at least personally I find it instructive to contrast them.

- The connection between the Bernstein–Szegő identity, or Rakhmanov's projection identity, and asymptotics is intuitively fairly obvious. The Fokas–Kitaev–Its identity seems more distant.
- Historically Bernstein–Szegő on $[-1, 1]$ and its cousin on the circle underlie Szegő asymptotics on the circle and $[-1, 1]$ and also on the real line. It has a proven record of giving Szegő or power asymptotics in fairly general situations, as well as pointwise asymptotics for orthogonal polynomials on the interval of orthogonality. Nevertheless, it does not provide good error terms.
- The Fokas–Its–Kitaev identity in the hands of Deift, Kriecherbauer, McLaughlin and others has proved to be a very powerful tool, yielding very precise asymptotics, with error terms, and even asymptotic expansions. Its applicability at the moment requires some sort of analyticity of Q.
- The Rakhmanov projection identity applies to very general weights but so far the estimation of the quantities that lead from the identity to the asymptotic also require some sort of analyticity of Q. It yields better error terms than does Bernstein–Szegő but at present does not yield the precision of the Deift–Kriecherbauer–McLaughlin approach based on the Fokas–Its–Kitaev identity.

In summary, my own feelings (which must be taken with a pinch of salt!) is that there is place for all three identities in the field of asymptotics of orthogonal polynomials. At present it seems that the Fokas–Its–Kitaev identity will lead to very precise asymptotics for restricted classes of weights, while I believe that Rakhmanov's projection identity and Bernstein–Szegő will lead to asymptotics for more general weights, but with weaker error estimates. It is too early to tell if Rakhmanov's projection identity will push out Bernstein–Szegő for general exponential weights – but then even the full potential of the Deift–Kriecherbauer–McLaughlin approach is not clear.

As we head into a new century and a new millenium, there is clearly plenty of scope for research into asymptotics of orthogonal polynomials!

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References

1. Achieser, N. I.: *Theory of Approximation* (translated by C. J. Hyman), Dover, New York, 1992.
2. Ambroladze, A.: On exceptional sets of asymptotic relations for general orthogonal polynomials, *J. Approx. Theory* **82** (1995), 257–273.
3. Aptekarev, A. I.: Asymptotics of orthogonal polynomials in a neighbourhood of the endpoints of the interval of orthogonality, *Russian Acad. Sci. Math. Sbornik* **76** (1993), 35–50.
4. Aptekarev, A. I.: Asymptotic behaviour of L_p norms and entropy for general orthogonal polynomials, *Russian Acad. Sci. Math. Sbornik* **185** (1994), 3–30.
5. Aptekarev, A. I., Dehesa, J. S. and Yanez, R. J.: Spatial entropy of central potentials and strong asymptotics of orthogonal polynomials, *J. Math. Phys.* **35** (1994), 4423–4428.
6. Aptekarev, A. I., Branquinho, A. and Marcellan, F.: Toda-type differential equations for the recurrence coefficients of orthogonal polynomials and Freud transformation, to appear.
7. Barrios, D., Lopez, G. and Saff, E. B.: Asymptotics of orthogonal polynomials inside the unit circle and Szegő–Padé approximants, manuscript.
8. Barrios, D., Lopez, G. and Torrano, E.: Polynomials generated by a three term recurrence relation with asymptotically periodic complex coefficients, *Math. USSR Sb.* **189** (1995), 629–659.
9. Beckermann, B.: On a conjecture of E. A. Rakhmanov, *Constr. Approx.* **16** (2000), 427–448.
10. Bleher, P. and Its, A.: Semiclassical asymptotics of orthogonal polynomials, Riemann–Hilbert problem, and universality in the matrix model, *Ann. of Math.* **150** (1999), 185–266.
11. Bo, R. and Wong, R.: A uniform asymptotic formula for orthogonal polynomials associated with $\exp(-x^4)$, *J. Approx. Theory* **98** (1999), 146–166.
12. Bonan, S. and Clark, D. S.: Estimates of the Hermite and the Freud Polynomials, *J. Approx. Theory* **63** (1990), 210–224.
13. Böttcher, A. and Silbermann, B.: *Analysis of Toeplitz Operators*, Springer/Akademie Verlag, Berlin, 1990.
14. Böttcher, A. and Silbermann, B.: *Introduction to Large Truncated Toeplitz Matrices*, Springer, Berlin, 1999.
15. Bultheel, A., Gonzalez-Vera, P., Hendriksen, E. and Njastad, O.: *Orthogonal Rational Functions*, Cambridge, 1999.
16. Buyarov, V. S.: Logarithmic asymptotics of polynomials orthogonal on the real axis, *Soviet Math. Dokl.* **43** (1991), 743–746.
17. Buyarov, V. S.: Logarithmic asymptotics of polynomials orthogonal with asymmetric weights, *Math. Notes Acad. Sci. USSR* **50** (1992), 789–795.
18. Buyarov, V. S. and Rakhmanov, E. A.: Families of equilibrium measures in an external field on the real axis, *Math. Sb.* **190** (1999), 791–802.
19. Chen, Y. and Ismail, M. E. H.: Hermitean matrix ensembles and orthogonal polynomials, *Stud. Appl. Math.* **100** (1998), 33–52.
20. Chen, Y., Ismail, M. E. H. and Van Assche, W.: Tau-function constructions of the recurrence coefficients of orthogonal polynomials, *Adv. in Appl. Math.* **20** (1998), 141–168.
21. Deift, P.: *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*, Courant Lecture Notes, 1999.
22. Deift, P., Kriecherbauer, T., McLaughlin, K., Venakides, S. and Zhou, X.: Asymptotics for polynomials orthogonal with respect to varying exponential weights, *Internat. Math. Res. Notices* **16** (1997), 759–782.

23. Deift, P., Kriecherbauer, T. and McLaughlin, K.: New results on the equilibrium measure for logarithmic potentials in the presence of an external field, *J. Approx. Theory* **95** (1998), 388–475.
24. Deift, P., Kriecherbauer, T. and McLaughlin, K.: New results for the asymptotics of orthogonal polynomials and related problems via the Lax–Levermore method, *Proc. Sympos. Appl. Math.* **54** (1998), 87–104.
25. Deift, P., Kriecherbauer, T., McLaughlin, K., Venakides, S. and Zhou, X.: Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.* **52** (1999), 1491–1552.
26. Deift, P., Kriecherbauer, T., McLaughlin, K., Venakides, S. and Zhou, X.: Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.* (to appear).
27. Erdős, P. and Turan, P.: On interpolation II, *Ann. of Math.* **39** (1938), 703–724.
28. Erdős, P. and Turan, P.: On interpolation III, *Ann. of Math.* **41** (1940), 510–553.
29. Faber, G.: *Ueber nach Polynomen fortschreitende Reihen*, Sitzungsber. Bayerischen Akad. Wiss., 1922, pp.157–178.
30. Fokas, A. S., Its, A. R. and Kitaev, A. V.: Isomonodromic approach in the theory of two-dimensional quantum gravity, *Uspekhi Mat. Nauk* **45** (1990), 135–136.
31. Fokas, A. S., Its, A. R. and Kitaev, A. V.: Discrete Painleve equations and their appearance in quantum gravity, *Comm. Math. Phys.* **142** (1991), 313–344.
32. Freud, G.: *Orthogonal Polynomials*, Akademiai Kiado/Pergamon Press, Budapest, 1971.
33. Garnett, J. B.: *Bounded Analytic Functions*, Academic Press, Orlando, 1981.
34. Geronimo, J. S., Smith, D. and Van Assche, W.: Strong asymptotics for orthogonal polynomials with regularly and slowly varying recurrence coefficients, *J. Approx. Theory* **72** (1993), 141–158.
35. Geronimo, J. S. and Van Assche, W.: Orthogonal polynomials with asymptotically periodic recurrence coefficients, *J. Approx. Theory* **46** (1986), 251–283.
36. Geronimo, J. S. and Van Assche, W.: Relative asymptotics for orthogonal polynomials with unbounded recurrence coefficients, *J. Approx. Theory* **62** (1990), 47–69.
37. Geronimus, Ya. L.: *Orthogonal Polynomials*, Consultants Bureau, New York, 1961.
38. Golinskii, L.: Schur functions, Schur parameters and orthogonal polynomials on the unit circle, *Z. Anal. Anwendungen* **12** (1993), 457–469.
39. Golinskii, L.: Reflection coefficients for generalized Jacobi weight functions, *J. Approx. Theory* **78** (1994), 117–126.
40. Golinskii, L.: Uniform asymptotics for second kind ultraspherical polynomials on the unit circle, *Constr. Approx.* **14** (1998), 599–608.
41. Golinskii, L.: Akhiezer's orthogonal polynomials and Bernstein–Szegő method for a circular arc, *J. Approx. Theory* **95** (1998), 229–263.
42. Golinskii, L., Nevai, P. and Van Assche, W.: Perturbation of orthogonal polynomials on an arc of the unit circle, *J. Approx. Theory* **83** (1995), 392–422.
43. Grenander, U. and Szegő, G.: *Toeplitz Forms and Their Applications*, Chelsea, New York, 1984.
44. Henrici, P.: *Applied and Computational Complex Analysis*, Vol. 3, Wiley, 1993.
45. Hernandez, M. B. and Lopez, G.: Ratio and relative asymptotics of polynomials orthogonal on an arc of the unit circle, *J. Approx. Theory* **92** (1998), 216–244.
46. Jha, Shing-Wu: The second order linear differential equations satisfied by the orthogonal polynomials associated with $w(x) = |x|^\rho \exp(-x^{2m})$ ($\rho > -1, m > 0$ integer), Preprint, 1992.
47. Jha, Shing-Wu: Asymptotics for orthogonal polynomials associated with $\exp(-x^{2m})$, $m \geqslant 2$, integer, Preprint, 1993.
48. Koosis, P.: *The Logarithmic Integral I*, Cambridge University Press, Cambridge, 1988.

49. Kriecherbauer, T. and McLaughlin, K. T.-R.: Strong asymptotics of polynomials orthogonal with respect to Freud weights, *Internat. Math. Res. Notices* **6** (1999), 299–333.
50. Kuijlaars, A.: Contracted zero distributions of extremal polynomials associated with slowly decreasing weights, Manuscript.
51. Kuijlaars, A.: On the finite gap ansatz in the continuum limit of the Toda lattice, Manuscript.
52. Kuijlaars, A.: Which eigenvalues are found by the Lanczos method?, Manuscript.
53. Kuijlaars, A. and Rakhmanov, E. A.: Zero distributions for discrete orthogonal polynomials, *J. Comp. Appl. Math.* **99** (1998), 255–274.
54. Kuijlaars, A. and Van Assche, W.: Extremal problems on discrete sets, *Proc. London Math. Soc.* **76** (1999), 191–221.
55. Kuijlaars, A. and Van Assche, W.: The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients, *J. Approx. Theory* **99** (1999), 167–197.
56. Landkof, N. S.: *Foundations of Modern Potential Theory* (translated by A. P. Doohovskoy), Springer, Berlin, 1972.
57. Levin, A. L. and Lubinsky, D. S.: Christoffel functions, orthogonal polynomials, and Nevai's conjecture for Freud weights, *Constr. Approx.* **8** (1992), 463–535.
58. Levin, A. L. and Lubinsky, D. S.: Christoffel functions and orthogonal polynomials for exponential weights on $[-1, 1]$, *Mem. Amer. Math. Soc.* 535 **111** (1994).
59. Levin, A. L. and Lubinsky, D. S.: Bounds for orthogonal polynomials for exponential weights, *J. Comp. Appl. Math.* **99** (1998), 475–490.
60. Levin, A. L. and Lubinsky, D. S.: Asymptotics associated with exponential weights, *Internat. Math. Res. Notices* **12** (1999), 673–683.
61. Levin, A. L. and Lubinsky, D. S.: *Orthogonal Polynomials for Exponential Weights*, to appear.
62. Lopez, G. L.: On the asymptotics of the ratio of orthogonal polynomials and convergence of multipoint Padé approximants, *Math. USSR Sb.* **56** (1987), 207–219.
63. Lopez, G. L.: Asymptotics of polynomials orthogonal with respect to varying measures, *Constr. Approx.* **5** (1989), 199–219.
64. Lopez, G. L.: Relative asymptotics for polynomials orthogonal on the real axis, *Math. USSR Sb.* **65** (1990), 505–529.
65. Lopez, G. L. and Rakhmanov, E. A.: Rational approximations, orthogonal polynomials, and equilibrium distributions, In: *Lecture Notes in Math.* 1329, Springer, Berlin, 1988, pp. 125–157.
66. Lubinsky, D. S.: *Strong Asymptotics for Extremal Errors Associated with Erdős-type Weights*, Pitman Research Notes 202, Longmans, Burnt Mill, Essex, 1989.
67. Lubinsky, D. S.: A survey of general orthogonal polynomials for weights on finite and infinite intervals, *Acta Appl. Math.* **10** (1987), 237–296.
68. Lubinsky, D. S.: The approximate approach to orthogonal polynomials for weights on $(-\infty, \infty)$, In: P. Nevai (ed.), *Orthogonal Polynomials: Theory and Practice*, NATO ASI Series C 294, Kluwer Acad. Publ., Dordrecht, 1990, pp. 293–310.
69. Lubinsky, D. S.: An update on orthogonal polynomials and weighted approximation on the real line, *Acta Appl. Math.* **33** (1993), 121–164.
70. Lubinsky, D. S., Mhaskar, H. N. and Saff, E. B.: Freud's conjecture for exponential weights, *Bull. Amer. Math. Soc.* **15** (1986), 217–221.
71. Lubinsky, D. S., Mhaskar, H. N. and Saff, E. B.: A proof of Freud's conjecture for exponential weights, *Constr. Approx.* **4** (1988), 65–83.
72. Lubinsky, D. S. and Saff, E. B.: *Strong Asymptotics for Extremal Polynomials Associated with Weights on* $(-\infty, \infty)$, Lecture Notes in Math. 1305, Springer, Berlin, 1988.
73. Magnus, A.: On Freud's equations for exponential weights, *J. Approx. Theory* **46** (1986), 65–99.
74. Mate, A., Nevai, P. and Totik, V.: Asymptotics for the zeros of orthogonal polynomials associated with infinite intervals, *J. London Math. Soc.* **33** (1986), 303–310.

75. Mate, A., Nevai, P. and Totik, V.: Extensions of Szegő's theory of orthogonal polynomials II, *Constr. Approx.* **3** (1987), 51–72.
76. Mate, A., Nevai, P. and Totik, V.: Extensions of Szegő's theory of orthogonal polynomials III, *Constr. Approx.* **3** (1987), 73–96.
77. Mate, A., Nevai, P. and Totik, V.: Strong and weak convergence of orthogonal polynomials, *Amer. J. Math.* **109** (1987), 239–282.
78. Mate, A., Nevai, P. and Zaslavsky, T.: Asymptotic expansion of ratios of coefficients of orthogonal polynomials with exponential weights, *Trans. Amer. Math. Soc.* **287** (1985), 495–505.
79. Mhaskar, H. N.: *Introduction to the Theory of Weighted Polynomial Approximation,* World Scientific, 1996.
80. Mhaskar, H. N. and Saff, E. B.: Extremal problems for polynomials with exponential weights, *Trans. Amer. Math. Soc.* **285** (1984), 204–234.
81. Mhaskar, H. N. and Saff, E. B.: Where does the sup norm of a weighted polynomial live?, *Constr. Approx.* **1** (1985), 71–91.
82. Mhaskar, H. N. and Saff, E. B.: Where does the L_p norm of a weighted polynomial live?, *Trans. Amer. Math. Soc.* **303** (1987), 109–124.
83. Mhaskar, H. N. and Saff, E. B.: On the distribution of zeros of polynomials orthogonal on the unit circle, *J. Approx. Theory* **63** (1990), 30–38.
84. Nevai, P.: Orthogonal polynomials, *Mem. Amer. Math. Soc.* **213** (1979).
85. Nevai, P.: Asymptotics for orthogonal polynomials associated with $\exp(-x^4)$, *SIAM J. Math. Anal.* **15** (1984), 1177–1187.
86. Nevai, P.: A new class of orthogonal polynomials, *Proc. Amer. Math. Soc.* **91** (1984), 409–415.
87. Nevai, P.: Geza Freud, orthogonal polynomials and Christoffel functions: A case study, *J. Approx. Theory* **48** (1986), 3–167.
88. Nevai, P. (ed.), *Orthogonal Polynomials and Special Functions: Theory and Practice*, NATO ASI Series 294, Kluwer Acad. Publ., Dordrecht, 1990.
89. Nevai, P. and Dehesa, J. S.: On asymptotic average properties of zeros of orthogonal polynomials, *SIAM J. Math. Anal.* **10** (1979), 1184–1192.
90. Pan, K.: On orthogonal polynomials with respect to varying measures on the unit circle, *Trans. Amer. Math. Soc.* **346** (1994), 331–340.
91. Pan, K.: Extensions of Szegő's theory of rational functions orthogonal on the unit circle, *J. Comp. Appl. Math.* **62** (1995), 321–331.
92. Pastur, L. A. and Scherbina, M.: Universality of the local eigenvalue statistics for a class of unitarily invariant random matrix ensembles, *J. Statist. Phys.* **86** (1997), 109–147.
93. Peherstorfer, F.: On Bernstein–Szegő orthogonal polynomials on several intervals, *SIAM J. Math. Anal.* **21** (1990), 461–482.
94. Peherstorfer, F.: Elliptic orthogonal and extremal polynomials, *Proc. London Math. Soc.* **70** (1995), 605–624.
95. Peherstorfer, F.: A special class of polynomials orthogonal on the unit circle including the associated polynomials, *Constr. Approx.* **12** (1996), 161–185.
96. Peherstorfer, F.: Explicit generalized Zolotarev polynomials with complex coefficients, *Constr. Approx.* **13** (1997), 261–270.
97. Peherstorfer, F.: Minimal polynomials on several intervals with respect to the maximum norm – a survey, In: A. Martinez *et al*. (eds), *Proc. of the Workshop "Methods of Complex Analysis in Approximation Theory"*, Publ. de la Universidad de Almeria, 1997, pp. 137–159.
98. Peherstorfer, F. and Steinbauer, R.: Orthogonal polynomials on arcs of the unit circle II: Orthogonal polynomials with periodic reflection coefficients, *J. Approx. Theory* **87** (1996), 60–102.
99. Peherstorfer, F. and Steinbauer, R.: Comparative asymptotics for perturbed orthogonal polynomials, *Trans. Amer. Math. Soc.* **348** (1996), 1459–1486.

100. Peherstorfer, F. and Steinbauer, R.: Asymptotic behaviour of orthogonal polynomials on the unit circle with asymptotically periodic reflection coefficients, *J. Approx. Theory* **88** (1997), 316–353.
101. Qiu, W.-Y. and Wong, R.: Uniform asymptotic formula for orthogonal polynomials with exponential weight, Manuscript.
102. Rakhmanov, E. A.: On the asymptotics of the ratio of orthogonal polynomials, *Math. USSR Sb.* **32** (1977), 199–213.
103. Rakhmanov, E. A.: On the asymptotics of the ratio of orthogonal polynomials II, *Math. USSR Sb.* **46** (1983), 105–117.
104. Rakhmanov, E. A.: On asymptotic properties of polynomials orthogonal on the real axis, *Math. USSR Sb.* **47** (1984), 155–193.
105. Rakhmanov, E. A.: On asymptotic properties of polynomials orthogonal on the circle with weights not satisfying Szegő's condition, *Math. USSR Sb.* **58** (1987), 149–167.
106. Rakhmanov, E. A.: Strong asymptotics for orthogonal polynomials associated with exponential weights on $\mathbb{R}$, In: A. A. Gonchar and E. B. Saff (eds), *Methods of Approximation Theory in Complex Analysis and Mathematical Physics*, Nauka, Moscow, 1992, pp. 71–97.
107. Rudin, W.: *Real and Complex Analysis*, McGraw-Hill, 1986.
108. Saff, E. B. and Totik, V.: *Logarithmic Potentials with External Fields*, Springer, Berlin, 1997.
109. Sheen, R. C.: Plancherel–Rotach type asymptotics for orthogonal polynomials associated with $\exp(-x^6/6)$, *J. Approx. Theory* **50** (1987), 232–293.
110. Stahl, H. B. and Totik, V.: *General Orthogonal Polynomials*, Cambridge University Press, Cambridge, 1992.
111. Stein, E. M.: *Harmonic Analysis: Real Variable Methods, Orthogonality and Singular Integrals*, Princeton University Press, Princeton, 1993.
112. Szegő, G.: *Orthogonal Polynomials*, Amer. Math. Soc. Colloquium Publications 23, Providence, 1975.
113. Totik, V.: *Weighted Approximation with Varying Weights*, Lecture Notes in Math. 1300, Springer, Berlin, 1994.
114. Totik, V.: Weighted polynomial approximation for convex external fields, *Constr. Approx.* **16** (2000), 261–281.
115. Totik, V.: Asymptotics for Christoffel functions for general measures on the real line, Manuscript.
116. Ullman, J. L.: Orthogonal polynomials associated with an infinite interval, *Michigan Math. J.* **27** (1980), 353–363.
117. Ullman, J. L.: Orthogonal polynomials associated with an infinite interval, In: E. W. Cheney (ed.), *Approximation Theory III*, Academic Press, New York, 1980, pp. 889–895.
118. Van Assche, W.: Weighted zero distribution for polynomials orthogonal on an infinite interval, *SIAM J. Math. Anal.* **16** (1985), 1317–1334.
119. Van Assche, W.: *Asymptotics for Orthogonal Polynomials*, Lecture Notes in Math. 1265, Springer, Berlin, 1987.
120. Van Assche, W.: Asymptotic properties of orthogonal polynomials from their recurrence formula II, *J. Approx. Theory* **52** (1998), 322–338.
121. Van Assche, W.: Norm behaviour and zero distribution for orthogonal polynomials with nonsymmetric weights, *Constr. Approx.* **5** (1989), 329–345.
122. Widom, H.: Extremal polynomials associated with a system of curves in the complex plane, *Adv. in Math.* **3** (1969), 127–232.
123. Zygmund, A.: *Trigonometric Series*, Vols. I and II, Cambridge University Press, Cambridge, 1959.