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Jöran Friberg
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New Mathematical Cuneiform Texts

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New Mathematical Cuneiform Texts

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Preface

The name of the book, New Mathematical Cuneiform Texts, is not entirely truthful. Many of the texts considered are really published here for the first time, but several others were published earlier by various authors but are treated here again because much of interest remains to be said about them.

Most of the new texts were found and copied for me by professor **Farouk N. H. Al-Rawi**, who also has made for this book quite a few new hand copies of previously published texts. Without his help the book could never have been written. Of the new texts found by Al-Rawi, some are from the **British Museum**, namely a Seleucid list of squares of many-place regular sexagesimal numbers in Chapter 1, a Neo-Babylonian factorization algorithm in Chapter 2, three fragments of Late Old Babylonian mathematical recombination texts from Sippar in Chapter 8, one of them with interesting problems for bricks, and a fragment of a catalog of rectangular-linear systems of equations in Chapter 9. Hand copies of the mentioned texts are published here with the kind permission of the Trustees of the British Museum. Other texts found by Al-Rawi are from the **Iraq Museum** in Baghdad, namely, in particular, nine texts from Old Babylonian Mê-Turran and Shaduppûm in Chapters 5-7, including fragments as well as large clay tablets, and in addition an interesting set of unusually brief Old Babylonian multiplication tables from Mê-Turran. Permission to publish texts from the Iraq Musum was kindly given by the Board of Archaelogy of Iraq. Most importantly, Al-Rawi found two early tables of reciprocals in the **Suleimaniyah Museum**, discussed in Chapter 13, one of which is the only known cuneiform table of reciprocals *not making use of sexagesimal place value numbers.*

By courtesy of **Vitali Bartash** a metro-mathematical text from Early Dynastic Umma with commercial exercises, the earliest of its kind, from the **Cuneiform Library at Cornell University**, is shown and discussed in Chapter 12.

Finally, Chapter 14, written in cooperation with **Anthony Phillips**, is devoted to a discussion of early examples of labyrinths and mazes. Among those are, in particular, several fragments from the **Iraq Museum** in Baghdad and the **Yale Babylonian Collection**, New Haven, CT, of apparently Neo-Sumerian drawings from Nippur of an earlier unknown kind of labyrinths in the form of house plans, probably with 180˚ rotational symmetry. These are by far the earliest known instances of labyrinths.

The initial part of the present volume, comprising Chapters 1-4, is devoted to a detailed study, for the first time ever, of a number of large mathematical and metrological cuneiform table texts from the 1st millennium BC. As a suitable introduction to the subject, Chapter 1 begins with a convenient survey of eleven categories of known Late Babylonian (Neo-Babylonian, Achaemenid, and Seleucid) mathematical tablets or tablet fragments inscribed with many-place regular sexagesimal numbers. Similarly, Chapter 3 begins with an additional survey of eight categories of known Neo-Assyrian, Neo-Babylonian, Achaemenid, and Seleucid mathematical or metrological exercises, problem texts, lists, and other kinds of tables.

In **Chapter 1**, five large many-place sexagesimal tables of reciprocals are discussed, the Seleucid Table B from Babylon, the Achaemenid Table U from Uruk, the Neo-Babylonian Table S from Sippar, the Seleucid Table V from Uruk, and the Achaemenid table W from Uruk. All the mentioned tables are more or less well preserved, with the exception of Table B, which can be reconstructed from a large number of small fragments. It is shown that Tables B, U, and S probably had a common ancestor, the hypothetical

Neo-Babylonian Table R, which had originally been constructed by use of the clever Old Babylonian doubling and halving algorithm. The younger Table V was constructed by use of a similar algorithm. Table W, on the other hand, is of another kind, and its construction was based on a triplicating and trisecting algorithm.

A new fragment of a list of squares of many-place regular sexagesimal numbers is published in Chapter 1. It is similar to the fragments which were used for the reconstruction of the pairs of reciprocals in Table B.

In **Chapter 2** is published a new Neo-Babylonian tablet where the reciprocal of the many-place regular sexagesimal number 1 01 02 06 33 45 is computed by use of another clever Old Babylonian algorithm, the last place factorization algorithm, and its inverse. The number was taken from the second line of Table R.

It is also shown in Chapter 2 how an understanding of the last place factorization algorithm allows the reconstruction of certain tables of many-place regular numbers of which only quite small fragments have been preserved.

Four metrological recombination texts from Achaemenid Uruk are discussed for the first time in **Chapter 3**. (A "recombination text" is a text of mixed content, with its various paragraphs copied from several older texts.) A common, Late Babylonian feature of these texts is that the great majority of the metrological sub-tables are written as if they were to be read from right to left. Another common feature is the following: In Old Babylonian metrological table texts sub-tables for different kinds of measures were always arranged in the order *C* (capacity), *M* (weight, metals), *A* (area), *L* (length). In Late Babylonian metrological tables, on the other hand, sub-tables are always arranged in the reverse order *L*, (*A*), *M*, *C*. In spite of these conspicuous differences, there are many indications that a prominent purpose of these metrological texts was to teach the fundamentals of Old Babylonian metrology to Late Babylonian students.

The first of the mentioned four texts is written on the obverse of a clay tablet which has Table W of many-place pairs of reciprocals on its reverse. It begins with a series of structure tables of a previously unknown kind for traditional (i. e. Old Babylonian) length measures with 1 rod = 12 cubits. It continues with a conversion table from area measures (Old Babylonian) to Kassite seed measures, and with a structure table for area measures. Then there is, somewhat out of place, a delightful account of the linear growth of a child in its mother's womb. The text ends with three parallel lists of length measures (*L*), weight measures (*M*), and capacity measures (*C)*, with a curious mix of Old and Late Babylonian metrology.

The second text begins with a non-metrological list of Gods' names and numbers and ends with two similarly non-metrological shadow length tables. In between, there are various conversion tables for length measures, area measures, and weight measures, and a range table of a previously unknown kind for the conversion of squares of length measures into area measures.

The third text is really only a sliver of a large clay tablet, with a structure table for traditional length measures on the obverse, making use of decimal numbers, and a metrological table for traditional (Old Babylonian) capacity measures, making use of sexagesimal numbers, on the reverse.

The fourth text is a large fragment with curious metrological conversion tables from decimal multiples of the grain (Old Babylonian) to sexagesimal fractions of the shekel, and from decimal multiples of one cubit to sexagesimal multiples of one 100-cubit. There is also a non-traditional table of reciprocals, where decimal reciprocals, written with number words, are expressed as sexagesimal fractions.

Chapter 4 is about an Achaemenid metrological text originally published by Hilprecht more than a century ago. As in the case of the four texts in Chapter 3, the lines of the metrological tables are intended to be read from right to left, and the sub-tables are arranged in the order *L*, *M*, *C*. However, there are four different sub-tables for length measures. In the first two sub-tables (*L*n and *L*c), traditional length measures

(Old Babylonian, with 1 cubit = 30 fingers) are equated with sexagesimal rod multiples and with sexagesimal cubit multiples. In the third and fourth sub-tables, common length measures (Late Babylonian, with 1 cubit = 24 fingers) are equated with sexagesimal multiples of one cubit or of one 100-cubit.

In the sub-table for weight measures, fractions or multiples of the shekel and decimal multiples of the grain are equated with sexagesimal multiples (of the shekel). In the last sub-table, Late Babylonian capacity measures (with 1 bán = 6 sìla and 1 sìla = 10 ninda) are equated with sexagesimal multiples (of the bariga).

A second part of the book, comprising Chapters 5-7 and also Chapter 10 is devoted to Old Babylonian mathematical texts from Mê-Turran (Tell Haddad) and Shaduppûm (Tell Harmal) in the Eshnunna region in the early 2nd millennium BC, some new and some previously published in the journal *Sumer*.

In **Chapter 5** are considered five texts from Old Babylonian Mê-Turran and Shaduppûm, all of them concerned with metric algebra problems for rectangles of a given form (sides in the proportion 1 : 2/3).

The first of those texts is a previously unpublished nearly intact large recombination text. It is shown that it is possible to rearrange the 19 exercises of that text into a systematically arranged theme text.

The second text is a previously unpublished hand tablet with a single exercise, and unusually interesting mathematical terminology.

The three remaining texts in this chapter were previously published by Bruins or Baqir, but with inadequate mathematical commentaries in both cases. One is a large fragment of recombination text, the other two are single problem texts. The text originally published by Baqir has also recently and independently been discussed in a book by Carlos Gonçalves.

In **Chapter 6** are considered three additional previously unpublished texts from Old Babylonian Mê-Turran. One of them is a fragment of a large text with three preserved mixed problems, each problem with its own illustrating diagram, and a table of constants. Interestingly, the Akkadian term *nalbanum* 'brick mold' appears here as a previously unknown but appropriate term for 'rectangle'. In its Sumerian form $^{\text{geš}}$ù.šub, this term for 'rectangle' appears also in a text from Sippar discussed in Chapter 8.

A second text from Mê-Turran is a well preserved large recombination text with metric algebra problems for squares and general quadrilaterals. The areas of the quadrilaterals are computed by use of the usual Sumerian/Old Babylonian inexact computation rule.

A third text from Mê-Turran is a relatively well preserved recombination text with metric algebra problems for circles and semicircles.

The text discussed in **Chapter 7** is a large fragment of a recombination text from Old Babylonian Shaduppûm. This text was first published by Bruins, in an inadequate transliteration without hand copies or photos, and with in most cases unsatisfactory mathematical interpretations. The text contains eight more or less well preserved problems about partners or brothers and their unequal shares, all leading to systems of linear equations. There is also an unrelated exercise with an *explicit* computation of a combined market rate, so far the only one of its kind known.

Texts belonging to Goetze's Group 6 of Old Babylonian mathematical texts without known provenance are discussed in Chapters 8 and 9. They are probably from Sippar in northern Mesopotamia, and there are indications that they were written after the abandonment of the cities in southern Mesopotamia, such as Ur, Larsa, and Uruk. Consequently, they can be assumed to be Late Old Babylonian.

The six Old Babylonian texts discussed in **Chapter 8** are shown by their repertoire of mathematical sumerograms to belong to Group 6. The first of these texts, previously unpublished, is a fragment of a mathematical recombination text with six preserved problems concerning bricks, three for extra thick rectangular bricks of length 3 sixtieths rods, and three for ordinary rectangular bricks of length 1/2 cubit. Particularly interesting is a problem for moist and dry bricks, closely related to an earlier published Late Babylonian(?) table of constants.

The two next texts, also previously unpublished, are small fragments of two-column clay tablets with homogeneous quadratic problems for two or three squares.

The fourth text in Chapter 8 is a fragment of an interesting theme text, originally published by Neugebauer in *MKT I*. It gives an explicit account of a combined factorization and doubling and halving algorithm, like the one in the well known table text CBS 1215 discussed in Section 2.2.

The fifth text is large recombination text, a join of three separately published fragments. An edition of this text has been published previously by Robson, unfortunately with unsatisfactory mathematical interpretations. Its general theme is seed measures and volumes of various kinds of whole and truncated pyramids and cones. All the computation rules applied in this text are correct. The only known parallel text is a single problem text from Late Old Babylonian Susa which is concerned with what may be called a ridge pyramid, a pyramid with a rectangular base and with uniformly sloping sides. A second related text is an interesting isolated exercise in another mathematical recombination text from Old Babylonian Sippar, published and partially explained by Neugebauer in *MKT I*, concerned with the volumes of two circular defense works, a ditch and a dike.

This part of the chapter ends with a discussion of how the Old Babylonian correct computation rules for the volumes of pyramids and cones may have been derived, with a brief discussion of similar computation rules in ancient Chinese, Greek, and Indian mathematical treatises.

The sixth text from Sippar treated in Chapter 8 is a join of two separately published fragments. An edition of this text, too, has been published previously by Robson, unfortunately again with unsatisfactory or insufficient interpretations. It is a mathematical recombination text, dealing with walls built of two kinds of extra thick rectangular or square bricks, with a square pyramid built of an unusual kind of ordinary rectangular bricks, and with the lengths of the diagonals of certain rectangles.

Chapter 9 begins with a survey of all known Old Babylonian mathematical texts without provenance but clearly belonging to Group 6, and therefore from Sippar. In addition to the five such texts discussed in Chapter 8, five more are discussed in Chapter 9. Four of them are simple catalogs of metric algebra problems, without solution procedures and answers. Two of them are previously unpublished.

Chapter 10 is devoted to a study of Goetze's well known "mathematical compendium", two clay tablets from Shaduppûm. In spite of being quite badly preserved these two texts are very interesting, containing a mix of catalogs of mathematical problems, a unique catalog of mathematical problem types, and a table of constants.

In the chapter is also discussed in some detail a pair of large catalogs of metric algebra problems, from Late Old Babylonian Susa.

Chapter 11 is devoted to three unusually interesting small recombination texts, together making up Goetze's Group 2b of Old Babylonian mathematical texts without known provenance. Two of them are numbered, one is not. One of the texts is exclusively concerned with commercial problems, with market rates (inverse unit prices) as the central concept. No solution procedures are given, but the the needed mathematical tool box includes the computation of combined market rates, as well as the solution of systems of linear equations or rectangular-linear systems of equations.

The second text contains an assortment of problems about bricks, mud, and reeds. A problem about carrying mud and molding bricks is particularly interesting. According to the interpretation of this problem proposed here, bricks reinforced with straw were made of five parts mud and one part straw and were therefore made extra thick, six fingers instead of the usual five fingers.

Also the third text considered in Chapter 11 contains a wide variety of problems. It begins with a systematic catalog of problems for cylindrical containers, with their capacities expressed in a special cylinder sìla. Then there are problems for a stair-case built of square bricks, a curious problem about how much silver is needed for making a thin rectangular silver foil of given dimensions, systems of linear equations for the ewes and lambs in two sheep folds, and a combined work norm problem for tearing out and carrying bricks.

Another Old Babylonian tablet with mixed mathematical problems, among them a problem for two sheep folds, was published by Neugebauer in *MKT I*. Exceptionally, in that text all the solution procedures for the various problems are chaotically organized. It is proposed here that that is because the solution procedures were jotted down by a student inattentively listening to the teacher's demonstration of how the stated problems should be solved. This proposed explanation is important, since it may explain how, in a number of known cases in mathematical cuneiform texts, a solution procedure not making much sense nevertheless produces the correct answer to a stated problem.

The text discussed in **Chapter 12** is a mathematical recombination text from Early Dynastic III or Early Sargonic Umma, just recently published by Vitali Bartash. It is the oldest such text known. The theme of the text is multiplication or division by fractions like 1 2/3 5 (shekels) or 1 2/3. Before the invention of sexagesimal place value numbers such operations were much more difficult than multiplication or division by the corresponding sexagesimal place value numbers 1 45 (non-regular) or 1 40 (regular).

The object considered in **Chapter 13** is a previously unpublished atypical table of reciprocals, apparently dating to the Neo-Sumerian Ur III period, before the invention of sexagesimal place value numbers. One of the reciprocal pairs listed in this table is igi 10 ⅔ gál.bi 5 ½ 7 ½, meaning that the reciprocal of 10 2/3 is 5 1/2 + 7 1/2 (sixtieths). (The corresponding reciprocal pair in sexagesimal place value numbers would be (10;40, 5;37 30), if a semi-colon is used to separate integers from fractions.) More precisely, all numbers appearing in this atypical table are of the type 'integer and basic fraction + integer and basic fraction times 1 shekel (= 1/60)'. The product of a number and its reciprocal in the table is always equal to 60.

The list of reciprocal pairs in this table is fundamentally different from the list of reciprocal pairs in the standard Old Babylonian tables of reciprocals and their Ur III predecessors making use of sexagesimal place value numbers. The pair (10 40, 5 37 30), for instance, is never present in standard tables of reciprocals.

An interesting new observation is that the list of head numbers in Old Babylonian multiplication tables may have been based not on the numbers in a standard table of reciprocals of the Old Babylonian type, but on an atypical table of reciprocals of the same type as the one mentioned above.

It is possible to show that the majority of the reciprocal pairs in this new atypical table of reciprocals were constructed by use of a systematic procedure involving doubling and halving, tripling and trisection, etc. Since this procedure did not produce enough pairs *n*, rec. *n* with *n* close to 60, a different method involving a kind of "regular twin numbers" was used to produce a few such pairs.

The only other known sexagesimal table of reciprocals not making use of sexagesimal place value numbers is Peter Hulin's curious table of reciprocals from Late Assyrian Sultantepe with Sumerian number words.

In Chapter 13 is also published a new table of reciprocals of the usual kind, probably Neo-Sumerian, and a comparative survey is made of all known Neo-Sumerian or Early Old Babylonian tables of reciprocals. The chapter ends with a discussion of the historical importance of the table of reciprocals without sexagesimal place value numbers as a missing link between metro-mathematical cuneiform texts from the 3rd millennium BC and mathematical cuneiform texts from the early second millennium BC.

Chapter 14, the last chapter of the book, was written in cooperation with Anthony Phillips. It begins with a renewed discussion of two Old Babylonian drawings of labyrinths of a completely new type, one rectangular, the other square, published by the present author in *MSCCT 1* in 2007. A new observation is that the rectangular labyrinth has a nearly perfect 180^{o} rotational symmetry, and that the square labyrinth would have had it if it had been only slightly different. This is important, because a large fragment of a Neo-Sumerian clay tablet with a drawing of another kind of labyrinth allows a complete reconstruction of the original drawing of a labyrinth if it is assumed that that labyrinth, too, had a perfect 180˚ rotational symmetry. The (reconstructed) Neo-Sumerian labyrinth has great similarities with earlier published Neo-Sumerian house plans.

Interestingly, the Old Babylonian labyrinths have a central square filled by a spiral pattern, while the Neo-Sumerian labyrinth has an open central square. In that respect it is similar to a number of known Roman mosaic floors with labyrinth patterns; these almost always have an open central court, with or without an explicit reference to the myth of the Minotaur in the palace at Knossos. It is also reminiscent of a number of known loom weights from a weaving house in Francavilla, Calabria, c. 800 BC, which are all adorned with patterns that can be shown to be the paths around labyrinths with a central open court, possibly referring to Ariadne's thread in the myth of the Minotaur.

Jöran Friberg, October 2016

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1. Late Babylonian Tables of Many-Place Regular Sexagesimal Numbers, from Babylon, Sippar, and Uruk

1.1 Late Babylonian Texts Concerned with Many-Place Regular Sexagesimal Numbers

For the notion of many-place regular sexagesimal numbers, and for many explicit examples, both Old and Late Babylonian, the reader is referred to Friberg, *MSCT 1* (2007), Sec. 1.4 and App. 9. In particular, it is important to recall that a sexagesimal number *n* is called "regular" if another sexagesimal number *n´* can be found such that *n* times *n´* equals some power of 60. (In Babylonian "relative" place value notation, every power of 60 is written as '1'.) The number *n´* is called the "reciprocal" of *n*. In the following, it is conveniently referred to as rec. *n*. Note that, since every power of 60 contains no other prime factors than 2, 3, 5, it is true exclusively for regular sexagesimal numbers in general that they and their reciprocals can contain no other prime factors than 2, 3, 5. That well known but important fact is basically the reason why it is possible to generate systematically various kinds of tables of many-place regular sexagesimal numbers.

This chapter is devoted to a detailed comparative discussion of five lexicographically ordered many-place "tables of reciprocals" (more precisely, tables of regular sexagesimal numbers and their reciprocals) from the 2nd half of the 1st millennium BC, namely 1) the Seleucid *Table B*, reconstructed from many small fragments of copies of a supposed original many-place table of reciprocals and from related tables of squares and squares of squares, 2) the enlarged Seleucid *Table B**, the existence of which is tentatively inferred here from two small fragments, one of a table of reciprocals and the other of a related table of squares, 3) the Achaemenid *Table U* = W 23283+, 4) the Neo-Babylonian *Table S* = Sippar 2175/12, and 5) the Seleucid *Table V* = AO 6456. Tables U and V are firmly dated by well preserved colophons. Of the five mentioned table texts, Tables B, U, and S appear to have had a common ancestor, a hypothetical *Table R*.

Tables B and B* are discussed in Sec. 1.2 below, Table U in Sec. 1.3, Table S in Sec. 1.4, and Table V in Sec. 1.5. Photos of Tables U and V are published here for the first time. Also hand copies of both the obverse and the reverse of Table S are published here for the first time, including a large piece of the tablet which is missing in a previously published photo of only the reverse. In addition, the previously unpublished text BM 35568, a Seleucid fragment of a table of squares, is presented with comments in Sec. 1.2.1 below.

Explanations of the errors in the various many-place tables of reciprocals yield important clues to how the tables were constructed. Thus, it is shown in Sec. 1.2.4 that the numerical algorithms used for the construction of such tables were based on the keen insight that complete and arbitrarily large many-place tables of reciprocals can be obtained by first constructing a table of reciprocals for consecutive powers of 3 (see Table 1.5 below, in Sec. 1.5.3), and then systematically generating, with departure from such a table, new pairs of reciprocals by the well known, originally Old Babylonian, method of "halving and doubling". Evidently, the final step of the construction was an extremely laborious sorting and copying, which in the case of Table V, but not in the case of Tables B, U, and S, was afflicted with numerous errors.

In Sec. 1.2.5 it is shown how the regular sexagesimal numbers appearing in the various many-place tables of reciprocals may be represented by their indexes in a certain "index grid". The device is a powerful visual aid, making it easier to understand the complete construction of the many-place tables of reciprocals.

The detailed study in this chapter of these Late Babylonian many-place tables of reciprocals is historically important because *it clearly illustrates the astonishing temporal continuity and regional unity of Babylonian mathematics, as well as the amazing ingenuity and perseverance of Babylonian mathematicians*.

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7_1

Here follows a convenient survey of 11 different categories of known Neo-Babylonian, Achaemenid, and Seleucid mathematical tablets or tablet fragments exhibiting examples of many-place regular sexagesimal numbers. It is an updated version of a similar survey in Friberg, *CTMMA II* (2005). Of the 34 listed texts, 11 will be discussed in Chs. 1-2 below.

I. Ten Seleucid fragments of many-place tables of reciprocals, with *n* from 1 to 2

A: BM 34577 = Sp. II 49 (Sachs, *LBAT* (1955) 1635; Vaiman, *SBM* (1961) F)	Babylon
B: BM 34596 = Sp. II 70 + 82-7-4, 128 (Sachs, *LBAT* 1633; Vaiman, *SBM* D)	Babylon
C: BM 34612 = Sp. II 91 (Sachs, *LBAT* 1631; Vaiman, *SBM* B)	Babylon
D: BM 34635 = Sp. II 118 (Sachs, *LBAT* 1634; Vaiman, *SBM* E)	Babylon
E: BM 34762 = Sp. II 255 (Sachs, *LBAT* 1632; Vaiman, *SBM* C)	Babylon
F: BM 76984+ = 83-1-18, 2356+ (Britton, *JCS* 43-45 (1991-93) A)	Babylon
G: BM 77051 = 83-1-18, 2427 (Britton, *JCS* 43-45 B)	Babylon?
H: BM 78079 = 86-5-12, (Britton, *JCS* 43-45 C)	Babylon
I: MMA 86.11.407+408+409 (Neugebauer and Sachs, *MCT* (1945) p. 36; Friberg, *CTMMA II* (2005))	Babylon
J: MMA 86.11.406+410+Liv. 29.11.77.34 (Neugebauer and Sachs, *MCT* 36; Friberg, *CTMMA II*)	Babylon

II. Eight Seleucid fragments of lists of squares of many-place regular numbers, with *n* from 1 to 2

A: BM 32178 = 76-11-7, 1905+ (Aaboe, *JCS* 19 III)	Babylon
B: BM 33567= Rm 4, 123 (Aaboe, *JCS* 19 II)	Babylon
C: BM 34578 = Sp. II 50 (Sachs, *LBAT* 1641; Vaiman, *SBM* J; Aaboe, *JCS* 19 VIII)	Babylon
D: BM 34714 = Sp. II 203 (Sachs, *LBAT* 1639; Vaiman, *SBM* I; Aaboe, *JCS* 19 VII)	Babylon
E: BM 34764 = Sp. II 257 (Sachs, *LBAT* 1640; Friberg, *Sumer* 42)	Babylon
F: BM 34875 = Sp. II 382 (Sachs, *LBAT* 1638; Vaiman, *SBM* H; Aaboe, *JCS* 19 VI)	Babylon
+ BM 45668 = SH. 81-7-6, 63 (Sachs, *LBAT* 1636; Vaiman, *SBM* G; Aaboe, *JCS* 19 V)	
G: BM 35568 (Sec. 1.2.1 below)	Babylon?
H: BM 99633 = 84-2-11, 1995 (Britton, *JCS* 43-45 D)	Babylon

III. Two Seleucid fragments of lists of squares of squares of many-place regular numbers, with *n* from 1 to 2

A: BM 32584 = 76-11-17, 2327 (Friberg, *MSCT 1* (2007), App. A9.1)	Babylon?
B: BM 55557 = 82-7-4, 147+ (Britton, *JCS* 43-45; Friberg, *MSCT 1*, App. A9.1)	Babylon

IV. A Seleucid fragment of a many-place table of reciprocals, with *n* from 4 to 8

A: BM 41101 = 81-4-28, 648 (Aaboe, *JCS* 19 I)	Babylon

V. A Seleucid fragment with an explicit computation of sq. sq. 3^{23} = sq. sq. 2 01 04 08 03 00 27

A: BM 34601 = Sp. II 76+759 (Sachs, *LBAT* 1644; Friberg, *MSCT 1*, App. A9.2)	Babylon

VI. One Neo-Babylonian, two Achaemenid, and three Seleucid examples of factorization algorithms for regular sexagesimal numbers

A: BM 46550 (Sec. 2.1 below)	
B: W 23021 (von Weiher, *Uruk* 4 (1993)174; Friberg, *BaM* 30 (1999); Fig. 2.1.6 below)	Uruk
C: W 23016 (von Weiher, *Uruk 5* (1998) 316; Friberg, *MSCT 1*, App. 9; Fig. 2.1.8 below)	Uruk
D: BM 34517 = Sp. 641 (Sachs, *LBAT* 1646; Sec. 2.3.1 below)	Babylon
E: BM 34907 = Sp. II 421 (Sachs, *LBAT* 1643; Sec. 2.3.3 below)	Babylon
F: BM 34958= Sp. II 479 (Sachs, *LBAT* 1642; Sec. 2.3.2 below)	Babylon

VII. A Seleucid table of reciprocals, resembling the standard OB table of reciprocals, from igi 2 30-*ú* to igi 1 21 gál.bi 44 26 40, followed by a table of squares of 3-place regular sexagesimal numbers, from sq. 1 02 30 to sq. 45

A: BM 34592 = Sp. II 65+ (Sachs, *LBAT* 1637; Vaiman *SBM* A; Aaboe, *JCS* 19 IV)	Babylon

VIII. An Achaemenid many-place table of reciprocals, with *n* from 1 to 4

U: W 23283+22905 = IM 76852 (von Weiher, *Uruk 4* 174; Sec. 1.3 below)	Uruk

IX. A Neo-Babylonian many-place table of reciprocals, with *n* from 1 to 3

S: Sippar 2175/12 (Al-Jadir, *Archeologia* 224; Sec. 1.4 below)	Sippar

X. An Achaemenid many-place table of selected reciprocals, with *n* from 1 to 1 00, and from 1 to 31

W: W 23281 = IM 76283, *rev.* (von Weiher, *Uruk 4* 173; Sec. 1.6 below)	Uruk

XI. A Seleucid many-place table of reciprocals, with *n* from 1 to 3

V: AO 6456 (Thureau-Dangin, *TCL 6* 31; Neugebauer, *MKT I* 14 ff; Sec. 1.5 below)	Uruk

1.2 Table B (Babylon). Fragments of a Seleucid Many-Place Table of Reciprocals, with *n* from 1 to 2

1.2.1 BM 35568. A List of Squares of Many-Place Regular Sexagesimal Numbers

BM 35568 is a Seleucid fragment of a list of squares of many-place regular sexagesimal numbers, of precisely the same kind as seven other known fragments of such texts, previously published by Sachs in 1955, by Aaboe in 1965, and by Britton in 1991-93. See category II of the list above (Sec. 1.1) of Late Babylonian texts exhibiting many-place regular sexagesimal numbers.

The fragment is relatively large, and by good luck it happens to include both the beginning and the end of the table once inscribed on the intact tablet. See Fig. 1.2.1 below. More precisely, the preserved square numbers on the fragment correspond to 29 lines at the beginning and 5 lines at the end of a table of squares closely related to the (reconstructed) Table B of many-place regular sexagesimal pairs of reciprocals. (See Friberg, *MSCT 1* (2007), 461-2.) The table of squares sq. *n* related to Table B of pairs of reciprocals *n*, rec. *n* may conveniently be called Table sq. B.

The longest square number recorded on the fragment BM 35568 is a 17-place sexagesimal number (*rev.* line 2´). Since most of the recorded square numbers are much shorter, it is not surprising that in several instances two consecutive square numbers from Table sq. B are inscribed after each other in one line of the table on the fragment. Thus, in eight cases two consecutive square numbers are inscribed in one line of the table, and in two cases three consecutive squares are inscribed in one line of the table. Consecutive square numbers inscribed in a single line of the table are separated from each other by a special "separation sign", in the transliteration below reproduced as a colon (:). In lines 7, 8, 9, this separating sign is written as the sign GAM (two diagonally placed oblique wedges). In lines 13, 14, 15, 18, the separating sign is instead written as three diagonally placed oblique wedges. Written like this, the sign is confusingly similar to the Seleucid simplified sign for 9, used for instance in 59 in the number on line 2 of the table.

A sign for an "internal double zero", appears in two of the square numbers inscribed on the fragment. The sign is written as a double Winkelhaken (two oblique wedges, one above the other), and is in the transliteration below reproduced as «.

Thus, in line 6 on the obverse of the table on the fragment, the number

1 04 29 50 06 57 01 26 24

is written with an internal double zero as

1 04 29 50 « 06 57 01 26 24.

To be more precise, if the special cuneiform signs for the tens 10, 20, 30, etc., are transliterated as 1°, 2°, 3°, etc., then the number in question can be more truthfully transliterated as

1 4 2° 9 5° « 6 5° 7 1 2° 6 2° 4.

Transliterating the number in question in this elaborate way makes it clear that the purpose of the double Winkelhaken in this particular case was to distinguish 5° « 6 (50 06) from 5° 6 (56). In other words, in this particular case, the sign « stands for an internal double zero of the form 0 0 ("an absent one followed by an absent ten"). (In general, the readers are asked to understand that, for instance, 06 stands for "an absent ten followed by 6 ones" in precisely the same way as 50 stands for "5 tens followed by an absent one".)

The sign for an internal double zero occurs also in line 12 on the obverse, where the number

1 13 23 00 45 14 28 50 18 14 24

is written with an internal double zero as

1 13 23 « 45 14 28 50 18 14 24.

Clearly, in this case the sign « stands for an internal double zero of the form 00 ("an absent ten followed by an absent one").

The table of squares on BM 35568 ends with a ruled line, followed by what appears to be a catch-line, the square of the number 2 01 04 08 03 00 27 with the line number 102a in Table 1.3 below. There are related catch-lines at the ends of one other fragment of a table of squares and at the end of a fragment of a table of reciprocals. See the discussion after Table 1.3 in Sec. 1.2.3.

On the edge, at the upper left corner of the reverse of BM 35568, some numbers are scrabbled. With some leap of the imagination, they may be read as 4 05 45 36, which is the square of 2 02 52 48, the number with the line number 105 in Table 1.3 below.

There is only one mistake in the list of squares on BM 35568. In line 10 of the list, 22 is incorrectly replaced by 24.

BM 35568, transliteration

	sq. *n*	***n***	**line numbers**
obv.	1 01 30 33 45	1 00 45	1
	1 02 05 17 25 04 28 03 59 03 45	1 01 02 06 33 45	2
	1 02 54 52 24 57 36	1 01 26 24	3
	1 03 03 24 11 33 50 « 03 45	1 01 30 33 45	4
	1 03 30 23 41 07 29 22 57 46 40	1 01 43 42 13 20	5
	1 04 29 50 « 06 57 01 26 24	1 02 12 28 48	6
	1 05 06 15 : 1 06 35 29 18 34 24 11 51 06 40	1 02 30 : 1 03 12 35 33 20	7, 8
	1 06 44 30 59 45 56 15 : 1 08 16 : 1 09 59 02 24	1 03 16 52 30 : 1 04 : 1 04 48	9, 10, 11
	1 10 38 33 09 03 45 : 1 11 44 40 19 33 36	1 05 06 15 : 1 05 36 36	12,13
	1 12 15 22 56 55 22 17 51 36 17 46 40	1 05 50 37 02 13 20	14
	1 12 25 10 42 58 2928 21 33 45	1 05 55 04 41 15	15
	1 13 23 « 45 14 28 50 18 14 24	1 06 21 18 43 12	16
	1 14 04 26 40 : 1 15 56 15 : 1 17 40 20 16	1 06 40 : 1 07 30 : 1 08 16	17, 18, 19
	1 17 50 52 05 23 26 15 : 1 19 37 34 27 50 24	1 08 20 37 30 : 1 09 07 12	20, 21
	1 20 22 31 51 06 40 : 1 21 37 45 36 55 17 45 36	1 09 26 40 : 1 09 59 02 24	22, 23
	1 22 [23 50 51 33 45 :] 1 24 16 47 24 26 40	1 10 18 45 : 1 11 06 40	24, 25
	1 24 [28 12 58 45 57 07] 44 03 45 : 1 26 24	1 11 11 29 03 45 : 1 12	26, 27
	[1 28 22 25 43 32 16 : 1 28 34 24] 36	1 12 49 04 : 1 12 54	[28], 29
	about 20 lines missing		—
rev.	about 20 lines missing		—
	[3 43 15 55 08 38 31 06 40 : 3 46 44 53 22 3]3 36	1 55 44 26 40 : 1 56 38 24	[96], 97
	3 48 [21 56 58 25 38 07 3]3 28 02 51 11 36 17 / 46 40	1 57 03 19 10 37 02 13 20	98
	3 48 52 54 36 33 45	1 57 11 15	99
	3 51 55 41 38 32 25 57 30 14 24	1 57 57 53 16 48	100
	3 54 06 38 21 14 04 26 40	1 58 31 06 40	101
	4 04 17 40 45 13 17 45 52 1[4 4]2 12 09	2 01 04 08 03 00 27	catch-line (102a)

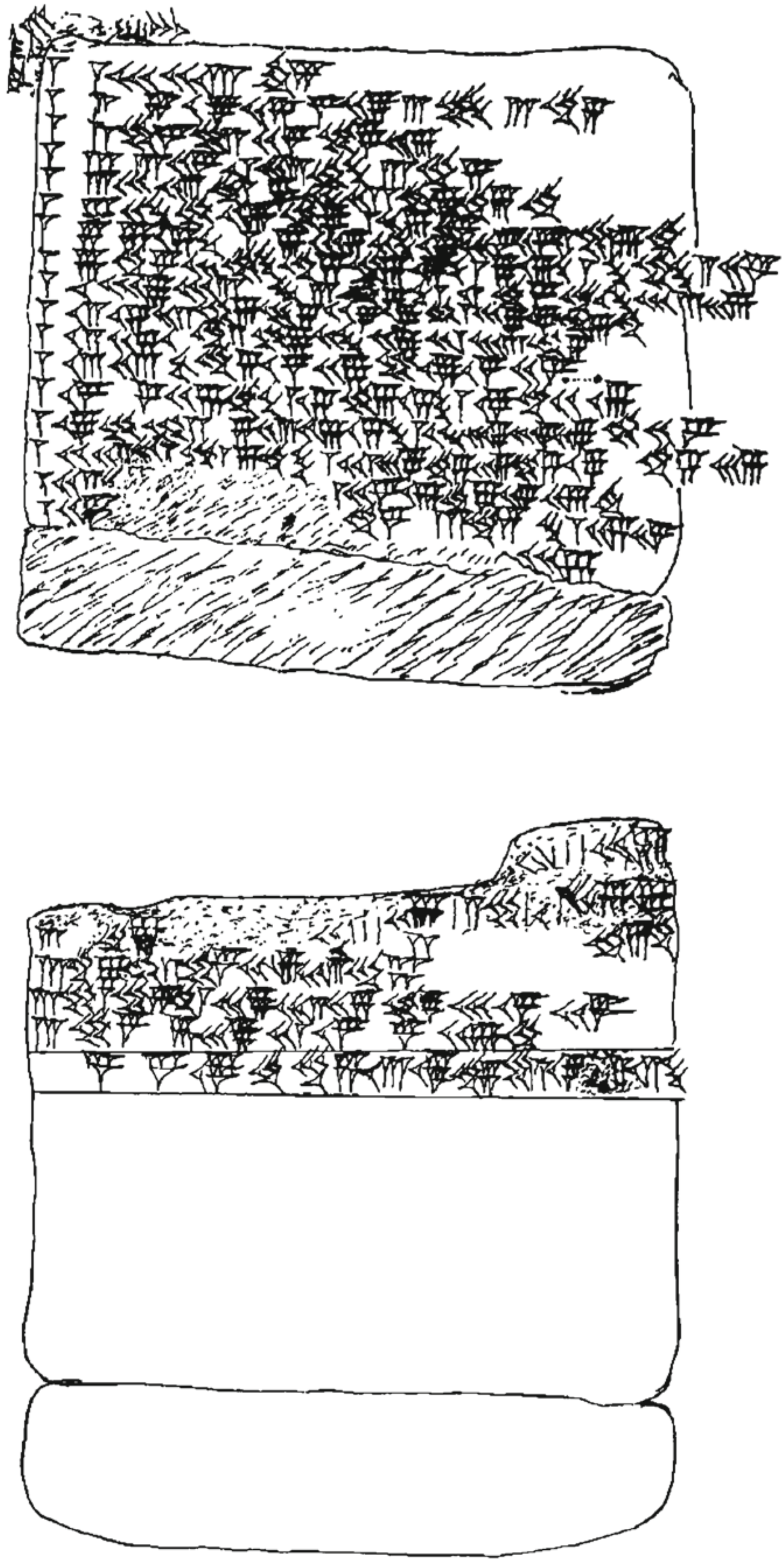


Fig. 1.1.1. BM 35568, hand copy.

1.2.2 Attested Many-Place Regular Numbers n in the Fragments and in Tables U and S

It is an interesting observation that most of the texts belonging to categories I-III in the survey in Sec. 1.1 can be shown to have a common origin. As will be shown below, this is clearly indicated already by the following initial part of a comparative table, borrowed from Friberg, *CTMMA II* (2005):

Table 1.1 (beginning). Attested and unattested many-place regular numbers *n* in texts belonging to categories I-III.

r	*s*	*p*	*q*	+	*ll.*	***n*, rec. *n***	**sq. *n***	**sq.sq. *n***
0	0	1	1		0	· · · · · · · · · · · U	· · · · ? · ·	· –
-18	-11	6	6		—	· · · · · · · · · · · –	· · · · · · ·	· –
15	-2	5	6		—	· · · · · · · · · · · –	· · · · · · ·	· –
-2	5	3	4		1?	· · · · · · · · · · · U	· · · · ? · ·	· –
-20	-6	6	5		2	· · · · · · · · · · · U	· · · · ? · ·	· B
13	3	4	4		3	· · · · · · · · · · · U	· · · · ? · ·	· B
-4	10	5	8	++	4	· · · · · · · · · · · U	· · · · E · ·	· B
-5	-8	6	3		5	· · · · · · · · · · · U	· B · · E · ·	· B
-22	-1	8	4	+	—	· · · · · · · · · · · –	· – · · – · ·	· –
11	8	5	4		6	· · · · · · · · · · · U	· B · · E · ·	· B
-7	-3	3	2		7	· · · · · · · · · · · U	· B · · ? · ·	· ·
8	-5	6	4		8	· · · · · · · · I · · U	· B · · E · ·	A B
-9	2	5	3		9	· · · · · · · · I · · U	· B · · · · ·	A B
6	0	2	2		10	· · · · · · · · I · · U	· B · · · · ·	A –
-12	-11	7	5	+	—	· · · · · · · · – · · –	· – · · · · ·	– –
4	5	3	3		11	· · · · · · · · I · · U	· B · · · · ·	A B
-14	-6	4	4		12	· · · · · · · · I · · U	· · · · · · ·	A B
19	3	5	6		—	· · · · · · · · – · · –	· ? · · · · ?	– –
2	10	4	7	+	13	· · · · · · · · I · · U	· · · · · · G	A B
1	-8	7	3	+	14	· · · · · · · · I · · U	· B · · · · G	A B
-16	-1	6	3		15	· · · · · · · · I · · U	· B · · · · G	A B
17	8	6	4		16	· · · · · · · · I · · –	· B · · · · G	– B
-1	-3	3	1		17	· · · · · · · · I · · U	· B · · · · G	? B
-3	2	3	2		18	· · · · · · · · I · · U	· B · · · · G	? B
-21	-9	6	6		—	· · · · · · · · – · · –	· – · · · · ?	– –
12	0	3	4		19	· · · · · · · · I · · U	· B · · · · ·	A B
-5	7	5	6		20	· · · · · · · · I · · U	· · · · · · ·	? B
-6	-11	8	4	+	—	· · · · · · · · – · · –	· · · · · · ·	· –
10	5	4	2		21	· · · · · · · · I · · U	· · · · · · ·	· B
-8	-6	4	3		22	· · · · · · · · I · S U	· · · · · F ·	· –
8	10	5	6		23	· · · ? · · · · I · – ?	· · · · · F ·	· B
-10	-1	4	2		24	· · · D · · · · I · S U	· · · · · F ·	· B
5	-3	4	3		25	· · · D · · · · I · S U	· · · · · F ·	· –
-12	4	6	5		26	· · · D · · · H I · S U	· · · · · F ·	· B
3	2	2	1		27	· · ? D · · · H · · S U	· · · · · F ·	· ?
-15	-9	5	5		—	· · ? ? · · · – · · – –	· · · · · – ·	· –
18	0	4	6		28	· · C D · · · H · · S U	· · · · · F ·	· ?
etc	etc.	etc.			etc.			

With respect to sorting, all the regular sexagesimal numbers *n* appearing, explicitly or implicitly, in texts of categories I-III in the survey in Sec. 1.1 above can be thought of as *numbers starting with 1 and ordered lexicographically* (as in the reconstructed Table R, see below, Table 1.3).

On the other hand, *with respect to indexing*, the same numbers *n* can be thought of as mixed powers of 2 and 3 (both positive and negative), as indicated by the list of indices (*r, s*) in col. *i* of Table 1.1 above. Recall that every regular sexagesimal number is a mixed power of 2, 3, and 5. However, for sexagesimal numbers in Babylonian relative place value notation, any power of 5 can also be understood as a corresponding power of rec. 12 = $2^{-2} \cdot 3^{-1}$. Therefore, in line 5, for instance, of col. *i* of Table 1.1, the *double index* (*r, s*) = (-20, -6) represents the following regular sexagesimal number

$$n = 1\ 01\ 02\ 06\ 33\ 45 = 3^{4} \cdot 5^{10} = \text{(in relative place value notation) } 3^{4} \cdot 12^{-10} = 2^{-20} \cdot 3^{-6}.$$

Expressed differently, 1 01 02 06 33 45 = $3^4 \cdot 5^{10}$ is the number with the *triple index* (0, 4, 10) in Gingerich's table of 11-place regular sexagesimal numbers and their reciprocals (Gingerich, *TAPS* 55(8) (1965)).

Similarly, in line 28 of col. *i* of Table 1.1, the double index (*r, s*) = (-6, -11) represents the regular sexagesimal number

n = 1 08 35 13 34 48 53 20 = $2^{16} \cdot 5^{11}$ = (in relative place value notation) $2^{16} \cdot 12^{-11} = 2^{-6} \cdot 3^{-11}$.

Hence, this is the number with the triple index (16, 0, 11) in Gingerich's table. And so on.

In col. *ii* of Table 1 are noted the numbers *p* and *q* of *sexagesimal (double-)places* in *n* and rec. *n*, respectively. In line 5 of Table 1.1, for instance,

$n = 2^{-20} \cdot 3^{-6}$ = 1 01 02 06 33 45 (p = 6), and rec. $n = 2^{20} \cdot 3^{6}$ = 58 58 56 38 24 (q = 5), so that (p, q) = (6, 5).

Now, it is easily observed that if (*n*, rec. *n*) is a pair of reciprocals recorded in one of the 10 known Seleucid fragments of many-place tables of reciprocals (category I in the survey above), and if (*p*, *q*) is the corresponding pair of sexagesimal places, then, in the great majority of cases, *p* and *q* are both not bigger than 6, while in almost all the remaining cases $p + q$ is not bigger than 12. Expressed more concisely, *with a few exceptions, pairs of reciprocals appearing in the Seleucid fragments belonging to category I above are usually (at most) "double 6-place", and otherwise (at most) "total 12-place".*

Plus signs in col. *iii* of Table 1.1 indicate cases when a pair of reciprocals is total 12-place but not double 6-place. Examples in Table 1.1 are the five cases when $p + q$ = 8 + 4 = 12, 7 + 5 = 12, 4 + 7 = 11, 7 + 3 = 10, or, again, 8 + 4 = 12.

Double plus signs in col. *iii* indicate cases when a pair of reciprocals is more than total 12-place. The only example in Table 1.1 is the case when $p + q$ = 5 + 8 = 13. Note that in that particular case the 5-place number *n* = 1 01 30 33 45 does not occupy more space than a normal 4-place number, if 1 01 is written as 1 1. Therefore, the corresponding pair of reciprocals does not occupy more space than a normal total 12-place pair, and may be called "essentially" total 12-place.

The inclusion of (essentially) total 12-place pairs can probably be explained as follows: In the case of the fragments belonging to category I above, *the corresponding original many-place table of reciprocals was, presumably, inscribed on a clay tablet (or in a column of a clay tablet) that was just wide enough to allow lines of two 6-place numbers side by side (double 6-place pairs). Such a clay tablet (or column) was then also just wide enough to allow lines of (essentially) total 12-place pairs.*

A dash (—) in col. *iv* of Table 1 indicates a case when an existing total 12-place pair of reciprocals *n*, rec. *n* does not appear in any of the fragments of tables of reciprocals (category I), nor in the tablets U, S (categories VIII-IX), nor, indirectly, in any of the fragments of tables of squares or squares of squares (categories II-III). In all other instances, a number in col. *iv* indicates a line number in a suggested hypothetical reconstruction of a Neo-Babylonian many-place table of reciprocals which may have been *a common ancestor,* directly or indirectly, to most of the fragments of categories I-III (Seleucid) as well as to the whole tablets U (Achaemenid) and S (Neo-Babylonian). A suitable name for the hypothetically reconstructed Neo-Babylonian ancestral table is *Table R*.

In Britton's paper *JCS* 43-45 (1991-93), 76-77, a reconstructed Seleucid table of reciprocals that may have been the common ancestor, directly or indirectly, of all the *fragments* from Babylon belonging to categories I-III, except the atypical fragments I D and II F, is called a "double 6-place table of regular numbers". This reconstructed Seleucid table of reciprocals will in the following be called *Table B*. The two atypical fragments can hesitantly be assumed to be derived from an "enlarged Table B*", about which not much more is known.

The tentative claim that all the Seleucid fragments and the Neo- and Achaemenid tablets belonging to categories I or VIII-IX may have a common ancestor in Table R is based on the observation that in almost all cases *when a total 12-place pair is "missing" (that is, for some reason does not appear) in Table U, it is also missing in Table S and in the fragments belonging to category I, and that if a total 12-place pair is missing in all the fragments, it is also missing in Tables U and S.*

An example is the total 12-place pair with the index (-12, -11), between lines 10 and 11 in Table 1.1, which is missing in Table U, as well as in fragment I I (a table of reciprocals), and, indirectly in fragments II B (a table of squares) and III A-B (two tables of squares of squares).

Another example is the (less than) double 6-place pair with the index (-15, -9), between lines 27 and 28 in Table 1.1, which is missing in fragments I C, I D, I H, II F, and in both Table S and Table U. Note, however, for instance, that line 16 in the reconstructed table is missing in Table U but not in fragments I I, II B, II G, and III B. Therefore the table of reciprocals inscribed on tablet U can only be thought of as a somewhat imperfect copy of the hypothetical Table R. Similarly, the fact that line 23 of the reconstructed original table is missing in Table S but not in some of the fragments means that also the table of reciprocals on tablet S may be a somewhat imperfect descendent of Table R.

Another possibility is, maybe, that the Neo-Babylonian and Achaemenid Tables S and U and the Seleucid Table B were not directly related but *were constructed by use of the same arithmetical algorithms,* with nearly identical imposed constraints.

In contrast to this, although the Seleucid many-place table of reciprocals V = AO 6456 seems to have been constructed by use of the same arithmetical algorithms, the imposed constraints in this case were clearly different. Therefore, quite a few of the pairs of reciprocals in Table V have no counterparts in Tables R, U, or S. (See Sec. 1.5 below, and compare Figures 1.2.3 and 1.5.1.)

A few many-place numbers appearing in one of the Seleucid fragments tables of reciprocals and in one of the Seleucid fragments of tables of squares do not seem to be derived from entries in Table R (see the reconstruction of that table below). This was first pointed out by Britton, in *JCS* 43-45 (1991-93), 82-83.

Two of the atypical entries occur in fragment *LBAT* 1634 (I D), where the two extra lines

59a	(-19, -10)	1 30 25 20 50	39 48 47 13 55 12
59b	(14, -1)	1 31 01 20	39 33 02 48 45

are inserted between the following lines from Table R (line numbers as in the reconstruction below):

59	(-1, 1)	1 30	40
60	(-3, 6)	1 31 07 30	39 30 22 13 20.

Note that the mentioned extra lines do not occur between lines 59 and 60 in the fragments of tables of reciprocals I A and I J, and that the corresponding extra lines do not occur in the table of squares fragment II C. Similarly, the extra lines are not present in Tables U and S.

Other atypical entries of a similar kind can be observed in five extra lines in the two fragments *LBAT* 1636 and *LBAT* 1638, both probably parts of the same large tablet with a table of squares (II F). The five extra lines in these two fragments are inserted between lines 52 and 53, between lines 54 and 55, and between lines 64 and 65, respectively, in the reconstructed Table R. They are

52a	(-9, 6)	1 25 25 46 52 30	42 08 23 42 13 20	
52b	(10, -12)	1 25 44 01 58 31 06 40	41 59 25 26 24	($p + q = 8 + 5 = 13$!)
52c	(23, -3)	1 26 18 09 11 06 40	41 42 49 22 21 12 39 22 30	($p + q = 7 + 9 = 16$!)
54a	(21, 2)	1 27 22 52 48	41 11 55 25 46 52 30	
64a	(8, 14)	1 34 28 42 14 24	38 06 14 12 40 29 37 46 40	($p + q = 6 + 9 = 15$!)

These lines are missing in fragments I E, and I J, in fragments II A, II C, and II D, and in Tables U and S.

Note that four of the mentioned extra lines are among only seven lines preserved on the fragment *LBAT* 1636. The remaining extra line is among only four preserved lines on the fragment *LBAT* 1638. What this implies is that there must have been a large amount of such extra lines in the table text (II F) of which *LBAT* 1636 and 1638 are very small parts. Obviously, that table text, for which a suitable name may be "the table of squares related to an enlarged Table B*", must have been constructed by use of imposed constraints that differed considerably from the restraints imposed on the arithmetical algorithms used to produce Table B and the related table of squares!

According to a reconstruction first made by Vaiman in *SVM* (1961), 225, and then confirmed by Britton in *JCS* 43-45 (1991-93), 83 and fn. 32, the mentioned large clay tablet, of which now only the two fragments *LBAT* 1636 and 1638 (II F) remain, was inscribed on the obverse with three columns of many-place regular square numbers. The number inscribed in the first line of the third column, still partly preserved, is

2 25 07 07 45 3[8 18 14 24] (= sq. 1 33 18 43 12) (line 63 in Table R).

From this observation it is possible to draw the tentative conclusion that the three columns on the obverse contained considerably more than 100 entries, and that they comprised squares of many-place numbers *n* from 1 to 2. If that is so, then the reverse of that tablet can be assumed to have comprised about equally many squares of many-place numbers *n* from 2 to 4.

1.2.3 Table R: A Reconstructed Common Ancestor to Tables B, U, and S

In the reconstructed many-place Table R of reciprocals (Table 1.3 below) all those many-place regular sexagesimal numbers *n* are arranged together which occur in at least one of the fragments of many-place tables of reciprocals (category I above), or, indirectly, in one of the fragments of the related lists of squares or squares of squares (categories II-III), or in the many-place tables of reciprocals on tablets S and U (categories VIII-IX). The tabulated numbers *n* are written in sexagesimal relative place value notation, and are ordered *lexicographically*. (Alternatively, they can be thought of as sexagesimal numbers in absolute place value notation, arranged in ascending order, from 1 and 1;00 16 53 53 20 to 3;57 02 13 20 and 4.) The ordering is indicated by line numbers in col. *i*.

Lines that are present *in Table V (Sec. 1.5) but not in the reconstructed Table R* are also included in the table, indicated not by line numbers but by a v or a V in col. *i*, where v refers to an at most total 12-place pair of reciprocals, while V refers to a more than total 12-place pair.

The two extra lines from fragment I D, between lines 59 and 60 in Table R, are included in the table, too, indicated not by line numbers but by a z in col. *i*. Similarly, pairs of reciprocals corresponding to the five extra lines from the atypical table of squares fragment II F are also included, indicated not by line numbers but by an x in col. *i* if the pairs are at most total 12-place, and by an X otherwise.

The numbers *n* attested, directly or indirectly, in at least one of the fragments of categories I-III include together all the entries provided with line numbers from 0 to 101 in the reconstructed Table R below (Table 1.3), except lines 0 and 42 (which, on the other hand, are present in Table S). This should be clear from the following survey:

Table 1.2. Attested line numbers from Table B in the fragments

	fragment ##	preserved lines on *obv./rev.*	missing lines	extra lines
I A	BM 34577	32-48, 57-66	42	
I B	BM 34596	—, 76-101, 102a		
I C	BM 34612	27-36, —		59a-b
I D	BM 34635	23-36, 58-68	30	
I E	BM 34762	29-41, 43-57	30, 42	
I F	BM 76984	33-40, 63-80	42	
I G	BM 77051	—, 77-87		
I H	BM 78079	26-31, 60-73		
I I	MMA 86.11.407+	8-26, —		
I J	MMA 86.11.410+	49-73, 83-89, —		
II A	BM 32178	33-55, 56-80	42	
II B	BM 33567	5-11, 14-19		
II C	BM 34578	52-54, 55-66		
II D	BM 34714	39-59, 60-79	42, 67, 69	
II E	BM 34764	[1]-8, ⌜100-101, 102a⌝		
II F	BM 34875	22-28, 32-35, 52a-55, 63-65, —		52a-c, 54a, 64a
II G	BM 99633	14-18, 85-90		
II H	BM 35568	1-29, 97-101, 102a		
III A	BM 32584	8-19, 76-88	16, 79, 86	
III B	BM 55557	2-28, 100-101	0, 1, 7, 10, 22, 25	
IV A	BM 41101	208-217, —	209	
V A	BM 34601	102a		

VI A VI B VI C	BM 46550 W 23021 W 23016	2 20-31 6	21, 25, 28, 30	
Table U	W 23283+	0-47, 58-106, 111-204	16, 23, 79, 86, 99, 118, 125, 149, 150, 178, 181, 188	67a

Actually, the lines preserved in just four of the fragments are sufficient to cover the whole range of lines from 0 to 101, with the mentioned exceptions (lines 0 and 42), as shown by the following diagram:

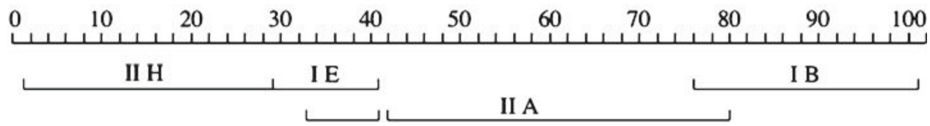


Fig. 1.2.2. Lines from Table R attested in at least one of the fragments.

As mentioned, the part of the reconstructed Table R below with *n* from 1 to 2 is identical with Britton's "Double 6-place table of regular numbers" in *JCS* 43-45 (1991-93), 76-77, with the following exceptions: The four lines that are present in Table U but not, directly or indirectly, in any of the fragments of categories I-III, are tentatively included in Table R but with question marks after the line numbers 0, 1, 42, and 67a. (The line given the line number 67a is problematic. It should almost certainly not be included in Table R. More about that below, in Sec. 1.3.5.)

The part of Table R below with *n* from 2 to 4 was tentatively constructed here through doubling and halving of each pair of reciprocals from the part of Table R with *n* from 1 to 2. That this is a reasonable procedure will be clear from the discussion in Sec. 1.5 below. Besides, all the pairs obtained in this way through doubling and halving are present in either Table S or Table U with only a few exceptions, namely that lines 118 and 125 are missing in both Table S and Table U, that lines 141 and 142 are missing in Table S and lost in Table U, and that lines 178, 181, 188, and 203a are missing in Table U and outside the range of Table S.

Here is the reason why it seems reasonable to let also the lines 0,1, and 42 be included in Table R. As mentioned, Britton's "double 6-place table" (or more correctly "total 12-place table"), Table B, is composed of all pairs of reciprocals *n,* rec. *n* attested, directly or indirectly, in at least one of the fragments of categories I-III (except the atypical fragments I D and II F). Importantly, the following similarities between Table B and Table U can be observed:

1. Apparently, about 100 total 12-place pairs with *n* from 1 to 2 were originally recorded in both Table B and Table U.
2. Of all the existing total 12-place pairs with *n* from 1 to 2, about 30 are missing simultaneously in both tables.
3. For *n* from 1 to 2 only five pairs in Table U are missing in Table B; four pairs in Table B are missing in Table U.
4. There are only two more than total 12-place pairs with *n* from 1 to 2 in Table B, namely

(-4, 10)	1 01 30 33 45	58 31 39 35 18 31 06 40	l. 4
(5, -10)	1 57 03 19 10 37 02 13 20	30 45 16 52 30	l. 98

The first of these pairs is attested, indirectly, in fragment III B, where the square of the square of 1 01 30 33 45 is recorded. Besides, the pair obtained from this pair by doubling and halving, twice, is recorded in fragment IV A. The second pair is attested in fragment I B. Both pairs are also attested in Table U, and there are no other more than 12-place pairs with *n* from 1 to 2 in Table U.

The noted similarities prove unequivocally that *Table U and Britton's reconstructed Table B must have had a common ancestor.* This common ancestor can be assumed to have contained, at least, all the pairs of reciprocals with *n* from 1 to 2 recorded in Table R below.

There are equally great similarities between Table B and Table S, obscured only by the fact that the obverse of Table S is quite poorly preserved. Thus, for instance, the first of the two mentioned more than total 12-place pairs with *n* from 1 to 2 in Tables B and U is not among the preserved pairs in Table S. Luckily, the second more than 12-place pair with *n* from 1 to 2 is present in Table U and well preserved. In addition the total 13-place pair in line 132 of Table R with index (1, -11), the only more than 12-place pair in Table R with *n* from 2 to 4, is present in both Table U and Table S. Therefore the conclusion must be that *also Table S must have had a common ancestor with Table U!*

Note that even if Table S may be older than Table U and Table B, it is not itself a direct ancestor of either one of them. This is clear because there are more “missing” pairs in Table S than in Table U and Table B. (Besides, only one pair that is missing in Table U is present in Table S.) *In the following, it will be assumed that the common ancestor to Tables S, U, and B is the reconstructed Table R in Table 1.3 below.*

In Table 1.3, errors in lines of Table U = W 23283+ or Table S = Sippar 2175/12 are indicated in cols. *iii -iv* by an underlining of the correct sexagesimal places in the lines in question, and by specifying the incorrect sexagesimal places in the columns headed Table U and Table S, respectively.

Errors in lines of Table V = AO 6456 are indicated by a double underlining of the correct sexagesimal places in the lines in question, and by specifying the incorrect sexagesimal places in the column headed Table V.

Lines that are present in Table U but not present, directly or indirectly, in any one of the fragments of categories I-III are indicated by a u (for total 12-place pairs) or a U (for a more than total 12-place pair), in the column headed Table U. As mentioned, there are only 4 such lines, namely lines 0, 1, 42, and 67a.

Table 1.3. The reconstructed Table R, with *n* from 1 to 4 (Tables R_1 and R_2)

line	*r*	*s*	*n*	rec. *n*	Table U	Table S	Table V
0?, v	0	0	1	1	u	lost	
v	-18	-11	1 00 16 53 53 20	59 43 10 50 52 48	—	—	
v	15	-2	1 00 40 53 20	59 19 34 13 07 30	—	—	
1 ?	-2	5	1 00 45	59 15 33 20	u	lost	
2	-20	-6	1 01 02 06 33 45	58 58 56 <u>38 24</u>		lost	33 45
3	13	3	1 01 26 24	58 35 37 30		lost	
4	-4	10	1 01 30 33 45	58 31 39 35 18 31 06 40		lost	
5	-5	-8	1 01 43 42 13 20	58 19 12		lost	
6	11	8	1 02 12 28 48	57 52 13 20		lost	
7	-7	-3	1 02 30	57 36		lost	
8	8	-5	1 03 12 35 33 20	56 57 11 15		lost	
9	-9	2	1 03 16 52 30	56 53 20		lost	
10	6	0	1 04	56 15		lost	
v	-12	-11	1 04 <u>18</u> 01 28 53 20	55 59 13 55 12	—	—	17
11	4	5	1 04 48	<u>55</u> 33 20		lost	45
12	-14	-6	1 05 06 15	55 17 45 36		lost	
v	19	3	1 05 32 09 36	54 55 53 54 22 30	—	—	
13	2	10	1 05 36 36	54 52 10 51 51 06 40		lost	
14	1	-8	1 05 50 37 02 13 20	54 40 30		lost	
15	-16	-1	1 05 55 04 41 15	54 36 48	06	lost	
16	17	8	1 06 21 18 43 12	54 15 12 30	missing	lost	missing
17	-1	-3	1 06 40	<u>54</u>	54 04	damaged	
V	0	15	1 06 25 48 27	54 11 32 12 41 35 28 23 42 13 20	—	—	
18	-3	2	1 07 30	53 20		damaged	
19	12	0	1 <u>08</u> 16	52 44 03 45	07		
20	-5	7	1 08 20 37 30	52 40 29 37 46 40			
21	10	5	1 09 07 12	<u>52 05</u>	52		
22	-8	-6	1 09 26 40	51 50 24			
—	—	—	(1 09 07 13)	(52)	corrupt	—	—
23	8	10	1 09 59 02 24	51 26 25 11 06 40	missing 15	missing	missing

24	-10 -1	1 10 18 45	51 12			
25	5 -3	1 11 06 40	50 37 30			
26	-12 4	1 11 11 29 03 45	50 34 04 26 40			
27	3 2	1 12	50			
28	18 0	1 12 49 04	49 26 18 30 56 15			17
29	1 7	1 12 54	49 22 57 46 40	27 49		
30	0 -11	1 13 09 34 29 08 08 53 20	49 12 27			
31	-17 -4	1 13 14 31 52 30	49 09 07 12			
32	16 5	1 13 43 40 48	48 49 41 15		missing	
33	-2 -6	1 14 04 26 40	48 36		missing	
34	14 10	1 14 38 58 33 36	48 13 31 06 40		missing	missing
35	-4 -1	1 15	48			
36	11 -3	1 15 51 06 40	47 27 39 22 30		missing	
37	-6 4	1 15 56 15	47 24 46 40			
38	9 2	1 16 48	46 52 30			
V	-8 9	1 16 53 12 11 15	46 49 19 40 14 48 53 20	—	—	54 58
39	-9 -9	1 17 09 37 46 40	46 39 21 36			
V	24 0	1 17 40 20 16	46 20 54 51 30 14 03 45	—	—	54
40	7 7	1 17 45 36	46 17 46 40		missing	missing
41	-11 -4	1 18 07 30	46 04 48			
42?	22 5	1 18 38 35 31 12	45 46 34 55 18 45	u	missing	missing
43	4 -6	1 19 00 44 26 40	45 33 45	corrupt		
v	-13 1	1 19 06 05 37 30	45 30 40	—	—	
44	2 -1	1 20	45			
V	-16 -12	1 20 22 31 51 06 40	44 47 23 08 09 36	—	lost	
V	17 -3	1 20 54 31 06 40	44 29 40 39 50 37 30	—	—	
45	0 4	1 21	44 26 40		—	
v	-18 -7	1 21 22 48 45	44 14 12 28 48	—	lost	
46	15 2	1 21 55 12	43 56 43 07 30		lost	45
47	-2 9	1 22 00 45	43 53 44 41 28 53 20		lost	
48	-3 -9	1 22 18 16 17 46 40	43 44 24	lost	lost	
v	-20 -2	1 22 23 50 51 33 45	43 41 26 24	—	lost	
49	13 7	1 22 56 38 24	43 24 10	lost	lost	51
50	-5 -4	1 23 20	43 12	lost	lost	
V	11 12	1 23 58 50 52 48	42 52 00 59 15 33 20	—	lost	
51	-7 1	1 24 22 30	42 40	lost	lost	
V	10 -6	1 24 16 47 24 26 40	42 42 53 26 15	—	lost	
52	8 -1	1 25 20	42 11 15	lost	—	13
x	-9 6	1 25 25 46 52 30	42 08 23 42 13 20	—	lost	
X	-10 -12	1 25 44 01 58 31 06 40	41 59 25 26 24	—	lost	31
X	23 -3	1 26 18 09 11 06 40	41 42 49 22 21 12 39 22 30	—	—	—
53	6 4	1 26 24	41 40	lost	lost	—
54	-12 -7	1 26 48 20	41 28 19 12	lost	lost	
x	21 2	1 27 22 52 48	41 11 55 25 46 52 30	—	lost	
55	4 9	1 27 28 48	41 09 08 08 53 20	lost	lost	—
56	3 -9	1 27 47 29 22 57 46 40	41 00 22 30	lost	damaged	43
57	-14 -2	1 27 53 26 15	40 57 36	lost	—	47

v	19 7	1 28 28 24 57 36	40 41 24 22 30	—	damaged	
58	1 -4	1 28 53 20	40 30	lost		
v	-16 3	1 28 59 21 19 41 15	40 27 15 33 20	—		
V	0 -22	1 29 12 19 26 34 23 19 49 -	40 21 22 41 00 09	—		42, ø
-	- -	38 08 36 52 20 44 26 40	-	—		43 20 12 20 34
V	0 19	1 29 40 50 24 27	40 08 32 44 57 28 29-		—	19, 00
-	- -	-	55 20 09 52 35 33 20	—	missing	45
59	-1 1	1 30	40	—		
z	-19 -10	1 30 25 20 50	39 48 47 13 55 12			—
z	14 -1	1 31 01 20	39 33 02 48 45			—
60	-3 6	1 31 07 30	39 30 22 13 20	—		23
61	12 4	1 32 09 36	39 03 45	—		
V	-5 11	1 32 15 50 37 30	39 01 06 23 32 20 44 26 40			
V	-23 0	1 32 41 49 43 00 28 07 30	38 50 10 08			ø, 00
62	-6 -7	1 32 35 33 20	38 52 48			
63	10 9	1 33 18 43 12	38 34 48 53 20	—	missing	15, 24
64	-8 -2	1 33 45	38 24		missing	
X	8 14	1 34 28 42 14 24	38 06 14 12 40 29 37 46 40		missing	—
65	7 -4	1 34 48 53 20	37 58 07 30			48
66	-10 3	1 34 55 18 45	37 55 33 20	U	missing	36, 45
67	5 1	1 36	37 30			
67a?	-12 8	1 36 06 30 14 03 45	37 27 27 44 11 51 06 40			—
68	-13 -10	1 36 27 02 13 20	37 19 29 16 48		—	37 27, 09
69	3 6	1 37 12	37 02 13 20	—		
70	-15 -5	1 37 39 22 30	36 51 50 24	lost	—	29
V	2 -12	1 37 32 45 58 50 51 51 06 40	36 54 20 15	lost	missing	
71	18 4	1 38 18 14 24	36 37 15 56 15	lost		
72	1 11	1 38 24 54	36 34 47 14 34 04 26 40	lost	missing	missing
73	0 -7	1 38 45 55 33 20	36 27	lost		
74	-17 0	1 38 52 37 01 52 30	36 24 32	lost	—	27
75	-2 -2	1 40	36			26
76	13 -4	1 41 08 08 53 20	35 35 44 31 52 30			
77	-4 3	1 41 15	35 33 20	missing		
78	11 1	1 42 24	35 09 22 30			
79	-7 -10	1 42 52 50 22 13 20	34 59 31 12		lost	
80	9 6	1 43 40 48	34 43 20		lost	
81	-9 - 5	1 44 10	34 33 36		lost	
82	6 -7	1 45 20 59 15 33 20	34 10 18 45		lost	43
83	-11 0	1 45 28 07 30	34 08		lost	
84	4 -2	1 46 40	33 45	missing	lost	
85	2 3	1 48	33 20		lost	
86	-16 -8	1 48 30 25	33 10 39 21 36		lost	
87	17 1	1 49 13 36	32 57 32 20 37 30		damaged	
88	0 8	1 49 21	32 55 18 31 06 40		damaged	31, 45
89	-1 -10	1 49 44 21 43 42 13 20	32 48 18	—	51 47	missing
90	15 6	1 50 35 31 12	32 33 07 30			32
V	-2 13	1 50 43 00 45	32 30 55 19 36 57 17 02 13 20	—	—	

91	-3 -5	1 51 06 40	32 24		—	
V	-20 2	1 51 14 11 39 36 33 45	32 21 48 26 40			
92	-5 0	1 52 30	32			
93	10 -2	1 53 46 40	31 38 26 15			25
94	-7 5	1 53 54 22 30	31 36 17 46 40			
95	8 3	1 55 12	31 15			
96	-10 -8	1 55 44 26 40	31 06 14 24	missing		
97	6 8	1 56 38 24	30 51 51 06 40			
98	5 -10	1 57 03 19 10 37 02 13 20	30 45 16 52 30			
99	-12 -3	1 57 11 15	30 43 12	—		
100	21 6	1 57 57 53 16 48	30 31 03 16 52 30			57, 06 13
101	3 -5	1 58 31 06 40	30 22 30	—		32
V	-14 2	1 58 39 08 26 15	30 20 26 40	—	—	36, 29
102	1 0	2	30			
V	0 -18	2 00 25 38 14 52 25 29 46 -	29 53 36 48 09		—	04
-	- -	00 29 37 46 40	-	lost	—	ø
102a	0 23	2 01 04 08 03 00 27	29 44 06 28 51 27 46 36 -	lost		
-	- -	-	32 42 52 17 26 54 48 53 20	lost		29 51 26 44 06
103	-1 5	2 01 30	29 37 46 40	lost		
104	-19 -6	2 02 04 13 07 30	29 29 28 19 12	lost		missing
105	14 3	2 02 52 48	29 17 48 45	lost		missing
106	-3 10	2 03 01 07 30	29 15 49 47 39 15 33 20			
107	-4 -8	2 03 27 24 26 40	29 09 36			
108	12 8	2 04 24 57 36	28 56 06 40			32
109	-6 -3	2 05	28 48			
110	9 -5	2 06 25 11 06 40	28 28 35 37 30			missing
111	-8 2	2 06 33 45	28 26 40			
112	7 0	2 08	28 07 30	59		
113	5 5	2 09 36	27 46 40	missing		
114	-13 -6	2 10 12 30	27 38 52 48		59	
115	3 10	2 11 13 12	27 26 05 25 55 33 20		missing	
116	2 -8	2 11 41 14 04 26 40	27 20 15			missing
117	-15 -1	2 11 50 09 22 30	27 18 24			00
118	18 8	2 12 42 37 26 24	27 07 36 15			missing
119	0 -3	2 13 20	27			missing
120	-2 2	2 15	26 40	missing		24
121	13 0	2 16 32	26 22 01 52 30			
122	-4 7	2 16 41 15	26 20 14 48 53 20		missing	21
123	11 5	2 18 14 24	26 02 30			00 , 00 18
124	-7 -6	2 18 53 20	25 55 12			missing
125	9 10	2 19 58 04 48	25 43 12 35 33 20			
126	-9 -1	2 20 37 30	25 36	18 55 38		missing
127	6 -3	2 22 13 20	25 18 45			
128	-11 4	2 22 22 58 07 30	25 17 02 13 20			missing
129	4 2	2 24	25	—	correct!	48
130	19 0	2 25 38 08	24 43 09 15 28 07 30		missing	
131	2 7	2 25 48	24 41 28 53 20		missing	missing

132	1 -11	2 26 19 08 58 16 17 46 40	24 36 13 30		—	missing missing
133	-16 -4	2 26 29 03 45	24 34 33 36			missing
134	17 5	2 27 27 21 36	24 24 50 37 30	lost	missing	27
V	0 12	2 27 37 21	24 23 11 29 42 42 57 46 40	lost		
135	-1 -6	2 28 08 53 20	24 18	—	missing	
136	15 10	2 29 17 57 07 12	24 06 45 33 20	lost		missing
137	-3 -1	2 30	24	—		missing
138	12 -3	2 31 42 13 20	23 43 45 41 15	lost	—	
139	-5 4	2 31 52 30	23 42 13 20	lost	missing	missing
140	10 2	2 33 36	23 26 15	lost	—	36
(V)	7 -9	(2 33 46 24 22 30)	(23 24 39 50 07 24 26 40)	—	missing	(57)
141	-8 -9	2 34 19 15 33 20	23 19 40 48	lost		
V	25 0	2 35 20 40 32	23 10 27 25 45 07 01 52 30	lost		52 00
142	8 7	2 35 31 12	23 08 53 20		—	missing
143	-10 -4	2 36 15	23 02 24		damaged	
144	5 -6	2 38 01 28 53 20	22 46 52 30	missing	damaged	missing
v	-12 1	2 38 12 11 15	22 45 20	missing	damaged	
145	3 -1	2 40	22 30		damaged	missing
146	1 4	2 42	22 13 20		damaged	missing
147	16 2	2 43 50 24	21 58 21 33 45		damaged	missing
148	-1 9	2 44 01 30	21 56 52 20 44 26 40		lost	missing
149	-2 -9	2 44 36 32 35 33 20	21 52 12		lost	missing
150	14 7	2 45 53 16 48	21 42 05	26	lost	
151	-4 -4	2 46 40	21 36	57, 30	lost	
152	-6 1	2 48 45	21 20		lost	
153	9 -1	2 50 40	21 05 37 30		lost	missing
154	7 4	2 52 48	20 50		lost	missing
155	-11 -7	2 53 36 40	20 44 09 36		lost	missing
156	5 9	2 54 57 36	20 34 34 04 26 40		lost	missing
157	4 -9	2 55 34 58 45 55 33 20	20 30 11 15		end lost	missing
158	-13 -2	2 55 46 52 30	20 28 48			
159	2 -4	2 57 46 40	20 15			00
160	0 1	3	20			end
161	-2 6	3 02 15	19 45 11 06 40	lost		
162	-20 -5	3 03 06 19 41 15	19 39 38 52 48	lost		
163	13 4	3 04 19 12	19 31 52 30			
164	-5 -7	3 05 11 06 40	19 26 24			
165	11 9	3 06 37 26 24	19 17 24 26 40			
166	-7 -2	3 07 30	19 12			
167	8 -4	3 09 37 46 40	18 59 03 45	27		
168	-9 3	3 09 50 37 30	18 57 46 40	21 [57]		
169	6 1	3 12	18 45			
170	-12 -10	3 12 54 04 26 40	18 39 44 38 24			
171	4 6	3 14 24	18 31 06 40	missing		
172	-14 -5	3 15 18 45	18 25 55 12			
173	19 4	3 16 36 28 48	18 18 37 58 07 30			
174	2 11	3 16 49 48	18 17 23 37 17 02 13 20	missing		

175	1 -7	3 17 31 51 06 40	18 13 30			
176	-16 0	3 17 45 14 03 45	18 12 16			
177	-1 -2	3 20	18	20		
178	14 -4	3 22 16 17 46 40	17 47 52 15 56 15			
179	-3 3	3 22 30	17 46 40			
180	12 1	3 24 48	17 34 41 15			
181	-6 -10	3 25 45 40 44 26 40	17 29 45 36	missing		
182	10 6	3 27 21 36	17 21 40			
183	-8 -5	3 28 20	17 16 48			
184	7 -7	3 30 41 58 31 06 40	17 05 09 22 30			
185	-10 0	3 30 56 15	17 04			
186	5 -2	3 33 20	16 52 30			
187	3 3	3 36	16 40			
188	-15 -8	3 37 00 50	16 35 19 40 48			
189	18 1	3 38 27 12	16 28 46 10 18 45			
190	1 8	3 38 42	16 27 35 15 33 20			
191	0 -10	3 39 28 43 27 24 26 40	16 24 09			
192	16 6	3 41 11 02 24	16 16 33 45			
193	-2 -5	3 42 13 20	16 12	39 22 30		
194	-4 0	3 45	16	21 21		
195	11 -2	3 47 33 20	15 49 13 07 30	01 28		
196	-6 5	3 47 48 45	15 48 08 53 20			
197	9 3	3 50 24	15 37 30	missing		
198	-9 -8	3 51 28 53 20	15 33 07 12	4 24		
199	7 8	3 53 16 48	15 25 55 33 20			
200	6 -10	3 54 06 38 21 14 04 26 40	15 22 38 26 15			
201	-11 -3	3 54 22 30	15 21 36			
202	22 6	3 55 55 46 33 36	15 15 31 38 26 15			
203	4 -5	3 57 02 13 20	15 11 15			
203a	-13 2	3 57 18 16 52 30	15 10 13 20			
204	2 0	4	15			

A reasonable conjecture seems to be that, *originally, Table R was written on three clay tablets, each containing about 100 entries.* If that is so, then n proceeded from 1 to 2 on the first of the three tablets, from 2 to 4 on the second tablet, and from 4 to 8 on the third tablet.

Indeed, it is important to notice that if n, rec. n is a given pair of reciprocals with n from 1 to 2, then 2 n, 1/2 rec. n is a pair of reciprocals with n from 2 to 4, while 4 n, 1/4 rec. n is a pair of reciprocals with n from 4 to 8. Therefore, the table with n from 2 to 4 can be constructed with departure from the table with n from 1 to 2 simply through the process of doubling and halving applied to all the pairs of reciprocals in that table. (For well known Old Babylonian examples of the application of the process of doubling and halving, see Friberg, *MSCT 1* (2007), App. 3.) The table with n from 4 to 8 can be constructed in a similar way with departure from the table with n from 2 to 4.

If n is greater than 8, then the inverse pair rec. n, n is a pair of reciprocals with rec. n smaller than 8, which, therefore, is an already recorded pair. Hence it would make no sense to continue the many-place table of reciprocals beyond the range of the pairs n, rec. n with n from 1 to 8, or even beyond the pair

(-8, -8) 7 42 57 46 40 7 46 33 36.

This is the last (at most) total 12-place pair n, rec. n with n smaller than rec. n. Between this pair and

(3, 0) 8 7 30

there are only the following four (at most) total 12-place pairs, all with *n* greater than rec. *n*:

(-10, -3)	7 48 45	7 40 48
(5, -5)	7 54 04 26 40	7 35 37 30
(-12, 2)	7 54 36 33 45	7 35 06 40.

Note, by the way, that since 7 42 57 46 is approximately equal to its reciprocal, it follows that 7;42 57 46 40 is an approximate square root of '1' (in this case by necessity meaning 60, not 1)!

The conjecture that the original Table R of many-place pairs of reciprocals may have been written on three tablets, one for *n* from 1 to 2 (R_1), another for *n* from 2 to 4 (R_2), and a third for *n* from 4 to 8 (R_3), is supported by the following series of observations:

1. In nearly all the Seleucid fragments of many-place tables of reciprocals *n*, rec. *n* (I A- I J), fragments of many-place tables of squares sq. *n* (II A - II H), and fragments of many-place tables of squares of squares sq. sq. *n* (III A - III B), *n* goes from 1 to 2. Hence these LB fragments may be excerpts from the hypothetical tablet R_1, or from closely related tables of squares or squares of squares.

2. In the fragment I B (see the hand copy in Sachs, *LBAT* 1633), the end of tablet R_1 is reached, with the last entry being

(3, -5)	[1 58] 31 06 40	30 22 30	*l.* 101

This entry is followed by a ruled line, and beneath it what appears to be a catch line, the number

(0, 23)	[2 01 04] 08 03 00 27	*l.* 102a

The reciprocal of this number is not given, probably for the following reason: The number itself is 7-place, and its reciprocal is 17-place, which means that the pair of reciprocals is total 24-place and certainly not the first entry of the total 12-place table of reciprocals R_2. (See Table 1.3 above.) Therefore, it is possible that fragment I B is part of (a copy of) a copy of tablet R_1 made by someone who did not own a copy of tablet R_2, but who knew that the 23rd power of 3 is only slightly greater than 2 (in the lexicographical sense used for the sorting of the entries of Table R).

The end of Table R_1 is reached also in fragment III B, Britton's fragment of a table of squares of squares, which ends with the square of the square of the number *n* in line 101 of Table R_1, followed by a ruled line.

Fragment II H, the new fragment of a table of squares (Sec. 1.2.1 above) ends with the square of the number *n* in line 101, followed by a ruled line and the square of the number *n* in line 102a. The small and badly preserved fragment II E (LBAT 1640) apparently ends, similarly, with traces of the squares of the numbers *n* in lines 100-101, followed by traces of the square of the number *n* in line 102a.

Note that the appearance of the number *n* in line 102a, or its square, or the square of its square, at the ends of the tables on so many fragments clearly indicates that this number must have been part of the Seleucid Table B itself, as a kind of catch line.

3. Fragment IV A contains part of the first column of a two-column tablet, with only traces remaining of the inscription in the second column. It contains pairs of reciprocals that can be obtained by doubling and halving of the pairs in lines 106 to 115 of Table R above, with the pair corresponding to line 107 missing. It is reasonable to assume that the fragment is part of a copy of the hypothetical tablet R_3. (Note by the way, that the total 12-place pair with *n* = 4 06 02 15, which is the first preserved pair in fragment IV A, is obtained by doubling and halving the more than total 12-place pair with *n* = 2 03 01 07 30 (Table R, line 106), which itself is obtained by doubling and halving the more than 12-place pair with *n* = 1 01 30 33 45 (Table R, line 4).

4. Table U, which is written on a relatively well preserved clay tablet, is an Achaemenid many-place table of reciprocals, with *n* from 1 to 4, while Table S, written on a much less well preserved clay tablet, apparently is a Neo-Babylonian many-place table of reciprocals with *n* from 1 to 3. Presumably, Table U, with about 200 pairs of reciprocals, is a copy of the tables of reciprocals on tablets R_1 and R_2. Table S, on the other hand, with originally perhaps 160 pairs of reciprocals, is, presumably, the first of two consecutive tables, on two clay tablets, together forming a complete copy of Tables R_1, R_2, and R_3.

Note, by the way that the number 9 is written in Tables U and S, and also in the table of squares VII A, in the traditional form with three times three upright wedges, while it is written in all the fragments (categories I-V) in the simplified form with three diagonally placed oblique wedges. What this implies is, presumably, that the fragments are of a younger date than Tables U and S, probably Seleucid. In the Seleucid Table V, the number 9 is, of course, written in the simplified form.

5. It is, evidently, no accident that in Table U there are 102 pairs of reciprocals with *n* from 1 to 2, presumably copied from Table R_1, and also 102 pairs of reciprocals with *n* from 2 to 4, presumably copied from Table R_2. As mentioned, the most logical explanation appears to be that each pair of reciprocals with *n* from 2 to 4 was constructed with departure from a corresponding pair of reciprocals with *n* from 1 to 2 through the already mentioned process of doubling and halving.

That this explanation is almost certainly correct is demonstrated by the following comparison of the beginnings of Tables R_1 and R_2 as they are preserved in Table R and the beginning of Table R_3 in fragment IV A. Note that each time that a total

12-place number is missing in Table R_1, the corresponding pairs are also missing in table R_2, and in fragment IV A, a part of Table R_3.

Table 1.4. A comparison of the beginnings of Tables R_1, R_2, and R_3

	R_1			R_2		R_3 (IV A)	
line	***r s***	***n***	**line**	***r s***	***n***	***r s***	***n***
0	0 0	1	102	1 0	2		
--	-18 -11	—	--	-17 -11	—		
--	15 -2	—	--	16 -2	—		
1	-2 5	1 00 45	103	-1 5	2 01 30		
2	-20 -6	1 01 02 06 33 45	104	-19 -6	2 02 04 13 07 30		
3	13 3	1 01 26 24	105	14 3	2 02 52 48	. . .	. . .
4	-4 10	1 01 30 33 45	106	-3 10	2 03 01 07 30	-2 10	4 06 02 15
5	-5 -8	1 01 43 42 13 20	107	-4 -8	2 03 27 24 26 40	-3 -8	missing
--	-22 -1	—	--	-21 -1	—	-20 -1	—
6	11 8	1 02 12 28 48	108	12 8	2 04 24 57 36	13 8	4 08 49 55 12
7	-7 -3	1 02 30	109	-6 -3	2 05	-5 -3	4 10
8	8 -5	1 03 12 35 33 20	110	9 -5	2 06 25 11 06 40	10 -5	4 12 50 22 13 20
9	-9 2	1 03 16 52 30	111	-8 2	2 06 33 45	-7 2	4 13 07 30
10	6 0	1 04	112	7 0	2 08	8 0	4 16
--	-12 -11	—	--	-11 -11	—	-10 -11	—
11	4 5	1 04 48	113	5 5	2 09 36	6 5	4 19 12
12	-14 -6	1 05 06 15	114	-13 -6	2 10 12 30	-12 -6	4 20 25
--	19 3	—	--	20 3	—	21 3	—
13	2 10	1 05 36 36	115	3 10	2 11 13 12	4 10	4 22 26 24
14	1 -8	1 05 50 37 02 13 20	116	2 -8	2 11 41 14 04 26 40	. . .	. . . . .
15	-16 -1	1 05 55 04 41 15	117	-15 -1	2 11 50 09 22 30		
16	17 8	1 06 21 18 43 12	118	18 8	2 12 42 37 26 24		
17	-1 -3	1 06 40	119	0 -3	2 13 20		
18	-3 2	1 07 30	120	-2 2	2 15		
--	-21 -9	—	--	-20 -9	—		
19	12 0	1 08 16	121	13 0	2 16 32		
20	-5 7	1 08 20 37 30	122	-4 7	2 16 41 15		
--	-6- 11	—	--	-5 -11	—		
21	10 5	1 09 07 12	123	11 5	2 18 14 24		
22	-8 -6	1 09 26 40	124	-7 -6	2 18 53 20		
etc.			etc.				

1.2.4 *The Numerical Algorithms Used for the Construction of Table R*

As will be shown below, in Sec. 1.3.4, numerical errors in Table U suggest that the lexicographically ordered list of (at most) total 12-place pairs of reciprocals in Table U, as well as in the hypothesized Table R, were computed by way of the following series of numerical algorithms:

1. First a table was constructed of 12 successive powers of 3, and their reciprocals, (Table 1.5 below in Sec. 1.5.3) continued for as long as the pair of reciprocals 3^n, rec. 3^n stayed at most total 12-place.

2a. Next, for each pair of reciprocals 3^n, rec. 3^n computed in step 1, a corresponding "preliminary algorithm table" was constructed, where a series of new pairs of reciprocals were computed by repeated use of a doubling and halving algorithm.

The table was (in principle) continued for as long as the calculated pairs of reciprocals stayed at most total 12-place (or until there was no more space available on the clay tablet on which the preliminary algorithm table was inscribed).

2b. Similarly, for each pair of reciprocals computed in step 1, a new table was produced, where a series of new pairs of reciprocals were computed by repeated use of the doubling and halving algorithm, with departure from the inverse pair rec. 3^n, 3^n. The table was continued for as long as the calculated pairs of reciprocals stayed at most total 12-place.

3. All the pairs of reciprocals *n*, rec. *n* produced in steps 2a-b with *n* smaller than rec. *n* (in the lexicographic sense), together with all inverse pairs of reciprocals rec. *n*, *n* with rec. *n* smaller than *n* (in the lexicographic sense), were collected together in one great table and ordered lexicographically with respect to the numbers *n* in the first column of the table.

To *compute* in this way the about 300 many-place pairs of reciprocals in Table R must have been a truly formidable task. The last of the pairs of reciprocals calculated in step 1 of the outlined algorithmic procedure was

3^{11}, rec. 3^{11} = 49 12 27, 1 13 09 34 29 08 08 53 20 (total 12-place) (line 30 in Table R)

Consequently, there were 12 preliminary algorithm tables of many-place pairs of reciprocals (A_0-A_{11}) produced in step 2a of the procedure, and 11 more preliminary algorithm tables with many-place pairs of reciprocals (B_1-B_{11}) produced in step 2b. Together, there were 23 preliminary algorithm tables produced in steps 2a-b.

To *rearrange and copy* more than 300 pairs of many-place sexagesimal numbers from so many subtables in order to get a combined table where *n* was made to ascend lexicographically from 1 to about 8 must have been an even more formidable task. Remember that when the medium was clay inscribed with cuneiform number signs *there was no easy way of repairing a mistake,* for instance by inserting a missed line or moving a misplaced line. It is likely that this is the explanation for some of the missing total 12-place pairs in Tables R, U, and S.

The first of the 23 preliminary algorithm tables is shown below. For the readers' convenience, a bold *n* indicates that smaller than rec. *n*. Similarly, a bold rec. *n* indicates that rec. *n* is smaller than *n*. Expressed differently, the bold numbers are all smaller than 7 42 57 46 40, and the remaining numbers are all greater than 7 42 57 46 40.

Table A_0. At most total 12-place pairs produced with departure from the pair 1, 1

r	*s*	lines		$n = 2^r \cdot 3^s$	rec. *n*		*p* + *q*
0	0	0		**1**	1		2
1	0	102		**2**	30		2
2	0	204		**4**	15		2
3	0	296		8	**7 30**		3
4	0	194		16	**3 45**		3
5	0	92		32	**1 52 30**		4
6	0	10		**1 04**	56 15		4
7	0	112		**2 08**	28 07 30		5
8	0	214		**4 16**	14 03 45		5
9	0	287		8 32	**7 01 52 30**		6
10	0	185		17 04	**3 30 56 15**		6
11	0	83		34 08	**1 45 28 07 30**		7
12	0	19		**1 08 16**	52 44 03 45		7
13	0	121		**2 16 32**	26 22 01 52 30		8
14	0	223		**4 33 04**	13 11 00 56 15		8
15	0	278		9 06 08	**6 35 30 28 07 30**		9
16	0	176		18 12 16	**3 17 45 14 03 45**		9
17	0	74		36 24 32	**1 38 52 37 01 52 30**		10
18	0	28		**1 12 49 04**	49 26 18 30 56 15		10
19	0	130		**2 25 38 08**	24 43 09 15 28 07 30		11
20	0	232		**4 51 16 16**	12 21 34 37 44 03 45		11
21	0	—		9 42 32 32	**6 10 47 18 52 01 52 30**		12
22	0	—		19 25 05 04	**3 05 23 39 26 00 56 15**		12
23	0	—		38 50 10 08	**6 10 47 18 52 01 52 30**		12

The line numbers in col. *ii* of the table above that are between 0 and 204 are identical with the line numbers in the reconstructed part of Table R, with *n* from 1 to 4. The numbers greater than 204 are conjectured line numbers for the essentially non-documented part of Table R with *n* from 4 to 8. Missing (at most) total 12-place pairs in Table R are indicated by dashes (—) in col. *ii*.

The 2nd and 3rd of the 23 tables produced in steps 2 a-b of the algorithmic procedure mentioned above are reproduced below. Note that in the 3rd table the first line, the one for (0, -1), is missing, since it would be a duplicate (although with the numbers in inverse order) of the first line in the 2nd table.

Table A$_1$. At most total 12-place pairs produced with departure from the pair 3^1, rec. 3^1 = 3, 20

0	1	160		**3**	20		2
1	1	262		**6**	10		2
2	1	239		12	**5**		2
3	1	137		24	**2 30**		3
4	1	35		48	**1 15**		3
5	1	67		**1 36**	37 30		4
6	1	169		**3 12**	18 45		4
7	1	271		**6 24**	9 22 30		5
8	1	228		12 48	**4 41 15**		5
9	1	126		25 36	**2 20 37 30**		6
10	1	24		51 12	**1 10 18 45**		6
11	1	78		**1 42 24**	35 09 22 30		7
12	1	180		**3 24 48**	17 34 41 15		7
13	1	282		**6 49 36**	8 47 20 37 30		8
14	1	219		13 39 12	**4 23 40 18 45**		8
15	1	117		27 18 24	**2 11 50 09 22 30**		9
16	1	15		54 36 48	**1 05 55 04 41 15**		9
17	1	87		**1 49 13 36**	32 57 32 20 37 30		10
18	1	189		**3 38 27 12**	16 28 46 10 18 45		10
19	1	291		**7 16 54 24**	8 14 23 05 09 22 30		11
20	1	—		14 33 48 48	**4 07 11 32 34 41 15**		11
21	1	—		29 07 37 36	**2 03 35 46 17 20 37 30**		12
22	1	—		58 15 15 12	**1 01 47 53 08 40 18 45**		12

Table B$_1$. At most total 12-place pairs produced with departure from the pair rec. 3^{-1}, 3^1 = 20, 3

1	-1	59		40	**1 30**		3
2	-1	44		**1 20**	45		3
3	-1	145		**2 40**	22 30		4
4	-1	248		**5 20**	11 15		4
5	-1	255		10 40	**5 37 30**		5
6	-1	152		21 20	**2 48 45**		5
7	-1	51		42 40	**1 24 22 30**		6
8	-1	52		**1 25 20**	42 11 15		6
9	-1	153		**2 50 40**	21 05 37 30		7
10	-1	255		**5 41 20**	10 32 48 45		7
11	-1	—?		11 22 40	**5 16 24 22 30**		8

12	-1	v		22 45 20	**2 38 12 11 15**		8
13	-1	v		45 30 40	**1 19 06 05 37 30**		9
14	-1	I D		**1 31 01 20**	39 33 02 48 45		9
15	-1	—		**3 02 02 40**	19 46 31 24 22 30		10
16	-1	—		**6 04 05 20**	9 53 15 42 11 15		10
17	-1	—		12 08 10 40	**4 56 37 51 05 37 30**		11
18	-1	—		24 16 21 20	**2 28 18 55 32 48 45**		11
19	-1	—		48 32 42 40	**1 14 09 27 46 24 22 30**		12
20	-1	—		**1 37 05 25 20**	37 04 43 53 12 11 15		12

It is possible to draw an important conclusion already from these first three examples of the preliminary algorithm tables produced in steps 2 a-b of the algorithmic procedure. Namely, if there is a missing pair of reciprocals in one of the preliminary algorithm tables, indicated by dashes (—) in col. *I*, then also all the ensuing pairs are missing. Thus, in table A_0 the last two pairs are missing, in table A_1 the last three pairs, and in table B_1 the last ten pairs. This conclusion is important because it implies that missing pairs in Table R are mostly due only to incomplete, prematurely ended, preliminary algorithm tables A_k or B_k, not to errors in the sorting process in step 3 of the mentioned algorithmic procedure.

Note, in the case of the preliminary table B_1, that it is not clear that the pair of reciprocals with the index (14, -1), which appears only in the "atypical" fragment I D, should be included at all in Table R. In Britton's version of Table R_1 (Table B), in *JCS* 43-45 (1991-93), 76-77, this pair is not included. For this reason, no line number in Table R is associated with this pair of reciprocals.

There would be no point in reproducing all the 23 tables of many-place pairs of reciprocals computed in steps 2a-b of the algorithmic procedure. However, two further examples are shown below.

Table A_{10}. At most total 12-place pairs produced with departure from the pair 3^{10}, rec. 3^{10}

0	10	191		16 24 09	**3 39 28 43 27 24 26 40**		11
1	10	89		32 48 18	**1 49 44 21 43 42 13 20**		11
2	10	13		**1 05 36 36**	54 52 10 51 51 06 40		11
3	10	115		**2 11 13 12**	27 26 05 25 55 33 20		11
4	10	217		**4 22 26 24**	13 43 02 42 57 46 40		11
5	10	283		8 44 52 48	**6 51 31 21 28 53 20**		11
6	10	181		17 29 45 36	**3 25 45 40 44 26 40**		11
7	10	79		34 59 31 12	**1 42 52 50 22 13 20**		11
8	10	23		**1 09 59 02 24**	51 26 25 11 06 40		11
9	10	125		**2 19 58 04 48**	25 43 12 35 33 20		11
10	10	227		**4 39 56 09 36**	12 51 36 17 46 40		11
11	10	272		9 19 52 19 12	6 25 48 08 53 20		11
12	10	170		18 39 44 38 24	**3 12 54 04 26 40**		11
13	10	68		37 19 29 16 48	**1 36 27 02 13 20**		11
14	10	34		**1 14 38 58 33 36**	48 13 31 06 40		11
15	10	136		**2 29 17 57 07 12**	24 06 45 33 20		11
16	10	238		**4 58 35 54 14 24**	12 03 22 46 40		11
17	10	—		9 57 11 48 28 48	**6 01 41 23 20**		11
18	10	—		19 54 23 36 57 36	**3 00 50 41 40**		11
19	10	I D		39 48 47 13 55 12	**1 30 25 20 50**		11
20	10	—		**1 19 37 34 27 50 24**	45 12 40 25		11
21	10	—		**2 39 15 08 55 40 48**	22 36 20 12 30		12
22	10	—		**5 18 30 17 51 21 36**	11 18 10 06 15		12

Table B_{10}. Up to 14-place pairs produced with departure from the pair rec. 3^{10}, 3^{10}

1 - 10	293		**7 18 57 26 54 48 53 20**	8 12 04 30	12
2 - 10	208		14 37 54 53 49 37 46 40	**4 06 02 15**	12
3 - 10	106		29 15 49 47 39 15 33 20	**2 03 01 07 30**	13
4 -10	4		58 31 39 35 18 31 06 40	**1 01 30 33 45**	13
5 -10	98		**1 57 03 19 10 37 02 13 20**	30 45 16 52 30	14
6 -10	200		**3 54 06 38 21 14 04 16 40**	15 22 38 26 15	14

In the preliminary algorithm table A_{10}, it is not clear that the pair of reciprocals with the index (19, 10), which appears only in the atypical fragment I D, should be included at all in Table R. In Britton's version of Table R_1 (Table B), in *JCS* 43-45 (1991-93), 76-77, this pair is not included. No line number in Table R is associated with this pair of reciprocals.

Also the preliminary algorithm table B_{10} is interesting, but for a different reason. Only the first two pairs of reciprocals in the table are 12-place, the four other pairs are more than 12-place, yet they are present in Table R! Perhaps the one who computed the six pairs of reciprocals thought that a preliminary algorithm table with only two pairs would be too brief?

1.2.5 The Double 6-Place Hexagon and the 12-Place Flower in the Index Grid

What follows here is an elaboration of a similar discussion in Friberg *CTMMA II* (2005).

As is well known, regular sexagesimal numbers are sexagesimal numbers that can be thought of as integers with no other prime factors than 2, 3, or 5. Consequently, in modern notations, every regular sexagesimal number *n* can be written in the form

$n = 2^r \cdot 3^s \cdot 5^t$ with a corresponding "triple index" (*r, s, t*).

To every such point corresponds a uniquely determined point with the coordinates *(r, s, t*) in a certain "triaxial index grid", originally introduced by Neugebauer. However, as mentioned above, 5 = rec. 12 in the Babylonian *relative* (or floating) place value notation for sexagesimal numbers, so that any power of 5 can be replaced by a corresponding negative power of 12, that is, by a (negative) mixed power of 2 and 3. Therefore, every regular sexagesimal number in relative place value notation can also be written in the simplified form

$n = 2^r \cdot 3^s$ with a corresponding "double index" (*r, s*).

Consequently, Neugebauer's triaxial index grid can, with advantage, be replaced by an equivalent "biaxial index grid", which is somewhat easier to understand. See Fig. 1.2.3 below.

Now, it can be shown (counting with logarithms) that all points in the index grid representing (at most) 6-place regular sexagesimal numbers n from 1 to 4 lie within a certain "6-place index triangle", which has its vertices roughly at the index points representing the three 6-place numbers

2^{31} = 2 45 42 03 14 08, 3^{20} = 4 29 02 31 13 21, $2 \cdot 5^{13}$ = 3 08 22 48 24 10.

The index points representing regular numbers such that their *reciprocals* are (at most) 6-place numbers lie within an inverse 6-place index triangle. Therefore, *double 6-place pairs of reciprocals* (where both the number and its reciprocal are 6-place) with *n* from 1 to 4 are represented by points in the index grid situated *simultaneously* in both 6-place index triangles. This means that the points common to the two triangles lie in a certain 6-sided figure, which may be called the "double 6-place index hexagon", colored white in Fig. 1.2.3 below.

A refined analysis shows that index points representing regular numbers *n* such that *n*, rec. *n* is (at most) a total 12-place pair can lie either in a "double 6-place index hexagon", or in a "5+7-place index hexagon", a "7+5-place index hexagon", etc. Taking all possible cases when $p + q = 12$ into consideration, one finds that all total twelve-place pairs with *n* from 1 to 4 must lie within a certain "total 12-place index flower". In Fig. 1.2.3, this complicated figure is divided into the white "double 6-place index hexagon" and the hatched areas which contain index points representing total twelve-place pairs which are outside the double 6-place hexagon.

Furthermore, in Fig. 1.2.3, *n tags* show the locations of all index points representing regular numbers *n* from 1 to 4 (in some cases 8) corresponding to (at most) total 12-place pairs of reciprocals *n*, rec. *n* in the reconstructed Table R of many-place pairs of reciprocals. More than total 12-place pairs of reciprocals *n*, rec, *n* in Table R with *n* from 1 to 4 are indicated by *N tags*. Question marks after *n* or *N* tags indicates the four cases when a pair of reciprocals is present in Table U but absent in the reconstructed Table R.

Note that if *n* is from 1 to 2, then 2 · *n* is from 2 to 4, and the point in the index grid representing 2 · *n* is one step to the right of the point representing *n*. Similarly, if *n* is from 1 to 2, then 4 · *n* is from 4 to 8, and the point in the index grid representing 4 · *n* is two steps to the right of the point representing *n*. That is the reason why in Fig. 1.2.3 the *n* tags usually occur in pairs, but occasionally also in triplets, with the third *n* tags representing pairs of reciprocals with *n* from 4 to 8 in fragment IV A = BM 41101.

Missing total (at most) 12-place pairs of reciprocals *n*, rec, *n* in Table R with *n* from 1 to 4 are indicated by *o tags*.

Index points representing total 12-place pairs missing in Table R but present in the atypical fragment I D = LBAT 1634 are indicated by *z tags*. Similarly, index points representing pairs *n*, rec. *n* missing in Table R but with sq. *n* present in the table of squares fragment II F = LBAT 1638 are indicated by *x tags* if the pair *n*, rec. *n* is at most total 12-place and by *X tags* if the pair is more than total 12-place.

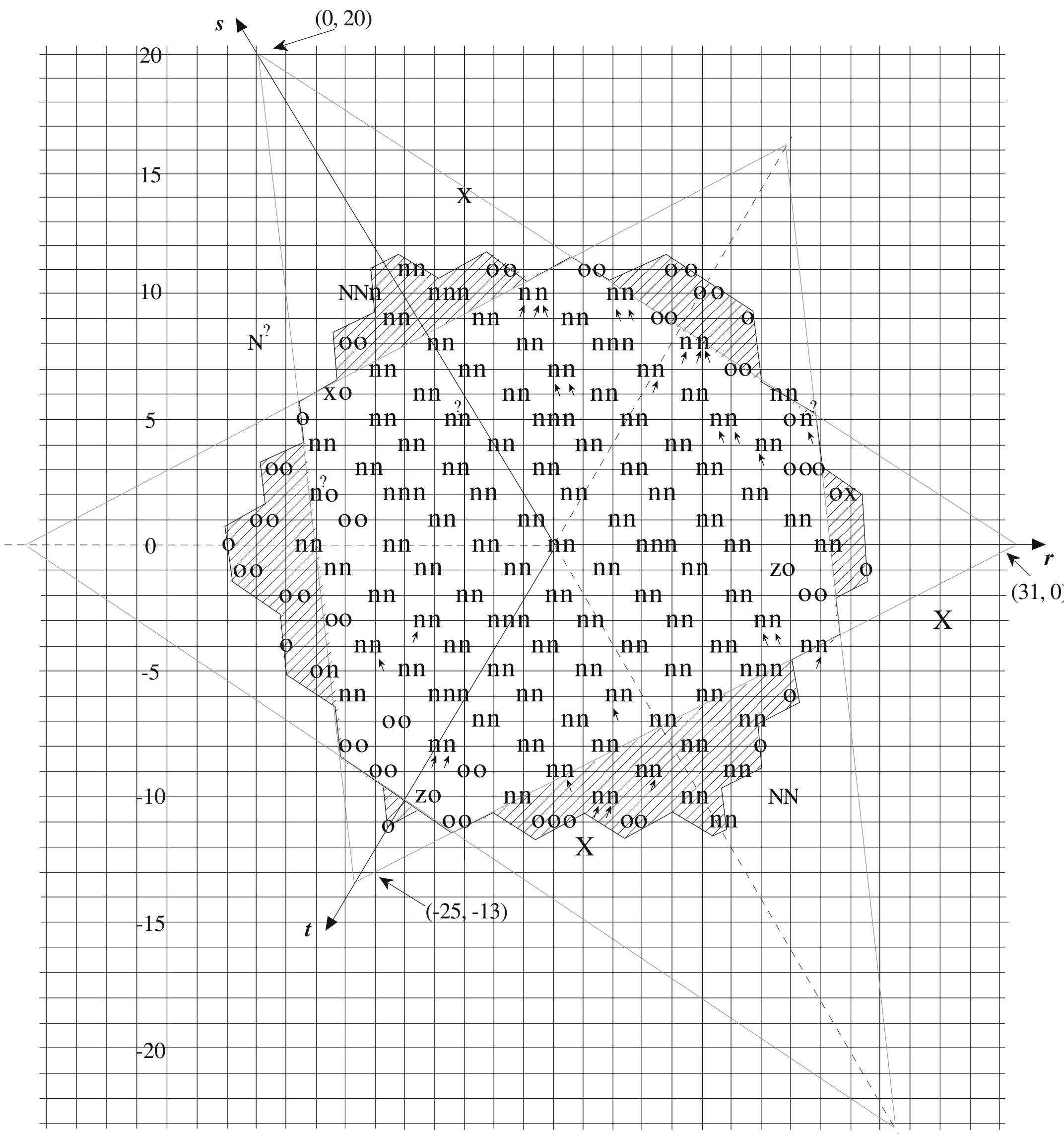


Fig. 1.2.3. Documented pairs of reciprocals in Table R. Representation in the index grid.

It is illuminating to consider how the *n* (and *N*) tags are distributed inside (and outside) the total twelve-place index flower. The following interesting patterns can be observed:

a) In each horizontal line of index points, the set of n tags is almost never interrupted by o tags. The only two exceptions are the o tags in the index points (-13, 2) and (21, 5) which both lie between an n tag and an $n^?$ tag.

b) No o tags are close to the 3-axis (representing positive powers of 3) or to the opposite, negative 3-axis (representing powers of rec. 3 = 20).

c) Thirteen of the o tags, those at the upper and lower borders of the index flower (s = 11), cannot be reached from the positive or negative 3-axis by a line in the direction of the 2-axis staying all the time inside the index flower.

d) Four N tags outside the index flower are closely related (*s* = 10 or -10).

These observed patterns agree perfectly with what could be observed above in connection with the explicitly exhibited preliminary algorithm tables A_0, A_1, B_1, and A_{10}, B_{10}.

The atypicality of the table of squares fragment II F is really obvious, in view of the three X tags far outside the total 12-place index flower. Note also that two *x* tags are separated from the *n* tags by *o* tags.

*1.2.6 An Error in a Table of Squares Related to the Enlarged Table B**

In his discussion in *SVM* (1961) of fragments I A - I E and II C - II F, all first published in Sachs *LBAT* (1955), Vaiman mentions 17 single-place errors, probably simple copying errors. In Aaboe's discussion in *JCS* 19 (1965) of fragments II A-F, only two single-place errors are mentioned, and in Britton's paper about fragments I F-H, II G, and III B only one single-place error is explicitly mentioned.

More interesting than these reported single-place errors in the fragments of many-place table texts is a double-place error in the atypical table of squares fragment LBAT 1636 (II F). Vaiman's explanation in *SVM*, 224 of that error is so ingenious that it deserves to be repeated here. The error in question is

sq. 1 25 44 01 58 31 06 40 = 2 02 30 17 <u>05 11</u> [02 25 13 18 21 14 04 26 40],

with 05 11 instead of 54 42. Vaiman considered the actual multiplication of the 8-place number 1 25 44 01 58 31 06 40 with itself, presuming that the operation was carried out on clay in the same way as we do multiplications on paper. That Vaiman's presumption was correct is known today, after the explanation in Friberg *MSCT 1* (2007), 459 of the explicit squaring on *LBAT* 1644 = BM 34601 of a 13-place regular sexagesimal number.

It is shown below what the multiplication may have looked like (with a slight modification of Vaiman's suggested layout of the multiplication to fit the layout of the multiplication in the case of *LBAT* 1644).

	1 25 44 01 58 31 06 40	8-place	
×			
1	1 25 44 01 58 31 06 40		
25	35 43 20 49 22 57 46 40		
44	1 02 52 17 26 54 48 53 20		
01	1 25 44 01 58 31 06 40		
58	1 22 52 33 54 34 04 26 40		
31	44 17 45 01 14 04 26 40		
06	8 34 24 11 51 06 40 00		
40	+ 57 09 21 19 00 44 26 40		
	2 02 30 17 54 42 02 25 13 18 21 14 04 26 40	15-place	(the correct result)
	– 49 31		
	2 02 30 17 05 11 02 25 13 18 21 14 04 26 40		(the result recorded in the text)

Vaiman observed that the difference between the correct and the erroneous double-place is 54 42 – 5 11 = 49 31. He could then explain the error in the recorded number by assuming that in the final step of the explicit multiplication algorithm, when the partial products were added together, two single-place numbers were overlooked, 49 in one of the partial products, and 13 in another one.

This visualization of how the ones who constructed the tables of squares and squares of squares related to Table B actually may have performed their calculations of squares of many-place sexagesimal numbers is quite astonishing. Not only was it *extremely laborious to compute all the pairs of reciprocals* in the mentioned preliminary algorithm tables and *to order lexicographically* the about 300 resulting (at most) total 12-place pairs of reciprocals. *It must, in addition to all this, have been exceedingly laborious to compute, by use of the same kind of multiplication algorithm as in the example above, the squares and the squares of squares of about 300 (at most) 9-place sexagesimal numbers!*

(Note, for instance, that in the two known fragments of the many-place table of squares of squares, III A - III B, there are five partially preserved 21-place numbers, squares of squares of 6-place numbers, and one partially preserved 25-place number, the square of the square of the 7-place regular sexagesimal number 1 05 50 3702 13 20 or, alternatively, the square of the 13-place number 1 12 15 22 56 55 27 17 51 36 17 46 40.)

1.3 Table U = W 23283+22905 (Uruk). An Achaemenid Many-Place Table of Reciprocals, with *n* from 1 to 4

1.3.1 Mathematical and Metrological Tablets from a House in Achaemenid Uruk

The following informative passage is borrowed from Robson, *MAI* (2008), Sec. 8.3:

> "Between 1969 and 1972 the German excavators of Uruk uncovered a house to the east of the city dating to the fifth and fourth centuries BC. Two different scribal families, the descendants of Šangû-Ninurta and Ekur-zakīr, had successively occupied it, both maintaining scholarly libraries there. Some 500 tablets and fragments were found in the house, around 180 of which can be associated with the Šangû-Ninurta family and 240 with the Ekur-zakīrs. The eighteen dateable legal and scholarly tablets belonging to the Šangû-Ninurtas span the sixth and fifth centuries BC, the latest nine of which are from early in the reign of the Persian king Darius II (r. 423-405). They suggest that the Šangû-Ninurtas left the house some time after 412 BC. Some time later, perhaps immediately afterwards, the Ekur-zakīrs moved in. . . .
>
> Three rooms and a courtyard have survived of the Šangû-Ninurtas house. Before they left, they carefully buried much of their household library, and whatever archival tablets they did not want to take with them, in clay jars in a rather strange room (locus 4) . . . . Excavators found other tablets scattered over the rooms . . ."

The contents of twelve percent of the scholarly tablets in the library of the Šangû-Ninurtas are astronomy, astrology, or mathematics (Robson, *op. cit.*, 229). The contents of the remainder are medical omens, prescriptions, and incantations; other incantations, rituals, and magic; hymns, literature, and lexical lists; omens; etc. The library includes, in particular, forty tablets which according to their colophons belonged to three generations of the Šangû-Ninurta family: one Šamaš-iddin; his sons Rīmūt-Anu and Anu-ikṣur; and the latter's son Anu-ušallim, all describing themselves as incantation priests.

One of the tablets owned by Šamaš-iddin, namely W 2391-x (Friberg, *et al.*, *BaM 21* (1990); Robson, *op. cit.*, Table 8.2.9) is a "metro-mathematical recombination text" (meaning simultaneously mathematical and metrological, with more or less closely related exercises borrowed from several older texts). Two tablets owned by Rīmūt-Anu are mathematical and metrological, respectively, namely W 23283+ (Robson, *op. cit.*, Table 8.2.11; below, in this section), and W 23273 (Friberg, *GMS 3* (1993), no. 11; Robson, Table 8.2.10). W 23273 and W 23283+ were both found in room 4 of the house of the Šangû-Ninurtas. W 23291-x may very well be from the same find spot, but in this case the records are unreliable.

Also from room 4, but without colophons, are W 23281= IM 76283 (Robson, *op. cit.*, Table B.20; Sec. 1.6 below), inscribed on the obverse with a metrological recombination text and on the reverse with a many-place table of reciprocals of an unusual kind, and the metro-mathematical recombination text W 23291 (Friberg, *BaM 28* (1997); Robson, *op. cit.*, Table B.20).

From the fill of the same level of the house of the Šangû-Ninurtas as the one on which room 4 is situated (Uruk UE XVIII/1, level 4) come two further mathematical texts, without colophons. They are W 23021 (Friberg, *BaM* 30 (1999)) and W 23016 (Friberg, *MSCT 1* (2007), 453), containing examples of factorization algorithms for many-place regular sexagesimal numbers (both in Robson, *op. cit.*, Table B.20). For a thorough discussion of such factorization algorithms, see Ch. 2 below.

Somewhat surprisingly, the only astronomical texts from the scholarly library in the house of the Šangû-Ninurtas are three fragments of astronomical diaries, dating from the reigns of Nabonidus to Darius I (late sixth century) and Artaxerxes I (mid-fifth century). There is no evidence of any interest in theoretical or mathematical astronomy (Robson, *op. cit.,* 237). This is an important observation, because it may imply that elaborate many-place tables of reciprocals like Tables S and U antedate Late Babylonian mathematical astronomy. Thus, *it is possibly not correct to assume that many-place arithmetical tables were constructed for the use of those who worked with mathematical astronomy. It is likely that the truth is instead that the previously acquired ability to construct and work with many-place arithmetical tables came in very handy for those who developed Late Babylonian mathematical astronomy, computing and constructing various kinds of astronomical many-place tables.*

Excellent hand copies of W 23283+ = *Uruk 4*, text 174, were published by von Weiher in 1993. The photos of the tablet in Figs. 1.3.1 and 1.3.3 below were generously presented to the author by von Weiher even before the publication of the hand copy, but are published here for the first time. The conform transliterations of the text in Figs. 1.3.2 and 1.3.4 are based on the photos but include also reconstructions of the lost or damaged parts of the text, based on the discussion of Table R in Sec. 1.2 above.

1.3.2 A Curious Incipit, and a Colophon

Table U = W 23283+ is inscribed with two columns of pairs of reciprocals on the obverse, and two further columns of pairs of reciprocals on the reverse. The total number of pairs of reciprocals is about 200. See col. *v* of the reconstructed Table R in Table 1.3, Sec. 1.2.3 above. Actually, W 23283+ is the most extensive of the known Late Babylonian many-place tables of reciprocals, extending as it is from 1 to 4.

The text of Table U begins with a kind of incipit, as follows:

Table U = W 23283+, incipit (col. *i*, 1-6)

1	[x] ma id ru	?
	1 1 íb.si	1 is 1 squared
	1 a.rá 1 a.rá	1 times 1 times
2	[50] a.rá 1 50	[50] times 1 is 50
	40 a.rá 1 40	40 times 1 is 40
3	[30] a.rá 1 30	[30] times 1 is 30
	20 a.rá 1 20	20 times 1 is 20
4	10 a.rá 1 10	10 times 1 is 10
	10 a.rá 6 1 a.⌜rá⌝ 1	[10] times 6 is 1 times 1
	x nam dub.sar *ana* ka-*šú* 6-*it*	?
5	1 $^2/_3$.bi 40	1, its 2/3 is 40
6	šu.ri.a.bi 30.àm	its half is 30

The meaning of this incipit is quite obscure. In particular, the meaning of the phrase

x ma id ru

is not at all clear. The phrase

1 1 íb.si

may be a reference to a preceding table of square roots. Similarly, the ensuing phrases,

from 1 a.rá 1 a.rá (obviously corrupt) to 10 a.rá 1 10

may be a reference to a preceding series of multiplication tables, clearly of an Achaemenid type quite different from the well known Old Babylonian series of multiplication tables.

The meaning of the next phrase,

10 a.rá 6 1 a.rá 1 (partly corrupt)

is also somewhat obscure, and so is the meaning of the phrase

x nam dub.sar *ana* KA-*šú* 6-*it.*

The final two phrases of the incipit,

1 $^2/_3$.bi 40 šu.ri.a.bi 30-àm

are, of course, identical with a form of the well known incipit of Old Babylonian tables of reciprocals. It is possible that it refers to an Old Babylonian standard table of reciprocals on a preceding tablet. Compare the way in which the (at most) 3-place table of squares on BM 34592 (VII A in the survey in Sec. 1.1 above) is preceded by a Seleucid table of reciprocals resembling a standard Old Babylonian table of reciprocals.

At the end of W 23283+, there is the following three-line colophon:

Table U = W 23283+, colophon (col. *iv*, 36-38)

36	[nu]al.til	[un]finished
	ki-i KA sumun-*šu*	according to an old original
	Id*Ri*-[*mut*-d*Ani*]	Rī[mūt-Anu]
37	dumu šá IdGIŠ.ŠIR-*id-di-nu*	son of Šamaš-iddin
	dumu lúÉ.BA[R-dNinurta]	descendant of Šangû-[Ninurta]
38	*ana* na áb mu di-*šu*	?
	in.sar-*ma* íb.rú x	wrote and checked it x

The table is correctly labeled as "unfinished" since it does not go all the way from 1 to 8 (or at least 7 42 57 46 40). It is, also labelled "according to an old original" possibly since it is a copy of Tables R_1 and R_2 of the original Table R of many-place pairs of reciprocals. As for the reference to the Šangû-Ninurta family and Rīmūt-Anu, son of Šamaš-iddin, see Sec. 1.3.1 above. The meaning of the phrase

ana na áb mu di-*šu,*

on the other hand, is not at all clear.

1.3.3 A Curious Use of the Technical Term íb.si *in Table U*

A curious use of the technical term íb.si can be observed in three preserved instances in Table U, namely in the following lines:

igi	1	1	íb.si
igi	1 12	50	íb.si
igi	[3]	20	íb.[si]

Compare with Old Babylonian arithmetical table texts, where the term íb.si or íb.sá appears only in tables of squares and tables of cubes, or in more sophisticated but much more rarely occurring kinds of arithmetical tables, while lines in Old Babylonian tables of reciprocals usually are of one of the following two forms:

igi	*n*	gál.bi	rec. *n*
n			rec. *n*

The mentioned curious use of the technical term íb.si in Table U can be compared with the following examples of a more understandable apparent use of the term àm (= it is) in the Neo-Babylonian Table S from Sippar (Sec. 1.4 below):

igi	1 30	[40]	⸢àm⸣
igi	2	30	⸢àm⸣

The curious use of the technical term íb.si in Table U can also be compared with many examples of a straightforward use of the term gál.bi in the Seleucid Table V (Sec. 1.5 below), such as

[igi]	1	gál.bi	1	àm
[igi]	1 04	gál.bi	56 15	
[igi]	1 06 40	gál.bi	54	
etc.				

1.3.4 Numerical Errors in Table U

The numerical errors in the table of reciprocals on W 23283+ (Table U) are indicated in col. *v* of the reconstructed Table R in Sec. 1.2.3 above, Table 1.3. Nine of the errors are simple and uninteresting 1-place errors, probably copying errors.

There is no obvious explanation for a curious error at the end of Table U,

(2, 0) 4 <u>24</u> instead of 4 *l.* 204

The following errors are much more interesting:

(0, -11)	1 13 09 27 49 08 08 53 20	instead of	1 13 09 34 29 08 08 53 20	*l.* 30
(1, -11)	2 26 18 55 38 16 1[7 46 40]	instead of	2 26 19 08 58 16 17 46 40	*l.* 132
(-1, -11)	[36 34 43 54 34 04 26 40]	instead of	36 34 47 14 34 04 26 40	[*l.* 72]
(-2, -11)	18 17 21 [57] 17 02 13 20	instead of	18 17 23 37 17 02 13 20	*l.* 174
(-6, 10)	15 22 39 22 30	instead of	15 22 38 26 15	*l.* 200

The error in line 30 can be explained as follows: Note that

34 29 – 27 49 = 6 40 = 20 · 20 (= 1/3 · 20).

Note also the entry

(0, -10) 3 39 28 43 27 24 26 40 = 20^{10} *l.* 191

Evidently, the value of *n* in line 30 should have been constructed as follows, with departure from the value of *n* in line 191:

(0, -11) 20 · 3 39 28 43 27 24 26 40 = 1 13 09 34 29 08 08 53 20 = 20^{11}

Instead, it was constructed in the following way:

(0, -11) 20 · 3 39 28 23 27 24 26 40 = 1 13 09 27 49 08 08 53 20 *l.* 30

Thus, the 2-place error in line 30 seems to be due to a simple miscalculation.

The error in line 132 is a typical "propagated error", being a direct consequence of the error in line 30. Indeed, the value of *n* in line 132 should have been calculated as

(1, -11) 2 · 1 13 09 34 29 08 08 53 20 = 2 26 19 08 58 16 17 46 40.

Instead, it was calculated as

2 · 1 13 09 27 49 08 08 53 20 = 2 26 18 55 38 16 17 46 40 *ll.* 30 and 132

Also the error in line 174 is a propagated error. It, too, is a consequence of the mentioned error in line 30. Indeed, the error in line 30 gave rise to the following error in line 72, a line which is lost in Table U:

(-1, -11) 30 · 1 13 09 27 49 08 08 53 20 = 36 34 43 54 34 04 26 40 instead of 36 34 47 14 34 04 26 40 [*l.* 72]

The suggested reconstruction of the entry in line 72 is certain, because the propagated error in line 72, in its turn, gave rise to the following propagated error in line 174:

(-2, -11) 30 · 36 34 43 54 34 04 26 40 = 18 17 21 [57] 17 02 13 20 instead of 18 17 23 37 17 02 13 20 *l.* 174

Thus, it has now been shown that the errors in lines 30, [72], 132, and 174 of Table U all must be due to a single incorrect computation of the 11th power of rec. 3 = 20, which occasioned corresponding errors in all entries of the two preliminary algorithm tables A_{11} and B_{11}!

There is no equally simple and obvious explanation for the error in line 200 of Table U. However, if it is assumed, somewhat arbitrarily, that 39 is a simple copying error for 38, then the incorrect value 15 22 38 22 30 can possibly(?) be explained as follows. The correct computation of the value in line 200 should have proceeded in the following way (cf. the preliminary algorithm table B_{10}):

(-4, 10)	1 01 30 33 45	*l.* 4
(-5, 10)	30 45 16 52 30	*l.* 98
(-6, 10)	15 22 38 26 15	*l.* 200

In other words, the value in line 200 should have been computed as

(-6, 10) 30 · 30 45 16 52 30 = 15 22 38 26 15.

Instead, it may have been computed, mistakenly, as

30 · 30 45 16 45 = 15 22 38 22 30.

This means that instead of halving the final places 52 30 of the number with the index (-5, 10), the one who calculated the value in line 200, exhausted after performing a very long previous series of laborious computations, may have halved the final place 45 of the number with the index (-4, 10).

Of the mentioned lines with interesting errors in Table U, only lines 30 and 72 are within the range of Table B. The first of these lines, line 30, is within the range of the four fragments I C, I D, I E, and I H. However, line 30 is missing in the two fragments I D and I E. In the other two fragments, it is not possible to discern if the error of Table U was repeated or not, namely

(0, -11) 1 13 09 27 49 08 08 53 20 instead of 1 13 09 34 29 08 08 53 20 *l.* 30

The reason is that what is preserved of the number in the two fragments is only

[. . . . .] 53 20 in I C = *LBAT* 1631

and

[. . . . .]3 20 in I H = *JCS* 43-45 C.

Line 72 is within the range of the three fragments I F, I H, and I J. Interestingly, *the error in line 72 of Table U is not repeated in I F and I H = JCS* 43-45 A and C. The line is not preserved in fragment I J.

In the case of Table S, line 30 is present but is badly damaged, so that only the last two places of the reciprocal 49 12 27 are preserved, and line 72 is not preserved at all. Therefore it is impossible to know if the errors in those two lines were repeated or not in Table S.

Of the other two lines with interesting errors in Table S, line 200 is outside the range of both Table B and Table S. Line 132, on the other hand, which is outside the range of Table B, is within the range of Table S, where it is perfectly preserved. Surprisingly, *the error in line 132 of Table U is not repeated in line 132 of Table S*!

It is hard to understand the reason for these differences between Table U on one hand and Tables B and S on the other. As mentioned, the errors in lines [72], 132, and 174 of Table U are propagated errors, and cannot be explained as simple copying errors. That these errors of a non-trivial nature are not repeated in Table B (represented by fragments I C, F, and H), or in Table S, seems to contradict the seemingly reasonable assumption that Tables B, U, and S all have a common ancestor in the hypothetical Table R.

If Table U is the oldest of the three Late Babylonian many-place tables of reciprocals, or, at least, a *copy* of some many-place table of reciprocals, older than Table S and, of course, Table B, then the existence of certain propagation errors in Table U but not in Tables S and B can be explained as follows. In the large fragment VI B = W 23021 (Friberg, *BaM* 39; Sec. 2.1.7 below; Fig. 2.1.6), an ingenious factorization method is used to compute the reciprocals of eight more or less consecutive reciprocals from a many-place table of reciprocals (eight of the lines between lines 20 and 31). The purpose of the computations can be assumed to have been to check that if a pair *n*, rec. *n* is listed in a many-place table of reciprocals, then rec. rec. *n* is equal to *n*, as it should be. In other words, a Babylonian school teacher may have given as a connected series of exercises to his students the task to check the correctness of various lines in some available copy of a many-place table of reciprocals with the mentioned non-trivial errors. In the course of those exercises one or several of the errors might have been discovered. The teacher then took upon himself to write a new copy of the table of reciprocals, with the discovered errors corrected. Over time, the episode may have been repeated, until all the errors were weeded out.

1.3.5 A Curious Extra Line in Table U

In the index grid in Fig. 1.2.3 in Sec. 1.2.5 above, the only index point with the $\mathrm{N}^{?}$ tag represents a quite exceptional pair of reciprocals in Table U, the “extra line”

(-12, 8) 1 36 06 30 14 03 45 37 27 27 44 11 51 06 40 *l.* 67a

This line does not appear in Tables B and S, and the index point for its total 15-place pair is far outside the total 12-place index flower, to which all other index points representing pairs of reciprocals in Tables B, U and S belong, except the previously mentioned four total 13- and 14-place pairs from the preliminary algorithm table B^{10} and the single total 13-place pair from the preliminary algorithm table B^{11}. (See Secs. 1.2.4 and 1.2.5 above.)

The presence of this interpolated extra line in Table U (line 67a) is very hard to explain, but a clue to where the extra line comes from may be the observation that (-12, 8) = 2 · (-6, 4). This means that there may be some curious kind of connection between the mentioned extra line in Table U and the following regular line in Table U:

(-6, 4) 1 15 56 15 47 24 46 40 *l.* 37

In particular, the pairs of reciprocals in the extra line can be written in the following form:

(-12, 8) sq. 1 15 56 15 sq. 47 24 46 40,

This way of writing the extra line in Table U can be taken as an excuse to consider again the Seleucid many-place table of squares derived from Table B (attested in fragments II A-II E, II G), and to ask how that table may have been constructed. It is, of course, possible that that table of squares, which may be called Table sq. B, was constructed simply by squaring separately all the numbers *n* in the first column of Table B, listing all those squares together in the same lexical order as the numbers *n* themselves. Apparently, this is how Vaiman believed that the many-place table of squares was constructed. See the reference in Sec. 1.2.6 above to Vaiman's explanation of an error in the (atypical) table of squares fragment II F.

However, that way of constructing the many-place table of squares would have been unnecessarily laborious and, by far, not as elegant as the algorithmic construction of the many-place table of reciprocals itself. Actually, however, also the many-place table of squares may have been constructed by use of an elegant algorithmic procedure, as will be shown below.

Recall that the many-place table of reciprocals R, a supposed common ancestor to Tables S, U, and B, was constructed in three major steps: first a total 12-place table of powers of 3 and their reciprocals was constructed (Table 1.5 below in Sec. 1.5.3). Then, for each of the computed powers of 3, the doubling and halving algorithm was used to produce a total 12-place preliminary algorithm table with pairs of reciprocals, and finally all computed numbers from 1 to 7 42 57 46 (the approximate square root of '1' = 60) were ordered lexicographically. The highest power of 3 computed was the 11th power and, consequently, there were altogether 23 computed preliminary algorithm tables. The preliminary algorithm tables were often incomplete, with several lines missing at the ends of the tables.

Now, a related table of squares can be constructed in three related steps, as well: first a table of powers of sq. 3 = 9 and their reciprocals is computed, up to the 11th power, then for each computed power of 9 an algorithm with multiplications and divisions by 4 (= sq. 2) is used to produce 23 preliminary algorithm tables with square pairs of reciprocals, and then the computed numbers are ordered lexicographically.

Each preliminary algorithm table with square pairs of reciprocals is allowed to proceed just as far as the corresponding preliminary algorithm table with pairs of reciprocals that it is related to, and only the numbers that are actually squares of numbers in Table R are ordered lexicographically. This somewhat vague description of the procedure is best explained by considering an explicit example of the computation of a preliminary algorithm table with square numbers.

Take, for instance the preliminary algorithm table A_1 (see above, Sec. 1.2.4). It proceeds from the pair of reciprocals with the index (0, 1) to the pair with the index (19, 1), where it stops prematurely. The related new preliminary algorithm table with square numbers, which may be called sq. A_1, is then allowed to proceed, in a corresponding way, from 2 · (0, 1) = (0, 2) to 2 · (19, 1) = (38, 2). The numbers in sq. A_1 corresponding to the numbers in A_1 from 1 to 7 42 57 46, which are bold in A_1, are bold also in sq. A_1. Note that if *n* goes from 1 to 2, then sq. *n* goes from 1 to 4, if *n* goes from 2 to 4, then sq. *n* goes from 4 to 16, and if *n* goes from 4 to 7 42 57 46, then sq. *n* goes from 16 to '1' (= 60). Therefore the bold numbers come in groups of three, alternately in the left and in the right column of the table.

Only the bold numbers are ordered lexicographically in the resulting table of squares, with sq. *n* getting the same line number as the corresponding number *n*.

Table sq. A_1. Square pairs of reciprocals produced with departure from the pair 3^2, rec. 3^2 = 9, 6 40

0 2	160		**9**	6 40
2 2	262		**36**	1 40
4 2	239		2 24	**25**
6 2	137		9 36	**6 15**
8 2	35		38 24	**1 33 45**
10 2	67		**2 33 36**	23 26 15
12 2	169		**10 14 24**	5 51 33 45
14 2	271		**40 57 36**	1 27 53 26 15
16 2	228		2 43 50 24	**21 58 21 33 45**
18 2	126		10 55 21 36	**5 27 08 23 26 15**
20 2	24		43 41 26 24	**1 22 23 50 51 33 45**
22 2	78		**2 54 45 45 36**	20 35 57 42 53 26 15
24 2	180		**11 39 03 02 24**	5 08 59 25 43 21 33 45
26 2	282		**46 36 12 09 36**	1 17 14 51 25 50 23 26 15
28 2	219		3 06 24 48 38 24	**19 18 42 51 27 35 51 33 45**
30 2	117		12 25 39 14 33 36	**4 49 40 42 51 53 57 53 26 15**
32 2	15		49 42 36 58 14 24	**1 12 25 10 42 58 29 28 21 33 45**
34 2	87		**3 18 50 27 52 57 36**	18 06 17 40 44 37 22 05 23 26 15
36 2	189		**13 15 21 51 31 50 24**	4 31 34 25 11 09 20 31 20 51 33 45
38 2	291		**53 01 27 26 07 21 36**	1 07 53 34 17 47 20 07 50 12 53 26 15

The following six numbers in the many-place table of squares from, for instance, sq. 1 to sq. 2 can be obtained from Table sq. A_1:

(-32, -2) 1 12 25 10 42 58 29 28 21 33 45 *l.* 15

(-20, -2) 1 22 23 50 51 33 45 *l.* 24

(-8, -2)	1 33 45	*l.* 35
(10, 2)	2 33 36	*l.* 67
(22, 2)	2 54 45 45 36	*l.* 78
(34, 2)	3 18 50 27 52 57 36	*l.* 87

Cf. Britton, *JCS* 43-45, 76-77.

Evidently, a similar algorithmic procedure can have been used for the production of a related table of squares of squares, as in the fragments III A-B. In that case the first step would have been the production of a table of powers of sq. sq. 3 = 1 21 and their reciprocals, up to the 11th power of 1 21, and the preliminary algorithm tables would have been obtained through multiplication and division by 16.

Now, return to the question about the origin of the curious "extra line" in Table U, which can be written in the following form:

(-12, 8)	1 36 06 30 14 03 45 (= sq. 1 15 56 15)	37 27 27 44 11 51 06 40 (= sq. 47 24 46 40)	*l.* 67a

A possible explanation for the interpolation of this extra line in a many-place table of reciprocals may be that, in the production of Table U, the one who was responsible for the procedure of sorting lexicographically the lines on the 23 preliminary algorithm tables A_0-A_{11} and B_1-B_{11}, in some way happened to include also a line from the related algorithm table sq. A_4.

Later, this extra line was eliminated in some way, since it is not present in Tables S and B. The elimination of the extra line may have been accidental, or it may have been caused by an observation that the pair of (square) reciprocals in line 67a is total 15-place, more than the preferred total 12-place. Note that the presence of the extra line 67a in Table U but not in Table S strengthens the assumption made above (in Sec. 1.3.4) that Table U is a copy of a table older than Table S.

The assumption that the (Seleucid) many-place table of squares, Table sq. B, was constructed by the method outlined above seems to be contradicted by Vaiman's explanation of an error in the table of squares fragment II F, an explanation based on the assumption that the entries in the many-place table of squares were constructed by squaring separately each number *n* in the left column of Table B. On the other hand, fragment II F is an atypical table of squares fragment, and nothing is known about any connection between fragment II F and Table B. Moreover, it is likely that Table sq. B was used by teachers as a source of exercises in squaring assigned to students, in the same way as Table B itself, and other many-place tables of reciprocals, are known to have been used as sources of exercises in computing reciprocals of given regular many-place numbers by use of the factorization method. (See the reference to W 23021 in Sec.1. 3.4 above.) It may have happened that an erroneous result of such a squaring exercise was incorporated into a many-place table of squares, of which fragment II F is an excerpt.

1.3.6 Table U. New Photos and Conform Transliterations

Fig. 1.3.1. W 23283+, *obv*. An Achaemenid many-place table of reciprocals, for *n* from 1 to 4. Photo by courtesy of E. von Weiher.

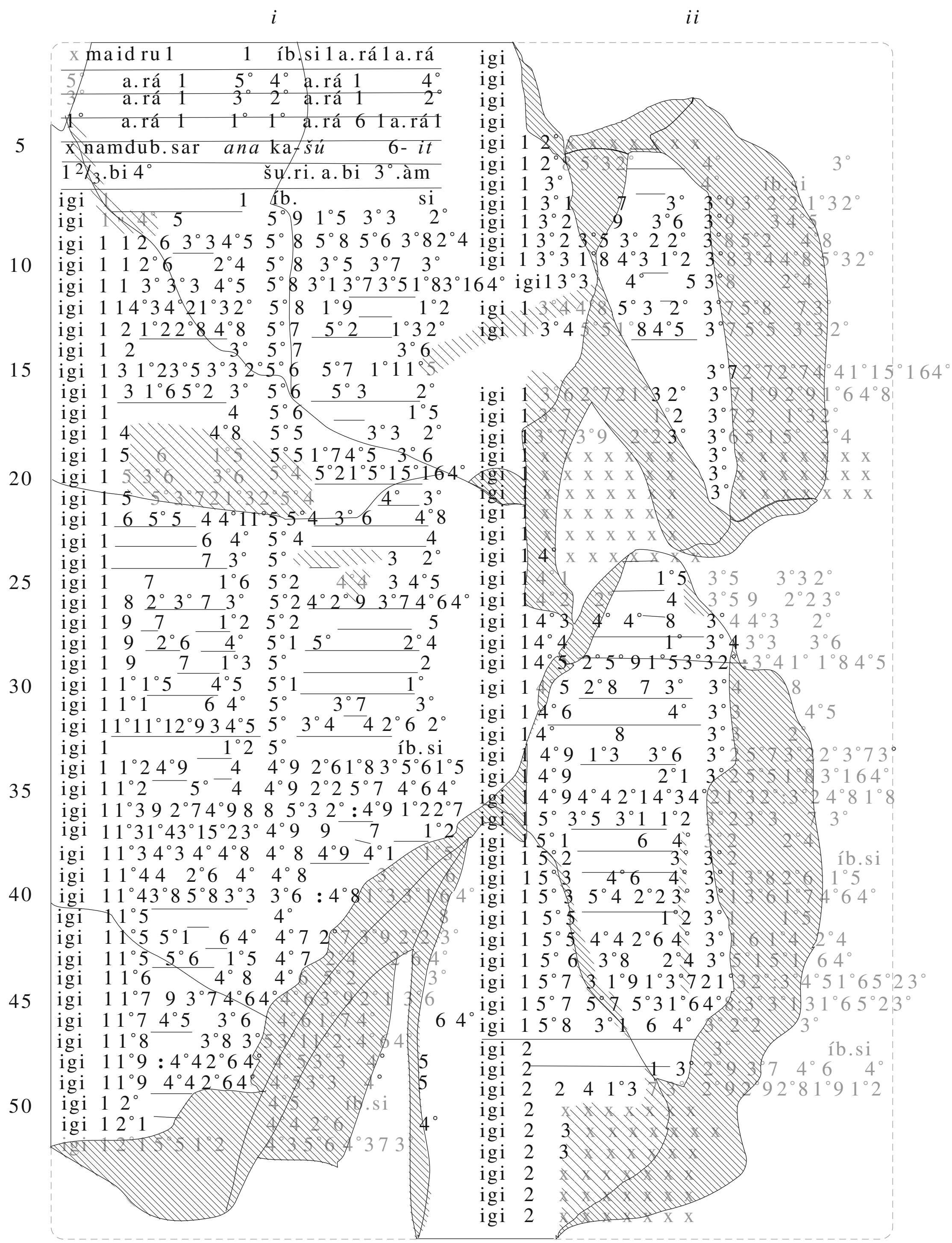


Fig. 1.3.2. W 23283+, *obv*. Conform transliteration with a reconstruction of damaged parts of the text.

Fig. 1.3.3. W 23283+, *rev*. Continuation of the many-place table, with a colophon. Photo: E. von Weiher.

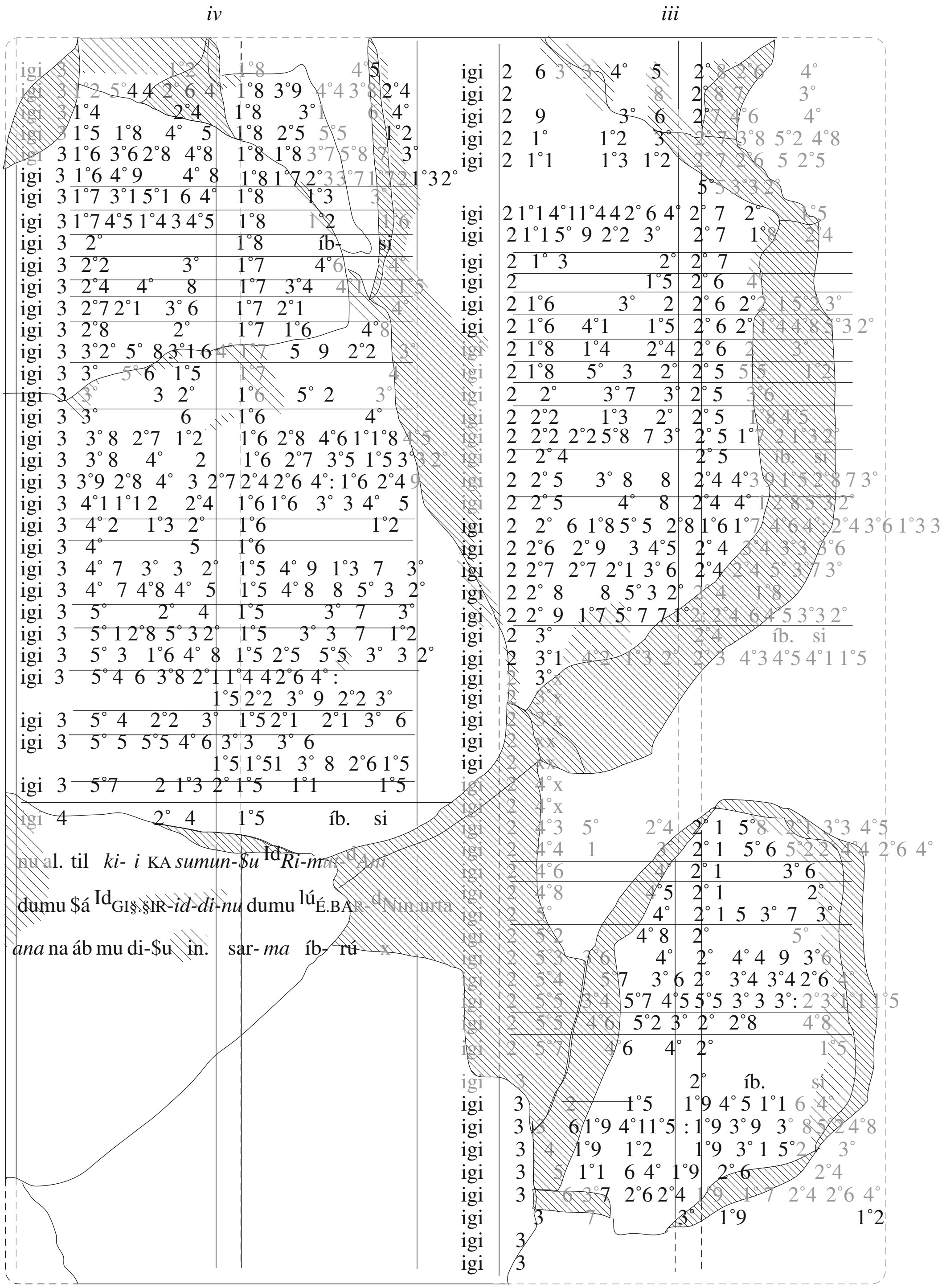


Fig. 1.3.4. W 23283+, *rev*. Conform transliteration with a reconstruction of damaged parts of the text.

1.4 Table S = Sippar 2175/12. A Neo-Babylonian Many-Place Table of Reciprocals, with *n* from 1 to 3

In the 8th campaign of the Archeological Institute of Baghdad in 1985-86, an intact library with hundreds of cuneiform tablets was found in room 355 of a Neo-Babylonian temple in Sippar, possibly devoted to the goddess Aya. Some of the floor slabs in the library are inscribed with the names of Nebuchadnezzar II (604-562 BC) and Nabonidus (555-539 BC), the last indigenous ruler of Babylonia before the Achaemenid conquest of Babylon.

One of the tablets found in this library is Sippar 2175/12 = Table S, a many-place table of reciprocals, with *n* from 1 to 3. There is no way of knowing if the tablet is Neo-Babylonian or Achaemenid. Physically, the tablet has a badly preserved obverse but a much better preserved reverse. See col. *vi* of the reconstructed Table R in Sec. 1.2.3 above. A photo of the reverse of the tablet, minus a large fragment on its lower left, was published by Al-Jadir in *Archeologia* 224 (1987). New hand copies and conform transliterations of both sides of the tablet are presented below, in Figs. 1.4.1-1.4.2.

1.4.1 Numerical Errors and Missing Pairs in Table S

Apparently there are very few numerical errors in the table text on Sippar 2175/12, although the reason for the absence of visible errors may be that so many of the pairs of reciprocals in the table are either lost or severely damaged. The only potentially interesting error that is clearly to be seen is

(-1, -10)	1 49 51 47 43 42 13 20	instead of	1 49 44 21 43 42 13 20	*l.* 89

Unfortunately, there is no obvious explanation to be offered for this error. Interestingly, the error does not appear in Table U, or in the only fragment where the number in question is preserved (I B).

With only one possible exception, all the reciprocals pairs present in Table S, even if only in damaged form, are also present in Table U or at least in the reconstructed Table R. The possible exception is the following: Between the partly preserved pairs

(-10, 3)	[1 34 55 18 45]	37 55 33 20	*l.* 66
(-13, -10)	[1 36 27 02 13 20]	3[7 19 29 16 48]?	*l.* 68

there seems to be a corrupt but only partly preserved form of the total 13-place pair

(23, 12)	[1 35 33 05 21 24 28 48]	37 40 33 40 50.

Quite a few pairs of reciprocals that are present in the reconstructed Table R are missing in Table S. Indices for such pairs are indicated by small black arrows, pointing upwards to the left, in the index flower in Fig. 1.2.3 in Sec. 1.2.5 above. (Indices for pairs from Table R missing in Table U are similarly indicated by small black arrows pointing upwards to the right.) Some of these missing pairs are, like the missing pairs in Table R, from the end of one or another of the preliminary algorithm tables.

Thus, the following pairs are missing in Table S, but not missing in Table R,

(14, 10)	1 14 38 58 33 36 48	13 31 06 40	*l.* 34
(15, 10)	2 29 17 57 07 12	24 06 45 33 20	*l.* 136

They are from the end of the preliminary algorithm table A_{10}, explicitly exhibited in Sec. 1.2.4 above.

Similarly,

(11, -3)	1 15 51 06 40	47 27 39 22 30	*l.* 36
(12, -3)	2 31 42 13 20	23 43 45 41 15	*l.* 138

are pairs from the end of the preliminary algorithm table B_3, missing in Table S but not in Table R What this means is not clear, the omissions in questions may or may not be significant. There are also other omissions of pairs from Table R in Table S, but those omissions are almost certainly due to simple copying errors.

1.4.2 Table S. Hand Copies and Conform Transliterations

Fig. 1.4.1. Sippar 2175/12, *obv*. Hand copy.

obv.

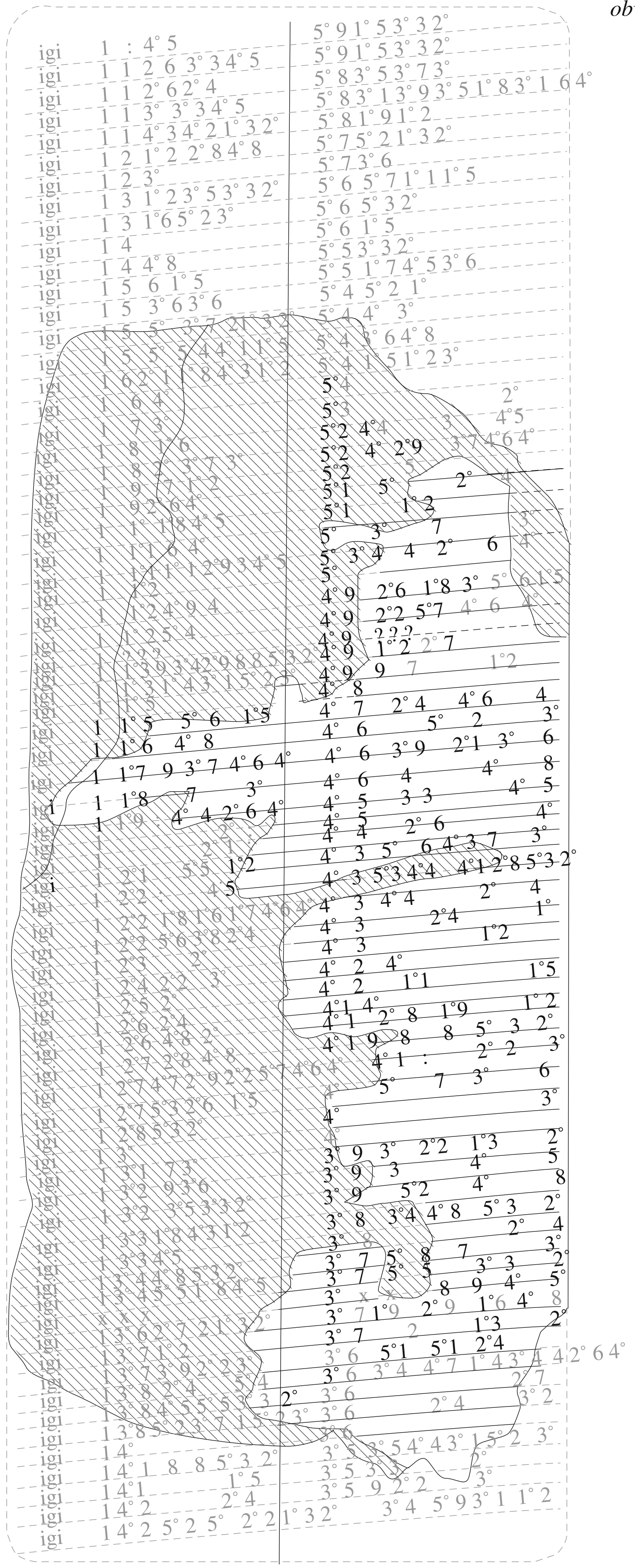


Fig. 1.4.2. Sippar 2175/12, *obv*. Conform transliteration.

rev.

Fig. 1.4.3. Sippar 2175/12, *rev*. Hand copy.

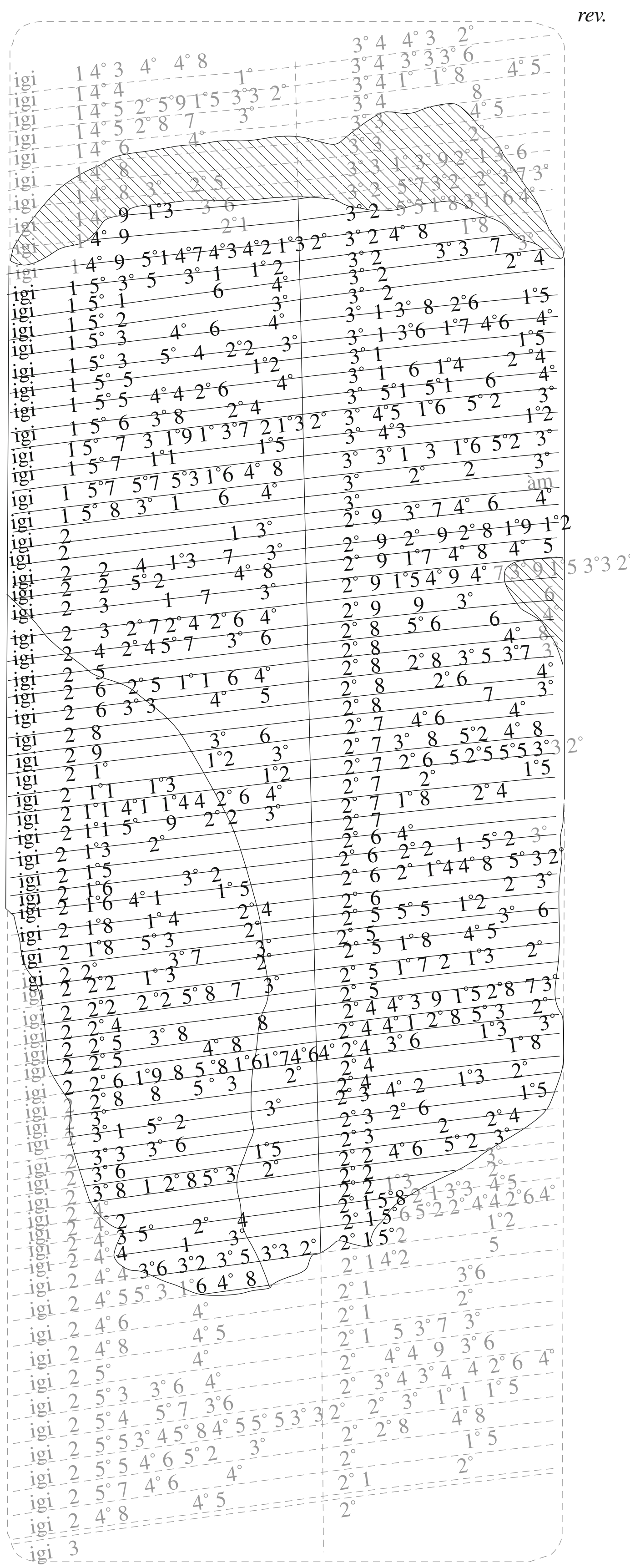


Fig. 1.4.4. Sippar 2175/12, *rev*. Conform transliteration.

1.5 Table V = AO 6456 (Uruk). A Seleucid Many-Place Table of Reciprocals, with *n* from 1 to 3

1.5.1 An Invocation, a Many-Place Table of Reciprocals, a Description, and a Colophon

Table V = AO 6456 is a Seleucid many-place table of reciprocals, with *n* from 1 to 3. It is inscribed in two columns on the obverse and two columns on the reverse of an excellently preserved clay tablet in landscape orientation. There is an invocation on the upper edge of the tablet, a description of the content of the table text below col. *iv*, and a colophon beginning under col. *iii* but continuing under part of col. *iv*.

The photos of the clay tablet, in Fig. 1.5.2 below, are new, but hand copies of the four columns of the tablet were published already by Thureau-Dangin in *TCL 6* (1922), no. 31. A transliteration of the whole text was published by Neugebauer in *MCT I* (1935), 14-22, together with a brief discussion of the text and the pointing out of surprisingly many numerical errors. Neugebauer characterized the text as an "essentially 6-place" table of reciprocals, in the sense that in each pair *n*, rec. *n* in the table either *n* or rec. *n* is at most 6-place. He indicated with a * the pairs that did not satisfy this condition. He also explicitly indicated all the essentially 6-place pairs with *n* from 1 to 3 that are missing in the table.

The text of AO 6456 begins with an invocation, directed to two tutelary gods of the city Uruk:

Table V = AO 6456, invocation (in one line on the upper edge)

i-na a-mat ᵈ*Anu u An-tum* *mim-ma ma-la epuš* (DÙ-*uš*) *ina qâti*(ŠU^II^)-*ia liš-lim*	By the word (= command) of Anu and Antu, whatever I do with my hands, may it fare well.

A discussion of invocations of this kind can be found in Roth, *JSS* 33 (1988).

In Table 1.3 in Sec. 1.2.3 above, Table V = the many-place table of reciprocals on the Seleucid tablet AO 6456 is compared with Table B = the reconstructed many-place table of reciprocals on Seleucid fragments from Babylon, as well as with the Achaemenid and Neo-Babylonian many-place tables of reciprocals from Uruk and Sippar, Table U = W 23283+ and Table S = Sippar 2175/12. What appears is that many of the pairs of reciprocals in Tables B, U, and S are missing in Table V and that, conversely, many of the pairs in Table V are missing in Tables B, U, and S. Therefore, it is clear that Table V was constructed independently of its predecessors. Moreover, for some reason, there are many more errors in Table V than in the other many-place tables of reciprocals. More about all this below.

The text ends, before the colophon, with a description of the content of the table:

Table V = AO 6456, description (in one line below col. *iv*)

pir-sú riš-tu-u : 1 : *a-mu-ú* : 2 : *a-mu-ú* *úl qatî* (NU AL.TIL)	First part, 1-x 2-x. Not finished.

What this means is clear, although the exact meaning of the term *a-mu-ú* is unknown: Table V = AO 6456 is a many-place table of reciprocals *n*, rec. *n* ordered lexicographically, with the leading place of *n* equal to either 1 or 2. It was (probably) the first of two consecutive tables, together covering the whole range from 1 to 8 (or at least 7 42 57 46 40). The phrase 'Not finished' probably refers to the fact that more pairs of reciprocals will follow on the second tablet, rather than to the fact that there are missing pairs in the table.

The colophon, finally, contains information about the owner of the tablet and about who wrote it:

AO 6456, colophon (in one line below col. *iii*)

ṭuppi([I]M) [I]*Nidinti*(NÍG.SUM.MU)-[d]*Anu*	Tablet of Nidinti-Anu,
māri(A) *šá* [I]*Ina-qí-bit-*[d]*Anu*	son of Ina-qibīt-Anu,
mār(A) [I]*Hun-zu-u*	descendant of Hunzû,
[lú]*maš-maš* [d]*Anu u An-tum* Uruk[ki] *-u*	incantation priest of Anu and Antu, Uruk.
qàt [I]*Ina-qí-bit-* [d]*Anu māri*(TUR.A.NI)-*šu*	Hand of Ina-qibīt-Anu, his son.

Robson, *MAI* (2008) Sec. 8.4 contains a very informative discussion of what is known about two scholarly families in Uruk in the Seleucid period, in particular that the owner of AO 6456 belonged to the Hunzû family and that the text can be dated to around the 90s SE, or 225-215 BC.

1.5.2 Numerical Errors and Missing Pairs in Table V

There are numerous numerical errors in the many-place table of reciprocals on AO 6456, all of them explicitly mentioned in Neugebauer, *MKT I*, 22. The errors are also mentioned in the last column of the reconstructed Table R in Sec. 1.2.3 above. As will be shown in some detail below, there are 61 such numerical errors, of various kinds. Some of them yield information about how the table was constructed.

1. *Simple 1-place errors (miscopied single places?)*. There are 38 errors of this kind, such as

17 instead of 18, 45 instead of 55 or 48, 51 instead of 50, 13 instead of 16, *etc.*

2. *Other kinds of easily understood copying errors*. There are 3 errors of this kind, namely

33 45	instead of	38 24	probably because the number in col. *i* ends with 33 45	*l.* 2, rec. *n*
57	instead of	57 57		*l.* 100, *n*
06 13	instead of	03 16		*l.* 100, rec. *n*

3. *Extra separation signs (or double zeros), written as a pair of diagonally placed oblique wedges, and below transliterated as «*. In most cases, the extra separation has been inserted intentionally, in order to avoid ambiguities and possible misreadings. There are 5 "errors" of this kind, namely

40 « 08	instead of	40 08	probably to avoid misreading as 48	*l.* 58+++, rec. *n*
10 « 08	instead of	10 08	probably to avoid misreading as 18	*l.* 61++, rec. *n*
20 « 15	instead of	20 15	probably to avoid misreading as 35	*l.* 116, rec. *n*
20 « 15	instead of	20 15	probably to avoid misreading as 35	*l.* 159, rec. *n*
41 « 15	instead of	41 15	reason unknown	*l.* 122, *n*

4. *Missing separation signs.* There are 3 errors of this kind, namely

43 28	instead of	43 « 28	in the number with the index (-23, 0)	*l.* 61++, *n*
41 09	instead of	41 « 09	in the number with the index (0, 22)	*l.* 58++, rec. *n*
46 29	instead of	46 « 29	in the number with the index (0, -18)	*l.*102+, *n*

5. *Propagated errors (telescoping errors)*. There are 3 errors of this kind, namely

54	instead of	30 14	in the number with the index (-24, 0)	*l.* 39+, rec. *n*
52 «	instead of	45 07	in the number with the index (-25, 0)	*l.* 141+, rec. *n*
54	instead of	40 14	in the number with the index (8, -9)	*l.* 38+, rec. *n*

6. *Misplaced lines (incorrect sorting)*. There are 3 errors of this kind.

7. *Unexplained errors (incorrect computations?)*. There are 6 errors of this kind, namely

31	instead of	08 23	(looks like a telescoping error)	*l.* 52+, rec. *n*
59 43 20 12 20 34	instead of	38 08 36 52 20 44		*l.* 58++, *n*
37 27	instead of	36 27 02		*l.* 68, *n*
29	instead of	30		*l.* 101+, rec. *n*
29 51 26 44 06	instead of	32 42 52 17 26		*l.* 102++, rec. *n*
« 18	instead of	14 48		*l.* 122, rec. *n*

In two of the enumerated cases, the propagated errors are particularly easy to explain. Indeed, compare the following three errors, here given in full detail:

(-23, 0)	1 32 41 49 43 28 07 30	instead of	1 32 41 49 43 « 28 07 30	*l.* 61++
(-24, 0)	46 20 54 51 54 03 45	instead of	46 20 54 51 30 14 03 45	*l.* 39+
(-25, 0)	23 10 27 25 52 « 01 52 30	instead of	23 10 27 25 45 07 01 52 30	*l.* 141+

(The points with the indices (-23, 0), (24, 0), (25, 0) are indicated with small black arrows in Fig. 1.5.2 below.) Without doubt the incorrect number with the index (-24, 0) was computed in the following way, with departure from the incorrect number with the index (-23, 0):

30 · 1 32 41 49 43 28 07 30 = 46 20 54 51 44 03 45, with the single place 44 at some later time miscopied as 54!

In other words, in the preliminary algorithm table A_0 (see the discussion of “preliminary algorithm tables” in Sec. 1.2.5 above), the missing zero-place in one number gave rise to a telescoping error (30 + 14 = 44 instead of 30 14) in the next number.

In a similar way, the incorrect number with the index (-25, 0) was computed in the following way, with departure from the incorrect number with the index (-24, 0) (without the erroneous 54!):

30 · 46 20 54 51 44 03 45 = 23 10 27 25 52 01 52 30, with an extra separation sign inserted after 52.

In other words, in the preliminary algorithm table A_0, the telescoping error in one number gave rise to a new telescoping error (45 + 07 = 52 instead of 45 07) in the next number. Since the juxtaposition of the erroneous 52 and the ensuing 01 could give rise to a possible misreading, 53 instead of 52 01, an extra separation sign was inserted between 52 and 01.

Another error of the telescoping type is

(-8, 9)	46 49 19 54 58 53 20	instead of	46 49 19 40 14 48 53 20	*l.* 38+

(See the small black arrow at the index point (-8, 9) in Fig. 1.5.1 below.) Here 58 instead of 48 is almost certainly a copying error, so that the error in the original text can be assumed to have been simply

(-8, 9)	46 49 19 54 48 53 20	instead of	46 49 19 40 14 48 53 20	*l.* 38+

It is likely that this is a propagated error, caused by the following error in a preceding number:

(-7, 9)	23 24 39 57 24 26 40	instead of	23 24 39 50 « 07 24 26 40

(That number appears to have been lost in the sorting procedure.) Indeed,

2 · 23 24 39 50 « 07 24 26 40 = 46 49 19 40 14 48 53 20,

but

2 · 23 24 39 57 24 26 40 = 46 49 19 54 48 53 20.

Thus, also the telescoping error in the number with the index (8, -9) can be explained as *caused by a missing separation sign* in a preceding line of the corresponding preliminary algorithm table.

In an alternative, more precise, transliteration, the cuneiform signs for the tens 10, 20, etc., are transliterated as, for instance, 1˚, 2˚, etc. With this kind of transliteration, the error (in a lost number) mentioned above may be better understood as 5˚7 instead of 5˚ « 7. Moreover, with this kind of more precise transliteration, also the nature of most of the “errors” of type 3 in the enumeration of errors in AO 6456 can be better understood. These “errors” can then be explained as

4˚ « 8	to avoid misreading as	4˚8	*l.* 58+++, col. *ii*
1˚ « 8	to avoid misreading as	1˚8	*l.* 61++, col. *ii*
2˚ « 1˚5	to avoid misreading as	3˚5	*l.* 116, col. *ii*
2˚ « 1˚5	to avoid misreading as	3˚5	*l.* 159, col. *ii*

Note that a difficulty for the proposed explanations of the error in the number with the index (8, -9) is that the “preceding” number with the index (7, -9) is missing in AO 6456. However, this is not surprising, since it is clear that Table V = AO 6456 was constructed in a much less systematic way than, for instance, Table U = W 23283+. In particular, there is only one observable sorting error in the construction of Table U, but many sorting errors (mostly omissions but also incorrect sorting) in the construction of Table V. Clearly, the two table texts were constructed independently of each other.

1.5.3 Representation in the Index Grid, Inside a 6-place Double Triangle

In Fig. 1.2.3 above in Sec. 1.2.5 was shown the locations of all points in a biaxial index grid representing regular numbers *n* from 1 to 4 corresponding to many-place pairs of reciprocals *n*, rec. *n in the reconstructed Table R*. It was noted that if *n* is from 1 to 2, then 2 · *n* is from 2 to 4, and the point in the index grid representing 2 · *n* is one step to the right of the point representing *n*. Similarly, if *n* is from 1 to 2, then 4 · *n* is from 4 to 8, and the point in the index grid representing 4 · *n* is two steps to the right of the point representing *n*. That is the reason why in Fig. 1.2.3 the *n* tags usually occur in pairs, but occasionally also in triplets (with the third *n* tags representing pairs of reciprocals with *n* from 4 to 8 in fragment IV A = BM 41101). It was also shown that almost all points *n* representing pairs *n*, rec. *n* in the reconstructed Table R lie inside the total 12-place index flower, indicating that most pairs of reciprocals in Table R are (at most) *total 12-place*.

In Fig. 1.5.1 below are considered, similarly, points representing regular numbers *n* from 1 to 3 corresponding to many-place pairs of reciprocals *n*, rec. *n in the Seleucid Table V*. In Fig. 1.5.1, *v tags* show the locations of all index points representing regular numbers *n* from 1 to 3 corresponding to *at most* total 12-place pairs of reciprocals *n*, rec. *n* in Table V. On the other hand, *V tag*s represent regular numbers *n* from 1 to 3 corresponding to *more than* total 12-place pairs of reciprocals *n*, rec. *n* in Table V. Finally, *o tags* represent regular numbers *n* from 1 to 3 corresponding to *missing* at most total 12-place pairs of reciprocals *n*, rec. *n* in Table V.

The following observations can be made:

a) All the v tags stay, by definition, inside the 12-place index flower. Almost all the V tags, on the other hand, are located outside the 12-place flower but inside the "6-place double triangle", which indicates that in pairs *n*, rec. *n* represented by V tags either *n* or rec. *n* is at most 6-place. The only exception is the V tag at the index point (0, 23), which represents the extreme pair of reciprocals 3^{23}, rec. 3^{23}.

b) If *n* is from 1 to 1 30, then 2 · *n* is from 1 to 3, but if *n* is from 1 30 to 2, then 2 · *n* is not from 1 to 3. Therefore, some of the v tags occur in pairs, while other v tags are single.

c) The V tags that are not located on the *s* axis, which they are only when *n* is a positive or negative power of 3, have indices with *s* between -12 and 13.

d) With only one exception, the *s* coordinates of successive V tags on the *s* axis differ by either 4 or 3.

e) With only one exception, all o tags occur in pairs or alone or are located to the right of a v tag.

These observations allow the following conclusions to be drawn about the construction of the table of many-place pairs of reciprocals on the Seleucid tablet AO 6456:

1. First a table was constructed of the first 23 successive powers of 3, and their reciprocals (see below).

2. Next, a number of preliminary algorithm tables were constructed by the method of doubling and halving, with departure from the individual pairs of reciprocals in the table of powers of 3 and their reciprocals. The preliminary algorithm tables extended from (at least) *s* = -12 to *s* = 13. Almost exclusively, only pairs of reciprocals *n*, rec. *n* with either *n* or rec. *n* at most 6-place were considered.

3. The pairs of reciprocals *n*, rec. *n* produced in steps 1-2 with *n* smaller than rec. *n* (in the lexicographic sense), together with all inverse pairs of reciprocals rec. *n*, *n* with rec. *n* smaller than *n* (in the lexicographic sense), were collected together in one great table and ordered lexicographically with respect to the numbers *n* in the first column of the table. The person responsible for the sorting procedures made a great number of copying errors, many omission errors, and a few sorting errors. In particular, after a relatively successfully sorting of the pairs with *n* from 1 to 2 he got tired and missed many of the pairs with *n* from 2 to 3. The pairs with *n* greater than 3 were, presumably, recorded on a second tablet.

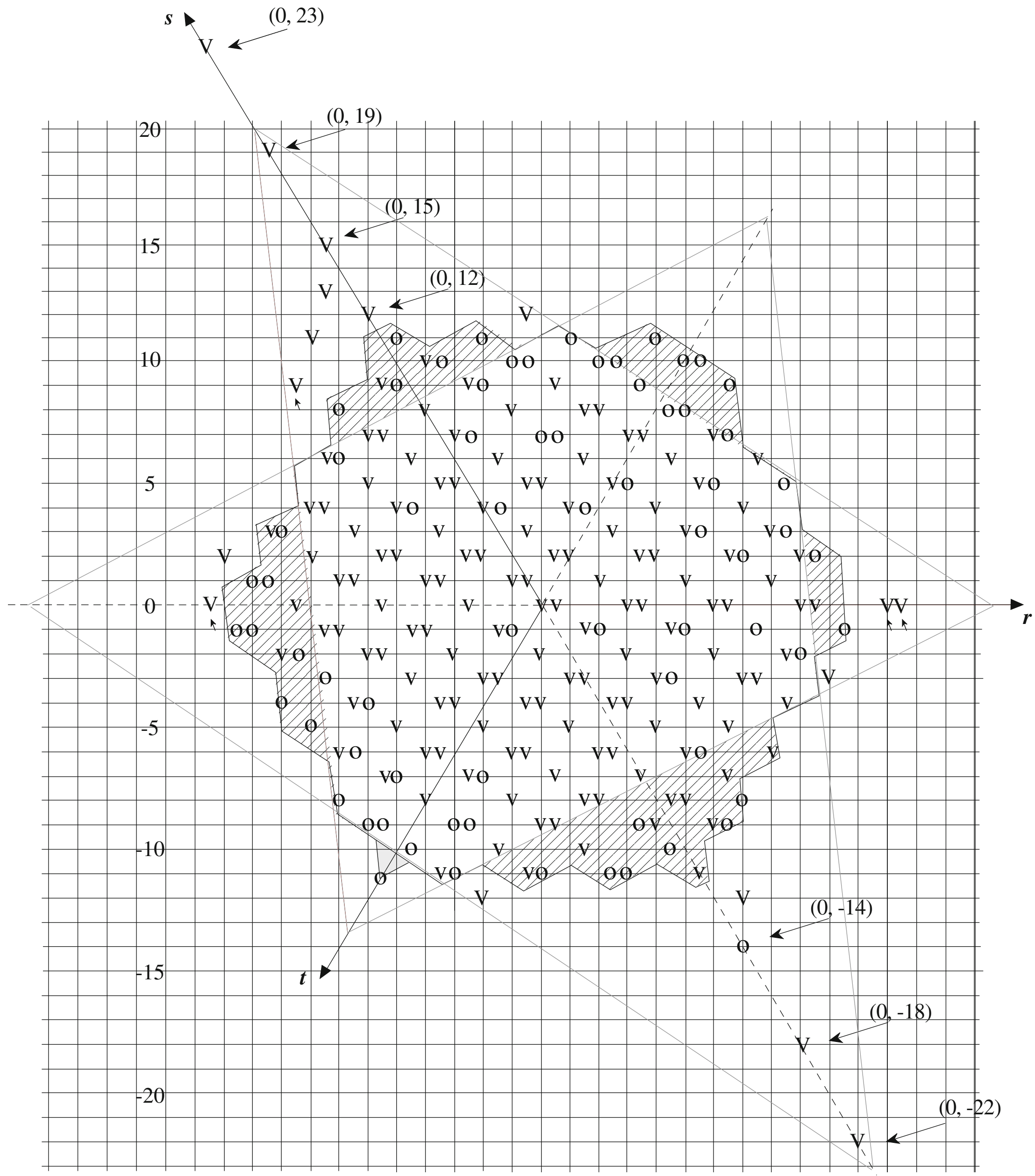


Fig. 1.5.1. Table V = AO 6456. Representation in the index grid of numbers *n* from 1 to 3.

A final interesting observation in this connection is that the number (3^{23} =) 2 01 04 08 03 00 27 appears not only in Table V but also in a catch line at the end of the table of reciprocals on fragment I B (BM 34596 = *LBAT* 1633; see Sec. 1.1 above), that is, in a fragment of a copy of the first part of the reconstructed Table R. The square of the number appears in a catch line at the end of the fragment of a table of squares II G (Sec. 1.1 above). The number itself appears again in fragment V A (BM 34601 = *LBAT* 1644; see Friberg, *MSCT 1*, App. A9.2), which contains a detailed computation of (3^{92} =) the square of the 13- place regular sexagesimal number 4 04 17 40 45 13 17 45 52 14 42 12 09, about which it is explicitly mentioned that it is the square of 2 01 04 08

03 00 27. A possible explanation for this apparent popularity of the number 2 01 04 08 03 00 27 is that it is a "funny number" with many internal vacant places (zeros). For the same reason, the 7-place number 2 01 04 08 03 00 27 is an "essentially" 6-place number!

In Table 1.5 below, a table of powers of 3 and their reciprocals is continued until it reaches the 23d power of 3, which is the mentioned funny number 2 01 04 08 03 00 27.

Table 1.5. The first 23 powers of 3, and their reciprocals. Numbers from 1 to 7 42 57 46 are marked as bold.

r	*s*	*n*	rec. *n*	*p*	*q*	*p*+*q*
0	0	1	1	1	1	2
0	1	**3**	20	1	1	2
0	2	9	**6 40**	1	2	3
0	3	27	**2 13 20**	1	3	4
0	4	**1 21**	44 26 40	2	3	5
0	5	**4 03**	14 48 53 20	2	4	6
0	6	12 09	**4 56 17 46 40**	2	5	7
0	7	36 27	**1 38 45 55 33 20**	2	6	8
0	8	**1 49 21**	32 55 18 31 06 40	3	6	9
0	9	**5 28 03**	10 58 26 10 22 13 20	3	7	10
0	10	16 24 09	**3 39 28 43 27 24 26 40**	3	8	11
0	11	49 12 27	**1 13 09 34 29 08 08 53 20**	3	9	12
0	12	**2 27 37 21**	24 23 11 29 42 42 57 46 40	4	9	13
0	13	**7 22 52 03**	8 07 43 49 54 14 19 15 33 20	4	10	14
0	14	22 08 36 09	**2 42 34 36 38 04 46 25 11 06 40**	4	11	15
0	15	**1 06 25 48 27**	54 11 32 12 41 35 28 23 42 13 20	5	11	16
0	16	**3 19 17 25 21**	18 03 50 44 13 51 49 27 54 04 26 40	5	12	17
0	17	9 57 52 16 03	**6 01 16 54 44 37 16 29 18 01 28 53 20**	5	13	18
0	18	29 53 36 48 09	**2 00 25 38 14 52 25 29 46 00 29 37 46 40**	5	14	19
0	19	**1 29 40 50 24 27**	40 08 32 44 57 28 29 55 20 09 52 35 33 20	6	14	20
0	20	**4 29 02 31 13 21**	13 22 50 54 59 09 29 58 26 43 17 31 51 06 40	6	15	21
0	21	13 27 07 33 40 03	**4 27 36 58 19 43 09 59 28 54 25 50 37 02 13 20**	6	16	22
0	22	40 21 22 41 00 09	**1 29 12 19 26 34 23 19 49 38 08 36 52 20 44 26 40**	6	17	23
0	23	**2 01 04 08 03 00 27**	29 44 06 28 51 27 46 36 32 42 52 17 26 54 48 53 20	7	17	24

1.5.4 Table V. New Photos

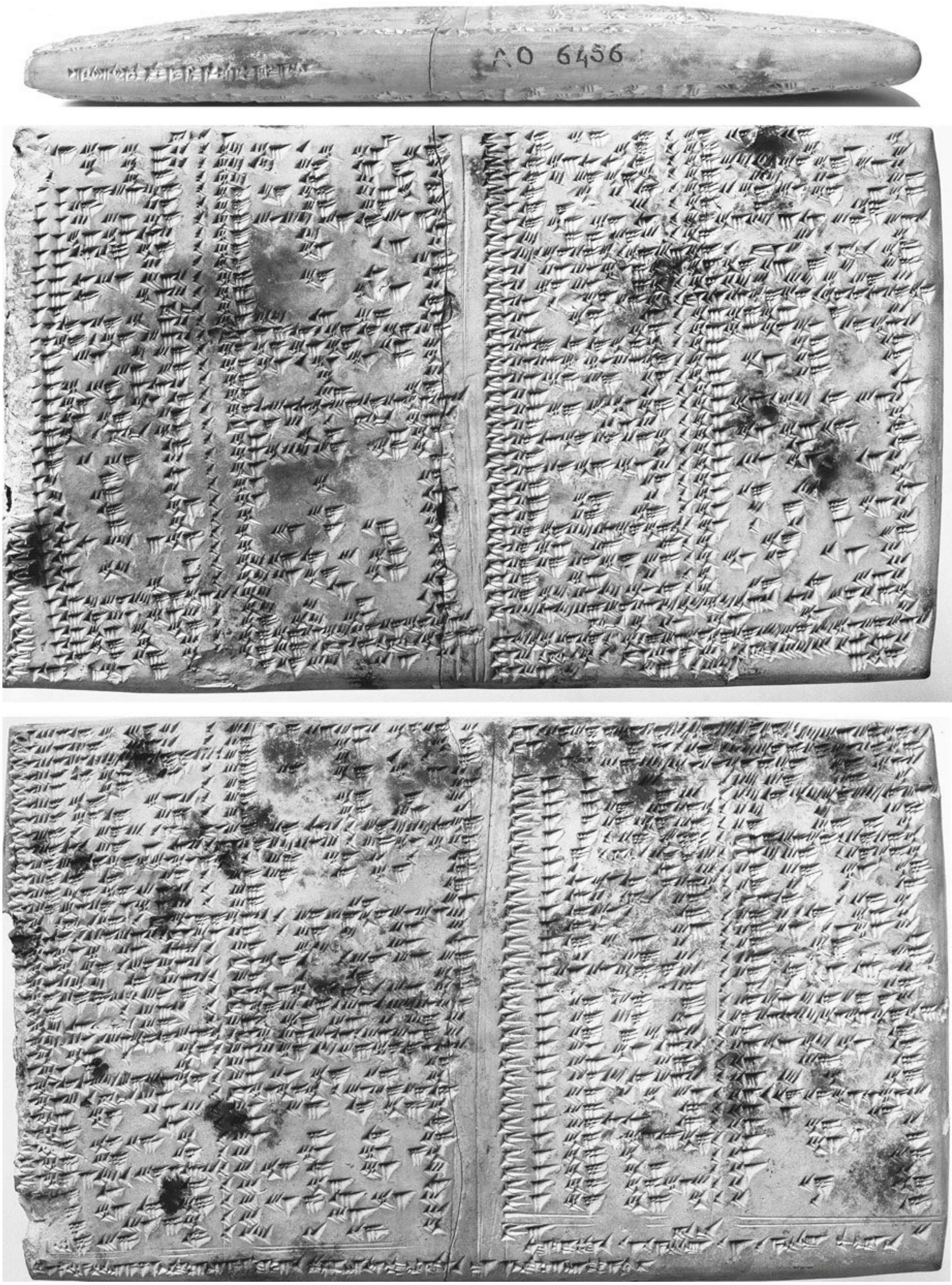


Fig. 1.5.2. Photos of AO 6456. Scale 3:4. Published with the kind permission of Béatrice André-Salvini, the Louvre.

1.6 Table W = W 23281, *rev.* (Uruk). An Atypical Achaemenid Many-Place Table of Reciprocals

1.6.1 An Atypical Many-Place Table of Reciprocals

W 23281 = IM 76283 (photos in Figs. 3.1.12-13) was found in the ruins of the same Achaemenid house in Uruk as Table U = W 23283. See the discussion in Sec. 1.3.1 above. The tablet is inscribed on the obverse with a metrological table (for details, see Sec. 3.1 below), and on the reverse with an atypical many-place table of reciprocals (Table W).

Excellent hand copies of W 23281 = *Uruk 4* (1993), text 173 were published by von Weiher in 1993. As is clear from these copies, the uppermost part of the reverse is lost, and with it several lines of text. The conform transliteration of the reverse of the text in Fig. 1.6.1 below is based on von Weiher's hand copy of the reverse but includes also reasonable reconstructions of lost or damaged parts of the text.

All available space on the reverse of W 23281 is occupied by a single mathematical table, with no relation to the metrological tables on the obverse. The table is divided into two columns, with the first column (col. *iii*) to the right and the second (col. *iv*) to the left, as is customary for columns on the reverse of a Babylonian clay tablet. Col. *iv* is continued onto the lower edge, and the two final lines of the table are inscribed on the left edge, together with the partly preserved colophon (the subscript). The whole text of the table is presented in a conventional transliteration in Table 1.6 below, with the lost lines reconstructed, and with errors in the original text corrected.

The many-place table of reciprocals on the reverse of Table W = W 23 281 has a couple of unusual properties, as the ensuing analysis of the table will show. This analysis is based on the easily understood assumption that exactly six whole lines are lost at the top of col. *iii* and five at the top of col. *iv*.

With suitably reconstructed lines inserted at the top of each column and with the line numbers expressly indicated, one of the unusual features of Table W becomes clearly visible. Indeed, consider the following lines of Table W, with their line numbers, and with the pairs of reciprocals written as if their numerical values were all between 1 and 1 (00) = 60:

...	... ...	... ...
6	6;24	9;22 30
7	7;12	8;20
8	8;40 50	6;54 43 12
9	9;36	6;15
...	... ...	... ...
57	57;36	1;02 30
58	58;35 37 30	1;01 26 24
59	59;15 33 20	1;00 45
1 (00)	1;01 43 42 13 20	58;19 12

This juxtaposition makes it easy to see that Table W was constructed so that *for the first 60 pairs (n, rec. n) of the table, the leading sexagesimal places of the regular numbers n were advanced by 1 at each count, with the result that those leading places of n were the same as the line numbers*.

A similar observation shows that *the next 30 pairs of Table W proceed in the same way, except that they start over again from 1*, in the following way:

1 01	1;36	37;30
1 02	2;18 53 20	25;55 12
...	... ...	... ...
1 29	29;09 36	2;03 27 24 26 40
1 30	30;22 30	1;58 31 06 40

It is interesting to note the following apparent restrictions on the construction of of Table W:

a) If a pair of reciprocals (*n,* rec. *n*) = (*n, m*) is present in Table W, then the pair in its inverse form (*m, n*) is not present there.

b) Although the counting starts over again at 1 in line 1 01 = 61, there are no duplicates in Table W!

c) Reciprocal pairs appearing in the Old Babylonian standard table of reciprocals are (almost) never present in Table W.

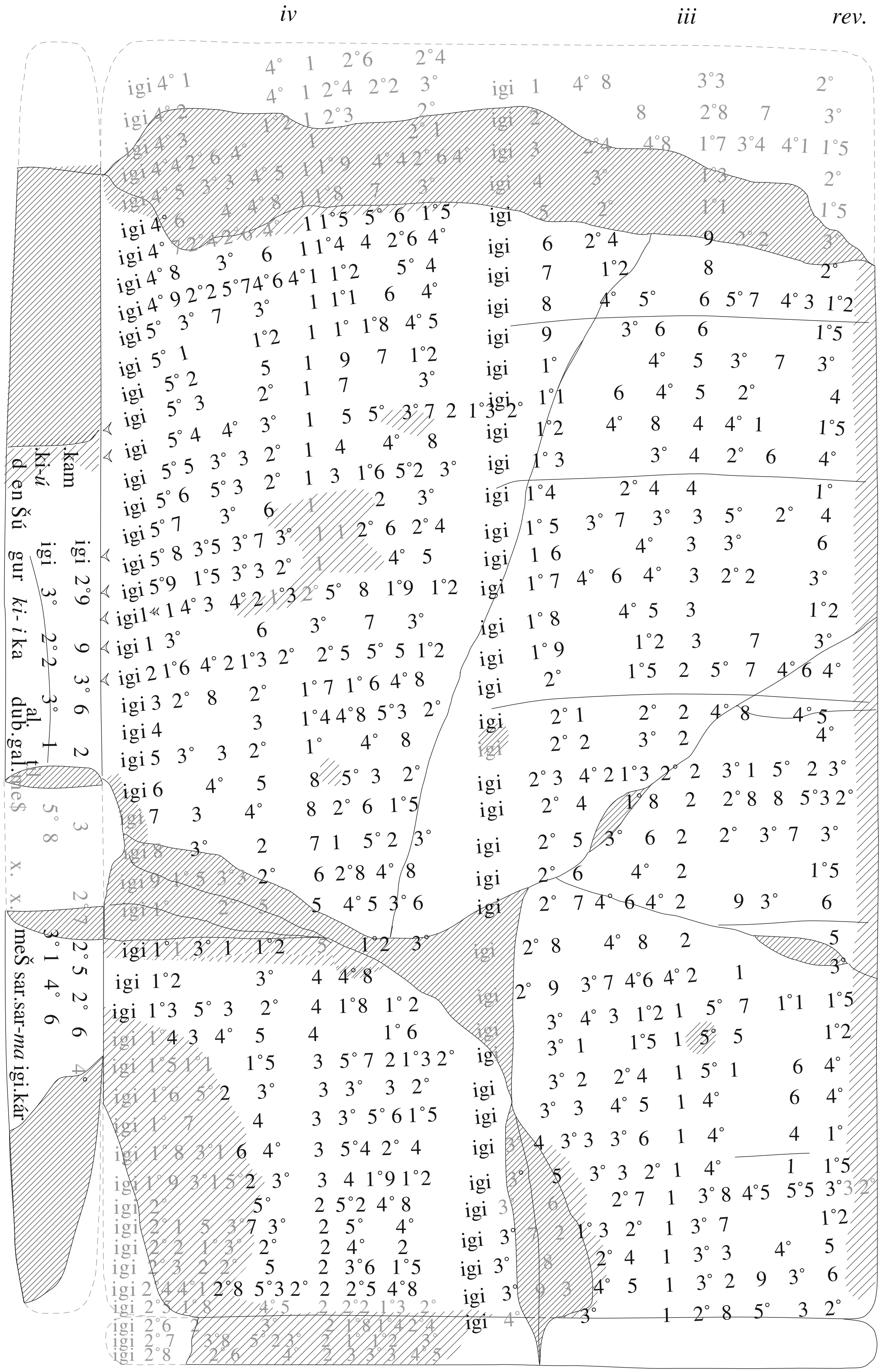


Fig. 1.6.1. Table W = W 23281, *rev.* A conform transliteration within an outline of the tablet.

Table 1.6. Table W = W 23281, *rev.*, transliteration, with indications of errors

(the corrected numbers are underlined, the incorrect numbers are noted to the right of the table)

iii

	[1 48]	[33 20]
	[2 08]	[28 07 30]
	[3 24 48]	[17 34 41 15]
	[4 30]	[13 20]
5	[5 20]	[11 15]
	6 24	[9 22 30]
	7 12	8 20
	8 40 50	6 54 43 12
	9 36	6 15
10	10 40	5 37 30
	11 06 40	5 2[4]
	12 48	4 41 15
	13 30	4 26 40
	14 24	4 10
15	15 37 30	3 50 24
	16 40	3 36
	17 46 40	3 22 30
	18 45	3 12
	19 12	3 07 30
20	20 15	2 5[7] 46 40
	21 20	2 48 45
	22 30	2 40
	23 42 13 20	2 31 52 30
	24 18	2 28 08 53 20
25	25 36	2 20 37 30
	26 40	2 15
	27 46 40	2 09 36
	28 48	2 05
	29 37 46 40	2 01 30
30	30 43 12	1 57 11 15
	31 15	1 55 12
	32 24	1 51 06 40
	33 45	1 46 40
	[3]4 33 36	1 44 10
35	[3] 5 33 20	1 41 15
	[36] 27	1 38 45 55 3[3 20]
	3[7 02] 13 20	1 37 12
	3[8] 24	1 33 45
	3[9 03] 45	1 32 09 36
40	[40] 30	1 28 53 20

iv

	[41 40]	[1 26 24]	
	[42 40]	[1 24 22 30]	
	[43 12]	[1 23 20]	
	[44 29 40 39 50 37 30]	[1 20 54 31 06 40]	
45	[45 33 45]	[1 19 00 44 26 40]	
	[46 04 48]	[1 18 07 30]	
	4[7 24 26 40]	1 15 56 15	
	48 36	1 14 04 26 40	
	49 22 57 46 40	1 12 54	
50	50 37 30	1 11 06 40	
	51 12	1 10 18 45	
	52 05	1 09 07 12	
	53 20	1 07 30	
	54 40 30	1 05 50 37 02 13 20	
55	55 33 20	1 04 48	
	56 53 20	1 03 16 52 30	
	57 36	[1] 02 30	
	58 35 37 30	[1 01] 26 24	
	59 15 33 20	[1 00] 45	
1 00	1 01 43 42 [1]3 [20]	58 19 12	
	1 36	37 30	
	2 18 53 20	25 55 12	16 42 13
	3 28 20	17 16 48	26
	4 03	14 48 53 20	
1 05	5 33 20	10 48	
	6 45	8 53 20	
	7 06 40	8 26 1[5]	03
	[8] 32	7 01 52 30	
	[9 15 33] 20	6 28 48	
1 10	[10 25]	5 45 36	
	1[1] 31 12	[5] 12 30	
	12 30	4 4[8]	
	13 53 20	4 19 12	43, 18
	[1]4 03 45	4 16	
1 15	[15 11] 15	3 57 02 13 20	30
	[16 5]2 30	3 33 20	
	[17] 04	3 30 56 15	
	[18 31 0]6 40	3 14 24	54
	[19 31 5]2 30	3 04 19 12	
1 20	[20]50	2 52 48	
	[21 05 3]7 30	2 50 40	
	[22 13] 20	2 42	
	[23 02 2]4	2 36 15	25
	[24 41] 28 53 20	2 25 48	
1 25	[25 18 45]	2 2[2 13 20]	
	[26 02] 30	2 1[8 14 24]	
	[2]7 [20 15]	2 [11 41 14 04 26 40]	
	[28 26 40]	2 06 [33 45]	
	29 09 36	[2 03 27] 24 26 [40]	25
1 30	30 22 30	[1 58 3]1 06 40	40 06

For the readers' convenience, the OB standard table of reciprocals is reproduced here:

2	30
3	20
4	15
5	12
6	10
8	7 30
9	6 40
10	6
12	5
15	4

16	3 45
18	3 20
20	3
24	2 30
25	2 24
27	2 13 20
30	2
32	1 52 30
36	1 40
40	1 30

45	1 20
48	1 15
50	1 12
54	1 06 40
1	1
1 04	56 15
1 12	50
1 15	48
1 20	45
1 21	44 26 40

This OB standard table of reciprocals comprises all reciprocal pairs *(n, m)* where *n* is a one-place regular sexagesimal number. In addition, it contains four reciprocal pairs where *n* is a two-place regular sexagesimal number. Three of these additional pairs, namely (1 12, 50), (1 15, 48), and (1 20, 45), are often omitted from copies of the standard table, probably because they are superfluous, being inverse copies of the pairs (50, 1 12), (48, 1 15), and (45, 1 20).

As mentioned, pairs of reciprocals appearing in the OB standard table of reciprocals almost never appear in Table W. A possible exception is the pair (44 26 40, 1 21), which is the inverse form of the last pair (1 21, 44 26 40) of the OB standard table, and which (as will be shown below) may be assumed to have been present in one of the lost lines of the text.

1.6.2 The Numerical Algorithms Used for the Construction of Table W

How was it possible for a Late Babylonian scribe to construct a table of reciprocals with the described properties? One possibility is, of course, that the author of Table W simply picked his entries from a complete many-place table of reciprocals with *n* from 1 to 1 (00) that happened to be available to him. A clue to a different and more likely answer to this question is provided by an analysis of the errors in Table W.

Those errors are indicated in Table 1.6 above by writing the incorrect digits of an entry in the margin of Table 1.6 and by simultaneously underlining the correct digits in the entry. In eight cases, the errors are simple writing errors, 26 instead of 16, 03 instead of 06, and so on. In one case the mistake is a simple transposition, 40 06 instead of 06 40. However, the remaining error is much more interesting. Indeed, in line 1 02 of Table W, the correct number 2 18 53 20 is replaced by the incorrect number 2 16 42 13 20. This is a *telescoping error* of a familiar type, due to insufficient attention to the difficulties of calculating with numbers in relative place value notation. It can be explained as follows: A quick look in Table R (Sec. 1.2.3 above) shows that the regular sexagesimal number 2 18 53 20 has the index -7, -6. In other words, 2 18 53 20 can be explained as $30^7 \cdot 20^6$. There are two possible ways in which this number can have been calculated. One way is as 30 times (or 1/2 of) 4 37 46 40 = $30^6 \cdot 20^6$. The other way is as 20 times (or 1/3 of) 6 56 40 = $30^7 \cdot 20^5$. The former way of calculating $30^7 \cdot 20^6$ cannot lead to a telescoping error of the observed type. However, in the other case, the telescoping error can have arisen as follows: Correctly,

20 · 6 56 40 = 20 · (6 50 00 + 6 40) = 2 16 40 00 + 2 13 20 = 2 18 53 20.

Without the benefit of a notation for zero, and with a faulty understanding of relative place value notation and orders of magnitude, the one who computed the incorrect number may have proceeded as follows:

20 · 6 56 40 = 20 · (6 50 + 6 40) = 2 16 40 + 2 13 20 = 2 16 42 13 20.

This is the incorrect number displayed in the text.

This explanation of the telescoping error in line 102 of Table W immediately suggests the following detailed conclusion about how Table W may have been constructed. (Note that the method probably used for the construction of Table W was *similar to, but different from,* the method apparently used for the construction of the much bigger and more complete Table R. See the discussion in Secs. 1.2.4-5 above.)

1. First a table was constructed of 14 successive *powers of 2*, and their reciprocals, all pairs at most total 8-place.
2. Next, 14 + 13 = 27 preliminary algorithm tables were constructed by the method of *triplicating and trisecting*, with departure from the individual pairs of reciprocals in the table of powers of 2 and the corresponding inverse pairs. Initially, only total 8-place pairs of reciprocals *n*, rec. *n* were considered. When this turned out to be insufficient, 4 total 9-place pairs were added to Table W.
3. From the pairs of reciprocals *n*, rec. *n* produced in steps 1-2, pairs were selected, one at a time, with either *n* or rec. *n* having the leading places 1, 2, 3, etc., up to 59 and 1 (00). All the time, pairs appearing in the OB standard table of reciprocals were avoided. Also repeated use of individual pairs was avoided. The selected pairs were immediately recorded as entries in Table W. Then the process was repeated until there was no more space available for the recording of any more entries.

It may be of interest to see how much work the implementation of the steps listed above actually required. In step 1 of the procedure, a table of the first 14 powers of 2 like the one below may have been calculated and inscribed on a small tablet. Proceeding beyond the 14th power would have given pairs of reciprocals with more than 8 sexagesimal places together, which clearly was not desirable in view of the limited widths of the columns on the reverse of W 23281.

Table 1.7. The first 14 powers of 2, and their reciprocals.

r	*s*	*n*	rec. *n*	*p*	*q*	*p+q*
0	0	1	1	1	1	2
1	0	2	30	1	1	2
2	0	4	15	1	1	2
3	0	8	7 30	1	2	3
4	0	16	3 45	1	2	3
5	0	32	1 52 30	1	3	4
6	0	1 04	56 15	2	2	4
7	0	2 08	28 07 30	2	3	5
8	0	4 16	14 03 45	2	3	5
9	0	8 32	7 01 52 30	2	4	6
10	0	17 04	3 30 56 15	2	4	6
11	0	34 08	1 45 28 07 30	2	5	7
12	0	1 08 16	52 44 03 45	3	4	7
13	0	2 16 32	26 22 01 52 30	3	5	8
14	0	4 33 04	13 11 00 56 15	3	5	8

In the outlined step 2 of the procedure, a number of preliminary algorithm tables were probably inscribed on a series of small clay tablets. The first of these algorithm tables may have been of the following form:

Table A_0

0	0		1	1
0	1		3	20
0	2		9	6 40
0	3		27	2 13 20
0	4		1 21	[A] ! 44 26 40
0	5	B	4 03	14 48 53 20
0	6		**12 09**	**4 56 17 46 40**
0	7	A	36 27	1 38 45 55 33 20

This algorithm table, Table A_0, proceeds with triplings and trisections until in the last line a total 8-place pair of reciprocals is reached. (The next pair would have been total 9-place.) The last pair, tagged with the letter A, is one of the 60 first pairs recorded in Table W. The pair tagged with the letter B is one of the pairs in the second round of pairs on Table W, recorded in line 1 04 of the reverse of W 23281. The pair tagged with an [A] , namely the pair 44 26 40, 1 21, is a suggested reconstruction of the pair in line 44 of Table W. The reason for the use of an exclamation mark here will be mentioned later.

Table A_1 below starts with the pair 2, 30 and proceeds with triplings and trisections until a total 8-place pair is reached. The two pairs tagged with an A are recorded in lines 24 and 49 of Table W, while the pair tagged with a B is recorded in line 1 22.

Table B_1 starts with the inverse pair 30, 20 and proceeds with triplings and trisections. Unexpectedly, it continues until it ends with the last pair recorded as line 54 in Table W. This is a total 10-place pair. It is not difficult to understand why this particular total 10-place pair was picked from Table B_1. In Table 1.3 (Sec. 1.2.3 above), there are five many-place pairs of reciprocals (with line numbers between 12 and 18) from which line 54 in Table W could have been picked. Three of these would have required an excessive amount of calculations, those with the indices (19, 3), (-16, -1), and (17, 8). The remaining two, with the indices (1, -8) and (2, 10), are both more than total 8-place. Anyway, since three triplings is the same as multiplication by 27, it was obvious that a number with the leading place 54 could be found by three cfbfbbds of the pair (2 01 30, 29 37 46 40) in Table B_1 with the index (-1, 5).

Table A_1

1	0		2		30
1	1		6		10
1	2		18		3 20
1	3		54		1 06 40
1	4		2 42	B	22 13 20
1	5		**8 06**		**7 24 26 40**
1	6	A	24 18		2 28 08 53 20
1	7		1 12 54	A	49 22 57 46 40

Table B_1

-1	0		30		2
-1	1		1 30		40
-1	2	[A]	4 30		13 20
-1	3	A	13 30		4 26 40
-1	4	A	40 30		1 28 53 20
-1	5		2 01 30	A	29 37 46 40
-1	6		**6 04 30**		**9 52 35 33 20**
-1	7		**18 13 20**		**3 17 31 51 06 40**
-1	8*	A	54 40 30		1 05 50 37 02 13 20

The procedure continues in the same way on the remaining preliminary algorithm tables, ending with Tables A_{14} and B_{14}. The pairs calculated and recorded on these preliminary algorithm tables are (in principle) at most total 8-place. However, in four cases, it was necessary to allow also total 9-place pairs in order to find numbers n or rec. n with the leading places 29, 45, 54, and 1 (01). (See the asterisk-marked pairs below with the indices (-1, 8), (4, 8), (-4, 6), and (5, 8).)

Table A_2

2	0		4		15
2	1		12		5
2	2		36		1 40
2	3	[A]	1 48		33 20
2	4		5 24	A	11 06 40
2	5		**16 12**		**3 42 13 20**
2	6	A	48 36		1 14 04 26 40
2	7		2 25 48	B	24 41 28 53 20

Table B_2

-2	0		15		4
-2	1		45		1 20
-2	2		2 15	A	26 40
-2	3	B	6 45		8 53 20
-2	4	A	20 15		2 57 46 40
-2	5		1 00 45	A	59 15 33 20
-2	6		**3 02 15**		**19 45 11 06 40**

Table A_3

3	0		8		7 30
3	1		24		2 30
3	2		1 12		50
3	3		3 36	A	16 40
3	4		10 48	B	5 33 20
3	5	A	32 24		1 51 06 40
3	6		1 37 12	A	37 02 13 20
3	7		4 51 36		12 20 44 26 40

Table B_3

-3	0		7 30		8
-3	1	A	22 30		2 40
-3	2		1 07 30	A	53 20
-3	3		3 22 30	A	17 46 40
-3	4		**10 17 30**		**5 55 33 20**
-3	5	B	30 22 30		1 58 31 06 40

Table A_4

4	0		16		3 45
4	1		48		1 15
4	2		2 24		25
4	3	A	7 12		8 20
4	4		**21 36**		**2 46 40**
4	5		1 04 48	A	55 33 20
4	6		3 14 24	B	18 31 06 40
4	7		**9 43 12**		**6 10 22 13 20**
4	8*	B	29 09 36		2 03 27 24 26 40

Table B_4

-4	0		3 45		16
-4	1		11 15	[A]	5 20
-4	2	A	33 45		1 46 40
-4	3		1 41 15	A	35 33 20
-4	4		**5 03 45**		**11 51 06 40**
-4	5	B	15 11 15		3 57 02 13 20
-4	6*	[A]	45 33 45		1 19 00 44 26 40

Table A_5

5	0		32		1 52 30
5	1	B	1 36		37 30
5	2		4 48	B	12 30
5	3	A	14 24		4 10
5	4	[A]	43 12		1 23 20
5	5		2 09 36	A	27 46 40
5	6		6 28 48	B	9 15 33 20
5	7		**19 26 24**		**3 05 11 06 40**
5	8*		58 19 12	A	1 01 43 42 13 20

Table B_5

-5	0		1 52 30		32
-5	1		5 37 30	A	10 40
-5	2	B	16 52 30		3 33 20
-5	3	A	50 37 30		1 11 06 40
-5	4		2 31 52 30	A	23 42 13 20

Table A_6

6	0		1 04		56 15
6	1		3 12	A	18 45
6	2	A	9 36		6 15
6	3	A	28 48		2 05
6	4		1 26 24	[A]	41 40
6	5		4 19 12	B	13 53 20
6	6		**12 57 36**		**4 37 46 40**
6	7		**38 52 48**		**1 32 35 33 20**

Table B_6

-6	0		56 15		1 04
-6	1		2 48 45	A	21 20
-6	2		8 26 15	B	7 06 40
-6	3	[B]	25 18 45		2 22 13 20
-6	4		1 15 56 15	A	47 24 26 40

Table A_7

7	0	[A]	2 08		28 07 30
7	1	A	6 24		9 22 30
7	2	A	19 12		3 07 30
7	3	A	57 36		1 02 30
7	4		2 52 48	B	20 50
7	5		**8 38 24**		**6 56 40**
7	6		25 55 12	B	2 18 53 20
7	7		**1 17 45 36**		**46 17 46 40**

Table B_7

-7	0		28 07 30		2 08
-7	1		1 24 22 30	[A]	42 40
-7	2		**4 13 07 30**		**14 13 29**
-7	3		**12 39 22 30**		**4 44 29 40**

Table A_8

8	0		4 16	B	14 03 45
8	1	A	12 48		4 41 15
8	2	A	38 24		1 33 45
8	3		1 55 12	A	31 15
8	4		5 45 36	B	10 25
8	5		17 16 48	B	3 28 20
8	7		**51 50 24**		**1 09 26 40**
8	8		**2 35 31 12**		**23 08 53 20**

Table B_8

-8	0		14 03 45		4 16
-8	1		**42 11 15**		**1 25 20**
-8	2		2 06 33 45	[B]	28 26 40
-8	3		**6 19 41 15**		**9 28 53 20**

Table A_9

9	0	B	8 32		7 01 52 30
9	1	A	25 36		2 20 37 30
9	2		**1 16 48**		**46 52 30**
9	3		3 50 24	A	15 37 30
9	4	B	11 31 12		5 12 30
9	5	A	34 33 36		1 44 10
9	6		**1 43 40 48**		**34 43 20**
9	7		**5 11 32 24**		**11 34 26 40**

Table B_9

-9	0		7 01 52 30		48 32
-9	1	B	21 05 37 30		2 50 40
-9	2		1 03 16 52 30	A	56 53 20

Table A_{10}

10	0	B	17 04		3 30 56 15
10	1	A	51 12		1 10 18 45
10	2		**2 33 36**		**23 26 15**
10	3		**7 40 48**		**7 48 45**
10	4	B	23 02 24		2 36 15
10	5		1 09 07 12	A	52 05
10	6		**3 27 21 36**		**17 21 40**
10	7		**10 22 04 18**		**5 47 13 20**

Table B_{10}

-10	0	3 30 56 15	17 04
-10	1	**10 32 48 45**	**5 41 40**
-10	2	**31 38 26 15**	**1 53 53 20**

Table A_{11}

11	0		**34 08**		**1 45 28 07 30**
11	1		**1 42 24**		**35 09 22 30**
11	2		**5 07 12**		**11 43 07 30**
11	3		**15 21 36**		**3 54 22 30**
11	4	[A]	46 04 48		1 18 07 30
11	5		2 18 14 24	[B]	26 02 30
11	6		6 54 43 12	A	8 40 50
11	7		**20 14 09 36**		**2 53 36 40**

Table B_{11}

-11	0	1 45 28 07 30	34 08
-11	1	**5 16 24 22 30**	**11 22 40**

Table A_{12}

12	0		**1 08 16**		**52 44 03 45**
12	1	[A]	3 24 48		17 34 41 15
12	2		**10 14 24**		**5 51 33 45**
12	3	A	30 43 12		1 57 11 15
12	4		1 32 09 36	A	39 03 45
12	5		**4 36 28 48**		**13 01 15**
12	6		**13 49 26 24**		**4 20 25**
12	7		**41 28 28 12**		**1 26 48 20**

Table B_{12}

-12	0	52 44 03 45	1 08 16
-12	1	**2 38 12 11 15**	**22 45 20**

Table A_{13}

13	0		**2 16 32**		**26 22 01 52 30**
13	1		**6 49 36**		**8 47 20 37 30**
13	2		**20 28 48**		**2 55 46 52 30**
13	3		1 01 26 24	A	58 35 37 30
13	4		3 04 19 12	B	19 31 52 30
13	5		**4 36 28 48**		**13 01 15**
13	6	[B]	27 38 52 48		2 10 12 30

Table B_{13}

-13	0	26 22 01 52 30	2 16 32

Table A_{14}

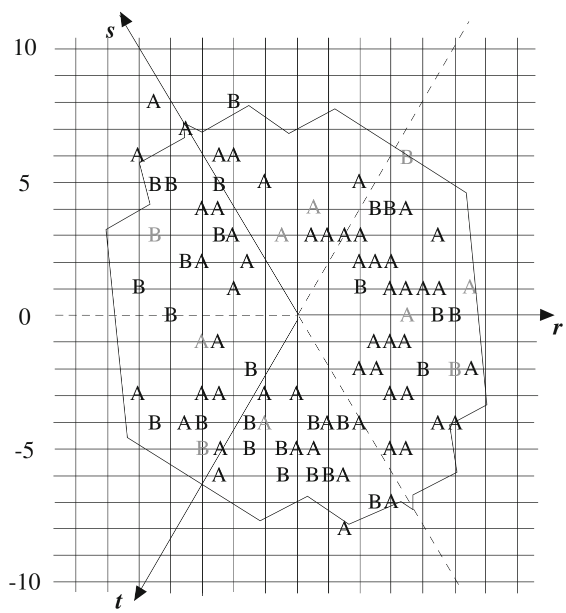
14	0	**4 33 04**	**13 11 00 56 15**
		13 39 12	**4 23 40 18 45**

Table B_{14}

-14	0	13 11 00 56 15	4 33 04

In the preliminary algorithm tables above, pairs of reciprocals selected as entries in Table W are tagged with either A or B, while unselected pairs are marked with bold type. Altogether, there are 47 unselected pairs, to be compared with 90 selected pairs. Among those 47 unselected pairs, only 9 have a leading place in either *n* or rec. *n* greater than 30. More precisely, among the 47 unselected pairs the only available leading places beyond 30 in either *n* or rec. *n* are 31, 34 (twice), 35, 41, 42, 46, 51, and 52. Hence, *it would not have been possible to continue Table W beyond line 1 30* without a heavy use of duplicates or a substantial extension of the total number of sexagesimal places allowed in the pairs of reciprocals! Talking about duplicates, it is interesting to note that there is no unselected pair with the leading place 44 in either *n* or rec. *n*. This observation explains why a pair of reciprocals from the OB standard table of reciprocals, otherwise avoided, had to be used in line 44 of Table W! (See Table 1.3 above, in Sec. 1.2.3, which shows that in the Achaemenid Table U = W 23283+, the only occurrence of the leading place 44 is in the pair 1 21, 44 26 40 from the OB standard table of reciprocals. Note, by the way, that according to Table 1.3 there is no telescoping error in the pair 2 18 53 20, 2 55 12. Therefore, *Table W with its telescoping error in line 102 cannot have been constructed by selecting pairs from the contemporary Table U!*)

In the index diagram below, gray tags A or B stand for lost but reconstructed entries in Table W.



1.6.2. Points representing Table W in the total 8-place index flower.

1.6.3 A Partly Preserved Colophon

On the left edge of W 23281 the final lines of W 23281, lines 1 29 and 1 30, are followed by the word al.t[il] 'finished'. This means that Table W ends with line 1 30 not because there was no more space for additional lines but because the author of Table W realized that he could not continue further without wasting much more time calculating additional or extended preliminary algorithm tables.

The first part of the colophon, inscribed in the left third of the left edge, is almost completely destroyed. What remains can be tentatively restored as

[dub.x].kam	the xth tablet
[Uruk]^{ki}-*ú*	the Urukian

Then follows a last line of the colophon, inscribed along the whole length of the edge:

[x x x x x] ^{d}en šú gur	a personal name
ki-i pî(ka) dub.gal.[meš libir.ra].meš	according to the wording of big [and old] tablets
sar.sar-*ma* igi.kár	wrote and wrote and checked

The double sar is seen here for the first time in a colophon but it may mean simply that the scribe felt overwhelmed by his laborious work with first a complicated metrological table on the obverse, and then a complex mathematical table on the reverse.

2. Direct and Inverse Factorization Algorithms for Many-Place Regular Sexagesimal Numbers

BM 46550 is a small Neo-Babylonian clay tablet, published for the first time in Sec. 2.1 below. On the obverse of the tablet is a teacher's model text, showing that the reciprocal of the 6-place regular sexagesimal number *n* = 1 01 02 06 33 45 is the 5-place sexagesimal number rec. $n = 2^8 \cdot 12^6$ = 58 58 56 38 24. The method used for the computation is a "direct and inverse algorithm": First *n* is factorized into the product of 14 single-place numbers, and then rec. *n* is computed as the product of the reciprocals of those numbers. On the obverse of the tablet, a student somewhat awkwardly tries to show by the same method that, conversely, the reciprocal of rec. *n* = 58 58 56 38 24 is $n = 30^8 \cdot 5^6$ = 1 01 02 06 33 45.

In Sec. 2.2, it is shown that the well known Old Babylonian arithmetical algorithm text CBS 12115 is closely related to BM 46550. Indeed, in each one of the 21 sub-algorithms on CBS 1215, the same method as the one applied on BM 46550 is used to compute first the reciprocal rec. *n* of a given number *n* and then its reciprocal rec. rec. *n,* which is always equal to the given number *n*.

In Sec. 2.3, the three small Late Babylonian tablet fragments BM 34517, BM 34958, and BM 34907, originally published by Sachs in *LBAT* (1955), are explained for the first time. It is shown, by use of a technically complicated reconstruction and representation in an index grid of the damaged regular sexagesimal numbers on the fragments, how the original texts of which the fragments are small parts may have been organized. In each case, the progressions of numbers in the original texts seems to have been computed by means of direct or inverse factorization algorithms. (For the meaning of the term "index grid", see Sec. 1.2.5 above)

In Sec. 2.4, it is argued that several earlier published Old Babylonian tables of powers can be viewed as progressions of numbers constructed by means of inverse factorization algorithms. IM 630174, previously presented in Arabic by Basima Jalil 'Abid in *Sumer* 53 (2005-6) is a new example of such an Old Babylonian table of powers. The text contains a number of interesting errors.

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7_2

2.1 BM 46550. A Neo-Babylonian Tablet with Direct and Inverse Factorization Algorithms

2.1.1 Description of the Tablet and New Photos

BM 46550 (Fig. 2.1.1 below; acquisition number 1881-08-30, 16) is a relatively well preserved clay tablet. The shape of the tablet and the form of the script firmly suggest that it is Neo-Babylonian. (The term refers to Babylonia under the rule of the 11th “Chaldean” dynasty, from the revolt of in 626 BC until the invasion of Cyrus the Great in 539 BC.) According to BM records, BM 46550 comes from a collection of Neo-Babylonian clay tablets excavated by Rassam at Babylon and Borsippa.

Fig. 2.1.1. BM 46550, photo. Published with the permission of the Trustees of the British Museum.

The tablet is inscribed on both sides with pairs of sequences of many-place sexagesimal numbers with successively decreasing numbers of sexagesimal places (tens and units together). Near the right edges of both the obverse and the reverse there are associated sequences of single-place sexagesimal numbers. The detailed explanation below of the arithmetical algorithms used for the construction of the sequences of many-place numbers on BM 46550 will be preceded by an introductory discussion of Babylonian many-place regular sexagesimal numbers and their reciprocals, and it will be followed by discussions of other known examples of both older and younger Babylonian arithmetical algorithm texts of related kinds.

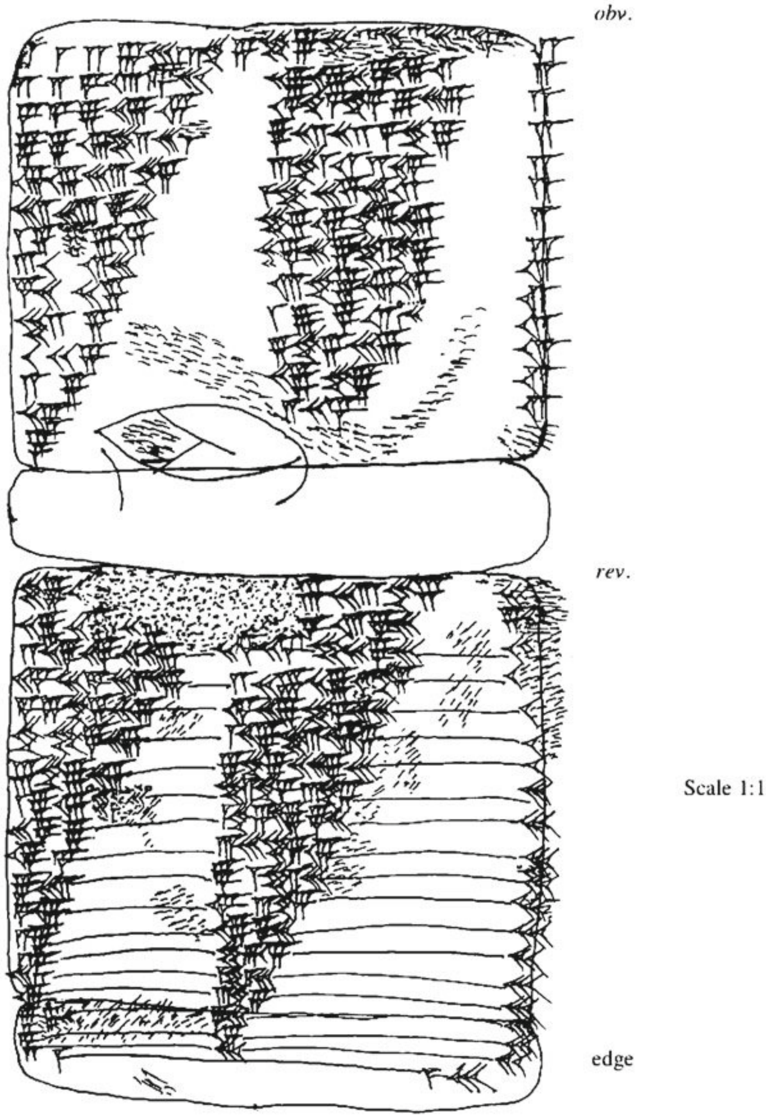


Fig. 2.1.2. BM 46550. Factorization algorithms for n = 1 01 02 06 33 45 = $30^8 \cdot 5^6$ and for rec. n = 58 58 56 38 24 = $2^8 \cdot 12^6$.

2.1.2 *A Spectacular Old Babylonian Example of a Many-Place Pair of Reciprocals*

Recall that a "regular sexagesimal number" is a sexagesimal number that is contained as a factor in some power of the sexagesimal base 60. In other words, a sexagesimal number n is regular if, and only if, there exists a "reciprocal" sexagesimal number, suitably called rec. n such that

$n \cdot$ rec. n = some power of 60.

Obviously, the reciprocal of a regular sexagesimal number is also regular, and it is always true that rec. rec. $n = n$. Moreover, since $60 = 3 \cdot 4 \cdot 5$, a sexagesimal number is regular if, and only if, it contains no other prime factors than 2, 3, and 5. In view of the symmetry of the relation between the two numbers n and rec. n, it is, perhaps, more correct to talk about "reciprocal pairs" of regular sexagesimal numbers than to talk about numbers and their reciprocals. (The Sumerian terms are igi and igi.bi, possibly meaning 'opponent' and 'its opponent'.)

In the Babylonian sexagesimal positional notation without zeros, where numbers have only "relative" (or floating) values, every power of 60 is written as '1'. Therefore, if n and rec. n are reciprocal sexagesimal numbers, then

$n \cdot$ rec. n = '1'.

Examples of hand copies of (probably) Old Babylonian clay tablets inscribed with reciprocal pairs of many-place regular sexagesimal numbers can be found in Friberg, *MSCT 1* (2007), Sec. 1.5 b. In one spectacular example (MS 3264 #2; *op. cit.*, Fig. 1.4.5; with a corrected error *op. cit.,* 35), the "8-place" regular sexagesimal number

n = 1 30 48 06 02 15 20 15

has the "18-place" reciprocal (called igi.bi)

rec. n = 39 38 48 38 28 37 02 08 43 27 09 43 15 53 05 11 06 40.

The following questions immediately arise: How was it possible for an Old Babylonian student of mathematics (in a wide sense of the word) to find such an 8-place (15-digit) regular sexagesimal number? Even more, how was it possible for him (or her?) to find its 18-place (36-digit) reciprocal?
The answer to the first question is easy and obvious. Since a regular sexagesimal number can have no other prime factors than 2, 3, and 5, every such number can be constructed by multiplication together a suitable number of 2s, 3s, and 5s. Thus, for instance, the regular number

n = 1 30 48 06 02 15 20 15 = $3^{25} \cdot 5$

can be computed by first computing the 25th power of 3 and then multiplying the result by 5.

Secondly, as soon as the factorization of n is known, it is easy, at least theoretically, to find a corresponding factorization of its reciprocal rec. n. With n as above, for instance, it is easy to see that

rec. $n = 2^{50} \cdot 5^{24}$, because then
$n \cdot$ rec. $n = (3^{25} \cdot 5) \cdot (2^{50} \cdot 5^{24}) = 2^{50} \cdot 3^{25} \cdot 5^{25} = (4 \cdot 3 \cdot 5)^{25} = 60^{25}$ = '1'.

2.1.3 An Application of a Last Place Factorization Algorithm

After these introductory remarks it is time to look at a transliteration of the cuneiform number signs on BM 46550 into modern number signs. For the moment, it will be enough to consider only the inscription on the obverse of the tablet. The inscription on the reverse is similar and will be considered later.

BM 46550, *obv.* (corrected)

1 01 02 06 33 45	58 ⌜58 56 38 24⌝	2	
2 02 04 13 07 30	29 ⌜29 28 19⌝ 12	2	error: 04 instead of 07
4 04 08 26 15	14 44 44 09 36	2	
8 08 16 52 30	7 22 22 04 48	2	error: a partly erased 1 after 30
16 16 33 45	3 41 11 02 24	2	
32 33 07 30	1 50 35 31 12	2	
1 05 06!15	55 17 45 36	2	
2 10 12 30	27! 38 52 48	2	error: 3 instead of 2
4 20 25	13 49 26 24	12	
52 05	1 09 07 12!	12	
10 25	5 45 36	12	
2 05	28 48	12	
25	2 24	12	
5	[12]	12	
1	[1]		

It is easy enough to see what is happening in the left column of numbers on the obverse. To begin with, it is clear that (after the indicated corrections) each number in the eight lines 2-9 of the left column is precisely double the number in the preceding line, which explains *the eight times repeated number 2* in the right margin.

However, it is important to note the use here of sexagesimal numbers in Babylonian relative place value notation without final double zeros. (The suppression of all final double zeros explains how a column of numbers with a *decreasing* number of places can be the result of a repeated doubling of the numbers!)

Subsequently, each number in the six lines 10-15 is 12 times as large as the immediately preceding number, which explains *the six times repeated number 12* in the right margin.

Thus, in the left column on the obverse of BM 46550, the initial 6-place number n = 1 01 02 06 33 45 is *multiplied* eight times by 2 and six times by 12, and the result is simply '1' (in Babylonian relative place value notation). However, since 30 is the reciprocal of 2, and since 5 is the reciprocal of 12, it is also true that if the initial number *n* is *divided* 8 times by 30 and six times by 5, the result is '1'. Therefore, it is shown quite clearly by the left column of many-place numbers on the obverse of M 46550, together with the column of single-place numbers in the right margin, that

$$n = 1\ 01\ 02\ 06\ 33\ 45 = 30^8 \cdot 5^6.$$

(This expression of *n* as the product of the 8th power of 30 and the 6th power of 5 is, of course, anachronistic. Less anachronistically, it can be understood as something like "*n* is the product of 8 copies of 30 and 6 copies of 5".) The result implies, clearly, that *n* = 1 01 02 06 33 45 is a regular many-place sexagesimal number.

The kind of factorization of the given number *n* = 1 01 02 06 33 45 achieved in the first column on the obverse of BM 46550 through repeated multiplications by 2 and 12 is, apparently, based on the following important insights:

a) every power of 5 is a sexagesimal number with the last place 05 or 25;
b) every power of 3 is a sexagesimal number with the last place 03, 09, 27, or 21;
c) every power of 3 or 5 (or, less specifically, every odd number) times a power of 30 is a sexagesimal number with the last place 30, 15, or 45.

and, conversely,

a) every sexagesimal number with the last place 5 or 25 has 5 as a factor, and a factor 5 can be removed from any such number through multiplication by 12;
b) every sexagesimal number with the last place 03, 09, 27, or 21 has 3 as a factor, and a factor 3 can be removed from any such number through multiplication by 20;
c) every sexagesimal number with the last place 15, 30, or 45 has 15 as a factor, and a factor 15 can be removed from any such number by means of two doublings (multiplications by 2).

For these reasons, it is clear, in particular, that every regular sexagesimal number that can be constructed as a power of 5 multiplied by a power of 30 ("a mixed power of 5 and 30") can be factorized by use of the same kind of "last place factorization algorithm" as the one used on the obverse of BM 46550 for the factorization of *n* = 1 01 02 06 33 45.

2.1.4 An Application of an Inverse Last Place Factorization Algorithm

Now consider the middle column of numbers on the obverse of BM 46550. While the numbers in the left column were written in a motion downwards from the first line, the numbers in the middle column were almost certainly written in a motion upwards from the last line. (Otherwise it would be difficult to explain why the top line of the middle column is not in line with the top line of the left column.) Thus, in lines 14-9, the sexagesimal numbers proceed from 12 to 12^6, and in lines 8-1, the sexagesimal numbers proceed from 12^6 to $2^8 \cdot 12^6$. The number in each line of the middle column will automatically be the reciprocal of the number in the corresponding line of the left column. In particular, the number in the first line of the middle column will be the reciprocal of the initial number, so that

$$n = 1\ 01\ 02\ 06\ 33\ 45 = 30^8 \cdot 5^6, \qquad \text{rec. } n = 58\ 58\ 56\ 38\ 24 = 2^8 \cdot 12^6.$$

Evidently, this construction of the reciprocal was based on the following clever observation:

After any factorization of a regular sexagesimal number, the reciprocal of that number can be computed through "inverse factorization", as the product of the reciprocals of all the obtained factors.

2.1.5 *Last Place Traces in a Triaxial Index Grid*

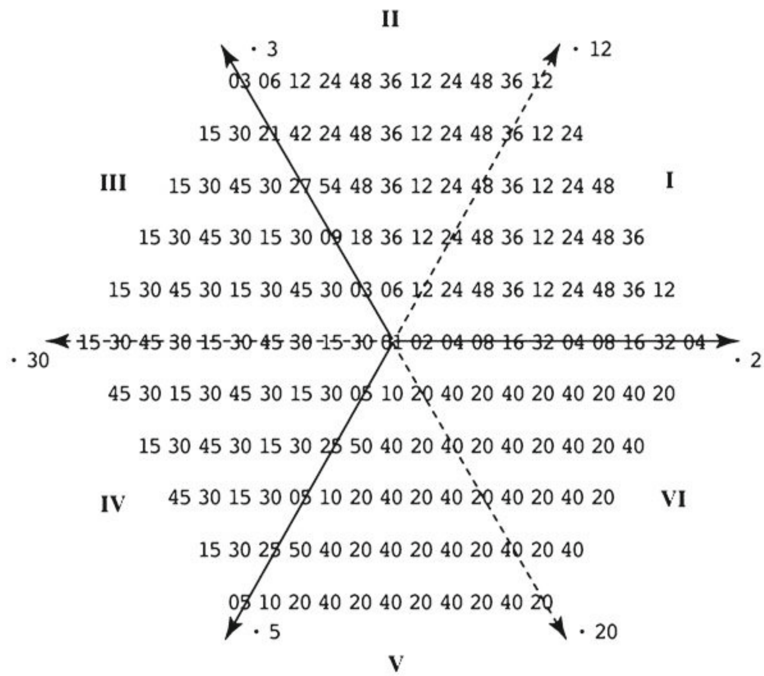


Fig. 2.1.3. Last places of regular sexagesimal numbers in a triaxial index grid.

In the left column of numbers on the obverse of BM 46550, all the numbers are what may be called “mixed powers” of 5 and 30. In the middle column of numbers, which are reciprocals of the numbers in the left column, all the numbers are correspondingly mixed powers of 2 and 12. This observation can be taken as an indication that the person who devised the exercise on BM 46550 may have been aware of the possibility to divide all regular sexagesimal numbers (in Babylonian relative place value notation without final double zeros) into six classes, namely those that can be written as

I: mixed powers of 2 and 12,
II: mixed powers of 3 and 12, or 2 times such mixed powers
III: mixed powers of 3 and 30,
IV: mixed powers of 5 and 30,
V: mixed powers of 5 and 20, or 2 times such mixed powers
VI: mixed powers of 2 and 20.

It is, indeed, easy to see that if the last place of a given regular sexagesimal number *n* is a multiple of 15 (thus, 15, 30, or 45), then *n* belongs to class III or IV and can be reduced to either a power of 3 or a power of 5 through multiplication by a power of 2. Similarly, if the last place of a given regular sexagesimal number *n* is a multiple of 6, other than 30 (thus, 06, 12, 18, 24, 36, 42, or 48), then *n* belongs to class I or class II and can be reduced to either a power of 2, or a power of 3, or 2 times a power of 3, through multiplication by a power of 5. Finally, if the last place of a given regular sexagesimal number *n* is a multiple of 10, other than 30 (thus, 10, 20, 40, or 50), then *n* belongs to class V or VI and can be reduced to either a power of 2 or a power of 5, or 2 times a power of 5, through multiplication by a power of 3. All this is visually demonstrated in Fig. 2.1.3 above. As a matter of fact, it was shown , in Friberg, *BaM* 30 (1999), following an idea due to O. Neugebauer, how regularly spaced points in a “triaxial index grid” can uniquely represent all regular sexagesimal numbers (without final double zeros). See Sec. 1.2.5 above, in particular Fig. 1.2.3.

In the example in Fig. 2.1.4 below it is shown how, for instance, points representing regular sexagesimal numbers belonging to class I (mixed powers of 2 and 12), marked by their last places, can be reached by proceeding first up along the “12 axis” (the inverse “5 axis”), then right parallel to the “2 axis”. Thus, all last places representing regular sexagesimal numbers belonging to class I lie in the sector of the index grid bounded by the 2 and 12 axes. All last places representing regular sexagesimal numbers belonging to class II lie in the sector of the index grid bounded by the 12 and 3 axes. And so on, for classes III-VI.

Note that the successive powers of 12, namely 12, 2 24, 28 48, 5 45 36, 1 09 07 12, *etc*., belong simultaneously to class I and class II. In the index grid in Fig. 2.1.3 above, they are marked along the "12 axis" (the inverse 5 axis) by their last places 12, 24, 48, 36, 12, *etc*. Similarly, the successive powers of 20, namely 20, 6 40, 2 13 20, 44 26 40, *etc*., belong simultaneously to class V and class VI. In the index grid in Fig. 2.1.3 above, they are marked along the "20 axis" (the inverse 3 axis) by their last places 20, 40, 20, *etc*.

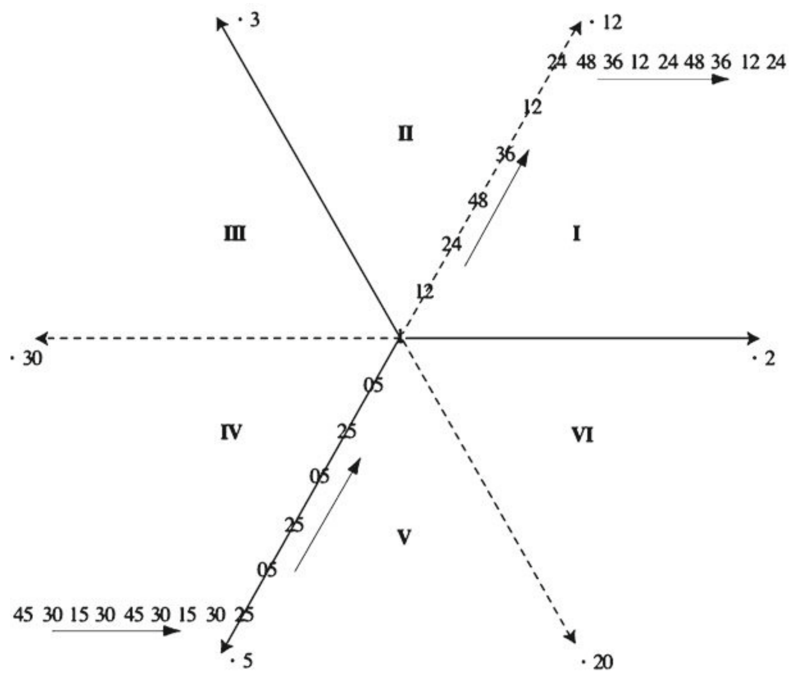


Fig. 2.1.4. The last place trace in the index grid of the algorithmic sequences on BM 46550, *obv*.

In the triaxial index grid in Fig. 2.1.4 above, the locations of the points representing the regular sexagesimal numbers of class IV in the left column on BM 46550, *obv*. are marked with the last places of those numbers, beginning with the last place 45 of n = 1 01 02 06 33 45 = $30^8 \cdot 56$. It is easy to see in the index grid how points representing the first nine numbers of the left column proceed in the direction of the 2 axis, and how the points representing the six last numbers of the column proceed in the direction of the 12 axis. The procedure stops when the point at the center of the index grid has been reached. It represents the number '1'.

The last places in the index grid representing the reciprocals in the middle column on BM 46550, *obv*., which are regular sexagesimal numbers of class I, proceed in a similar way, first along the 12 axis, then in the direction of the 2 axis. Note that the locations of the last places representing regular sexagesimal numbers and their reciprocals, respectively, are mirror images of each other with respect to the central point of the index grid.

Taken together, all the last places in the index grid for the numbers in algorithmic sequences of regular sexagesimal numbers, like the numbers in the left and middle columns on BM 46550, *obv*., will be called the "last place trace" in the index grid of those sequences.

2.1.6 A Faulty Application of the Last Place Factorization Algorithm

The columns of numbers on the obverse of BM 46550 are neatly written, probably copied from a teacher's model text, while the text on the reverse is much messier, obviously computed and written without a model text, by a rather inept student.

Consider the transliteration below of the text on the reverse of BM 46550. Remember that on the obverse the given number and its computed reciprocal were

n = 1 01 02 06 33 45 = $30^8 \cdot 5^6$ rec. n = 58 58 56 38 24.

On the reverse, the given number and its computed reciprocal are instead

rec. n = 58 58 56 38 24 = $2^8 \cdot 12^6$ rec. rec. n = 1 01 02 06 33 45.

BM 46550, *rev*. (corrected)

58 [58 56 38 24]	[1 01 02] 06 33 45		
30			
29 29 [28 19 12]	[2 02] 04 13 07 30		
5			error: 13 instead of 03
2 27 27 21 36 —	24 24 50 37 30 —	10!	
24 34 33 36 —	2 26 29 03 45 —	10	
4 05 45 36 —	14 38 54 22 30 —	10	
40 57 36 —	1 27 53 26 15 —	10	
6 49 36 —	8 47 20 37 30 —	10	error: 37 instead of 3
1 08 16 —	52 44 03 45 —	30	
34 08 —	1 45 28 07 30 —	30	
17 04 —	3 30 56 15 —	30	
8 32 —	7 01 52 30 —	30	
4 16 —	14 03 45 —	30	
2 08 —	28 07 30 —	30	
1 04 —	56 15 —	30	
32 —	1 52 30 —	30	
16 —	3 45 —	30	
8! —	7 30 —	30	
4 —	15 —	30	
2 —	30 —	30	
1 —	1–30		

This is, obviously, an example of the easily observed rule, mentioned above, that

The reciprocal of the reciprocal of a regular sexagesimal number is that number.

Clearly, the left column of numbers on BM 46550, *rev*., can be explained as a factorization algorithm for rec. n = 58 58 56 38 24, while the middle column of numbers can be explained as a corresponding inverse factorization algorithm, used for the construction of rec. rec. n, the reciprocal of the reciprocal of n = 1 01 02 06 33 45.

Surprisingly, however, the factorization in the left column of numbers on the reverse proceeds in a quite unsystematic way. This is obvious already from the column of single-place multipliers listed close to the right edge. There are three different multipliers, first 30 and 5, then 10, five times, and then 30, 13 times.

Note that the last notation of 30 is a mistake. In the last line of the inscription on the reverse there should only be a number 1, finishing the left column of numbers, and a second number 1, starting the middle column of numbers (going upwards), but no number 30.

The factorization of rec. *n* in the left column on the reverse, in 19 steps, can be understood as follows:

rec. n = 58 58 56 38 24 = $2 \cdot 12 \cdot 6^5 \cdot 2^{12}$. (Cf. the simpler factorization rec. $n = 2^8 \cdot 12^6$ on the obverse.)

Similarly, the factorization of rec. rec. *n* in the middle column can be understood as

rec. rec. n = 1 01 0206 33 45 = $30 \cdot 5 \cdot 10^5 \cdot 30^{12}$. (Cf. The simpler factorization $n = 30^8 \cdot 5^6$ on the obverse.)

The unsystematic procedure on the reverse is clearly visible in the corresponding last place trace in the triaxial index grid (Fig. 2.1.5 below). The correct way to proceed in this situation would have been to factorize rec. *n* = 58 58 56 38 24 through a series of six multiplications by 5, followed by eight multiplications by 30. The corresponding last place trace in the index grid (in gray in Fig. 2.1.5) would have proceeded first in the direction of the 5-axis, then of the 30-axis, in only 14 steps altogether instead of 19!

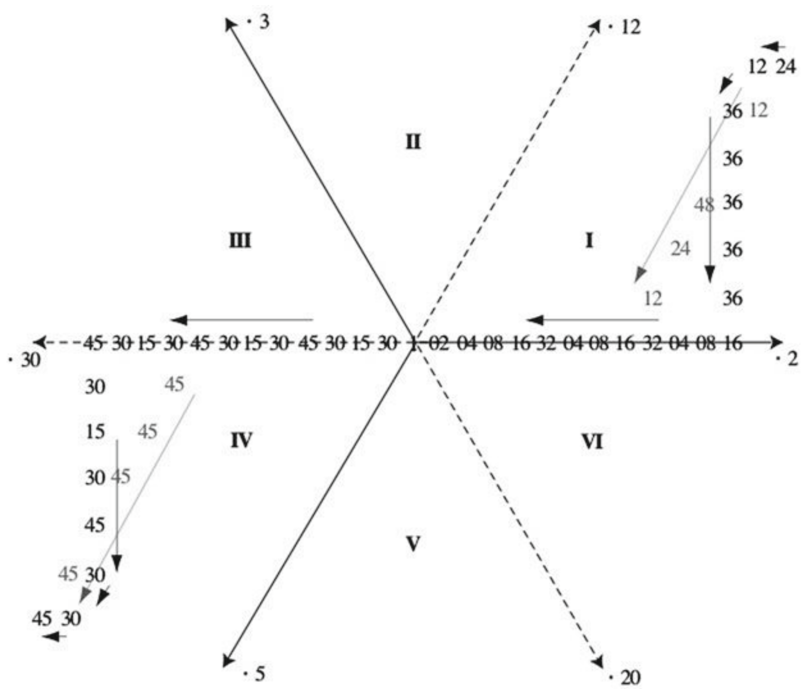


Fig. 2.1.5. The last place trace in the index grid of the algorithmic computations on BM 46550, *rev*.

2.1.7 On the Choice of the Initial Number n = 1 01 02 06 33 45

It is clear that the choice of *n* = 1 01 02 06 33 45 as the initial regular sexagesimal many-place number on the reverse of BM 46550 cannot have been accidental. Indeed, this number is closer to 1 than nearly all other 6-place regular sexagesimal numbers (in relative place value notation), since it is the number *n* in the *second* line of the reconstructed Late Babylonian Table R of many-place reciprocal pairs! (Sec. 1.2.3.)

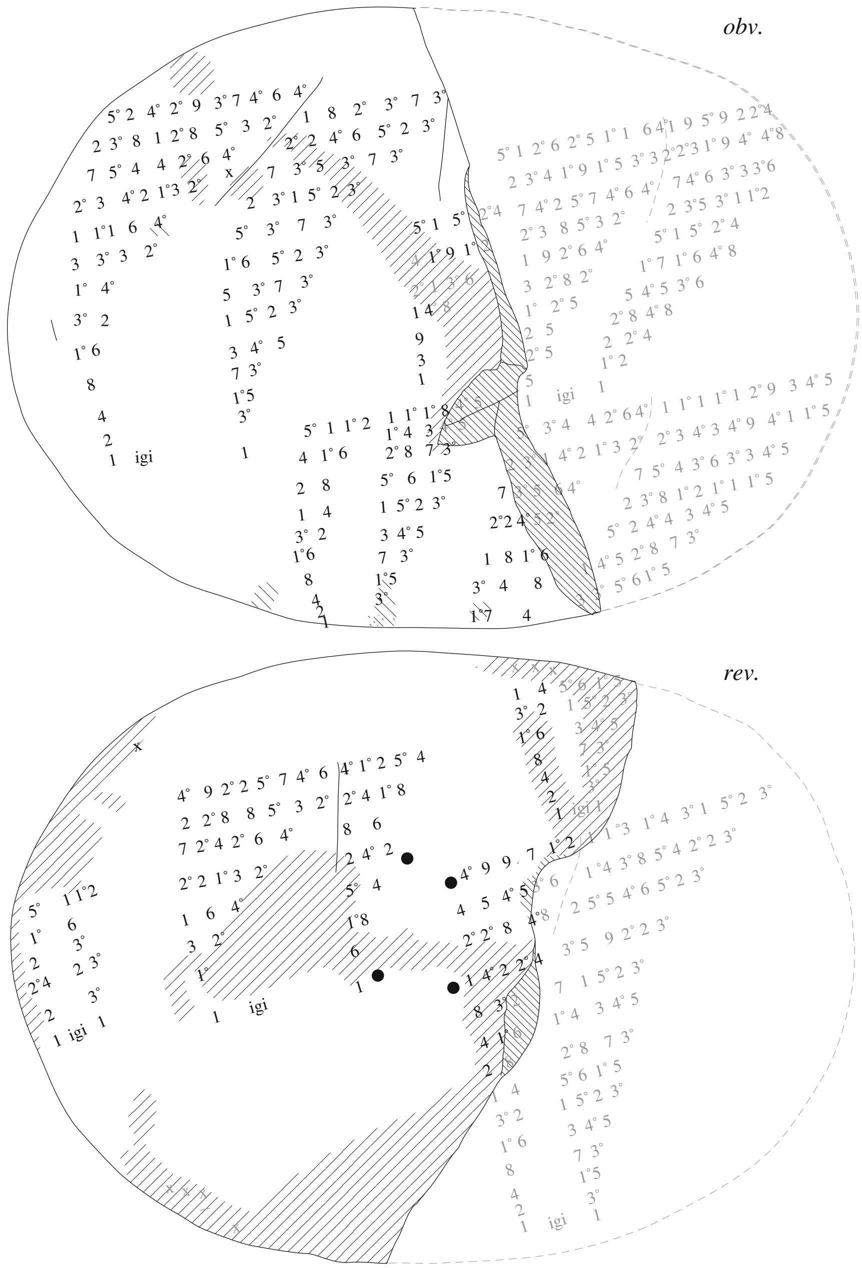


Fig. 2.1.6. W 23021. Conform transliteration, with a plausible reconstruction of lost parts of the text.

In this connection it is natural to mention also W 23021 (Friberg, *BaM* 30 (1999); Fig. 2.1.6 above) and W 23016 (Friberg, *MSCT 1* (2007), 453; Fig. 2.1.8 below). These are two Late Babylonian round tablets from Uruk, first published by E. von Weiher as *Uruk 4,* 175 (1993) and *Uruk 5*, 316 (1998), respectively.

In the original version of the larger text W 23021, there appears to have been eight algorithmic calculations of reciprocals (of reciprocals) of many-place regular sexagesimal numbers, all of the same carefully considered kind as the calculation on BM 46550, *obv.* According to the reconstruction of the text proposed in Friberg, *BaM* 30 (1999), the eight pairs of reciprocals computed on W 23021 are, in this order,

52 40 29 37 46 40	1 08 20 37 30	A
51 50 24	(1 09 26 40)	B

[51 26 25 11 06 40	1 09 59 02 24]	[C]
51 12	1 10 18 45	D
50 34 04 26 40	1 11 11 29 03 45	E
50	1 12	F
49 22 57 47 40	1 12 54	G
49 09 07 12	1 13 14 31 52 30	H

These eight pairs of reciprocals (of reciprocals) were almost certainly picked from the mentioned Late Babylonian Table R of many-place reciprocal pairs. Indeed, consider the following 11 consecutive pairs of reciprocals near the beginning of Table R (see again Table 1.3 in Sec. 1.2.3 above):

1 08 20 37 30	52 40 29 37 46 40	$= 2^5 \cdot 20^7$	class VI	A		*l.* 20
1 09 07 12	52 05	$= 5^5$	IV		K	*l.* 21
1 09 26 40	51 50 24	$= 3^2 \cdot 12^4$	II	B		*l.* 22
1 09 59 02 24	51 26 25 11 06 40	$= 5^4 \cdot 20^6$	V	C		*l.* 23
1 10 18 45	51 12	$= 2^8 \cdot 12^1$	I	D		*l.* 24
1 11 06 40	50 37 30	$= 3^3 \cdot 30^5$	III		L	*l.* 25
1 11 11 29 03 45	50 34 04 26 40	$= 2^1 \cdot 20^4$	VI	E		*l.* 26
1 12	50	$= 2 \cdot 5^2$	V	F		*l.* 27
1 12 49 04	49 26 18 30 56 15	$= 30^{18}$	III		M	*l.* 28
1 12 54	49 22 57 47 40	$= 2 \cdot 5^1 \cdot 20^6$	V	G		*l.* 29
1 13 09 34 29 08 08 53 20	49 12 27	$= 3^{11}$	III		N	*l.* 30
1 13 14 31 52 30	49 09 07 12	$= 2^9 \cdot 12^4$	I	H		*l.* 31

An explanation of why only the pairs A-H but not the pairs K-N in the list above appear in the calculations on W 23021 was suggested in Friberg, *BaM* 30 (1999), 156-157. The explanation is based on the observation that in all the excluded pairs, the initial number (52 05, etc.) is a power of 3 or 5 multiplied by a power of 30, hence of class III or IV. In other words, only pairs with initial number belonging (exclusively) to the classes I, II, V, or VI are considered on W 23021. Presumably, the excluded pairs belonging to classes III and IV had already been dealt with in some other way, on other tablets.

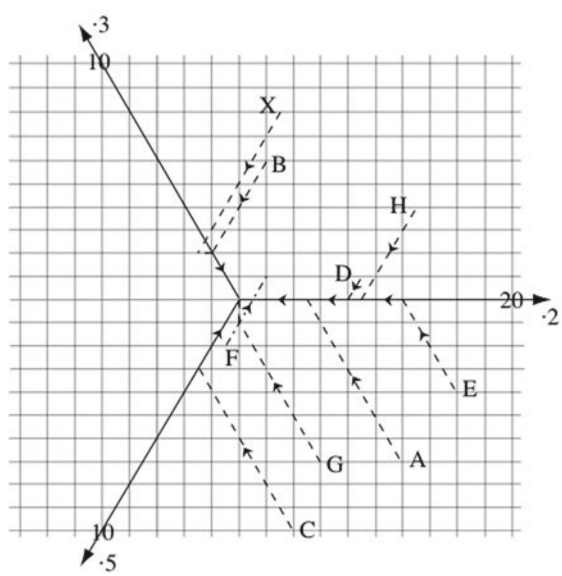


Fig. 2.1.7. Last place traces in the triaxial index grid of the eight factorizations on W 23021.

In Fig. 2.1.7 above are shown the last place traces in the triaxial index grid of the eight factorizations on W 23021 (called A-H), and of the single factorization on W 23016 (called X; see below, Fig. 2.1.8). All these traces take the best way to the central point in the index grid, the point representing the number ‘1’, with one or two exceptions: In the trace tagged X, as will be mentioned below, the two steps from 54 = 6 · 9 to 9 can be replaced by a single step (a multiplication by 10), and in the trace tagged F, the four steps from 10 to 1 can be replaced by a single step (a multiplication by 6), or possibly two steps (multiplications by 12 and 30).

Note: In Robson, *MAI* (2008), 367, fn. 41, it is groundlessly claimed that the reconstruction of the pair [C] proposed in Friberg, *BaM* 30 is incorrect and based on a “much later” table of reciprocals, whatever that is supposed to mean. Robson’s idea that the reconstructed number [51 26 25 11 06 40] (class V) should be replaced by the shorter number [52 05] (class IV) violates the explanation given above, namely that numbers belonging to classes III and IV were avoided, and does not consider the space available in the upper right quarter of the obverse of W 23021 for the corresponding reconstructed columns of numbers. Actually (see Fig. 2.1.6 above), the reconstructed columns for the initial number [51 26 25 11 06 40] (pair [C]) fill out the entire upper right quarter of the obverse of W 23021. The single column for the initial number 51 50 25 (B) was squeezed in between the columns for pairs A and [C], probably as a correction to an unintended omission. Note the vertical line pointing down to this brief, squeezed-in, column of numbers, and note that there was no space available for a second column of numbers, with the computation of rec. 51 50 25!

1 02 12 28 48!	· 5	
5 11 02 24	· 5	
25 55! 12	· 5	
2 09! 36!	· 5	
10 48!	· 5	
54	· 5	· 10
4 30	· 2	
9 (erasure)	· [20]	
3	· 20!	
1!		

Fig. 2.1.8. W 23016. A Late Babylonian factorization algorithm for n = 1 02 12 28 48 = 2 · $125^{5} \cdot 3^{2}$.
Hand copy: von Weiher.

The small Late Babylonian tablet W 23016 (von Weiher, *Uruk 5*, 316; Fig. 2.1.8 above) is inscribed with a last place factorization algorithm for the 5-place regular sexagesimal number n = 1 02 12 28 48 (Table R, line 6). The multipliers used in the successive steps of the algorithm are not indicated on W 23016, as they were in the case of the similar factorization algorithms on the obverse and the reverse of BM 46550. (For the readers’ convenience, however, they are indicated in the transliteration to the right in Fig. 2.1.8)

The steps of the factorization algorithm correspond to the factorizations

n = 1 02 12 28 48 = $12^6 \cdot 30 \cdot 3^2$, rec. $n = 5^6 \cdot 2 \cdot 20^2$.

Incidentally, the factorization algorithm in this case is not optimal. The reduction of 54 into 9 could have been achieved in one step, through multiplication by 10, instead of in two steps through multiplication first by 5 and then by 2!

Note that the many numerical errors in the text of W 23016 obviously are copying errors, since they do not influence the course of the computation.

2.1.8 Errors in BM 46550

Apparently, the student who wrote BM 46550 found the job of copying numbers from the teacher's model text so boring that he listlessly drew a doodle at the bottom of the obverse, and also scratched away the smooth surface of the lower obverse in a bow, obliterating the numbers 12 and 1 at the two bottom lines of the middle column.

In the first column on the obverse of BM 46550 there are two errors: The number in line 2 ends with 4 30 instead of 7 30, and the number in line 8 begins with 3 instead of 2. The fact that these errors do not give rise to propagated errors in the following lines implicates that the numbers on the obverse were copied from another clay tablet, not computed just before they were written down. Similarly, the numerical error in line 9 of the first column on the reverse does not give rise to a propagated error in line 10 of the first column, and the numerical error in line 4 of the second column on the reverse does not give rise to a propagated error in line 3 of the second column. The only conceivable explanation for this phenomenon is that the student had first performed all the necessary computations on a scratch pad before he copied the result onto the reverse of BM 46550.

The inscription on the obverse of the tablet is clearly much neater than the awkwardly executed inscription on the reverse. Therefore, as mentioned above, it is likely that while the columns of numbers on the obverse, with the algorithms for the calculation of rec. *n*, were copied from the teacher's model text, the columns on the reverse with a similar computation of rec. rec. *n*, were written by the student without any help from the teacher. Proceeding unsystematically, as mentioned above, the student produced an algorithmic computation of rec. rec. *n* in 19 steps on the reverse instead of the 14 steps in the computation of rec. *n* on the obverse. As a result, the lines of numbers in the left column on the reverse were written in a cramped and messy fashion. Moreover, when the student wrote down the numbers in the third column on the reverse, he went one step too far, and wrote down a superfluous number 30 in the last line of the column. Note, in this connection, that on the lower edge of the reverse of the tablet, the number 1 which should rightly be written at the bottom of the middle column is written too far to the right, close to the superfluous number 30.

Finally, realizing that it might be difficult to see which lines of the left and right columns should correspond to lines of the middle column, the student connected related numbers to each other by awkwardly drawing lines in the clay with a finger nail.

2.2 CBS 1215 and the Old Babylonian Trailing Part Factorization Algorithm

The Neo-/Late Babylonian “last place factorization algorithm” discussed above, exemplified by the computations on BM 46550, is a simplified version of a closely related Old Babylonian “trailing part factorization algorithm”, used for the same purpose, namely to calculate the reciprocal of a given many-place regular sexagesimal number as the product of the reciprocals of factors of that number.

The trailing part factorization algorithm is discussed in Friberg, *MSCT 1* (2007), App. A3.3. In Fig. A3.4 there, a transliteration to modern number signs is given of all the 21 sub-algorithms of the Old Babylonian algorithm table CBS 1215. In CBS 1215 # 1, for instance, the first and simplest of the 21 sub-algorithms, it is shown in the following concise way that the reciprocal of the regular number 2 05 (= 5^3) is 28 48, and that, conversely, the reciprocal of 28 48 is 2 05.

CBS 1215 # 1

2 05	12
25	2 24
28 48	1 15
36	1 40
2 05	

rec. 5 = 12,	2 05 · 12 = 25
rec. 25 = 2 24,	2 24 · 12 = 28 48 = rec. 2 05
rec 48 = 1 15	28 48 · 1 15 = 36
rec. 36 = 1 40	1 40 · 1 15 = 2 05 = rec. rec. 2 05

A “trailing part” of a sexagesimal number in Babylonian relative place value notation consists of the last one or two sexagesimal places (sometimes more), depending on the circumstances. In the first line of this small algorithm table, 5 (or, rather, 05) is a “trailing part” (as well as the last sexagesimal place) of the regular number 5^3 = 2 05, and 12 is the reciprocal of 5. Therefore, in Babylonian sexagesimal notation with relative values,

2 05 / 5 = 2 05 · 12 = 2 (00) · 12 + 5 · 12 = 24 + 1 = 25.

In the next step of the algorithm, it is noted that rec. 25 = 2 24. (This was well known, of course, since it is a line in the Old Babylonian standard table of reciprocals.) Therefore, as indicated to the left in line 3,

rec. 2 05 = rec. (5 · 25) = rec. 5 · rec. 25 = 12 · 2 24 = 28 48.

To the right in the same line is written the reciprocal 1 15 of the trailing part 48 of 28 48. The next step of the algorithm is to multiply 28 48 by 1 15. The result is that

28 48 · 1 15 = 28 (00) · 1 15 + 48 · 1 15 = 35 + 1 = 36.

In the fourth line of the small algorithm table above, it is noted to the left that 28 48 · 1 15 = 36, and to the left that the reciprocal of 36 is 1 40. Therefore,

rec. rec. 2 05 = rec. 28 48 = rec. 48 · rec. 36 = 1 15 · 1 40 = 2 05.

The reciprocal of 28 48, computed in this way, is written in the final line of the small algorithm table. It is, of course, equal to the initial number 2 05.

CBS 1215 ## 2-21 are just like CBS 1215 # 1, but with the initial regular sexagesimal number 5^3 = 2 05 multiplied by increasing powers of 2. In the last of these examples, # 21 (see below), the initial number is

$5^3 \cdot 2^{20}$ = 10 06 48 53 20.

Through application of the trailing part algorithm, it is shown that this number can be factorized as follows:

$5^3 \cdot 2^{20}$ = 10 06 48 53 20 = 3 20 · 2 40 · 16 · 16 · 16.

Consequently, since rec. 16 = 3 45 (three times), rec. 2 40 = 22 30, and rec. 3 20 = 18, the reciprocal number can be computed as follows, in inverse order:

rec. 10 06 48 53 20 = 3 45 · 3 45 · 3 45 · 22 30 · 18 = 5 55 57 25 18 45.

In the second half of CBS 1215 # 21, the reciprocal of the reciprocal is computed in a similar way.

CBS 1215 # 21

1	**10 06 48 53 20**	18	rec. 3 20 = 18,	18 · 10 06 48 53 20 = 3 02 02 40
2	3 02 02 40	22 30	rec. 2 40 = 22 30,	22 30 · 3 02 02 40 = 1 08 16
3	1 08 16	3 45	rec. 16 = 3 45,	3 45 · 1 08 16 = 4 16
4	4 16	3 45	rec. 16 = 3 45,	3 45 · 4 16 = 16
5	16	3 45	rec. 16 = 3 45,	3 45 · 3 45 = 14 03 45
6	14 03 45			3 45 · 14 03 45 = 52 44 03 45
7	52 44 03 45			22 30 · 52 44 03 45 = 19 46 31 24 22 30
8	19 46 31 24 22 30			18 · 19 46 31 24 22 30 = **5 55 57 25 18 45**
9	**5 55 57 25 18 45**	16	rec. 3 45 = 16,	16 · 5 55 57 25 18 45 = 1 34 55 18 45
10	1 34 55 18 45	16	rec. 3 45 = 16,	16 · 1 34 55 18 45 = 25 18 45
11	25 18 45	16	rec. 3 45 = 16,	16 · 25 18 45 = 6 45
12	6 45	1 20	rec. 45 = 1 20,	1 20 · 6 45 = 9
13	9	6 40	rec. 9 = 6 40,	6 40 · 1 20 = 8 53 20
14	8 53 20			16 · 8 53 20 = 2 22 13 20
15	2 22 13 20			16 · 2 22 13 20 = 37 55 33 20
16	37 55 33 20			16 · 37 55 33 20 = **10 06 48 53 20**
17	**10 06 48 53 20**			

An alert reader may notice that in lines 9-11 the indicated multiplier 16 is neither the reciprocal of the trailing part 18 45, nor the reciprocal of the last place 45. What is going on here is that the author of this text seems to have noticed that 16 is the reciprocal of 3 45, and that 3 45 is a factor in 18 45. Indeed,

18 45 = 15 · 60 + 3 45 = 15 · 15 · 4 + 3 45 = 3 45 · (4 + 1) = 3 45 · 5.

Therefore, in Babylonian relative place value notation without final zeros, 16 · 18 45 = 5, so that 16 is a reciprocal of 18 45 in a generalized sense. It follows that, for instance, as in lines 9-10,

16 · 5 55 57 25 18 45 = 16 · 5 55 57 25 (00 00) + 16 · 18 45 = 1 34 55 18 40 + 5 = 1 34 55 18 45.

Note that, despite the different layouts, and despite the difference between the last place algorithm and the trailing part algorithm, there is still a striking similarity between a combination of the calculations on the obverse and on the reverse of the Neo-Babylonian text BM 46550 on one hand and in the Old Babylonian exercise CBS 1215 # 21 on the other hand. In both cases there are similar chains of four successive algorithmic computations:

1. The successive eliminations of factors in the given number *n*, with the reciprocals noted in the right margin.
2. The computation of rec. *n* as the product of the reciprocals of the eliminated factors.
3. The successive eliminations of factors in rec. *n*, with the reciprocals of the eliminated factors noted in the right margin.
4. The computation of rec. rec. *n* as the product of the reciprocals of the eliminated factors.

It is tempting to assume that there must have been a historical connection of some kind between the two algorithmic procedures, which so obviously share a common idea!

Note, by the way, that Robson, in her *MAI* (2008), 237, is of a diametrically opposite opinion, when she writes as follows, in her commentary to the Late Babylonian last place factorization algorithm in W 23021: “Thus once again, a favorite Old Babylonian school problem is solved using a new method: not cut-and-paste procedures but repeated factorization.” However, the alleged cut-and-paste procedure in question is Robson’s own proposed explanation in geometric terms (*op. cit.*, 109) of the computations in the Old Babylonian text CBS 1215, an explanation which is totally hypothetical, unwarranted, and unpractical. Besides, since the Neo- or Late Babylonian last place factorization algorithm is just a simplified variant of the Old Babylonian trailing part factorization algorithm, Robson’s hypothetical cut-and-paste procedure can in no way make a distinction between Old and Late Babylonian factorization algorithms.

Note, by the way, that an independent interesting discussion of CBS 1215, BM 46550, and other related texts can be found in Proust (2012) “Interpretation of reverse algorithms in several Mesopotamian texts”.

2.3 Reconstructions of Factorization Algorithms on Three Seleucid Tablet Fragments

The hand copies in Fig. 2.3.1 below of the three small fragments *LBAT* 1642, 1643, and 1646 = fragments VI D - VI F, presumably from some Seleucid site in Babylon or Sippar, were made by Pinches between 1895 and 1900, and were published by Sachs in *LBAT* (1955). Sachs presumed that the three fragments were "mathematical" and suspected that *LBAT* 1642 was an "analysis of reciprocal (factor: 12)", but had no explanation to offer for *LBAT* 1643 and 1646. (The fragment *LBAT* 1644 = fragment V A, which Sachs also classified as an "analysis of reciprocal", was ultimately shown to be an explicit multiplication algorithm for the computation of the 25-place square of the square of a 7-place regular sexagesimal number. See Friberg, *MSCT 1* (2007), App. A9.2.)

A necessary prerequisite for any attempted explanation of small fragments with preserved traces of many-place (presumably) regular sexagesimal numbers is an extensive and systematically constructed table of many-place regular sexagesimal numbers. Neugebauer produced a table of precisely this kind, namely a table of 6-place regular numbers between 1 and 2, and their reciprocals, computed by hand, for his study in *MKT I* (1935) of the Seleucid many-place table of reciprocals AO 6456 = Table V, discussed in Sec. 1.5 above.

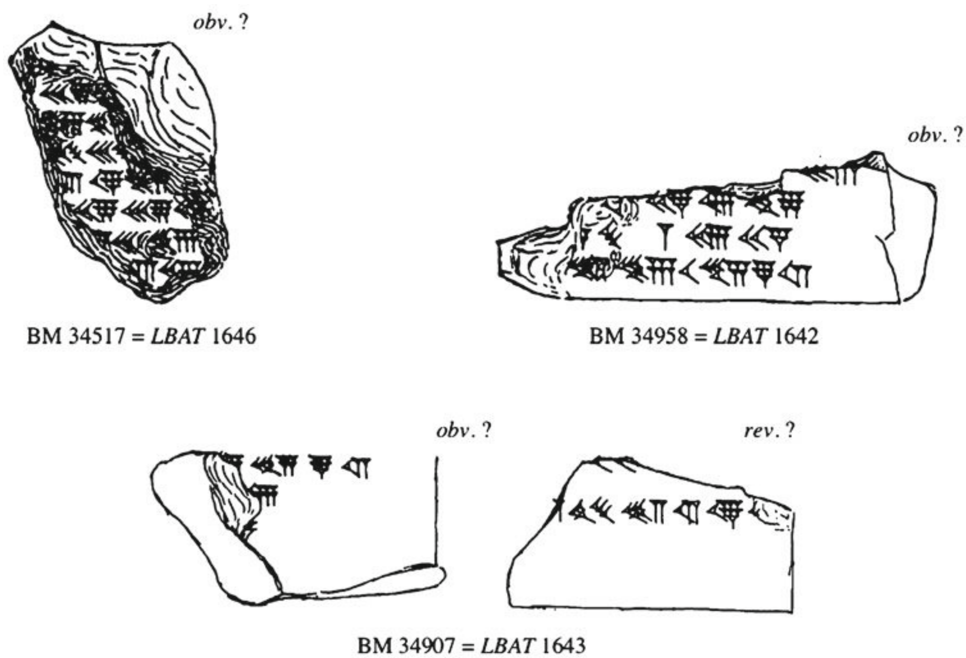


Fig. 2.3.1. Three small fragments of Late Babylonian texts with many-place sexagesimal numbers.

Later, an even more extensive lexicographically ordered table of 11-place regular sexagesimal numbers and their reciprocals, computed electronically (long before the advent of personal computers!), was published by Gingerich in *TAPS* 55 (1965). Note, however, that in a large number of cases the reciprocals of the 11-place regular numbers in Gingerich's table may be (much) more than 11-place!

It is now, of course, possible for anyone to produce, with some effort, a new extensive table of a similar kind by use of a suitable computer program, even if a considerable remaining difficulty is to order the very large number of computed many-place numbers in some convenient way (for instance lexicographically). However, this added difficulty did not discourage the anonymous Neo- or Late Babylonian student of mathematics who computed, by hand, the world's first systematically ordered table of 6- place regular sexagesimal numbers, and their reciprocals (Table R, discussed in Sec. 1.2.3 above).

2.3.1 BM 34517. A Descending Table of Powers of 9

Consider the fragment VI D = BM 34517, shown in Fig. 2.3.1, top left. It is inscribed with parts of 7 many-place sexagesimal numbers. Unfortunately, only 3 places (or less) in the middle of these many-place numbers are preserved. Nevertheless, a sensible approach could be to look in an extensive table of many-place regular sexagesimal numbers, such as Gingerich's, for many-place regular sexagesimal numbers containing such 2- or 3-place pieces of many-place sexagesimal numbers as, in this case,

[...] 56 50 [...], [...]9 39 [...], [...]3 17 55 [...], [...] 28 38 [...], [...] 30 56 [...], and [...]3 26 [...].

Once a list of possible candidates has been found, another task is to find which of these candidates can be explained as consecutive numbers, in some mathematically appropriate sense.

The reader probably doesn't want to be bothered with too many details about how the search proceeded. Instead, here is without delay the result of the analysis of the fragment BM 34517. Thus, for instance, the number, of which a small piece is preserved in the last line of the fragment, is

[20 06 23 36 04] 03 26 [14 40 13 21] = the 20th power of 9.

This 11-place number is actually present in Gingerich's table. Its reciprocal is a 30-place number! All the other numbers of which traces are preserved on BM 34517 are also powers of 9, but are more than 11-place. Indeed,

[49 28 10 48 09 26] 32 [50 19 46 38 45 37 21]	$= 9^{26}$	14-pl.
[5 29 47 52 01 02] 56 5[8 55 31 50 58 24 09]	$= 9^{25}$	14-pl.
[36 38 39 06 46] 59 39 [52 50 12 19 49 21]	$= 9^{24}$	13-pl.
[4 04 17 40 45 1]3 18 45 [52 14 42 12 09]	$= 9^{23}$	13-pl.
[27 08 37 51 41] 28 38 25 [48 18 01 21]	$= 9^{22}$	12-pl.
[3 00 57 32 24] 36 30 56 [12 02 00 09]	$= 9^{21}$	12-pl.
[20 06 23 36 04] 03 26 [14 40 13 21]	$= 9^{20}$	11-pl.

(Note the confirmation of the doubtful reading 32 in line 1.)

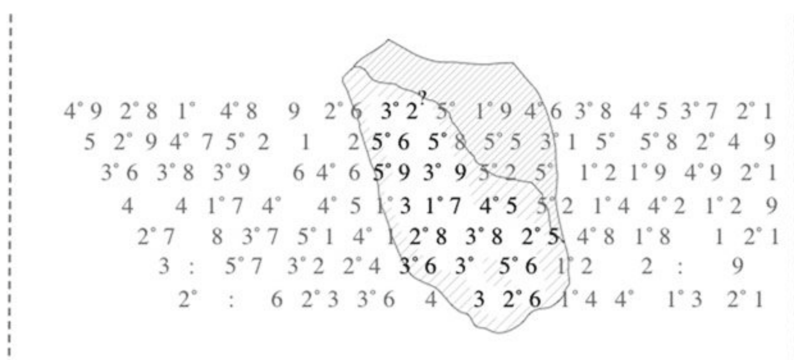


Fig. 2.3.2. Fragment VI D = BM 34517. Proposed reconstruction.

Thus, all the sexagesimal places preserved on the fragment BM 34517 can be explained as traces of a descending table of powers of 9, from the 26th to the 20th. The 26th power of 9 is a 14-place regular sexagesimal number. It is, of course, impossible to know if the computation partially preserved on BM 34517 was just a factorization of some very high power of 9, or if it also comprised the computation of the reciprocal of that power, which can be estimated to be a more than 30-place regular sexagesimal number.

Note, by the way, that a modified version of the "last place factorization algorithm" must be used here, since the numbers representing the descending table of powers of 9 have alternately the last sexagesimal places 09 and 21, where 21 is a non-regular sexagesimal number! However, any number with its last place equal to 21 and with the sexagesimal place before it equal to 37, 49, 01, or 13, as in lines 1, 3, 5, and 7 on the fragment, obviously contains 9 as a factor.

The reason is that

[.....] 37 21 = [.....] 36 00 + 1 21
[.....] 49 21 = [.....] 48 00 + 1 21
[.....] 01 21 = [.....] 00 00 + 1 21
[.....] 13 21 = [.....] 12 00 + 1 21

where each of the terms into which the numbers are split contains 9 as a factor.

2.3.2 BM 34958. A Last Place Factorization Algorithm for a Large Number of Class I

Apparently, the small fragment VI F = BM 34958 (Fig, 2.3.1, top right) represents the lower right corner of the obverse(?) of a clay tablet originally inscribed with several many-place, presumably regular, sexagesimal numbers. Only the trailing parts of 4 sexagesimal numbers are preserved, but since the last places are 36, 48, 24, 12 in this order, a reasonable conjecture is that BM 34958 was inscribed with a last place factorization algorithm for a regular sexagesimal number of class I or II. Indeed, the last of the 4 partially preserved numbers is of the form

[.....] 5[5]$^?$ 56 10 45 07 12.

Now, if a number with the trailing part 56 10 45 07 12 is multiplied by 12, the result is

12 · [.....] 56 10 45 07 12 = [.....] 14 09 01 26 24.

This agrees well with the preserved portion of the preceding number on the fragment, which is

[.....4]$^?$ 09 01 26 24.

Similarly, if a number with the trailing part 14 09 01 26 24 is multiplied by 12, the result is

12 · [.....] 14 09 01 26 24 = [.....] 49 48 17 16 48.

This agrees well with the preserved portion of the next preceding number on the fragment, which is

[..... 17]$^?$ 16 48.

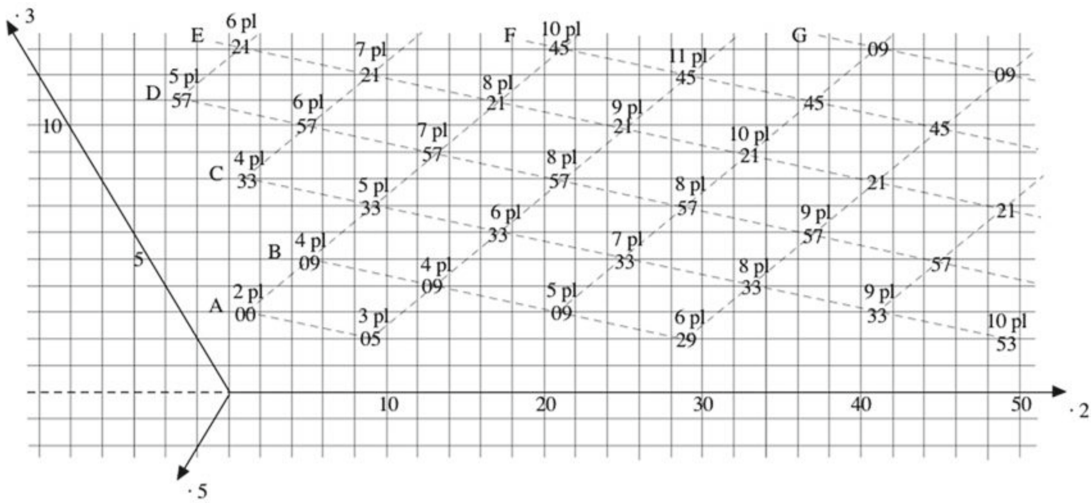
Therefore, as observed already by Sachs, in *LBAT* (1955), p. xxxviii, the last couple of steps of the algorithmic computation on the obverse(?) of the fragment clearly were a couple of divisions by 12 (or multiplications by 5).

To proceed with the analysis, a possible approach is to try to find a regular sexagesimal number with the trailing part 56 10 45 07 12. As it turns out, there are very few such numbers, proportionally, so a better idea is to start looking for regular sexagesimal numbers with the trailing part 10 45 07 12, or 45 07 12, or even only 07 12. Such numbers are not difficult to find, for instance by looking through Gingerich's table of 11-place regular sexagesimal numbers. A result of such a search for numbers with the trailing part 07 12 is the observation that the points in the index grid representing regular sexagesimal numbers with this trailing part are distributed evenly across the space between the 2-axis and the 3-axis.

In Fig. 2.3.3 below, all such points that can be found with a reasonable amount of work are represented in each case by the value of the "preceding place" (the sexagesimal place preceding 07 12). Evidently, all points representing numbers with the trailing part 07 12 are distributed in a two-dimensional lattice. Moreover, all such points situated on the line called B, for instance, have their "preceding places" equal to 09 (or in one case 29). Points on the line C have their preceding places equal to 33 (or in one case 53), points on the line D have their preceding places equal to 57, points on the line E have their preceding places equal to 21, and points on the line F have their preceding places equal to 45, the desired value. Note the regular increase of the values of the preceding places: the steps from 09 to 33, from 33 to 57, from 57 to [...] 21, and from 21 to 45 are all equal to 24.

So far, it has been established, in this heuristic way, that points representing regular sexagesimal numbers with the desired trailing part 45 07 12 are situated along the line F in the index grid, and, presumably, along other similar lines further out in the index grid. One of the points on the line F, the one with the index (34, 13), even represents a number with the desired trailing part 10 45 07 12.

The reasoning can be taken one or two steps further. In order to find more points representing numbers with the desired trailing part 10 45 07 12, one has to consider the “preceding places” of numbers with the trailing part 45 07 12, represented by points in the index grid. As shown in Fig. 2.3.4 below, the points representing numbers with the trailing part 45 07 12 are distributed in a two-dimensional lattice of a kind similar to the two-dimensional lattice in Fig. 2.3.3. Along the parallel lines called F, K, P in this lattice, the preceding places take (with a few exceptions) alternatingly the values 46, 34, 22, 10, and 58, each one of these values exceeding the next one by 12.



A	4, 3, (0)	7 12
	11, 2	5 07 12
B	10, 5	1 09 07 12
	17, 4	49 09 07 12
	24, 3	34 57 09 07 12
	31, 2	24 51 18 29 07 12
C	9, 8	15 33 07 12
	16, 7	11 03 33 07 12
	23, 6	7 51 51 33 07 12
	30, 5	5.35 32 39 33 07 12
	37, 4	3 58.36 33 27 33 07 12
	44, 3	2 49 40.39 47 35 33 07 12
	51, 2	2 00 39 34.57 50 36 53 07 12

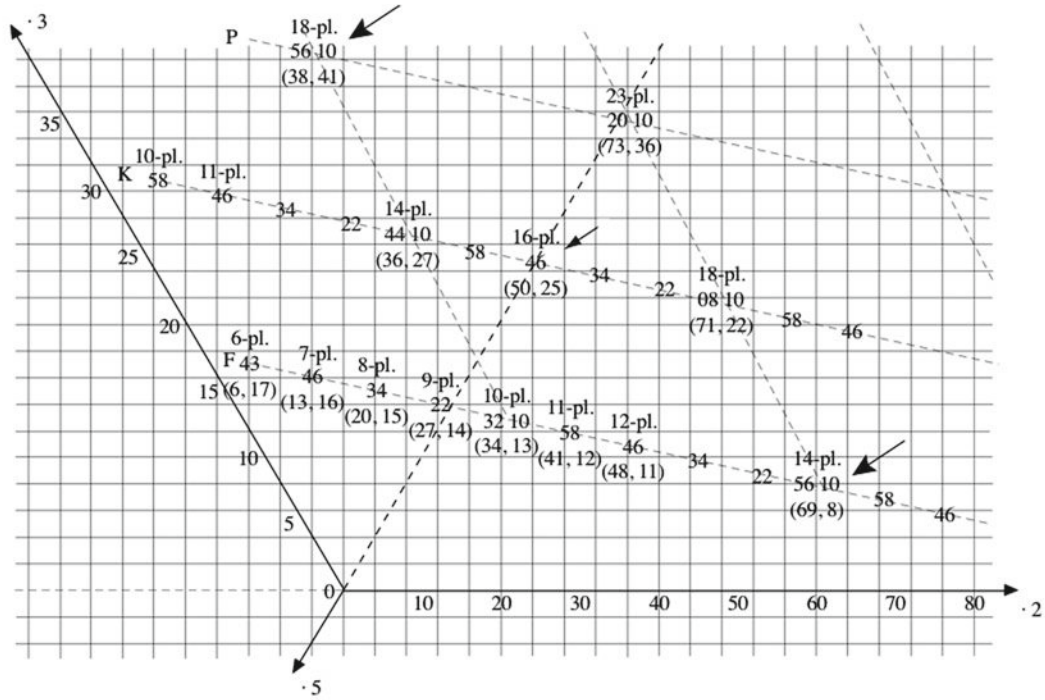
D	8, 11, (0)	3 29 57 07 12
	15, 10	2 29 17 57 07 12
	22, 9	1.46 10 05 57 07 12
	29, 8	1 15.29 50 53 57 07 12
	36, 7	53 41.13 31 41 57 07 12
	43, 6	38 10 38.57 12 29 57 07 12
E	14, 13	33 35 32 21 07 12
	21, 12	23 53 16 20 21 07 12
	28, 11	16 59.12 57 08 21 07 12
	35, 10	12 04 46.32 37 56 21 07 12
	42, 9	8 35 23 45.52 18 44 21 07 12
F	34, 13	2 43 04 28.30 32 **10 45 07 12**
	41, 12	1 55 57 50 49.16 12 58 **45 07 12**

Fig. 2.3.3. The lattice of points representing regular sexagesimal numbers with the trailing part 07 12.

It is also clear from Fig. 2.3.4 that points in the index grid representing numbers with the trailing part 10 45 07 12 are distributed in yet another two-dimensional lattice, and that in this case the preceding places alternatingly take the values 32, 44, 56, 08, 20, each one of these values being 12 less than the following one. The procedure manages to single out 6 points in the index grid representing regular sexagesimal numbers with the trailing part 10 45 07 12. In addition, two of these, marked with big black arrows, correspond to the trailing part 56 10 45 07 12. One of these, finally, with the index (69, 8) corresponds to a trailing part of precisely the desired form, namely 58[!] 56 10 45 07 12.

Thus, the procedure has not only produced a regular sexagesimal number of the desired form, it has also demonstrated (heuristically) that there is no other such number of a reasonable size! Consequently, the damaged number in the last line of the fragment BM 34958 can with certainty be reconstructed as

[29 39 11 59 28 12 08 12] 5[8] 56 10 45 07 12 = $2^{69} \cdot 3^{8} = 2^{53} \cdot 12^{8}$ 14-pl



F	6, 17, (0)	10 37 43 45 07 12
	13, 16	7 33 29 46 45 07 12
	20, 15	5 22 29 10 34 45 07 12
	27, 14	3 49 19 24 51 22 45 07 12
	34, 13	2 43 04 28 20 32 **10 45 07 12**
	41, 12	1 55 57 50 49 16 12 58 45 07 12
	48, 11	1 22 27 48 08 22 11 53 46 45 07 12
	55, 10	– – – –
	62, 9	– – – –
	69, 8	29 39 11 59 28 12 08 12 **58 56 10 45 07 12**

K	8, 31, (0)	15 41 25 54 26 15 58 45 07 12
	15, 30	11 09 27 45 22 40 41 46 45 07 12
	22, 29	– – – –
	29, 28	– – – –
	36, 27	4 00 44 04 58 03 51 14 57 44 **10 45 07 12**
	43, 26	– – – –
	50, 25	2 01 44 04 10 09 14 15 31 07 19 05 46 45 07 12
P	38, 41	5 55 22 42 58 02 59 33 21 58 14 52 34 **56 10 45 07 12**
	73, 36	– – – –

Fig. 2.3.4. The two-dimensional lattice of points representing regular sexagesimal numbers with the trailing part 45 07 12.

Repeatedly multiplying this reconstructed 14-place number by 12, one gets the corresponding reconstructed numbers in the two preceding lines:

12 · [29 39 11 59 28 12 08 12] 5[8] 56 10 45 07 12 = [5 55 50 23 53 38 25 38 35 47 1]4 09 01 26 24 = $2^{53} \cdot 12^{9}$ 15-pl

12 · [5 55 50 23 53 38 25 38 35 47 1]4 09 01 26 24 = [1 11 10 04 46 43 41 07 43 09 26 49 48] 17 16 48 = $2^{53} \cdot 12^{10}$ 16-pl

See the suggested reconstruction in Fig. 2. 3.5 below of the damaged text on the fragment.

If the four preserved trailing parts of many-place numbers on the fragment VI F = BM 34958 originally were positioned in the lower right corner of the *obverse* of some intact tablet, then it is possible that there originally was a continuation of the algorithmic computation on the reverse of the tablet. In that case, the number in the 8th line on the reverse, for instance, would have been the 53rd power of 2, namely

[53 37 35 32 22 29 43 36 32] = 2^{53} 9-pl

Consequently, the 9th line on the reverse would, presumably, have been

30 · [53 37 35 32 22 29 43 36 32] = [26 48 47 46 11 14 51 48 16] = 2^{52} 9-pl

And so on.

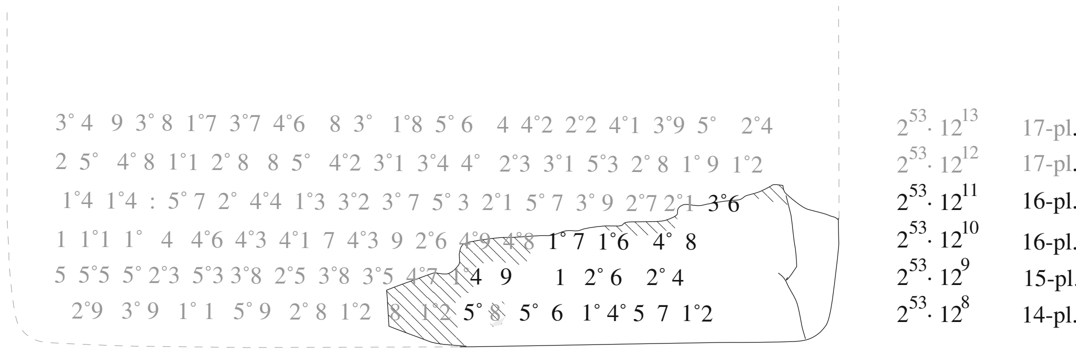


Fig. 2.3.5. Fragment VI F = BM 34958. Proposed reconstruction.

2.3.3 BM 34907. A Direct and an Inverse Last Place Factorization Algorithm

The two sides of the small fragment VI E = BM 34907 (Fig, 2.3.1, bottom) apparently represent the upper right corner of the obverse(?) and the lower right corner of the reverse(?) of a clay tablet originally inscribed with a long list of many-place sexagesimal numbers. On the side of the fragment which appears to be an upper right corner, the following brief trailing parts of 3 sexagesimal numbers are preserved:

[.....]x 45 07 12, [.....]6, [.....]x.

On the side which appears to be a lower right corner, a single, somewhat longer trailing part is preserved, albeit only imperfectly:

[.....]x 49 52 12 17 [.....].

For the moment, nothing more can be said about the number with the trailing part 45 07 12, other than that it may be one of the numbers represented by their "preceding places" in the two-dimensional lattice in Fig. 2.3.4 above. Fortunately, a lot more can be said about the number with the trailing part 49 52 12 17 [.....]. Indeed, a rapid visual search in Gingerich's table of 11-place regular sexagesimal numbers yields surprisingly few examples of such numbers in which the pair 12 17 appears near the end of the number, namely only

12 17 16 48	(15, 4, 0)	4-pl.
18 08 23 28 12 17 16 48	(17, 18, 0)	8-pl.

This result immediately suggests that it might be a good idea to search somewhat more carefully in Gingerich's table for numbers with the more complete trailing part 12 17 16 48, or at least 17 16 48. The numbers turning up this time are (see Fig. 2.3.6):

17 16 48	(8, 5, 0)	3-pl.
12 17 16 48	(15, 4, 0)	4-pl.
25 30 33 00 17 16 48	(10, 19, 0)	7-pl.
18 08 23 28 12 17 16 48	(17, 18, 0)	8-pl.
12 53 58 01 23 24 17 16 48	(24, 17, 0)	9-pl.
9 10 22 35 39 18 36 17 16 48	(31, 16, 0)	10-pl.
6 31 22 44 01 17 13 48 17 16 48	(38, 15, 0)	11-pl.
37 39 26 10 39 02 21 00 17 16 48	(12, 33, 0)	11-pl.

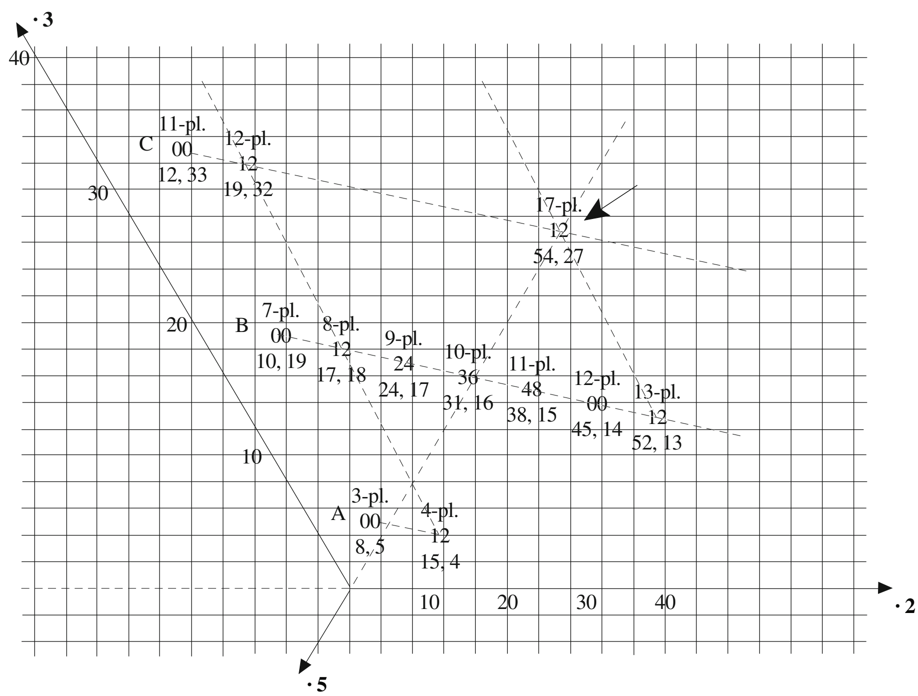
(A look in Gingerich's table reveals that the last of these numbers has a 23-place reciprocal!) The pattern that appears in the index grid is quite clear (see again Fig. 2.3.6). Therefore, it is easy to find, through simple extrapolation, that the points with the indices (19, 32, 0) and (52, 13, 0) must be 12- and 13-place regular sexagesimal numbers, respectively, with the trailing part 12 17 16 48. A third such number is

4 52 09 46 00 22 10 13 14 41 33 49 52 12 17 16 48 = 12^{27}	(54, 27, 0)	17-pl.

Thus, the procedure has managed to produce a single regular sexagesimal number with the desired trailing part 49 52 12 17 [.....]! The number in question is 17-place and not equal to any one of Gingerich's 11-place numbers. Incidentally, however, its reciprocal

12 19 18 49 59 29 17 42 10 02 05 = 5^{27} (0, 0, 27) 11-pl.

is only 11-place. Therefore, the number with the trailing part 49 52 17 16 48 that was found above can also be found in Gingerich's table, as the reciprocal of an 11-place number!



A	8, 5, (0)	17 16 48
	15, 4	**12 17** 16 48
B	10, 19	25 30 33 00 17 16 48
	17, 18	18 08 23 28 **12 17** 16 48
	24, 17	12 53 58 01 23 24 17 16 49
	31, 16	9 10 22 35 39 18 36 17 16 48
	38, 15	6 31 22 44 01 17 13 48 17 16 48

C	12, 33, (0)	37 39 26 10 39 02 21 00 17 16 48
	– – – –	– – – –
	54, 27	4 52 09 46 00 22 10 13 14 41 33 **49 52 12 17** 16 48

2.3.6. The two-dimensional lattice of points representing regular sexagesimal numbers with the trailing part 17 16 48.

It is now possible to make the following reconstruction of the two sequences of regular sexagesimal numbers on the obverse and the reverse, respectively, of the tablet of which only the corner piece BM 34907 is preserved: It was shown above that the number in the last line of the reverse(?) must have been 12^{27}. Moreover, it is obvious that there were about the same number of lines on the obverse as on the reverse of the tablet, and it is reasonable to assume that there was a simple connection between the two sequences of sexagesimal numbers on the obverse and on the reverse, respectively. Therefore, there can be little doubt that fragment VI E = BM 34907 is a small fragment of a Late Babylonian tablet which was originally inscribed with an algorithmic double table similar in some way to the Neo-Babylonian algorithmic double table on both sides of BM 46550 (Fig. 2.1.2), and also to the Old Babylonian algorithmic double table CBS 1215 # 21 (Sec. 2.2 above). Indeed, the most likely interpretation is that BM 34907 originally was part of a tablet inscribed on the obverse(!) with a descending table of powers of 12, constituting a last place factorization algorithm for the regular sexagesimal number

[2 01 44 04 10 09 14 15 31 07 19 05 4]6 45 07 12 = 12^{25} (50, 25, 0) 16-pl.

Furthermore, the reverse(!) of the tablet was inscribed with an ascending table of powers of 12, constituting what was meant to be an inverse last place factorization algorithm for the same number. However, by mistake, the table of powers of 12 was not stopped in time, but was allowed to progress two steps too far, ending with the regular sexagesimal number

[4 52 09 46 00 22 10 13 14 41 33] 49 52 12 17 [16 48] = 12^{27} (54, 27, 0) 17-pl.

See the proposed reconstruction in Fig. 2.3.7 below.

Possibly, an even more serious mistake made by the person who wrote the text is that the number computed on the reverse is not a product of reciprocals of factors in the given number on the obverse. More precisely, it is possible that the teacher's intention was that what the student should compute on the reverse was not 12^{27}, which is what the student actually computed, but instead the reciprocal of the given number on the obverse, which is

rec. $12^{25} = 5^{25}$ = 29 34 21 11 58 46 18 29 12 05 10-pl.

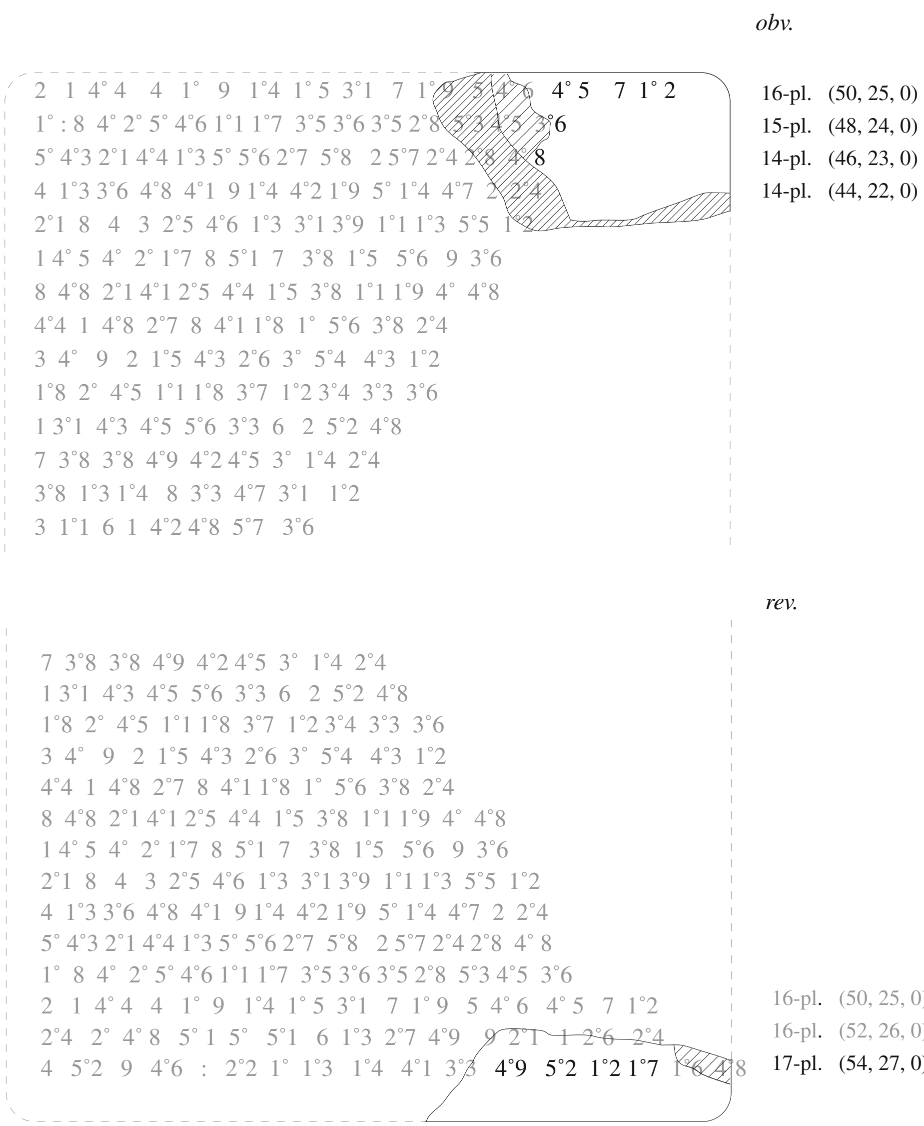


Fig. 2.3.7. Fragment VI E = BM 34907. Proposed reconstruction.

2.4 Old Babylonian Ascending and Descending Tables of Powers

According to the suggested reconstructions above of the Late Babylonian fragments VI D = BM 34517 (Fig. 2.3.2) and VI E = BM 34907 (Fig. 2.3.7), the former was once part of a descending table of powers of 9, and the latter was once part of a descending table of powers of 12, followed by an ascending table of powers of 12.

Several Old Babylonian predecessors of such descending or ascending tables of powers are known. Thus, MS 2242 (Friberg, *MSCT 1* (2007), Fig. 1.4.1) is a descending table of powers of 3 45, starting with the 6th power of 3 45, and MS 3037 (*ibid.*) is a descending table of powers of 12, starting with the 12th power of 12.

Previously published Old Babylonian ascending tables of powers are listed in Friberg, *op. cit.*, 26, fn. 1, namely the first 10 powers of 3 45 on four tablets from Kish, namely Ist. O 3816, 3826, 3862, 4583 (Neugebauer *MKT I,* 77-78), the first 10 powers of 3 45, followed by the first 10 powers of 16 on IM 73355, a tablet from Larsa (Friberg, *op. cit.*, Fig. A5.7), and the first 10 powers of 1 40, followed by the first 10 powers of 5 on BM 22706 (Nissen/Damerow/Englund, *ABK* (1993), 150).

2.4.1 IM 630174. An Ascending Table of Powers from Old Babylonian Bikasi

A new Old Babylonian ascending table of powers is IM 630174, an ascending table of 10 successive powers of 40, from the 2nd to the 11th power. It is from the third stratum of Bikasi, modern Tell Abu Intiek. A hand copy of the tablet is published here with the kind permission of Basima Jalil 'Abid, who first published the text in *Sumer* 53 (2005-6).

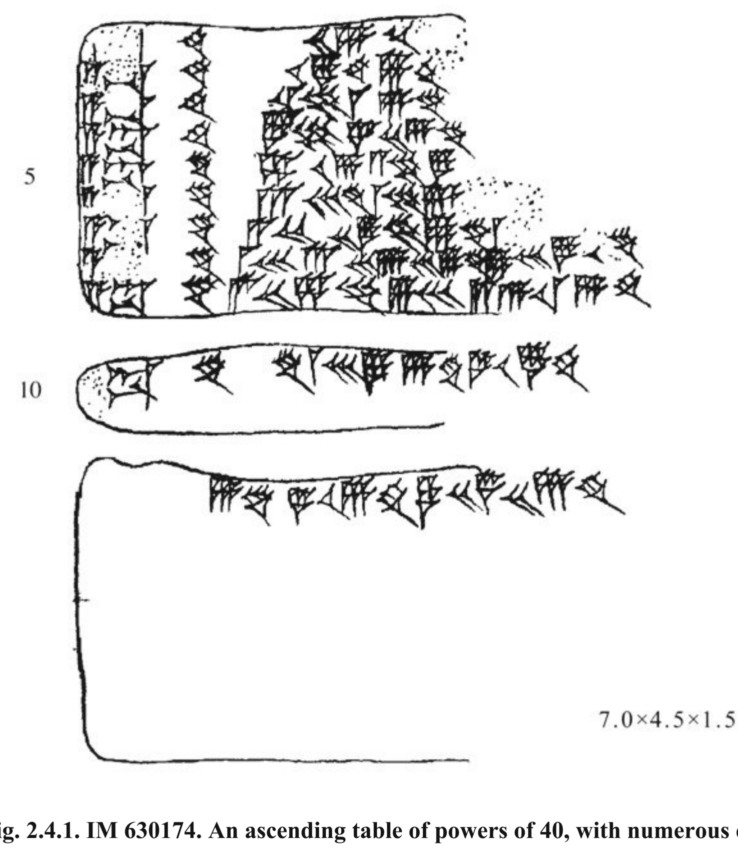


Fig. 2.4.1. IM 630174. An ascending table of powers of 40, with numerous errors.

There are several errors in this text, marked by underlining in the transliteration below.

	IM 630174	corrected
1	[a.rá] 40 26 40	40^2 = 26 40
	a.rá 40 17 46 40	40^3 = 17 46 40
	a.rá 40 11 51 06 40	40^4 = 11 51 06 40
	a.rá 40 7 54 26 40	40^5 = 7 54 04 26 40
5	a.rá 40 5 16 02 57 [46 40]	40^6 = 5 16 02 57 46 40
	a.rá 40 3 30 41 58 [31 06 40]	40^7 = 3 30 41 58 31 06 40
	a.rá 40 2 20 27 59 44? [26 40]	40^8 = 2 20 27 59 00 44 26 40
	a.rá 40 1 33 38 39 49 37 [46]? 40	40^9 = 1 33 38 39 20 29 37 46 40
	a.rá 40 3 25 46 33 05 11 06 40	40^{10} = 1 02 25 46 13 39 45 11 06 40
10	[a].rá 40 41 37 06 44 17 40	40^{11} = 41 37 10 49 06 30 07 24 26 40
	6 44 16 47 24 26 40	

The first error is in line 4, where the person who wrote IM 630174 seems to have incorrectly copied the number 7 54 04 26 40, written on an older tablet serving as a model, not noticing that the 4 in 54 should be followed by a second 4. The error is not propagated to the number in line 5, which, however, seems to be missing the last two aces, 46 40, probably because they were written out of sight on the edge of the older tablet.

Similarly, in line 6 the last three places, 31 06 40, seem to be missing, probably for the same reason. Again, in line 7, the last two places, 26 40, seem to be missing, and possibly the preceding place, 44, is incorrectly copied as 41. There is no indication, like a special sign or a gap, for the vacant sexagesimal place, transliterated as 00 in the corrected version above.

The missing indication of the vacant place in line 7 is not due to a miscopying by the one who wrote IM 630174. This is obvious because the error in line 7, the missing indication of the vacant place, gives rise to a propagated error in line 8, a so called telescoping error. Indeed, the incorrect number in line 8 is

40 · 2 20 27 59 44 26 40 = 1 33 38 39 49 37 46 40,

instead of the correct number

40 · 2 20 27 59 00 44 26 40 = 1 33 38 39 20 29 37 46 40.

Note that 49 = 20 + 29.

The propagated error in line 8 gives rise to a new propagated error in line 9. Indeed, in line 9,

40 · 1 33 38 39 49 37 46 40 = 1 02 25 46 33 05 11 06 40,

instead of the correct number

40 · 1 33 38 39 20 29 37 46 40 = 1 02 25 46 13 39 45 11 06 40.

This propagated error must have been present already on the older tablet serving as a model for IM 630174. However, in addition to this propagated error there is also an interesting copying error, namely that the first two places of the number on line 9 of the older tablet, 1 02, which should be written as a single upright wedge followed by a pair of upright wedges, was copied in line 9 of IM 630174 as 3, written as a triple of upright wedges!

It is not clear what is going on in lines 10-11 of IM 630174. One would have expected to see here 40 times the (incorrect) number in (the original version of) line 9, that is, with a propagated error, 40 · 1 02 25 46 33 05 11 06 40 = 41 37 11 02 03 27 24 26 40 (9-place)

Instead one sees in line 10 the 6-place number

41 37 06 44 17 40,

followed in line 11 by the 7-place number

6 44 16 47 24 26 40.

The number in line 10 begins in the expected way, with 41 37, and the number in line 11 ends in the expected way, with 7 24 26 40, but if the number in line 11 is thought of as a continuation of the number in line 10, the two numbers together would make a 12-place number instead of the expected 9-place number! A possible clue to what is going on here is that the two places 06 44 in the number in line 10 reappears at the beginning of the number in line 11. So, maybe, the author of the text discovered that he had made a mistake at the end of the number in line 10 and continued in line 11 with what he thought was a correction of the mistake. In that case, the intended number in lines 10-11 could be

41 37 <u>06 44 16 47</u> 24 26 40.

This is, as expected, a 9-place number, and it begins, as expected, with 41 37, and ends, as expected, with 7 24 26 40. However, there is no obvious explanation for the four places in the middle of the number, 06 44 16 47, instead of the expected four places 11 02 03 27.

Added in the proofs:

In *JCS* 66 (2014), M. Ossendrijver published several new fragments of arithmetical table texts from Late Babylonian Babylon, namely

Text A: BM 34249 + 32401 + 34517, a factorization table for 9^{46},
Text B: BM 42744 + 34958, a factorization table for $9^{11} \cdot 12^n$ ($n \geq 39$),
Texts C-H: BM 36065, 37095, 33447, 32681, 42980, 36917, six many-place tables of reciprocals,
Texts I-J, X: BM 37020, 32178, 45884, 37338, four many-place tables of regular square numbers.

The offered reconstructions of Tables A and B were made possible to some extent by recourse to a computer-generated table of all regular sexagesimal numbers up to thirty digits, as well as a well known table of fourth powers of many-place regular sexagesimal numbers.

Two of the fragments discussed by Ossendrijver were also, independently, discussed above in the present chapter, namely fragment A3 = BM 34517 and fragment B2 = BM 34958. Ossendrijver's results confirm and extend the results obtained in this chapter.

Indeed, Ossendrijver shows that his three fragments A1-A3 are parts of a single clay tablet with a descending table of powers of 9, starting with the number 9^{46} = sq. sq. 3^{23}, while fragment A3 was discussed above in Sec. 2.3.1 *BM 34517. A Descending Table of Powers of 9*. (Recall that it was shown in Friberg, *MSCT 1*, App. A9.2 that the Seleucid fragment BM 34601 contains an explicit computation of 9^{46} = sq. sq. 3^{23}!)

Similarly, Ossendrijver shows that his two fragments B1-2 are parts of a single clay tablet with a factorization table for a many-place regular number of the type $9^{11} \cdot 12^n$ ($n \geq 39$). In particular, he is able to show that the last number on the fragment B2 can be factorized as $9^{11} \cdot 12^{19}$, while it was shown above in Sec. 2.3.2 *BM 34958. A Last Place Factorization Algorithm for a Large Number of Class I* that the same number can be factorized as $2^{38} \cdot 3^{41}$ (= $9^{11} \cdot 12^{19}$).

3. Metrological Table Texts from Achaemenid Uruk

Mathematical and/or Metrological Cuneiform Texts from the 1st Millennium BC

The corpus of published *mathematical* and/or *metrological* cuneiform texts from the 1st millennium BC is only moderately extensive. In Sec. 1.1 above, 34 Late Babylonian (Neo-Babylonian, Achaemenid, or Seleucid) texts dealing with many-place regular sexagesimal numbers were listed, and 11 of them were discussed in Chs. 1-2. In addition to those texts, the corpus comprises the following 47 Neo-Assyrian, Neo-Babylonian, Achaemenid, and Seleucid items, of which 11 will be discussed in Chs. 3-4 below:

I. Five large mathematical recombination texts

A: AO 6484 (Neugebauer, *MKT I*, 96)	Seleucid	Uruk
B: BM 34568 (Neugebauer, *MKT III*, 14; Høyrup, *LWS*, 391)	Seleucid	
C: VAT 7848 (Neugebauer and Sachs, *MCT* Y)	Seleucid	
D: W 23291-x (Friberg, et al., *BaM* 21)	Achaemenid	Uruk
E: W 23291 = IM 75985 (Friberg, *BaM* 28)	Achaemenid	Uruk

II. Three small mathematical single problem texts

A: BM 47431 (Robson, *FS Slotsky*, 213; Friberg, *AfO* 52, 125)	Neo-Babylonian	
B: BM 78084 (Friberg, *ToV* 4, 4; Nemet-Nejat, *NABU* 2001:10)		Babylon?
C: BM 78822 (Jursa, *AfO* 40/41; Sec. 4.1.3 below)	Neo-Babylonian	

III. Eleven small fragments or excerpts with mathematical exercises

A: BM 34081+ (Neugebauer, *ACT* 813:5)	Seleucid	Babylon
B: BM 324724 (Sachs, *LBAT* 1648; Muroi, *SBM* 1)	Seleucid	Babylon
C: BM 34757 (Neugebauer, *ACT* 817:4)	Seleucid	Babylon
D: BM 34800 (Sachs, *LBAT* 1647; Friberg and George, *PGC*, 146)		Babylon
E: BM 34901 (Sachs, *LBAT* 1645)	Seleucid	Babylon
F: BM 64696 (Friberg, et al., *BaM* 21, 502)	Neo-Babylonian?	Sippar
G: BM 67314 (Friberg, *BaM* 28, 296; Sec. 4.1.3 below)	Neo-Babylonian?	Sippar
H: MM 86.11.404 (Neugebauer and Sachs, *MCT* X; Friberg, *CTMMA II*, 306)	Seleucid	Babylon
I: N 2873 (Robson, *Sciamvs* 1, no. 20)		Nippur
J: Sm 162 (King, *BBS*, pl. 11)	Neo-Assyrian	Niniveh
K: Sm 1113 (Weidner, *GD*, 393)	Neo-Assyrian	Niniveh

IV. Nine mathematical table texts

A: Ash. 1924.796+ (Neugebauer, *MKT I*, 73; Robson, *SCIAMVS* 5, no. 28)	Neo-Babylonian	Kish
B: BM 36849+ (Aaboe, *AACD*, 178)		Babylon
C: BM 141493 (Nemet-Nejat, *BiOr* 58; Friberg, *MSCT 1*, 97)	Neo-Babylonian	Uruk?
D: CBS 1535 (Neugebauer and Sachs, *MCT* 34)	Achaemenid?	Nippur
E: Ist U 91+ (Aaboe, *AACD*, 178)		Uruk
F: K 2069 (Neugebauer, *MKT I*, 30 and *II*, pl. 10)	Neo-Assyrian	Nineveh
G: SU 52/5 (Hulin, *JCS* 17)	Neo-Assyrian	Huzurīna (Sultantepe)
H: W 22715/2 = IM 76930 (von Weiher, *Uruk 4*)	Achaemenid?	Uruk

V. Two fragmentary texts with "mystical" mathematics

A: BM 47860 (Livingstone, *MMEW*, 35)		
B: BM 64696 (Friberg, et al., *BaM* 21, 502)		Sippar

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7_3

VI. Four fragmentary texts with tables of constants

A: BM 36776 (Robson, *MMTC*, 206; Friberg, *MSCT 1*, 171)		Babylon
B: BM 37096 (Robson, *MMTC*, 206)		Babylon
C: BM 47860 (Livingstone, *MMEW*, 35)		
D: CBS 10996 (Kilmer, *Or* 29; Friberg, *AfO* 52, 140)		Nippur

VII. Six mixed metrological table texts

A: AO 6555 (Scheil, *BE 20/1* 30; Friberg, *BaM* 28, 298 and Fig. 3.1.10 below)	Neo-Babylonian	Uruk/Borsippa
B: CBS 8539 (Hilprecht, *BE 20/1* 30; Sec. 4.1 below)	Achaemenid	Uruk
C: W 22260 = IM 74428+ 74429+74430 Hunger, *Uruk 1*; Sec. 3.4 below)	Achaemenid	Uruk
D: W 22309 = IM 74409 (Hunger, *Uruk 1*; Sec. 3.3 below)	Achaemenid	Uruk
E: W 23273 = IM 76821 (von Weiher, *Uruk 4*; Sec. 3.3 below)	Achaemenid	Uruk
F: W 23281 = IM 76283, *obv*. (von Weiher, *Uruk 4*; Sec. 3.2 below)	Achaemenid	Uruk

VIII. Six single metrological lists or tables

A: Ash. 1924.1278 (Robson, *Sciamvs* 5, no. 28)	Neo-Babylonian	Kish
B: Ass. 13956dr (Thureau-Dangin, *RA* 23; Fig. 3.1.5 below)	Neo-Assyrian	Assur
C: BM 51077 (Friberg, *GMS 3*, no. 6)		Sippar?
D: CBS 11019 (Sachs, *JCS* 1; Sec. 3.4 below)	Neo-Babylonian	Nippur
E: CBS 11032 (Thureau-Dangin, *JCS* 1; Sec. 3.4 below)	Neo-Babylonian	Nippur
F: VAT 9840+ (Schroeder *KAV*, no. 184; Friberg, *GMS 3*, no. 1)	Neo-Assyrian	Assur

Six fragments/excerpts of Neo-Babylonian metrological tables were published recently in Steele, *SCIAMVS* 16 (2015).
Additional fragments/excerpts of Neo-Babylonian metrological lists or tables are enumerated in Robson, *MAI*, Table B.18.
In Leichty's *Catalogue of the Babylonian Tablets in the British Museum*, *VI-VIII*, are listed at least two additional unpublished mathematical or metrological texts from the 1st millennium BC, namely BM 77951, a standard table of reciprocals of regular sexagesimal numbers, and BM 65238, a small table of squares

The known metrological tables from the 2nd millennium BC are *all of one type:* In combined metrological tables, the sub-tables appear in the order *C, M, A, L, SR, CR*, and in each (sub-)table units of measure (placed to the left) are equated with sexagesimal fractions or multiples of some given basic unit (sìla/barig, gín/ma.na, sar, nindan or kùš). All multiples are expressed in terms of *sexagesimal* numbers. (Even in Nougayrol, *Ugaritica* 5 no. 144, the lists of measures appear in the order *C, M, A*.)

The cuneiform metrological tables from the 1st millennium BC are of various types but have a pronounced tendency to share *some common novel traits*: In combined tables, the sub-tables in most cases appear in the "reverse" order *L, (A,) M, C,* and in each line of the tables the units of measure and the multiples of the basic unit appear, again in most cases, in the "reverse" order with the units of measure to the right. Furthermore, in most cases multiples of units are expressed in terms of *decimal* numbers. Completely new features are structure tables for length or area measures, a range table for length measure and area measure, and equivalence tables for area measure and traditional(!) seed measure. New are also the tables for common reed measure, for named shekel fractions, and for grain multiples as shekel fractions.

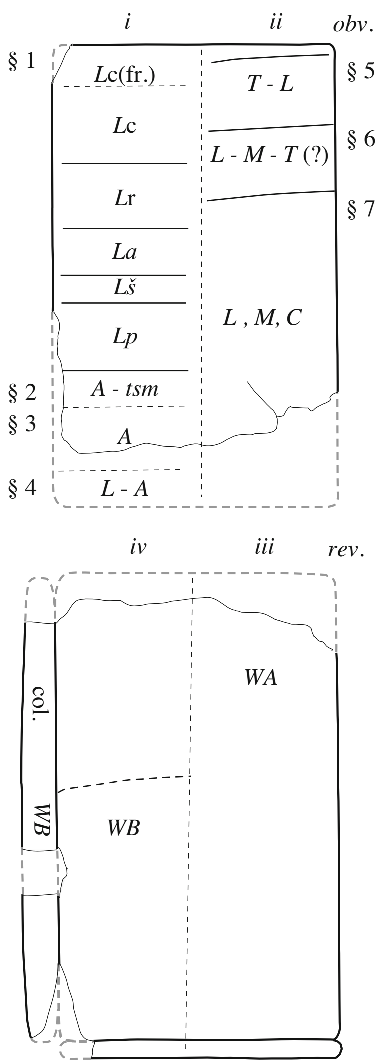
In spite of all those new features and conspicuous deviations from the formerly fixed layout, the metrological tables from the 1st millennium BC have several traits in common with their counterparts from the preceding millennium. Noteworthy is, in particular, the strange persistence in the late metrological tables of the (by then obsolete) grain as a small shekel fraction, of the "traditional" (Kassite) seed measure, and of the equally "traditional" systems of capacity, area, and length measure.

3.1 W 23281, *obv.* A Metrological Recombination Text from Achaemenid Uruk

(A "recombination text" is a text of mixed content, with its various paragraphs copied from several older texts.) The many-place table of reciprocals on the *reverse* of W 23281 (von Weiher, *Uruk 4* (1993), 173) was discussed in Sec. 1.6 above. See also the discussion in Sec. 1.3.1 of the provenance of W 23281 from a house in Achaemenid Uruk.

A table of contents for W 23281 was exhibited in Friberg, *GMS 3* (1993), no. 12. The table of contents is reproduced below, together with a conform transliteration of the metrological table text on the obverse, within an outline of the tablet.

W 23281: Table of contents



Note: All sub-tables except §§ 9, 13, and 14 *read from right to left*.

§ 1a *Lc*(fr.): Structure table for fractions of the cubit,
from [2]? threads? = 1 grain to 30 fingers = 1 cubit

§ 1b *Lc*: Conversion of large length numbers to multiples of cubits,
from *nik-ka-su* = 3 cubits to *bé-e-ri* = 21,600 cubits

§ 1c *Lr*: Conversion of large length numbers to multiples of reeds,
from *ṣu-up-pan* = 10 reeds to *bé-e-ri* = 3,600 reeds

§ 1d *La*: Conversion of large length numbers to multiples of ropes,
from 'sixty' = 6 ropes to *bé-e-ri* = 180 ropes

§ 1e *Lš*: Conversion of large length numbers to multiples of *šuššān*,
from 'half' = 15 'sixties' to *bé-e-ri* = 30 `sixties'

§ 1f *Lp*: Conversion of large length numbers to multiples of *purīdu*,
from *ṣu-up-pan* = 20 legs to *bé-e-ri* = 7,200 legs

§ 2 *A - Ksm*: Conversion of area numbers to Kassite seed measure
from 1 iku = *ṣi-mi-id* (3 bán) to *šá-a-ri* = 108 gur
and 18 shekels (tsm) = *mu-sa-ru*

§ 3 ***A***: Structure table for area measure
from *mu-sa-ru* = 18 shekels
and 50 *mu-sa-ru* = *ú-bi* to [60 bur_x = *šá-a-ri* (šár)]

§ 4 *L - A*: Range table for length and area numbers
from [fingers] to [grain] to from 6 uš to šár

§ 5 *T - L*: Linear growth of a child,
from 1 day = 1/2 grain to 10 months = 1 cubit

§ 6 *L - M - T*: Star distances(?), weights, time(?)
180 šár leagues = 1 talent = ...

§ 7 *L, M, C* : Parallel descending tables of length, silver, and seed measure;
multiples and fractions of uš , gín and bán

§ 8 *WA*: Many-place table of reciprocals
with leading places from 1, 2, 3, ... to 59, and 1

§ 9 *WB*: Many-place table of reciprocals
with leading places from 1, 2, 3, ..., to 29, and 30

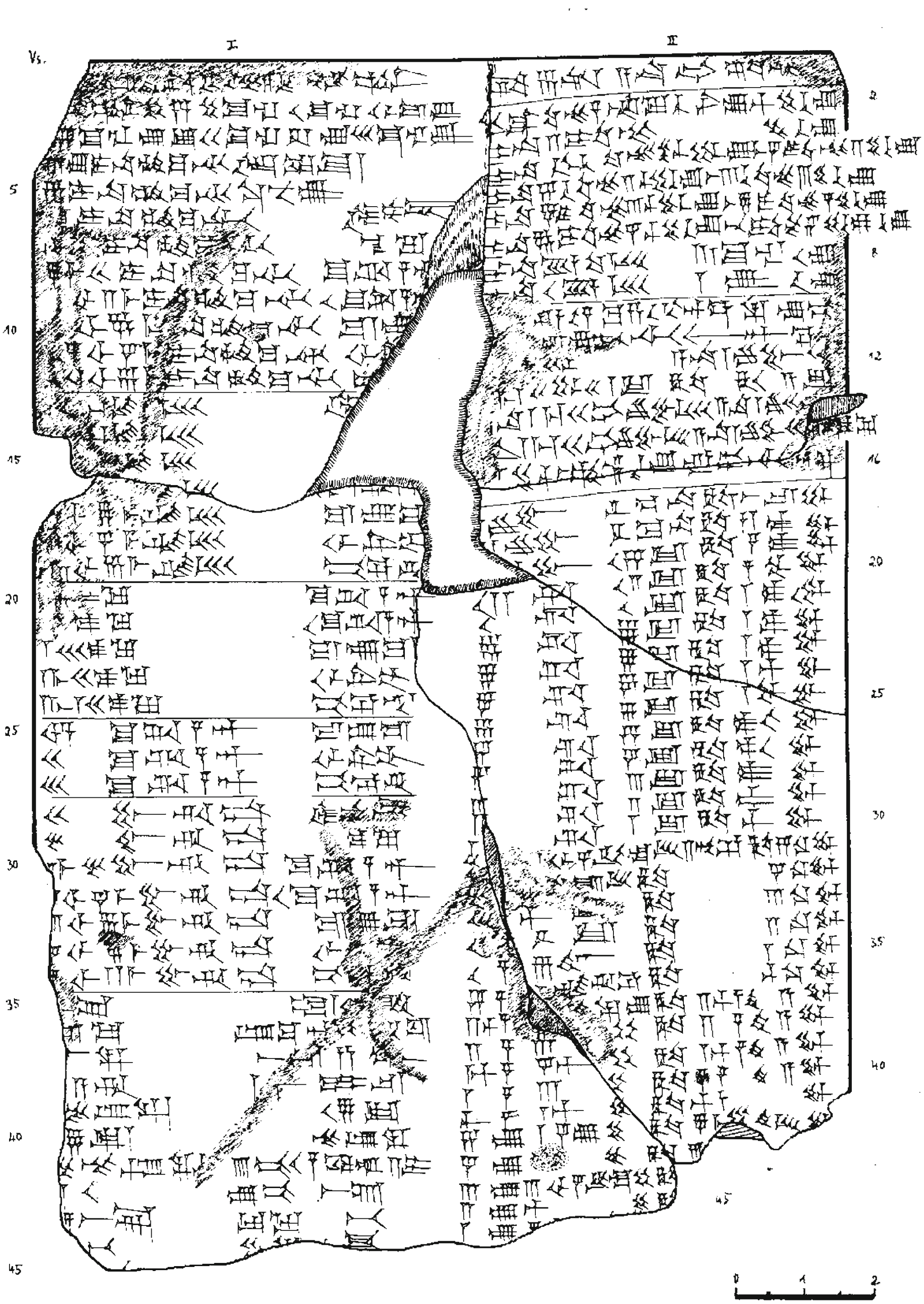


Fig. 3.1.1. W 23281 *obv*. Hand copy by E. von Weiher.

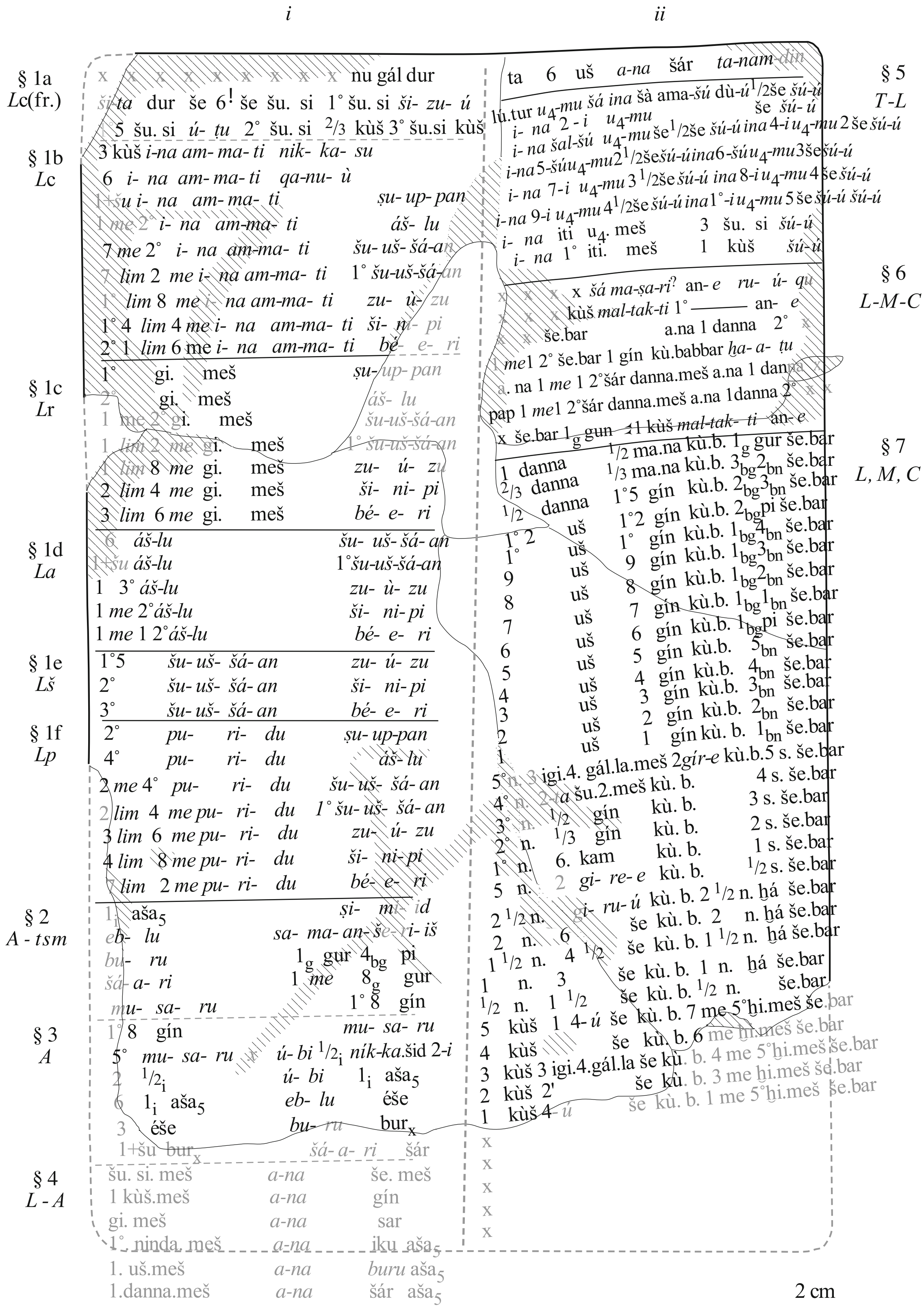


Fig. 3.1.2. W 23281 *obv*. Conform transliteration with a partial reconstruction of lost lines.

3.1.1 § 1. A Series of Structure Tables for Traditional Length Measure

W 23281 *obv.* begins with a progressive series of “structure tables” for the “traditional” (= Old Babylonian) system of length measure. (A structure table for a system of measure contains information about the *names for the units* of the system, and about the *relations between successive units* of the system.)

W 23281 *i*: 1-34. § 1: Tables *L*c(fr.), *L*c, *L*r, etc. (to be read *from right to left!* The translation in reverse order.)

§ 1a	x x x x x x x x nu gál		dur /	a knot(?) /	=	[definition] non-existing	
	[*ši-t*]*a*	dur	še	a grain	=	[tw]o	knots(?)
	⸢5⸣	še	šu.si	a finger	=	⸢5⸣	grains
	10	šu.si	*ši-zu-ú* /	a *šizû* /	=	10	fingers
	[1]5	šu.si	*ú-ṭu*	a *ūṭu*	=	[1]5	fingers
	20	šu.si	2/3 kùš	a 2/3 cubit	=	20	fingers
	30	šu.si	kùš /	a cubit /	=	30	fingers
§ 1b	3 kùš	*i-na am-ma-ti*	*nik-ka-su* /	a *nikkas* /	=	3 cubits	in cubits
	6	*i-na am-ma-ti*	*qa-nu-ú* /	a reed /	=	6	in cubits
	[1+*š*]*u*	*i-na am-ma-ti*	*ṣu-up-pan* /	a *ṣuppān* /	=	[six]ty	in cubits
	[1 *me* 20]	*i-na am-ma-ti*	*áš-lu* /	a rope /	=	[1 hundred 20]	in cubits
	7 *me* 20	*i-na am-ma-ti*	*šu-uš-šá-an* /	a *šuššān* /	=	7 hundred 20	in cubits
	[7] *lim* 2 me	*i-na am-ma-ti*	10 *šu-uš-šá-a*[*n*] /	a 10-*šuššān* /	=	[7] thousand 2 hundred	in cubits
	[10] *lim* 8 *me*	*i-na am-ma-ti*	*zu-ú-*[*zu*] /	a *zūzu* (1/2) /	=	[10] thousand 8 hundred	in cubits
	[1]4 *lim* 4 *me*	*i-na am-ma-ti*	*ši-n*[*i-pi*] /	a *šinipu* (2/3) /	=	[1]4 thousand 4 hundred	in cubits
	21 *lim* 6 me	*i-na am-ma-ti*	*b*[*é-e-ri*] /	a league /	=	21 thousand 6 hundred	in cubits
§ 1c	10	gi.meš	*ṣu-*[*up-pan*] /	a *ṣuppān* /	=	10	reeds
	[20]	gi.meš	[*áš-lu*] /	a [rope] /	=	[20]	reeds
	[1 *me* 20]	[g]i.meš	[*šu-uš-šá-an*] /	a [*šuššān*] /	=	[1 hundred 20]	reeds
	[1 *lim* 2 me]	[gi].meš	⸢10 *šu-uš-šá*⸣-[*an*] /	a 10 *šuššān* /	=	[1 thousand 2 hundred]	reeds
	[1 *lim*] 8 *me*	gi.meš	*zu-ú-z*[*u*] /	a *zūzu* (1/2) /	=	[1 thousand] 8 hundred	reeds
	[2] *lim* 4 *me*	gi.meš	*ši-ni-pi* /	a *šinipu* (2/3) /	=	[2] thousand 4 hundred	reeds
	3 *lim* 6 *me*	gi.meš	*bé-e-ri* /	a league /	=	3 thousand 6 hundred	reeds
§ 1d	[6]	[*á*]*š-lu*	*šu-uš-šá-an* /	a *šuššān* /	=	[6]	ropes
	[1+*šu*]	*áš-lu*	10 *šu-uš-šá-an* /	a 10-*šuššān* /	=	sixty	ropes
	1 30	*áš-lu*	*zu-ú-zu* /	a *zūzu* (1/2) /	=	1 30	ropes
	1 *me* 20	*áš-lu*	*ši-ni-pi* /	a *šinipu* (2/3) /	=	4 hundred 20	ropes
	1 *me* 1 20	*áš-lu*	*bé-e-ri* /	a league /	=	1 hundred 1 20	ropes
§ 1e	15	*šu-uš-šá-an*	*zu-ú-zu* /	a *zūzu* (1/2) /	=	15	*šuššān*
	20	*šu-uš-šá-an*	*ši-ni-pi* /	a *šinipu* (2/3) /	=	20	*šuššān*
	30	*šu-uš-šá-an*	*bé-e-ri* /	a league /	=	30	*šuššān*
§ 1f	20	*pu-ri-du*	*ṣu-up-pan* /	a *ṣuppān* /	=	20	paces
	40	*pu-ri-du*	*áš-lu* /	a rope /	=	40	paces
	2 *me* 40	*pu-ri-du*	*šu-uš-šá-an* /	a *šuššān* /	=	2 hundred 40	paces
	⸢2⸣ *lim* 4 *me*	*pu-ri-du*	10 *šu-uš-šá-an* /	a 10-*šuššān* /	=	⸢2⸣ thousand 4 hundred	paces
	[3] *lim* 6 *me*	*pu-ri-du*	*zu-ú-zu* /	a *zūzu* (1/2) /	=	[3] thousand 6 hundred	paces
	⸢4⸣ *lim* 8 *me*	*pu-ri-du*	*ši-*[*ni-pi*] /	a *šinipu* (2/3) /	=	4 thousand 8 hundred	paces
	7 *lim* 2 *me*	*pu-ri-du*	*bé-*[*e-r*]*i*	a league	=	7 thousand 2 hundred	paces

The metrological table text in § 1 of W 23281 *obv*. is clearly divided by ruled lines into at least five sub-tables. (The first of these could have been further divided.) The table text with its sub-tables can be characterized as a progressive series of structure tables for length measure. Structure tables were an innovation in metrological texts appearing in texts from the 1st millennium BC. (Three further examples are studied, in Fig. 3.1.5 below, and in Secs. 3.1.3 and 3.3.1.) There are no known Old Babylonian parallels.

As mentioned, the nature of the structure tables in § 1 can best be understood if it is assumed that the lines of the tables were intended to be read partly in inverse order, from right to left. Thus, the second line

[*ši-t*]*a* dur   še   [tw]o knots(?)   a grain,

can be understood as an equation defining the length unit 'grain' as equal to two 'knots'(?):

A grain (is equal to) 2 knots(?).

The reading dur = *riksu* 'knot'(?) for an obscure term in lines 1 and 2 of § 1 should be compared with the second line of the parallel text W 22309 = *Uruk 1*, 102 *obv*. (below, Sec. 3.3), which clearly says

2 *qu-*˹*ú*˺   [še]   2 threads   [a grain].

In the second line of the table,

˹5˺ še   šu.si   ˹5˺ grains   a finger,

the length unit 'finger' is defined as being equal to 5 grains.

At the next hierarchical level, three fractions of the cubit, and the cubit itself, are expressed in terms of fingers:

10 šu.si	*ši-zu-ú*	10 fingers	a *šizû* (1/3 cubit)
15 šu.si	*ú-ṭu*	15 fingers	an *ūṭu* (1/2 cubit)
20 šu.si	2/3 kùš	20 fingers	a 2/3 cubit
30 šu.si	kùš	30 fingers	a cubit.

Thus, this brief sub-table defines three small units of length, *šizû, ūṭu*, and kùš, as being equal to 10, 15, and 30 fingers, respectively. Alternatively, of course, the *šizû* and the *ūṭu* can be thought of as 1/3 and 1/2 cubit, respectively. For the third cubit fraction, '2/3 cubit', the table does not have a particular name but notes, nevertheless, that it is equal to 20 fingers. The origin of the measure names *šizû* (OB/jB) and *ūṭu* (OA, according to *CDA*) is unknown.

Note, in particular, the defining equation

A cubit (is equal to) 30 fingers.

It shows that *the system of length measure considered here is the "traditional" Sumerian/Old Babylonian system, which differs from the Late Babylonian system where 1 cubit = 24 fingers*.

The first seven lines of the table, here called *L*c(fr.), with the defining equations for the the grain, the finger, and the cubit with its main fractions, is not of the same format as the rest of the table (*L*c - *L*p), in which appear units of length larger than the cubit. Indeed, these first six lines form what may be called a "recursive structure table" for fractions of the cubit, in which successive hierarchical levels are reached through a recursive procedure. The purpose of this kind of structure table seems to have been to introduce directly the various "conversion factors" of the system.

The information contained in a basic structure table like *L*c(fr.) can be condensed into a *factor diagram*, exhibiting typically the names of successive units of a given system of measure, together with the associated string of conversion factors. The factor diagram for *L*c(fr.) is shown below.

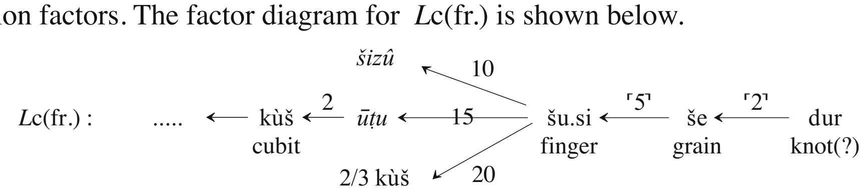
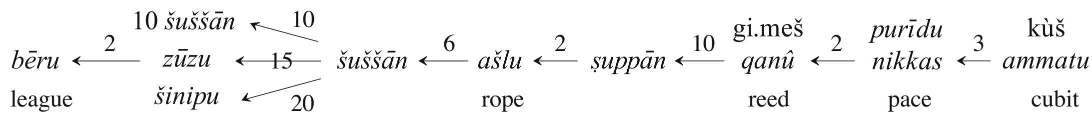


Fig. 3.1.3. W 23281, § 1. Factor diagram for fractions of the cubit. *Traditional* length measure (1 cubit = 30 fingers).

The information contained in the series of structure tables *L*c, *Lp*, *L*r, *La*, and *Lš* in § 1 of W 23281 is exhibited in a condensed but comprehensive form in the *expanded factor diagram* below. Note, in particular, that *1 reed = 6 cubits here, as in the traditional Sumerian/Old Babylonian system, which differs from the Late Babylonian system, where 1 reed = 7 cubits.*



*L*c :	21,600	7,200 10,800 14,400	720	120	60	6	3	1
*L*p :	7,200	2,400 3,600 4,800	240	40	20	2	1	
*L*r :	3,600	1,200 1,800 2,400	120	20	10	1		
La :	180	60 90 120	6	1				
Lš :	30	10 15 20	1					

Fig. 3.1.4 W 23281, § 1. Factor diagram for the larger units of *traditional* length measure (1 reed = 6 cubits). A progressive series of cumulative structure tables.

The series of structure tables *L*c - *Lp* in § 1 of W 23281 gives an almost exhaustive account of how all units of the "traditional" (essentially Old Babylonian) system of length measure can be expressed as multiples of smaller units of the system. Each one of these sub-tables has a "cumulative" rather than "recursive" form, with all the units larger than a fixed basic unit expressed as multiples of that unit. Thus, in the first of the mentioned structure tables, Table *L*c, the following units:

nikkas, *qanû* 'reed', *ṣuppān*, *ašlu* 'rope', *šuššān*, *bēru* `league', and three basic fractions of the *bēru* ,

are expressed as decimal multiples of the cubit, referred to both as *ammatu* (Akkadian) and kùš (Sumerian). The equations in Table *L*c can be reformulated as follows:

A *nikkas*	(is)	3 cubits,
a reed	(is)	6 cubits,
a *ṣuppān*	(is)	60 cubits,
a rope	(is)	120 cubits,
a *šuššān*	(is)	1,200 cubits,
a 10-*šuššān*,	(is)	7,200 cubits,
a *zūzu* (1/2)	(is)	10,800 cubits,
a *šinipu* (2/3)	(is)	14,400 cubits,
a *bēru*	(is)	21,600 cubits.

It is interesting to notice that although the metrological table in § 1 of W 23281 deals with traditional, old Babylonian style length measure, it is distinctly Late Babylonian in character in that it makes use of *decimal* numbers to count multiples of the various units of length measure. Compare with other Achaemenid metrological tables for traditional length measure, in W 23273 § 2 (Sec. 3.2.2 below) and CBS 8539 § 1 (Sec. 4.1.1 below), which count by use of *sexagesimal* numbers.

The next larger length unit after the cubit is the *nikkas*, also called *purīdu* 'leg, pace?'. For some reason, the sub-table *Lp*, in which the basic unit is the *purīdu,* is inserted, out of order, after the other sub-tables, *L*r, *La*, and *Lš*, with the basic units gi.meš 'reeds', *ašlu* 'rope', and *šuššān*, respectively. A sub-table *Lṣ* with the basic unit *ṣuppān* is missing, probably by mistake.

The term *nikkas* in this text had a long history. A *nikkas* equal to 1 1/2 double-cubits or 90 fingers, written níg.šid, appeared already in the 2000 years older Early Dynastic metro-mathematical table text CUNES 50-08-01, a series of tables of square areas. (See Friberg, *MSCT 1* (2007), App. 7.)

The meaning of the word *ṣuppān* is not clear. (There is no known corresponding Sumerian term.)

The length measure *šuššān* is defined here by the phrase

7 *me* 20 *i-na am-ma-ti šu-uš-šá-an*	a *šuššān* (equals) 7 hundred 20 in cubits

A parallel phrase in W 2309, *obv*. (Sec. 3.3.1 below) is

[7 *me* 20 ta *am-mat* 1 uš.g]i.1.nindan	a [sixty]-nindan-[re]ed (equals) 7 hundred 20 of cubit.

Actually, it is true that in traditional length measure 720 cubits = 12 · 60 cubits = 60 nindan. (Compare with the well known term *ginindanakku*, 'a sixty-nindan-measuring-reed'.)

As for the meaning of the word *šuššān*, *CDA* (Black, George, and Postgate) states that it is the dual of the Akkadian word *šuššu* 'one-sixth', so that the term would mean 'two-sixths', that is, 'one-third'. According to the Chicago Assyrian Dictionary the term *šuššān* (Sum. šušana) otherwise occurs only in lexical texts, always spelled *šu/šú-uš-šá-an*, as the Akkadian word for the fraction 1/3. However, this interpretation cannot be correct here, since the *šuššān* is not one-third of any Babylonian unit of length. The precise etymology of *šuššān* with the meaning 'sixty nindan' is not known, but the most likely interpretation seems to be that the term is a derivative of *šūš*(*i*), the Akkadian word for '60'. (Cf. the OB/jB term *šuššār* < *šūš* + *šār*, Sum. šár × diš , meaning 60 šár = 60 × 3,600 = 21,600.) By the way, it may not be as strange as it seems that the same term may mean simultaneously 'one-third' and 'sixty' . Indeed, in the Sumerian/ Babylonian traditional system of weight measure, 1/3 shekel is equal to 60 grains.

Two of the terms used in Table *L*c for basic fractions of the *bēru*, sometimes arbitrarily translated as 'stage' or 'league', are *šinipu* (Sum. šanabi), here and everywhere else in OB/jB texts meaning simply 'two-thirds', and *zūzu* here and elsewhere in MB/jB texts used with the general meaning 'half-unit'. For one-third of the *bēru*, no special term seems to have been available, so the surrogate 10 *šuššān* was used instead.

There are two known parallel texts to W 23281 § 1. One of them is the obverse of W 22309 (Hunger, *Uruk 1* (1976), no. 102). W 22309 is a small fragment, found at Uruk in the same excavation square as W 23281 (Ue XVIII/1) but without any informative context. (For details, see Sec. 3.3 below.) Another parallel text is Ass. 13956dr (Thureau-Dangin, *RA 23* (1926); Fig. 3.1.5 below). It is a small tablet with an almost perfectly preserved, brief and concise structure table for length measure. It was found in library N4 in Neo-Assyrian Assur (see Pedersén, *ALA* (1985-6), II 68 no. 374), and is clearly older than W 23281. Like the structure tables in § 1 on W 23281, the structure table on Ass. 13956dr is written in reverse order, with the successive units of length measure to the right and multiples of smaller units to the left. In four lines of the structure table, Akkadian terms for units of length measure are interpolated as glosses between two parts of Sumerian terms for the same units.

5	še	šu. si
1°	šu. si	*ši- zu- u*
1°5	šu. si	1/2 *û-tu* kùš
3°	šu. si	1 kùš
3	kùš	*nik- kàs*
6	kùš	1/2 *qa-nu* nindan
1°	gi	5 *šu-up-pan* nindan
2°	gi	1° *áš- lu₄* nindan
6	*áš- lu₄*	1 uš
1°5	uš	1/2 danna
2°	uš	2/3 danna
3°	uš	1 danna

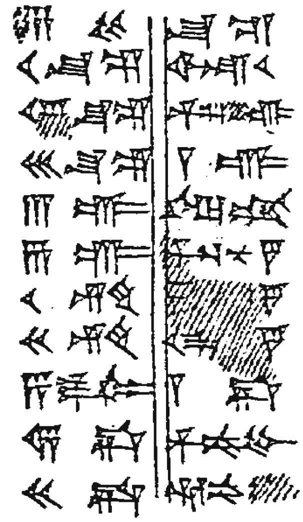


Fig. 3.1.5. Ass. 13956dr. A brief structure table for traditional length measure. From a Neo-Assyrian library in Assur.

The first part of the structure table on Ass. 13956dr, for fractions of the cubit, is just like the corresponding part of W 23281 § 1, except for the omission of the knot = 1/2 grain and of 2/3 cubit.

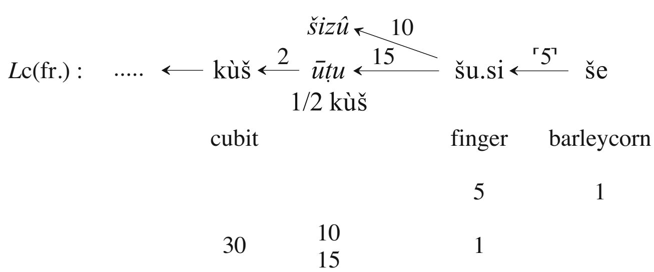


Fig. 3.1.6. Ass. 13956dr. Factor diagram for fractions of the cubit. Traditional length measure (1 cubit = 30 fingers).

The remainder of the structure table, for multiples of the cubit, is of the same form as the corresponding part of W 23281 § 1, but with different terminology. Most interesting is the omission of the strange and possibly incorrectly used term *šuššān* in favor of the familiar term 1 uš for 60 nindan. The fraction 1/3 danna is not mentioned.

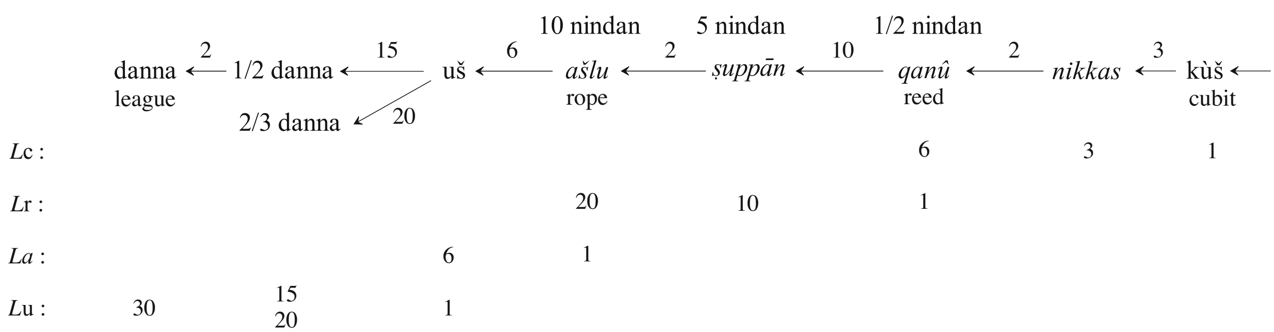


Fig. 3.1.7. Ass. 13956dr. Factor diagram for the larger units of traditional length measure (1 reed = 6 cubits).

3.1.2 § 2. A Conversion Table from Area Measure to Kassite Seed Measure

This brief metrological table shows how to convert the various units of traditional Sumerian/Old Babylonian area measure into the "Kassite seed measure" (Ksm) first used in Kassite kudurrus (boundary stones). The less suitable term used in Friberg, *BaM* 28 (1997), § 3, was "traditional seed measure".

W 23281 § 2: Table *A – Ksm*

⸢1(iku)⸣ ašag	*ṣi-m*⸢*i-i*⸣*d* /	⸢1 iku⸣ (area measure)	= a *ṣimdu* (= 3 bán) /
[*e*]*b-lu*	*sa-ma-an-š*[*e-r*]*i-iš* /	an [*e*]*blu*	= an eighteenfold(?) /
[*b*]*u-ru*	1_{gur} gur 4_{bariga} bariga /	a *būru*	= 1 gur 4 barig
[*šá*]*-a-ri*	1 [*m*]*e* 8_{gur} gur /	a *šāru*	= 1 hundred 8 gur /
mu-sa-ru	18 gín	a *mūsaru*	= 18 shekels

Note that this brief table clearly was intended to be read from left to right, unlike the structure tables in W 23281 § 1 and Ass 13956dr!

The meaning of W 23281 § 2, Table *A – tsm* will be explained later, as it can most easily be understood in view of what is offered in the structure table for units of area measure W 23281 § 3 (Table *A* below, in Sec. 3.1.3). Incidentally, unlike W 23281 § 2, W 23281 § 3 was intended to be read from right to left!

The last line of Table *A – tsm* above seems to be the result of an afterthought, since it appears to be misplaced. Logically, it should have been placed at the top of the table. However, it is possible that the one who copied this table from some other clay tablet, on his own made the following little computation:

If 1 iku = 100 *mūsaru* and 1 iku = 3 bán = 18 sìla (Ksm), then 1 *mūsaru* = 18 100th of a sìla = 18 shekels (Ksm).

Note that here was no traditional counterpart to the kind of shekel appearing here, a 100th of a sìla. In Sumerian/Old Babylonian texts a shekel was always 'a 60th'.

A factor diagram for the system *C* of capacity measure appearing in Table *A – tsm* (used to measure grain and other commodities) is displayed immediately below:

C(LB) : gur ←5— barig ←6— bán ←6— sìla ←100— gín

kurru *parsiktu* *sūtu* *qû* *šiqlu*

(c. 1 liter)?

Fig. 3.1.8. W 22381, § 2. Factor diagram for the Late Babylonian system of capacity measure (1 bán = 6 sìla, 1 sìla = 100 gín).

The basic rule for the conversion from area measure to seed measure in Table *A - tsm* says, in the form in which it appears repeatedly in Kassite *kudurrus* (boundary stones):

še.numun *šá* 1(iku) ašag 3(bán) *ina* kùš gal-*ti*
'seed (measure) such that 1 iku in area (measure) is 3 bán, (with lengths measured) in the great cubit'.

(Here 1 iku = 100 square rods, and the 'great cubit' is 1 1/2 ordinary cubits.) The calculations needed to establish the remaining relations in Table *A – tsm* are explained below, right after Fig. 3.1.9 in Sec. 3.1.3.

Note the term *sa-ma-an-š*[*e-r*]*i-iš* in the second line of Table *A - tsm*. The term is not in the Assyriological dictionaries, but it is obviously in some way related to **samānēšeret* 'eighteen' and stands here for '18 bán'.

3.1.3 § 3. A Structure Table for Area Measure

W 23281 § 3: Table *A*

[1]8	gín		*mu-sa-ru* /	*mūsaru*	=	[1]8	shekels /
⸢50⸣	*mu-sa-ru* ⸢x⸣	*ú-bi* $^1/_2$ (iku)	*ník-ka*.šid 2-*i* /	*ubû*,1/2 iku, *nikkas*? 2nd	=	⸢50⸣	*mūsaru* /
⸢2⸣	$^1/_2$(iku) *ú-bi*		1(iku) ašag /	1(iku)	=	⸢2⸣	*ubû*, 1/2(iku) /
⸢6⸣	1(iku) ašag	*eb-lu*	1(èše) /	*eblu*, 1(èše)	=	⸢6⸣	1(iku) /
[3]	1(èše)	⸢*bu-ru*⸣	1(buru)	*būru*, 1(bùr)$_x$	=	[3]	1(èše) /
[60]	⸢1(buru)$_x$⸣	[*šá-a-ri*]	[1(šár)]	*šāru*, [1(šár)]	=	[60	1(bùr)]

In the structure table in W 23281 § 3 (above), five consecutive units of Sumerian/Babylonian area measure are defined recursively as multiples of smaller units of area measure. The *mūsaru* alone is defined in a different way, as equal to 18 shekels (Ksm), precisely as in the last line of W 23281 § 2.

In the second line of the structure table, 1/2 iku, with the Akkadian name *ubû*, is equated with 50 *mūšaru*. The 1/2 iku is written with its own well known cuneiform sign, an oblique wedge, while *ubû* is written syllabically. The line ends with a curious phrase, *ník-ka*, followed by a smaller sign šid and 2-*i* '2nd'. With some imagination, one may read here *nikkas* (remember the mentioned Early Dynastic spelling níg.šid). If *nikkas* is what is meant, it may be a (previously unknown) *second Akkadian name* for the unit 1/2 iku.

In the third line of the structure table, 1 iku is equated with 2 times 1/2 iku. The writer of the text seems to have made a small mistake here, deviating from the chosen format for his structure table. Indeed, the mention of *ú-bi* seems to be superfluous, while a syllabic spelling of 1 iku is missing. The expected form of the third line would have been

2 1/2(iku) *i-ki* 1(iku) ašag.

In the fourth line, 1 èše, written both with its own cuneiform sign and in syllabic Akkadian as *eb-lu*, is equated with 6 times 1 iku.

In the fifth line, 1 bùr is written with its own cuneiform sign and syllabically as *bu-ru*. However, the cuneiform sign used here for 1 bùr was in Old Babylonian metrological texts used for '10 bur'. The explanation for the mixup may be that the Sumerian name for 10 bùr was bur'u, pronounced just like *bu-ru*!

The sixth line of the structure table is almost completely lost, but it is likely that it stated that 1 šár, written both with its own cuneiform sign and syllabically as *šá-a-ri* was equal to 60 bùr.

In the factor diagram below for System *A* are displayed the Sumerian and Akkadian names and the special cuneiform signs for the successive units of area measure, as well as the corresponding conversion factors, and the equivalents in Kassite seed measure. All this corresponds to the information offered by the structure table in W 23281 § 3, plus additional information provided by the conversion table in W 23281 § 2 above and by a related conversion table inserted in the middle of the colophon of the well known "Esagila tablet" AO 6455 (Thureau-Dangin, *TCL 6* (1922), no. 32), a Neo-Babylonian metro-mathematical text (Fig. 3.1.10 below).

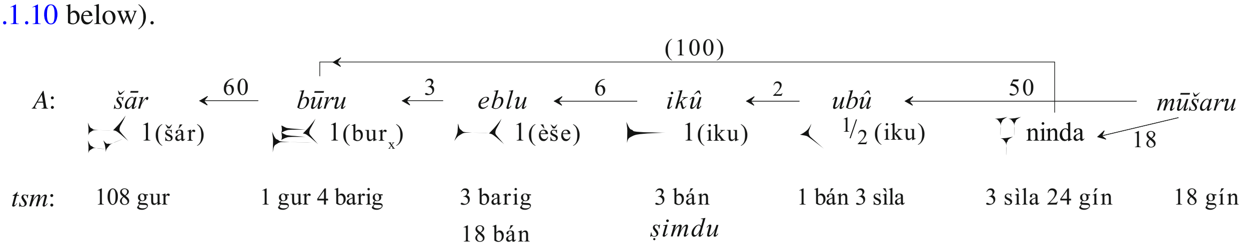


Fig. 3.1.9. W 23281 §§ 2-3 and AO 6555. Factor diagram for System *A* (in the big cubit), with conversion to Kassite seed measure.

The capacity numbers in Kassite seed measure corresponding to the successive units of area measure can be calculated as follows.

1 iku	=	3 bán (Ksm) according to the Kassite rule
1 èše	=	6 iku = 6 times 3 bán = 18 bán = 3 bariga (Ksm)
1 bùr	=	3 èše = 3 times 3 bariga = 9 bariga = 1 gur 4 barig (Ksm)
1 šár	=	60 bùr = 60 times 1 gur 4 bariga = 60 gur 240 bariga = 108 gur (Ksm)
1/2 iku	=	1/2 times 3 bán = 1 bán 3 sìla (Ksm)
1 ninda	=	1/100 bùr = 1/100 times 1 gur 4 bariga = 1/100 times 324 sìla = 3 24/100 sìla = 3 sìla 24 gín (Ksm)
1 *mušaru*	=	1/100 iku = 1/100 times 3 bán = 1/100 times 18 sìla = 18/100 sìla = 18 gín (Ksm)

The use of ninda as a name for 1/100 bùr is not known from any other Late Babylonian metrological text.

A parallel to the conversion and structure tables in W 23281 §§ 2-3 is, as mentioned, a brief final table in the Esagila tablet AO 6555. It has come down to us in the form of a later copy made at Uruk in 229 BC (SE 83), after an original from Borsippa.

In § 4 of that text, the surface extent of the ki.gal ‘base’ of the ziqurrat Etemenanki is said to be a square of side 3 · 1+*šu* (= 3 · 60) ‘small’ cubits. Its area is therefore 9 (00 00) square cubits. This area is multiplied by the constant ‘2’, meaning *2 bán of seed on 1 (00 00) sq. cubits, in the small cubit*. This is the Late Babylonian *common seed measure (csm)*. *Se*e the discussion of § 1 in the Achaemenid mathematical recombination text W 23291 Friberg, *BaM* 28 (1997), § 1, where the ‘seed constant’ is ‘20’, meaning ;20 bariga = 2 bán of seed on 1 (00 00) sq. cubits. Consequently, in § 4 of the Esagila tablet, the extent of the ki.gal is calculated to be 9 · ‘2’ = ‘18’, where ‘18’ is explained to be (18 bán =) 3 bariga (csm).

In § 5 of the same text, the (same) ki.gal is described as a square of side 10 rods in the ‘great’ cubit (equal to 1 1/2 small cubit). Hence, its area is 1 40 *mūsaru*. (The text does not mention that this is 1 iku in the great cubit.) This area is multiplied by the constant ‘18’, meaning *18 shekels (Ksm) on 1 mūsaru,* where the shekel is 1/100 sìla (as in W 23281 § 2). Thus the extent of the ki.gal is 1 40 · ‘18’ = ‘30’, where ‘30’ is explained as ‘1 iku 3 bán [in the great cubit]’. Indeed, in Kassite seed measure, 1 40 · 18 shekels (Ksm) = 30 (00) shekels = 18 · 100 shekels = 18 sìla = 3 bán (Ksm), or 1 iku in the great cubit.

The final paragraph of AO 6555 is in a strange way inserted in the middle of the colophon. (This may be an indication that this particular paragraph was not copied from the same older tablet as the main part of the text of AO 6555.) It is a brief metrological text that has drawn the futile attention of several scholars, like F. H. Weissbach (1914), F. Thureau-Dangin (1922), and M. Powell (1982). It was first recognized as a combined structure and conversion table for area measure and Kassite seed measure in Friberg, *Survey* (1982), 144 and *GMS 3* (1993), 397.

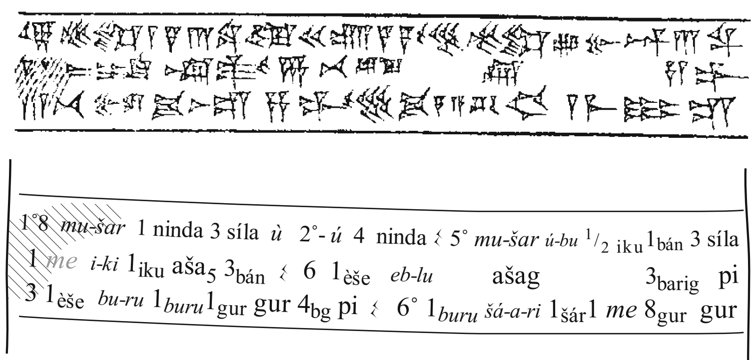


Fig. 3.1.10. AO 6555. A combined structure table for area measures (in the great cubit) and conversion table *A – Ksm* inserted as a subscript in the middle of the colophon.

This brief but puzzling metrological text can be rewritten in the following tabular form:

18	*mu-sar*		1 ninda	3 sìla *ú* 20-*ú* 4 ninda :
50	*mu-sar*	*ú-bu*	1/2(iku)!	1(bán) 3 sìla /
1 [*me*]	<*mu-sar*>	*i-ki*	1(iku) ašag	3(bán) :
6	<1(iku) ašag>	1(èše) *eb-lu* ašag		3(barig) pi /
3	1(èše)	*bu-ru*	1(*buru*)	1(gur) gur 4(barig) pi :
60	1(*buru*)	*šá-a-ri*	1(šár)	1 *me* 8(gur) gur

The organization of the metrological text is now clear: It is a combined structure and conversion table for the following units of area measure:

1 ninda (= 1/100 bùr), 1/2 iku, 1 iku, 1 èše, 1 bùr, 1 sar.

With the exception of ‘1 ninda’, these units are written with their own cuneiform number signs.

To the right of these names for the units of area measure are written *their equivalents in Kassite seed measure,* as in Fig. 1.3.9 above.

To the left of the names of the units of area measures, again with the exception of 1 ninda, are written *their syllabic spellings*. There is an irregularity in the line for 1 èše, which will be explained later. *Further to the left* are indicated *the definitions of the successive units of area measure as multiples of smaller units*.

In the line for 1 iku, the word *mu-sar* is omitted. It is possible that the author of the table was fooled by the circumstance that the word *me* (hundred) came to be written in the place meant for *mu-sar.* In the line for 1 èše, by mistake the the syllabic spelling *eb-lu* is written to the right instead of to the left of 1(èše).

1 ninda as a name for 1/100 bùr probably is not documented elsewhere. The reading of its equivalent in traditional seed measure as 3 sìla *ú* 20-*ú* 4 ninda is conjectural and is based on the mentioned calculation of 1/100 of 1 gur 4 bariga (= 9 bariga = 54 bán = 324 sìla) as equal to 3 sìla 24 gín. It is likely that ninda here instead of gín is a mistake, in particular since normally 1 ninda is 1/10 of a sìla, not 1/100! (See the discussion of W 23281 § 7 in Sec. 3.2.7 below.) Instead of 4 ninda one can read *šá* ninda, in which case 3 sìla and 20 ninda would be an approximation of the correct value. Moreover, it is not clear why 20-*ú* is written instead of simply 20.

It is interesting to note that in the last line of the table *60 is written with six oblique wedges, which confirms that the writer counted with decimal numbers*.

A striking observation is that if the combined structure and conversion table in the subscript of AO 6555 were to be separated into a conversion table followed by a structure table, the result would automatically be a conversion table to be read from left to right as the table in W 23281 § 2, followed by a structure table to be read from right to left as the table in W 23281 § 3!

3.1.4 § 4. A Badly Preserved Range Table for Lengths and Square Areas

Only the last line of this paragraph is preserved:

W 23281 *i*: 46 - *ii*: 1. § 4: Table *L-A*

[... ... ...]	[... ... ...]
ta 6 uš *a-na* šár *ta-nam-*[*din*]	from 6 rods to šár you will [give it]

This badly damaged paragraph in W 23281 appears to be a precise parallel to a much better preserved paragraph in W 23273 (Sec. 3.2.6 below).

3.1.5 § 5. The Linear Growth of a Child in its Mother's Womb

In this text, time measure (counted in days and months) is related to length measure.

W 23281: Table *T-L*, linear growth (Hunger, *NABU* 1994/34)

lú.tur	A child,
u$_4$*-mu šá ina* šag$_4$ ama! -*šú* dù-*ú* 1/2 še *šú-ú* /	the day it is created inside its mother, 1/2 grain it is,
i-na 2-*i u*$_4$*-mu* še *šú-ú* /	in the 2nd day, a grain it is, /
i-na šal-šú u$_4$*-mu* še 1/2 še *šú-ú*	in the third day, a grain 1/2 grain it is,
ina 4-*i u*$_4$*-mu* 2 še *šú-ú* /	in the 4th day, 2 grains it is, /
i-na 5-*šu u*$_4$*-mu* 2 1/2 še *šú-ú*	in its 5th day, 2 1/2 grains it is,
ina 6! -šu *u*$_4$*-mu* 3 še *šú-ú* /	in its 6th! day, 3 grains it is, /
i-na 7-*i u*$_4$*-mu* 3 1/2 še *šú-ú*	in the 7th day, 3 1/2 grains it is,
ina 8-*i u*$_4$*-mu* 4 še *šú-ú* /	in the 8th day, 4 grains it is, /
i-na 9-*i u*$_4$*-mu* 4 1/2 še *šú-ú*	in the 9th day, 4 1/2 grains it is,
ina 10-*i u*$_4$*-mu* 5 še šú.si *šú-ú* /	in the 10th day, 5 grains, a finger it is, /
i-na iti ud$^{\text{meš}}$ 3 šu.si *šú-ú*	in a month of days, 3 fingers it is,
i-na 10 iti$^{\text{meš}}$ 1 kùš *šú-ú*	in 10 months, 1 cubit it is. /

This small metro-mathematical table is, apparently, the answer to an imaginative question of the following type: If the length of an unborn child is 1/2 grain long on the day of its conception and if the child increases its length by 1/2 grain each day in the womb, how long will it be when it is born, after 10(!) months?

According to the table, the length of the child is 5 grains, or 1 finger, on the 10th day. (See line 3 of W 23281 § 1 above, in Sec. 3.1.1.) After a month the length is 3 times more, or 3 fingers, and after 10 months 10 times more, that is 30 fingers or 1 cubit.

There is an interesting spelling error in this text, in the line for the 10th day, with the incorrect spelling šú.si instead of the otherwise ubiquitous spelling šu.si 'finger'.

As observed by Hunger in *NABU* 1996/2, there is another text about the linear growth of a child in one of the paragraphs of W 22646 (von Weiher, *Uruk 2,* text 43).

W 2246 § 2 (*Uruk 2*, 43; Hunger, *NABU* 1996/2)

lú.tur *u_4-mu šá* <*ina*> šag$_4$ ama$^!$-*šú* dù-[*ú*]	A child, the day it is created <in>side its mother,
[*mi*]-*šil uṭ-ṭa-at* / *šú-ú*	[ha]lf a grain / it is.
10 $^1/_2$ še	10 (is) 1/2 grain.
10 a.rá 3[0 x x] ⸢x⸣ túm-*ma* / 5	10 steps 3[0 x x] ⸢x⸣ carry, then / 5 (00).
5 15 : 3 šu.si	5 (is) 15 : 3 fingers.
5 a.rá 10 ⸢x⸣ itu / túm-*ma* 50	5 steps 10 ⸢x⸣ month(s) / carry, then 50.
50 1 *me* 50 še^{meš} 30 šu.si^{meš}	50 (is) 1 hundred 50 grains (is) 30 fingers.
[*ina*] *muḫ-ḫi im-mal-lad*	On account of that, (she) will give birth.

Note the syllabic spelling *mi-šil uṭ-ṭa-at* in line 1, but the logographic spelling 1/2 še in line 2.

There is an interesting metrological error in this text, where 1/2 grain is equated with '10'. Since 5 grains is a finger, 30 fingers a cubit, 6 cubits a reed, and 2 reeds a rod, it is easy to see that

1/2 grain = ;06 finger = ;00 12 cubit = ;00 02 reed = ;00 01 rod.

So, clearly, there is no way in which 1/2 grain could correctly be equated with '10'. The explanation for the error is not difficult to find. Indeed, as a unit of weight measure, a grain is 1/180 shekel, with 1/180 = ;00 20. Therefore, as a unit of weight measure, 1 grain = '20' and 1/2 grain = '10'. This means that the mistake made by the author of this brief text was to confuse a grain as a unit of length measure with a grain as a unit of weight measure!

With 1/2 grain mistakenly equated with '10', the text continues, more or less, as follows:

After one month, on the 30th day, the length of the unborn child is '10' times 30 = '5'.
It is also 1/2 grain times 30 = 15 grains = 3 fingers.
After 10 months, the length of the child is '5' times 10 = '50'.
It is also 15 grains times 10 = 150 grains = 30 fingers.
However, by that time, the child is born.

Apparently, this second text about the growth of a child was written by a more advanced student than the first one, a student who could find a speedier way towards the final result, and also a student who tried (unsuccessfully) to count with sexagesimal numbers representing the length measures he encountered.

3.1.6 § 6. A Catalog of Equations from the 1st Millennium BC

The text of § 6 is damaged, with the beginning or the end or both of the 7 lines in the paragraph often missing. This is unfortunate in view of the fact that part of the terminology in the text is unfamiliar and hard to understand.

One added difficulty is how one should interpret the word spelled a.na which occurs frequently in the lines of this paragraph. Is it the Akkadian preposition *a-na* or is it the Sumerian indeterminate pronoun a.na?

As a matter of fact, this is a difficult question in many known mathematical cuneiform texts, where frequently the word a.na seems to be used as a Sumerian alternative to the Akkadian words *ma-la* or *ki-ma*.

W 23281 *ii*:10-16. § 6:

[x x] x x *ma-ṣa-ri*? an-*e* *ru-ú-q*[*u*] /	[x x] x x of the circumference of the distant sky /
[x x x] kùš *mál-tak-ti* an-*e* /	[x x x] cubits on the water-clock of the sky /
[x x x] ⌜še⌝.bar a.na 1 danna : /	[x x x] of grain as much as 1 league : /
[1] *me* 1 20 še.meš 1 gín kù.babbar *ḫa-a-ṭu* /	[1] hundred 1 20 grains, 1 shekel silver to weigh out /
[a].na 1 *me* 1 20 šár danna.meš a.na 1 dan[na] /	[as] much as 1 hundred 1 20 šár leagues as much as 1 league /
[x] x 1 *me* 1 20 šár danna.meš a.na 1 danna 20 x x x x /	[x] x 1 hundred 1 20 šár leagues as much as 1 league 20 x x x x /
[x] še.bar 1$_{gun}$ gun : 1 kùš *mál-tak-ti* ⌜an-*e*⌝	[x] grain 1 talent : 1 cubit of the water-clock of the sky

Note that in an Old Babylonian problem text from Sippar (Sec. 8.5.5 below) the term *ma-ṣa-rum* 'the go-around' clearly has the meaning 'circumference (of the circular base of a cone)'.

There are unfortunate damages to crucial numerical data in the left parts of all the seven lines. Only the fourth line is completely clear; it says that, as always, 1 shekel equals 180 grains.

The only clue to the meaning of this isolated catalogue of metrological equations is offered by a comparison with the text K9794/AO 6478 (Thureau-Dangin, *RA* 10 (1913); Horowitz, *MCG* (1998)), where the sum of the distances between the *ziqpu*-stars in the path of Enlil is given as 1 + 1/90 times

1 talent's weight (measured by the water-clock) or 12 leagues (= 360 degrees) 'on the ground' or
6 hundred 48 thousand (= 1 hundred 1 20 šár) leagues 'in the sky'.

3.1.7 § 7. Parallel Metrological Lists for Length, Silver, and Grain Numbers

This paragraph contains three *descending* metrological lists of units in Systems *L*, *M*, and *C*, in this order, which is the reverse of the order between the sub-tables in Old Babylonian mixed metrological tables. Therefore, this sub-table, too, was probably intended to be read from right to left.

The three lists progress strictly in parallel. Notably, in one of the lines of the triple list, 1 uš of length (System *L*) corresponds to 1 gín of silver (System *M*) and to 1 bán of grain (System *C*). As for *multiples* of the uš, the gín, and the bán, from 1 to 10 uš correspond to from 1 to 10 gín and to from 1 bán to 1 bariga 4 bán = 10 bán. Similarly, 12 uš, 1/2, 2/3, and 1 danna = 12, 15, 20, and 30 uš correspond to 12, 15 gín, 1/3, 1/2 ma.na = 12, 15, 20, and 30 gín, and to 2 barig, 2 bariga 3 bán, 3 bariga 2 bán and 1 gur = 12, 15, 20 and 30 bán. And so on, even if in the lower part of the text the correspondences are slightly more complicated for *fractions* of the uš, the gín, and the bán.

The pedagogical merits of a text like this are obvious.

W 23281 § 7: Three metrological lists displayed in parallel.

lengths		**weights**			**capacity measures**	
⌜1⌝	danna	1/2	ma.na	kù.babbar	1(gur) gur	še.bar
⌜2/3⌝	danna	1/3	ma.na	kù.babbar	3(barig) 2(bán)	še.bar
[1/2]	danna	15	gín	kù.babbar	2(barig) 3(bán)	še.bar
12	uš	12	gín	kù.babbar	2(barig)! pi	še.bar
10	uš	10	gín	kù.babbar	1(barig) 4(bán)	še.bar
9	uš	9	gín	kù.babbar	1(barig) 3(bán)	še.bar
8	uš	8	gín	kù.babbar	1(barig) 2(bán)	še.bar
7	uš	7	gín	kù.babbar	1(barig) 1(bán)	še.bar
6	uš	6	gín	kù.babbar	1(barig) pi	še.bar
5	uš	5	gín	kù.babbar	5(bán)	še.bar

4	uš	4	gín	kù.babbar	4(bán)		še.bar
3	uš	3	gín	kù.babbar	3(bán)		še.bar
2	uš	2	gín	kù.babbar	2(bán)		še.bar
1	uš	1	gín	kù.babbar	1(bán)		še.bar
⌜50⌝	[nindan]	[3] igi. 4 .gál.la.meš	2 *gír-e*	kù.babbar	5	sìla	še.bar
⌜40⌝	[nindan]	[2]-⌜*ta*⌝ šu$^{\text{II}}$.meš		kù.babbar	4	sìla	še.bar
⌜30⌝	[nindan]	1/2	gín	kù.babbar	3	sìla	še.bar
⌜20⌝	⌜nindan⌝	1/3	gín	kù.babbar	2	sìla	še.bar
10	nindan	6.kam		kù.babbar	1	sìla	še.bar
5	nindan	⌜2	*gi*⌝*-re-e*	kù.babbar	1/2	sìla	še.bar
2 1/2	nindan		⌜*gi*⌝*-ru-ú*	kù.babbar	2 1/2	ninda.ḫá	še.bar
2	nindan	⌜6⌝	še	kù.babbar	2	ninda.ḫá	še.bar
1 1/2	nindan	4 1/2	še	kù.babbar	1 1/2	ninda.ḫá	še.bar
1	nindan	3	še	kù.babbar	1	ninda.ḫá	še.bar
1/2	nindan	1 1/2	še	kù.babbar	1/2	ninda	še.bar
5	kùš	1 4-*ú*	še	kù.babbar	7 *me* 50	ḫi$^{\text{meš}}$	še.⌜bar⌝
4	kùš	(1)	še	kù.babbar	6 [*me*	ḫi$^{\text{meš}}$	še.bar]
3	kùš	3 igi.4.gál.la	še	kù.⌜babbar⌝	[4 *me* 50	ḫi$^{\text{meš}}$	še.bar]
2	kùš	1/2	še	kù.⌜babbar⌝	[3 *me*	ḫi$^{\text{meš}}$	še.bar]
1	kùš	4-[*ú*	še	kù.babbar]	[1 *me* 50	ḫi$^{\text{meš}}$	še.bar]

New names for fractions appearing in the lines for fractions of the gín and the bán are the *girû*, the 3/4 shekel and 3/4 grain, the ninda, and the ḫi. Their respective values are given in the series of factor diagrams below. (The suffixes ḫá and meš can be regarded as just two kinds of plural endings.) The fraction ḫi is previously undocumented, at least in known metro-mathematical cuneiform texts. It is unrealistically small.

L: danna ←30— uš ←60— nindan ←12— kùš
league, length, rod, cubit
10.8 km, 360 m, 6 m, 0.5 m

M: 1/2 ma.na ←30— gín ←24— *girû* ←7 1/2— še
1/2 mina, shekel, carob-seed, grain
250 g., 8 1/3 g., 1/3 g., 1/20 g.

C: gur ←5— barig ←6— bán ←6— sìla ←10— ninda ←1800— ḫi
180 liter, 1 liter, 0.1 liter

Fig. 3.1.11. W 23281 § 7. Factor diagrams for Systems *L*, *M*, and *C*.

In the first line of fractions, 50 nindan = 5/6 uš corresponds to

3 igi.4.gál.la.meš 2 *gir-e* = 3/4 (gín) 2/24 gín = 3/4 1/12 gín = 10/12 gín = 5/6 gín,

and to 5 síla = 5/6 bán. In the next line, 40 nindan = 2/3 uš corresponds to 2-*ta* šu$^{\text{IImeš}}$ = 2/3 (gín) and to 4 síla = 2/3 bán. And so on. In particular, 10 nindan = 1/6 uš corresponds to 6.kam = 1/6 (gín) and to 1 síla = 1/6 bán, and 5 nindan = 1/12 uš corresponds to 2 *gi-re-e* = 2/24 gín = 1/12 gín and to 1/2 síla = 1/12 bán.

One half as much is 2 1/2 nindan = 1/24 uš, *gi-ru-ú* = 1/24 gín, and 2 1/2 ninda = 2 1/2 · 1/60 bán = 1/24 bán. Next follow 2 nindan = 1/30 uš, 6 še = 6/180 gín = 1/30 gín, and 2 ninda = 2/60 bán = 1/30 bán.

Counting with fractions of the uš, the gín, and the bán is now getting increasingly laborious, so it is simpler to proceed as follows for the calculations in the final few lines, with departure from the line

1 nindan 3 še kù.babbar 1 ninda.ḫá še.bar

First, 5 kùš = 5/12 nindan. Correspondingly 1 4-*ú* še = 1 1/4 še = 5/4 še = 5/12 · 3 še, and 750 ḫi = 750 · 1/1800 ninda = 5/12 ninda. And so on.

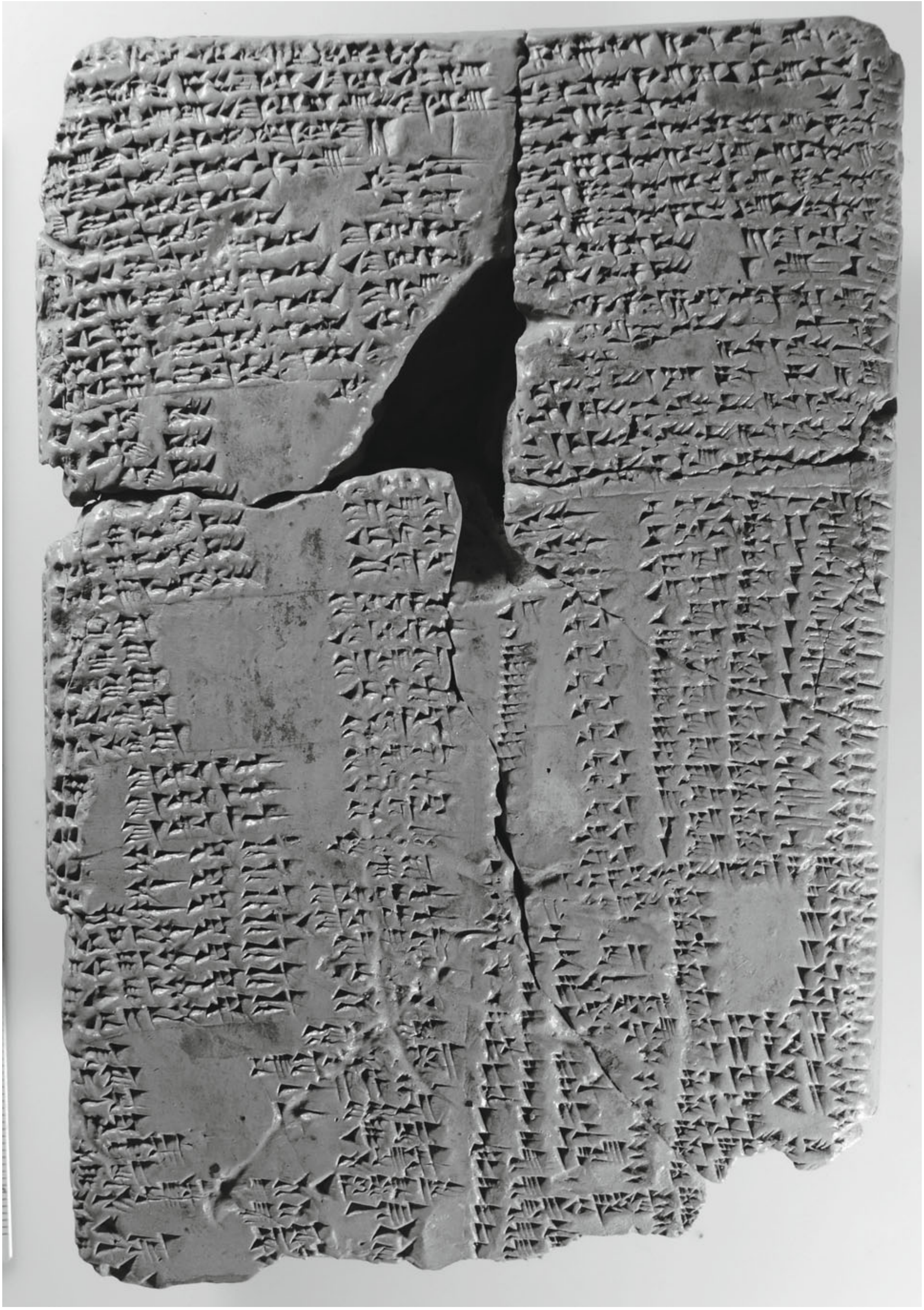


Fig. 3.1.12. W 23281, *obv*., photo (Anmar A. Fahdil/DAI).

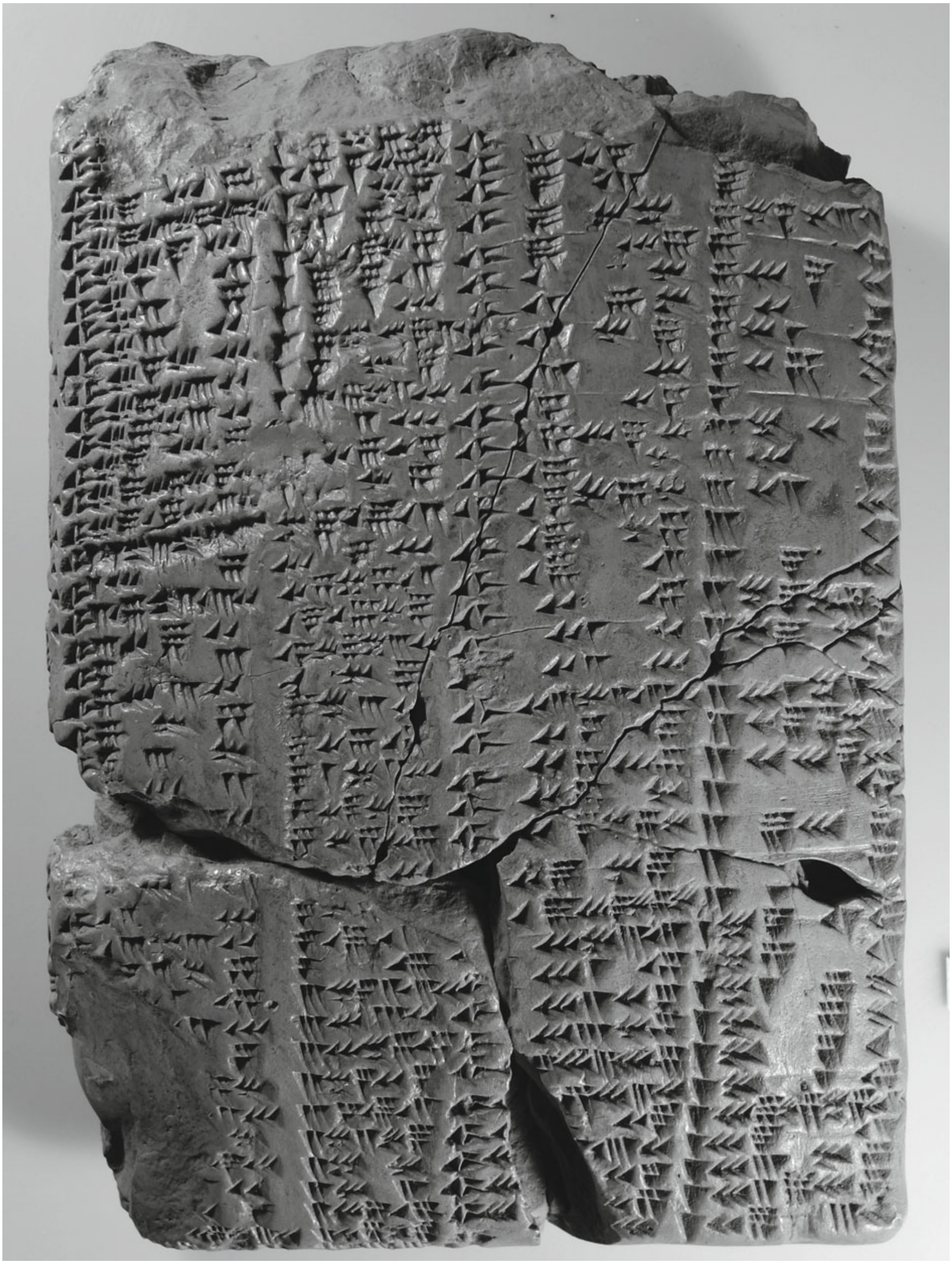


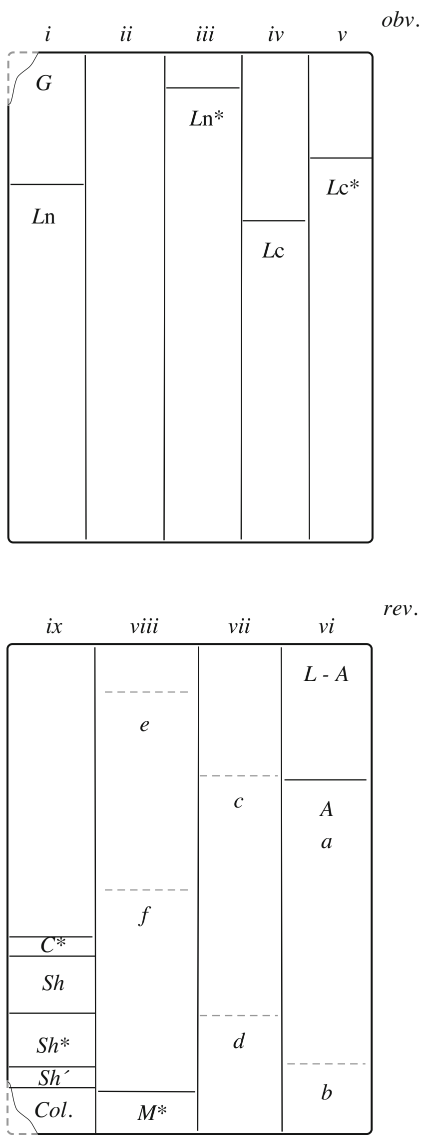
Fig. 3.1.13. W 23281, *rev*., photo.

3.2 W 23273. Another Metrological Recombination Text from Achaemenid Uruk

See the discussion in Sec. 1.3.1 of the provenance of W 23273 from a house in Achaemenid Uruk.

A table of contents for the metrological recombination text W 23273 was exhibited in Friberg, *GMS 3* (1993), no. 11. It is reproduced below. After that follows a conform transliteration of the metrological table text, within outlines of the obverse and reverse of the tablet.

W 23273: Table of contents



Note: Each line of each sub-table except § 6 *reads from right to left*.

§ 1 ***G*:** Gods' names – Gods' numbers
from *Anu* = 1
to En.líl = 50

§ 2 ***Ln*:** Length numbers – Sexagesimal rod multiples
from 1 finger = 10 (60^{-2} rods)
to 1 hundred leagues = 50 (60^{2} rods)

§ 3 ***Ln** *:** Sexagesimal rod multiples – Length numbers
from 10 (60^{-2} rods) = 1 finger
to 1 (60^{2} rods) = 2 leagues

§ 4 ***Lc*:** Length numbers – Sexagesimal cubit multiples
from 1 finger = 2 (60^{-1} cubits)
to 1 length = 12 (60^{1} cubits)

§ 5 ***Lc** *:** Sexagesimal cubit multiples – Length numbers
from [2 (60^{-1} cubits) = 1 finger]
to 12 (60^{1} cubits) = 1 length

§ 6 ***L - A*:** Range table: (squares of) length numbers into area numbers
from 'fingers into grains'
to 'dannas into šárs'

§ 7 ***A*:** Area numbers – Sexagesimal sar multiples
from 1/2 grain = 10 (· 60^{-3} sar)
to 19 šár ašag = 9 30 (· 60^{2} sar)

§ 8 ***M**** Weight numbers – Sexagesimal mina multiples
from 10 (· 60^{-3} mina) = 1/2 grain
to 1 (· 60^{2} minas) = 1 talent = 3 uš šár grains

§ 9 ***C´*:** Catch-line for a table for capacity measure on the next tablet

§ 10 ***Sh*:** Months' names – Shadow lengths at noon in cubits
from month IV – [x] cubit
to [month X – 1 30 (· 60^{-1} cubit)]?

§ 11 ***Sh**** Shadow lengths at noon in cubits – Months' names
from [x (· 60^{-1} cubit) shadow x – month IV]
to 1 30 (· 60^{-1} cubit) shadow x – month X

§ 11 ***Sh´*:** Catch line for a shadow length table on another tablet

§ 11 ***Col.*:** The usual information about owner, etc.

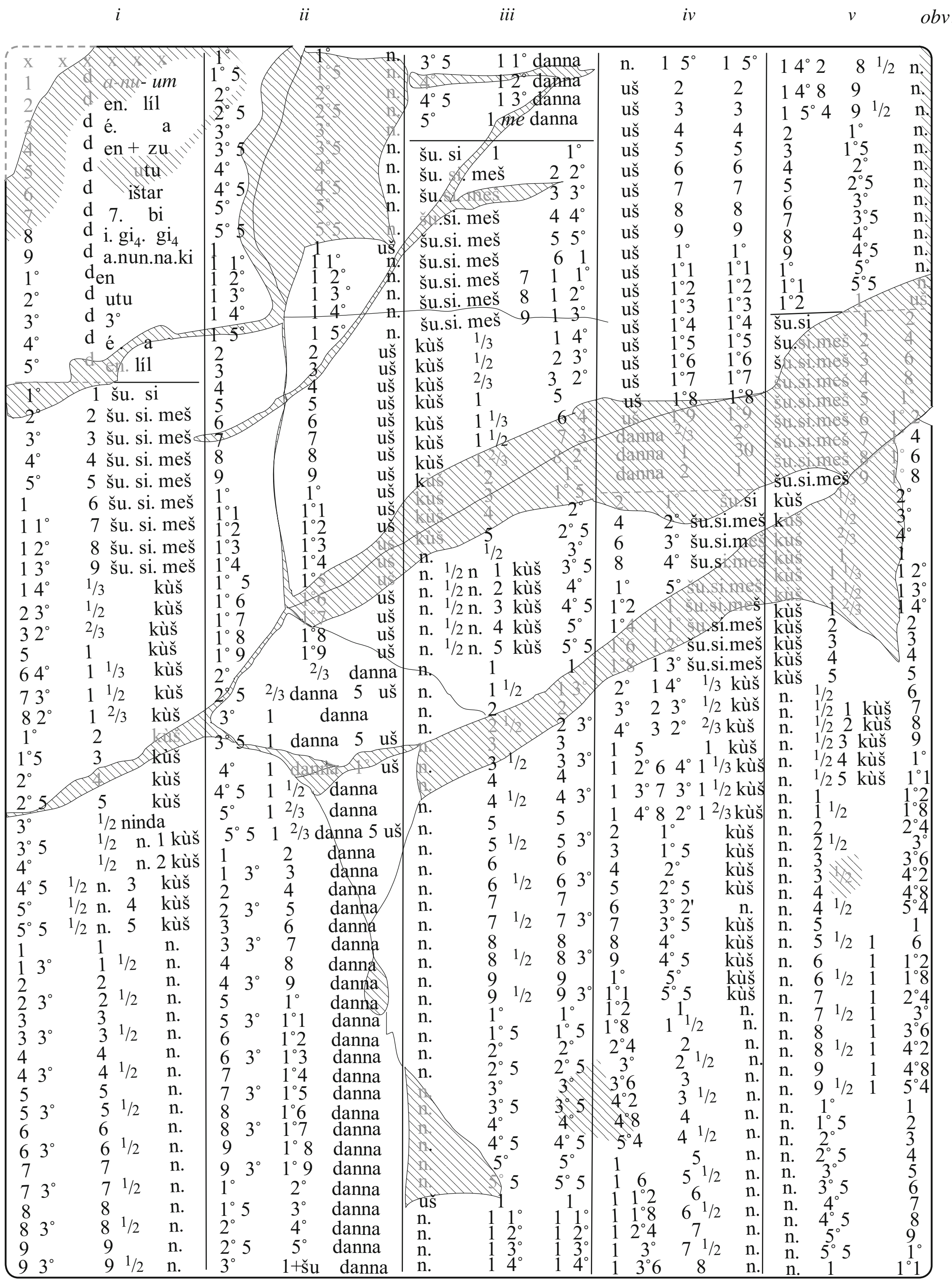


Fig. 3.2.1. W 23273, *obv*. Conform transliteration.

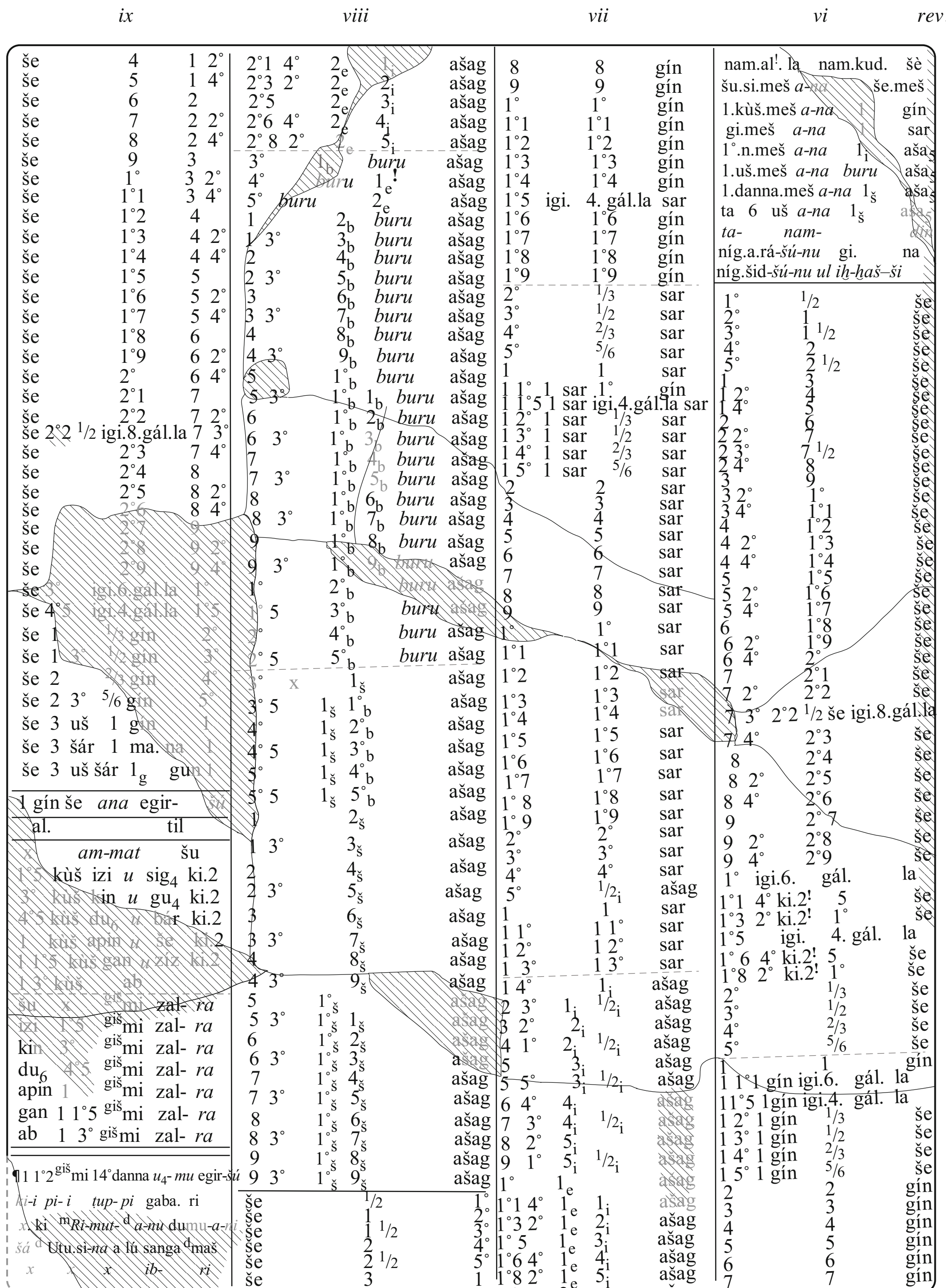


Fig. 3.2.2. W 23273, *rev*. Conform transliteration.

Both in form and content the metrological table text W 23273 has several conspicuous features, characteristic for Late Babylonian metrological tables. (See Friberg *RlA 7* (1990), the last half of § 5.1). In particular, the interior organization of each sub-table of the text, as well as of the organization of the whole set of sub-tables is exactly the opposite of what one is used to see in the case of Old Babylonian "mixed metrological tables". Nevertheless, both the metrological details and the use of sexagesimal numbers in place value notation show that the purpose of §§ 2-8 of W 23273 was to enable Achaemenid students of mathematics to read and understand Old Babylonian mathematical and metrological texts.

To be more specific, just as in the case of §§ 1a-f and 2 of W 23281 above (Sec. 3.1), it is clear that each line of each sub-table on W 23273 was meant to be *read from right to left*. Moreover, (complete) mixed Old Babylonian combined metrological tables always contain the sub-tables *C*, *M*, *A*, *L* (for capacity measure, weight or metal measure, area measure, and length measure), in this order. See Friberg, *MSCT 1* (2007), 396-398. Mixed Late Babylonian metrological tables, on the other hand contain the corresponding sub-tables in the reverse order *L*, (*A*), *W*, *C*. Note that the inclusion of a sub-table for area measure is an anachronistic trait, since in cuneiform texts from the first millennium surface content was measured in reed measure or seed measure, not in area measure. (See Friberg, et al., "Seed and Reeds" *BaM* 21 (1990), and Friberg "Seed and Reeds Continued" *BaM* 28 (1997).)

The reason for the reverse order is easy to understand. In Mesopotamia in the Late Babylonian period, the traditional writing on clay tablets using the cuneiform script had largely been replaced by writing in Aramaic on some more perishable material using the Aramaic script, which was written and read from right to left. Someone used to reading texts and tables in the Aramaic way from right to left might have found it natural to write both the sub-tables and the individual lines in those sub-tables in the reverse order when copying from one or several older metrological table texts, written from left to right.

3.2.1 § 1. Gods' Names and Gods' Numbers (reads from right to left)

W 23273 § 1: Table *G*

[1]	[^{d}*a-nu*]-*um*	[*An*]*u*	=	[1]
[2]	[^{d}]en.lil	Enlil	=	[2]
[3]	^{d}é.a	*Ea*	=	[3]
[4]	^{d}en+zu	*Sîn* (the Moon)	=	[4]
[5]	^{d}[u]tu	*Šamaš* (the Sun)	=	[5]
[6]	^{d}ištar	*Ishtar*	=	[6]
[7]	^{d}7.bi	the Seven	=	[7]
8	^{d}i.gi$_4$.gi$_4$	the Igigi	=	8
9	^{d}a.nun.na.ki	the Anunaki	=	9
10	^{d}en	*Bēl*	=	10
20	^{d}utu	*Šamaš*	=	20
30	^{d}30	*Sîn*	=	30
40	^{d}é.a	*Ea*	=	40
50	^{d}en.lil	Enlil	=	50

In this pseudo-mathematical table, the main Mesopotamian gods Anu, Enlil, and Ea are mentioned first, then the Moon, the Sun, and the goddess Ishtar. After them are listed the Seven, the Igigi, and the Anunaki, various Mesopotamian groups of gods or heavenly spirits. In reverse order, from the end of the table, come again Enlil, Ea, the Moon, and the Sun. (In this reverse order, the mention of Anu with the number 1 (= 60) has been regarded as superfluous.)

As is well known, a similar enumeration of gods' names and numbers appears in the Niniveh tablet K 170 (King, *CT 25* (1909), 50.) In that text, which is much more elaborate than the simple table above, the enumeration of gods and numbers starts with 1 for Anu, 50 for Enlil, 40 for Ea, 30 for the Moon, 20 for the Sun. The ensuing numbers in K 170 for other gods are 6, 10, 15, 50, 14, and 10, with no relation at all to the gods and numbers in W 23273 § 1.

More about Mesopotamian gods' numbers can be found in Röllig's article "Götterzahlen" in *Reallexikon der Assyriologi*e, vol. 3.

3.2.2 § 2. A Conversion Table from Length Numbers to Sexagesimal nindan *Multiples (reads from right to left)*

W 23273 § 2: Table *Ln*

[10]	1 šu.si	[1 finger	=	10] (· 60^{-2} rods)
20	2 šu.si.meš	2 fingers	=	20 (· 60^{-2} rods)
…	………	………		………
1 30	9 šu.si.meš	9 fingers	=	1 30 (· 60^{-2} rods)
1 40	$^{1}/_{3}$ kùš	1/3 cubit	=	1 40 (· 60^{-2} rods)
2 30	$^{1}/_{2}$ kùš	1/2 cubit	=	2 30 (· 60^{-2} rods)
3 20	$^{2}/_{3}$ kùš	2/3 cubit	=	3 20 (· 60^{-2} rods)
5	1 kùš	1 cubit	=	5 (· 60^{-1} rods)
…	………	……		……
25	5 kùš	5 cubits	=	25 (· 60^{-1} rods)
30	$^{1}/_{2}$ nindan	1/2 rod	=	30 (· 60^{-1} rods)
35	$^{1}/_{2}$ nindan 1 kùš	1/2 rod 1 cubit	=	35 (· 60^{-1} rods)
…	………	…………		……
55	$^{1}/_{2}$ nindan 5 kùš	1/2 rod 5 cubits	=	55 (· 60^{-1} rods)
1	1 nindan	1 rod	=	1
1 30	1 $^{1}/_{2}$ nindan	1 1/2 rod	=	1 30 (· 60^{-1} rods)
…	………	………		……
9 30	9 $^{1}/_{2}$ nindan	9 1/2 rod	=	9 30 (· 60^{-1} rods)
10	10 nindan	10 rods	=	10
15	15 nindan	15 rods	=	15
…	………	……		……
55	55 nindan	55 rods	=	55
1	1 uš	1 length	=	1 (· 60^{1} rods)
…	………	……		……
19	19 uš	19 lengths	=	19 (· 60^{1} rods)
20	$^{2}/_{3}$ danna	2/3 league	=	20 (· 60^{1} rods)
…	………	………		………
55	1 $^{2}/_{3}$ danna 5 uš	1 2/3 leagues 5 lengths	=	55 (· 60^{1} rods)
1	2 danna	2 leagues	=	1 (· 60^{2} rods)
…	………	……		………
30	1+*šu* danna	sixty leagues	=	30 (· 60^{2} rods)
…	………	………		………
45	1 30 danna	1 30 (90) leagues	=	45 (· 60^{2} rods)
50	1 *me* danna	1 hundred leagues	=	50 (· 60^{2} rods)

This extensive conversion table with 157 entries is almost precisely of the same kind as the corresponding Old Babylonian Table *Ln* (*n* for nindan) described in Friberg *MSCT 1* (2007), 392-293, which has 133 entries, and which begins with the following entries:

1	šu.si	10	1	finger	=	10 (· 60^{-2} rods)
1 $^{1}/_{2}$	šu.si	15	1 $^{1}/_{2}$	fingers	=	15 (· 60^{-2} rods)
2	šu.si	20	2	fingers	=	20 (· 60^{-2} rods)
…	……	…	…	……		…

The two tables differ only in minor details, with the following significant exceptions:

1) The Old Babylonian Table *Ln* reads *from left to right*, the Achaemenid Table *Ln from right to left*.
2) The Old Babylonian table *ends with the entry for sixty leagues*, the Achaemenid table with *the entry for a hundred leagues*.

The basic idea is the same in both cases, namely that the table should be constructed in such a way that

1) The sequence of listed length numbers *increases in small steps*, and *mentions all the main units* (fingers, cubits, rods, lengths, leagues) of the Old Babylonian system of length measures.
2) Only length numbers are listed which are *integral or half-integral multiples* of the main length units, or *a basic fraction* (1/3, 1/2, 2/3), or *1 plus a basic fraction*, times such main length units.
3) The sexagesimal multiples of the (silently understood) basic length unit, the nindan, are numbers with only one digit (1, 2, …, 9, 1°, 2°, 3°, 4°, 5°) or two digits (*a ten and a one*, as in 1°1, 1°2, …, or *a one and a ten*, as in 1 3°, 1 4°, 2 3°, 3 2°).

With rules such as these in mind, a student could easily on his own construct a new copy of Table *Ln*, or any other metrological table of the Old Babylonian kind, without having to copy someone else's table text! (This is probably the reason why there appear to be no known couples of cuneiform metrological tables that are equal in all details.)

Anyone intending to construct metrological tables of the same kind as Table *Ln* above had also to be familiar with the *conversion rules* for the systems of measures figuring in the metrological tables. In the case of Table *Ln*, the needed conversion rules are simply and succinctly described by the following *factor diagram* for Old Babylonian units of length measure, with the rod (nindan) as the basic unit:

	league		length		rod		cubit		finger
Ln(OB):	danna	← 30	uš	← 60	nindan	← 12	kùš	← 30	šu.si
	30 00 n.		1 00 n.		1 n.		;05 n.		;00 10 n.

Fig. 3.2.3. W 23273. Factor diagram for System *Ln*(OB).

What this factor diagram implies is that *in cuneiform texts from the Old Babylonian period*,

30 fingers = 1 cubit, 12 cubits = 1 rod, 60 rods = 1 length, 30 lengths = 1 league.

In cuneiform texts from the first millennium BC, these rules were no longer operative, except of course in school texts like W 23273 which had as a goal to show how Old Babylonian systems of measures were constructed.

3.2.3 § 3. *A Reverse Conversion Table from Sexagesimal* nindan *Multiples to Length Numbers (reads from right to left)*

W 23273 § 3: Table *Ln**

šu.si	1	10	10 (· 60^{-2} rods)	=	1	finger
šu.si.meš	2	20	20 (· 60^{-2} rods)	=	2	fingers
……	…	…	…………			…………
šu.si.meš	9	1 30	1 30 (· 60^{-2} rods)	=	9	fingers
kùš	$^{1}/_{3}$	1 40	1 40 (· 60^{-2} rods)	=	$^{1}/_{3}$	cubit
kùš	$^{1}/_{2}$	2 30	2 30 (· 60^{-2} rods)	=	$^{1}/_{2}$	cubit
kùš	$^{2}/_{3}$	3 20	3 20 (· 60^{-2} rods)	=	$^{2}/_{3}$	cubit
kùš	1	5	5 (· 60^{-1} rods)	=	1	cubit
……	…	…	…………			………
kùš	5	25	25 (· 60^{-1} rods)	=	5	cubits
nindan	$^{1}/_{2}$	30	30 (· 60^{-1} rods)	=	$^{1}/_{2}$	rod
……	…	…	……	…	…	
nindan	$^{1}/_{2}$ nindan 5 kùš	55	55 (· 60^{-1} rods)	=	$^{1}/_{2}$ rod 5 cubits ((rod))	
nindan	1	1	1 (rod)	=	1	rod
nindan	1 $^{1}/_{2}$	1 30	1 30 (· 60^{-1} rods)	=	1 $^{1}/_{2}$	rod
……	…	…	……	…	…	
nindan	9 $^{1}/_{2}$	9 30	9 30 (· 60^{-1} rods)	=	9 $^{1}/_{2}$	rods
nindan	10	10	10 (rods)	=	10	rods

nindan	15	15	15 (rods)	=	15	rods	
……	…	…	……	…	…		
nindan	55	55	55 (rods)	=	55	rods	
uš	1	1	1 (· 60^1 rods)	=	1	length	
uš	1 10	1 10	1 10 (rods)	=	1 10	lengths	(should be 1 length 10 rods)
……	…	…	……	…	…		
uš	1 50	1 50	1 50 (rods)	=	1 50	lengths	(should be 1 length 50 rods)
uš	2	2	2 (· 60^1 rods)	=	2	lengths	
……	…	…	……	…	…		
[uš	19	19]	[19 (· 60^1 rods)	=	19	lengths]	
[danna	$^2/_3$	20]	[20 (· 60^1 rods)	=	$^2/_3$	leagues]	
[danna	1	30]	[30 (· 60^1 rods)	=	1	leagues]	
[danna	2	1]	[1 (· 60^2 rods)	=	2	leagues]	

Note that there is no known Old Babylonian reverse conversion table from sexagesimal numbers to any kind of measure numbers! So, a table of this kind seems to be an innovation (of doubtful value).

3.2.4 § 4. A Conversion Table from Length Numbers to Sexagesimal Cubit Multiples (reads from right to left)

Examples of Old Babylonian conversion tables from length numbers to sexagesimal cubit multiples (Table *Lc*) are well known. See Friberg, *MSCT 1* (2007), 394-395. Such tables were obviously useful when counting, for instance, with volume measure, since the length and width of an object was usually expressed in multiples of the rod, while the height of the object was expressed in multiples of the cubit.

W 23273 § 4: Table *Lc*

[2	10	1	šu.si]	[1	finger	=	10 (· 60^{-2} rods)	=	2	(· 60^{-1} cubits)]
4	20		šu.si.meš	<2>	fingers	=	20 (· 60^{-2} rods)	=	4	(· 60^{-1} cubits)
…	…		……	…	……		…………		……	
[18]	1 30		šu.si.meš	<9>	fingers	=	1 30 (· 60^{-2} rods)	=	[18]	(· 60^{-1} cubits)
20	1 40	$^1/_3$	kùš	1/3	cubit	=	1 40 (· 60^{-2} rods)	=	20	(· 60^{-1} cubits)
30	2 30	$^1/_2$	kùš	1/2	cubit	=	2 30 (· 60^{-2} rods)	=	30	(· 60^{-1} cubits)
40	3 20	$^2/_3$	kùš	2/3	cubit	=	3 20 (· 60^{-2} rods)	=	40	(· 60^{-1} cubits)
1	5	1	kùš	1	cubit	=	5 (· 60^{-1} rods)	=	1	(cubit)
1 20	6 40	1 $^1/_3$	kùš	1 1/3	cubits	=	6 40	=	1 20	(· 60^{-1} cubits)
1 30	7 30	1 $^1/_2$	kùš	1 1/2	cubits	=	7 30	=	1 30	(· 60^{-1} cubits)
1 40	8 20	1 $^2/_3$	kùš	1 2/3	cubits	=	8 20	=	1 40	(· 60^{-1} cubits)
2	10		kùš	<2>	cubits	=	10 (· 60^{-1} rods)	=	2	(cubits)
3	15		kùš	<3>	cubits	=	15 (· 60^{-1} rods)	=	3	(cubits)
…	…		……	…	……		…………		……	
11	55		kùš	<11>	cubits	=	55 (· 60^{-1} rods)	=	11	(cubits)
12	1		nindan	1	rod			=	12	(cubits)
18	1 $^1/_2$		nindan	1 1/2	rods			=	18	(cubits)
…	…		……	…	…				………	
1 54	9 $^1/_2$		nindan	9 1/2	rods			=	1 54	(cubits)
2	10		nindan	10	rods			=	2	(· 60^1 cubits)
…	…		……	…	…				………	
11	55		nindan	55	rods			=	11	(· 60^1 cubits)
12	1		uš	1	length			=	12	(· 60^1 cubits)

This fourth sub-table on W 23273 is *corrupt*, with several incorrect entries. More precisely, the entries in the last part of this table are correct, from the entry ‘1 rod = 12’ (cubits) to the final entry ‘1 length = 12 (00 cubits)’. All the preceding entries incorrectly list not only sexagesimal multiples of the cubit, but also sexagesimal multiples of the rod, as for instance the entry ‘1/3 cubit = ;01 40 (rods) = ;20 (cubits)’. It is likely that this double standard was in place already in the original table text of which W 23273 § 4 is a copy. If that is so, then the copyist may have been trying to delete superfluous numbers in several of the entries. Unfortunately, he did not understand what he was doing. For instance, in the entry ‘<2> fingers = ;00 20 rods = ;04 cubits’ the number 2 before ‘fingers’ has been falsely deleted.

3.2.5 § 5. A Reverse Conversion Table from Sexagesimal Cubit Multiples to Length Numbers (reads from right to left)

In the corresponding reverse conversion table, the unwarranted multiples of the rod are no longer there. The only incorrect entries are the last 11 ones, where the integers from 1 to 11 must be replaced by the integers from 2 to 12. Moreover, in the last line, 1 (· 60) rods should be instead 1 uš = 1 length.

W 23273 § 5: Table *Lc**

šu.si	[1	2]	[2 (· 60^{-1} cubit)	=	1]	finger	
[šu.si.meš	2	4]	4 (· 60^{-1} cubit)	=	2	fingers]	
……	…	…	…		…	……	
[šu.si.meš	9	1]8	[1]8 (· 60^{-1} cubit)	=	[9	fingers]	
kùš	1/3	20	20 (· 60^{-1} cubit)	=	1/3	cubit	
…	…	…	…		…	……	
kùš	1 2/3	1 40	1 40 (· 60^{-1} cubit)	=	1 2/3	cubits	
kùš	2	2	2 (cubits)	=	2	cubits	
…	…	…	…		…	……	
kùš	5	5	5 (cubits)	=	5	cubits	
nindan	1/2	6	6 (cubits)	=	1/2	rod	
nindan	1/2 1 kùš	7	7 (cubits)	=	1/2 rod 1 cubit		
…	………	…	…		……………		
nindan	1/2 5 kùš	11	11 (cubits)	=	1/2 rod 5 cubits		
nindan	1	12	12 (cubits)	=	1	rod	
nindan	1 1/2	18	18 (cubits)	=	1 1/2	rods	
…	…	…	…		…	……	
nindan	9 1/2	1 54	1 54 (cubits)	=	9 1/2	rods	
nindan	10	1	$2^{!}$ (· 60 cubits)	=	10	rods	
nindan	15	2	$3^{!}$ (· 60 cubits)	=	15	rods	
…	…	…	…		…	…	
nindan	55	10	$11^{!}$ (· 60 cubits)	=	55	rods	
nindan	1	11	$12^{!}$ (· 60 cubits)	=	1 (· 60)	rods	should be: 1 length

3.2.6 § 6. A Range Table from (Squares of) Length Measures to Area Measures

W 23273 § 6: Table *L-A*

nam.al$^{?}$.la nam.kud.šè			For hoeing$^{?}$ and cutting:		
šu.si.meš	*a-na*	še.meš	fingers	to	grains
1.kùš.meš	*a-na*	gín	cubits	to	shekels
gi.meš	*a-na*	sar	reeds	to	sars
10.nindan.meš	*a-na*	iku ašag	tens of rods	to	ikus
1.uš.meš	*a-na*	*buru* ašag	lengths	to	*burus*
1.danna.meš	*a-na*	šár ašag	leagues	to	šárs

ta 6 uš *a-na* šár ašag	from 6 lengths to šárs
ta-nam-[*din*]	you will gi[ve].
níg.a.rá-*šu-nu* gi.na	Their multiplications will be correct.
níg.šid-*šu-nu ul iḫ-ḫaš-ši*	Their account will not be obscure.

The basic connection between Old Babylonian length measure and Old Babylonian area measure is that the area of a square with the side 1 nindan (a rod) is 1 sar (garden plot). More briefly:

1 sq. nindan = 1 sar.

Other units of Old Babylonian area measure are either multiples or fractions of the sar. The structure of the Old Babylonian system of area measure is simply and succinctly described by the following *factor diagram* for Old Babylonian units of area measure, with the sar as the basic unit:

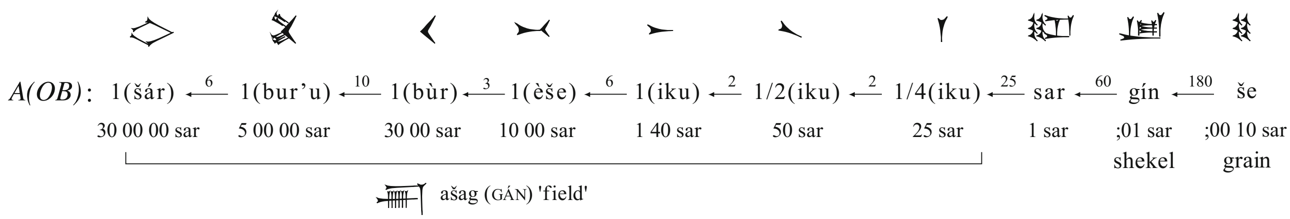


Fig. 3.2.4. W 23273 § 6a. Factor diagram for the Old Babylonian system of area measure.

What this factor diagram implies is that *in cuneiform texts from the Old Babylonian period*,

100 sar = 1 iku, 6 iku = 1 èše, 3 èše = 1 bùr, 10 bùr = 1 bur'u, 6 bur'u = 1 šár

180 grains = 1 shekel, 60 shekels = 1 sar

Multiples of area units above the sar are always followed by the determinative ašag (GÁN) 'field'.

Note that in §§ 6 and 7 of W 23273, a variant of the Old Babylonian area number sign 1(bur'u) is used phonetically instead of the Old Babylonian sign 1(bùr) as a logogram for bùr (Akkadian *buru*).

In Old Babylonian mathematical texts, computations of areas of rectangular fields were computed in the following way: First the given lengths of the two sides of a rectangle were converted from ordinary length numbers to two sexagesimal numbers (silently understood as multiples of 1 rod), in the Babylonian relative place value notation for sexagesimal numbers, without zeros. Then the two sexagesimal numbers were multiplied with each other. The result was a new sexagesimal number (silently understood as a multiple of 1 square rod = 1 sar). Finally, this sexagesimal number was converted back into an area number. This way of computing areas with the help of a detour into the world of sexagesimal numbers was, of course, the reason for the existence of Old Babylonian metrological tables for length and area measure.

A complication with this clever procedure was that the computed sexagesimal multiple of 1 sar was only given in *relative* place value notation, without any indication of the actual size of the number. It is in this situation that a range table like W 23273 § 6 can be useful. (For some reason, no Old Babylonian range tables of this kind are known. However, this does not necessarily mean that they did not exist.)

The meaning of the mentioned range table can be explained as follows. It is intended to be of help in calculating areas of field, for instance in order to be able to estimate the work needed for hoeing the field, or the yield obtained when cutting something growing in the field. For this purpose, a good start is to make the following recursive sequence of calculations:

sq. (1 reed)	= 1/4 sq. (1 rod)	= 1/4 sar	
sq. (10 rods)	= 1 40 · sq. (1 rod)	= 1 iku	
sq. (1 length)	= 36 · sq. (10 rods)	= 36 iku	= 2 bùr
sq. (1 league)	= 15 00 · sq. (1 length)	= 30 00 bùr	= 30 šár
sq. (6 lengths)	= 36 · sq. (1 length)	= 1 12 bùr	= 1 šár 12 bùr

In view of these simple calculations, it is clear that, as stated in the range table

for a square with a side length of between 1 reed and 10 rods, the area can be measured in multiples of 1 sar,
for a square with a side length of between 10 rods and 1 length, the area can be measured in multiples of 1 iku,
for a square with a side length of between 1 length and 1 leagues, the area can be measured in multiples of 1 bùr,
for a square with a side length of 1 league and above, the area can be measured in multiples of 30 šár.

Since the last of these estimates is too imprecise, the range table makes the following amendment:

for a square with a side length of 6 lengths and above, the area can be measured in multiples of 1 šár.

Going in the other direction, one finds that, in a similar way,

sq. (1 cubit)	= 1/36 sq. (1 reed)	= 1/36 · 1/4 sar	= 15/36 · 1 shekel	= appr. 1/2 shekel,
sq. (1 finger)	= 1/(15 00) sq. (1 cubit)	= 1/(30 00) · 1 shekel	= 1/10 grain.	

Consequently, as stated in the range table,

for a square with a side length of between 1 finger and 1 cubits, the area can be measured in multiples of 1 grain,
for a square with a side length of between 1 cubit and 1 reed, the area can be measured in multiples of 1 shekel.

L-sq. :	1.danna.meš	← 25	6.uš.meš	← 36	1.uš.meš	← 36	10.ninda.meš	← 400	gi.meš	← 36	1.kùš.meš	← 900	šu.si.meš
	sq. league		sq. (6 lengths)		sq. length		sq. ropes		sq. reed		sq. cubit		sq. finger
A :			šár	← 30	2 bùr	← 36	iku	← 400	1/4 šar	← 30	1/2 gín	← 900	1/10 še.meš
									garden		shekel		barleycorns

Fig. 3.2.4. W 23273 § 6b. An explanation of the range table in terms of factor diagrams.

3.2.7 § 7. A Conversion Table from Area Numbers to Sexagesimal sar *Multiples (reads from right to left)*

Note that in this table, the sar and its fractions (the shekels and the grains) are counted with ordinary sexagesimal number signs (counting numbers), while multiples of area units above the sar are written with special *area number signs*, repeated as many times as needed.

In Fig 3.2.2 above (just before Sec. 3.2.1), *cols.* viii-ix, the conform transliteration of Table *A* (§ 7) on W 23273 *rev.* has been considerably simplified, for the readers' convenience, by writing 1_i instead of 1(iku), 2_i instead of 1(iku) 1(iku), 3_i instead of 1(iku) 1(iku), 1(iku), etc. A similar use has been made of the abbreviated notations 1_e instead of 1(èše), 2_e instead of 1(èše) 1(èše), and so on, and so forth.

W 23273 § 7 is a very long metrological table, almost filling three columns on the reverse of W 23273. For the readers' convenience, the table is divided below into sub-tables, roughly corresponding to the division into ranges suggested by the range table in the preceding paragraph: grain multiples, shekel multiples, and so on.

W 23273 § 7 a: Table *A*, shekel fractions

10	1/2	še	1/2	grain	=	10	(· 60^{-3} sar)
20	1	še	1	grain	=	20	(· 60^{-3} sar)
30	1 1/2	še	1 1/2	grains	=	30	(· 60^{-3} sar)
…	…	…	…	…		………	
7 20	22	še	22	grains	=	7 20	(· 60^{-3} sar)
7 30	22 1/2	še igi.8.gál.la	an 8th = 22 1/2	grains	=	7 30	(· 60^{-3} sar)
7 40	23	še	23	grains	=	7 40	(· 60^{-3} sar)
…	…	…	…	…		………	

9 40	29	še	29	grains	=	9 40	(· 60^{-3} sar)
10	igi.6.gál.la		a 6th		=	10	(· 60^{-2} sar)
11 40	ki.2 5	še	ditto 5	grains	=	11 40	(· 60^{-3} sar)
13 20	ki.2 10	še	ditto 10	grains	=	13 20	(· 60^{-3} sar)
15	igi.4.gál.la		a 4th		=	15	(· 60^{-2} sar)
16 40	ki.2 5	še	ditto 5	grains	=	16 40	(· 60^{-3} sar)
18 20	ki.2 10	še	ditto 10	grains	=	18 20	(· 60^{-3} sar)
20	$^{1}/_{3}$	še	1/3	grains	=	20	(· 60^{-2} sar)
30	$^{1}/_{2}$	še	1/2	grains	=	30	(· 60^{-2} sar)
40	$^{2}/_{3}$	še	2/3	grains	=	40	(· 60^{-2} sar)
50	$^{5}/_{6}$	še	5/6	grains	=	50	(· 60^{-2} sar)

Here are examples of how the entries in this sub-table can be computed, somewhat anachronistically,

1/2 grain =	1/360 shekel =		10 (· 60^{-3} sar)
1 grain =	1/180 shekel =		20 (· 60^{-3} sar)
22 grains =		22 · 20 (· 60^{-3} sar) =	7 20 (· 60^{-3} sar)
1/8 shekel =	22 1/2 grains =	22 1/2 · 20 (· 60^{-3} sar) =	7 30 (· 60^{-3} sar)
1/6 shekel =	;10 shekel =		10 (· 60^{-2} sar)
1/6 shekel 5 grains =	35 grains =	35 · 20 (· 60^{-3} sar) =	11 40 (· 60^{-3} sar)
1/4 shekel =	;15 shekel =		15 (· 60^{-2} sar)
1/4 shekel 5 grains =	50 grains =	50 · 20 (· 60^{-3} sar) =	16 40 (· 60^{-3} sar)
1/3 shekel =	;20 shekel =		20 (· 60^{-2} sar)
5/6 shekel =	;50 shekel =		50 (· 60^{-2} sar)

Note that, confusingly, '1/3 grains' does not stand for 1/3 grain but for 1/3 shekel (in the range of the grains), and so on. Such oddities are not unusual in Old Babylonian mathematical/metrological texts.

W 23273 § 7 b: Table *A*, shekel multiples

1	1	gín	1 shekel		=	1	(· 60^{-1} sar)
1 10	1 gín igi.6.gál.la		1 shekel a 6th		=	1 10	(· 60^{-2} sar)
1 20	1 gín $^{1}/_{3}$	še	1 shekel 1/3	grains	=	1 20	(· 60^{-2} sar)
1 30	1 gín $^{1}/_{2}$	še	1 shekel 1/2	grains	=	1 30	(· 60^{-2} sar)
1 40	1 gín $^{2}/_{3}$	še	1 shekel 2/3	grains	=	1 40	(· 60^{-2} sar)
1 50	1 gín $^{5}/_{6}$	še	1 shekel 5/6	grains	=	1 50	(· 60^{-2} sar)
2	2	gín	2 shekels		=	2	(· 60^{-1} sar)
…	…	…	………			…	
14	14	gín	14 shekels		=	14	(· 60^{-1} sar)
15	igi.4.gál.la	sar	a 4th	sar	=	15	(· 60^{-1} sar)
…	…	…	…			…	
19	19	gín	19 shekels		=	19	(· 60^{-1} sar)

Here, too, '1/3 grains' does not stand for 1/3 grain but for 1/3 shekel (in the range of the grains), and so on.

W 23273 § 7 c: Table *A*, sar multiples

20	$^{1}/_{3}$	sar	1/3	sar	=	;20	(sar)
…	…	…	…	…		…	
50	$^{5}/_{6}$	sar	5/6	sar	=	;50	(sar)
1	1	sar	1	sar	=	1	(sar)
1 10	1 sar 10	gín	1 sar 10	shekels	=	1 10	(· 60^{-1} sar)

1 15	1 sar igi.4.gál.la	sar	1 sar a 4th	sar	=	1 15	(· 60^{-1} sar)
1 20	1 sar 1/3	sar	1 sar 1/3	sar	=	1 20	(· 60^{-1} sar)
…	………	…	…	…		…	
2	2	sar	2	sar	=	2	(sar)
…	…	…	…	…		…	
40	40	sar	40	sar	=	40	(sar)
50	1/2(iku)	ašag	1/2 iku		=	50	(sar)
1	1	sar	1 (· 60^1 sar)	sar	=	1	(· 60^1 sar)
…	…	…	…	…		…	
1 30	1 30	sar	1 30	sar	=	1 30	(sar)

Note the use of a special number sign for 1/2 iku.

W 23273 § 7 d: Table *A*, iku multiples

1 40	1(iku)	ašag	1 iku	=	1 40	(sar)
2 30	1(iku) 1/2(iku)	ašag	1 iku 1/2 iku	=	2 30	(sar)
…	…	…	…	…………		
9 10	5(iku) 1/2(iku)	ašag	5 iku 1/2 iku	=	9 10	(sar)
10	1(èše)	ašag	1 èše	=	10	(· 60^1 sar)
11 40	1(èše) 1(iku)	ašag	1 èše 1 iku	=	11 40	(sar)
…	……	…	…	…………		
18 20	1(èše) 5(iku)	ašag	1 èše 5 iku	=	18 20	(sar)
20	2(èše)	ašag	2 èše	=	20	(· 60^1 sar)
…	…	…	…	……		
28 20	2(èše) 5i(ku)	ašag	2 èše 5 iku	=	28 20	(sar)

Here special number signs are used for multiples of the iku and the eše. The entries in the sub-table are easily calculated as in the following examples:

1(iku) =	100 sar =	1 40 (sar)
5(iku) 1/2(iku) =	5 1/2 · 1 40 sar =	9 10 (sar)
2(èše) 5(iku) =	17 · 1 40 sar =	28 20 (sar)

W 23273 § 7 e: Table *A*, *buru* multiples

30	[1(bùr)] *buru* ašag	[1 bùr]	=	30	(· 60^1 sar)
40	[1(bùr)] 1(èše)! ašag	[1 bùr] 1 èše	=	40	(· 60^1 sar)
50	*buru* 2(èše) ašag	1 bùr 2 èše	=	50	(· 60^1 sar)
1	2(bùr) *buru* ašag	2 bùr	=	1	(· 60^2 sar)
…	……… …	……			…
4 30	9(bùr) *buru* ašag	9 bùr	=	4 30	(· 60^1 sar)
[5]	10(bùr) *buru* ašag	10 bùr	=	[5]	(· 60^2 sar)
…	……… …	……			…
9 30	10(bùr) [9(bùr) *buru*] ašag	1[9] bùr	=	9 30	(· 60^1 sar)
10	20(bùr) [*buru* ašag]	20 bùr	=	10	(· 60^2 sar)
…	……… …	……			…
25	50(bùr) *buru* ašag	50 bùr	=	25	(· 60^2 sar)

Here special number signs are used for the èše and the bùr. The entries in this sub-table are easily calculated as in the following examples:

1(bùr) =	3 · 1(èše) =	3 · 10 00 sar =	30	(· 60 sar)
10(bùr) =	10 · 30 00 sar =	5 00 00 sar =	5	(· 60^2 sar)
50(bùr) =	50 · 30 00 sar =	25 00 00 sar =	25	(· 60^2 sar)

W 23273 § 7 f: Table *A*, sar multiples

[30]	1(šár)		ašag	1 šár	=	[30] (· 60^2 sar)
35	1(šár)	10(bùr)	ašag	1 šár 10 bùr	=	35 (· 60^2 sar)
…	……		…	………..		………..
55	1(šár)	50(bùr)	ašag	1 šár 50 bùr	=	55 (· 60^2 sar)
1	2(šár)		ašag	2 šár	=	1 (· 60^3 sar)
…	……		…	……		………..
9 30	10(šár)	9(šár)	ašag	19 šár	=	9 30 (· 60^2 sar)

Here special number signs are used for multiples of the šár and the bùr, and the entries are calculated as in the following examples:

1(šár) =	(60 bùr =)	30 00 00 sar =	30	(· 60^2 sar)
19(šár) =	(19 · 30 00 00 sar =)	9 30 00 00 sar =	9 30	(· 60^2 sar)

The form of the original Old Babylonian version of the metrological table for system *A* (that is, the conversion table from area numbers to sexagesimal multiples of 1 sar) is described in Friberg, *MSCT 1* (2007), 390-391. It is similar to W 23273 § 7, but does not contain any initial sub-tables for shekel fractions and shekel multiples (above, sub-tables § 7a-b). On the other hand, at the very end of the table there are six lines that are missing in W 23273 § 7, namely

20(šár) =	10	(· 60^3 sar)
……	…	……
50(šár) =	25	(· 60^3 sar)
1(šár)×1.gál =	30	(· 60^3 sar)
1(šár)×2.gál =	1	(· 60^4 sar)

3.2.8 § 8. A Reverse Conversion Table from Sexagesimal Mina Multiples to Weight Numbers and Grain Multiples

W 23273 § 8: Table *M**

še	1/2	10	grain	1/2	10 (· 60^{-3} mina)
še	1	20	grain	1	20 (· 60^{-3} mina)
…	…	…	…	…	…
še	22	7 20	grains	22	7 20 (· 60^{-3} mina)
še 22 1/2	igi.8.gál.la	7 30	grains 22 1/2	an 8th	7 30 (· 60^{-3} mina)
še	23	7 40	grains	23	7 40 (· 60^{-3} mina)
…	…	…	…	…	…
še	29	9 40	grains	29	9 40 (· 60^{-3} mina)
še [30	igi.6.gál.la	10]	grains [30	a 6th	10] (· 60^{-2} mina)
še 4[5	igi.4.gál.la	15]	grains 4[5	a 4th	15] (· 60^{-2} mina)
še 1	[1/3 gín	20]	grains 1 (00)	[1/3 shekel	20] (· 60^{-2} mina)
še 1 [30	1/2 gín	30]	grains 1 [30	1/2 shekel	30] (· 60^{-2} mina)
še 2	[2/3 gín	40]	grains 2 (00)	[2/3 shekel	40] (· 60^{-2} mina)
še 2 30	5/6 [gín	50]	grains 2 30	5/6 [shekel	50] (· 60^{-2} mina)

še 3 uš	1 g[ín	1]	grains 3 · 60	1 she[kel	1] (· 60^{-1} mina)
še 3 šár	1 ma.[na	1]	grains 3 · 60 · 60	1 mi[na	1] (mina)
še 3 uš šár	1(gun) gun	[1]	grains 3 ·60 · 60 · 60	1 talent	[1] (· 60^{1} mina)

This Achaemenid metrological table for weight numbers is much briefer and less detailed than the corresponding Old Babylonian table. See Friberg, *MSCT 1* (2007), 387-389. On the other hand, the Old Babylonian table lists grain multiples only as far as 29 grains.

3.2.9 § 9. A Catch Line Referring to Tables for Capacity Measures

W 23273 § 9: Catch line for Table *C*

1 gín še *ana* egir-*šú*	1 shekel of grain thereafter.
al.til	Finished

This brief notice tells the reader that the combined metrological table which begins with conversion tables for length numbers, area numbers, and weight numbers in §§ 2-8 of W 23273 (with an interpolated range table for length and area measure in § 6) continues with one or two final conversion tables for capacity numbers, in the Old Babylonian style, on another clay tablet. The first line of those final tables would be

1	1 gín še	1	1 shekel of grain

This is also, essentially, the way in which the corresponding Old Babylonian metrological table for capacity measure begins. See, for instance, the conform transliteration of the table on the prism MS 2723 in Friberg, *MSCT 1* (2007), Fig. 3.1.3.

3.2.10 § 10. A Shadow Length Table: Months' Names and Cubit Multiples (reads from right to left)

W 3273 § 10: Shadow Length Table (noon shadow)

[x]	*am-mat*	šu			
[15]	kùš	izi	*u*	sig$_4$	ki.min
[30	kùš	kin]	*u*	gu$_4$	ki.min
[45	kùš	du$_6$	*u*]	⸢bára⸣	ki.min
[1	kùš	apin	*u*	še	ki].min
[1 15	kùš	gan	*u*	zíz	ki.min]
[1 30	kùš	ab]			

[no]	cubit	month IV			
[15]	cubits	month V	and	month III	the same
[30	cubits	month VI]	and	month II	the same
[45	cubits	month VII	and]	month I	the same
[1	cubit	month VIII	and	month XII	the] same
[1 15	cubits	month IX	and	month XI	the same]
[1 30	cubits	month X]			

Note that all the numbers in the leftmost column in this naive astronomical table are lost and only partially and tentatively reconstructed here. The reconstruction proposed here is based on the assumption that the table in § 11 of W 23273 (see below) is, essentially, the reverse of the table in § 10.

The month names appearing in this brief table are abbreviations of the Sumerian forms of the Assyrian month names. According to Neugebauer, *ACT 1* (1955), 38, the names of the months in astronomical cuneiform texts are as follows (see also Appendix 4 in Labat's *Manuel d'épigraphie* (1976)):

I	*nisannu*	bára	March-April
II	*aiaru*	gu$_4$	April-May
III	*simānu*	sig$_4$	May-June
IV	*dūzu*	šu	June-July
V	*ābu*	izi	July-August
VI	*ulūlu*	kin	August-September

VII	*tašrītu*	du$_6$	September-October
VIII	*araḫsamna*	apin	October-November
IX	*kislīmu*	gan	November-December
X	*tebētu*	ab	December-January
XI	*šabaṭu*	zíz	January-February
XII	*adaru*	še	February-March

In view of this list, the structure of the table in § 10 appears to be clear. *The shadow at noon* of a gnomon in a sundial (if that is what the table is dealing with) is shortest at the height of the summer, in month IV (June-July). The shadow is longer by equal amounts in months V and III (July-August and May-June). And so on. The shadow is longest in the middle of the winter, in month X (December-January).

A difficulty with this proposed reconstruction of the lost numbers in the leftmost column of § 10 is that in the first line of the paragraph [x] *am-mat* would have to stand for '[0] cubits', which is somewhat unlikely, since zero as an abstract number to my knowledge does not appear anywhere else in published cuneiform texts. Perhaps a better alternative is '[no] cubit'.

As a matter of fact, *an alternative reconstruction of the missing numbers* was suggested recently in Steele, *Sciamvs* 14 (2013). Steele takes his departure from the shadow length scheme which is found towards the middle of the second tablet of the astronomical compendium mul.apin (II ii 21–42 in Hunger's edition 1989). According to that scheme, on the 15th day of months I and VII, i. e. at the equinoxes, when the day and night have equal lengths, the shadow (of a 1 cubit long gnomon) is 1 cubit at 2 1/2 *bēru* = 1 15 uš after sunrise. It is 2 cubits at 37;30 uš after sunrise, and 3 cubits at 25 uš after sunrise. (Here 1 *bēru* = danna 'league, double-hour' = 30 uš = 2 hours, and 1 uš = 4 minutes.) Clearly, then, the shadow length, counted in cubits, multiplied by the time after sunrise, counted in uš, is constantly equal to 1 15. Similarly, on the 15th days of months IV and X, i. e. at the summer and winter solstices, the shadow length multiplied by the time after sunrise is constantly 1 00 and 1 30, respectively. The shadow length scheme ends with an explicit rule saying, essentially, that *the time after sunrise when the shadow length is 1 cubit* changes by 5 for every month. In other words, that time varies as a zigzag function with the maximum 1 30 in month X and the minimum 1 in month IV. Conversely, *the length of the shadow 1 00 uš after sunrise* varies in precisely the same way. Steele's proposition is now that *it is the values of this zigzag function that are mentioned in § 10 of W 23273*. Therefore, following Steele, the lost numbers in that paragraph should, alternatively, be reconstructed as follows:

W 3273 § 10: Shadow Length Table (the shadow 2 *bēru* = 1 00 uš after sunrise)

[1]	*am-mat*	šu			
[1 05]	kùš	izi	*u*	sig$_4$	ki.min
[1 10	kùš	kin]	*u*	gu$_4$	ki.min
[1 15	kùš	du$_6$	*u*]	⌜bára⌝	ki.min
[1 20	kùš	apin	*u*	še	ki].min
[1 25	kùš	gan	*u*	zíz	ki.min]
[1 30	kùš	ab]			

[1]	cubit		month IV		
[1;05]	cubits	month V	and	month III	the same
[1;10	cubits	month VI]	and	month II	the same
[1;15	cubits	month VII	and]	month I	the same
[1;20	cubit	month VIII	and	month XII	the] same
[1;25	cubits	month IX	and	month XI	the same]
[1;30	cubits		month X]		

3.2.11 § 11. A Reverse Shadow Length Table (reads from right to left)

W 3273 § 11: Reverse Shadow Length Table (noon shadow)

[šu	x	^{geš}mi	zal-*ra*]
[izi	15]	^{geš}mi	zal-*ra*
kin	30	^{geš}mi	zal-*ra*
du$_6$	45	^{geš}mi	zal-*ra*

[month IV	x	<cubits>	shadow	x]
[month V	15]	<cubits>	shadow	x
month VI	30	<cubits>	shadow	x
month VII	45	<cubits>	shadow	x

apin	[1]	^{geš}mi	zal-*ra*	month VIII	[1]	<cubits>	shadow	x
gan	1 15	^{geš}mi	zal-*ra*	month IX	1 15	<cubits>	shadow	x
ab	1 30	^{geš}mi	zal-*ra*	month X	1 30	<cubits>	shadow	x

In this reverse table, but not in the table in § 10, the word 'shadow' appears explicitly. The table is somewhat abbreviated, in that it mentions only months IV-X, omitting the second member of each pair of months mentioned in the table in § 10. Obviously, the only preserved numbers in this table, 1 15 and 1 30, must be interpreted as 1;15 and 1;30 cubits.

The reading and meaning of the word zal-*ra* is problematic. Steele (*op. cit.*), referring to a discussion of the word with Hunger, reads zal = *uḫurru* and translates zal-*ra* as 'delayed', without explaining what that would mean. Importantly, however, he compares W 23273 § 11 with lines *obv.* 6´ -12´ in BM 45721, an undated and badly preserved fragment of unclear content. What is written in those seven lines is far from clear, since they contain two words of unknown meaning (ḫé.gál and zal, the latter word apparently used just as in W 23273 § 11), nonsense calculations, and an incorrect metrology with 12 fingers in the cubit instead of the usual 24 or 30. Anyway, disregarding the nonsense calculations and the incorrect counting of fingers, and leaving the obscure words untranslated, what the seven lines seem to say is (essentially) that

in month IV	2 00 uš after sunrise	(i. e. at noon)	ḫé.gál	the shadow length is	[0]	cubit	zal
in month V	1 50 uš after sunrise	(i. e. at noon)	ḫé.gál	the shadow length is	;15	cubits	zal
in month VI	1 40 uš after sunrise	(i. e. at noon)	ḫé.gál	the shadow length is	;30	cubits	zal
in month VII	1 30 uš after sunrise	(i. e. at noon)	ḫé.gál	the shadow length is	;45	cubits	zal
in month VIII	1 20 uš after sunrise	(i. e. at noon)	ḫé.gál	the shadow length is	1	cubits	zal
in month IX	1 20 uš after sunrise	(i. e. at noon)	ḫé.gál	the shadow length is	1;15	cubits	zal
in month X	1 00 uš after sunrise	(i. e. at noon)	ḫé.gál	the shadow length is	1;30	cubits	zal

Without doubt, these seven lines of BM 45721 describe the same zigzag function as the table in W 23273 § 11. In addition, it says, quite explicitly, that the shadow lengths are noon shadows, the longest shadows of each day. Indeed, it is clear that the seven lines of BM 45721 are based on the assumption that the ratio of the longest night to the shortest night is 2 : 1. Then at the summer solstice, in month IV, the length of the night is 8 hours, while the length of the day is 16 hours = 8 double-hours = 4 00 uš, and noon is 2 00 uš after sunrise. Correspondingly, at the winter solstice, in month X, the length of the night is 16 hours, while the length of the day is 8 hours = 4 double-hours = 2 00 uš, and noon is 1 00 uš after sunrise.

In Steele 2013, Fig. 5, the zigzag function tabulated in W 23273 § 11 and BM 45271 lines *obv.* 6´ -12´ is compared with the modern precisely calculated wave-formed curve for the length of the noon shadow (at the latitude of Baghdad). Steele draws the conclusion that

> "The zigzag function is in remarkable agreement with the actual variation of the shadow length at noon, better than noon shadows deduced from the mul.apin scheme. ……… The assumption that the shadow is equal to zero at the summer solstice is of course incorrect, but was probably seen as a small price to pay for an otherwise excellent and simple function for the length of noon shadow."

What looks like the first line of the colophon in W 23273 is actually a second catch line. The first catch-line, inserted just before the two shadow length tables, referred to a continuation on another tablet of the combined metrological table in §§ 2-9 of W 23273. The second catch line refers to a continuation elsewhere of the shadow length tables. It has the following form:

¶ 1 12 ^{geš}mi 1 40 danna *u*$_4$*-mu* egir-*šú*	¶ 1 12 <cubits> the shadow, at 1 40 leagues of day, thereafter.

In Steele 2013, the sign ¶ is used a translation of the vertical diš sign, indistinguishable from the number sign for 1, which often in the shadow length table in mul.apin and also in BM 45271 is used as a textual marker at the beginning of new sections of the text.

Furthermore, in Steele 2013, the catch line after the shadow length tables in W 23273 is explained as follows:

> "A similar statement is found in part of the first line of BM 29371, a text that presents the length of shadow at 1;40 *bēru* after sunrise at different dates during the year. Although BM 29371 cannot be the text referred to in this catchline, as the wording is slightly different, this statement must refer to the same scheme."

Interested readers are referred to Steele 2013 for a detailed discussion of BM 29371, which is an almost completely preserved tablet listing weights (of water in a water clock) and shadow length for every five days in the ideal 360-day calendar. Here it will be enough to cite the top and bottom lines of BM 29371, which state (twice) that

¶ ina ^{itu}šu ud.15.kam 1 ki.lá 1 12 1 kuš ^{geš}mi 1 2/3 danna *u*$_4$*-mu*
¶ ina ^{itu}ab ud.15.kam 1 30 <ki.lá> 1 48 1 kuš ^{geš}mi 1 2/3 danna *u*$_4$*-mu*

¶ In month IV, day 15 (the summer solstice) 1 the weight 1;12 cubits the shadow at 1 2/3 double-hours of daytime.
¶ In month X, day 15 (the winter solstice) 1;30 <the weight> 1;48 cubits the shadow at 1 2/3 double-hours of daytime.

These two lines reveal the structure of the whole table of weights (times) and shadow lengths on BM 29371:

The ratio of the longest night to the shortest night is 1;30 : 1 = 3 : 2.
The ratio of the longest shadow to the shortest shadow is 1;48 : 1;12 = 9/5 : 6/5 = 3 : 2.

Between the extremes, the listed numbers representing the length of the night and the length of the shadow 1 2/3 (= 1;40) double-hours after sunrise form linear zigzag functions.

3.2.12 § 12. A Colophon

W 3273 § 112: Colophon

[*ki*]*-i pi-i ṭup-pi* gaba.ri /	According to an old tablet /
[x].ki ^{m}*ri-mut-*^{d}*a-nu* dumu /	[from x.] Rīmūt-Anu, son /
[*šá* ^{d}]utu.si-*na* a lú sanga.^{d}maš /	[of] Šamaš-iddin, descendant of Šangû-ninurta /
[*x x*] *x ib-ri*	[wrote and] checked it.

For an interesting discussion of the meaning and significance of this colophon, the interested reader is referred to Ch. 8.3 in Robson, *MAI* (2008), in particular the enumeration of colophons in Table 8.2: Scholarly Tablets of the Šangû-Ninurta Family in Fifth-Century Uruk.

The detailed discussion above of the text of W 23273 §§ 1-11 demonstrates the complexity of the text. Furthermore, it clearly shows that the mathematical/metrological training in the Achaemenid period in fifth-century Uruk was on a relatively high level, and that it was greatly influenced by Old Babylonian mathematics and metrology, while it also possessed several new, independent and interesting features. This result confirms what was learned from a discussion of the varied contents of the mathematical recombination texts W 23291-x in Friberg, et al., *BaM* 21 (1990) and W 23291 in Friberg, *BaM* 28 (1997). Also W 23291-x is one of the cuneiform texts from fifth-century Uruk mentioned in Robson's Table 8.2.

3.2.13 W 23273. Photos of the Tablet

Fig. 3.2.5. W 23273, *obv*., photo by courtesy of E. von Weiher.

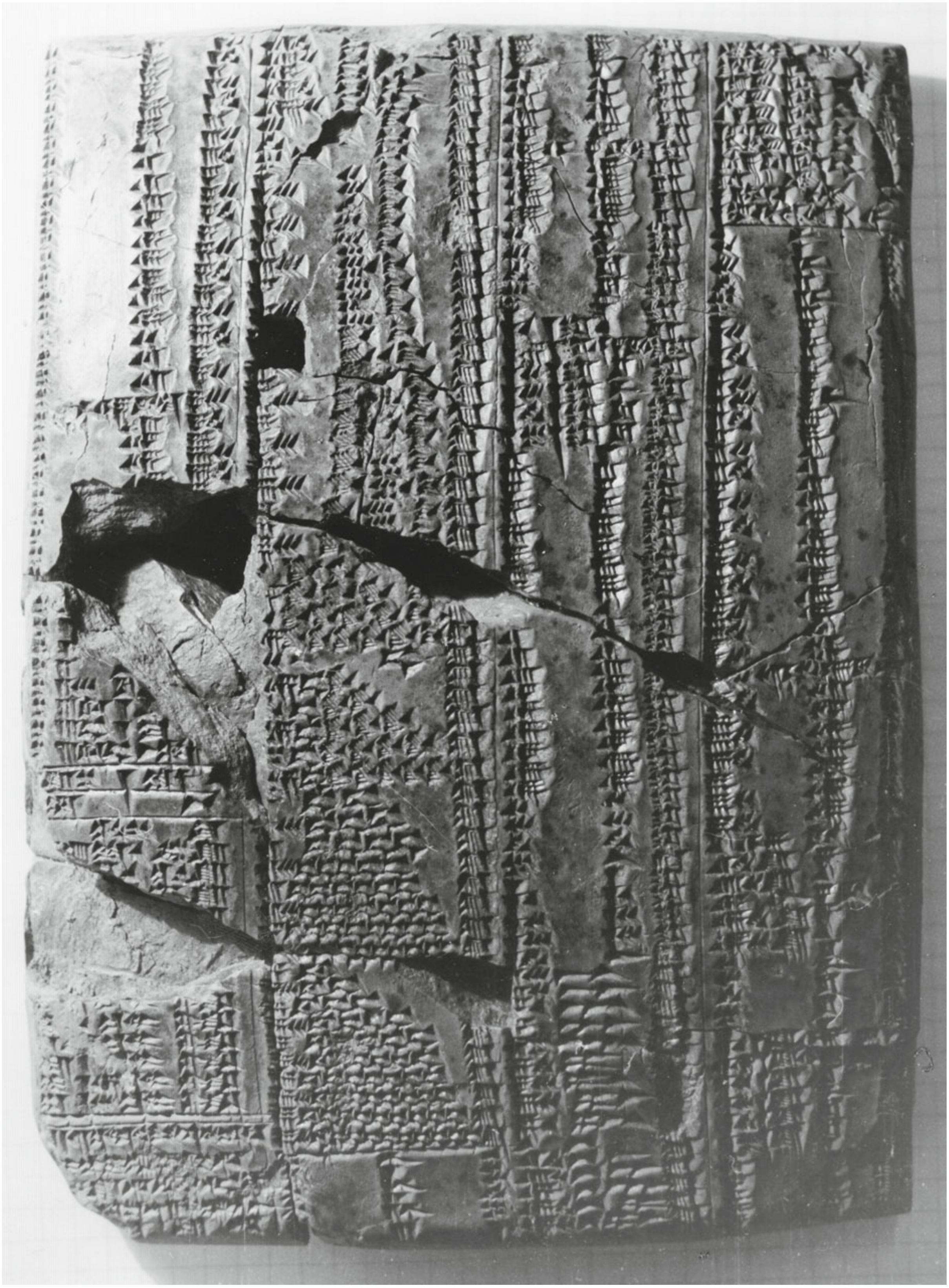


Fig. 3.2.6. W 23273, *rev*.

3.3 W 22309. A Small Fragment of a Metrological Recombination Text from Achaemenid Uruk

3.3.1 W 22309, obv. A Structure Table for Traditional Length Measures

W 22309 (Hunger, *Uruk 1* (1976), no. 102) is a small fragment of what was once a relatively large metrological table text. It was found at Uruk in the same excavation square as W 23281 (Ue XVIII/1) but without any informative context. (See Robson, *MAI* (2008), 337.)

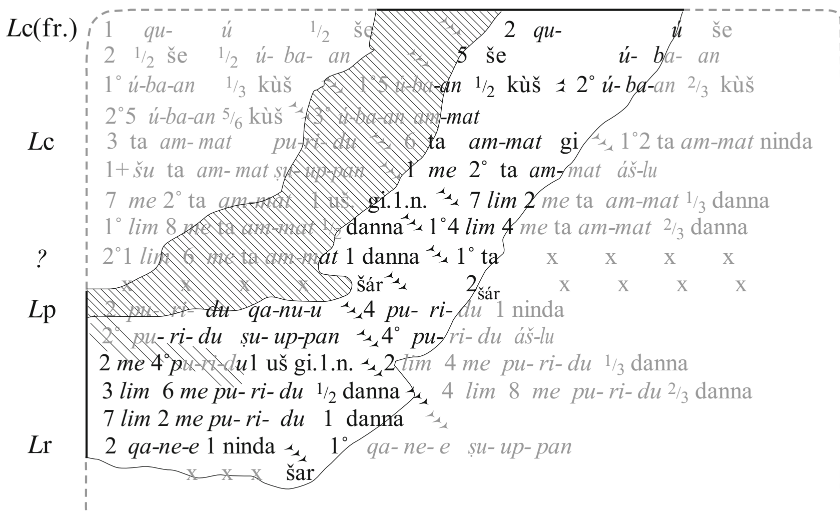


Fig. 3.3.1. W 22309, *obv.* Conform transliteration with a partial reconstruction of lost parts of the text.

The progressive series of cumulative structure tables in what remains of W 22309, *obv.* is of the same kind as the series of structure tables in § 1 of W 23281 in Sec. 3.1.1 above.

However, the organization of the text into a proper table has been abandoned, and the terminology W 22309, *obv.* differs thoroughly from that in W 23281 § 1, with

> *qú-ú* instead of dur, *u-ba-an* instead šu.si, 1/2 kùš instead of *ú-ṭu,* yet *am-mat* instead of kùš*, ta am-mat* instead of *i-na am-ma-ti,* 1/2 danna instead of *zu-ú-zu,* danna instead of *bé-é-ri,* gi instead of *qa-nu-ú,* yet *qa-ne-e* instead of gi.meš, and 1.uš.gi.1.nindan (1 uš *ginindanakku*) instead of *šu-uš-šá-an*.

In W 22309, *obv.*, but not in W 23281 § 1, there is a line saying that 12 cubits = 1 rod, and the sub-table *Lp*, in which the basic unit is the *purīdu,* is not misplaced. Only the beginning of the subtable *Lr* is preserved. The sub-tables *La* and *Lš* are lost.

W 22309, *obv.* Tables *Lc*(fr.), *Lc*, *Lp*, *Lr*, ... (reads from right to left)

[1	*qú-ú*]	[1/2 še «]	[1/2 grain]	=	[1		thread]
2	*qú-*⸢*ú*⸣	[še] /	[a grain]	=	2		threads
[2 1/2	še]	[1/2 *u-ba-an* «]	[1/2 finger]	=	[2 1/2		grains]
⸢5⸣	še	*u-ba-*[*an*] /	a finger	=	⸢5⸣		grains
[10	*u-ba-an*]	[1/3 kùš «]	[1/3 cubit]	=	[10		fingers]
[15	*u-ba*]*-an*	1/2 kùš «	1/2 cubit	=	[15		fing]ers
20	*u-ba-*[*an*]	[2/3 kùš] /	[2/3 cubit]	=	20		finge[rs]
[25	*u-ba-an*]	[5/6 kùš «]	[5/6 cubit]	=	[25		fingers]
[30	*u-ba-an*]	[*am*]*-mat* /	[a cu]bit	=	[30		fingers]
[3	ta *am-mat*]	[*pu-ri-du* «]	[a *purīdu*]	=	[3		of a cubit]
[6]	ta *am-mat*	g[i «]	a re[ed]	=	[6]		of a cubit

[12	ta *am-mat*]	nindan /	a rod	=	[12	of a cubit]
[60	ta *am-mat*]	[*ṣu-up-pan* «]	[a *ṣuppān*]	=	[60	of a cubit]
1 *me* 20	ta *am*-[*mat*] /	[*áš-lu*] /	[a rope]	=	1 hundred 20	of a cu[bit]
[7 *me* 20	ta *am-mat*]	[1 uš.g]i.1.nindan «	[a sixty]-nindan-[re]ed	=	[7 hundred 20	of a cubit]
7 *lim* 2 [*me*	ta *am-mat*]	[$^{1}/_{3}$ danna] /	[1/3 danna]	=	7 thousand 2 [hundred	of a cubit]
[10 *lim* 8 *me*	ta *am-mat*]	[$^{1}/_{2}$] danna «	[1/2] danna	=	[10 thousand 8 hundred	of a cubit]
14 *lim* 4 [*me*	ta *am-mat*]	[$^{2}/_{3}$ danna] /	[2/3 danna]	=	14 thousand 4 [hundred	of a cubit]
[21 *lim* 6 *me*	ta *am-m*]*at*	1 danna «	1 danna	=	[21 thousand 6 hundred	of a cub]it
10	ta [.......]	[.......] /	[.......]	=	10	of a [.......]
[...]	[.......]	šár «	a šár	=	[...]	[.......]
⌜2⌝ šár	[.......] /	[.......] /	[.......]	=	⌜2⌝	šár
[2	*pu-ri*]-*du*	*qa-nu-u* «	a reed	=	[2	*purīdu*
4	*pu-ri*-[*du*]	[1 nindan] /	a rod	=	4	*purīdu*
20	*pu-ri-du*	[*ṣu-up-pan*] /	a *ṣu-up-pan*	=	20	*purīdu*
40	*pu*-[*ri-du*]	[*áš-lu*] /	a rope	=	40	*purīdu*
2 *me* 40	*p*[*u-ri-d*]*u*	1 uš.gi.1.nindan /	a sixty-nindan-reed (?)	=	2 hundred 40	*purīdu*
2 [*lim* 4 *me*	*pu-ri-du*]	[$^{1}/_{3}$ danna] /	1/3 danna	=	2 thousand 4 hundred	*purīdu*
3 *lim* 6 *me*	*pu-ri-du*	$^{1}/_{2}$ danna	1/2 danna	=	3 thousand 6 hundred	*purīdu*
[4 *lim* 8 *me*	*pu-ri-du*]	[$^{2}/_{3}$ danna] /	2/3 danna	=	4 thousand 8 hundred	*purīdu*
7 *lim* 2 *me*	*pu-ri-du*	1 danna	1 danna	=	7 thousand 2 hundred	*purīdu*
2	*qa-ne-e*	1 nindan /	1 rod	=	2	reeds
10	[*qa-ne-e*]	[*ṣu-up-pan*] /	a *ṣuppān*	=	10	reeds
[...]	[.......]	[.......]	[..]		[.......]	[.......]

Factor diagrams for the sub-table *L*c(fr.) and for the larger units of the system of traditional length measure according to W 22309 are shown below. They should be compared with the corresponding factor diagrams in the case of W 22381, § 1 (Figs. 3.1.3-4 above). Note, in particular, the (conjectured) addition of a line for 5/6 cubit = 25 fingers (unfortunately lost in the fragment).

What is going on in the damaged part between the cubit section and the *purīdu* section is not clear.

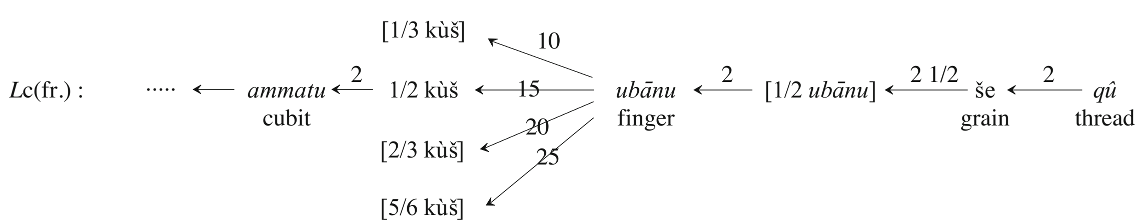


Fig. 3.3.2. W 22309, *obv*. Factor diagram for fractions of the cubit. Traditional length measure (1 cubit = 30 fingers).

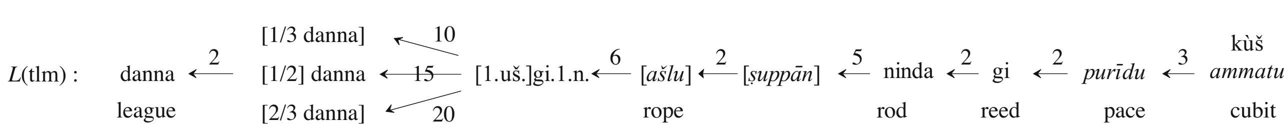


Fig. 3.3.3. W 22309, *obv*. Factor diagram for the larger units of traditional length measure (1 reed = 6 cubits).

3.3.2 *W 22309, rev. A Metrological Table for Traditional Capacity Measures*

The three columns of the table are inscribed from left to right, which is very unusual on the reverse(?) of a clay tablet.

Like almost all other known Late Babylonian metrological cuneiform texts, the obverse of W 22309 is written with each metrological equation in reverse order, as if the cuneiform text was a copy in cuneiform of an original written in Aramaic, from right to left. Surprisingly, however, what remains of the text on the *reverse* (Fig. 3.3.4 below) is written in the ordinary Old Babylonian way, from left to right. In each line of the text, the capacity number is to the left and its value, as a sexagesimal multiple of the barig, to the right. In addition, the system of capacity measure in the table on the reverse is clearly *the traditional, Old Babylonian system* with a bán of 10 sìla, and with multiples of the gur written as non-positional sexagesimal numbers followed by the sign gur. The sexagesimal number 10(géš) = 10 · 60, for instance, is written with the cuneiform sign *nēr*. Therefore, W 22309 *rev*. seems to be, exceptionally, *a Late Babylonian direct copy of an Old Babylonian metrological table*.

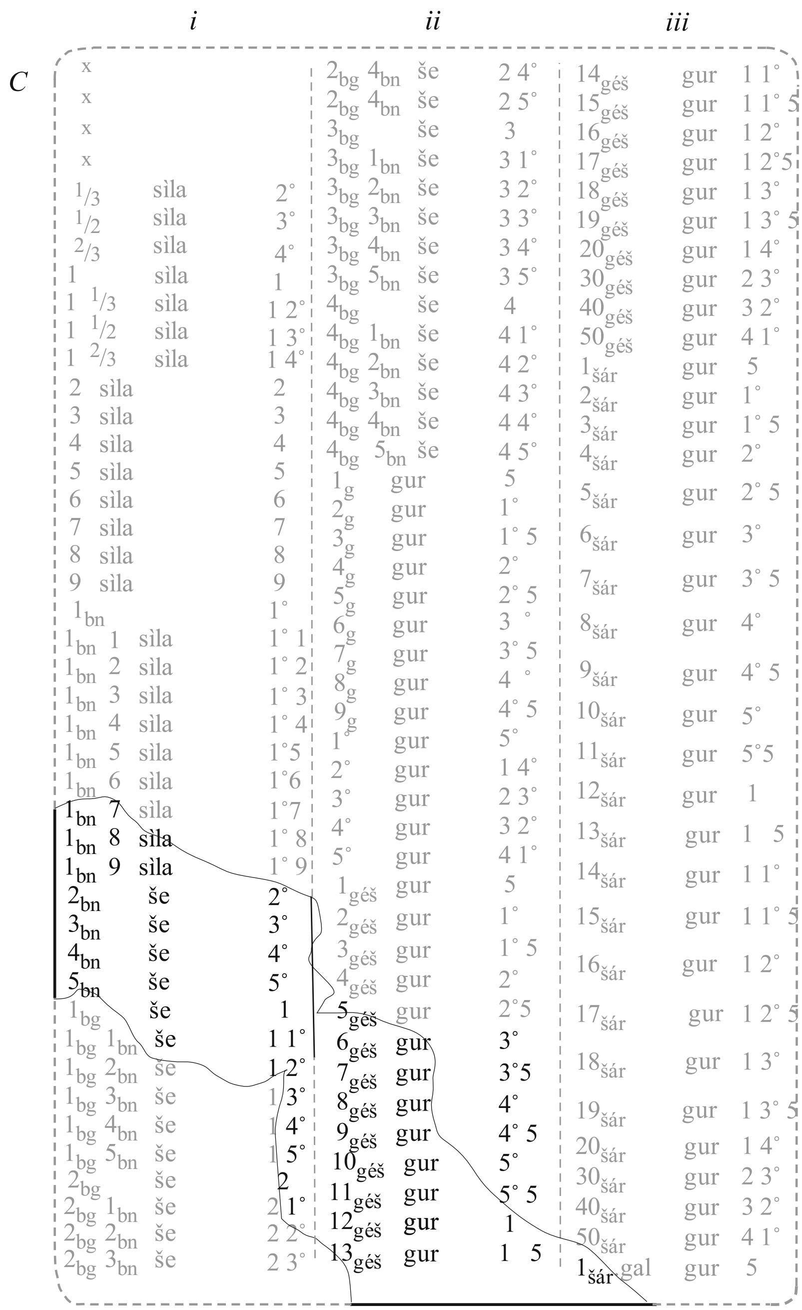


Fig. 3.3.4. W 22309, *rev.* Conform transliteration with a proposed partial reconstruction of lost parts of the text.

3.4 W 22260. A Large Fragment of a Metrological/Mathematical Recombination Text from Achaemenid Uruk

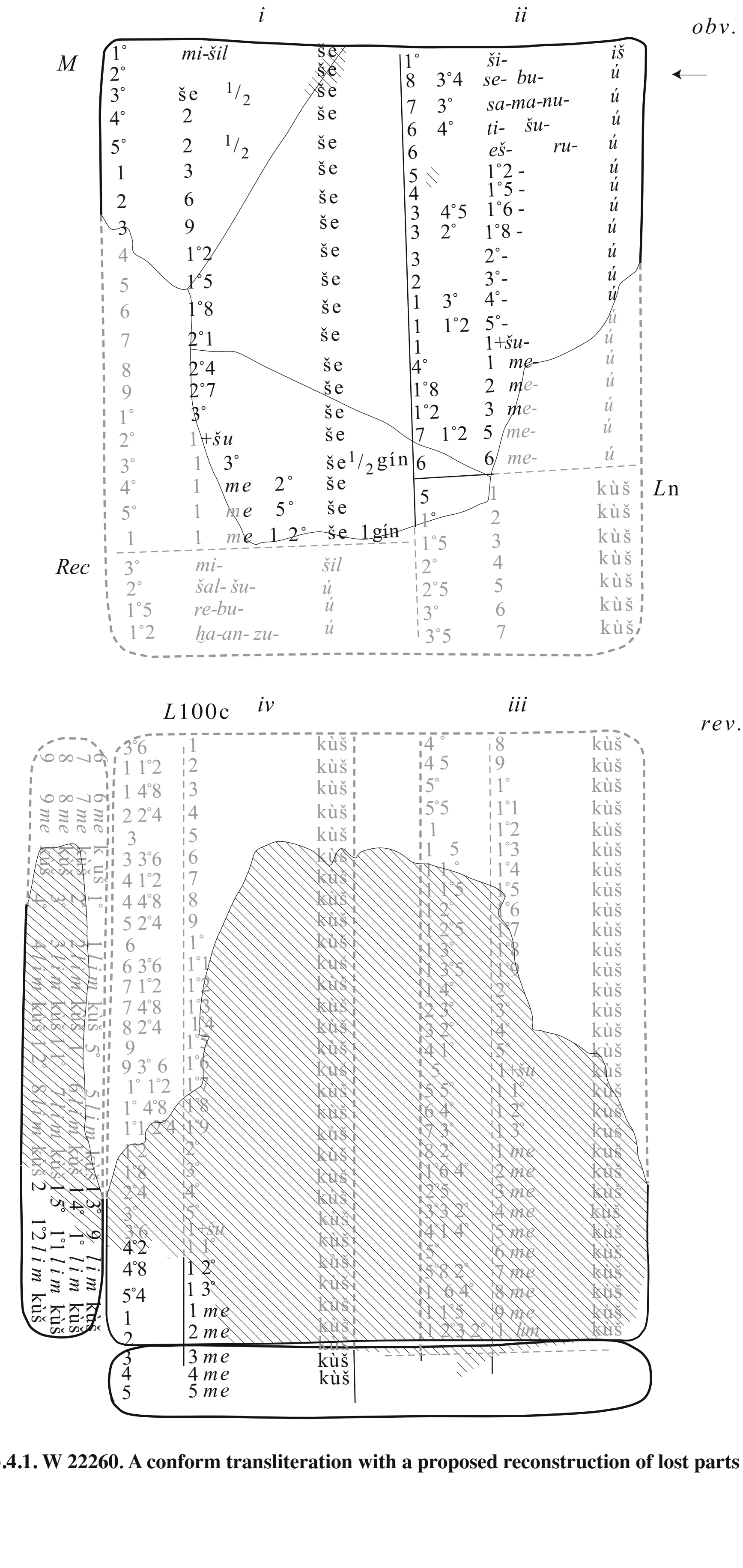


Fig. 3.4.1. W 22260. A conform transliteration with a proposed reconstruction of lost parts of the text.

W 22260 (Hunger, *Uruk 1* (1976), no. 101) is, like W 22309 above, a fragment of a relatively large metrological table text, found at Uruk in the same excavation square as W 23281 (Ue XVIII/1) but without any informative context. What remains of the obverse is well preserved, but almost all the text on the reverse is lost, fortunately with the exception of the continuation of the text onto the lower and left edges.

The organization of the subtables of the W 22260 is somewhat chaotic: A metrological table for weight measure (System *M*) is followed by a table of reciprocals (*Rec*) and (if the proposed reconstruction of the text is correct) by two metrological tables for length measure (Tables *L*n and *L*100c).

3.4.1 § 1. A Conversion Table from Decimal Grain Multiples to Sexagesimal Mina Fractions (reads from right to left)

The table for weight measure is straightforward. It goes from 1/2 grain (1/360 shekel) to 1 hundred 1 20 grains (180 grains) or 1 shekel, and equates these weight numbers with sexagesimal fractions of a shekel, or a mina.

W 22260 § 1: Table *M*

10	*mi-šil*		še		half a		grain	=	10	($\cdot$ 60^{-3} minas)
20			še				grain	=	20	($\cdot$ 60^{-3} minas)
30	še	$^1/_2$	še		a grain	1/2	grain	=	30	($\cdot$ 60^{-3} minas)
40	2		še		2		grains	=	40	($\cdot$ 60^{-3} minas)
50	2 $^1/_2$		še		2 1/2		grains	=	50	($\cdot$ 60^{-3} minas)
1	3		še		3		grains	=	1	($\cdot$ 60^{-2} minas)
2	6		še		6		grains	=	2	($\cdot$ 60^{-2} minas)
…	…		…		…		……	……		
9	22 $^1/_2$		še		22 1/2		grains	=	27	($\cdot$ 60^{-2} minas)
10	30		še		30		grains	=	10	($\cdot$ 60^{-2} minas)
20	1+*šu*		še		sixty		grains	=	20	($\cdot$ 60^{-2} minas)
30	1 30		še		1 30		grains	=	30	($\cdot$ 60^{-2} minas)
40	1 me 20		še		1 hundred 20		grains	=	40	($\cdot$ 60^{-2} minas)
50	1 me 50		še		1 hundred 50		grains	=	50	($\cdot$ 60^{-2} minas)
1	1 me 1 20		še	1 gín	1 hundred 1 20		grains 1 shekel	=	1	($\cdot$ 60^{-1} minas)

3.4.2 § 2. A Decimal-Sexagesimal Table of Reciprocals (reads from right to left)

This Achaemenid table of reciprocals differs from the standard Old Babylonian table of reciprocals (see Friberg, *MSCT 1* (2007), 68) in a number of ways:

- It lacks the two initial lines of an Old Babylonian table of reciprocals.
- From ‘a half’ to ‘a tenth’ it spells out the number words.
- It includes an approximate reciprocal for the non-regular (sexagesimal) number 7 (7 · 8 34 = 59 58).
- However, it cleverly avoids considering the reciprocals of 70 and of 7 hundred.
- It does not make use of the word igi for ‘reciprocal’.
- After the reciprocal of 18 it continues with reciprocals of the tens, from 20 to 60, and of the hundreds, up to 6 hundred.

W 22260 § 2: Table *Rec*

[30	*mi-šil*]	[a half	=	30]	($\cdot$ 60^{-1})
[20	*šal-šu-ú*]	[a third	=	20]	($\cdot$ 60^{-1})
[15	*re-bu-ú*]	[a fourth	=	15]	($\cdot$ 60^{-1})
[12	*ḫa-an-zu-ú*]	[a fifth	=	12]	($\cdot$ 60^{-1})

10	*ši-iš*	a sixth	=	10	(· 60^{-1})	
8 34	*se-bu-ú*	a seventh	=	8 34	(· 60^{-2})	approximation
7 30	*sa-ma-nu-ú*	an eighth	=	7 30	(· 60^{-2})	
6 40	*ti-šu-ú*	a ninth	=	6 40	(· 60^{-2})	
6	*eš-ru-ú*	a tenth	=	6	(· 60^{-1})	
5	12-*ú*	a 12th	=	5	(· 60^{-1})	
4	15-*ú*	a 15th	=	4	(· 60^{-1})	
3 45	16-*ú*	a 16th	=	3 45	(· 60^{-2})	
3 20	18-*ú*	an 18th	=	3 20	(· 60^{-2})	
3	20-*ú*	a 20th	=	3	(· 60^{-1})	
2	30-*ú*	a 30th	=	2	(· 60^{-1})	
1 30	40-*ú*	a 40th	=	2	(· 60^{-1})	
1 12	50-[*ú*]	a 50[th]	=	1 12	(· 60^{-2})	
1	1+*šu*-[*ú*]	a sixtie[th]	=	1	(· 60^{-1})	
40	1 *me*-[*ú*]	a 1 hundred[th]	=	40	(· 60^{-2})	error
18	2 *m*[*e-ú*]	a 2 hund[redth]	=	18	(· 60^{-2})	
12	3 *m*[*e-ú*]	a 3 hu[ndredth]	=	12	(· 60^{-2})	
<9	4-*me-ú*>	<a 4 hundredth	=	9>	(· 60^{-2})	gap
7 12	5 [*me-ú*]	a 5 [hundredth]	=	7 12	(· 60^{-3})	
6	6 [*me-ú*]	a 6 [hundredth]	=	6	(· 60^{-2})	

The omission of a line for the reciprocal of 4 hundred may have been unintentional. The reciprocal of 1 hundred should have been given as 36 instead of 40, which is the reciprocal of 1 30 (90). This mistake may have happened because the table of reciprocals on W 22260 was copied from a more extensive table, including the reciprocal of 90.

3.4.3 *§ 3.* [*A Conversion Table from Decimal Cubit Multiples to Sexagesimal* nindan *Multiples*] *(reads from right to left)*

This metrological table in § 3 of W 22260 is almost completely lost. See the conform transliteration of W 22260 with a proposed reconstruction of the text in Fig. 3.4.1 above. Originally the table in § 3 extended from the last third of col. *ii* on the obverse to (probably) the whole col. *iii* on the reverse. The only part of the table that still remains is the beginning of the first line, with the single number '5'. The whole reconstruction of this almost completely lost table is based on the assumption that the metrological table for length measure in §3 was organized in the same way as the better preserved alternative metrological table for length measure in § 4. See below, in Sec. 3.4.4.

3.4.4 *§ 4. A Conversion Table from Decimal Cubit Multiples to Sexagesimal 100-Cubit Multiples (reads from right to left)*

The first half of this table, or more, is completely lost. See again Fig. 3.4.1. Luckily, parts of the end of the table are preserved in the lower left corner of the reverse, on the lower edge, and on the left edge. This is enough for a fairly trustworthy reconstruction of the whole table

W 22260 § 4: Table *L*100c

[36	1	kùš]	[1	cubit	=	36]	(· 60^{-2} 100-cubits)
[1 12	2	kùš]	[2	cubits	=	1 12]	(· 60^{-2} 100-cubits)
……	…	……	…………	………		…	

[30	50	kùš]	[50	cubits	=	30]	(· 60^{-1} 100-cubits)
[36	1+*šu*	kùš]	[sixty	cubits	=	36]	(· 60^{-1} 100-cubits)
42	[1 10	kùš]	[1 10	cubits]	=	42	(· 60^{-1} 100-cubits)
48	1 20	[kùš]	1 20	[cubits]	=	48	(· 60^{-1} 100-cubits)
54	1 30	[kùš]	1 30	[cubits]	=	54	(· 60^{-1} 100-cubits)
1	1 *me*	[kùš]	1 hundred	[cubits]	=	1	(100-cubit)
……	…	……	…………	………		…	
5	5 *me*	<kùš>	5 hundred	<cubits>	=	5	(100-cubits)
[6	6 *me*	kùš]	[6 hundred	cubits	=	6]	(100-cubits)
……	…	……	…………	………		…	
[9	9 *me*	kùš]	[9 hundred	cubits	=	9]	(100-cubits)
[10	1 *lim*	kùš]	[1 thousand	cubits	=	10]	(100-cubits)
……	…	……	…………	………		…	
[1 20	8 *lim*	kùš]	[8 thousand	cubits	=	1 20]	(100-cubits)
1 30	9 *lim*	kùš	9 thousand	cubits	=	1 30	(100-cubits)
1 40	10 *lim*	kùš	10 thousand	cubits	=	1 40	(100-cubits)
1 50	11 *lim*	kùš	11 thousand	cubits	=	1 50	(100-cubits)
2	12 *lim*	kùš	12 thousand	cubits	=	2	(· 60^{1} 100-cubits)

Note that in the Late Babylonian system of "mixed sexagesimal-decimal numbers" integers less than 1 hundred were written as sexagesimal numbers, while multiples of a hundred were written as multiples of me 'hundred', of lim 'thousand', me lim 'hundred thousand', and even lim lim 'thousand thousand', the last of these expressions in Ash. 1924.1278 (Robson, *Sciamvs* 5 (2004) no. 28).

Here are some examples of how the entries in this table can have been calculated:

1 cubit = rec. 100 · 100 cubits = ;36 · 100 cubits = 36 100-cubits (· 60^{-2}).
1 10 cubits = 1 10 · ;36 · 100 cubits = (36 + 6) · 100 cubits = 42 100-cubits.
1 hundred cubits = 1 100-cubit.
11 thousand cubits = 11 · 10 · 100 cubits = 1 50 100-cubits.

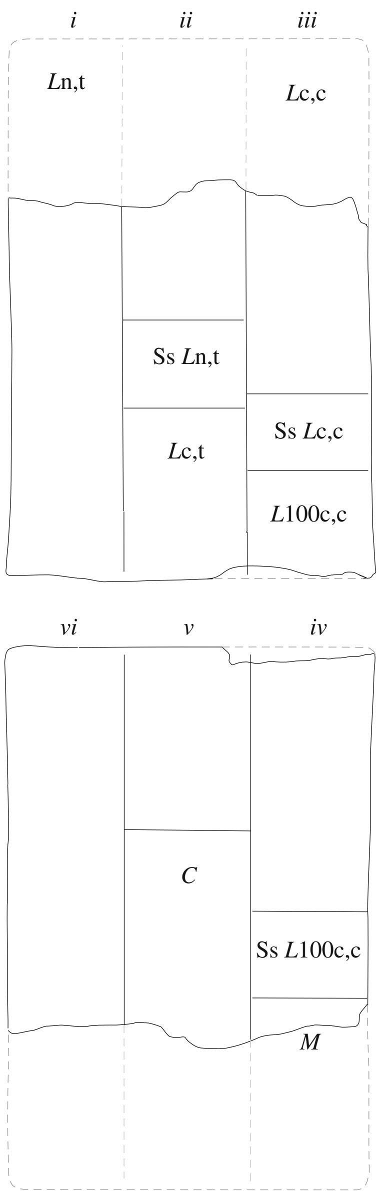
4. CBS 8539. A Mixed Metrological Table Text from Achaemenid Nippur

4.1 CBS 8539. Metrological Tables for Systems *L* (4 variants), *M*, and *C*

CBM 8539 is a fragment of a large combined metrological table, with sub-tables for length measure, of four kinds, for weight measure, and for capacity measure. It was first published by Hilprecht in *BE 20/1* (1906), text no. 30. Excellent photographs of the tablet are now available online, at cdli.ucla.edu/P230041.

A table of contents for the table text was first published in Friberg, *GMS 3* (1993), no. 10. The table of contents is reproduced below. All lines of all the tables in the text are to be read from right to left.

CBS 8539: Table of contents



obv.	**§ 1**	***L*n,t** :	Traditional length numbers – Sexagesimal rod multiples			
			from	[1 finger	=	10 ($\cdot 60^{-2}$ rods)]
			to	[1 cubit]	=	5 ($\cdot 60^{-1}$ rods)
				a *ṣuppān*	=	5 (rods)
			and	3 leagues	=	1 30 ($\cdot 60^{1}$ rods)
		Ss *L*n,t :	Subscript to table ***L*n,t**			
	§ 2	***L*c,t** :	Traditional length numbers – Sexagesimal cubit fractions			
			from	1 finger	=	2 ($\cdot 60^{-1}$ cubits)
			to	[30 fingers	=	1] (cubit)
			(No subscript to table ***L*c,t !**)			
	§ 3	***L*c,c** :	Common length numbers – Sexagesimal cubit fractions			
			from	[1 finger	=	2 30] ($\cdot 60^{-2}$ cubits)
			to	24 fingers	=	1 (cubit)
		Ss *L*c,c :	Subscript to table ***L*c,v**			
	§ 4	***L*100c,c** :	Common length numbers – Sexagesimal 100-cubit fractions			
			from	1 finger	=	1 30 ($\cdot 60^{-3}$ 100-cubits)
			to	24 fingers	=	36 ($\cdot 60^{-2}$ 100-cubits)
		Ss *L*100c,c :	Subscript to table ***L*100c,c**			
rev.	§ 5	***M*** :	Weight numbers–Sexagesimal mina multiples – Weight names			
			from	[22 1/2 grains	=	7 30 ($\cdot 60^{-3}$ mina)
					=	*bit-qa*]
			to	180 grains	=	1 ($\cdot 60^{-1}$ mina)
					=	1 shekel
			and	1/2 mina	=	30 ($\cdot 60^{-1}$ mina)
	§ 6	***C*** :	Capacity numbers – Sexagesimal bariga multiples			
			from	a ninda	=	10 ($\cdot 60^{-3}$ barig)
			to	[1(barig)	=	1] (barig)
				1_g gur	=	5 (barig)
			and	[100 gur	=	8 20] (barig)

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7_4

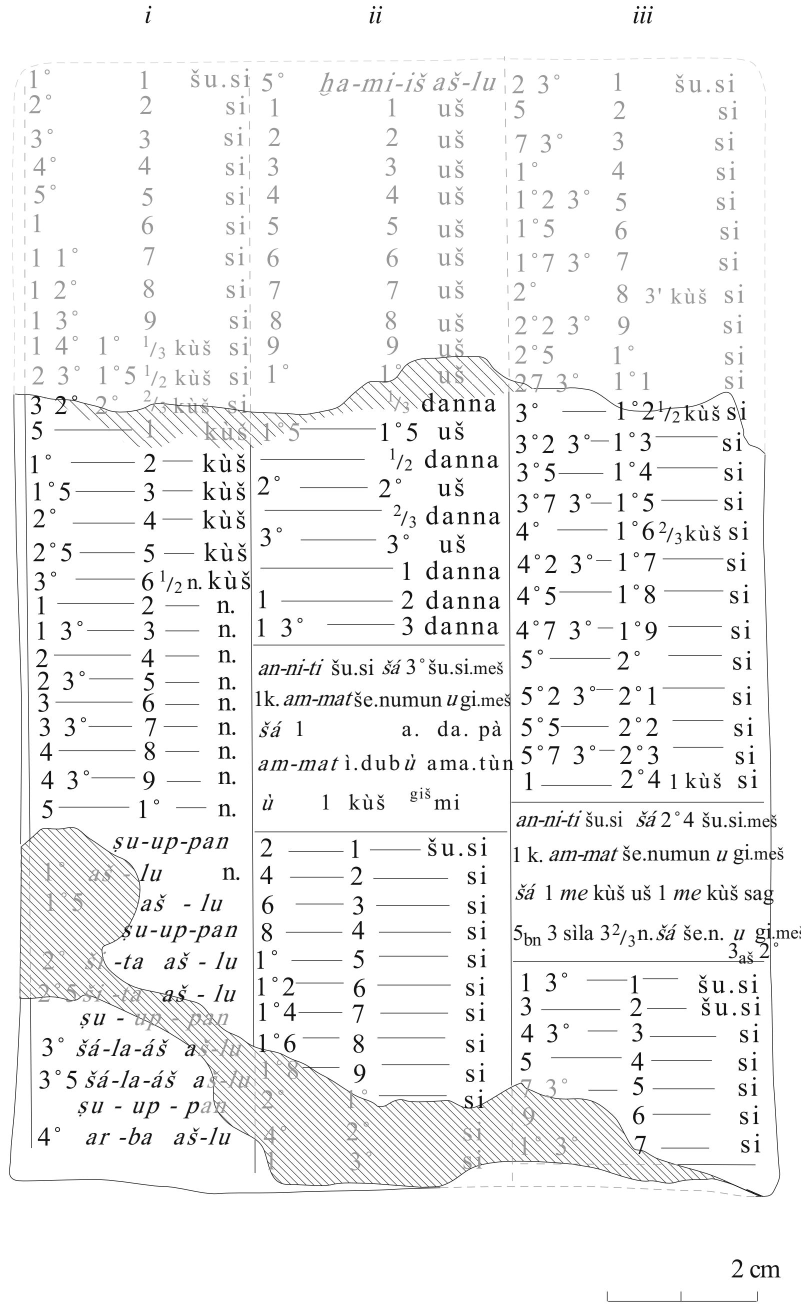


Fig. 4.1.1. CBS 8539, *obv*. Conform transliteration.

In *BE 30/1* (1906) Hilprecht claimed that the tablet CBS 8539 had been found in the ruins of "a small temple library from the Kassite period", and dated it to around 1350 BC. In *JA 13* (1909), Thureau-Dangin found this claim invalid and stated that the script made him think instead of the tablet as Neo-Babylonian. Here, it is tentatively classed as Achaemenid(?), for the following reasons:

First, CBS 8539 is just like W 23281 and W 23273 and other metrological texts from Achaemenid Uruk discussed in Ch. 3 above in that it arranges sub-tables for length measure, weight measure, and capacity measure in the reverse order compared with Old Babylonian mixed metrological tables, and writes each line of each sub-table to be read from right to left.

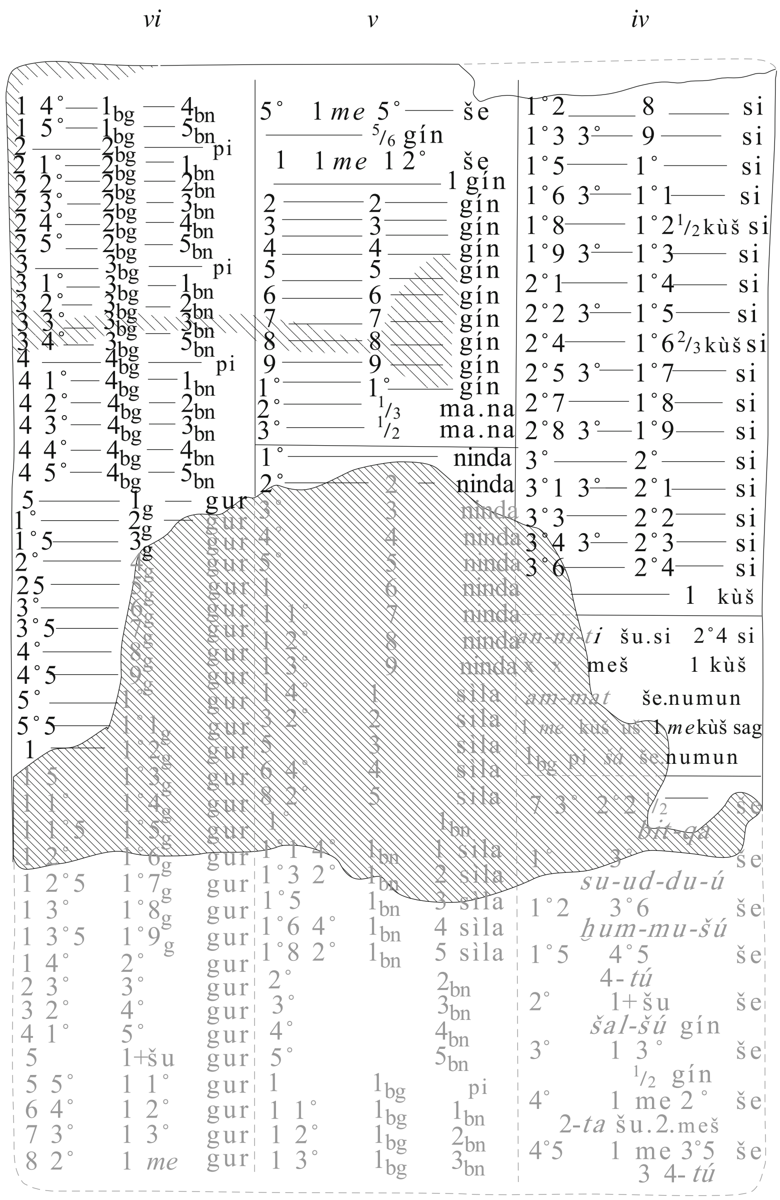


Fig. 4.1.2. CBS 8539, *rev*. Conform transliteration.

Secondly, just like the mentioned Achaemenid metrological texts, it contains a mix of "traditional" (Old Babylonian) metrology and metrology characteristic for texts from the 1st millennium BC. Thirdly, the way in which it shows side by side metrological tables for traditional length measure with either the rod (nindan) or the 30-finger cubit as the basic length measure, and metrological tables for Late Babylonian length measure with either the 24-finger cubit or the 100-24-finger cubit as the basic length unit, is reminiscent of the way in which the mathematical recombination texts W 23291 and W 23291-x, both from Achaemenid Uruk, show side by side examples of various kinds of traditional and Late Babylonian metrology. (See Friberg, et al., *BaM* 21 (1990) and Friberg, *BaM* 28 (1997).) Common to both CBS 8539 and in W 23291-x is the expression

še.numun *u* gi.meš 'seed and reeds'

which, apparently, is a reference to Late Babylonian "seed measure" and "reed measure".

Besides, the writing on CBS 8539 could very well be a product of the Achaemenid period (Andrew George, private communication).

4.1.1 *§ 1. A Conversion Table from Traditional Length Measures to Sexagesimal* nindan *Multiples (reads from right to left)*

CBS 8539 § 1a: Table *L*n,t

[10	1	šu.si]	[1	finger	=	10]	(· 60^{-2} rods)	
[20	2	si]	[2	(fing)ers	=	20]	(· 60^{-2} rods)	
…	………	……	…	………	……			
[1 30	9	si]	[9	(fing)ers	=	1 30]	(· 60^{-2} rods)	
[1 40	$^{1}/_{3}$	kùš]	[1/3	cubit	=	1 40]	(· 60^{-2} rods)	
[2 30	$^{1}/_{2}$	kùš]	[1/2	cubit	=	2 30]	(· 60^{-2} rods)	
3 20	[$^{2}/_{3}$	kùš]	[2/3	cubit	=]	3 20	(· 60^{-2} rods)	
5	[1	kùš]	[1	cubit	=]	5	(· 60^{-1} rods)	
…	………	……	…	………	……			
25	5	kùš	5	cubits	=	25	(· 60^{-1} rods)	
30	6	$^{1}/_{2}$ nindan kùš	6 1/2 rod	cubits	=	30		(interpolation)
1	2	nindan	(· 60^{-1} rods)					(reeds!)
…	………	……	2	rods	=	1	(rod)	………
5	10	nindan / *ṣu-up-pan*	…	…	……			(reeds!)
[10	*aš*]-*lu*	{nindan}	10	rods / a *ṣuppān*	=	5	(rods)	
[15]	*aš-lu* / *ṣu-up-pan*		[a ro]pe		=	[10]	(rods)	
[20	*ši*]-*ta aš-lu*		a rope / a *ṣuppān*		=	[15]	(rods)	
[25	*ši-ta*] *aš-lu* / *ṣu-up-pan*		[t]wo ropes		=	[20]	(rods)	
30	*ša-la-aš a*[*š-lu*]		[two] ropes / a *ṣuppān*		=	[25]	(rods)	
35	*ša-la-aš a*[*š-lu*] / *ṣu-up-p*[*an*]		three ro[pes]		=	30	(rods)	
40	*ar-ba aš-lu*		three ro[pes] / a *ṣupp*[*ān*]		=	35	(rods)	
[50	*ḫa-mi-iš aš-lu*]		four ropes		=	40	(rods)	
[1	1	uš]	[five ropes		=	50	(rods)]	
…	………	……	[1	length	=	1]	(· 60^{1} rods)	
[10	10 uš /	$^{1}/_{3}$] danna	…	………	……			
[15]	15 uš /	$^{1}/_{2}$ danna	[1/3] league / = [10 lengths]		=	[10]	(· 60^{1} rods)	
20	20 uš /	$^{2}/_{3}$ danna	1/2 league / = 15 lengths		=	[15]	(· 60^{1} rods)	
30	30 uš /	1 danna	2/3 league / = 20 lengths		=	20	(· 60^{1} rods)	
1		2 danna	1 league / = 30 lengths		=	30	(· 60^{1} rods)	
1 30		3 danna	2 leagues		=	1	(· 60^{2} rods)	
			3 leagues		=	1 30	(· 60^{1} rods)	

Note the sequence of errors where 2, 3, 4, 5, … rods are falsely equated with 1, 1;30, 2, 2;30, … (rods). This may be a copying error from an older text in which it was stated, correctly, that 2, 3, 4, 5, … reeds are equal to 1, 1;30, 2, 2;30, … (rods). Indeed, in the Old Babylonian metrology, 1/2 rod = 1 reed = 6 cubits.

In Table *L*n,t above (*n for nindan = rod as the silently assumed basic length unit and t for traditional, essentially Old Babylonian length measure*), it is clear that 10 fingers = 1/3 cubit, so that 30 fingers = 1 cubit. Here is the factor diagram for the system of length measure figuring in Table *L*n,t:

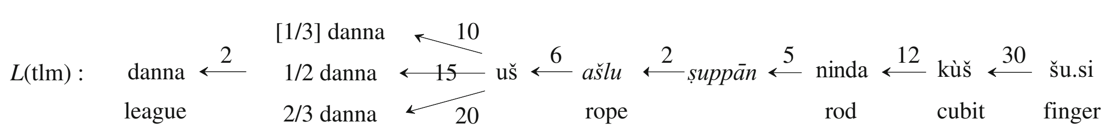


Fig. 4.1.3. CBS 8539, §§ 1-2. Factor diagram for the system of traditional length measures (tlm: 1 cubit = 30 fingers).

In the following subscript to Table *L*n,t it is stated explicitly that 30 fingers = 1 cubit:

CBS 8539 § 1b: Subscript to Table *L*n,t

an-ni-ti šu.si *šá* 30 šu.si.meš / 1 kùš	This is the finger such that 30 fingers / are 1 cubit,
am-mat še.numun *u* gi.meš /	the cubit of seed and reeds, /
šá 1 a.da.pà /	that of xxx, /
am-mat ì.dub *ù* ama.tùn /	the cubit of the pile and the pit, /
ù 1 kùš $^{\text{geš}}$mi	and the 1 cubit shadow.

In a self-contradictory way, after stating that the cubit figuring here is the traditional cubit of 30 fingers, the subscript goes on to associate the same cubit with še.numun *u* gi.meš 'seed and reeds'. This expression is known from the first line of the colophon to the mathematical recombination text W 23291-x, which says:

še.numun *u* gi.meš al.til Seed and reeds finished.

Actually, this is the conclusion of a text dealing with *various ways of measuring the size of fields:* Thus, for instance, in W 23291-x §§ 1-5 exercises are concerned with *traditional area measure*, in § 10 exercises are dealing with two kinds of *reed measure*, and § 11 is an exercise comparing reed measure with *seed measure*. (A detailed table of contents for W 23291-x can be found in Friberg, *BaM* 28 (1997), 253.) As a matter of fact, the problem in the mentioned exercise in § 11 of W 23291-x is stated in the following words:

25 gi.meš / ḫé en še.numun 25 are the reeds / what is the seed?

Therefore, there can be no doubt that the 'cubit of seed and reeds' mentioned in the subscript to CBS 8539 § 1 is *the cubit used when measuring fields*, without regard to how the size of the fields are calculated, whether it is in area measure, in seed measure, or in reed measure. The discussion of this issue will be continued below, in the proposed explanation of the subscript to CBS 8539 § 3.

The meaning of the next line of the subscript, mentioning 1 a.da.pà, is totally obscure. (The Babylonian sage Adapa was held to have imparted learning to the human race in the remote past.)

Next follows a line specifying the cubit in CBS 8539 § 1 as the cubit of 'the pile and the pit'. I owe to Andrew George the explanation that ì.dub *ù* ama.tùn 'the pile and the pit' is a figurative expression for *piled and excavated volumes*. (Aptly enough, ì.dub = *našpaku* means 'store room' or 'storage vessel', while it is more mysterious that normally ama.tùn = *agarinnu* means 'beer mash'.) Anyway, as is well known, in Old Babylonian metrology, volumes above or below ground were calculated using rods (nindan) for their horizontal extension (length and width) but cubits for their vertical extension (height). This, by the way, is the reason why in CBS 8539 §§ 1-2, just as in W 23273 §§ 2-5 (Sec. 3.2 above), and usually in Old Babylonian mixed metrological tables, there are, in parallel, metrological tables for length measure with first rods and then cubits as the basic length measure.

In the last line of the subscript to CBS 8539 § 1, the cubit of 30 fingers is said to be (the cubit of) the '1 cubit shadow'. This must be a reference to shadow length tables such as W 23273 §§ 10-11 (Secs. 3.2.10-11 above), in which the length of the gnomon apparently is assumed to be precisely 1 cubit.

4.1.2 § 2. A Conversion Table from Traditional Length Numbers to Sexagesimal Cubit Fractions (reads from right to left)

In §§ 1-2 of CBS 8539 there are in parallel metrological tables for length measure, with first rods, then cubits as the silently assumed basic length measure, although the table in § 2 is much shorter than the one in § 1. It doesn't go beyond the level 1 cubit, and there is no subscript.

Below follows an abbreviated version of the text of § 2, here called Table *L*c,t (*c for cubit, t for traditional*). Note the repeated use in the text of the table of the abbreviation si for šu.si.

CBS 8539 § 2: Table *L*c,t

2	1	šu.si	1	finger	=	2	(· 60^{-1} cubits)
4	2	si	2	(fing)ers	=	4	(· 60^{-1} cubits)
…	…	…	…	…	………		
[20	10]	si	[10]	(fing)ers	=	[20]	(· 60^{-1} cubits)
[40	20	si]	[20	(fing)ers]	=	[40]	(· 60^{-1} cubits)
[1	30	si]	[30	(fing)ers]	=	[1]	(cubit)

4.1.3 § 3. A Conversion Table from Common Length Numbers to Sexagesimal Cubit Fractions (reads from right to left)

In § 3, Table *L*c,c (c for cubit as the basic length measure, and also c for "common", Late Babylonian length measure), just like Table *L*c,t in § 2, doesn't go beyond the level 1 cubit. In its last line, the table clearly states that 24 fingers = 1 cubit, although the message is somewhat muddled by an interpolation. In the first line of the table, reconstructed here, it is stated that 1 finger = '2 30', where 2 30 is the reciprocal of 24. (Indeed, 24 · 2 30 = 48 00 + 12 00 = 1 00 00.)

CBS 8539 § 3a: Table *L*c,c

[2 30	1		šu.si]	[1		finger	=	2 30]	(· 60^{-2} cubits)	2 30 = rec. 24
……	…		…	…		………		……		
[27 30	11		si]	[11		(fing)ers	=	27 30]	(·	
30	12	1/2 kùš	si	60^{-2} cubits)						(interpolation)
32 30	13		si	12	1/2 cubit	(fing)ers	=	30	(· 60^{-1} cubits)	
……	…		…	13		(fing)ers	=	32 30	(· 60^{-2} cubits)	
40	16	2/3 kùš	si	…		………		……		(interpolation)
42 30	17		si	16	2/3 cubit	(fing)ers	=	40	(· 60^{-1} cubits)	
……	…		…	17		(fing)ers	=	42 30	(· 60^{-2} cubits)	
57 30	23		si	…		………		……		
1	24	1 kùš	si	23		(fing)ers	=	57 30	(· 60^{-2} cubits)	(interpolation)
				24	1 cubit	(fing)ers	=	1	(cubit)	

CBS 8539 § 3b: Subscript to Table *L*c,c

an-ni-ti šu.si *šá* 24 šu.si.meš / 1 kùš	This is the finger such that 24 fingers / are 1 cubit,
am-mat še.numun *u* gi.meš /	the cubit of seed and reeds, /
šá 1 *me* kùš uš 1 *me* kùš sag /	such that 1 hundred length 1 hundred front /
5(bán) 3 sìla 3 2/3 ninda *šá* še.numun *u* gi.meš /	is 5(bán) 3 sìla 3 2/3 ninda of seed and reeds. /
3(aš) 20	3 20

The subscript to Table *L*c,c in § 3 starts by saying that the cubit in this table is the (Late Babylonian) cubit of 24 fingers, and that this cubit, too, can be used for 'seed and reeds', that is, for calculations of the extents of fields. Next, it goes on to tell precisely how this particular cubit is used when the extent of a field is given in terms of what may be called "common seed measure" (Friberg, *BaM* 28 (1997), 273, 293). It does this by prescribing, essentially, that

The common seed measure of a square with the side 100 cubits is 5 bán 3 sìla 3 2/3 ninda.

Presumably, the system of capacity measure implied here is the same as the one described by the metrological table for capacity measures in § 6 of CBS 8539 (see below, in Sec. 4.1.6), for which the factor diagram is as follows:

$$C\text{(LB)}: \quad \text{gur} \xleftarrow{5} \text{barig} \xleftarrow{6} \text{bán} \xleftarrow{6} \text{sìla} \xleftarrow{10} \text{ninda}$$

Fig. 4.1.4. CBS 8539, § 6. Factor diagram for the Late Babylonian system *C* (1 bán = 6 sìla, 1 sìla = 10 ninda).

It is interesting to calculate the equation for common seed measure in terms of sexagesimal rather than decimal numbers. Clearly, the common seed measure of a square of side 60 cubits ought to be 36/100 of the common seed measure of a square of side 100 cubits. Now, in view of the factor diagram above,

6 · 5(bán) 3 sìla 3 2/3 ninda = 30 bán 18 sìla 20 ninda = 200 sìla, 6 · 200 sìla = 200 bán.

Therefore,

> The common seed measure of a square with the side 60 cubits is 2 bán.

In terms of the rod (= 12 cubits), the corresponding equation would be

> The common seed measure of a square with the side 60 rods is 48 barig. (Indeed, sq. 12 · ;20 bariga = 48 barig.)

Precisely 2 bán as the *common seed measure* of a square with the side 60 cubits occurs in § 4 of the famous Esagila tablet. (See Friberg, *BaM* 28 (1997), 298, and Sec. 3.1.3 above.) There it is said that since the base of the step-pyramid Etemenanki is a square with the side 3 · 60 cubits, the area of the base is 9 (· sq. (60 cubits)). Consequently, the seed measure of the base is '9' · '2' = '18', where '18' (bán) is explained as 3 bariga of seed. In § 5 of the same text, the *Kassite seed measure* of the base is calculated (see again Sec. 3.1.3 above), and is found to be 3 bán.

In the Achaemenid mathematical recombination text W 23291 (Friberg, *BaM* 28 (1997)), §§ 1a-1g are devoted to *metric algebra problems formulated in terms of common seed measure.* One example is enough to show what is going on there. Below is shown the text of W 23291 § 1c, which is a "square side problem", namely *to find the length of the side of a square with a given common seed measure (csm)*.

W 23291 § 1c: A square-side problem

a.šag$_4$ en a.an *lu-m*[*aḫ*]-*ḫir-ma*	A field, what each way shall I make equalsided
ḫé 1(gur) gur 2(bán) še.numun /	so that 1 gur 2(bán) will be the seed? /
mu nu zu-*ú*	If you do not know:
še.numun *ša* e-*ka* e /	The seed that was said to you, /
⌜*a-na*⌝ igi.gub-*bé-e* še.[n]um[un]	⌜to⌝ the <reciprocal of the> seed constant
du-*ak le-qé-ma ši*[*d-d*]*u* /	go (and) take <each way>, then the length. /
šum-ma 5 *am-mat-ka*	If 5 is your cubit:
5 20 a.rá 1 15 du-[*ma*] 6 [40].e /	5 20 steps 1 15 go, [then] 6 40 it is, /
20 a.an *ti-qé* 20 nindan a.an *tu-m*[*aḫ*]-*ḫar* /	20 each way take, 20 rods each way you will make equalsided.
šum-ma 1 *am-mat-ka*	If 1 is your cubit:
5 20 a.|rá] 3 du-*ma* 16.e / [4] a.an ti-*qé*	5 20 steps 3 go, then 16 it is, / 4 (00) each way take.
2 me 40 kùš a.an *tu-m*[*aḫ*]-*ḫar*	2 hundred 40 cubits each way you will make equalsided.šag$_4$

Since the literal translation is somewhat unintelligible, here is an explanation of what is going on in this exercise. First the *question*:

> What are the sides of a square with the (common) seed measure 1 gur 2 bán?

Then an explanation of the procedure:

> If you don't know (how to proceed, then do as follows).
> Multiply the given seed by the reciprocal of the seed constant. Compute the square side, then you will find the length.

Next comes the actual computation, first in the case when, as always, the basic unit of capacity measure is the barig, and *when the assumed (traditional) basic unit of length measure is the rod*. It proceeds as follows (here in modernized and slightly anachronistic form):

> Assume that (lengths are measured in rods so that) 1 cubit = ;05 (rods).
> The given (common) seed measure *S* of the square is 1 gur 2 bán = 5;20 (bariga).
> The known seed constant c is 48 (barig/sq. (60 rods)), and rec. c = ;01 15 (sq. (60 rods)/barig).
> Therefore, the area of the square is *A* = *S* · rec. c = 5;20 · ;01 15 = ;06 40 (sq. 60 rods) = 6 40 (sq. rods).
> Consequently, the length of the side of the square is *s* = sqs. 6 40 = 20 (rods).

The actual computation in the case when the basic unit of capacity measure is, as before, the barig, but *when the assumed (contemporary) basic unit of length measure is the cubit*, proceeds as follows:

> Assume that (lengths are measured in cubits so that) 1 cubit = 1 (cubit).
> The given (common) seed measure *S* of the square is still 1 gur 2 bán = 5;20 (barig).
> The known seed constant c is now ;20 (barig/sq. (60 cubits)), and rec. c = 3 (sq. (60 cubits)/barig).
> Therefore, the area of the square is *A* = *S* · rec. c = 5;20 · 3 = 16 (sq. 60 cubits).
> Consequently, the length of the side of the square is *s* = sqs. 16 = 4 · (60 cubits) = 240 cubits.

Common reed measure (crm) is, as the name suggests, the commonly (and exclusively?) used type of reed measure in Late Babylonian economic and administrative cuneiform texts. It is associated with the common, Late Babylonian system of length measure (clm). The two factor diagrams below clearly demonstrate the differences between common, Late Babylonian length measure and traditional, essentially Old Babylonian length measure (tlm).

$$L(\text{tlm}):\ \text{ninda} \xleftarrow{2} \text{gi} \xleftarrow{6} \text{kùš} \xleftarrow{30} \text{šu.si}$$

rod reed cubit finger

$$L(\text{clm}):\ \text{ninda} \xleftarrow{2} \text{gi} \xleftarrow{7} \text{kùš} \xleftarrow{24} \text{šu.si}$$

rod reed cubit finger

Fig. 4.1.5. CBS 8539. Factor diagrams for the traditional (OB) and common (LB) systems of length measure.

See, for instance, text no. 3 (on BM 46703) in Nemet-Nejat, *LBFP* (1982). There the ground plan of an Achaemenid house in Babylon shows a nearly rectangular room with

the long sides 23 cubits and 22 cubits 12 fingers and the short sides 20 1/2 cubits and 20 cubits 12 fingers.

The ground plan of the house is claimed to have the reed measure 9 reeds 3 cubits 15 fingers. (It is well known that *reed measure was most often used for urban objects, while seed measure was more suitable for agricultural fields.*) What this means is that

(23 c. + 22 c. 12 f.)/2 · (20 1/2 c. + 20 c. 12 f.)/2 = (9 r. 3 c. 15 f.) · 1 r.

This equation is correct, provided that one counts with the common system of length measure, because then both sides of the equation can be shown to be equal to 466 3/8 square cubits.

The field plan in text no. 64 in *LBFP* (on the fragment BM 46588) is particularly interesting. Although much of the field plan is lost, plans of two sub-fields remain, complete with their inscribed data. The first sub-field is a nearly rectangular field with

the long sides 1,402 and 1,320 cubits, and the short sides 153 1/2 and 150 cubits.

The corresponding area, calculated in the usual way as the average long side times the average short side, is

A = 1,361 c. · 151.75 c. = 22 41 c. · 2 31;45 c. = 57 22 11; 45 sq. cubits (= 206,531.75 sq. cubits).

The corresponding *common seed measure* can be calculated as

S = (;20 barig/sq. 60 cubits)) · 57 22 11; 45 sq. cubits = ;20 · 57;22 11 45 bariga = 19;07 23 55 bariga
= approximately 19;07 bariga = 3 gur 4 bariga 4 sìla 2 ninda.

This is precisely the seed number inscribed in the middle of the field plan for the first sub-field.

The second sub-field in field plan no. 64 is another nearly rectangular field, with

the long sides 6,170 and 6,150 cubits, and both the short sides 150 cubits.

The area, calculated in the usual way, is

A = 6,160 c. · 150 c. = 1 42 40 c. · 2 30 c = 4 16 40 00 sq. cubits (= 924,000 sq. cubits).

The corresponding seed measure, inscribed in the middle of the field plan for the second sub-field, is

S = 15 gur 2 bariga [x x]

Surprisingly, this cannot be common seed measure, because the common seed measure for a field with the area 4 16 40 00 square cubits is

; 20 · 4 16;40 bariga = 1 25;33 20 bariga = 17 gur 3 bán 2 sìla.

It is not difficult to find instead the correct multiplication, which is precisely

S = ; 18 · 4 16;40 bariga = 1 17 bariga = 15 gur 2 bariga (csm).

This means that the seed measure used for the second sub-field in field plan no. 64 is what is called “major seed measure” in Friberg, *BaM* 28 (1997).

In the text of *LBFP* no. 64, *the common seed measure for the first field is tagged with the word [zaq]-pa 'planted', while the major seed measure for the second field is tagged with the word me-re-šu 'cultivated'.* (The terminology is somewhat strange, since in text no. 64 it is stated that 8 date palms are planted in the 'cultivated' field!) Anyway, the same tags appear in the Neo-Babylonian mathematical exercise BM 78822 (Jursa *AfO* 40/41 (1993/94); Friberg, *BaM* 28 (1997), 295; Robson, *MAI* (2008), 186). The text of the exercise is as follows, in translation:

50 (cubits) are the fronts. How much for my length do I set so that 1 gur planted and 1 gur cultivated?
5 steps of 3 is 15, 5 steps of 3 20 is 16 40. Heap them, then it is 31 40.
What to 50 do I go for 31 40? 38 steps of 50 go, then 31 40. 38 are the lengths.

Here, precisely as in *LBFP* no. 4, *zaq-pa* 'planted' seems to stand for fields measured in common seed measure, while *me-re-šu* 'cultivated' stands for fields measured in major seed measure.

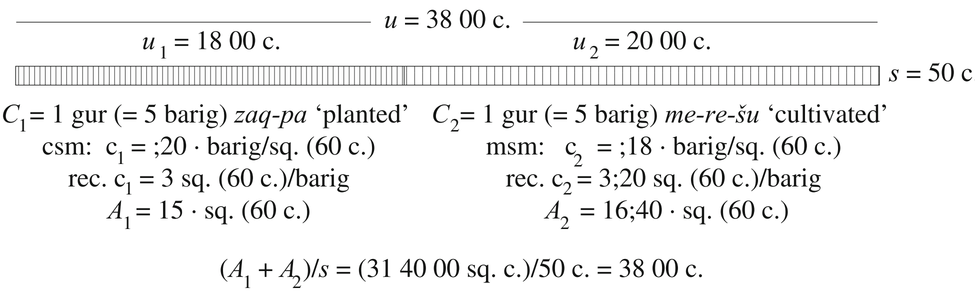


Fig. 4.1.6. BM 78822. A metric algebra problem for a field measured in terms of common and major seed measure.

Major seed measure appears also in two Late Babylonian administrative texts, and in two mathematical exercises in the partly published text BM 67314 (see Friberg, *BaM* 28 (1997), 294 ff.). In the first of the mentioned exercises, for instance, a circle with the diameter 8 (· 60 c.?) has the area 48 (· sq. (60 c.)?), which corresponds to a major seed measure of

S = ;18 · 48$^?$ bariga = 14;24 barig.

Common seed measure appears also in the extremely interesting mathematical exercise BM 47431(Robson, *FS Slotsky* (2007); Friberg, *AfO* 52 (2013), 125). There four circles are inscribed in a square of given side 1 (00) cubits, and therefore given seed measure 2 bán (csm). The common seed measure is computed for each circle and for the variously shaped parts of the square outside the circles.

Common reed measure appears in the mathematical exercise W 23291-x § 10 (Friberg, et al., *BaM* 21 (1990), Sec. 10), which contains a clever procedure for the conversion of square cubits into common reed measure. The exercise in W 23291-x § 10 is so interesting that it is motivated to reproduce it in full below:

W 23291-x § 10: Conversion of square cubits into common reed measure

[gi.meš *ša*] 7 kùš uš 7 kùš sag 1 gi	[Reeds such that] 7 cubits length 7 cubits front is 1 reed.
kùš.meš *ša* tur.ki.e /	Cubits that are small(?). /
[*mi-ḫi-i*]*l-tu*$_4$ a.rá ki.2 *ù* 1 12 du-*ma*	[Li]ne steps the same and 1 12 go, then
šá a-na igi-*ka* e$_{11}$.a /	what comes up for you to see /
⸢sá⸣ *mi-nu-ti*	in ⸢equal⸣ numbers
gi.meš kùš.meš tur.meš *te-eṣ-ṣip*	of reeds and small cubits you will pair.
1 10 kùš ur.a / ⸢ḫé⸣ en gi.meš	1 10 cubits equalsided, / what shall the reeds be?
1 10 a.rá 1 10 1 21 40	1 10 steps 1 10 (is) 1 21 40.
1 21 40 a.rá 1 12 / 1 38 gi.meš	1 21 40 steps 1 12 / (is) 1 38 reeds.
1 38 kùš.meš tur.meš 2 gi.meš	1 38 small cubits (is) 2 reeds.
2 gi.meš *a-na* / 1 38 *te-eṣ-ṣip-ma* 1 me gi.meš	2 reeds to / 1 38 you will pair, then 1 hundred reeds.

This exercise begins by mentioning that (by definition) *a square of side 7 cubits (= 1 reed (clm)) has the (common) reed measure 1 reed*. Next it states that it will count with "small cubits".

Then comes the following *explicit computation rule*:

> Multiply the given length by itself and by 1 12. Express the result as an equal number of reeds and small cubits.

The computation rule is illustrated by a *numerical example*, which (essentially) says the following:

> A square of side 1 10 cubits has the area sq. (1 10 c.) = 1 21 40 sq. c. What is its reed measure?
> Since 1 21 40 · 1 12 = 1 38, the reed measure is 1 38 reeds + 1 38 small cubits.
> Since 1 38 small cubits is 2 reeds, the reed measure is (1 38 + 2) reeds = 1 40 reeds.

The computation rule is, apparently, based on the observation that

> 1 square reed = 7 cubits · 7 cubits = 49 square cubits.
> 1 square cubit = 1/7 cubit · 7 cubits = 1/7 cubit · 1 reed = 1/7 cubit (crm), which may be called a "small cubit".
> Consequently, 50 square cubits = (1 reed + 1 small cubit) (crm).

Since rec. 50 = ;01 12, it follows that

> 1 square cubit = ;01 12 · (1 reed + 1 small cubit) (crm).

This is, essentially, the explanation of the computation rule in W 23291-x § 10. Incidentally, the question in the numerical example in the same exercise can be answered more quickly in the following way:

> sq. (1 10 cubits) = sq. (10 reeds) = 100 sq. reeds = 100 reeds · 1 reed = 100 reeds (crm).

Now return to the subscript of Table *L*c,c (CBS 8539 § 3 b). The single number 3 20 (where 3 is written as three horizontal wedges) is inscribed below the last line of the subscript. It was suggested by Thureau-Dangin already in *JA* 13 (1909) that this number may stand for the reed measure of the mentioned square of side 100 cubits. This is probably correct since, in view of the computation rule just discussed,

> sq. (100 cubits) = 100 · 100 · ;01 12 · (1 reed + 1 small cubit) (crm) = 200 · (1 reed + 1 small cubit) (crm)
> = 3 20 · (1 reed + 1 small cubit) (crm) = (approximately) 3 20 reeds in common reed measure.

Closely connected with W 23291-x §10 is the next paragraph, W 23281-x § 11 (Friberg, et al., *BaM* 21 (1990), § 11; Friberg, *GMS 3* (1993)). It is simultaneously a multiplication table for length measure into common reed measure and a structure table for common reed measure (crm).

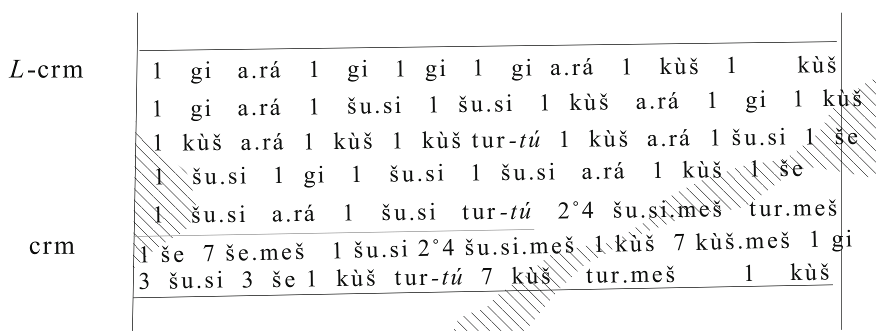


Fig. 4.1.7. W 23291-x § 11. A conform transliteration of the text.

This text is easier to understand if it is arranged as below so that its structure becomes clearly visible:

W 23291-x § 11. A multiplication table and structure table.

1 gi a.rá 1 gi	1 gi	1 reed	× 1 reed	=	1 reed	(crm)
1 gi a.rá 1 kùš	1 kùš		× 1 cubit	=	1 cubit	(crm)
1 gi a.rá 1 šu.si	1 šu.si		× 1 finger	=	1 finger	(crm)
1 kùš a.rá 1 gi	1 kùš	1 cubit	× 1 reed	=	1 cubit	(crm)
[1] kùš a.rá 1 kùš	1 kùš-tur-*tú*		× 1 cubit	=	1 small-cubit	(crm)
1 kùš a.rá 1 šu.si	1 še		× 1 finger	=	1 grain	(crm)
[1 š]u.si <a.rá> 1 gi	1 šu.si	1 finger	× 1 reed	=	1 finger	(crm)
1 šu.si a.rá 1 kùš	1 [še]		× 1 cubit	=	1 grain	(crm)
[1 š]u.si a.rá 1 šu.si	<> tur-*tú*		× 1 finger	=	1 small-finger	(crm)

24 šu.si-⸢meš tur⸣-meš	[1] še	24 small-fingers	(crm)	=	1 grain	(crm)
7 še-meš	1 šu.si	7 grains	(crm)	=	1 finger	(crm)
24 šu.si-meš	1 kùš	24 fingers	(crm)	=	1 cubit	(crm)
7 kùš-meš	1 gi	7	(crm)	=	1 reed	(crm)
3 šu.si 3 še	1 kùš-tur-*tú*	cubits	(crm)	=	1 small-cubit	(crm)
7 kùš-tur-meš	1 kùš	3 fingers 3 grains	(crm)	=	1 cubit	(crm)
		7 small-cubits				

Actually, the nine equations in the "multiplication table" above are a kind of structure table, giving the *definitions* of the following *six units of common reed measure*:

reed, cubit, finger, small-cubit, grain, and small-finger.

The next six equations form a further structure table for common reed measure, related to the structure table for length measure in W 23281 § 1 (Sec. 3.2.1 above). Five of the equations are obviously true. Only the equation 3 fingers 3 grains (crm) = 1 small-cubit (crm) needs a comment. It can be verified as follows:

3 fingers 3 grains (crm) = 3 fingers · (1 reed 1 cubit) = 3 fingers · 8 cubits = 24 fingers · 1 cubit =
1 cubit · 1 cubit = 1 small-cubit (crm).

4.1.4 § 4. A Conversion Table from Variant Length Numbers to Sexagesimal 100-Cubit Multiples (reads from right to left)

CBS 8539 § 4a: Table *L*100c,c

1 30	1		šu.si	1		finger	=	1 30	(· 60^{-3} 100-cubits)	
3	2		šu.si	2		fingers	=	3	(· 60^{-2} 100-cubits)	
4 30	3		si	3		(fing)ers	=	4 30	(· 60^{-3} 100-cubits)	
……	…		…	…				………		
16 30	11		si	11		(fing)ers	=	16 30	(· 60^{-3} 100-cubits)	
18	12	1/2 kùš	si	12	1/2 cubit	(fing)ers	=	18	(· 60^{-2} 100-cubits)	(interpolation)
19 30	13		si	13		(fing)ers	=	19 30	(· 60^{-3} 100-cubits)	
……	…		…	…				………		
24	16	2/3 kùš	si	16	2/3 cubit	(fing)ers	=	24	(· 60^{-2} 100-cubits)	(interpolation)
25 30	17		si	17		(fing)ers	=	25 30	(· 60^{-3} 100-cubits)	
……	…		…	…				………		
34 30	23		si	23		(fing)ers	=	34 30	(· 60^{-3} 100-cubits)	
36	24		si / 1 kùš	24		(fing)ers / 1 cubit	=	36	(· 60^{-2} 100-cubits)	

Table *L*100c,c, in § 4 doesn't go beyond the level 1 cubit, just like Tables *L*c,t in § 2, and *L*c,c in § 3. In the last line of the table, 1 cubit is 1/100 = ;00 36 times the basic 100-cubits unit. Similarly, in the first line of the table, 1 finger is 1/24 · 1/100 = 1/(40 00) = ;00 01 30 times the basic 100-cubits unit.

CBS 8539 § 4b: Subscript to Table *L*100c,c

[*an-ni-t*]*i* šu.si 24 si / [x x].meš kùš /	This is the finger such that 24 fingers / are [x x] cubit, /
[*am-mat*] še.numun /	[the cubit of] seed, /
[1 *me* kùš uš] 1 *me* kùš sag /	[1 hundred length] 1 hundred front /
[1(barig) pi *šá* še].numun	[1 bariga of se]ed.

It is unfortunate that this subscript is damaged. In particular, it is not clear at all what may have been written in the beginning of the second line of this subscript. However, the proposed reconstruction of the beginning of the last line of the subscript is fairly certain, since it fits the space available, and since it makes perfect sense. (Note that pi = *pānu* 'basket', and that 1(barig) pi was the standard way of writing '1 barig'. See, for instance the metrological table for capacity measures in Sec. 4.1.6 below.)

The kind of seed measure (probably) mentioned in this subscript is called "variant seed measure" in Friberg, *BaM* 28 (1997), 300 ff. It is characterized by the following simple equation:

The variant seed measure of a square with the side 100 cubits is 1 barig.

An alternative equation, in terms of sexagesimal numbers, is

The variant seed measure of a square with the side 60 cubits is ;21 36 barig.

Variant seed measure appears in four exercises in two Seleucid mathematical recombination texts (from the third century BC or later). See Neugebauer and Sachs, *MCT* (1945), 141 ff., and Friberg, *op. cit*., 300. One of these is the Uruk text AO 6484, where §§ 5 and 7 are exercises devoted to the calculation of the variant seed measure of unusually small geometric objects. In § 5, a square has the side 1 2/3 cubit = 1/60 · 100 cubits. Consequently, the variant seed measure is *S* = 1 bariga / sq. 60. Now, in traditional, Old Babylonian capacity measure, at least,

1 sìla = 1 00 shekels = 3 00 00 grains.

Assuming that this equation still holds for Late Babylonian capacity measure, one finds that

1 bariga = 6 bán = 36 sìla = 36 00 shekels = 1 48 00 00 grains,

so that

S = 1 bariga / sq. 60 = 1 48 grains (vsm).

The given answer in AO 6484, § 5 is, correctly, 1 me 8 še.numun '1 hundred 8 (grains) is the seed'.

In § 7 of AO 6484, a small symmetric triangle with the sides 5, 5, and 6 (cubits) is found to have the area 12 (sq. cubits). Consequently, the variant seed measure of the triangle is

S = 12 · ;21 36 (bariga / sq. 60) = 4;19 12 (bariga / sq. 60) = 4;19 12 · 1 48 (grains) = 7 46;33 36 (grains).

The answer is given as, approximately, 4 me 1 06 *ù mi-šil* še.meš '4 hundred 1 06 and a half (grains)'.

In the Seleucid mathematical recombination text VAT 7848 (Friberg, *op. cit.*, 302), §§ 1 and 2-4 are exercises dealing with variant seed measure. In § 3, for instance, the area of a trapezoid is calculated and is found to be 12 48 sq. cubits. The corresponding seed measure is calculated as '12 48' · '21 36' = '4 36 28 48', and the meaning of this number is claimed to be '2 1/2 sìla 2 ninda 1/2 ninda a 10th […]'. Indeed,

12 8 sq. cubits · ;21 36 bariga / sq. (60 cubits) = ;04 36 28 48 bariga = approximately 22 1/2 sìla 2 1/2 1/10 ninda.

4.1.5§ 5. A Conversion Table from Weight Names and Grain Multiples to Sexagesimal Mina Fractions (reads from right to left)

CBS 8539 § 5: Table *M*

[7 30	22 $^{1}/_{2}$	še /	*bit-qa*]
[10	30	še /	*su-ud-du-ú*]
[12	36	še /	*ḫum-mu-šú*]
[15	45	še /	4-*tú*]
[20	1+*šu*	še /	*šal-šú* gín]
[30	1 30	še /	1/2 gín]
[40	1 me 20	še /	2-*ta* šu^{II}.meš]
[45	1 me 35	še /	3 4-*tú*]
50	1 me 50	še /	5/6 gín
1	1 me 1 20	še /	1 gín
2	2		gín
…	…		……
10	10		gín
20	$^{1}/_{3}$		ma.na
30	$^{1}/_{2}$		ma.na

[22 1/2	grains /	*bitqu*	=	7 30]	(· 60^{-3} minas)
[30	grains /	a sixth (*suddû*)	=	10]	(· 60^{-2} minas)
[36	grains /	a fifth (*ḫumšu*)	=	12]	(· 60^{-2} minas)
[45	grains /	a 4th (*rebūtu*)	=	15]	(· 60^{-2} minas)
[sixty	grains /	a third shekel (*šalšu*)	=	20]	(· 60^{-2} minas)
[1 30	grains /	1/2 shekel	=	30]	(· 60^{-2} minas)
[1 hundred 20	grains /	2/3 ('two hands')	=	40]	(· 60^{-2} minas)
[1 hundred 35	grains /	3 4-th (shekel)	=	45]	(· 60^{-2} minas)
1 hundred 50	grains /	5/6 shekel	=	50	(· 60^{-2} minas)
1 hundred 1 20	grains /	1 shekel	=	1	(· 60^{-1} minas)
2	shekels		=	2	(· 60^{-1} minas)
…	…	………			
10	shekels		=	10	(· 60^{-1} minas)
1/3	mina		=	20	(· 60^{-1} minas)
1/2	mina		=	30	(· 60^{-1} minas)

In this metrological table for weight measure, shekel fractions from 1/8 shekel to 1 shekel are expressed in three ways, first with individual, Late Babylonian names for the shekel fractions, then as multiples of a grain (in typically Late Babylonian mixed sexagesimal-decimal numbers), and finally as sexagesimal fractions of the mina (= 60 shekels). The proposed reconstruction of the first half of this metrological table is based on the parallel texts which will be discussed in Sec. 4.2 below, where also more will be said about the Late Babylonian names for shekel fractions.

In the second half of the table, multiples of 1 shekel and basic fractions of the mina are equated with sexagesimal mina fractions.

4.1.6 § 6. A Conversion Table from Capacity Numbers to Sexagesimal bariga Multiples (reads from right to left)

The units of capacity measure appearing in this metrological table are the Late Babylonian units shown in the factor diagram in Fig. 4.1.4 above. Much of the beginning and much of the last part of the table are lost (see Fig. 4.1.2 above), but the proposed reconstruction seems to be reasonable. If the table really ends with ‘1 hundred gur = 8 20’, that would be a parallel to W 23273 § 2, a metrological table for length measures which goes as far as to ‘1 hundred leagues = 50’.

CBS 8539 § 6: Table *C*

10		nindan	a nindan	=	10	($\cdot$ 60^{-2} barig)
20	[2]	nindan	[2] nindan	=	20	($\cdot$ 60^{-2} barig)
…	…	……	……		……	
[1 30	9	nindan]	[9 nindan	=	1 30]	($\cdot$ 60^{-2} barig)
[1 40	1	sìla]	[1 sìla	=	1 40]	($\cdot$ 60^{-2} barig)
…	…	……	……		……	
[8 20	5	sìla]	[5 sìla	=	8 20]	($\cdot$ 60^{-2} barig)
[10		1(bán)]	[1 bán	=	10]	($\cdot$ 60^{-1} barig)
[11 40	1(bán)	1 sìla]	[1 bán 1 sìla	=	11 40]	($\cdot$ 60^{-2} barig)
…	…	………	…………		……	
[18 20	1(bán)	5 sìla]	[1 bán 5 sìla	=	18 20]	($\cdot$ 60^{-2} barig)
[20		2(bán)]	[2 bán	=	20]	($\cdot$ 60^{-1} barig)
…	…	………	……		…	
[50		5(bán)]	[5 bán	=	50]	($\cdot$ barig)
[1	1(barig)	pi]	[1 barig	=	1]	($\cdot$ 60^{-1} barig)
[1 10	1(barig)	1(bán)]	[1 bariga 1 bán	=	1 10]	($\cdot$ 60^{-1} barig)
……	……	……	…………		……	
1 50	1(barig)	5(bán)	1 bariga 5 bán	=	1 50	($\cdot$ 60^{-1} barig)
2	2(barig)	pi	2 barig	=	2	($\cdot$ barig)
2 10	2(barig)	1(bán)	2 bariga 1 bán	=	2 10	($\cdot$ 60^{-1} barig)
……	……	……	…………		……	
4 50	4(barig)	5(bán)	4 barig	=	4 50	($\cdot$ 60^{-1} barig)
5	1(gur)	gur	1 gur	=	5	(barig)
…	……	…	……		…	
[1 35	19(gur)	gur]	19 gur	=	1 35]	(barig)
[1 40	20	gur]	[20 gur	=	1 40]	(barig)
…	…	…	……		……	
[5	1+*šu*	gur]	[sixty gur	=	5]	($\cdot$ 60^{1} barig)
…	…	…	……		…	
[8 20	1 *me*	gur]	[1 hundred gur	=	8 20]	(barig)

4.2 CBS 11032 and 11019. Two Metrological Tables for Shekel Fractions from Nippur

CBS 11032 and CBS 11019 (photo online at cdli.ucla.edu/P266196) are two small, allegedly Neo-Babylonian cuneiform tablets from Nippur with tables for shekel fractions. The conform transliterations presented here of these texts are based on the hand copies made by Sachs in *JCS* 1 (1947). The format of both clay tablets is similar to the format regularly used for first millennium *letters*. According to Sachs both tablets appear to be from the "Persian" period, in other words Achaemenid.

In CBS 11032, *Late-Babylonian terms for the standard fractions 3/4(!), 2/3, 1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/12, and 1/24 of a shekel* are equated with sexagesimal shekel fractions. The same terms for shekel fractions are known also from Late-Babylonian economical texts, except that one of the known Late-Babylonian shekel fractions is missing in the present table, namely the *ḫalluru* 'chick pea', equal to 1/10 of a shekel.

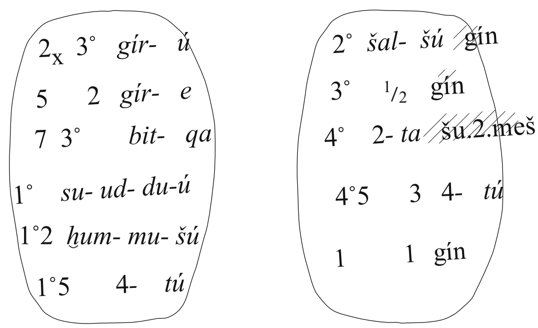


Fig. 4.2.1. CBS 11032. Conform transliteration.

2 30	*gír-ú*	*girû* (carob seed)	=	2 30	(· 60^{-2} shekel)
5	2 *gír-e*	2 *girû*	=	5	(· 60^{-1} shekel)
7 30	*bit-qa*	*bitqu*	=	7 30	(· 60^{-2} shekel)
10	*su-ud-du-ú*	a sixth (*suddû*)	=	10	(· 60^{-1} shekel)
12	*ḫum-mu-šú*	a fifth (*ḫumšu*)	=	12	(· 60^{-1} shekel)
15	4-*tú*	a 4th (*rebūtu*)	=	15	(· 60^{-1} shekel)
20	*šal-šú* gín	a third shekel (*šalšu*)	=	20	(· 60^{-1} shekel)
30	1/2 gín	1/2 shekel	=	30	(· 60^{-1} shekel)
40	2-*ta* šu^{II}.meš	2/3 ('two hands')	=	40	(· 60^{-1} shekel)
45	3 4-*tú*	3 4-th shekel	=	45	(· 60^{-1} shekel)
1	1 gín	1 shekel	=	1	(shekel)

CBS 11019 is the second small cuneiform tablet from Nippur with a table for fractions of the shekel. The obverse of the tablet is difficult to read, due to the incorrect positioning of the line intended to separate two consecutive columns of text. In the first half of the table, multiples of the grain are equated with *sexagesimal* fractions of the shekel. The second half of the table is an expanded version of the table on CBS 11032, in which both *Late-Babylonian terms for standard shekel fractions* and *decimal grain multiples* are equated with *sexagesimal shekel fractions*.

As noted by Sachs (*op. cit.* p.71) the appearance of the 'grain' in this table "is surprising since only one text of thousands of economic texts of the period uses it". The same remark applies also, of course, to Tables *M* in CBS 8539 § 5 (Sec. 4.1.5 above) and W 22260 (Sec. 3.4.1 above).

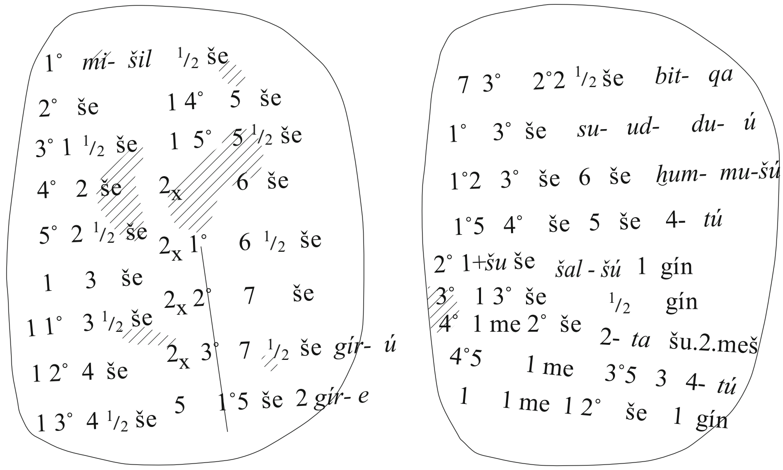


Fig. 4.2.2. CBS 11019. Conform transliteration.

10	*mi-šil*	1/2 še		a half 1/2	grain		=	10	(· 60^{-2} shekel)
20		še		a	grain		=	20	(· 60^{-2} shekel)
30	1 1/2	še		1 1/2	grains		=	30	(· 60^{-2} shekel)
40	2	še		2	grains		=	40	(· 60^{-2} shekel)
50	2 1/2	še		2 1/2	grains		=	50	(· 60^{-2} shekel)
1	3	še		3	grains		=	1	(· 60^{-1} shekel)
1 10	3 1/2	še		3 1/2	grains		=	1 10	(· 60^{-2} shekel)
1 20	4	še		4	grains		=	1 20	(· 60^{-2} shekel)
1 30	4 1/2	še		4 1/2	grains		=	1 30	(· 60^{-2} shekel)
1 40	5	še		5	grains		=	1 40	(· 60^{-2} shekel)
1 50	5 1/2	še		5 1/2	grains		=	1 50	(· 60^{-2} shekel)
2	6	še		6	grains		=	2	(· 60^{-1} shekel)
2 10	6 1/2	še		6 1/2	grains		=	2 10	(· 60^{-2} shekel)
2 20	7	še		7	grains		=	2 20	(· 60^{-2} shekel)
2 30	7 1/2	še	*gír-ú*	7 1/2	grains	*girû* (carob seed)	=	2 30	(· 60^{-2} shekel)
5	15	še	2 *gír-e*	15	grains	2 *girû*	=	5	(· 60^{-1} shekel)
7 30	22 1/2	še	*bit-qa*	22 1/2	grains	*bitqu*	=	7 30	(· 60^{-2} shekel)
10	30	še	*su-ud-du-ú*	30	grains	a sixth (*suddû*)	=	10	(· 60^{-1} shekel)
12	30 še	6še	*ḫum-mu-šú*	36	grains	a fifth (*ḫumšu*)	=	12	(· 60^{-1} shekel)
15	40 še 4-*tú*	5	še	45	grains	a 4th (*rebūtu*)	=	15	(· 60^{-1} shekel)
20	1+*šu*	še	*šal-šú* 1 gín	sixty	grains	a third shekel (*šalšu*)	=	20	(· 60^{-1} shekel)
30	1 30	še	1/2 gín	1 30	grains	1/2 shekel	=	30	(· 60^{-1} shekel)
40	1 me 20	še	2-*ta* šu^{II}.meš	1 hundred 20	grains	2/3 (‘two hands’)	=	40	(· 60^{-1} shekel)
45	1 me 35	<še>	3 4-*tú*	1 hundred 35	grains	3 4-th shekel	=	45	(· 60^{-1} shekel)
1	1 me 1 20	še	1 gín	1 hundred 1 20	grains	1 shekel	=	1	(shekel)

Note: The number sign 2_x (2 vertical wedges, one on top of the other) is employed in both CBS 11019 and CBS 11032 with the unexpected value ‘2 00’, rather than the customary ‘2 barig’.

5. Five Texts from Old Babylonian Mê-Turran (Tell Haddad), Ishchali and Shaduppûm (Tell Harmal) with Rectangular-Linear Problems for Figures of a Given Form

5.1 IM 121613. A Recombination Text from Mê-Turran with Metric Algebra Problems for Rectangles

IM 121613 (see the hand copies in Figs. 5.1.20-21 below) is a large and fairly well preserved Old Babylonian clay tablet from ancient *Mê-Turran* (the site Tell Haddad, situated in the Himrin basin near Diyala). The various fragments of the text were gathered together by Farouk Al-Rawi, who also made the hand copies of the text. Thanks are due to the excavators Dr. Nail Hanoun and Mr. Burhan Shakir for their permission to publish and for their support during the copying of the text.

IM 121613 contains 19 mathematical exercises, complete with questions (statements of problems), solution procedures, and answers. It is a *mathematical recombination text*, by which is meant that it is a somewhat disorganized collection of more or less closely related mathematical exercises copied from various source texts. In this particular case, all the exercises have *a common theme*, namely *rectangular-linear problems for rectangles of a given form* (the sides are always in the ratio 1 : 2/3). Actually, a subscript following the last exercise on the clay tablet mentions [19 x x x 2/3] uš sag.ki '[19 exercises (with)] the front 2/3 of the length'. On the edge, the clay tablet is signed im.gid$_2$.da *Da-du-ra-bi* 'long tablet of Dadurabi'.

The first step of all the solution procedures in IM 121613 is to assume that the sides of the rectangle of a given form are 1 (00) and 40 (probably thought of as multiples of a measuring reed of unknown length). The result is that *the stated problem is transformed into a quadratic problem* (presumably, for the length of the measuring reed). In each case the quadratic equation is either homogeneous or of one of three "basic types".

In this chapter, geometric explanations are given for all the solution procedures. The accompanying explanatory diagrams are made by the author and are included for the readers' benefit. For obvious reasons, there was not enough space on the clay tablet itself for such diagrams. On the other hand, it is likely that similar diagrams occasionally were drawn and demonstrated by Old Babylonian teachers in classroom situations, perhaps on a floor or on a wall.

It is shown that if the 19 exercises in IM 121613 are regrouped in a suitable way, the result is a reconstruction of (a substantial part of) *a well organized and fairly exhaustive theme text with the mentioned theme*, *rectangular-linear* problems for rectangles of a given form.

It would be reasonable to assume that all the exercises in the recombination text are copies of (copies of) exercises in some original theme text. However, it must be observed that a difficulty with this assumption is that the mathematical terminology in the 19 exercises is not quite uniform. The following tentative explanation can be proposed for this curious property of the text.

It is possible that the mentioned surmised Old Babylonian theme text with *rectangular-linear* problems for *rectangles* of a given form was derived from *a corresponding theme text with problems for either homogenous quadratic problems or quadratic problems of three basic types*. The one time existence of such a theme text was previously only vaguely indicated by a series of problems without solution procedures in section 4 of the "catalog text" *TMS V* (Sec. 8.9 below).

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For some obscure reason, *explicitly solved quadratic problems of the three basic types are exceedingly rare in published mathematical cuneiform texts.* A rapid survey has located *only two examples*, namely the exercises in section 1 of the Old Babylonian recombination text BM 13901 (Friberg, *AT* (2007), 36), and exercise # 1 in the (probably) Kassite recombination text MS 5112 (Friberg, *MSCT 1* (2007), 309).

The author is also aware of the existence of *only three other Old Babylonian texts with rectangular-linear problems for rectangles of a given form*, all of them in the collections of the Iraq Museum, just like IM 121316. One of them is the single problem text IM 43993 (Sec. 5.2 below), unprovenanced but apparently from somewhere in Eshnunna. Interestingly, the mathematical terminology in IM 43993 is quite unusual and differs markedly from the mathematical terminology in IM 121613. The other two are the exercise IM 31247 # 4´ (Baqir, *Sumer* 7 (1951); Sec. 5.3 below) and the badly preserved and previously badly published recombination text IM 54559 (Bruins, *Sumer* 9 (1953); Sec. 5.4 below), both from Shaduppûm.

In addition, IM 53963 (Bruins, *op. cit.*; Sec. 5.5 below) is a badly preserved text with a simple *rectangular-linear* problem for a *triangle* of a given form.

For the readers' convenience, in the following the individual Old Babylonian exercises in IM 121613 are *transliterated and translated one by one*, and, in each case, *sentence by sentence,* rather than line by line. In both transliterations and translations, the end of each line in the original text is marked by a slash (/).

5.1.1 # 1. A Form and Magnitude Problem for a Rectangle

The first exercise in the recombination text IM 121613 is well preserved. It is also exceedingly simple, a suitable initial exercise in a theme text dealing with *rectangular-linear* problems for the sides of a rectangle. The explicitly formulated geometric solution procedure can be interpreted in two different ways, either in terms of "quadratic and linear scale factors", or in terms of a "reed of unknown length".

IM 121613 # 1 (col. *i:* 1-8)

1 $^2/_3$ uš sag.ki
1(èše) ašag a.šag$_4$-*lam ab-ni*
2 uš *ù* sag.ki / *mi-nu*
3 za.e 1 uš 40 $^2/_3$ *šu-ku-un* 1 *ù* (erasure) 40 / *šu-ta-ki-il-ma*
40 a.šag$_4$ *sa-ar-ra-am ta-mar* /
4 igi 40 a.šag$_4$ *sa-ar-ri-im* duḫ
5 *a-na* 10 a.šag$_4$ / gi.na *i-ši-ma* 15 *ta-mar*
6 ba.sá.e 15 / *šu-li-ma* 30 *ta-mar*
7-8 30 *a-na* 1 *ù* 40 / *ma-ni-a-tim i-ta-aš-ši-ma* 30 uš *ù* 20 sag.ki / *ta-mar*
ki-a-am ne-pé-šum

1 2/3 of the length (the long side), the front (the short side).
1 èše field measure a field I built.
2 The length and the front, / what?
3 You: 1, the length, 40, 2/3, set. 1 and 40 / let eat each other (multiply together), then
40, the false field (area), you will see. /
4 The opposite (reciprocal) of 40, the false field, release (compute),
5 to 10, the true field (the given area), / carry (multiply), then 15 you will see.
6 The equalside (square-side) of 15 / let come up (compute), then 30 you will see.
7-8 30 to 1 and 40, / the measures, always carry, then 30, the length, and 20, the front, / you will see.
Such is the procedure.

In this exercise, the stated problem is to find the sides of a rectangle (implied by the mention of the 'length' and the 'front'), when it is given that

The short side is $^2/_3$ of the long side. The area is 1 èše.

The nindan (appr. 6 meters) is the most commonly used Old Babylonian length measure, usually not explicitly mentioned. The èše = 10 (00) sq. nindan is a related common Old Babylonian area measure. The quasi-modern number notation 10 (00), meaning 10 · 60, would in Old Babylonian texts be written either in full as ‘10 sixties’, or in relative sexagesimal place value notation with floating values simply as ‘10’. (Note, by the way, that in this work, *double quotes “…” indicate convenient terms introduced by the author, while single quotes ‘…’ indicate translations of terms in the Old Babylonian text.*)

In quasi-modern symbolic notations, the stated problem can be replaced by the following simple “rectangular-linear” problem for the unknowns u = the length (uš), and s = the front (sag.ki) of a rectangle, with A (a.šag$_4$) denoting the corresponding unknown area:

$s = 2/3\ u, \qquad u \cdot s = A = 10\ (00)$ sq. n. (n. = nindan) 2/3

A *modern, algebraic, way* of solving this equation would be to let s in the second equation be replaced by the value of s in the first equation. The operation would result in the following quadratic equation for the single variable u:

2/3 sq. u = 10 (00) sq. n.

(Note the use here of the quasi-modern symbolic notation sq. u, rather than the modern u^2, which would be unnecessarily anachronistic in the discussion of an Old Babylonian mathematical text! Correspondingly, sq. n. means, of course, square rods.)

The next couple of steps in the quasi-modern algebraic solution procedure would be as follows:

sq. u = 1/(2/3) · 10 (00) sq. n. = 3/2 · 10 (00) sq. n. = 15 (00) sq. n.
u = sqs. (15 (00) sq. n.) = 30 n.

(Here sqs. is a convenient abbreviation for “square-side”, a quasi-modern geometric expression which is meant to be less anachronistic than the modern algebraic expression “square-root” .)

As is well known (see Høyrup *LWS* (2002)), in Old Babylonian mathematics, problems for the sides of rectangles or squares were generally interpreted and solved *geometrically*, rather than algebraically, as in the (quasi-)modern way exemplified above. What this means in the case of the exercise IM 121613 # 1 is shown in Fig. 5.1.1 below.

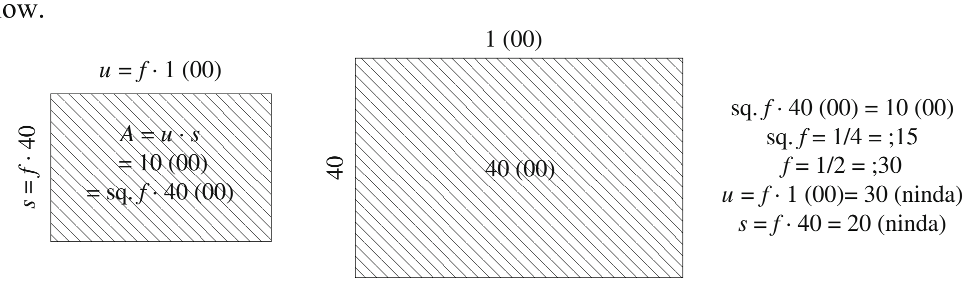


Fig. 5.1.1. IM 121613 # 1: $s = 2/3\ u$, $A = 10$. A proposed geometric solution, in terms of a scale factor.

The solution procedure starts, in lines 2-3 of the exercise, by (apparently) considering an initial rectangle with the length ‘1’, the front ‘40’, and the corresponding area ‘40’. In the text, the sides 1 and 40 of the initial rectangle are referred to as the ‘measures’ (*maniātu*, pl. of *minītu*), while the area 40 is referred to as the ‘false field’ (a.šag$_4$ *sarru*).

In view of the floating values of sexagesimal numbers in Old Babylonian relative place value notation, there are various possible interpretations of what this means. The simplest interpretation seems to be that the mentioned initial rectangle was understood as a “reference rectangle” with the length 1 (00) n., with the front 2/3 · 1 (00) n. = 40 n., and with the ‘false field’ 1 (00) n. · 40 n. = 40 (00) sq. n. (Terms like “reference rectangle”, *etc*., were introduced by Proust in *TMN* (2007), Ch. 7.)

In lines 4-5 of the text, the area ‘10’ of the given rectangle, called a.šag$_4$ gi.na, the ‘true field’, is multiplied by the reciprocal ‘igi 40’ of the ‘false field’. The result is ‘15’. A possible interpretation of this operation is that the area of the given rectangle is compared with the area of the reference rectangle, and that it is found that the area of the given rectangle is 15 sixtieths (written as ‘15’) of the area of the reference rectangle. Indeed,

igi (40 (00) sq. n.) · 10 (00) sq. n = 10 (00) / 40 (00) = 1/4 = 15 sixtieths.

This number, 15 sixtieths, or simply ;15, can be thought of as a “quadratic scale factor”.

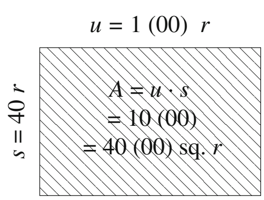
In lines 5-6, the square-side of ‘15’ is computed and stated to be ‘30’. This result can be interpreted as saying that the quadratic scale factor ;15 = 1/4 corresponds to a “linear scale factor” f = ;30 = 1/2.

In lines 7-8, finally, the ‘measures’ ‘1’ and ‘40’ are multiplied by the linear scale factor ‘30’ which has just been computed. This gives the result that u = ‘30’ and s = ‘20’. Apparently, what this means is that the sides of the given rectangle are f = ;30 times the sides of the reference rectangle, so that

$u = f \cdot$ 1 (00) n. = ;30 · 1 (00) n. = 30 n., and $s = f \cdot$ 40 n. = ;30 · 40 n. = 20 n.

It is hardly a coincidence that in this exercise, as in many similar Old Babylonian mathematical exercises, the linear scale factor turns out to be equal to ‘30’, probably meaning ;30 = 1/2. Actually, an alternative to explaining the solution procedure in IM 121613 #1 in terms of quadratic and linear scale factors is to explain it instead in terms of a “reed of unknown length”. (Reeds of unknown length appear explicitly in several Old Babylonian mathematical exercises. See, for instance, Friberg *MSCT 1* (2007), Secs. 10.1 a-c.)

In terms of a reed of unknown length, the solution procedure in IM 121613 # 1 can be explained as follows: If the length of the given rectangle is assumed to be 1 (00) = 60 such reeds, the reeds not explicitly mentioned, then the front is 2/3 · 1 (00) = 40 (reeds), and the area is 40 (00) (square reeds). On the other hand, the given area is 10 (00) sq. n. (See Fig. 5.1.2 below.) Therefore, 40 (00) square reeds = 10 (00) sq. n., and it follows that 1 square reed = 1/4 sq. n. Consequently, the length of the unknown reed is 1/2 n., and the sides of the rectangle are 1 (00) reeds = 30 n. and 40 reeds = 20 n., respectively. Actually, it is well known that a reed equal to 1/2 rod was one of the basic units in the Old Babylonian system of length measures.



40 (00) sq. r = 10 (00)
1 sq. r = 1/4 = ;15
1 r = ;30 (ninda) = 1 reed
u = 1 (00) r = 30 (ninda)
s = 40 r = 20 (ninda)

Fig. 5.1.2. IM 121613 # 1. Another geometric solution, in terms of a reed of unknown length.

Note. “Form and magnitude problems” of essentially the same type as the problem in the Old Babylonian exercise IM 121613 # 1 are known to appear also in classical Greek mathematics, namely in Euclid’s *Elements* VI.25 and in Euclid’s *Data* 55. (See Friberg *AT* (2007), Sec. 10.4). Form and magnitude problems may have been behind also several known applications of a certain “field expansion procedure” in cuneiform texts from the Old Babylonian period, and also from the Old Akkadian period, five hundred years earlier. (See Friberg *MSCT 1*, Sec. 8.1 b, in particular Fig. 8.1.3, and App. A6.1.)

Note: Related to the problem in IM 121613 # 1 is the series of 23 problems in the catalog text YBC 6492. See Fig. 8.5.1 below.

5.1.2 # 2. A Rectangle of a Given Form with a Given Sum of the Field and the Length

Exercise # 2 in IM 121613 is a suitable second exercise in a theme text dealing with *rectangular-linear* problems for the sides of a rectangle of a given form. The text of the exercise is relatively well preserved. (Here, and in the following, proposed reconstructions of missing or unreadable parts of the text are, as usual, indicated by straight brackets, both in the transliteration, and in the translation.) The geometric solution procedure is straightforward. It will be interpreted below in terms of a reed of unknown length.

IM 121613 # 2 (col. *i:* 9-21)

1 2/3 uš sag.ki
uš *a-na* a.šag$_4$ daḫ-*ma* 40
2 uš *ù* / sag.ki *mi-nu*
za.e 1 uš 40 sag.ki /
3-4 uš *ù* sag.ki *šu-ta-ki-il-ma* 40 a.šag$_4$ *sa-ar-ra-* / *am* / *ta-mar*
5 40 a.šag$_4$ *a-na* 40 a.šag$_4$ gi.na *i-ši-ma* / 26 40 *ta-mar re-eš-ka li-ki-il*
6 *tu-úr* / 1 uš *i*$^!$-*ta*$^!$ <*ḫe-pe*> *me-eḫ-ra-am* <*i-di*>
7 *šu-ta-ki-il-ma* / 15 *ta-mar*
8 15 *a-na* 26 40 *ša re-eš-ka* / *ú-ki-lu* daḫ-*ma* 41 40 *ta-mar*
9 ba.sá.e 41 40 / *šu-li-ma* 50 *ta-mar*
10 *i-na* šag$_4$ ba.sá.e 30 / *ta-ki-il-ta-ka* zi.zi-*ma* [20] *ta-mar*
11-12 *tu-úr* / igi 40 a.šag$_4$ *sa-ar-ri-im a-na* 20 *i-ši-ma* / 30 uš *ta-mar*
13 30 *a-na* 40 sag.ki *i-ši-ma* / 20 sag.ki *ta-mar*

1 2/3 of the length, the front.
The length to the field I added on, then 40.
2 The length and / the front, what?
You: 1, the length, 40, the front, touch. /
3-4 The length and the front, let them eat each other (multiply each other), then 40, the false field, / you will see.
5 40, the field, to 40, the true field, carry, then / 26 40 you will see. Let it hold your head.
6 Return. / 1 uš (= 1 00 n.), the adjacent, <break> a copy <lay down>,
7 let them eat each other, then / 15 you will see.
8 15 to 26 40 that held your head / add on, then 41 40 you will see.
9 The equalside of 41 40 / let come up, then 50 you will see.
10 Out of the equalside 30, / your held-on-to, tear off, then [20] you will see.
11-12 Return. / The reciprocal of 40, the false field, to 20 carry, then / 30, the length, you will see.
13 30 to 40, the front, carry, then / 20, the front, you will see

The stated problem, in line 1 of this text, is to find the sides of a rectangle when it is given that

a) The front is 2/3 of the length. b) The field plus the length is equal to '40'.

What the second condition means is not immediately obvious. However, as will be clear from the form of the solution procedure, it is almost certain that what is meant is

a) The front is 2/3 of the length. b) The area-plus-'1' times the length is equal to '40'.

The geometric formulation of this rectangular-linear problem is shown in Fig. 5.1.3 a below. There u and s are the length and front of a rectangle, both unknown, and p = 1 uš = 1 (00 rods) is a given "extension" of the front, in line 6 possibly called *i-ta*, 'next to, adjacent', a term known from cuneiform field plans.

B = 40(00 sq. n.) is the given area of the correspondingly "extended rectangle" with the length u and the front $s + p$.

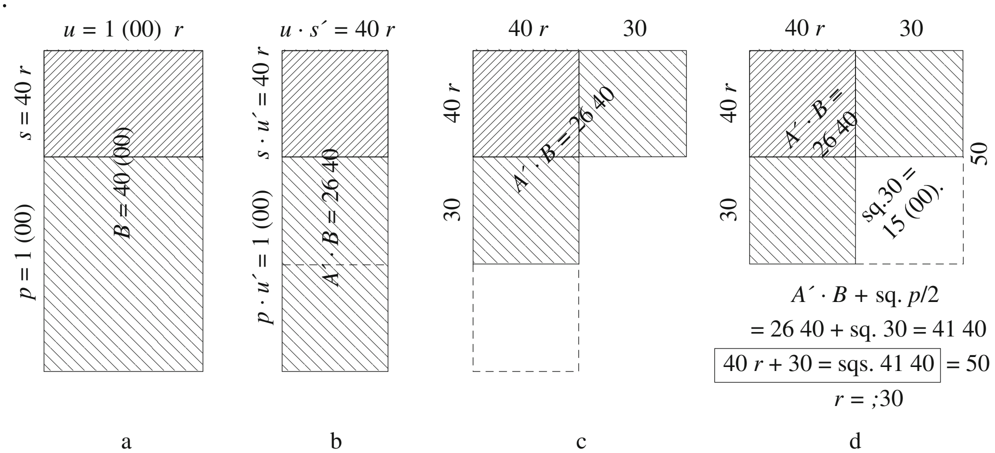


Fig. 5.1.3. IM 121613 # 2: $s = 2/3\ u$, $A + u = 40$.
Geometric formulation of the problem, followed by the processes of equalizing, balancing, and completing.

The solution procedure begins in lines 2-4 of the text by setting u = '1' and s = '40', possibly thought of as 1 (00) r and 40 r, respectively, where r stands for a silently understood measuring reed of unknown length. The corresponding product '1' · '40' = '40', possibly thought of as 40 (00) sq. r, is called the 'false field'. It is conveniently referred to here as A'.

The next step of the solution procedure, in lines 4-5, is to multiply the 'true field 40', meaning the given area B of the extended rectangle in Fig. 5.1.3 a, by the 'false field 40'. The result is written as '26 40'.

What this is supposed to mean is not explained in the text., but it is likely that the purpose of the operation is to "equalize" the sides of the initial rectangle, the one with the unknown sides u and s. This can be done through multiplication of u = 1 (00) r by s' = 40 and of s = 40 r by u' = 1 (00), with the result that the initial rectangle is transformed into a square, while the extended rectangle is transformed into a corresponding "extended square" with the area $A' \cdot B$, where A' = 1 (00) · 40. However*, in view of the floating values of sexagesimal numbers in Babylonian relative place value notation*, the operation can just as well be thought of as multiplication of u by s' = ;40 = 2/3 and of s by u' = 1. The result of the operation will then be an extended square with the area $A' \cdot B$, where B = 40 (00), and where A' = 1 · ;40 = ;40. Consequently, $A' \cdot B$ = 26 40, as indicated in Fig. 5.1.3 b above.

The next step of the solution procedure, in lines 6-7 of the text, is to "balance" the extended square, by cutting off half the extension and placing the cut-off half as an extension in another direction, with the result that the once extended square is replaced by a balanced, twice extended square, as in Fig. 5.1.3 c above. To this balanced, twice extended square is added a square with the side $p/2$ = 30, that is, a square with the area sq. 30 = 15 (00). See Fig. 5.1.3 d. (In the text, $p/2$ is called *takiltum* possibly meaning 'held-on-to', that is a quantity that you have computed once before and hold in you memory for further use. See the discussion of the term in Friberg and George, *PGS* (2010), 124. See also Robson, *MMTC* (1999), 232 fn. 4.) The result of this *geometric* "completion of the square", in line 8, is an enlarged square with the area

$A' \cdot B$ + sq. $p/2$ = 26 40 + 15 (00) = 41 40.

In lines 8-9, the side of this enlarged square is found to be sqs. 41 40 = 50.
Consequently,

40 r + 30 = 50

Hence, in lines 9-10, 40 r = 50 – 30 = 20, so that, in lines 11-12*,* r = 20/40 = '30'. Therefore,

u = 1 (00) r = 30 (rods), and s = 40 r = 20 (rods).

Note that there is a small mistake in the text, in line 11, where 20 is multiplied by 'igi 40' and '40' is called the 'false field'. A correct identification would have been to call '40' the 'false front' which would be an appropriate name for s'= ;40 in Fig. 5.1.3 b. Note also the short-cut taken in line 12, where 20/40 = '30' is immediately identified with the length u, instead of first with the length of the "unknown reed" r*,* and then with the length u.

What is going on in IM 121613 # 2 can be described in perhaps more familiar forms as follows: The stated problem has to be interpreted as meaning that

a) The front s is 2/3 of the length u. b) The area $u \cdot s$ plus 1 (00) · the length u is equal to 40 (00).

Here the equation a) is linear, while the equation b) can be called "rectangular".

More concisely, in quasi-modern symbolic notations,

s = ;40 · u, $u \cdot s$ + 1 (00) · u = 40 (00) (*)

This "rectangular-linear" problem for the two unknowns u and s is the algebraic equivalent of the geometric interpretation in Fig. 5.1.3 a of the problem stated in IM 121613 # 2. The geometric transformation of Fig. 5.1.3 a into Fig. 5.1.3 b corresponds to an algebraic transformation of the rectangular-linear problem (*) into the following "quadratic-linear" system of equations (one linear, the other quadratic):

s = ;40 · u, sq. s + 1 (00) · s = 26 40 (**)

All that is needed to achieve the transformation from (*) to (**) is to multiply both sides of the second equation in (*) by the factor ;40.

Through the customary completion of the square, the quadratic equation for s in (**) can be reduced to the equivalent quadratic equation

sq. (s + 30) = 26 40+ 15 (00) = 41 40 = sq. 50.

Consequently,

s + 30 = 50, so that s = 20 (rods) and u = 20/;40 = 30 (rods).

5.1.3 # 3. A Rectangle of a Given Form with a Given Sum of the Field and Twice the Front

The question in exercise # 3 in IM 121613 is a simple variation of the question in exercise # 2, with 2(00) · the *front* instead of 1 (00) · the *length* added to the (area of the) field. The change leads to a slight complication of the solution procedure, namely that in this case the "equalization" step of the solution procedure affects the size of the extension of the front.

IM 121613 # 3 (col. *i:* 22-35)

1 [2/3 uš s]ag.ki a.šag$_4$ *ab-ni*
2 *a-tu-úr* sag.ki *a-na* / [*ši-na*] *e-ṣi-ma a-na* šag$_4$ a.šag$_4$ daḫ-*ma* 5[0]
3 [u]š / [*ù* sag.ki] *ki-ia*
4-5 1 uš 40 *ši-ni-pa-tim* / [*lu-pu-ut* 1 *a-na* 40] *i-ši-ma* 40 a.šag$_4$-*lam* / [*sa*-<*ar*>-*ra-am ta-mar*]
6 [40 a.š]à *sa*-<*ar*>-*ra*-[*am*] / [*a-na* 50 a].šag$_4$ gi.na *i-š*[*i-m*]*a* [3]3 20 *ta-mar* /
7 [*x x*] *x ù a-na* 2 40 *e-ṣi*-[*ma* 1 20 *t*]*a-mar* /
8-9 [1 20 *ḫe-p*]*e šu-ta-ki-il-ma* 26 40 / [*ta-mar*]
10 2[6 4]0 [erasure] *a-na* 33 20 daḫ-*ma* 1 *ta-mar* / {[1 *ta*-]*mar*}
ba.sá.e-*šu* 1 *ta-mar*
11 40 *ta-ki-il*-[*ta-ka*] / [*i-na*] šag$_4$ ba.sá.e *ḫu-ru-iṣ-ma* 20 *ta-mar*
12-13 igi / [4]0 a.šag$_4$ *sa-ar-ri-im* duḫ *a-na* 20 *i-ši-ma* / 30 uš *ta-mar*
14 30 *a-na* 40 sag.ki *i-ši-ma* / 20 sag.ki *ta-mar* 30 uš 2[0 sa]g.ki

1 [2/3 of the length] the front. A field I built.
2 I turned back. The front to / [two] I repeated, onto the field I added it, then 50.
3 The length / [and the front] how much (each)?
4-5 1 the length, 40 of the two-thirds, / [touch. 1 to 40] carry, then 40, the [false] field, / [you will see].
6 [40], the false [field / to 50], the true field, carry, then [3]3 20 you will see. /
7 [x x] x and to 2 40 repeat, [then 1 20] you will see. /
8-9 [1 20 break], let eat itself, then 26 40 / [you will see].
10 2[6 4]0 to 33 20 add on, then 1 you will see. / {[1 you] will see.}
Its equalside 1 you will see.
11 40, [your] holder, / out [from] the equalside break off, then 20 you will see.
12-13 The reciprocal / of 40, the false field, release, to 20 carry, then / 30, the length, you will see.
14 30 to 40, the front, carry, then / 20, the front, you will see. 30 the length, 20 the front.

In this exercise, the stated problem is to find the sides of a rectangle when it is given that

a) The front is 2/3 of the length. b) The field plus two times the front is equal to '50'.

This almost certainly means that

a) The front is 2/3 of the length. b) The area plus 2 (00) · the front is equal to 50 (00).

The geometric interpretation of this rectangular-linear problem is shown in Fig. 5.1.4 a below. There u and s are the length and the front of a rectangle, both unknown, and q = 2 (00) is the given extension of the length.

The first step of the solution procedure, in lines 4-6 of the text, the equalization, is to (silently) multiply the extended length by s' = '40' (2/3) and the front by u' = '1'. Consequently, the area B = 50 (00) (silently understood as 50 (00) sq. rods) is multiplied by the 'false field' '40'. The result is that $A' \cdot B$ = 33 20 (meaning 33 20 sq. rods), as shown in Fig. 5.1.4 b. In addition (see line 7 of the text), it follows that the new extended length is

;40 · (u + q) = ;40 · (1 (00) r + 2 (00)) = 40 r + 1 20.

The remaining steps are precisely as in the case of exercise # 2: A "balancing" step lets the extended square with the side 40 r be replaced by a balanced, twice extended square, with both fronts expanded by the amount $q \cdot s'/2$ = 1 20/2 = 40. Next, a completion of the square transforms the balanced, twice extended square into a larger square with the side 40 r + 40, and the area

33 20 + sq. 40 = 33 20 + 26 40 = 1 (00 00) = sq. 1 (00).

Therefore,

40 r + 40 = 1 (00), so that 40 r = 20 and r = 20/40 = 1/2 = ;30 (rods) = 1 reed.

Therefore, as in most similar cases, u = 30 (rods), and s = 20 (rods).

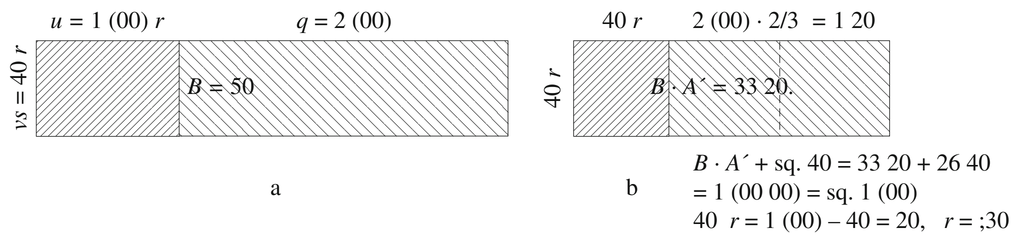


Fig. 5.1.4. IM 121613 # 3: $s = 2/3\ u$, $A + 2\ s = 50$.

The modern, algebraic counterpart to the geometrically formulated problem in Fig. 5.1.4 a is the rectangular-linear problem

s =;40 · u, $u \cdot s$ + 2 (00) · s = 50 (00) (*)

The geometric transformation of Fig. 5.1.4 a into Fig. 5.1.4 b corresponds to an algebraic transformation of the rectangular-linear problem (*) into the quadratic-linear problem:

s =;40 · u, sq. s + 1 20 · s = 33 20.

This transformation can be achieved through multiplying the second equation in (*) by ;40.

5.1.4 # 4. A Rectangle of a Given Form and a Given Sum of Two Rectangular Fields

Crucial parts of the question in exercise # 4 of IM 121613 are missing, as well as the initial part of the solution procedure, which makes it difficult to reconstruct the exact original form of the question. Anyway, it is clear that the given problem can be reduced to a *homogeneous* quadratic problem (without a linear term), which can be solved, like the problem in exercise # 1, without the use of any completion of the square.

IM 121613 # 4 (col. *i:* 36-45, col. *ii*: 1-6)

1 $^2/_3$ uš sag.ki a.šag$_4$ *ab-ni*
2 *a-tu-úr* [$^2/_3$] / *i-na* uš zi.z[*i-ma x x x x ša*] /
3-4 *i-na* uš zi.zi *a-n*[*a* s]ag.ki [*x x x x*] / a.šag$_4$ *ab-ni*
5 *a-*[*xxx*] *ù s*[*a-a*]*r-r*[*a-a*]*m* / *ak-mur-ma* 20
[*x x x* uš *ù* sag.ki] *mi-nu* /
6-7 za.e [*x x x x x x x x x x*] / *x x sa-ar-ra-am x* [*x x*] *ta-mar* /
8-9 [*x x*] *i-na* 1 40 $^2/_3$ zi.zi-*ma* 20 / *ta-mar*]
10 40 *ša ta-aḫ-ru-uṣ a-na* [*ši-na*] / [*e-ṣ*]*i-ip-ma* 1 20 *ta-mar*
11 1 20 *a-na* 40 / *ša ki-ma* sag.k[i] *ta-al-pu-tu* daḫ-*ma* 2 *ta-mar* /
12 2 *a-na* 20 *ši-ta-at* uš *i-ši-ma* 40 *ta-mar*
13 40 *a-na* / 40 a.šag$_4$ *sa-ar-ri-im* daḫ-*ma* 1 20 *ta-mar*
14 igi / 1 20 duḫ *a-na* 20 a.šag$_4$-*lim i-ši-ma* 15 *ta-mar*
15 ba.sá.e-*šu* <*šu-li-ma*> / 30 *ta-mar*
16 30 *a-na* 1 *ù* 40 *ma-ni-a-ti-k*[*a i-š*]*i-ma* / 30 uš 20 sag.ki a.šag$_4$ *x* [*x x*] *x*
1 2/3 of the length, the front. A field I built.
2 I returned. [2/3] / from the length I tore off, then [x x x x x x that] /
3-4 from the length I tore off to the front [x x x x x x x x x x]. / A field I built.
5 [x x x x] and the false <field> / I heaped, then 20.
[The length and the front], what? /

6-7 **You:** [x x x x x x x x x x] / x x false [x x x] you will see
8-9 [x x] From 1, the length, 40, 2/3, tear off, then 20 / [you will see].
10 40 that you cut off, to [two] / repeat, then 1 20 you will see.
11 1 20 to 40 / that you as front touched, add on, then 2 you will see. /
12 2 to 20, the left-over of the length, carry, then 40 you will see.
13 40 to / 40, the false field, add on, then 1 20 you will see.
14 The reciprocal / of 1 20 release, to 20, the field, carry, then 15 you will see.
15 Its equalside <let come up>, / 30 you will see.
16 30 to 1 and 40, of your measures, carry, then / 30 the length, 20, the front. The field [x x x x x].

As mentioned, not much is preserved of the question. However, it seems to be clear that, in addition to the rectangular field formed by the length *u* and the front *s* (line 1 of the text), another rectangular field is formed in some way (lines 2-4), and that the sum of the (areas of the) two rectangular fields is 20 (line 5). In the formation of the second field, something that is subtracted from the length *u* seems to play some role.

In the solution procedure, the first step (lines 6-7) appears to be the computation of the 'false field' in the usual way, as $1 \cdot 40 = 40$. With the false values 1 and 40 for *u* and *s*, the solution procedure continues as follows: In the second step of the procedure (line 8), the length minus 2/3 of the length is found to be 20. Next, 2 times 2/3 of the length plus the front is seen to be equal to 2 (lines 9-11). Then 2 times 20 = 40 is computed (line 12), and this number is added to the number 40, representing the false area (lines 12-13). The result, 1 20, is found to be 15 times larger than 20, the given area (lines 14-15). And so on.

The fairly well preserved solution procedure, described above, makes it possible to reconstruct the question, which, apparently, in quasi-modern symbolic notations can be expressed as follows:

$$s = ;40 \cdot u, \qquad u \cdot s + (u - 2/3\ u) \cdot (2 \cdot 2/3\ u + s) = 20\ (00).$$

With $u = 1\ (00)\ r$ and $s = 40\ r$, where *r* is a reed of unknown length, the second equation can be reformulated as

$$1\ (00) \cdot 40 \text{ sq. } r + (1\ (00) - 40) \cdot (2 \cdot 40 + 40) \text{ sq. } r = (1\ (00) \cdot 40 + 20 \cdot 2\ (00)) \text{ sq. } r = 20\ (00).$$

Consequently,

$$1\ 20\ (00) \text{ sq. } r = 20\ (00), \quad \text{so that} \quad \text{sq. } r = 20\ (00) / 1\ 20\ (00) = ;15, \quad \text{and} \quad r = \text{sqs. } ;15 = ;30.$$

Therefore, as usual, $u = 1\ (00)\ r = 30$, and $s = 40\ r = 20$

5.1.5 # 5. A Rectangle of a Given Form with a Given Sum of the Field, the Length, and the Front

In exercise # 5, the theme from exercises ## 1-3 is continued. The question in this exercise is, in a way, a combination of the questions in exercises ## 2 and 3. This time, the considered rectangle with unknown sides is extended in two directions. The change leads to only a slight complication of the solution procedure.

IM 121613 # 5 (col. *ii*: 7-16)

1 2/3 uš sag.ki
a.šag$_4$ uš *ù* sag.ki *ak-mur-ma* 1 3[7 30] /
2 uš *ù* sag.ki *mi-nu*
3 za.e 40 sag.ki *a-na* 1 uš *šu-ta-ki-*[*il*] / 40 a.šag$_4$ *sa-ar-ra-am ta-mar*
4 40 a.šag$_4$ *sa-ar-ra-a*[*m*] / *a-na* 1 37 30 *i-ši* 1 05 *ta-mar*
5 uš *ù* sag.ki / *ku-mur*$^{!}$ *ḫe-pé šu-ta-ki-il-ma* 41 40 *ta-mar* /
6 41 40 *a-n*[*a* 1] 05 daḫ-*ma* 1 46 40 *ta-mar*
7 ba.sá.e-*šu* / [*šu-li* 1 20 *ta-mar*]
8 *i-na* šag$_4$ 1 20-*e* 50 *ta-ki-il-ta-ka* / z[i.zi-*m*]*a* 30 *ta-mar*
9 igi 40 a.šag$_4$ *sa-ar-ri-im* duḫ / *a-*[*na* 30] *i-ši-ma* 45 uš *ta-mar*
10 45 *a-na* 40 / [*i-ši-ma* 30] sag.ki *ta-mar*
45 uš 30 sag.ki
1 2/3 of the length, the front.
The field, the length, and the front I heaped (added together), then 1 3[7 30]. /
2 The length and the front, what?
3 You: 40, the front, to(sic!) 1, the length, let eat each other, / 40, the false field, you will see.

4 40, the false field, / to 1 37 30 carry, 1 05 you will see.
5 The length and the front / heap, break, let eat each itself, then 41 40 you will see. /
6 41 40 [to 1] 05 add on, then 1 46 40 you will see.
7 Its equalside / [let come up, 1 20 you will see].
8 Out from 1 20, 50, your holder, / [tear off, then] 30 you will see.
9 The reciprocal of 40, the false field, release, / [to 30] carry, then 45, the length, you will see.
10 45 to 40 / [carry, then 30], the front, you will see.
45 is the length, 30 is the front.

In this exercise, the stated problem is to find the sides of a rectangle when it is given that

a) The front is 2/3 of the length. b) The field plus the length and the front is equal to '1 37 30'.

This almost certainly means that

a) The front is 2/3 of the length. b) The area plus 1 (00) · (the length and the front) is equal to 1 37 30.

The geometric interpretation of this rectangular-linear problem is shown in Fig. 5.1.5 a. There u = 1 (00) r and s = 40 r are the length and front of a rectangle, both unknown, p = 1 (00) is the given extension of the length, and q = 1 (00) the given extension of the front. B = 1 37 30 is the given area of the twice extended rectangle.

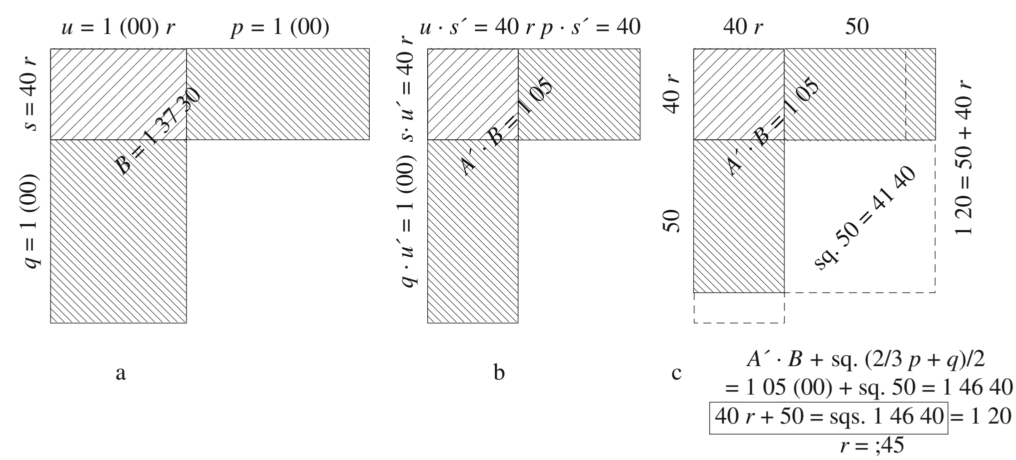


Fig. 5.1.5. IM 121613 # 5: $s = 2/3\,u$, $A + u + s$ = 1 33 20.

The first step of the solution procedure, the equalization in lines 2-4 of the text, is to (silently) multiply the extended length by s' = ;40 and the extended front by u' = 1. Consequently, the area B = 1 37 30 is multiplied by the 'false field' ;40. The result is that $A' \cdot B$ = 1 05 (00), as shown in Fig. 5.1.5 b. In addition, it follows that the new extended length is

;40 · (u + q) = ;40 · (1 (00) r + 1 (00)) = 40 r + 40.

This fact is not explicitly mentioned in the text, possible because the computation '40' · '1' = '40' is so trivial in Babylonian relative place value notation with floating values.

In the next step of the solution procedure, the balancing in lines 4-5, the unequal extensions $q \cdot s'$ = 40 of the length and p = 1 (00) of the front are both replaced by their mean-value (40 + 1 00)/2 = 50, which is the side of a square with the area sq. 50 = 41 40. (Here the author of the text makes a curious mistake and calls p = 1 (00) and $q \cdot s'$ = 40 'the length and the front' (line 4 right) instead of 'the extensions of the length and the front', obviously deceived by the coincidence that the extensions are 1 (00) and 40, just like the assumed values of the length and the front!)

Consequently, a completion of the square transforms the balanced, twice extended square into a larger square with the side 40 r + 50, and the area

1 05 (00) + sq. 50 = 1 05 (00) + 41 40 = 1 46 40 = sq. 1 20.

Therefore, as in Fig. 5.1.5 c,

40 r + 50 = 1 20, so that 40 r = 1 20 – 50 = 30 and r = 30 /40 = ;45 (rods) = 1 1/2 reed.

This unexpected result implies that, atypically,

u = 45 (rods), s = 30 (rods).

5.1.6 # 6. A Rectangle of a Given Form with a Given Sum of Many Terms

The text of exercise # 6 in IM 121613 is badly preserved, but it has been possible, with considerable effort, to reconstruct most of the text in a satisfactory way.

The problem in the exercise is quite strangely formulated. Yet, it is, in a way, a continuation of the straightforward problem in exercise # 5. The author of the exercise seems to have had in mind to show, in a playful way, that *many kinds of generalizations* can be made of problems like the ones in exercises ## 1-3 and 5 in IM 121613, without any major complication of the solution procedure.

IM 121613 # 6 *(col. ii*: 17-32)

1 2/3 uš sag.ki

2 a.[š]à uš sag.ki 1/3!-*ti* uš *ri*!-*ba-at* / sag.ki *ši*-[*ta* u]š

3 *ši-ta* sag.ki *ù ku-la ša* uš / ugu sag.ki dirig *ak-mu-ur-ma* 3 05

4 uš *ù* / sag.ki *mi-nu*

5 za.e 40 sag.ki [*a-n*]*a*! 1! uš *i-ši-ma* / 40 a.šag$_4$ *sa-ar-ra-am ta-mar*

6 [a.šag$_4$] *sa-ar-ra-am* / *a-na* 3 05 *i-ši-ma* 2 03 20 *ta-mar*

7 *tu-úr*! 1 uš / [40] sag.ki 20 *ša-lu-uš-ti*!(written as 20) uš 10 *ri*!-*ba-at* sag.ki /

8 [2] *ši*!-*ta*! uš 1 20 *ši-ta* sag.ki *ù* < 10> *ku-la ša* uš /

9 ugu sag.ki dirig <*ku-mur*> *ḫ*[*e-pé-ma* 2 55 *ta*]-*mar* /

10-11 [*ku-mu-u*]*r ḫe-pé-ma* 2 55 *mé*-[*eḫ-ra-am*]? *šu-ta-ki-il* / [8 3]1? 25 *ta-mar*

12 8 30 2[5 *a*]-*na* 2 03 20 / [x] a.šag$_4$ daḫ-*ma* 10 3[3 4]5 *t*[*a-m*]*ar*

13 [ba.sá.e-*šu šu-li* 3 15] / [*ta*]-*mar*

14 2 55 *ša* [*re-eš-ka ú-ki-lu i-na* 3 15] / zi.zi-*ma* 20 *ta-ma*[*r*]

15 [x igi 40 a.šag$_4$ *sa-ar-ri-im* duḫ] / [*a-na* 20 x] *i-ši-ma* [30 *ta-mar*]

16 [30 *a-na* 1 uš *ù* 40 sag].ki / *i-ši-ma* [30 uš 20 sag.ki *ta-ma*]*r*

1 2/3 of the length, the front.

2 The field, the length, the front, the third of the length, the fourth / of the front, two lengths,

3 two fronts, and all that the length / over the front was beyond, I heaped, then 3 05.

4 The length and / the front, what?

5 You: 40, the front, to 1, the length, carry, then / 40, the false field, you will see.

6 The false [field] / to 3 05 carry, then 2 03 20 you will see.

7 Return. 1, the length, / [40] the front, 20 the third of the length, 10 the fourth of the front,

8 [<2>, two] lengths, 1 20, two fronts, and <10>, all that the length /

9 over the front is beyond, <heap>, break, [then 2 55 you] will see. /

10-11 Heap, break, then 2 55, [a copy]?, let them eat each other, / 8 30! 25 you will see.

12 8 30 2[5 to] 2 03 20, / the field, add on, then 10 3[3 45 you will see].

13 [Its equalside let come up, then 3 15 / you will] see.

14 2 55 that [held your head from 3 15] / tear off, then 20 you will see.

15 [The reciprocal of 40, the false field, release, / to 20] carry, then [30 you will see].

16 [30 to 1, the length, and 40, the fr]ont, / carry, then [30 the length, 20 the front, you will see].

Precisely as in the related exercises considered so far, ## 1-3, and 5, the stated conditions were almost certainly intended to mean that (in quasi-modern symbolic notations)

a) $s = 2/3\ u$, b) $u \cdot s + 1\ (00) \cdot (u + s + u/3 + s/4 + 2\ u + 2\ s + (u - s)) = 3\ 05\ (00)$.

In view of condition a), condition b) can be thought of as

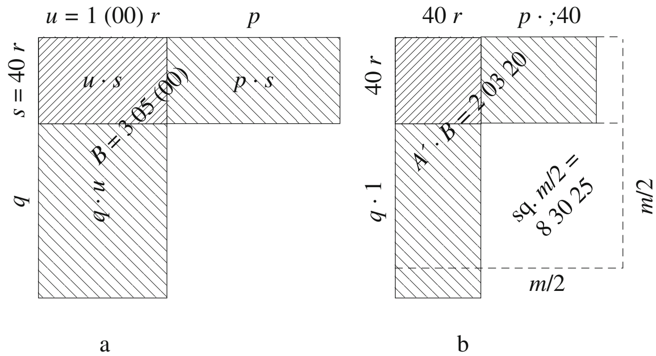
$u \cdot s + m \cdot u = 3\ 05\ (00)$, where $m = 1\ (00) + 40 + 20 + 10 + 2\ (00) + 1\ 20 + 20 = 5\ 50$.

The computation in this way of m and, subsequently, of $m/2 = 5\ 50\ /2 = 2\ 55$ is carried out in lines 6-9 of the text of exercise # 6.

Since now the stated problem in exercise # 6 has been reduced to the problem

a) $s = 2/3\ u$, b) $u \cdot s + m \cdot u = 3\ 05\ (00)$, where $m = 5\ 50$,

it can be solved without trouble in the same way as the simpler problem in exercise # 2. Additional details of the solution procedure are shown in Fig. 5.1.6 below. As usual, it turns out that the reed of unknown length is ;30 = 1/2 (rod), so that $u = 30$ (rods), $s = 20$ (rods).



$q \cdot 1$ = 1 00 + 20 + 2 00 + 1 00
$p \cdot ;40$ = 40 + 10 + 1 20 – 40

$m/2 = (q \cdot 1 + p \cdot ;40)/2$ = 2 55

$A´ \cdot B$ + sq. $m/2$
= 2 03 20 + 8 30 25
= 10 33 45

40 r + 2 55 = sqs. 10 33 45

= 3 15
r = 20/40 = ;30

Fig. 5.1.6. IM 121613 # 6: s = 2/3 u, $A + u + s + u/3 + s/4 + 2\,u + 2\,s + (u - s)$ = 3 05.

5.1.7 # 7. A Rectangle of a Given Form with a Given Sum of the Field and the Squares on the Length and the Front

Almost all of the question in this exercise is lost, as well as the beginning of the solution procedure. Therefore, the reconstruction below of the question is partly tentative.

Apparently, the sum of (the area of) a rectangle of given form and the sum of the (areas of the) squares on the length and the front of the rectangle is given. The problem can easily be reduced to a quadratic bproblem of the simplest kind, one which can be solved, like the problems in exercises ## 1 and 4, without the use of any completion of the square.

IM 121613 # 7 (col. *ii:* 33-46)

1 $^2/_3$ uš [sag.ki a.šag$_4$ *ab-ni*]
2 [*a*]*-tu-úr* / [sag.ki *a-na x x x x x x x x x x x x x x x x x*] a.šag$_4$ /
3 [*x x x x x x x x x x x x x x x*]
4 [uš *ù*] sag.ki / [*mi-nu*]
[za.e *x x x x x x x x x x x x x x*] *ta-mar* /
5-6 [1 uš *a-na*] 40 sag.ki [*i-ši-ma* 40 *ta-mar*] / 40 *re-eš-ka li-ki-il*
7 *tu-úr x x x / x x x x x* 40 *lu-pu-ut šu-ta-ki-il*
8-9 *a-na* / 40 *ša re-eš-ka ú-ki-lu* daḫ*-ma* 2 06$^!$ 40 / *ta-mar*
10 *mi-nam a-na* 2 06 40 *lu$^!$-uš-ku-un / ša* 31 40 *i-na-ad-di-nam*
11 15 *lu$^!$-uš$^!$-ku$^!$-un$^!$* / 15 *a-na* 2 06 40 *i-ši-ma* 331 40 *ta-mar* /
12 ba.sá.e 15 *šu-li-ma* 30 *ta-mar*
13 30 *a-na* {1} / 1 uš *ù* 40 sag$^!$ *i-ši-ma*
14 30 uš *ù* 20 sag.ki / *ta-mar*
30 uš 20 sag.ki

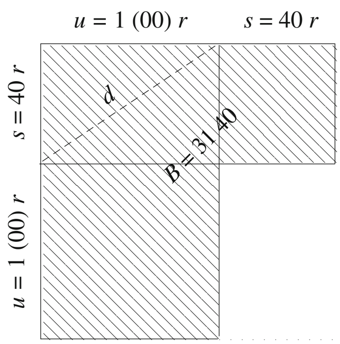
1 23 / of the length, [the front. A field I built.]
2 [I] returned. / x x x x x x x x x x x x x x x x x x x x x] the field. /
3 [x x x x x x x x x x x x x x x x x]
4 [The length and] the front, / what?
[You: x x x x x x x x x x x x x] you will see. /
5-6 [1, the length, to] 40, the front, [carry, then 40 you will see.] / Let 40 hold your head.
7 Return. x x x / x x x x x 40 touch, let eat each other,
8-9 to / 40 that holds your head add on, then 2 06 40 / you will see.
10 What to 2 06 40 should I set / that will give me 31 40 ?
11 15 I should set. / 15 to 2 06 40 carry, then 31 40 you will see. /
12 The equalside of 15 let come up, then 30 you will see.
13 30 to {1} / 1, the length, and 40, the front, carry, then
14 30, the length, and 20, the front, / you will see.
30 the length, 20 the front.

As mentioned, the question in this exercise is lost. Also the first and the third steps of the solution procedure are lost, in lines 4 and 7 of the text, respectively, although it is clear that a multiplication is performed in line 7. In line 5, apparently, the area of the rectangle is computed as '1' · '40', probably meaning 1 (00) r · 40 r = 40 (00) (sq. r). Then, in line 8, this area of the rectangle is added to a new 'false field', apparently the result of the multiplication in line 7. The sum of the two fields is 2 06 40 (sq. r). Therefore the result of the multiplication in line 7 must have been

2 06 40 (sq. r) – 40 (00) (sq. r) = 1 26 40 (sq. r).

The simplest explanation of this area number is that it is the sum of the areas of the squares on the length and the front of the rectangle with the sides 1 (00) r and 40 r. Indeed,

sq. 1 (00) r + sq. 40 r = 1 (00 00) sq. r + z26 40 sq. r = 1 26 40 sq. r.



A + sq. u + sq. s
= 1 (00) r · 40 r + sq. 1 (00) r + sq. 40 r
= (40 (00) + 1 (00 00) + 26 40) sq. r
= 2 06 40 sq. r
= B = 31 40

sq. r = 31 40 / 2 06 40 = 1/4
r = 1/2 = ;30

Fig. 5.1.7. IM 121613 # 7: s = 2/3 u, A + sq. u + sq. s = 31 40.

Consequently, it seems to be reasonable to propose the following reconstruction of the question in exercise # 7 of IM 121613:

a) the front is 2/3 of the length,
b) the sum of the areas of the rectangle and the squares on the length and the front is 31 40.

The meaning of the last few steps of the solution procedure is clear (see the well preserved lines 9-14 of the text), namely that if the sum of the areas of the rectangle and of the squares on the length and the front is 1 26 40 (sq. r), then

1 26 40 sq. r = 31 40, so that sq. r = 31 40 / 1 26 40 = 1/4 = ;15 and r = ;30.

Consequently, as usual, u = 1 (00) r = 30 (rods) and s = 40 r = 20 (rods).

5.1.8 # 8. A Rectangle of a Given Form with a Given Product of the Field and the Length

The text of this brief exercise is well preserved. The stated problem, to find the sides of a rectangle of a given form when the product of its length and its area is given, is easy to understand and easy to solve. The problem can be understood as finding the sides of a rectangular block of a given form and a given volume. (A similar problem in exercise # 1 was to find the sides of a rectangle of a given form and a given area.) In modern terms, the problem is solved, essentially, through the computation of a cube root.

IM 121613 # 8 (col. *iii*: 1-8)
1 $^{2}/_{3}$ uš sag.ki
2 uš *a-na* a.šag$_{4}$-*ia aš-ši-ma* / 1 28 53 20
uš *ù* sag.ki *mi-nu* /
3-4 za.e 40 $^{2}/_{3}$ uš *a-na* 1 *i-ši-ma* / 40 *ta-mar*
5 igi 40 duḫ *a-na* 1 28 53 20 / *i-ši-ma* 2 13 20 *ta-mar*
6 íb.sá.e / [2 1]3 20 *šu-li-ma* 20 *ta-mar*
7-8 20 *a-na* / [1 uš] *ù* 40 $^{2}/_{3}$ *i-ši-ma* 20 uš / [13 20 sag].ki *ta-mar*
ki-a-am ne-pé-šum

1 2/3 of the length, the front.
2 The length to my field I carried, then / 1 28 53 20.
The length and the front, what? /
3-4 You: 40, 2/3 of the length, to 1 carry, then / 40 you will see.
5 The reciprocal of 40 release, to 1 28 53 20 / carry, then 2 13 20 you will see.
6 The equalside of / [2 1]3 20 let come up, then 20 you will see.
7-8 20 to / [1, the length], and 40, 2/3, carry, then 20, the length, / [and 13 20, the front,] you will see.
Such is the procedure.

The stated problem can be expressed as follows, in quasi-modern symbolic notations:

a) $s = 2/3\ u$, b) $A \cdot u = u \cdot s \cdot u$ = '1 28 53 20'.

In the text of the solution procedure, it is implicitly assumed that u = 1 (00) r and s = 40 r, where r is the unknown length of a measuring reed. Therefore,

'1 28 53 20' = $A \cdot u = u \cdot s \cdot u$ = 4 (00) sq. $r \cdot$ 1 (00) r = 40 (00 00) sq. $r \cdot r$.

The computation is not explicitly mentioned. However, in lines 4-5 of the text, the given value '1 28 53 20 'is divided by the number '40', which of course stands for the mentioned coefficient 40 (00 00). The result of the computation is that

sq. $r \cdot r$ = '1 28 53 20' / '40' = '2 13 20'.

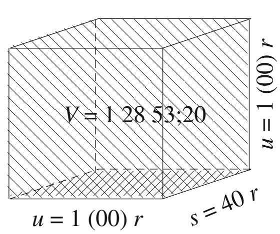
Then, in lines 5-6, the 'equalside' of '2 13 20 'is computed. It is found to be '20'. Clearly, this must mean that the unknown measuring reed has the length r = ;20 (rods). Working backwards from this result, one can draw the conclusion that

'2 13 20' = sq. ;20 · ;20 = ;02 13 20 (sq. rods · rod).

Consequently, the given number '1 28 53 20' must be interpreted as meaning

40 (00 00) · ;02 13 20 = ;40 · 2 13 20 = 1 28 53;20 (sq. rods · rod).

Since in this exercise the unknown reed r has the atypical value ;20 (rods), rather than the standard value ;30 (rods), it follows (in lines 7-8 of the text), that the sides of the rectangle of a given form have the unusual values u = 20 (rods) and s = 13;20 (rods).



40 (00) sq. $r \cdot$ 1 (00) r = 1 28 53;20
sq. $r \cdot r$ = ;02 13 20
r = ;20
u = 20, s = 13;20

Fig. 5.1.8. IM 121613 # 8: $s = 2/3\ u$, $h = u$, $h \cdot A$ = 1 28 53 20.

Remark: The cube root of 2 13 20 was probably well known. Otherwise, it can have been computed in the following way: Since the last place of the number is 20, it is obvious that 20 is a factor in the number. This factor can be removed through multiplication by 3, the reciprocal of 20. Indeed, 3 · 2 13 20 = 6 40. Then, since 6 40 = 20 · 20, it follows immediately that 2 13 20 = 20 · 20 · 20.

5.1.9 # 9. A Rectangle of a Given Form with a Given Sum of the Field and Half the Length

Exercise # 9 in IM 121613 is a simple variant of exercise # 2, where the sum of the area and the whole length of a rectangle was given. The text of the exercise is relatively well preserved. The geometric solution procedure is straightforward and follows the same path as the solution procedure in exercise # 2.

IM 121613 # 9 (col. *iii:* 9-24)
1 $^2/_3$ [uš sag.ki]
2 [*ba-m*]*a-at* uš *a-na* a.šag$_4$-*ia* / daḫ-*ma* 25
uš *ù* sa[g.k]i *m*[*i-n*]*u* /
3 za.e [1 uš 40 sag.k]i *lu-pu-ut*
4-5 40 [sag.ki] / *a-na* [1 uš] *i-ši-ma* 40 a.šag$_4$ *sa-ar-ra-am* / *ta-mar*
6 a.šag$_4$ *sa-ar-ra-am a-na* / 25 [*i*]-*ši-ma* 16 40 *ta-mar re-eš-ka li-ki-il* /
7-8 *tu-úr* 1 uš *ra-ba-an-ti-ka le-qé* / *ḫe-pé* 30 *ta-mar*
9 *tu-úr* 30 *ḫe-pé* 15 / *ta-mar*
10 15 *mé-eḫ-r*[*a*]-*am* [*i-di šu*]-*ta-ki-il-ma* / 3 45 *ta-mar*
11 3 45 *a*-[*na* 16 40] daḫ-*ma* / 21 25 *ta-mar*
12 ba.[sá-*e* 20 25] *šu-li-ma* / 35 *ta-mar*
13 15 [*ta-ki*]-*il-ta-ka* / zi.zi-*ma* 20 *ta-mar*
14 [igi 40 a].šag$_4$ *sa-ar-ri-im* / duḫ *a-na* 20 *i*-[*ši-ma* 30] *ta-mar* /
15-16 30 *a-na* 1 *ù* [40 *ma-ni-a-ti-ka*] / *i-ta*-[*aš-ši-ma*] 30 uš 20 sag.ki *ta-mar*

1 2/3 of the length, [the front.]
2 [The half-part] of the length to my field / I added on, then 25.
The length and the front, what? /
3 You: [1, the length, and 40, the front,] touch.
4-5 40, the front, / to [1, the length] carry, then 40, the false field, / you will see.
6 The false field to / 25 carry, then 16 40 you will see. Let it hold your head. /
7-8 Return. 1 uš, your enlargement, take, / break, 30 you will see.
9 Return. 30 break, 15 / you will see.
10 15, a copy, lay down, let them eat each other, then / 3 45 you will see.
11 3 45 to [16 40] add on, then / 20$^{!}$ 25 you will see.
12 The equalside [of 20 2]5 let come up, then / 35 you will see.
13 15, your [holder], / tear off, then 20 you will see.
14 [The reciprocal of 40], the false field, / release, to 20 carry, [then 30] you will see. /
15-16 30 to 1 and [40, your numbers], / always li[ft,] [then 30, the length and 20, the front, you will see.

5.1.10 # 10. A Rectangle of a Given Form with a Given Square of the Sum of the Length and the Front

A crucial word is missing in the question in this exercise. Fortunately, however, the missing word appears in a partial rephrasing of the problem in the first line of the solution procedure, so the nature of the problem is completely clear.

The problem is exceedingly simple. Essentially, the (area of the) square on the sum of the length and the front of a rectangle of a given form is given. The seemingly strange formulation of this problem, to find the sides of a rectangle of a given form if the sum of length and front times the sum of front and length is given, is easily explained if one remembers that the problem almost certainly was understood geometrically. See Fig. 5.1.9 below, where the sides of the square are constructed as *the length extended by the front* and *the front extended by the length*, respectively.

IM 121613 # 10 (col. *iii:* 25-34)
1 $^2/_3$ uš sag.ki
2 uš *a-na* sag.ki sag.ki / *a-na* uš daḫ-*ma* <*uš-ta-ki-il-ma*> 41 40
uš *ù* sag *mi-nu* /
3 za.e <1 uš 40 sag.ki *lu-pu-ut*>
4-5 uš *a-na* sag.ki sag.ki *a-na* / u[š] da[ḫ-*m*]*a šu-ta-ki-il-ma* 2 46 40 / *ta-mar*
6 igi 2 46 40 duḫ 21 3[6] / *ta-mar*
7 21 36 *a-na* 41 40 *i*$^{!}$-*ši-m*[*a*] / 15 *ta-mar*
8 ba.sá.e-*šu šu-li-ma* / 30 *ta-mar*
9 30 *a-na* 1 *ù* 40 *i-ši-ma* / <30 uš> 20 sag.ki *ta-mar*
30 uš 20 sag.ki

1 2/3 of the length, the front.
2 The length to the front, the front / to the length, I added on, then <I let eat each other, then> 41 40.
The length and the front, what? /
3 You: <1, the length, 40, the front, touch.>
4-5 The length to the front, the front to / the [length] add on, then let eat each other, then 2 46 40 / you will see.
6 The reciprocal of 2 46 40 release, 21 3[6] / you will see.
7 21 36 to 41 40 carry, then / 15 you will see.
8 Its equalside let come up, then / 30 you will see.
9 30 to 1 and 40 carry, then / <30, the length>, 20, the front, you will see.
30 the length, 20 the front.

The stated problem is, essentially, expressed in the following way, in lines 1-2 of the text:

a) The front is 2/3 of the length.
b) The length plus the front and the front plus the length make a rectangle with the area 41 40.

The solution procedure begins by (silently) assuming that the length is 1 (00) and the front is 40 (times a reed of unknown length). Then, in quasi-modern symbolic notations,

$(u + s) \cdot (s + u) = (1\ (00) + 40)\ r \cdot (40 + 1\ (00))\ r = 1\ 40\ r \cdot 1\ 40\ r = 2\ 46\ 40$ sq. r.

See lines 3-4. Simultaneously, it is given that

$(u + s) \cdot (s + u) = 41\ 40.$

Consequently, as in lines 5-8,

sq. $r = 41\ 40 / 2\ 46\ 40 = 1/4 = ;15,$ so that $r = 1/2 = ;30.$

It follows, in the usual way, that $u = 1\ (00)\ r = 30$ (rods), $s = 40\ r = 20$ (rods).

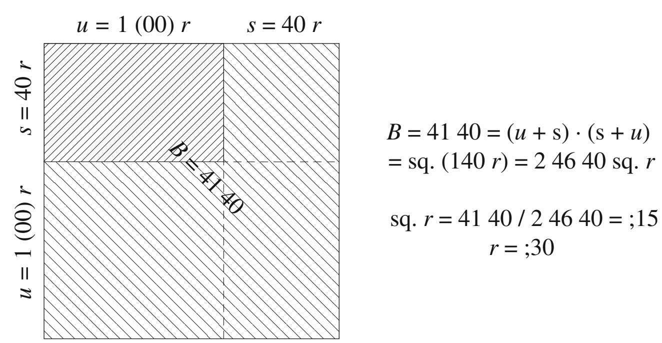


Fig. 5.1.9. IM 121613 # 10: $s = 2/3\ u$, $(u + s) \cdot (s + u) = 41\ 40$.

5.1.11 # 11. A Rectangle of a Given Form with a Given Excess of the Length over the Field

The problem in exercise # 11 of IM 121613 is a continuation of the series of related problems in exercises ## 1-3, 5-6, and 9, which are all rectangular-linear problems that can be reduced to (essentially) quadratic problems of the basic type B4a (see Friberg, *AT* (2007), 6). The problem in exercise # 11, in its turn, can be reduced to (essentially) a quadratic problem of the basic type B4c (*loc. cit.*). The geometric solution procedure for a problem of type B4c (Fig. 5.1.10 below) is related to, but different from, the geometric solution procedure for a problem of type B4a (Fig. 5.1.3 above).

IM 121613 # 11 (col. *iii*: 35-46)

1 $^2/_3$ uš sag.ki
2 a.šag$_4$ *i-na* uš *ás-su-ḫa-a*[*m-m*]*a* / 20
uš *ù* sag.ki *mi-nu*
3 za.e 1 uš / *a-na* 40 $^2/_3$ *i-ši-ma* 40 *ta-mar* /

4 40 *a-na* 20 *i-ši-ma* [13 2]0 *ta-mar* /
5-6 *ka-al* 1 uš *l*[*e-q*]*é* *ḫ*[*e-p*]*é* *šu-ta-ki-il-ma* / 15 *ta-mar*
7 13 20 *i-na* 15 zi.zi-*ma* / 1 40 *ta-mar*
8 ba.sá.e 1 40 *šu-li-ma* / 10 *ta-mar*
9 10 *i-na* 30 *ta-ki-il-t*[*a*!-*k*]*a*! / zi.zi-*ma* 20 *ta-mar*
10 igi 40 a.šag$_4$ *sa-ar-r*[*i-im*] / duḫ *a-na* 20 *i-ši-ma* 30 *ta-mar* /
11-12 30 *a-na* 1 uš *ù* 40 *ši-ni-pa-tim* / *i-ši-ma* 30 *uš* 20 sag.ki *ta-mar*

1 2/3 of the length, the front.
2 The field from the length I tore out, then / 20.
The length and the front, what?
3 You: 1, the length, / to 40, 2/3, carry, then 40 you will see. /
4 40 to 20 carry, then [13 2]0 you will see. /
5-6 The whole 1 uš take, break, let eat itself, then / 15 you will see.
7 13 20 from 15 tear off, then / 1 40 you will see.
8 The equalside of 1 40 let come up, then / 10 you will see.
9 10 from 30, your holder, / tear off, then 20 you will see.
10 The reciprocal of 40, the false field, / release, to 20 carry, then 30 you will see. /
11-12 30 to 1, the length, and 40, two-thirds, / carry, then 30, the length, 20, the front, you will see.

In this exercise, the stated problem is to find the sides of a rectangle when it is given that

a) The front is 2/3 of the length. b) The field subtracted from the length is equal to '20'.

This almost certainly means that

a) The front is 2/3 of the length. b) 1 (00) · the length minus the field is equal to 20 (00).

The geometric interpretation of this rectangular-linear problem is shown in Fig. 5.1.10 a. There u and s are the length and front of a rectangle, both unknown, and B = 20 (00) is the area of what remains of a larger rectangle with the sides u and q = 1 (00) after subtraction of the (unknown) area A of the initial rectangle.

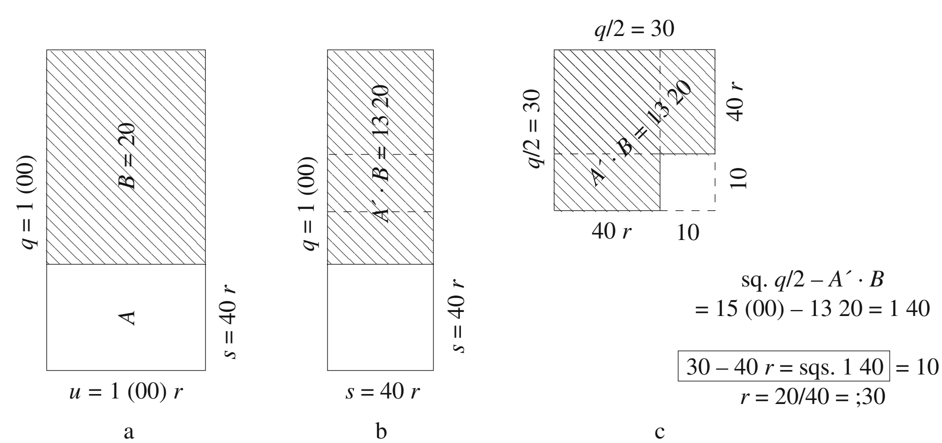


Fig. 5.1.10. IM 121613 # 11: $s = 2/3\ u$, $u - A = 20$.

In the present case, just as in the previously considered cases of rectangular-linear problems that can be reduced to (essentially) quadratic problems, the first step of the solution procedure is to equalize the figure, transforming the initial rectangle into a square. This is done in lines 3-4 of the text, and the result is as shown in Fig. 5.1.10 b.

Note that the large rectangle in Fig. 5.1.10 b can be divided into two equal halves, each half consisting of a square and a small rectangle. Obviously, the two small rectangles are equal, and therefore the gray rectangle with the area 13 20 can be transformed into a "square corner" as in Fig. 5.1.10 c, a square of side 1 (00) /2 = 30 from which has been subtracted a square of unknown side and unknown area. In lines 5-7 of the text, the whole step from Fig. 5.1.10 b to Fig. 5.1.10 c is described quite concisely as taking one half of the length 1 uš = 1 (00) of the large rectangle, multiplying this half side with itself to form a square of area sq. 30 = 15 (00), and then subtracting from this square the area 13 20 (of the square corner), obtaining in this way the information that the area of the remaining small square is 15 (00) – 13 20 = 1 40.

In lines 7-8 it is observed that the side of the remaining small square is sqs. 1 40 = 10. Therefore, the side of the small square in Fig. 5.1.10 c can be expressed in two ways, as 10 and as 30 – 40 r. It follows that 40 r = 30 – 10 = 20, so that r = 1/2 = ;30. Thus, u = 30 (rods) and s = 20 (rods), as usual.

In quasi-modern symbolic notations, the quadratic problem in Fig. 5.1.10 b can be expressed as the quadratic equation

$$p \cdot s - \text{sq. } s = 13\ 20, \qquad \text{with} \qquad p = 1\ (00).$$

The solution to this quadratic equation is obtained through algebraic completion of the square. First it is observed that $p/2$ = 1 (00)/2 = 30. Therefore,

$$\text{sq. } 30 - \text{sq. } (30 - s) = 1\ (00) \cdot s - \text{sq. } s = 13\ 20,$$

so that

$$\text{sq. } (30 - s) = \text{sq. } 30 - 13\ 20 = 15\ (00) - 13\ 20 = 1\ 40 = \text{sq. } 10.$$

Therefore,

$$30 - s = 10, \qquad \text{so that} \qquad s = 30 - 10 = 20, \qquad \text{and consequently} \qquad u = s\ /(2/3) = 30.$$

Clearly, this quasi-modern algebraic solution procedure is closely related to, but not entirely parallel with, the geometric solution procedure outlined in Fig. 5.1.10 a-c. Note that the geometric solution procedure appears to be somewhat easier to comprehend.

5.1.12 # 12. A Rectangle of a Given Form with a Given Excess of the Front over the Field

IM 121613 # 12 (col. *iv*: 1-13)

1 $^2/_3$ uš sag.ki
2 a.šag$_4$ *i-na* sag.ki zi.zi-*ma* / 7 30
uš *ù* sag.ki *mi-nu-um* /
3 za.e 1 uš 40 $^2/_3$ *lu-pu-ut*
4 40 *a-na* 1 uš *i-ši-ma* / 40 a.šag$_4$ *sa-<ar>-ra-am ta-mar*
5 40 a.šag$_4$ *sa-<ar>-ra-am* / ⸢*a-na* 7 30 *i-ši-ma* 5 *ta-mar*
6-7 *tu-úr* / [*x x x x x x x x x x x x x x xx x x*] *x-ma* / [*x x ta-mar*]
[*x x x x x x x x x x x x x x x x x*] *x* /
8 [*x x x x x x x x x x x x x x x x x x x x x x x x*] *x* /
9 *x x x* [ba.sá.e 1 40 *šu-li-ma* 10] *ta-mar* /
10-11 [10 ba.sá.e *i-na* šag$_4$ 20 *ḫu-ru-iṣ*]*-ma* / 10 *ta-mar*
[igi 40 a.šag$_4$ *sa-ar-ri-im* duḫ] /
12 *a-na* 1[0 x x *šu*]*-ta-ki-*[*il-ma*] 15! uš! /
13 15 *a-na*! [40! *i-ši-ma* 10]? sag.ki *ta-mar*

1 2/3 of the length, the front.
2 The field from the front I tore off, then / 7 30.
The length and the front, what? /
3 You: 1, the length, 40, 2/3, touch,
4 40 to 1, the length, carry, then / 40, the false field, you will see.
5 40, the false field, / [to] 7 [30 carry, then] 5 [you] will see.
6-7 Return, / [x x x x x x x x x x] then / [x x you will see].
[x x x x x x x x x x x x x x x x x] x /
8 [x x x x x x x x x x x x x x x x x x x x x x x x] x /
9 x x x [the equalside of 1 40 let come up, then 10] you will see. /
10-11 [10, the equalside, from 20 break off], then / 10 you will see.
[The reciprocal of 40, the false field, release], /
12 to(sic!) 10 [x x] let them eat each other, [then] 15, the length.
13 15 to [40 carry, the 10] the front, you will see.

According to the well preserved first few lines of exercise # 12, the problem in this exercise is closely related to the problem in the preceding exercise, # 11. Unfortunately, most of the text of the solution procedure is very poorly preserved. Nevertheless, it is likely that the geometric solution procedure proceeded, essentially, as in Fig. 5.1.11 a-c below. Note the “square corner” in Fig. 5.1.11 c, the gray figure with the area 5 (00), which is in the form of a square of side 20, from which has been subtracted a square of unknown side and unknown area.

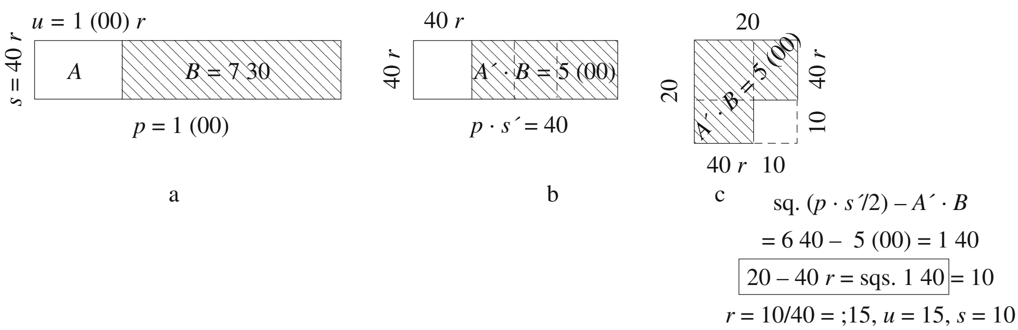


Fig. 5.1.11. IM 121613 # 12: *s* = 2/3 *u*, *s* – *A* = 7 30.

The answer, $u = 15$ and $s = 10$ is atypical. However, *it is easy to see why this problem must have an answer different from the standard r = ;30, u = 30, s = 20.* Indeed, when r = ;30, then $40\ r = 20$, so that in Fig. 5.1.11 b the area of the rectangle with sides $40\ r$ and 40 is precisely twice the area of the square with the side $40\ r$. That means that the rectangle is no longer divided into two squares and two small rectangles but only into two squares. The transition to Fig. 5.1.11 c is then unnecessary, and there is no longer any need for a completion of the square. What is needed to get a situation which leads to the use of a completion of the square in step c precisely as in exercise # 12 is, obviously, that *s = 40 r is less than 20.* This condition is satisfied, for instance, when, as here, r = ;15.

The problem in exercise # 12 is of the form

a) The front is 2/3 of the length. b) 1 (00) · the front, minus the field, is equal to 7 30.

In quasi-modern symbolic notations:

a) $s = 2/3\ u$, b) $1\ (00) \cdot s - u \cdot s = B = 7\ 30$.

After multiplication by ;40, the second equation becomes a quadratic equation for s:

$40 \cdot s - \text{sq. } s = A´ \cdot B = 5\ (00)$.

And so on, as in exercise # 12. Now look, instead, at the general case when the value of B is left unspecified. The quadratic equation

$40 \cdot s - \text{sq. } s = A´ \cdot B$

can then be reduced, by use of *algebraic* completion of the square, to

$\text{sq. } (20 - s) = \text{sq. } 20 - A´ \cdot B$ with the solution $20 - s = \text{sqs. } (\text{sq. } 20 - A´ \cdot B)$ if s is smaller than 20.

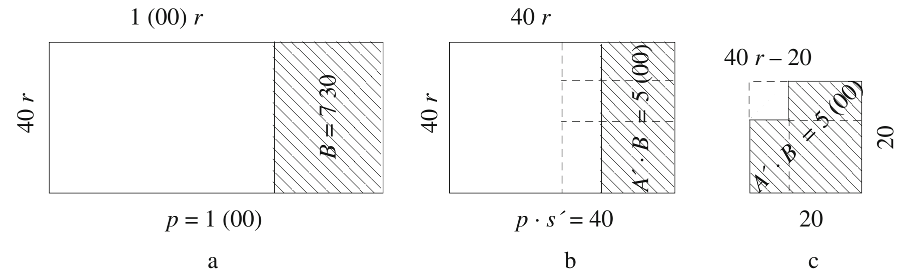
Alternatively, it can be reduced to

$\text{sq. } (s - 20) = \text{sq. } 20 - A´ \cdot B$ with the solution $s - 20 = \text{sqs. } (\text{sq. } 20 - A´ \cdot B)$ if s is greater than 20.

However, when $s = 20$, that is when $40 \cdot s - \text{sq. } s = 6\ 40$, then the quadratic equation can be reduced instead to the degenerate form

$\text{sq. } (s - 20) = 0$.

A geometric interpretation of the alternative solution to the problem in exercise # 12, the solution with s greater than 20, is shown in Fig. 5.1. 13 below. It is interesting to note that this alternative solution, which the author of IM 121613 probably was unaware of, has a geometric interpretation which is markedly different from the geometric interpretation in Fig. 5.1.11.



sq. $(p \cdot s´/2) - A´ \cdot B$
$= 6\ 40 - \ 5\ (00) = 1\ 40$
$40\ r - 20 =$ sqs. $1\ 40 = 10$
$r = 30/40 = ;45,\ u = 45,\ s = 30$

Fig. 5.1.12. An alternative solution to IM 121613 # 12: *s* = 2/3 *u*, *s* – *A* = 7 30.

5.1.13 # 13. A Rectangle of a Given Form with a Given Excess of the Length over the Field

Exercise # 13 is essentially a duplicate of exercise # 11. However, there are several minor differences in the texts of the two exercises, so they cannot both be direct copies of the same original. In particular, in lines 10-12 of exercise # 11, the length of the unknown reed is computed first as 20/40 = 30, and then *u* and *s* are computed as 30 times 1 and 40, while in lines 9-10 of exercise 13, *u* is computed directly as 20/40 = 30.

IM 121613 # 13 (col. *iv*: 14-23)

1 ²⁄₃ uš sag.ki
2 [a.šag₄ *i-na* uš] *a-*[*su-ḫa-am*] / 20
uš *ù* sag.ki [*mi-nu*]
[za.e] 1 uš [*ù*] 40 ²⁄₃ *lu-pu-ut* /
3 *tu-úr* 40 sag.ki *a-*[*na* 1 uš *i-ši-ma*] 40 a.šag₄ *sa-<ar>-ra-am ta-mar* /
4 40 a.šag₄ *sa-<ar>-ra-am a-na* 20 [*i-ši-ma*] 13 20 *ta-mar* /
5 1 uš *le-qé ḫe-pé* ⸢*šu-ta-ki-il-ma*⸣ 15 *ta-mar* /
6 *i-na* 15-*e* 13 20 zi.zi-*ma* 1 40 *ta-mar* /
7 ba.sá-*šu* 10 *ta-mar*
8 10 *i-na ta-ki-il-ta-ka* / zi.zi-*ma* 20 *ta-mar*
9 igi 40 a.šag₄ *sa-<ar>-ri-im* duḫ / *a-na* ⸢20 *i-ši-ma*⸣ 30 uš *ta-mar*
10 30 *a-na* 40 sag.ki / *i-ši-ma* 2[0 sag.ki *ta*]-*mar*
ki-a-am né-pé-šum

1 2/3 of the length, the front
2 [The field from the length] I [tore out, then] / 20
The length and the front, [what]?
[You]: 1, the length, [and] 40, 2/3, touch. /
3 Return. 40, the front, [to 1, the length, carry, then] 40, the false field, you will see. /
4 40, the false field, to 20 [carry, then] 13 20 you will see. /
5 1 uš (= 1 00 n.) take, break, let eat itself, then 15 you will see. /
6 From this 15, 13 20 tear off, then 1 40 you will see. /
7 Its equalside, 10, you will see.
8 10 from your holder / tear off, then 20 you will see.
9 The reciprocal of 40, the false field, release, / to 20 carry, then 30, the length, you will see.
10 30 to 40, the front, / carry, then 2[0, the front, you] will see.
Such is the procedure.

5.1.14 # 14. A Rectangle of a Given Form with a Given Excess of the Field over Half the Front

The problem in exercise # 14 of IM 121613 is a continuation of the series of related problems in exercises ## 1-3, 5-6, and 9, which are all rectangular-linear problems that can be reduced to (essentially) quadratic problems of the basic type B4a, and the problems in exercises ## 11-13, which can be reduced to (essentially) quadratic problems of the basic type B4c (see Friberg, *AT* (2007), 6). The problem in exercise # 14 is the only problem in IM 121613 that can be reduced to (essentially) a quadratic problem of the basic type B4b (*loc. cit.*). The geometric solution procedure for a problem of type B4b (Fig. 5.1.13 below) is related to, but different from, the geometric solution procedures for problems of types B4a and B4c (Figs. 5.1.3 and 5.1.11 above).

IM 121613 # 14 (col. *iv*: 24-33)
1 $^{2}/_{3}$ uš sag.ki
2 $^{1}/_{2}$ sag [*i*]-*na* a.šag$_4$ [zi.zi]-*ma* / 7 30
uš *ù* sag.ki *mi-nu*
za.e [1 uš *ù* 40 $^{2}/_{3}$ *lu-pu-ut*] /
3 40 *a-na* 1 *i-ši-ma* 40 a.šag$_4$ *sa-<ar>-ra-am ta-*[*mar*] /
4 *a-na* 7 30 *i-*[*ši-ma* 5 *t*]*a-mar*
tu-úr [1] uš *ù* 40 $^{2}/_{3}$ /
5 *mi-nam* [*x x x x x x x x x x*]-*ma* 10 *ta-mar* /
6 *t*[*u-úr x x x x x x x x x x*]
7 1 40 *a-na* 5 daḫ-*ma* / [6 40 *ta-mar*]
<ba.sa-*šu šu-li-ma* 20 *ta-mar*>
8 [20] *a-na* 10 *ta-ki-il-ta-ka* / [daḫ-*ma* 30 *ta-mar*]
[igi] 40 [a.šag$_4$ *sa-ar*]-*ri-im* duḫ /
9 *a-na* 30 *i-ši-ma* [4]5 uš *ta-mar* /
10 45 *a-na* 40 sag.ki *i-ši-ma* [30] sag.ki *ta-mar*

1 2/3 of the length, the front.
2 1/2 the front from the field [I tore off, then] / 7 30.
The length and the front, what?
You: [1, the length, and 40, 2/3, touch.] /
3 40 to 1 carry, then 40, the false field, you will [see]. /
4 To 7 30 [carry, then 5 you] will see.
Return. [1], the length, and 40, 2/3. /
5 What [*x x x x x x x x x x*] then 10 you will see. /
6 Return. [*x x x x x x x x x x*]
7 1 40 to 5 add on, then / [6 40 you will see].
<Its equalside let come up, then 20 you will see.>
8 [20] to 10, your holder, / [add on, then 30 you will see].
[The reciprocal] of 40, the false [field], release, /
9 to 30 carry, then [4]5, the length, you will see. /
10 45 to 40, the front, carry, then [30], the front, you will see.

In this exercise, the stated problem is to find the sides of a rectangle when it is given that

a) The front is 2/3 of the length. b) The field minus half the front is equal to '7 30'.

This almost certainly means that

a) The front is 2/3 of the length. b) The field minus 1 (00) · half the front is equal to 7 30.

The geometric interpretation of this rectangular-linear problem is shown in Fig. 5.1.13 a. There u and s are the length and the front of a rectangle, both unknown, and B = 7 30 is the area of what remains after subtraction of 30 times the unknown front from the unknown area of the initial rectangle.

The solution procedure begins in lines 2 right to 3 by setting, as usual, u = '1' and s = '40', possibly thought of as 1 (00) r and 40 r, respectively, where r stands for a silently understood measuring reed of unknown length. The corresponding product '1' · '40' = '40', possibly thought of as 40 (00) sq. r, is called the 'false field', referred to below as A'.

The next step, in line 4, is to multiply the 'true field 40', meaning the given area B of the diminished rectangle in Fig. 5.1.13 a, by the 'false field 40'. The result is written as '5'. The purpose of the operation was, probably, in the usual way, to equalize the figure through crosswise multiplication by 1 and 2/3, replacing the initial rectangle by a square with the side 40 r.

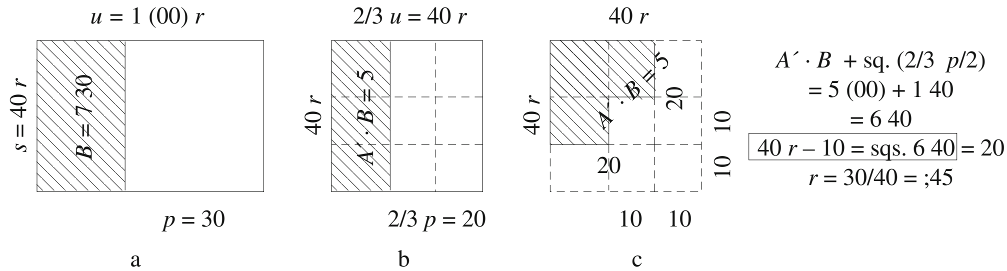


Fig. 5.1.13. IM 121613 # 14: $s = 2/3\ u$, $A - s/2 = 7\ 30$.

The third step of the solution procedure, in the badly preserved lines 5-6, was certainly to balance the diminished square, by cutting off half the extension, of length 20, of the small square in the upper left corner (see Fig. 5.1.13 b), placing the cut-off half of he extension as an extension in another direction, with the result that the once extended square is replaced by a balanced, twice extended square, as in Fig. 5.1.13 c. To this balanced, twice extended square is then added a square with the side 2/3 p/2 = 10, that is, a square with the area sq. 10 = 1 40. (See lines 6 right to 7 left). The result of this geometric completion of the square is an enlarged square with the area

$A' \cdot B$ + sq. (2/3 p/2) = 5 00 + 1 40 = 6 40.

In an unintentionally omitted part of the solution procedure, the side of this enlarged square is then found to be 20. Consequently, as in lines 7 right to 8 left of the solution procedure,

40 r – 10 = sqs. 6 40 = 20.

Hence, in lines 8 right to 9, r = 30/40 = '45' (probably meaning ;45 rods = 1 1/2 reed). Therefore, in lines 9-10 of the text,

u = 1 (00) r = 45 (rods), and s = 2/3 u = 30 (rods).

In quasi-modern symbolic notations, what is going on in IM 121613 # 14 can be described as follows: The stated problem can be interpreted as meaning that

a) The front s is 2/3 of the length u. b) The area $u \cdot s$ minus 1 (00) · the half the front s is equal to 7 30.

More concisely,

s = ;40 · u, $u \cdot s$ – 30 · s = 7 30.

This rectangular-linear problem for the two unknowns u and s is the modern, algebraic equivalent of the geometric interpretation in Fig. 5.1.13 a above. The geometric transformation of Fig. 5.1.13 a into Fig. 5.1.13 b corresponds to an algebraic transformation of the rectangular-linear problem above into the following quadratic-linear system of equations:

s = ;40 · u, sq. s – 20 · s = 5 (00).

Through the customary *algebraic* completion of the square the quadratic equation for s can be reduced to the equivalent quadratic equation

sq. (s – 10) = 5 (00) + 1 40 = 6 40 = sq. 20.

Consequently,

s – 10 = 20, so that, atypically, s = 30 (rods) and u = 20 /;40 = 45 (rods).

Note that in the case of exercise # 14, the use of data that would lead to the preferred Old Babylonian form of the solution to a problem for the sides of a rectangle with s = 2/3 u, namely u = 30, s = 20, was not an option, for the following reason: *If u = 30 and s = 20, then u · s – 30 s = 0, so that in Fig. 5.1.13 a above the gray rectangle is reduced to nothing.*

From a modern, algebraic point of view, the equation $u \cdot s$ – 30 s = 0 can immediately be reformulated as s · (u – 30) = 0, and has trivially the solution u = 30.

5.1.15 # 15. A Rectangle of a Given Form with a Given Field of a Certain Kind

Exercises ## 15-17 are the only exercises in IM 121613 where the term $u \cdot s$ in a rectangular equation has a coefficient that is different from 1. However, in spite of this apparent added complexity in the question, the solution procedures are of the same kind in these exercises as in all the other exercises of related types discussed above.

IM 121613, # 15 (col. *iv*: 34-46*)*

1 $^2/_3$ uš sag.ki
2 *i-na* uš $^1/_3$ zi.zi *a-na* [sa]g 15 daḫ / 11 40 a.šag$_4$
uš *ù* sag.ki *mi-nu-um*
3 za.⸢e⸣ [1 uš *ù* 40 $^2/_3$] / *lu-pu-ut i-na* 1 uš 20 zi.zi-*ma* 40 [*ta-mar*] /
4 40 *šu-ta-ki-il-ma* 26 40 *ta-mar*
5 26 40 / ⸢*a*⸣-*na* 11 40 a.šag$_4$ *i-ši-ma* 5 11 ⸢06⸣ [40 *ta-mar*] /
6 40 *a-na* 15 *i-ši-ma* 10 *ta-mar*
7 10 *ḫ*[*e-pé*] / 5 *ta-mar*
5 *šu-ta-ki-il-ma* 25 *t*[*a*]-*mar* /
8-9 25 *a-na* 5 11 06 40 daḫ-*ma* 5 36 06 40 / *ta-mar*
ba.sá-*šu* 18 20 *ta-mar*
10-11 *i-na* 18 20 / 5 *ta-ki-il-ta-ka* zi.zi-*ma* 13 20 / *ta-mar*
igi 26 40 duḫ-*ma* 2 15 *ta-mar* /
12 2 15 *a-na* 13 20 *i-ši-ma* 30 *ta-mar*
13 30 a.na uš *lu-pu-ut* / 30 *a-na* 40 sag *i-ši-ma* 20 sag.ki *ta-mar*

1 2/3 of the length, the front.
2 From the length 1/3 I tore off, to the [front] 15 I added on, / 41 40 was the field.
The length and the front, what?
3 You: [1, the length, and 40, 2/3], / touch. From 1, the length, 20 tear off, then 40 [you will see]. /
4 40 let eat itself, then 26 40 you will see.
5 26 40 / to 11 40, the field, carry, then 5 11 06 [40 you will see]. /
6 40 to 15 carry, then 10 you will see.
7 10 br[eak, then] / 5 you will see.
5 let eat itself, then 25 you will see. /
8-9 25 to 5 11 06 40 add on, then 5 36 06 40 / you will see.
Its equalside, 18 20, you will see.
10-11 From 18 20, / 5, your holder, tear off, then 13 20 / you will see.
The reciprocal of 26 40 release, then 2 15 you will see. /
12 2 15 to 13 20 carry, then 30 you will see.
13 30 as the length touch. / 30 to 40, the front, carry, then 20, the front, you will see.

In this exercise, the stated problem is to find the sides of a rectangle when it is given that

a) The front is 2/3 of the length.
b) If the length is diminished by a third, and if 15 is added to the front, the area is 41 40.

In quasi-modern symbolic notations this means that

a) $s = 2/3\ u,$ b) $(u - 1/3\ u) \cdot (s + 15) = 41\ 40.$

The second equation can, of course, immediately be simplified to

b´) $2/3\ u \cdot (s + 15) = 41\ 40.$

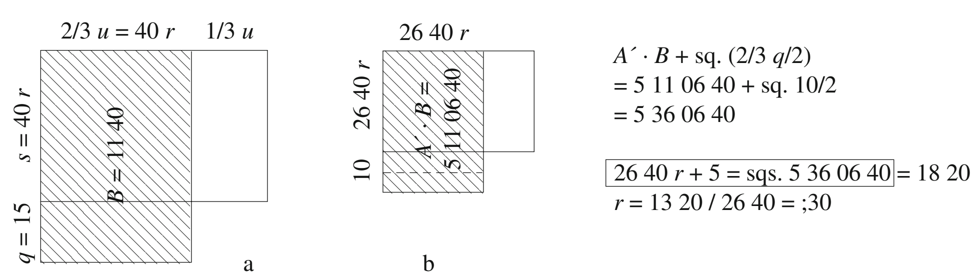


Fig. 5.1.14. IM 121613 # 15: $s = 2/3\ u$, $(u - 1/3\ u) \cdot (s + 15) = 41\ 40$.

The geometric interpretation of this rectangular-linear problem is shown in Fig. 5.1.14 a. The solution procedure begins in lines 2 right to 3 by setting, as usual, u = '1' and s = '40', possibly thought of as 1 (00) r and 40 r, respectively, where r stands for a measuring reed of unknown length. The corresponding expression for u – 1/3 u is then found to be '1' – '20' = '40', which can be thought of as 1 (00) r – 20 r = 40 r. Consequently, the initial rectangle with the unknown sides u and s has been replaced by a new rectangle with the sides 40 r and 40 r + 15, and the given area 11 40.

Actually, this rectangle is already an extended square, but this fact was overlooked, possibly deliberately, by the author of the exercise. Indeed, the solution procedure continues routinely in the same way as it would have in the case of an extended rectangle with the sides a r and b r + c, namely through equalization by way of a crosswise multiplication, where the side a r is multiplied by b and the side b r + c is multiplied by a. In the present case, a = b = 40, so that A' = a · b = 40 · 40 = 26 40. Consequently, the given area B = 11 40 is replaced by the new area $A' \cdot B$ = 11 40 · 26 40 = 5 11 06 40 (lines 4-5 of the text), while the extension 15 is replaced by 40 · 15 = 10 (line 6). The resulting transformation of the given extended "rectangle" into an extended square with the sides 26 40 r and 26 40 r + 10 is shown in Fig. 5.1.14 b.

The next step of the solution procedure is, in the usual way, a geometric completion of the square (lines 8-9). The side of the resulting square can be expressed in two ways, either as 26 40 r + 5, or as 18 20, both in relative sexagesimal numbers. In absolute numbers, this result can be interpreted as the equation

26;40 r + 5 = sqs. (5 11;06 40 + 25) = sqs. 5 36;06 40 = 18;20.

This is an equation for the unknown length r of the measuring reed. The solution is, of course, as indicated in lines 9-12, that

26;40 r = 18;20 – 5 = 13;20, so that r = 13;20 /26;40 = ;30.

Consequently, as in most similar cases, u = 30 and s = 20.

5.1.16 # 16. Another Rectangle of a Given Form with a Given Field of a Certain Kind

This exercise is, essentially, of the same type as the preceding exercise, only with the front being diminished by its fourth instead of, as in the preceding exercise, the length being diminished by its third. Since the question is perfectly preserved, it should not matter much that most of the text of the solution procedure is lost. Note that the expression 30 *ta-mar ki-ma* sag.ki *sa-ar-tim* '30 you will see as the false front' occurs only in this exercise. The expression 40 *ša ki-ma* sag.ki *ta-al-pu-tu* '40 that as the front you touched (= pointed to?)' occurs also in exercise # 4. Compare with the expression 30 a.na uš *lu-pu-ut* in exercise # 15, line 12, where the Sumerian a.na takes the place of the Akkadian *ki-ma*.

IM 121613, # 16 (col. *v*: 1-17*)*

1 $^2/_3$ uš sag.ki
2 *a-na* uš 10 daḫ *i-na* sag.ki *ra-bi-a-tim* / zi.zi 10 a.šag$_4$
uš *ù* sag.ki *mi-nu*
3 za.e 40 $^2/_3$*-tim* / *ša ki-ma* sag.ki *ta-al*!*-pu-tu*
4 *a-na* 15 *ra-bi-a-tim* / *i-ši-ma* 10 *ta-mar*
5 10 *i-na* šag$_4$ 40 *ša ki-ma* sag.ki / *ta-al*!*-pu-tu* zi.zi 30 *ta-mar*
6 *ki-ma* sag.ki / *sa-ar-tim*
7 30 *a-na* 1 uš ⸢*i-ši-ma* 30 a.šag$_4$ *sa-<ar>-ra-am*⸣ / *ta-mar*
8 30 a.šag$_4$ *sa-<ar>-ra-am* [*a-na* 10 *i-ši-ma* 5 *ta-mar*] / [*re-eš-ka li-ki-il*]
[*tu-úr x x x x*] /
9 30 [*x x x x x x x x x x x x x x*] /
10-11 *x* [*x x x x x x x x x x*] / 6 15 *t*[*a-mar*]
12 [6 15 *a-na* 5 *ša*] *re-eš-ka* / *ú-ki-lu* [daḫ*-ma* 5 06] 15 *ta-m*[*ar*]
13 ba.sa-*šu* / *šu-li-*[*ma* 17] 30 *ta-mar*
14 [*i-na* 17 30] 2! 30 *ta-ki-il-ta*!*-ka*! / [zi.zi-*ma* 15] *ta-mar*
[igi 30 a.šag$_4$ *sa-ar-ri-im* duḫ *a-na* 15] *i-ši-ma* /
15 [30 *ta-mar*] *x a-na* 15 *ša* [*x x x x x*]
16 *x* 1 *ù* 40 / [*x x x x x*] *x x x* 30 a.šag$_4$ [*x x x x x*]
17 *x x uš* / [*x x x x x x x x x i*]*t-ta-ši-ma* [*x x x x x x t*]*a-mar* /

1 2/3 of the length, the front.
2 To the length, 10 I added on, from the front, a quarter / I tore off, 10 the field.
The length and the front, what?
3 You: 40, of 2/3, / that as the front you touched,
4 to 15, the quarter, / carry, then 10 you will see.
5 10 out from 40 that as the front / you touched, tear off, 30 you will see,
6 as the / false front.
7 30 to 1 length, [carry, then 30, the false field], / you will see.
8 30, the false field, [to 10 carry, then 5 you will see]. / [Let it hold your head.]
[Return. x x x x x x x x x x x x x x x x x] /
9 30 [x x x x x x x x x x x x x x x x x x x x x] /
10-11 x [x x x x x x x x x x x x x x x x x] / 6 15 you [will see].
12 [6 15 to 5 that] your head held [add on, then 5 06] 15 you will see.
13 Its equalside / let come up, [then 17] 30 you will see.
14 [From 17 30] 2 30, your holder, / [tear off, then 15] you will see.
[The reciprocal of 30, the false field, release, to 15] carry, then
15 [30 you will see]. x to 15 that [x x x x]
16 x 1 and 40 / [x x x x x x x x x x] x x x 30, the field [x x x x x]
17 x x the length / [x x x x x x x x x x] always carry, then [x x x x x x] you will see. /

The badly preserved solution procedure in exercise # 18 appears to be of the same kind as the solution procedure in exercise # 15 , and corresponds to that of a basic quadratic equation of type B4a.

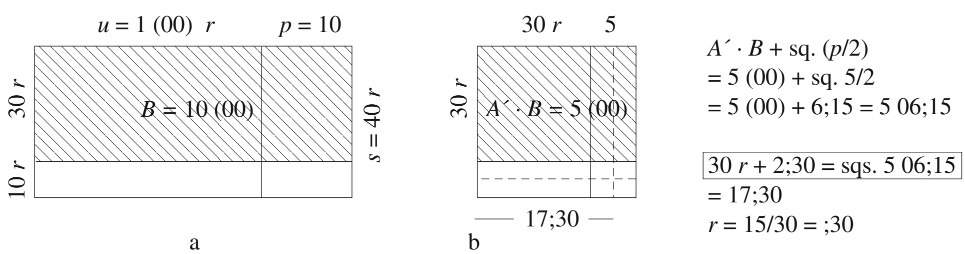


Fig. 5.1.15. IM 121613 # 16: $s = 2/3\,u$, $(u + 10) \cdot (s - s/4) = 10$.

5.1.17 # 17. A Rectangle of a Given Form with a Given Excess of a Certain Kind

The problem in exercise # 17 is of the same type as the problems in exercises ## 15 and 16. There is, however, an interesting error in the last few lines of the solution procedure.

IM 121613, # 17 (col. *v*: 18-30*)*
1 $^{2}/_{3}$ uš sa[g.k]i
[$^{2}/_{3}$ uš *i-na* uš zi.zi]-*ma* /
2 *ši-ta-at* [*x x x x x x x x x x*]
3 uš sag / *uš-ta-ki-il-ma* 2
a.šag$_4$ *a-na* sag uš *ap-lu-uk*! /
4 uš *ù* sag.ki *mi-nu*
za.e 1 uš 40 $^{2}/_{3}$ *šu-ku-un* /
5 *ši-ni-ip* uš *i-na* uš zi.zi-*ma* [*x*] 20 *ta-mar* /
6 20 *a-na* 40 *i-ši-ma* 13 20 *ta-mar*
7 13 20 *a-na* 2 / *i–si-ma* 26 40 *ta-mar*
8 40 *a-na* 4 sag *i-ši-ma* / 2 40 *ta-mar*
2 40 *ḫe-pé-ma* 1 20 *ta-mar* /
9-10 *šu-ta-ki-il a-na* 26 40 daḫ-*ma* 28 26 40 / *ta-mar*
ba.sá-*šu* 5 20 *ta-mar*
11 *a-na* 5 20 [1 20 daḫ-*ma*] / 6 40 *ta-mar*
ba.sá-*šu* 20 *ta-mar*
12 igi 40 a.šag$_4$ [*x*] / duḫ *a-na* 20 *i-ši-ma* 30 uš *ta-mar*
13 30 *a-na* [40] / $^{2}/_{3}$ <*ši-ni*>-*pa-tim i-ši-ma* 20 sag.ki *ta-mar*

1 2/3 of the length, the f[ront.]
[2/3 of the length from the length I tore off], then /
2 the remainder [x x x x x x x x x x].
3 The length and the front / I let eat each other, then 2.
The field for front and length I marked out. /
4 The length and the front, what?
You: 1, the length, 40, 2/3, set.
5 Two-thirds of the length from the length tear off, then 20 you will see.
6 20 to 40 carry, then 13 20 you will see.
7 13 20 to 2 carry, then 26 40 you will see.
8 40 to 4 <that was subtracted from> the front carry, then 2 40 you will see.
2 40 break, then 1 20 you will see.
9-10 Let eat itself, to 26 40 add on, then 28 26 40 you will see.
Its equalside, 5 20, you will see.
11 To 5 20, [1 20 add on, then] 6 40 you will see.
Its equalside, 20, you will see.
12 The reciprocal of 40, the field, release, to 20 carry, then 30, the length, you will see.
13 30 to [40], 2/3, <two>-thirds, carry, then 20, the front, you will see.

In this exercise, most of the question is destroyed. What remains is only the statement in lines 2 to 3 that a certain length and a certain front together define a rectangle of area 2 (00). Fortunately, the solution procedure is well preserved, so that it is easy to reconstruct the lost question, which can be formulated in the following way:

a) The front is 2/3 of the length.
b) If the length is diminished by 2/3 of itself and by 4, then the product of the length and the front is ‘2’.

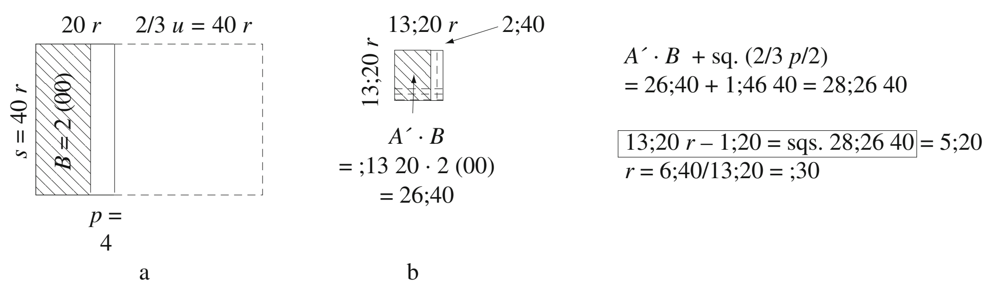


Fig. 5.1.16. IM 121613 # 17: $s = 2/3\, u$, $(u - 2/3\, u - 4) \cdot s = 2$.

In quasi-modern symbolic notations, the question can be rephrased as

a) $s = 2/3\, u$, b) $(u - 2/3\, u - 4) \cdot s = 2$ (00).

The solution procedure in exercise # 17 starts, as usual, by setting $u = 1$ (00 r) and $s = 40$ (r). The length diminished by its 2/3 will then be 20 r, as in line 5. Correspondingly, the rectangle with the sides 20 r – 4 and 40 r has the given area 2 (00). (See Fig. 5.1.16 a.)

Next, the rectangle with the sides 20 r and 40 r is equalized in the usual way through crosswise multiplication. The result is that the contracted rectangle with the given area 2 (00) is replaced by a new rectangle with the sides ;40 · (20 r – 4) = 13;20 r – 2;40 and ;20 · 40 r = 13;20 r, and with the given area $A' \cdot B$ = ;40 · ;20 · 2 (00) = ;13 20 · 2 (00) = 26;40. (See lines 6 to 8 left of the text, and Fig. 5.1.16 b.)

This rectangle can be seen as an extended square and can be treated in the usual way with a completion of the square. The side of the completed square is found to be

sqs. ($A' \cdot B$ + sq. 2;40/2) = sqs. (26;40 + 1;46 40) = sqs. 28;26 40 = 5;20.

(See lines 8 right to 10 left.)

Consequently, the square with the side 13;20 r resulting from the equalization is now seen to have the side 5;20 + 1;20 = 6;40. (See Fig. 5.1.16 b and lines 10 right to 11 left.) The next step of the solution procedure would, obviously, have been to compute the value of r in the following way:

13;20 r = 5;20 + 1;20 = 6;40, r = 6;40/13;20 = ;30.

And so on.

Instead, however, the author of the exercise, recalling that 6 40 is the square of 20, thoughtlessly divided 20 = sqs. 6 40 by 40, which gave him the expected result, namely r = 30. (Note the counting here with floating numbers, without regard to the absolute sizes of the numbers. Ironically, 6;40 is not even a square number!)

5.1.18 # 18. A Rectangle of a Given Form with a Given Excess of the Length and the Front over the Field

IM 121613 # 18 (col. *v*: 31-44)?

1 $^{2}/_{3}$ uš sag.ki
2 a.šag$_4$ *i-na ku-mur-ri* uš *ù* [sag.ki] / zi.zi-*ma* 40
uš *ù* sag.ki *mi-nu*
3 za.e [1 uš] / 40 $^{2}/_{3}$ *šu-ku-un šu-ta-ki-il*-[*ma*]
4 [40 a.šag$_4$] / *sa-ar-ra-am ta-mar*
5 40 [a.šag$_4$ *sa-*<*ar*>*-ra-am*] / *a-na* 40 a.šag$_4$-*lim i-ši-ma* 26 [40 *ta-mar*] /
6-7 1 uš *ù* 40 *ma-ni-a-ti-*[*ka le-qé ḫe-pé* 50] / *ta-mar*
8 50 *me-eḫ-ra-a*[*m i-di šu-ta-ki*]*-il-ma* / 41 40 *ta-mar*
9 *i-na* [41 40]-*e* 26 40 zi.zi-*ma* / 15 *ta-mar*
ba.sá.e *šu-li-ma* 30 *ta-mar* /
10-11 30 *i-na* 50 *ta-ki-il-ti-ka* zi.zi-*ma* / 20 *ta-mar*
12 igi 40 a.šag$_4$ *sa-ar-ri-im* duḫ / *a-na* 20 *ša ta-am-ru i-ši-ma*
13 30 *mín-da-nam* / *ta-mar*
14 30 *a-na* 1 *ù* 40 *ma-ni-a-ti-ka* / *i-ta-aš-ši-ma* 30 uš 20 sag.ki *ta-mar*

1 2/3 of the length, [the front].
2 The field from the heap of length and front / I tore off, then 40
The length and the front, what?
3 You: [1, the length], / 40, 2/3, set, let eat each other, [then]
4 [40], / the false [fiel]d, you will see.
5 40, [the false field], / to 40, the field, carry, then 26 [40 you will see]. /
6-7 1, the field, and 40, [your] numbers, [take and break, 50] / you will see.
8 50, a copy [draw, let eat each other], then / 41 40 you will see.
9 From [41 40], 26 40 tear off, then / 15 you will see.
Let the equalside come up, then 30 you will see.
10-11 30 from 50, your holder, tear off, then / 20 you will see.
12 The reciprocal of 40, the false field, release, / to 20 that you saw carry, then
13 30, the count / you will see.
14 30 to 1 and 40, your numbers, / always carry, then 30, the length, 20, the front, you will see.

Exercise # 18 in IM 121613 is a combination of exercises ## 11 and 12, in the same way as exercise # 5 is a combination of exercises ## 2 and 3. The question in this exercise is well preserved. It is of the form

a) The front is 2/3 of the length. b) The sum of the length and the front minus the area is equal to '40'.

What this means is almost certainly, in quasi-modern symbolic notations,

a) $s = 2/3\ u$, b) $1\ (00) \cdot (u + s) - A = 40\ (00)$.

Geometrically, $1\ (00) \cdot (u + s)$ can be interpreted as the sum of two partly overlapping rectangles, one with the sides 1 (00) and u, the other one with the sides 1 (00) and s. See Fig. 5.1.17 a, where, because of the overlap, the area A of the rectangle with the sides u and s is counted twice. Therefore, when A is subtracted from $1\ (00) \cdot (u + s)$, the result is simply the whole area of the twice extended rectangle in Fig. 5.1.17 a.

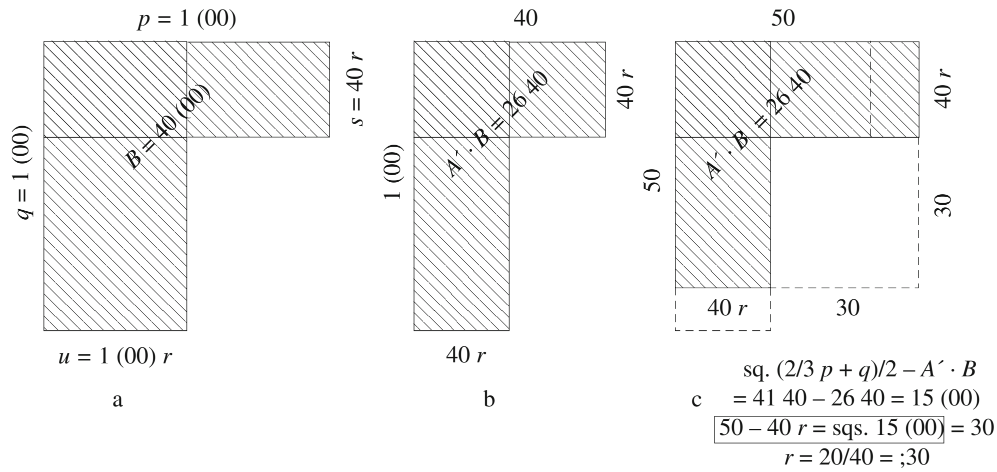


Fig. 5.1.17. IM 121613 # 18: $s = 2/3\ u$, $(u + s) - A = 40$.

The solution procedure in exercise # 18 starts, in the usual way, with the computation of the false area $A´ = 1 \cdot 2/3 = ;40$ and the computation of $A´ \cdot B = ;40 \cdot 40\ (00) = 26\ 40$ (lines 2-5). This step corresponds to an equalization, by which the diagram in Fig. 5.1.17 a is transformed into the diagram in Fig. 5.1.17 b. In the next step, this diagram is balanced (line 6). The result is the “square corner” in Fig. 5.1.17 c, a figure with the area 26 40, consisting of a square of side (1 (00) + 40)/2 = 50, from which has been subtracted a square of unknown side. However, since the area of the subtracted square is

$$\text{sq. } 50 - A´ \cdot B = 41\ 40 - 26\ 40 = 15\ (00) = \text{sq. } 30.$$

Therefore, the subtracted square has the side 30 (lines 7-9). Consequently, as in lines 10-12,

$$40\ r = 50 - 30 = 20, \quad \text{so that} \quad r = 20/40 = ;30.$$

And so on.

5.1.19 # 19. A Rectangle of a Given Form with a Given Shortened and Broadened Field

This is, in a sense, the culmination of the series of exercises where, for a rectangle of given form, the area plus or minus a multiple of the length or the front is given. In this exercise, *the length is shortened* at the same time as *the front is broadened*.

IM 121613, # 19 (col. *vi*: 1-19)

1 [$^2/_3$ uš sag.ki]
2 *i-na* uš 10 zi.zi *a-na* sag.ki / [5 daḫ 8] 20 a.šag$_4$
uš *ù* sag.ki *mi-nu* /
3 za.e 1 uš 40 <sag.ki> *lu-pu-ut*
4 10 *ša i-na* [šag$_4$] / uš zi.zi [*x x x*]
5 5 [*ša a-na*] sag.ki daḫ / *šu-ta-ki-il-ma* 50$^!$ *ta-mar*
6 5[0 *a-n*]*a* 8 20 / daḫ-*ma* 9 10 *ta-mar*
7 *tu-úr* 40 [$^2/_3$ *a-na*] / 1 uš *i-ši-ma* 40 a.šag$_4$ *sa-ar-ra-am* [*ta-mar*] /
8 40 *a-na* 9 10 *i-ši-ma* 6 06 40 *ta-mar* /
9-10 *tu-úr* 40 sag.ki *a-na* 10 *ša i-na* uš zi.zi / *i-ši-ma* 6 40 *ta-mar*
11 5 *ša a-na* sag.ki / daḫ *a-na* 1 uš *i-ši-ma* 5 *ta-mar*
12 *tu-úr* / [6] 40 ugu 5 *mi-nam* dirig 1 40 *x-ia* /
13 [1] 40 *ḫe-pe šu-ta-ki-il-ma* 41 40 *ta-mar* /
14-15 [41] 40 *a-na* 6 06 40 daḫ-*ma* 6 [07] 21 40 / *ta-mar*
ba.sá.e-*šu* 19 10 *ta-mar*
16-17 *a-na* [19 10] / [ba.sá]-*e* 50 *ta-ki-il-ta-ka* [daḫ-*ma*] / [20 *ta*]-*mar*
18 igi 40 a.šag$_4$ *sa-ar-ra-*[*am a-na*] / [20 *i*]-*ši-ma* 30 uš *ta-mar*
19 30 [*a-na* 40] / [*x x x x x x*] *i-ši-ma* 20 sag ki [*ta-mar*]

1 [2/3 of the length, the front.]
2 From the length 10 I tore off, to the front / [5 I added on, 8] 20 the field.
The length and the front, what? /
3 You: 1, the length, and 40 <the front> touch.
4 10 that [out] from / the length you tore, [x x and
5 5 [that to] the front you added on / let eat each other, then 50 you will see.
6 5[0 to] 8 20 / add on, then 9 10 you will see.
7 Return. 40, [the front, to] / 1, the length, carry, then 40, the false field, [you will see]. /
8 40 to 9 10 carry, then 6 06 40 you will see. /
9-10 Return. 40, the front, to 10 that you tore off from the length, / carry, then 6 40 you will see.
11 5 that to the front you added on, to 1, the length, carry, then 5 you will see.
12 Return. / [6] 40 over 5 what is it beyond? 1 40 x x.
13 [1] 40 break, let eat itself, then 41 40 you will see /
14-15 [4]1 40 to 6 06 40 add on, the 6 [07] 21 40 / you will see
Its equalside 19 10 you will see.
16-17 To [19 10], / [the equalside], 50, the holder, [add on, then] / [20 you] will see.
18 The reciprocal of 40, the false field, to / [20] carry, then 30, the length, you will see.
19 30 [to 40], / [x x x x x x] carry, then 20, the front, [you will see].

In this exercise, the stated problem is to find the sides of a rectangle when it is given that

a) The front is 2/3 of the length.
b) The length shortened by 10 combined with the front broadened by 5 is a rectangle with the area 8 20.

In quasi-modern symbolic notations, the problem can be expressed as follows:

a) $s = 2/3\ u$, b) $(u - 10) \cdot (s + 5) = 8\ 20$.

The geometric interpretation is the one shown in Fig. 5.1.18 a.

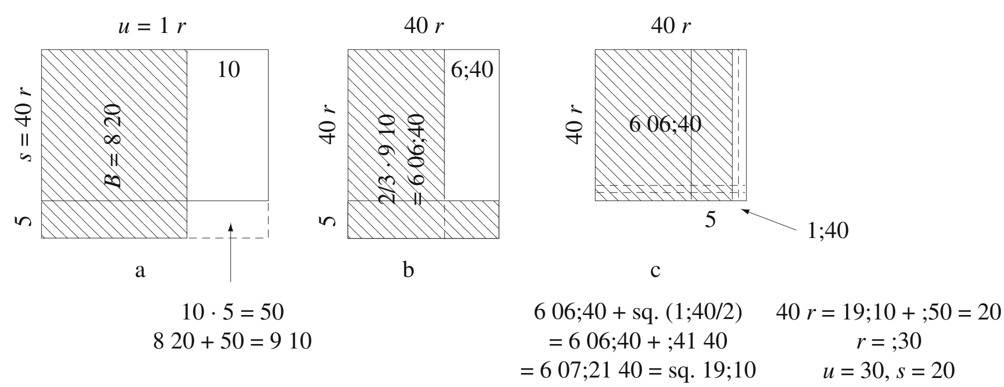


Fig. 5.1.18. IM 121613 # 19: $s = 2/3\ u$, $(u - 10) \cdot (s + 5) = 8\ 20$.

Since $(u - 10) \cdot (s + 5) = u \cdot s + 5\ u - 10\ s - 50$, which is obvious both geometrically and arithmetically, the problem can be simplified to

a) $s = 2/3\ u$, b) $A + 5 \cdot u - 10 \cdot s = 8\ 20 + 50 = 9\ 10$.

See lines 3-6 of the solution procedure.

Note that in this situation there is no reason to assume that $5 \cdot u$ and $10 \cdot s$ should be understood as $5\ (00) \cdot u$ and $10\ (00) \cdot s$!

Next, the diagram in Fig. 5.1.18 a is transformed into the diagram in Fig. 5.1.18 b through the usual kind of equalization. The result is a figure with the area ;40 · 9 10 = 6 06;40, consisting of a square with the side 40 r, extended by a rectangle with the sides 40 r and 5, and diminished by a rectangle with the sides 40 r and 6;40. (Lines 6 right to 11.)

A simpler way of characterizing the figure resulting from the equalization is to say that a square of side 40 *r*, diminished by a rectangle of the sides 40 *r* and 6;40 – 5 = 1;40, has the area 6 06; 40, as in Fig. 5.1.18 c. (See line 12 of the solution procedure.)

And so on, in the usual way.

5.1.20 Subscripts

Right below the text of the last exercise on IM 121613 there is the following subscript:

[19 x x x 2/3] uš sag.ki	[19 cases of "2/3] of the length (was) the front".

The suggested reconstruction implies that the subscript gives an account of how many exercises there are in the text, all concerning a rectangle of a given form.

A second subscript is written on the lower edge of the reverse. It simply gives the name of the one who wrote the tablet:

im.gíd.da *Da-du-ra-bi*	long tablet, Dadurabi

5.1.21 The Vocabulary of IM 121316

ašag (GÁN)	field, area-measure	# 1
a.šag$_4$	field, area Akk.: *eqlu*	*passim*
a.šag$_4$-*lam,* a.šag$_4$-*lim*	——	## 1, 3, 4, 18
a.šag$_4$-*ia*	——	## 8, 9
ba.sá.e-(*šu*), ba.sá-*šu*	equal(side) (= square-side, square root)	## 1, 3, 11, 14, 17, 19
ba.sa-*šu*	——	# 16
daḫ	add on	*passim*
gi.na	true, correct	## 1, 2, 3
íb.sá.e	equal(side) (= cube-side, cube root!)	# 8
igi (*n*) duḫ	release the opposite of (*n*) (compute rec. *n*)	*passim*
sag.ki	front (= short side)	*passim*
a-na šag$_4$	onto, into (šag$_4$ = the interior)	
i-na šag$_4$	out of	## 2, 3, 5, 16
(*a*) ugu (*b*) (*c*) dirig	*a* over *b* exceeds by *c* ($a - b = c$)	## 6, 8, 9, 19
uš	length (= long side)	*passim*
za.e	you	*passim* (but not in # 3)
zi.zi, zi	tear off (= subtract)	*passim*
ki-a-am ne-pé-šum	such is the procedure	## 1, 8, 13
re-eš-ka li-ki-il	let it hold your head (= keep this in mind)	## 2, [3, 6,] 7, 9
ša re-eš-ka ú-ki-lu	that held your head (= that you kept in mind)	## 2, [6,] 7
šu-ta-ki-il, uš-ta-ki-il	< *akālum* Št to make eat each other (= to multiply (two sides))	*passim*
ta-mar, ta-am-ru	< *amārum* to see (= to find a result)	*passim*
ab-ni	< *banûm* to build, to construct (a rectangle)	##1, 3, 4
šu-li	< *elûm* Š to come up, to rise (= to result (of a square root))	*passim*
e-ṣi-ip	< *eṣēpum* to double	## 3, 4
ḫe-pé	< *ḫepûm* to break (= to halve)	*passim*
ḫu-ru-uṣ, ta-aḫ-ru-uṣ	< *ḫarāṣum* to break off (= to subtract)	## 3, 4
ak-mur, ku-mur, ak-mu-ur-ú	< *kamārum* to heap, to pile up (= to add together)	## 5, 6
li-ki-il, ú-ki-lu	< *kullum* to hold	## 2, [3,] 7, 9
lu-pu-ut	< *lapātum* to touch (= to point at?)	## 2, 9, 12, 13, 14, 15, 19
ta-al-pu-tu	< *lapātum* to touch	## 4, 16
le-qé	< *leqûm* to take	## 5, 11, 13
i-na-di-nam	< *nadānum* to give	# 7

i-di	< *nadūm* to lay down (= to draw?)	# 9
ás-su-ḫa-am	< *nasāḫum* to tear out (= to subtract)	## 11, 13
i-ši, aš-ši	< *našūm* to carry (= to multiply (by a number))	*passim*
i-ta-(aš)-ši	< *našūm* Gt to carry, repeatedly (= to multiply (two lengths by a number))	## 1, 9, 16, 18
ap-lu-uk	< *palākum* to mark out	# 17
lu-uš-ku-un, šu-ku-un	< *šakānum* to put down, to set (= to make a note of?)	## 1, 7, 17, 18
tu-úr, a-tu-úr	< *tārum to return (= start again)*	*passim*
[*ba-ma-a*]*t*	*bātu* half, half-part	# 9
i-ta	*ita* adjacent to, next to	# 2
ka-al, ku-la	*kalūm* all (of)	## 6, 11
ki-ia	*kiyā* how much each?	# 3
ki-ma	*kīma* as, like	## 4, 16
ku-mur-ri	*kumurrûm* heap, piling up (= sum)	# 18
ma-ni-a-tim, ma-ni-a-ti-ka	*minītum, pl. maniātum* number	## 1, 4, [9,] 18
mé-eḫ-ra-am,	*meḫrum* copy, duplicate	## 6, 9
me-eḫ-ra-am	——	## 2, 18
mín-da-nam	*middatum, mindatum* count	# 18
mi-nu-(um), mi-nam	*mīnum* what?	*passim*
ra-ba-an-ti-ka	enlargement (?), < *rabūm* to be great (?)	# 9
ri-ba-at, ra-bi-a-tim	*rebūtum* (f.) fourth, quarter, 1/4	## 6, 16
sa-(ar)-ra-am, sa-ar-ri-(im)	*sarrum* false	##ṭ 1, 4, 5, 6, 9, 19
sa-ar-tim	*sarru* false (f.)	# 16
1/3*-ti, ša-lu-uš-ti*	*salšum*, f. *šaluštu* one-third, 1	# 6
ši-ni-pa-tim, ši-ni-ip	*šinipum* two-thirds	## 3, 11, 17
ši-ta	*šina* f. *šitta* two, 2	# 6
ši-ta-at	*šittum* remainder	# 4
ta-ki-il-ta	*takiltum* that which is held	## 2, 5, 11,13, 14, 15, 16,19

The only unusual mathematical terms in this text are *ma-ni-a-ti-ka* and *ra-ba-an-ti-ka* (or should it be *ra-ma-an-ti-ka*?), both of which appear here for the first time in a published mathematical cuneiform text. A somewhat surprising feature is that the term ba.sá.e (once with the spelling ba.sa.e) is used repeatedly for ‘equalside’, in the sense of square-side (square root), while the term íb.sá.e is used, once, for ‘equalside’, in the sense of cube-side (cube-root), instead of the other way round, as in many other mathematical cuneiform texts.

5.1.22 A Proposed Reconstruction of the Original Theme Text

It is clear that 18 of the 19 exercises on IM 121613 are basically concerned with “*rectangular-linear* problems” for a rectangle of a given form. On the other hand, there is no clear organization of the text as a whole. The reason is, of course, that IM 121613 is an Old Babylonian “mathematical recombination text” of a kind that is quite well known from several previously known examples. See, for instance, the list of 11 such texts in Friberg, *AT* (2007), 459. A mathematical recombination text is, by definition, “a somewhat disorganized collection of more or less closely related mathematical exercises from a number of sources” (*op. cit*., 35). See also Friberg, *UL* (2005), 93-94.

The reason for the designation “mathematical recombination text” is the following deliberation. It is likely that Old Babylonian mathematics had its origin in the activities of a relatively small number of particularly gifted but anonymous teachers of mathematics from several important Mesopotamian cities. Building upon old ideas inherited from the pre-Babylonian mathematics of the 3rd millennium BC, but also exploiting new insights, these talented teachers composed various “theme texts”. (Cf. the discussion in Sec. 9.4 below.) By that is meant cuneiform texts with long series of closely related exercises (with or without explicit solution procedures and answers), often *beginning with simple cases and then continuing in an orderly fashion with permutations of the data and of the unknowns, and with other variations and expansions of the theme.*

Theme texts were, of course, a great teaching tool, and they seem to have been used as rich sources of writing exercises or mathematical assignments, by teachers who wanted to give their students *related but non-identical* mathematical writing exercises to work with or mathematical problems to solve. That is probably why there exist in the various published collections of cuneiform mathematical texts so many small cuneiform tablets with single multiplication tables for different head numbers, so many different brief excerpts of metrological tables for measures of capacity, or weight, or area, or length, and that is why there exist so many small clay tablets with only one or a few related mathematical exercises.

Apparently, now and then an only moderately talented teacher who was not in possession of one of the original theme texts wanted instead to make his own collection of exercises to be used as a source of individual assignments. For that purpose, he collected as many small mathematical cuneiform texts as he could find, sorted them in some half-hearted way, and copied them together into one or several large mathematical recombination texts.

That IM 121613 is a mathematical recombination text of the mentioned kind is clear from the following reorganization of its table of contents:

1.1	# 1	$s = 2/3\ u$,	$A = 10\ (00)$	$u = 30, s = 20$
1.2	# 7	$s = 2/3\ u$,	$A +$ sq. $u +$ sq. $s = 31\ 40$	$u = 30, s = 20$
1.3	# 10	$s = 2/3\ u$,	$(u + s) \cdot (s + u) = 41\ 40$	$u = 30, s = 20$
1.4	# 4	$s = 2/3\ u$,	$A + (u - s) \cdot (2 \cdot 2/3\ u + s) = 20\ (00)$	$u = 30, s = 20$
2.1.1	# 2	$s = 2/3\ u$,	$A + u = 40\ (00)$	$u = 30, s = 20$
2.1.2	# 9	$s = 2/3\ u$,	$A + u/2 = 25\ (00)$	$u = 30, s = 20$
2.2.1	# 3	$s = 2/3\ u$,	$A + 2\ s = 50\ (00)$	$u = 30, s = 20$
2.3.1	# 5	$s = 2/3\ u$,	$A + u + s = 1\ 33\ 20$	$\boldsymbol{u = 45, s = 30}$
2.3.2	# 6	$s = 2/3\ u$,	$A + u + s + u/3 + s/4 + 2\ u + 2\ s + (u - s) = 3\ 05\ (00)$	$u = 30, s = 20$
3	# 14	$s = 2/3\ u$,	$A - s/2 = 7\ 30$	$\boldsymbol{u = 45, s = 30}$
4	# 19	$s = 2/3\ u$,	$A + 5\ u - 10\ s - 50 = (u - 10) \cdot (s + 5) = 8\ 20$	$u = 30, s = 20$
5.1	## 11, 13	$s = 2/3\ u$,	$u - A = 20\ (00)$	$u = 30, s = 20$
5.2	# 12	$s = 2/3\ u$,	$s - A = 7\ 30$	$\boldsymbol{u = 15, s = 10}$
5.3	# 18	$s = 2/3\ u$,	$(u + s) - A = 40\ (00)$	$u = 30, s = 20$
6.1	# 15	$s = 2/3\ u$,	$;40\ A + 10\ u = (u - 1/3\ u) \cdot (s + 15) = 41\ 40$	$u = 30, s = 20$
6.2	# 16	$s = 2/3\ u$,	$;45\ A + 7;30\ s = (u + 10) \cdot (s - s/4) = 10$	$u = 30, s = 20$
6.3	# 17	$s = 2/3\ u$,	$;20\ A - 4\ s = (u - 2/3\ u - 4) \cdot s = 2$	$u = 30, s = 20$
7	# 8	$s = 2/3\ u$,	$h = u, h \cdot A = 1\ 28\ 53\ 20$	$\boldsymbol{u = 20, s = 13\ 20}$

This reorganized table of contents for IM 121613 starts with the problems in the four exercises of group 1 (1.1-1.4). With $u = 1\ (00)\ r$ and $s = 40\ r$ in the usual way, all the problems in this group of exercises can be reduced to *homogeneous quadratic equations* of the simple form sq. $r = B$. In geometric terms, all the problems in this group can be reduced to the simple problem of finding the side of a square when the area of the square is given. Consequently, the solution procedures contain nothing more difficult than the computations of square-sides (square roots). It is clear that this is a deliberately simple initial group of exercises in a theme text with *rectangular-linear* problems for rectangles of a given form.

The problems in the five exercises of group 2 (2.1 1-2.3.2) are all of the kind where for a rectangle of a given form the area plus the length (or a multiple of the length), or the area plus the front (or a multiple of the front), or the area plus multiples of both the length and the front, is given. With $u = 1\ (00)\ r$ and $s = 40\ r$ in the usual way, they can all be reduced to quadratic equations of the form sq. $r + a \cdot r = B$. These are *basic quadratic equations of type B4a* (Friberg, *AT* (2007), 6.) In geometric terms, all the problems in this group can be reduced to problems where the area of an “extended square” is given. The geometric solution procedures typically consist of an “equalization”, followed by a “balancing”, a “completion of the square”, and the computation of a square-side.

The problem in the single exercise of group 3 is of the form where for a rectangle of a given form the area minus a multiple of the front is given. Setting u = 1 (00) r and s = 40 r, the problem can be reduced to a quadratic equation of the form sq. $r - a \cdot r = C$. This is *a basic quadratic equation of type B4b* (Friberg, *loc. sit.*). In geometric terms, the area of a “contracted square” is given. The solution procedure in this case, too, consists of an “equalization”, followed by a “balancing”, a “completion of the square”, of a second kind, and the computation of a square-side.

The problem in the single exercise of group 4 is of the form where for a rectangle of a given form the area plus a multiple of the length minus a multiple of the front is given. In geometric terms, this kind of problem can be reduced to the case when the area is given of a square which is extended in one direction and contracted in another direction. As the given example shows, such problems can be reduced to problems for either an extended square or a contracted square, as in the exercises belonging to groups 2 and 3.

The problems in the three exercises of group 5 (5.1-5.3) are of the form where for a rectangle of a given form the area is subtracted from some linear combination of the length and the front. With u = 1 (00) r and s = 40 r in the usual way, they can all be reduced to quadratic equations of the form $a \cdot r$ – sq. $r = C$. These are *quadratic equations of the basic type B4c* (Friberg, *loc. sit.*). In geometric terms, all the problems in this group can be reduced to problems where the area of a new kind of “extended square” is given.

Finally, the problems in the three exercises of group 6 (6.1-6.3) are of the same forms as the problems in the exercises of groups 2-3, with the only difference that the some multiple $c\ A$ of the area of the rectangle appears in the problem (after a suitable simplification of the question) instead of simply the area A itself.

The problem in the single exercise of group 7 is of a kind of its own, concerning a three-dimensional object (a rectangular box with a rectangle of a given form as a base).

Ordered in the way suggested above, 18 of the exercises in IM 121613 clearly make up an excellently organized theme text with examples of all conceivable simple variations of the theme “rectangular-linear problems for a rectangle of a given form”. The 19th exercise, with the single problem of the odd group 7, has the appearance of a bridge from the mentioned theme text to a related theme text with problems for three-dimensional objects.

It is tempting to conjecture that IM 121613 is a descendant of a single theme text of the mentioned kind, in the sense that all the individual exercises collected together and copied in IM 121613 were once single problem texts copied directly from that original theme text (or copies of such single problem texts). There are, however, some difficulties with this hypothesis.

One difficulty is that the answers to most but not all of the 18 first problems (in the reorganized table of contents above) is the standard solution u = 30, s = 20. However, in two cases (## 5 and 14) the solution is instead u = 45, s = 30, and in one case (# 15) it is u = 15, s = 10. This seems to indicate that these three exercises have another origin than the remaining 15 exercises. However, as was shown above, in two of the mentioned cases (## 12 and 14), a problem of the considered type would work poorly in the case of the standard solution, which means that the switch to a non-standard solution was imperative. As for the remaining case of a non-standard solution (# 5), a possible explanation is that the author of the theme text included a case with a non-standard solution in order to discourage cheating among students who would otherwise know beforehand the answer to all the assignments. Note that such cheating actually occurred in the case of exercise # 17. It is, of course, also possible that the original theme text contained short lists of identically formulated problems differing from each other only by different choices of the data. See, for instance, the text BM 80209 (Friberg, *AT* (2007), 29; Sec. 8.8 below), where in problems 4.1-4.4 the given area of a circle takes the four values 8 20, 2 13 20, 3 28 20, and 5. (The corresponding values for the circumference can then be shown to be 10, 40, 50, and 1 (00), respectively.)

Another difficulty with the mentioned hypothesis is that there are some obvious differences in the mathematical terminology used in the individual texts. Here is a list of the most obvious differences:

1. The most common form of the question is

uš *ù* sag.ki *mi-nu* — 'the length and the front, what?' — *passim*

This question is replaced in two cases by the mimated variant

uš *ù* sag.ki *mi-nu-um* — —— — ## 12, 15

and in one case by

uš *ù* sag.ki *ki-ia* — 'the length and the front, how much each?' — # 3

2. At the beginning of the solution procedure, the most commonly occurring phrase is

1 uš (*ù*) 40 ($^2/_3$) *lu-pu-ut* — 1, the length, and 40, 2/3, touch — ## 2, 9, 12, 13, 15, 19

This phrase is replaced in two instances by the following more elaborate variant

40 $^2/_3$-*tim ša ki-ma* sag.ki *ta-al-pu-tu* — 40, of 2/3, that as the front you touched — ## 4, 16

In four cases, another verb is used instead:

1 uš 40 $^2/_3$ *šu-ku-un* — 1, the length, and 40, 2/3, set. — ## 1, 7, 17, 18

In one case only, the word *lu-pu-ut* 'touch' appears in the answer, namely in the phrase

30 a.na uš *lu-pu-ut* — 30 as the length touch — # 15

The following phrases have no verb taking the place of *lapātu* or *šakānu*:

1 uš *a-na* 40 $^2/_3$ *i-ši* — 1, the length, to 40, 2/3, carry — # 11

40 sag.ki *a-na* 1 uš *i-ši* — 40, the front, to 1, the length, carry — # 6

40 $^2/_3$ uš *a-na* 1 *i-ši* — 40, 2/3 of the length, to 1 carry — # 8

Note that in the following two phrases 1 and 40 are interpreted as length measures rather than dimensionless numbers, which is clear because the verb for the multiplication used is the one normally used for geometric combination of the sides of a rectangle:

40 sag.ki *ù* 1 uš *šu-ta-ki-il* — 40, the front, and 1, the length, let eat each other — # 5

1 uš 40 $^2/_3$ *šu-ku-un šu-ta-ki-il* — 1, the length, 40, 2/3, set, let eat each other — # 18

The following phrase appears in four instances, but never at the beginnings of the solution procedures:

1 (uš) *ù* 40 *ma-ni-a-ti(m)* — 1 (the length) and 40, the measures — ## 1, 4, [9], 18

3. The following quite explicit phrase appears in only one of the exercises:

30 *ta-mar ki-ma* sag.ki *sa-ar-tim* — 30 you will see as the false front — # 16

4. The following concluding phrase occurs in only three of the exercises:

ki-a-am ne-pé-šum — such is the procedure — ## 1, 8, 13

5. The following phrase occurs in only one of the exercises (here for the first time in any published mathematical cuneiform text), in the question:

a.šag$_4$ *a-na* sag uš *ap-lu-úk* — the field for front and length I marked out — # 17

These more or less conspicuous differences in the mathematical terminology of the individual exercises in IM 121613 can, possibly, be explained as follows. Suppose that the original theme text was in the form of a *list of assignments* (stated problems without solution procedures), intended to be handed out to a number of students to solve. Then each student would have formulated his solution procedure in his own way, in his own words, recording it as a single problem text. Nineteen such single problem texts, with their varying terminologies, can then have been collected and copied together in the recombination text IM 121613.

Interestingly, IM 43993, an unpublished and unprovenanced single problem text, apparently from Eshnunna (Sec. 5.2 below), contains a stated problem similar to the one in IM 121613 # 5, with an explicit solution procedure. Unlike the non-standard solution $u = 45$, $s = 30$ in IM 121613 # 5, the solution in IM 43993 is of the standard form $u = 30$, $s = 20$. The predominantly Akkadian mathematical terminology used in IM 43993 differs markedly from the mathematical terminology used in IM 121613. In spite of this difference in terminology, IM 43993 is the only other known Babylonian mathematical text, besides IM 121613, in which the term *ma-ni-a-ti(-ka)* appears.

Another published Old Babylonian mathematical text related in some way to the theme of *rectangular-linear* problems for a rectangle of a given form is the assignment in col. *iv*, 17-21 (exercise # 7) of the mathematical recombination text AO 8862 from Larsa (Høyrup *LWS* (2002), 338; Neugebauer, *MKT I* (1935), 117). The question in that brief text is of the following form:

If, from a distance of 30 ropes [x x x x x].
I heaped my bricks, men, and days, then 2 20.
Two-thirds of the men were my days.
Single out for me my bricks, men, and days.

There is no solution procedure given, apart from a rather chaotic and obscure tabular array of numbers. The solution, which is explicitly given, is that there were '1 30' bricks, 30 men, and 20 days.

A work norm for carrying bricks is given in a preceding assignment on AO 8862, in col. *iv*. 4-6, where it is stated that

For thirty ropes one man brought me 9 sixties of bricks.

In other words, the work norm for carrying bricks over a distance of thirty ropes was *9 sixties of bricks per man-day* (per man and day). Therefore the stated problem can be rephrased as follows, in quasi-modern symbolic notations:

a) The days d are 2/3 of the men m. b) $9\ (00) \cdot m \cdot d + m + d =$ '2 20'.

However, when $m = 30$, $d = 20$, as in the stated solution to the problem, then

$$9\ (00) \cdot m \cdot d + m + d = 9\ (00) \cdot 10\ (00) + 30 + 20 = 1\ 30\ (00\ 00) + 50 = 1\ 30\ 00\ 50.$$

This is not the number given in the question. The author of the assignment seems to have counted, incorrectly, as follows:

$$9 \cdot 30 \cdot 20 + 1\ (00) \cdot 30 + 1\ (00) \cdot 20 = 1\ 30\ (00) + 50\ (00) = 2\ 20\ (00).$$

Or, with total disregard for the correct absolute values of the numbers in relative sexagesimal place value notation without zeros:

$$9 \cdot 30 \cdot 20 + 30 + 20 = 9 \cdot 10 + 50 = 1\ 30 + 50 = 2\ 20.$$

In the reorganized table of contents for the recombination text IM 121613 (see above) a problem of this kind would fit in nicely as problem 6.4, a variant of problem 2.3.1.

5.1.23 Solution Procedures in Terms of a Square Band (a Ring of Rectangles)

Solutions to quadratic equations of the three basic types B4a, B4b, and B4c were explained above, in the course of the discussions of exercises ## 2, 14, and 12, respectively, by use of of three different kinds of "completions of the square". See Figs. 5.1.3 c, 5.1.13 c, and 5.1.11 c. The three variants of a completion of the square all take their departure from geometric expressions of the three basic quadratic equations in terms of "square corners". (Incidentally, square corners appear also, but with no explicit reference to quadratic equations, in Euclid's *Elements* II:4-8. See Friberg, *AT* (2007), Secs. 1.3-1.5.)

No published mathematical cuneiform texts contain any drawings of what can be interpreted as square corners. Actually, it is not known precisely which kind of geometric diagrams Old Babylonian teachers of mathematics may have been using when they, presumably, in front of their students demonstrated how to solve quadratic equations. However, it is not unlikely that what they used for this purpose was "square bands" and "square corners". (Cf. Friberg *CDLJ 2009:3*, Fig. 12).

Demonstrations of solutions to basic quadratic equations by use of square bands have the advantage that they are simpler and visually more obvious than demonstrations by use of square corners (Greek: *gnomons*) alone. Indeed, consider *two concentric squares with parallel side*s, the larger square with the side m and the smaller square with the side n, as in Fig. 5.1.19 left, below. The difference between the two squares can be called "a square band with the outer side m and the inner side n". It is worth noting that problems for the 'outer' and 'inner' sides of square bands appear, without explicit solutions, in the Old Babylonian mathematical catalog text *TMS V* (Friberg, *AT* (2007), 32; Sec. 8.9 below). Problems of the same type, but with explicit solutions, appear in the unpublished Old Babylonian mathematical recombination text IM 121565 (Sec. 6.2 below).

A crucial observation is that a square band can be divided into a "ring" of four equal figures in two different ways, either a ring of four equal rectangles as in Fig. 5.1.19 left (below), or a ring of four equal square corners as in Fig. 5.1.19 right. Consequently, the area of each one of the four rectangles is equal to the area of each one of the four square corners. Admittedly, no published mathematical cuneiform text contains any drawing of what can be interpreted as a ring of four equal rectangles or a ring of four equal square corners. However, one text is known on which is depicted a "triangular band" divided into a ring of four equal trapezoids (Friberg, *AT* (2007), 80).

Five numerical parameters are connected with every given square band divided into a chain of four equal rectangles. They are the outer and inner sides m and n of the square band, with m greater than n, the sides a and b of each rectangle, with a greater than b, and the area C of each rectangle. See Fig. 5.1.19 left. There are two obvious connections between the sides m and n of the square band on one hand and the sides a and b of each rectangle on the other hand, namely the two linear equations

$a - b = n$ and $a + b = m$.

Moreover, if n and C are given, then Fig. 5.1.19 left can be interpreted as the geometric representation, simultaneously, of the following two basic quadratic equations, the first one for the unknown b, and the second one for the unknown a:

B4a: $sq.\ b + n \cdot b = C$. B4b: $sq.\ a - n \cdot a = C$.

The corresponding solutions are, according to the geometric representation in Fig. 5.1.19 right:

B4a: $b = \text{sqs. (sq. } n/2 + C) - n/2$.
B4b: $a = \text{sqs. (sq. } n/2 + C) + n/2$.

In a similar way, if m and C are given, then Fig. 5.1.19 left can be interpreted as the geometric representation, simultaneously, of the following two basic quadratic equations, the first one for the unknown b, and the second one for the unknown a:

B4c: $m \cdot b - sq.\ b = C$ (b smaller than $m/2$).
B4c: $m \cdot a - sq.\ a = C$ (a greater than $m/2$).

The corresponding solutions are, according to the geometric representation in Fig. 5.1.19 right:

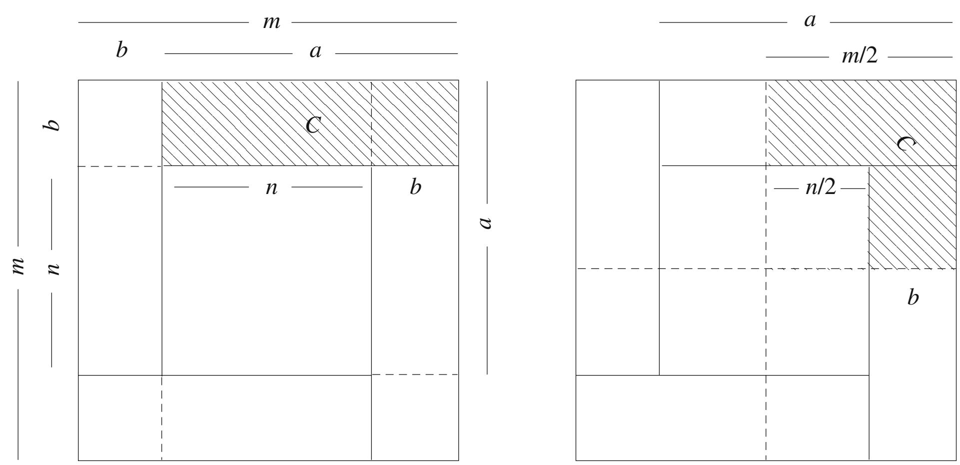
B4c: $b = m/2 - \text{sqs. (sq. } m/2 - C)$
B4c: $a = m/2 + \text{sqs. (sq. } m/2 - C)$.

Thus, *the single couple of diagrams in Fig. 5.1.19 is all that is needed to express geometrically both the three types of basic quadratic equations and the corresponding solutions!*

In addition, *the same couple of diagrams is all that is needed to express geometrically the two basic rectangular-linear systems of equations of types B1a and B1b and their solutions.* (See Friberg, *AT* (2007), 6) It is not altogether unlikely that it was the discovery of this remarkable fact that triggered the pronounced preoccupation in Old Babylonian mathematics with all kinds of quadratic equations and rectangular-linear systems of equations.

Note the following interesting difference between the modern algebraic and the Babylonian geometric way of looking at quadratic equations and their solutions. As is well known, from the modern algebraic point of view a quadratic equation always has a pair of solutions (in exceptional cases coinciding). From the geometric point of view, on the other hand, a quadratic equation always has only one solution. The reason for the different points of view is, for equations of type B4a or B4b, that geometric solutions are interpreted as sides of a geometric figure, and therefore cannot be negative. In the case of equations of type B4c, the form of the solution will depend on what has been assumed, beforehand, about the size of the solution.

A rectangular-linear systems of equations, for obvious reasons, always has a pair of solutions.



B4a: sq. $b + n \cdot b = C$ solution: $b =$ sqs. (sq. $n/2 + C) - n/2$

B4b: sq. $a - n \cdot a = C$ solution: $a =$ sqs. (sq. $n/2 + C) + n/2$

B1b: $a \cdot b = C$
$a - b = n$

B4c: $m \cdot b -$ sq. $b = C$
$b < m/2$
solution: $b = m/2 -$ sqs. (sq. $m/2 - C)$

B4c: $m \cdot a -$ sq. $a = C$
$a > m/2$
solution: $a = m/2 +$ sqs. (sq. $m/2 - C)$

B1a: $a \cdot b = C$
$a + b = m$

Fig. 5.1.19. A pair of diagrams illustrating four types of quadratic equations and their solutions, and simultaneously two types of rectangular-linear systems of equations and their solutions.

An article by Aldo Bonet with the title “The diagram of clay: the dawn of scientific thought (in Italian)”, published on-line at lanostra-matematica.org/2014/05/il-diagramma-di-argilla-lalba-del.html, is devoted to a discussion of square bands of the type displayed in Fig. 5.1.19 b above and their conjectured role for the development of algebraic thinking. An interesting observation in Fig. 17 of Bonet’s article is that the votive reliefs on a well known stone slab in the Louvre showing king Ur-Nanshe of Lagash (Early Dynastic III, 2500 BC) presiding over the ceremonies at the foundation and inauguration of a shrine (see below) *seems to be divided into four panels in a way similar to the division of a square band into four rectangles*. See the reproduction below of a photo of the slab.

Fig. 5.1.19a. The plaque mentioned by Aldo Bonet. Photo: Philipp Bernard (RNM).

5.1.24 IM 121613. Hand Copies of the Tablet

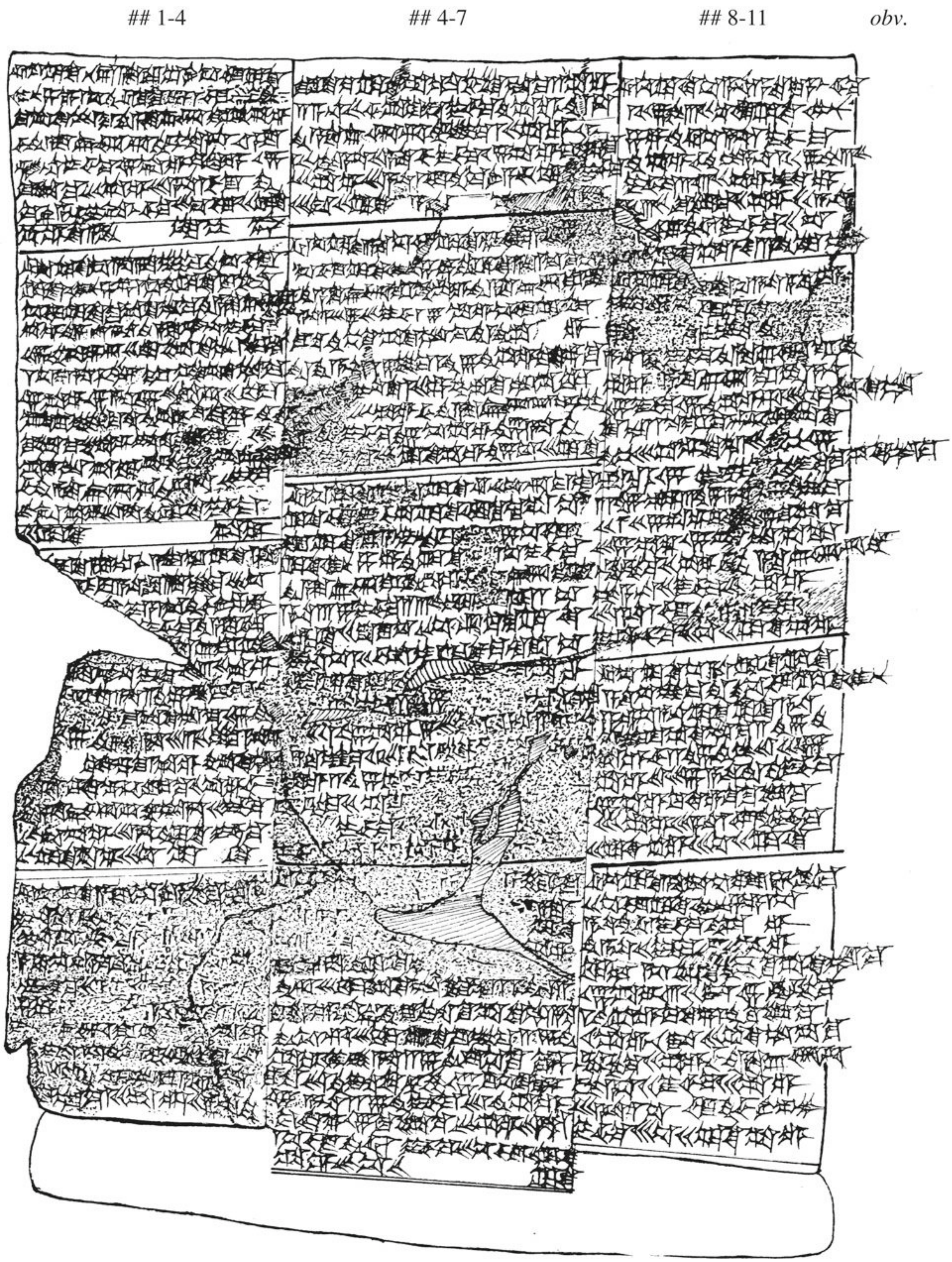


Fig. 5.1.20. IM 121613, *obv.* Hand copy.

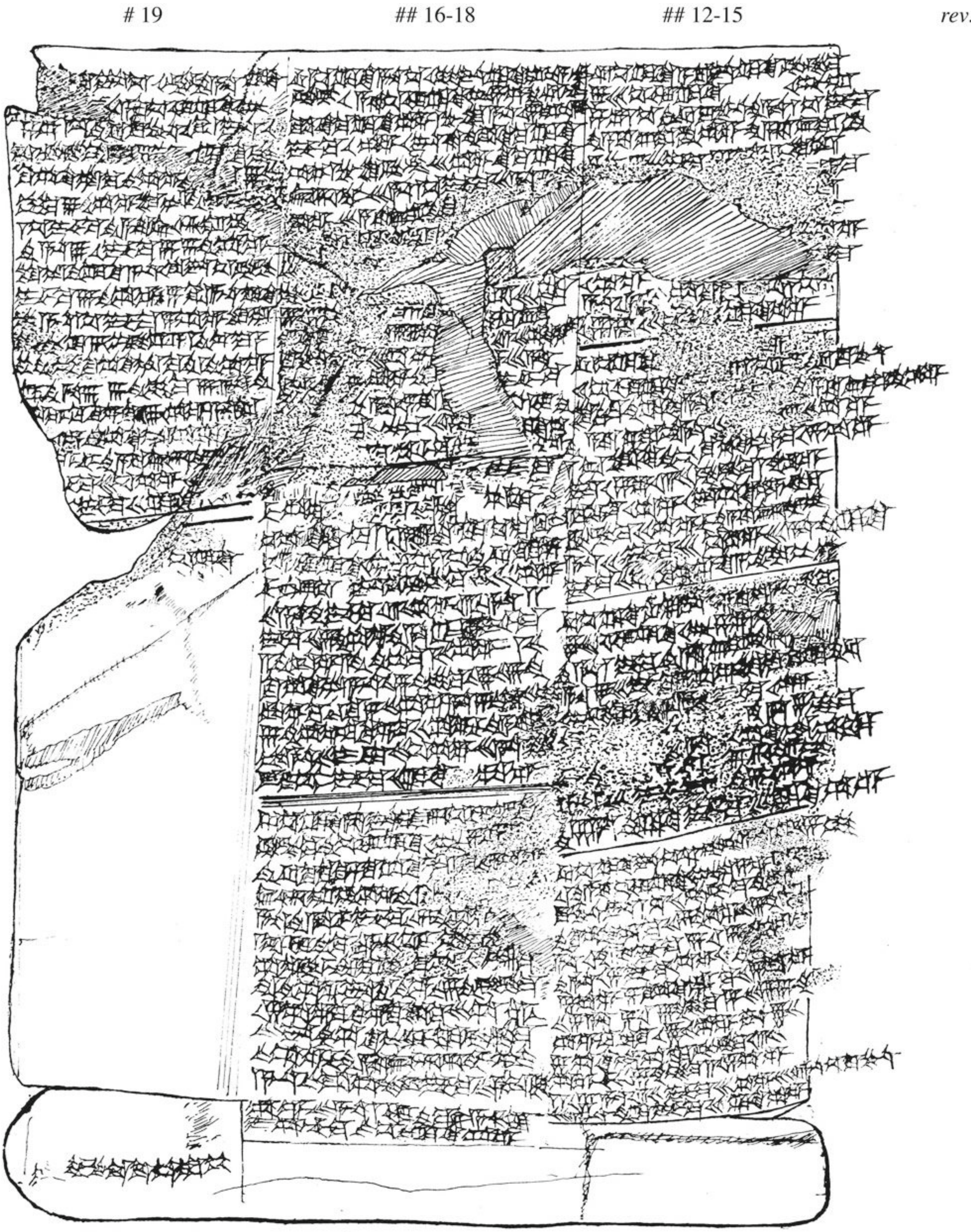


Fig. 5.1.21. IM 121613, *rev.* Hand copy.

5.2 IM 43993. A Small Text from Shaduppûm(?) with an Interesting Metric Algebra Problem for a Rectangle

IM 43993 is an Old Babylonian mathematical text without known provenance. However, the initial phrase *šum-ma ki-a-am i-ša-al-ka um-ma šu-ma* 'If thus he asks you saying this' is typical for mathematical texts from Tell Harmal (ancient *Shaduppûm* of the Eshnunna kingdom, situated in east Baghdad within the so called Diyala region), such as the 10 single problem texts published by Baqir In *Sumer* 7 (1951). See, for instance, IM 54559 (Sec. 5.4 below), Baqir's text No. 8, where the initial phrase is *šum-ma ki-a-am i-ša-al um-ma šu-ú-ma* 'If so he asked, saying this'. Initial phrases of a similar type do not appear in any other published mathematical cuneiform texts. Therefore, it seems to be safe to conclude that IM 43993, too, is from Shaduppûm (Tell Harmal).

The text of IM 43993 is exceptionally well preserved. The question in the single problem text IM 43993 is a close parallel to the question in exercise # 5 of the Tell Haddad text IM 121613 (Sec. 5.1.5 above). However, the solution procedure in IM 43993 is not a close parallel to the solution procedure in IM 121613 # 5. In addition, the mathematical terminology used in IM 43993 is quite different from the terminology used in IM 121613 # 5. It is, as a matter of fact, in several ways unique, without precedents in previously published mathematical cuneiform texts.

5.2.1 A Rectangle of a Given Form with a Given Sum of the Area and the Two Sides

	IM 43993
1	*šum-ma ki-a-am i-ša-al-ka um-ma šu-ma* /
2-4	*ši-ni-ip ši-di-im-mi pu-*[*ta*]*-am* / a.šag$_4$ *ši-di ù pu-ti-mi ak-m*[*u-u*]*r-ma* / 1-*mi*
	a.šag$_4$ *ši-di ù pu-ti mi-*[*nam ši-d*]*i-im* /
5	*at-ta i-na e-pí-ši-ka*
5b-7	1 1 *ù* 40 *m*[*a-n*]*i-a-ti-* / *ka* / *šu-ku-un-ma* 30 *li-gi-nam e-ru-ub* / 30 *a-na* 1 *i-ši-ma* 30 *i-li* /
8	*ù* 30-*ma* / *a-na* 40 *i-ši-ma* 20 *i-li*
8b-9	30 *ši-da-am ù* 20 *pu* / -*ta-am* / *šu-ta-ki-il-ma* 10 a.šag$_4$-*ka i-li* /
10-11	30 20 *ù* 10 *ku-mu-ur-ma* 1-*ma i-li* / *qi-bi-am*
12	***t**a-*[*ás-sa-ḫ*]*a-ar* / *šum-ma* [*wa*]*-ar-ka-tam* / *ta-pa-ra-ás*
13	1 *ù* 40 *ma-ni-a-ti-ka* / *šu-ta-ki-il-ma* 40 *i-li*
14	*pa-ni* 40 *pu-ṭur-ma* / 1 30 *i-li*
15-16	1 30 *a-na* 1 *ši-di-im pu-ti-im* / *ù* a.šag$_4$-*im ma-la ka-am-ru i-ši-ma* / 1 30 *i-li*
	re-eš-ka li-ki-il /
17	*pa-ni* 40 *ma-ni-a-ti-ka pu-ṭur-ma* 1 30 *i-li* /
18-19	1 30 *re-eš ma-ni-ti-ka šu-ku-un* /
20	*ù* 1 30-*ma a-na* 40-*ma ma-ni-a-ti-ka* / *i-ši-ma* 1 *i-li*
21	1 30 *ù* 1 *ku-mu-ur* / *ù le-te-e-ma* 1 15 *i-li*
22	*me-ḫe-*[*er*]*-šu* / *šu-ku-un šu-ta-ki-il-ma* 1 3[3 45] *i-li* /
23-24	*a-na* 1 33 45 *ša i-lu-kum* 1 30 *ša* [*ú*]*-ka-l*[*u*] / *ru-di-ma* 3 03 45 *i-li*
25	*ip-s*[*a*]*-šu* [*šu*]*-li-ma* / 1 45 *ip-su-šu i-li*
26	*i-na* 1 45 [*ša*] *i-lu-kum* / 1 15 *ša tu-uš-ta-ki-lu ú-su-uḫ-ma*
27	30 *li-gi-nam* / *ša te-ru-bu i-li*
	ki-am ne-pí-šum /
28	*ki-*[*am ne-pí*]- / *eš* /
29-31	30 uš / 20 sag / 1(èše)$^{\text{ašag}}$ a.šag$_4$

1	If so he asked you, saying this: /
2-4	Two-thirds of the length is the front. / The field, my length, and my front I added and / (it was) 1
	The field, my length and my front, were wh[at of le]ngth? /

5 You, in your procedure:
5b-7 1, 1 and 40 your / measures / set, then 30 on the tablet enter / 30 to 1 carry, then 30 will come up, /
8 and this 30 / to 40 carry, then 20 will come up.
8b-9 30, the length, and 20, the / front, / let eat each other, then 10, your field, will come up; /
10-11 30, 20, and 10 heap, then 1 will come up, / my prescribed (sum).
12 You [shall turn ar]ound. / If you shall examine it / more closely:
13 1 and 40, your dimensions, / let eat each other, then 40 will come up.
14 The reciprocal of 40 release, then / 1 30 will come up;
15-16 1 30 to 1 of the length, the front, / and the field, whatever heaped, carry, then / 1 30 will come up;
Let it keep your head. /
17 The reciprocal of 40, of your measures, release, then 1 30 will come up. /
18-19 1 30 as the head of your dimensions set, /
20 and this 1 30 to this 40 of your measures / carry, then 1 will come up.
21 1 30 and 1 heap / and split, then 1 15 will come up.
22 Its copy / set, let eat each other, then 1 33 45 will come up. /
23-24 To 1 33 45 that came up for you 1 30 that kept (your head) / add, then 3 03 45 will come up.
25 Its square-side let come up,then / 1 45, its square-side, will come up.
26 From 1 45 [that came up for you, / 1 15 that you let eat each other tear off,then
; 27 30 that you ent]ered / on the tablet will come up.
Such is the procedure. /
28 Such is the procedure: /
29-31 30 the length, / 20 the front, / 1 rope the field.

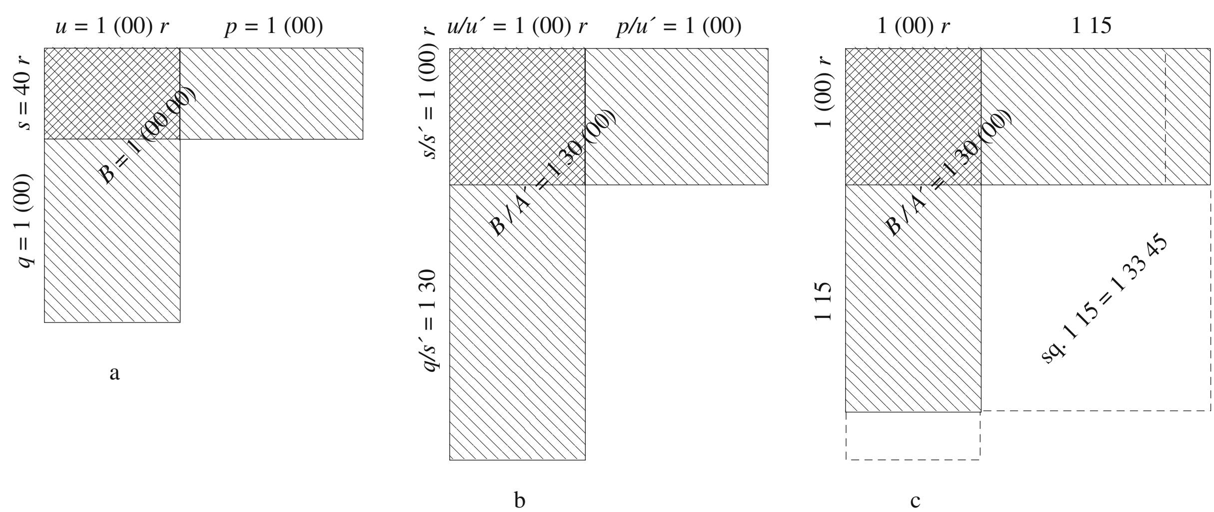
In this exercise, the stated problem is to find the sides of a rectangle when it is given that

a) The front is 2/3 of the length. b) The field plus the length and the front is equal to '1'.

This almost certainly means that

a) The front is 2/3 of the length. b) The area plus 1 (00) · (the length and the front) is equal to 1 (00 00).

The geometric interpretation of this problem is shown in Fig. 5.2.1 a below. There u = 1 (00) r and s = 40 r are the length and front of a rectangle, both unknown, q = 1 (00) is the given extension of the length, and p = 1 (00) the given extension of the front. B = 1 (00 00) is the given area of the twice extended rectangle.



B / A´ + sq. (1 30 + 1 00)/2
= 1 30 (00) + sq. 1 15 = 3 03 45
1 (00) r + 1 15 = sqs. 3 03 45 = 1 45
r = (1 45 – 1 15)/1(00) = ;30

Fig. 5.2.1. IM 43993: *s* = 2/3 *u*, *A* + *u* + *s* = 1.

A unique feature of this text is that the solution procedure is divided in two parts, first a presentation of the answer to the stated problem and a verification of the correctness of the answer, and only after that the main body of the solution procedure with the needed computations.

Indeed, the solution procedure begins (in line 5-6) by introducing the numbers 1, 1, and 40, called *ma-ni-a-ti-ka* 'your measures'. What this means is not absolutely clear, but it is likely that the author of the text thought of '1' as the given area, '1' (times the unknown reed) as the length, and '40' (times the unknown reed) as the front. Then he reveals, prematurely, (in line 6), that the length of the reed is '30'. Consequently (as is shown in lines 7-9) the length is '30' · '1' = 30, the front is '30' · '40' = 20, and the field (the area) is 30 · 20 = '10'. The result is verified (in lines 10-11) by showing that '30' + '20' + '10' = '1'.

The second part of the solution procedure starts in line 12 with the phrase *šum-ma wa-ar-ka-tam ta-pa-ra-ás* 'If you shall examine it more closely' (literally 'afterwards decide'), a phrase without any parallel in previously published mathematical cuneiform texts. The first step of this second part of the solution procedure is the equalization in lines 13-20. Interestingly, the equalization in this exercise does not proceed in precisely the same way as in the parallel text IM 121613 # 5 (see Sec. 5.1.5, in particular Fig. 5.1.5). In IM 121613 # 5, the extended length is *multiplied* by s' = ;40, and the extended front by u' = 1. In this exercise, the extended length 1 (00) r + 1 (00) is instead *divided* by u' = 1, and the extended front 40 r + 1 (00) by s' = ;40. As a result, the given area 1 (00 00) of the twice extended rectangle is divided by '1' · '40' = '40', and becomes equal to 1 30 (00) (see lines 12-16 of the text). The extended front is transformed into rec. ;40 · (40 r + 1 (00)) = 1 (00) r + 1 30 (lines 17-20), while the extended length remains unchanged. In the process, the twice extended rectangle has been replaced by a twice extended square, as in Fig. 5.2.1 b. The remainder of this second part of the solution procedure is a completion of the square of the usual kind, as in Fig. 5.2.1 c. The obtained result is that

1 (00) r + 1 15 = sqs. (1 30 (00) + sq. 1 15) = sqs. 3 00 45 = 1 45,

so that r = (1 45 – 1 15) / 1 (00) = ;30 (lines 20-26). Finally, this value for r is "entered on the tablet", which brings the reader of the text back to the first part of the solution procedure (line 6)!

The end result of the computation, that u = 30, s = 20 (and $u \cdot s$ = 10 (00 square rods) = 1 èše) is written on the left edge of the tablet.

This text and its interesting equation for "the sum of the field, the length, and the front" is mentioned in Høyrup, *LWS* (2002), 372, with references to related problems in IM 121613 # 5 (Sec. 5.1.5 above), *TMS* IX ## 2-3 (LWS; , 89), AO 8862 # 7 (mentioned in Sec. 5.1.22 above), YBC 4668 #A9, and BM 80209 § 5 (Sec. 8.8 below). See also the comparison in Friberg, *AT* (2007), 428 of BM 80209 § 5 with the question in the pseudo-Heronic *Geometrica*, 24.46 (the only known Greek mathematical text with an explicit solution of a quadratic equation).

5.2.2 The Vocabulary of IM 43993

a.šag$_4$, a.šag$_4$^{im}	field (area)
sag	front
uš	length
šum-ma ki-a-am i-ša-al-ka um-ma šu-ma	beginning of the question
at-ta i-na e-pí-ši-ka	you, in your procedure (beginning of the procedure)
ki-am ne-pí-šum, ki-am ne-pí-eš	such is the procedure (end of the procedure)
šum-ma wa-ar-ka-tam ta-pa-ra-ás	beginning of the more detailed procedure

li-gi-nam e-ru-ub, li-gi-nam ša te-ru-bu	writing down the result
ip-sa-šu šu-li, ip-su-šu i-li	computation of the square-side
pa-ni (*a*) *pu-ṭur*	compute the reciprocal of (*a*)
-mi	in citation of direct speech
-ma	and (then)
-ma	to stress single words
šu-ta-ki-il, tu-uš-ta-ki-lu	< *akālum* Št to eat (= to multiply (two sides))
i-li, i-lu, šu-li	< *elûm* Š to come up, to rise (= to result)
e-ru-ub, te-ru-bu	< *erēbum* to enter (= to write down)
ak-mu-ur, ku-mu-ur, ka-am-ru	< *kamārum* to heap (= to add together)
le-te-e	< *letûm* to split (= to halve)
i-ši	< *našûm* to carry (= to multiply (by a number))
ta-pa-ra-ás	< *parāsum* to divide, to decide
pu-ṭur	< *paṭārum* to release (= to compute (a reciprocal))
ru-di	< *redûm* D *t*o add
šu-ku-un	< *šakānum* to put down, to set (= to make a note of?)
ma-la	*mala* as much as, whatever
mi-[*nam*]	*mīnum* what
li-gi-nam	*liginnum* tablet
ma-ni-ti, ma-ni-a-ti	*minītum, pl. maniātum* dimensions
me-ḫe-er-šu	*meḫrum* copy, duplicate
pa-ni	*pāni n* = igi *n* the opposite (reciprocal) of *n*
pu-ta-am, pu-ti-im, pu-ti	*pūtum* front (short side)
qi-bi-am	*qībum* command, statement
re-eš	*rešum* head, beginning
ši-di, ši-di-im	*šiddum* length (long side)
ši-ni-ip	*šinipum* two-thirds
wa-ar-ka-tam	*warkatam* later, soon afterwards

It is noteworthy that the text of IM 43993 is written entirely in Akkadian, with the exception of the terms uš, sag, èše (all three only in the answer on the left edge), and a.šag$_4$. In the main text, the Akkadian terms *šiddum* and *pūtum* are used for 'length' and 'front'.

The text of IM 43993 has several terminological peculiarities. Thus, the expression *li-gi-nam e-ru-ub* 'enter on the tablet' and *šum-ma wa-ar-ka-tam ta-pa-ra-ás* ' if you shall examine it (more closely)' is not known to appear in any previously published mathematical cuneiform texts. The only other known mathematical cuneiform text in which the term *maniātu* 'dimensions' appears is IM 121613 (see Sec. 5.1.21 above). The use of the term *pāni n* instead of igi *n* for 'the reciprocal of *n*' is quite unusual, although it appears also in IM 31247 (see Sec. 5.3 below). Quite unusual in mathematical cuneiform texts is also the use of the enclitics *-mi* (in citations) and *-ma* (to stress single words).

5.2.3 IM 43993. A Hand Copy of the Tablet

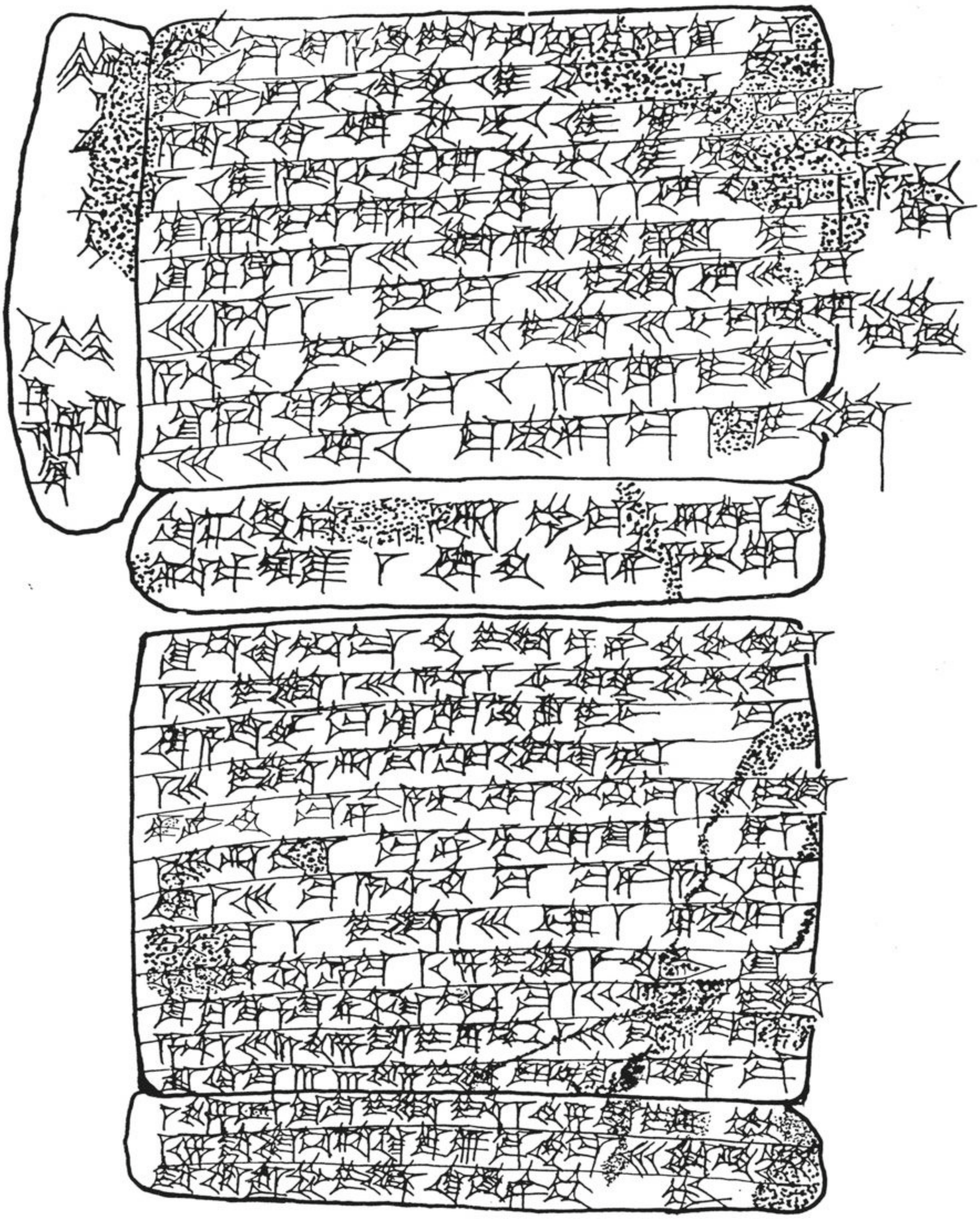


Fig. 5.2.2. IM 43993 Hand copy.

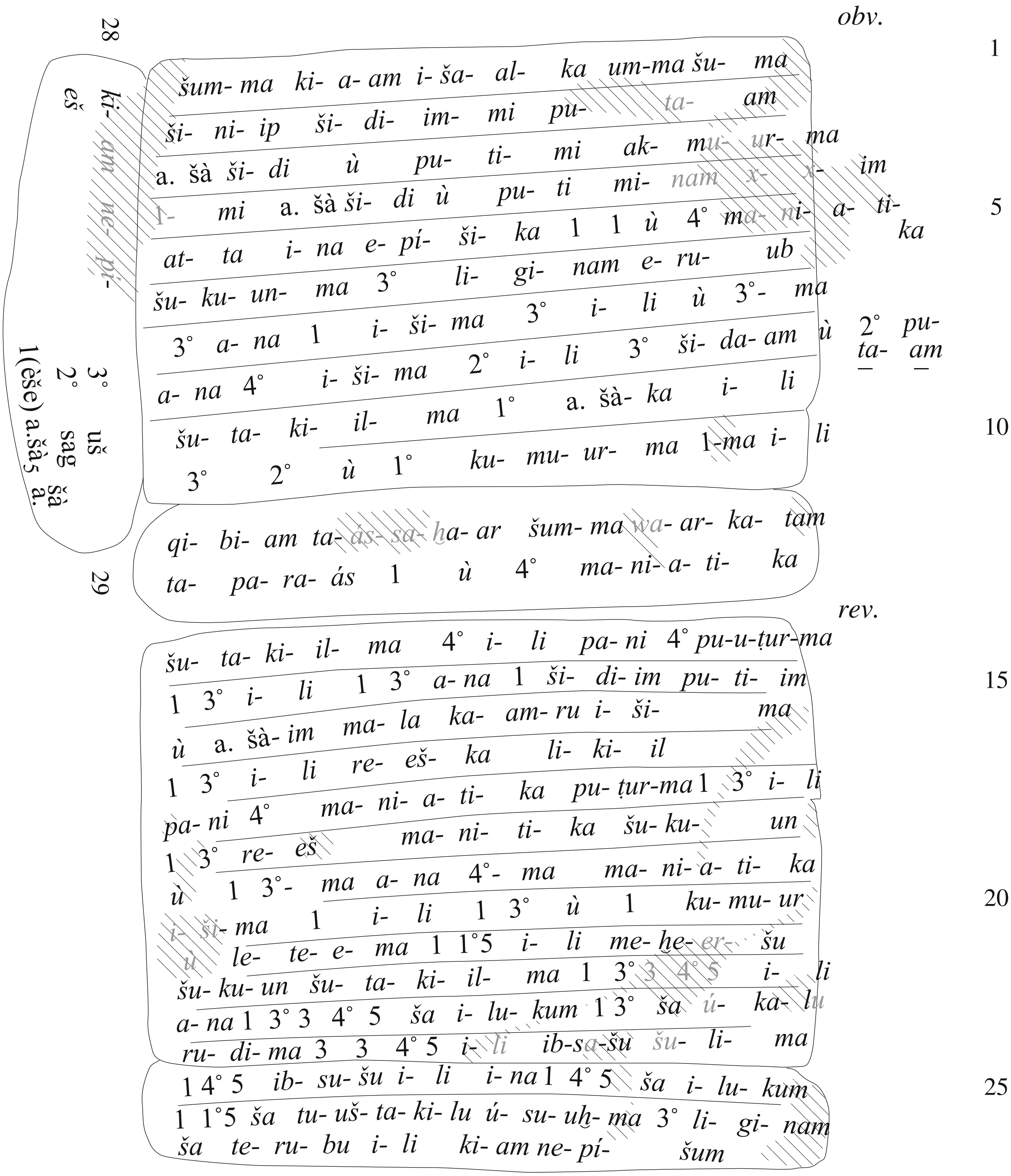


Fig. 5.2.3. IM 43993. Conform transliteration.

5.3 IM 31247. A Recombination Text from Ishchali with Metric Algebra Problems for Rectangles

IM 31247 is a large fragment of a badly preserved clay tablet from Ishchali in an unusual landscape format. The obverse of the tablet is completely destroyed, and less than half of the reverse is preserved. See Figs. 5.3.2-3 below. The text inscribed on the tablet was inadequately published and poorly explained by Bruins in *Sumer* 9 (1953), 249-252. Actually, as will be shown below, IM 31247 seems to have been a mathematical recombination text with the same theme as the Mê-Turran text IM 121613 (Sec. 5.1 above), quadratic problems for rectangles of a given form. Of the nine(?) exercises originally inscribed on the reverse (here numbered from 1′ to 9′), only four are preserved well enough to allow a valid interpretation.

5.3.1 # 1′. The Last Few Lines of a Badly Preserved Exercise

IM 31247 # 1′ (col. *v*)
1′ [*x x x x x x x a-na x x*] a.šag$_4$-*ka i-ši-i-m*[*a*] /
2′ [*x x x x x x x x x x x*] *x* 15 *ku-mu*!-*ur*!-*ma* /
3′ [*x x x x x x x x a*]-*na* 10 *i-ši-i-ma* 5 *i-l*[*i*] /
4′ [*x x x x x x x x x* sag]-*ka*
ki-a-am né-[*pé-šum*]

1′ [x x x x x x x to x x], your field, carry, then /
2′ [x x x x x x x x x x x] x 15 heap, then /
3′ [x x x x x x x x x] to 10 carry, then 5 will come up /
[x x x x x x x x x x] your.
Such is the procedure.

There is so little preserved of this exercise that nothing can be said about it.

5.3.2 # 2′. A Rectangle of a Given Form with a Given Square of the Length and the Front

In this exercise, the question is lost. Fortunately, however, the solution procedure is well preserved, so the content (even if not the exact wording) of the question is easily reconstructed. Besides, it is clear that the problem in IM 31247 # 2′ is a close parallel to the problem in IM 121613 # 10 (Sec. 5.1.10 above), in spite of several interesting dissimilarities.

IM 31247 # 2′ (col. *v*)
1 [*šum-ma i-ša-al-lu-ka š*]*i-ni-ip* uš s[ag] /
2 [*x x x x x x x x x x x*] sag-*k*[*a*] /
[possibly additional lines lost]
3-4 [*x x x x x x x x x x x*] *x x* / [41 40]
[u]š *ù* sag *k*[*i m*]*a-ṣi* /
5 [*at-t*]*a i-na e-pé-ši-i-ka* /
6-7 [1 u]š-*ka a-na* 40 sag-*ka r*[*u-d*]*i-i-ma* / [1 40 *i*]-*li*
8 40 sag-*ka a-na* 1 [u]š-*ka* / [*ru-di-i*]-*ma* 1 40 {1 40} *i-li* /
9-10 [1 40 *me-eḫ-r*]*a-am i-di šu-ta-ki-il-ma* / [2] 46 [40 *i*]-*li*
11 *pa-ni* 2 46 40 *pu-ṭú*[*r-ma*] / 2[1] 36 *i-li*
12 21 3[6] *a-na* 41 40 / [a].šag$_4$-*ka i-ši-i-ma* 15 *i-li-i* /
13 í[b].sa 15 *šu-li-i-ma* 30 *i-li* /
14 3[0] *a-na* 1 40 *i-ši-i-ma* 50 *i-li* /
15 50 [uš]-*ka* 50 sag-*ka*
ki-a-am ne-pé-šum

1 [If they asked you, saying this: Two-thirds of the length, the front.] /
2 [x x x x x x x x x x x] x the front x /
3-4 [x x x x x x x x x x x] x x / [41 40]
The length and the front, how much? /

5 [You, i]n your procedure: /
6-7 [1], your length, to 40, your front, [add on],then / [1 40] will come up.
8 40, your front, to 1, your [length], / [add on], then 1 40 {1 40} will come up. /
9-10 [1 40, a copy], lay down, let eat each other, then / [2] 46 [40] will come up.
11 The reciprocal of lines 2 46 40 re[lease, then] / 2[1] 36 will come up.
12 21 [36] to 41 40, / your field, carry, then 15 will come up. /
13 The equalside of 15 let come up, then 30 will come up. /
14 [30] to 1 40 carry, then 50 will come up. /
15 50, your [length], 50, your front.
Such is the procedure.

The solution procedure begins, in lines 6-10 left, with the following simple computations:

$u + s = 1\ (00) + 40 = 1\ 40, \qquad s + u = 40 + 1\ (00) = 1\ 40, \qquad (u + s) \cdot (s + u) = \text{sq. } 1\ 40 = 2\ 46\ 40.$

Here it is silently understood (as more explicitly stated in similar cases in IM 121613) that since it was said (in the lost question) that the front is 2/3 of the length, the length and the front can be assumed to be equal to 1 (00) and 40, respectively (times a measuring reed of unknown size).

The area of a field is mentioned in line 12 as '41 40, your field'. Therefore, it is clear that the question can be reconstructed as follows (in quasi-modern symbolic notations):

$s = 2/3\ u, \qquad (u + s) \cdot (s + u) = 41\ 40.$

It is also clear that the result of the mentioned computations in lines 6-10 left can be understood as

$(u + s) \cdot (s + u) = \text{sq. } 41\ 40 = 2\ 46\ 40 \text{ sq. } r.$

Like its parallel IM 121613 # 10, the problem in this exercise is exceedingly simple. Essentially, the square of the sum of the length and front of a rectangle of a given form is given. The quaint formulation of the problem, to find the sides of a rectangle of a given form if the sum of the length and front times the sum of the front and length is given, is easily explained if one remembers that the problem almost certainly was understood geometrically. See Fig. 5.1.9 above, where the sides of an extended square are constructed as the length extended by the front and the front extended by the length, respectively.

In lines 10 right-13, the equation 2 46 40 sq. r = 41 40 is solved as follows:

$\text{sq. } r = 41\ 40 / 2\ 46\ 40 = 21\ 36 \cdot 41\ 40 = \text{‘15’}, \qquad r = \text{sqs. ‘15’} = \text{‘30’}.$

It is easy to see that the correct interpretation is that, as in most similar cases, r = ;30 = 1/2.

In the final step of the solution procedure, in lines 14-15, the (extended) length and the (extended) front of the extended square are computed as follows:

$30 \cdot 1\ 40 = 50, \qquad \text{the “length”} = 50, \qquad \text{the “front”} = 50.$

This is, of course, a mistake. See the end of the parallel exercise IM 121613 # 10, where the length and front of the initial rectangle are computed as follows:

30 to 1 and 40 carry, then / <30, the length>, 20, the front, you will see. 30 the length, 20 the front.

5.3.3 # 3´. A Few Lines of a Badly Preserved Exercise

IM 31247 # 3´ (col. *v*)

1 *šu*[*m-m*]*a i-ša-al-lu-ka ši-ni-ip* uš [s]ag /
2 [*x x x x x x x x x x x x x x x*] /
3-4 [*x x x x x x x x x x x x x x x*] / a.šag$_4$ [8 20]
[u]š *ú* sag *ki ma*-[*ṣí*] /
[*at-ta i-na*] *e-pé-ši-i*-[*ka*] /
[*x x x x a*]-*na* 8 20 a.šag$_4$-*k*[*a i-ši-i-ma x x x*] /
[*x x x x x x x x x x x x x x x*] /
[*x x x x x x x x x x x x x x x*]
. . . .

Except for much of the initial line, very little is preserved of this exercise. The only meaningful observation is that 8 20 a.šag$_4$-*ka* ‘8 20, your field’ is mentioned in line 7 of the text, the first line of the solution procedure.

5.3.4 # 4´. A Rectangle of a Given Form with a Given Area Plus 10 Lengths Minus 10 Fronts

Nearly one half of the text of this exercise is lost, including the question and the beginning of the solution procedure. The only trace that remains of the question is the condition (in line 3 of the text) that the (area of a certain) field should be equal to 20 (00). Note that in line 15 the area of another field is given as 13 20, which, presumably, has to be interpreted as 2/3 of 20 (00). This circumstance, together with what remains of the solution procedure, can be utilized to make a satisfactory reconstruction of the lost question, and of the whole exercise. As it turns out, there is an interesting error in line 22 and in the answer in line 24 to the (reconstructed) problem.

IM 31247 # 4´ (col. ***vi***)

1 [*šum-ma i-ša-al-lu-ka ši-ni-ip* uš sag] /
2-3 [*x x x x x x x x x x x x x x x x*] / a.šag$_4$ 20
u[š *ú* sag *ki ma-ṣí*] /
4 *at-ta i-*[*na e-pé-ši-i-ka*] /
5 40 sag-*ka* [*x x x x x x x x x x*] /
6 [*x x x x x x x x x x x x x x x x*] /
7-8 [*x x x x x x x x x x x x x x x x*] / *i-ši-i-m*[*a x x*] /
9-10 1 uš-*ka* [*x x x x x x x*] / *i-ši-i-*[*ma x x x x*] /
11-12 *i-ṣa-am i-na ma-di-im ú-*⸢*su-uḫ*⸣*-*[*ma*] / 3 20 *i-li*
13 *ba-a* 3 20 gaz-[*ma*] / 1 40 *i-li*
14 1 40 *me-eḫ-ra-am i-*[*di*] / *šu-ta-ki-il-ma* 2 46 40 *i-li* /
15-16 [2 46] 40 *a-na* 13 20 *ru-ud-di-i-ma* / 16 06 40 *i-li*
17 íb.sa 16 06 40 */ šu-li-i-ma* 28 20 *i-li*
18 28 20 *x* / [*me-eḫ-ra*]*-am i-di-i-ma*
18 b 1 40 *ša tu-*[*x x*] */ šu-ku-*<*un*> /
19-19 b [*i-n*]*a iš-te-en ú-su-uḫ-ma 26 40 / i-li* /
20 [*pa-n*]*i* 40 sag *pu-ṭùr* 1 30 *i-li*
21 1 30 / [*a-n*]*a* 26 40 *i-ši-i-ma* 40 *i-li* /
22-23 [1 40 *ša tu-x x*] *a-na ša-ni-i-im* daḫ*-ma /* [30 *i-li*]
40 uš-*ka* 30 sag-*ka* /
24 [*ki-a-am ne*]*-pé-šum*

1 If they asked you: Two-thirds of the length, the front. /
2-3 [x x x x x x x x x x x x x x x x] / The field, 20
The length and the front, how much? /
4 You, in your procedure: /
5 40, your front, [x x x x x x x x x x] /
6 [x x x x x x x x x x x x x x x x] /
7-8 [x x x x x x x x x x x x x x x x] / carry, then [x x] /
9-10 1, your length, [x x x x x x x] / carry, then [x x x x] /
11-12 the smaller (number) from the greater (number) tear off / 3 20 will come up.
13 The half-part of 3 20 break, [then] / 1 40 will come up.
14 1 40, a copy lay down / let them eat each other, then 2 46 40 will come up. /
15-16 [2 46] 40 to 13 20 add on, then / 16 06 40 will come up.
17 The equalside of 16 06 40 / let come up, then 28 20 will come up.
18 28 20, / a [copy] lay down, then

18 b 1 40 that you [x x] / set. /
19-19 b [From] one tear off, then 26 40 / will come up. /
20 [The reciprocal] of 40, the front, release, 1 30 will come up.
1 30 / [to] 26 40 carry, then 40 will come up. /
22-23 1 40 that you [x x] to the second add on, then / [30 will come up].
40 your length, 30 your front. /
24 [Such is the pro]cedure.

Very little is preserved of the beginning of the solution procedure in lines 5-10 of the text, only the information that the front is 40 and the length 1 (00) (presumably times a reed of unknown length). In lines 11-12, a "smaller number" subtracted from a "larger number" gives the result '3 20'. Therefore, presumably, the lost question was of the following form:

$s = 2/3\ u, \qquad u \cdot s + 10\ u - 10\ s = 20\ (00).$

The geometric form of this presumed question is shown in Fig. 5.3.1 a below. Note that in this situation '10' cannot mean 10 (00), because 10 (00) (rods) is too much to be subtracted from the length of the rectangle.

The usual equalization of the problem is achieved through multiplication of the length 1 (00) r – 10 by 2/3 = ;40. The result is that, as in Fig. 5.3.1 b, a square of side 40 r plus 10 times the side, minus 6;40 times the side, now is required to have the area 2/3 · 20 (00) = 13 20. In Fig. 5.3.1 c, the problem is simplified to say that a square with the side 40 r plus the side times 3;20, explained as the "larger number" (10) minus the "smaller number" (6;40), is equal to 13 20. This is a quadratic equation of type B4a (see Fig. 5.1.19 in Sec. 5.1.24), which can be solved in the usual way by a completion of the square. See Fig. 5.3.1 d.

At this stage of the solution procedure, there is a numerical error in the text. The unknown length of the measuring reed should have been computed as follows:

40 r + 1;40 = sqs. (13 20 + sq. 1;40) = sqs. (13 20 + 2;46 40) = sqs. 13 22;46 40 = 28;20.

40 r = 28;20 – 1;40 = 26;40, 1 (00) r = 26 40 / 40 = 40.

However, working with sexagesimal numbers in relative place value notation, in lines 15-16 the author of the text made the incorrect addition '13 20' + '2 46 40' = '16 06 40', instead of '13 22 46 40'. This error did not prevent him from finding the correct square-side, '28 20'. *This is one of several known cases of cheating, where the author of a mathematical cuneiform text has obtained the correct answer in spite of an incorrect computation, clearly because he knew beforehand what the answer should be.*

Having come this far, the author of the text could have announced the correct answer to the stated problem (as reconstructed above), namely that

u = 1 (00) r = 40, s = 40 r = 26;40.

Instead, he makes another mistake, forgetting that he is solving a quadratic equation of type B4b for one unknown (r) and proceeding as if he is solving a related rectangular-linear system of equations of type B1b for two unknowns (u and s). (See again Fig. 5.1.19.) The result is, in lines 22-23, the spurious solution

s = sqs. (13 20 + sq. 1;40) + 1;40 = 28; 20 + 1;40 = 30.

The correct solution u = 40 in line 21 together with this spurious solution is announced in line 23 as the final answer to the problem, with the following words:

40 uš-*ka* 30 sag-*ka* '40 your length, 30 your front'.

This is a remarkably inept solution to a relatively simple mathematical problem. What is even more remarkable, however, is that there exists a parallel to this exercise in another text from Shaduppûm, with the same incorrect answer, namely the single problem text IM 54559 (Sec. 5.4 below)!

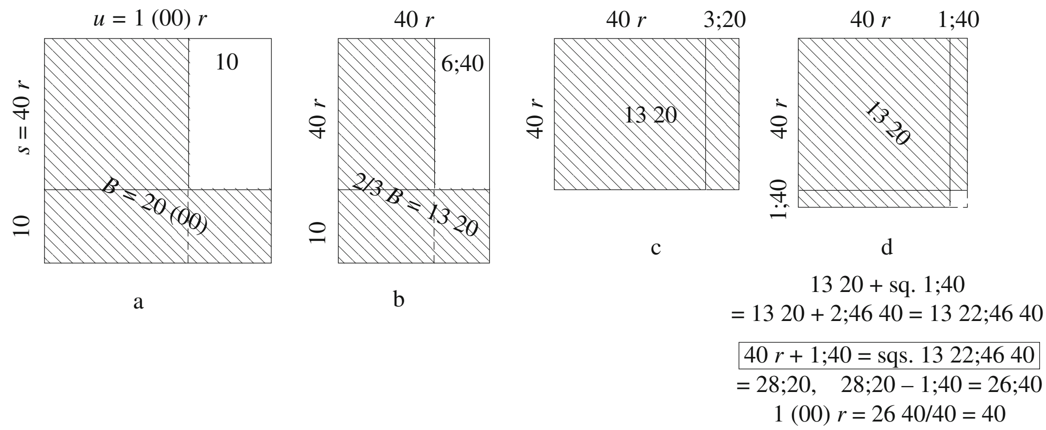


Fig. 5.3.1. IM 31247 # 4′: $s = 2/3\ u$, $u \cdot s + 10\ u - 10\ s = 20$.

5.3.5 # 5′. The Last Few Lines of Another Badly Preserved Exercise

IM 31247 # 5′ (col. *vi-vii*)
[*šum-ma i-ša-al-lu-ka ši-ni-ip* uš sag] /
[*x x x x x x x x x x x x x x x x*] / [*x x x x x x x x x x x x x x x x*] /
[*x x x x x x x x x x x x x x x x*] / [*x x x x x x x x x x x x x x x x*] /
1′-2′ [*x x x*] *x x x x* [*x x x x*] / [*x x*] *i-li* [*x x x*] /
3′ [*x*] *ku-mur-ma* 10 *i-li*
4′-5′ [*x x x x*] / [*ša t*]*u-uš-ta.ki-lu a-n*[*a x x x x*] / [*i-ši-i*]*-ma* 13 20 *i-li* /
6′ [*n*]*a-às-ḫi-ir-ma*
7′ *pa-n*[*i* 40] sag *pu-t*[*úr*]*-ma* / 1 30 *i-li*
8′ 1 30 [*a-na*] 13 20 *i-ši-*[*i-ma*] / 20 *i-li*
20 u[š-*ka*] 10 sag-*ka*
9′ *ki-a-am* / *né-pé-šum*

[If they asked you: Two-thirds of the length, the front.] /
[x x x x x x x x x x x x x x x x] / [x x x x x x x x x x x x x x x x] /
[x x x x x x x x x x x x x x x x] / [x x x x x x x x x x x x x x x x] /
1′-2′ [x x x] x x x x x x x x x [x x x x] / [x x] will come up [x x x] /
3′ [x] heap, then 10 will come up. [x x x] /
4′-5′ [x x x x] / [that] you let eat itself, to [x x x x] / [carry], then 13 20 will come up. /
6′ Turn back, then
7′ the reciprocal of [40], the front, release, then / 1 30 will come up.
8′ 1 30 [to] 13 20 carry, [then] / 20 will come up.
20 [your] length, 10 your front,
9′ Such is / the procedure.

There is so little preserved of this exercise that nothing much can be said about it, except that the solution to the problem appears to be

$s = 40\ r = 13;20, \quad u = 1\ (00)\ r = 20.$

However, for some reason, the proposed answer is $u = 20$, $s = 10$ (instead of 13;20).

5.3.6 # 6′. A Rectangle of a Given Form with a Given Sum of the Squares on the Length and on the Front

The text of this exercise is well preserved, and the interpretation offers no difficulties. The problem in the exercise leads to a quadratic problem of the simplest possible type, one which can be solved without the use of any completion of the square. It is similar to the problem in the exercise IM 121316 # 7 (Sec. 5.1.7 above).

IM 31247 # 6´ (cols. *vii*)

1 *šum-ma i-š*[*a*]*-al-lu-k*[*a š*]*i-ni-ip* [uš sag] /
2 uš *it*$_{x}$-<*ti*> uš [s]ag *it*$_{x}$*-ti* sag
3 *uš-*[*ta-ki-il-ma*] / *ú* a.šag$_4$ [*a*]*k-mur-ma* 21$^!$ 40
uš *ú* sa[g *ki ma-ṣí*] /
4 *at-t*[*a i*]*-na e-pé-ši-ka*
5 1 u[š-*ka*] *me-eḫ-ra-am* / *šu-k*[*u-u*]*n-ma*
6 *it*$_{x}$*-ti* 1 uš*-ka šu-ta-ki-il-ma* / 1 [*i-l*]*i*
7 40 *me-eḫ-ra-am id-*<*i*>*-ma*
8 *it*$_{x}$*-ti* / [40 s]ag*-ka šu-ta-ki-il-ma* / 26 40 *i-li* /
9-10 26 40 *ú* 1 *ku-mu-ur-ma* 1 26 40 / *i-li*
*pa-n*i 1 26 40 *pu-ṭúr-ma* /
11 <*a-na* 21 40 *i-ši-i-ma*> 15 *i-li*
12 íb.si 15 *šu-l*[*i*]*-m*[*a*] / [30] *i-li*
13 30 uš*-ka a-na* 40 sag*-ka* / [*i-ši-i*]*-ma* 20 *i-li*
13 b 30 uš*-ka* 20 sag- / *ka* /
14 [*n*]*é-pé-šum*

1 If they asked you: Two-thirds of [the length, the front]. /
2 The length with the length, the front with the front,
3 I let [eat each other], / and the fields [I] added together, then 21 40.
The length and the front, [how much]? /
4 You, in your procedure:
5 1, [your] length, a copy / lay down, then
6 with 1, your length, let eat each other, then / 1 [will come up].
7 40, a copy lay down,
8 with / [40], your front, let eat each other, then / 26 40 will come up. /
9-10 26 40 and 1 add together, then 1 26 40 / will come up.
The reciprocal of 1 26 40 release, then /
11 <to 21 40 carry, then> 15 will come up.
12 The equalside of 15 let come up, then / [30] will come up.
13 30, your length, to 40, your front, / [carry], then 20 will come up.
13 b 30, your length, 20, / your / front.
14 The procedure.

In this exercise, the stated problem is to find the sides of a rectangle when it is given that

a) the front is 2/3 of the length, b) the sum of the areas of the squares on the length and the front is 21 40.

The solution procedure is simple and straightforward. In lines 4-9 of the text, the sum of the squares on the length and on the front is computed as

sq. 1 (00) + sq. 40 = 1 (00 00) + 2 46 40 = 1 26 40, presumably times the square of a measuring reed of unknown length r.

Consequently, the the length r of the measuring reed can be computed as follows (in lines 10-12):

sq. r = 21 40 / 1 26 40 = '15', r = sqs. '15' = '30'.

Interestingly, the details of this computation are omitted. Indeed, there is a troublesome gap in the text at this point, after the sentence *pa-n*i 1 26 40 *pu-ṭúr-ma* 'compute the reciprocal of 1 26 40'. The reason for the existence of this gap is obvious: 1 26 40 = (sq. 3 + sq. 2) · sq. 20 = 13 · sq. 20 is not a regular sexagesimal number. Consequently, the reciprocal of 1 26 40 does not exist. Apparently, the author of the text discovered this difficulty too late, so he chose to bluff and wrote down the expected result anyway. (He should have asked instead, in the usual way: 'What times 1 26 40 is 21 40?', with the answer '15 times 1 26 40 is 21 40'.)

After the computation of r, the computation of u and s is straightforward (in lines 12-13).

5.3.7 ## 7´- 8´. Two Badly Preserved Brief Exercises

IM 31247 # 7´ (col. *vii*)
[*šum-ma i-ša-al*]*-lu-ka* [*ši-ni*]*-ip uš sa*[*g*] /
[the remainder of the exercise is lost]

IM 31247 # 8´ (col. *viii*)
[*šum-ma i-ša-al-lu-ka ši-ni-ip* uš sag] /
[several lines lost]
[50 *x*] *a-na li*-[*bi x x x x*] / a.šag$_4$ 13 20
uš *ú* sag *ki* [*ma-ṣí*] /
at-ta i-na e-pé-ši-ka
[*x x x*] 53 20 *i-li*
sag [*x x x*] / [*x x x*] *i-li*
[*x x x*] / [*x x x*] 30 *a-na* 40 *i-ši-ma* 20 *i-li* /
30 uš-*ka* 20 sag-*ka*
ki-a-am né-pé-šum

[If they asked you: Two-thirds of the length, the front.] /
[several lines lost]
[50 x] into [x x x x]. /
The field, 13 20.
The length and the front, how [much]? /
You, in your procedure:
[x x x] 53 20 will come up.
The front [x x x] / [x x x] will come up.
[x x x] / [x x x] 30 to 40 carry, then 20 will come up. /
30, your length, 20, your front.
Procedure.

5.3.8 # 9´. A Rectangle of a Given Form with a Given Sum of the Length and the Front

This brief exercise is well preserved. Exceptionally, the stated problem is linear, not quadratic. (It is possible that the preceding two badly preserved exercises, in view of their briefness, may have been of the same kind.)

IM 31247 # 9´ (col. *viii*)
1 *šum-ma i-ša-lu-ka ši-ni*-[*ip*] uš sag /
2 [u]š / *ú* sag *ak-mur-ma* 50
u[š *ú*] sag [*ki*] *ma-ṣí* /
3 *at-ta i-na e-pé-ši*-[*i*]-*ka* /
4 uš *ú* sag *ku-mu*-[*u*]*r*
5 <*pa-ni pu-ṭúr-ma*> *a-na* a.šag$_4$-*ka* / [*i-ši*]-*ma* 30 *i-li*
6 30 *a-na* 40 sag / [*i-ši*]-*ma* 20 *i-li*
30 uš-*ka* 20 sag-*ka* /
7 [*ki-a-am*] *né-pé-šum*

[*x x x*] *Nidaba*

1 If he asked you: two-thirds of the length, the front. /
2 The length / and the front I heaped, then 50.
The length [and] the front, [how] much? /
3 You, in your procedure: /
4 The length and the front heap,
5 <the reciprocal release,> to your field / [carry], then 30 will come up.
6 30 to 40, the front, / [carry], then 20 will come up.
30, your length, 20, your front. /
7 [Such is] the procedure.

[*x x x*] Nidaba

In this exercise, the stated problem is to find the sides of a rectangle when it is given that

a) the front is 2/3 of the length, b) the sum of the length and the front is 50.

In modern symbolic notations:

$s = 2/3\ u, \quad u + s = 50.$

This is a system of two linear equations for two unknowns. The intended solution was probably to set *u* and *s* equal to 1 (00) and 40 (times the reed of unknown length *r*), and to divide the given sum 50 by the sum 1 40 in order to find the value of *r*. In lines 4-5 of the text, several parts of this simple procedure are omitted. Furthermore, in line 4, the area of the rectangle is mentioned incorrectly, instead of the sum of the sides. Nevertheless, the computed value, $r = 30$, is correct, and it follows that *u* and *s* are equal to 30 and 20.

5.3.9 The Subscript

[*x x x*] *Nidaba* [x x x] Nidaba

Nidaba was a Sumerian goddess of agriculture and writing, although in the Old Babylonian period her functions were taken over by the god Nabu. Sumerian literary compositions often end with the doxology 'Praise to Nidaba!'.

5.3.10 Table of Contents

On the reverse of IM 31247 there are more or less well preserved traces of nine exercises. Four of the exercises, ## 2´, 4´, 7´, and 9´, are so well preserved that both the question and the answer can be read or reconstructed. Two of these exercises have, in addition to a correct answer, also a false answer. Two of the badly preserved exercises, ## 5´ and 8´ have preserved answers but no preserved questions. One of these, 5´, has both a correct and a false answer.

			correct answer	false answer
# 1´:	?		?	
# 2´:	$s = 2/3\ u$,	$(u + s) \cdot (s + u) = 41\ 40$	$u = 30, s = 20$	$u = 50, s = 50$
# 3´:	?		?	
# 4´:	$s = 2/3\ u$,	$u \cdot s + 10\ u - 10\ s = 20\ (00)$	$u = 40, s = 26;40$	$u = 40, s = 30$
# 5´:	?		$u = 20, s = 13;20$	$u = 20, s = 10$
# 6´:	$s = 2/3\ u$,	sq. u + sq. s = 21 40	$u = 30, s = 20$	
# 7´:	?		?	
# 8´:	?		$u = 30, s = 20$	
# 9´:	$s = 2/3\ u$,	$u + s = 50$	$u = 30, s = 20$ (incorrect solution method)	

5.3.11 The Original Theme Text

Evidently, the Shaduppûm text IM 31247 was originally a large mathematical recombination text, written in the same landscape format as the ten single problem texts published by Baqir in *Sumer* 7 (1951). It appears to have contained about 18 exercises, 9 on the obverse and 9 on the reverse, of which now only 4 are relatively well preserved. All the well preserved problems are about rectangles of a given form, but only three of them are rectangular-linear systems of equations which in the solution procedures are reduced to quadratic equations. The remaining well preserved problem is a system of two linear equations. It is likely that also the badly preserved problems ## 7´ and 8´ were systems of linear equations.

The conclusion that can be drawn from these observations is that, presumably, the exercises on IM 31247 were taken from one or two well organized theme texts with the combined theme *linear or quadratic problems* *for the sides of a rectangle of given form*. However, somewhere along the line from the assumed original theme text to the final recombination text, a long series of mistakes of various kinds were introduced in the texts of the exercises, in particular several quite curious false answers.

5.3.12 The Vocabulary of IM 31247

ašag (GÁN), a.šag$_4$-*ka*	field (= area), your field
daḫ	to add on
gaz	break
íb.sa, íb.si	equal(side) (= square-side, square root))
sag, sag-*ka*	front, your front (= short side)
uš, uš-*ka*	*length, your length (= long side)*

at-ta i-na e-pé-ši-(*i-*)*ka*	you, in your procedure
i-ṣa-am i-na ma-di-im	the greater (in number) from the smaller (in number)
(*ki-a-am*) *né-pé-šum*	(such is) the procedure
pa-ni (*a*) *pu-ṭúr*	compute the opposite (reciprocal) of (*a*)
(*a*) *it$_x$-ti* (*b*) *šu-ta-ki-il*	multiply (*a*) with (*b*)
šum-ma i-ša-al-lu-ka	if they asked you

šu-ta-ki-il, uš-ta-ki-il	< *akālum* Št to make eat each other (= to multiply (two sides))
šu-li, i-li	< *elûm* (Š) to come up, to rise (= to result (of a square root))
ak-mur, ku-mu-ur	< *kamārum* to heap, to pile up (= to add together)
i-di	< *nadüm* to lay down (= to draw?)
ú-su-uḫ	< *nasāḫum* to tear off (= to subtract)
i-ši-i	< *našüm* to carry (= to multiply (by a number))
pu-ṭúr	< *paṭārum* to loosen, release (= compute reciprocal)
ru-di-i, ru-ud-di-i	< *redûm* D to add on
na-às-ḫi-ir	< *saḫārum* N to turn back
šu-ku-un	< *šakānum* to put down, to set (= to make a note of?)
i-ša-al-lu	< *šâlum* to ask

ba-a	*bā'u, bû* half
i-ṣa-am	*(w)īṣum* few, small (in number)
iš-te-en	*išten* (number) one
it$_x$-ti	*itti* with
a-na li-bi	*ana libbi* into
ki ma-ṣí	*kī maṣi* how much?
ma-di-im	*mādum* many, numerous
me-eḫ-ra-am	*meḫrum* copy, duplicate
ša-ni-im	*šanûm* second
ši-ni-ip	*šinipum* two-thirds

5.3.13 IM 31247. A Hand Copy and a Conform Transliteration of the Fragment

Fig. 5.3.2. IM 31247, *rev*. Hand copy. The obverse is destroyed.

Fig. 5.3.3. IM 31247, *rev*. Conform transliteration within a reconstructed outline of the text.

5.4 IM 54559. A Text from Shaduppûm with a Single Metric Algebra Problem

IM 54559 is text No. 8 of ten Old Babylonian mathematical texts from Shaduppûm (Tell Harmal), published by T. Baqir in *Sumer* 7 (1951). Concerning these tablets, Baqir wrote:

> "The ten tablets comprising the new mathematical texts have been chosen from the collection of tablets excavated during the fourth season of work at Tell Harmal (1949). They were with the exception of one tablet found in a room of one of the private houses of the site (Room No. 252) — a remarkable room, the floor of which was strewn with some 225 tablets, all from a single occupational level. According to our field registration, nine of these new mathematical tablets were found 10 cms. beneath the pavement of Level II (from the top). The tenth tablet (No. 8: IM 54559) was found in another room (Room No. 256), also 10 cms. beneath the pavement of Level II. By means of the date formulae on several tablets of the same provenance, our mathematical tablets can be dated to the end of the reign of Dadusha, and to the reign of his son, Ibalpiel II, the two well known kings of Eshnunna, during the middle of the Old Babylonian period, about 1800 BC."

(All the ten Old Babylonian mathematical texts from Shaduppûm published by Baqir in 1951 plus two other texts of a similar kind published by Baqir one year earlier are discussed in great detail in the recently published book Gonçalves *MTTH* (2015).)

Baqir's presentation of IM 54559 includes photos, a hand copy, transliteration, and translation. However, Baqir's attempted mathematical interpretation was left unfinished, obviously due to pronounced difficulties (serious errors in the original formulation of the text), which will be pointed out below.

The text of IM 54559 is fairly well preserved, unfortunately with the exception of the second half of line 2, which happens to be a crucial part of the question.

5.4.1 An Exercise with a Remarkably Large Number of Errors

IM 54559 (Baqir, *Sumer* 7 (1951) No. 8; Muroi, *HS* 5-3 (1996))

obv.	1-2	*šum-ma ki-a-am i-ša-al um-ma šu-ú-ma* / *ši-ni-ip* uš sag.ki
		a-na ⸢a.šag$_4$ 10⸣ uš 10 ⸢sag.ki⸣ [*ú*]-⸢*ṣí-ib*⸣ (erased?) /
	3	*a-na* sag.ki 10 *ú-ṣí-*⸢*ib*⸣ a.šag$_4$ 20
	4	uš *ú* sag.ki ⸢*a-tu-úr*⸣ (erased) / *mi-nu-um*
		at-ta i-na e-pé-ši-ka /
	5	ša.na.bi *a-na* 10 *i-ši-ma* 6 40 *ta-mar* /
	6	*na-ás-ḫi-ir* 40 *a-na* 1 *i-ši-ma* 40$^{!}$ *ta-mar* /
	7-8	10 *e-li* 6 40 *mi-na-am wa-ta-ar* / 3 20 (written over *mi-na-am*) *wa-ta-ar*
		na-ás-ḫi-ir-ma
	9	3 20 *ḫe-pé-m*[*a*] / ⸢1⸣ 40 *ta-mar*
	10	1 40 *šu-ta-ki-il-ma* / 2 46 40 *i-li*
edge	11-12	2 46 40 / *a-na* 13 20 *ṣí-im-ma* / 13 22 46 40 *i-li* /
rev.	13-14	13 22 46 40 *mi-na-am* íb.si.⸢e⸣ / 28 20 íb.si.e
		28 20 *mé-eḫ-ra-am* ⸢*i-dí*⸣-[*ma*] /
	15-16	1 40 *ša tu-<uš>-ta-ki-lu a-na iš-te-*⸢*en*⸣ / *ṣí-ib i-na iš-te-en ḫu-ru-uṣ* /
	17	*iš-te-en* 30 *i-li iš-te-en* 26 40 /
	18	*na-ás-ḫi-ir-ma*
		40 uš 30 ⸢sag⸣.ki /
	19	*ki-a-am né-pé-šum*

obv.	1-2	If so he asked, saying this: / Two-thirds of the length, the front.
		To ⸢the field, 10⸣ lengths, 10 ⸢fronts I added on⸣ (erased). /
	3	To the front 10 I added on, the field, 20.
	4	The length and the front return (erased) / what?
		You, in your procedure: /
	5	Two-thirds to 10 carry, then 6 40 you will see. /
	6	Turn back. 40 to 1 carry, then 40 you will see. /
	7-8	10 over 6 40 what is it more? / 3 20 it is more.
		Turn back.
	9	3 20 break, then / ⸢1⸣ 40 you will see.
	10	1 40 let eat itself, then / 2 46 40 will come up.

edge 11-12 2 46 40 / to 13 20 add on, then / 13 22 46 40 will come up. /
rev. 13-14 13 22 46 40, what is it equalsided? / 28 20 it is equalsided.
28 20, a copy ⌜lay down⌝ , [then] /
15-16 1 40 that you let eat itself to one / add on, from one break off, /
17 one, 30 will come up, one, 26 40 will come up. /
18 Turn back, then
40, the length, 30, ⌜the front⌝. /
19 Such is the procedure.

The question in this text begins with the condition that the front should be two-thirds of the length. Therefore, it is reasonable to assume that the exercise in the Shaduppûm text IM 54559 is a parallel to the exercises in the Mê-Turran text IM 121613 (above Sec. 5.1). In particular, the mathematical problem in the exercise can be seen to be a quadratic problem for a rectangle of a given form.

Unfortunately, a part of the question in this quadratic problem, written in the second half of line 2, is partly erased, so the suggested transliteration/translation above is only tentative. In addition, various parts of the question do not at all agree with what goes on in the subsequent solution procedure.

In view of the mentioned obscurity of the question, it may be a good idea to start by looking at the solution procedure, and to postpone until later the discussion of the question.

The solution procedure begins in line 5 with the computation of two-thirds of 10 = 6;40, and in line 6 with the computation ‘40’ times ‘1’ = ‘40’. Next, the difference 10 – 6;40 = 3;20 is computed in lines 7-8, after which the solution procedure goes on to a completion of the square. All this suggests that the quadratic problem solved in this exercise was to find the sides of a rectangle when it is given that

a) The front is 2/3 of the length. b) The field increased by 10 lengths and decreased by 10 fronts is 20 (00).

The geometric way of formulating these two conditions is shown in Fig. 5.4.1 a below.

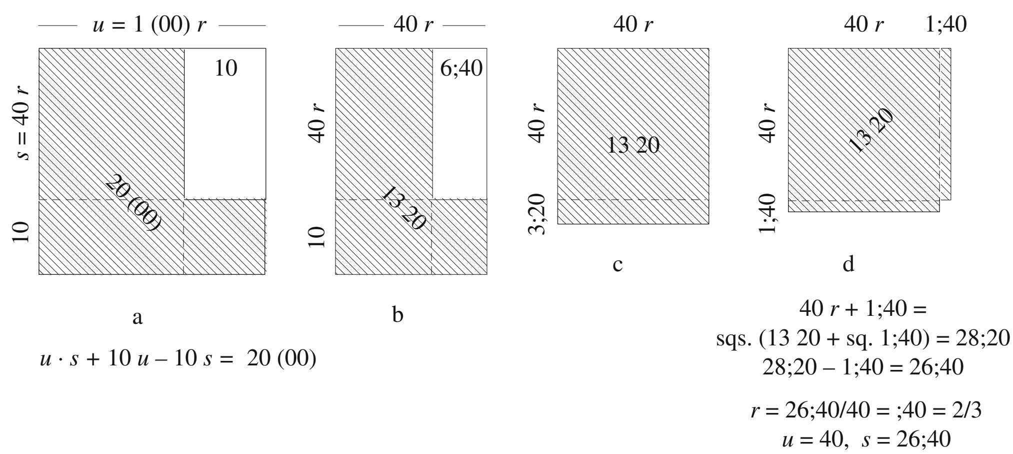


Fig. 5.4.1. IM 54559: $s = 2/3\ u$, $u \cdot s + 10\ u - 10\ s = 20$. (Corrected to agree with the solution procedure.)

It is clear that the first step of the solution procedure, as in all the related exercises in IM 121613, was to assume (in this case silently) that the length *u* is equal to 1 (00) (reeds of some unknown length *r*), and that, correspondingly, the front *s* is equal to 40 (reeds of the same unknown length).

It is also clear that the next step of the solution procedure, the equalization, was to transform the rectangle with the sides 1 (00) *r* and 40 *r* (as in Fig. 5.4.1 a) into a square with the side 40 *r* (as in Fig. 5.4.1 b). This was done through cross-wise multiplication of the length by *ša-na-bi* ‘2/3’ and of the front by 1, in lines 5-6. (The explicit mentioning of ša.na.bi ‘2/3’ in a solution procedure, rather than ‘40’, as in line 6, is probably unique in a known mathematical cuneiform text!) In the process, the subtracted 10 is replaced by a subtracted 2/3 · 10 = 6;40 (see line 5). That the length 1 (00) *r* was replaced by 2/3 · 1 (00) r = 40 *r*, and that the front 40 *r* is replaced by 1 · 40 *r* = 40 r, is indicated in line 6. However, the author of the text forgot to mention that, as a result of the shrinking of the length, the given area 20 (00) is reduced to 2/3 · 20 (00) = 13 20. Nevertheless, the number 13 20 is correctly mentioned in line 11.

The initial rectangular-linear system of equations for the two unknowns u and s has now been replaced by a quadratic equation for the single unknown $s = 40\ r$. The geometrical interpretation in Fig. 5.4.1 b is that if the area of a square of side s is increased by 10 times the side in one direction and decreased by 6;40 times the side in another direction, then the result is a figure of area 13 20.

Equivalently, as in Fig. 5.4.1c, if the area of a square of side s is increased in one direction only by 10 – 6;40 = 3;20 times the side (see lines 7-8), then the result is an extended square of area 13 20.

The remaining part of the solution procedure is a standard completion of the square, starting with a balancing where the square extended in one direction only is replaced by a square extended symmetrically in two directions. See Fig. 5.4.1 d and lines 8-9 of the text. In lines 9-12 it is shown that the completed square has the area

13 20 + sq. 1;40 = 13 20 + 2;46 40 = 13 22;46 40.

Consequently, the side of the completed square is, as in lines 13-14,

sqs. 13 22;46 40 = 28;20.

Therefore,

s + 1;40 = 28;20, hence s = 28;20 – 1;40 = 26;40, and u = rec. 2/3 · s = 1;30 · s = 40.

However, the last step of the solution procedure, the computation of u, is missing in the text. Apparently, the author of the text was less familiar with solution procedures for quadratic equations with one unknown (type B4a) than with solution procedures for rectangular-linear systems of equations with two unknowns (type B1b)! (See Fig. 5.1.19 above, in Sec. 5.1.24.) Therefore, after the computation of the side 28;20 of the completed square, he continued by computing, in lines 14-17,

28;20 + 1;40 = 30, 28;20 – 1;40 = 26;40.

Possibly the author of the text at this stage of the attempted solution procedure realized that this could not be the correct answer. That may be the reason why he concluded his efforts (in line 18) by cheating, incorrectly, stating, without further arguments, that the solution to the problem is

u = 40, s = 30.

Note that the value u = 40 is correct (it was probably known beforehand), while the value s = 30 is incorrect.

The multiple errors at the end §of the solution procedure in this exercise have, as already mentioned, a counterpart in some serious errors also in the question at the beginning of the text of IM 54559. Apparently, the question in line 2 of IM 54559 can be reformulated as follows:

a) The front is 2/3 of the length. b) The field increased by 10 lengths and by 10 fronts is ……… . (*)

This part of the text of IM 54559 was possibly borrowed from the question in an adjacent exercise of the text from which IM 54559 was copied. The formulation of that question in geometric terms and the corresponding solution algorithm is shown in Fig. 5.4.2 below (assuming that u is still equal to 40).

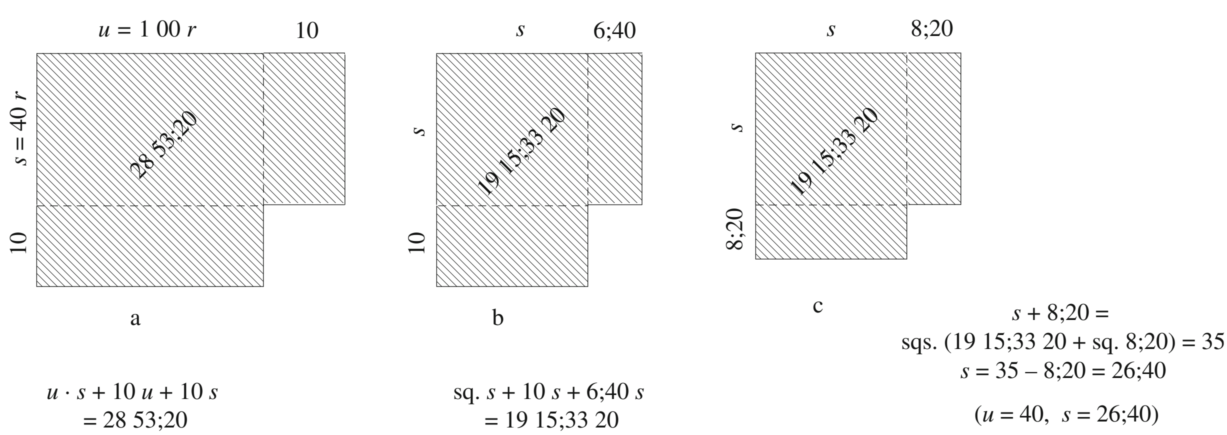


Fig. 5.4.2. IM 54559(*): s = 2/3 u, $u \cdot s$ + 10 u + 10 s = 28 53 20.

At the beginning of line 3 it is stated that

a-na sag.ki 10 *ú-ṣí-ib* — To the front, 10 I added on.

This statement agrees neither with the ensuing solution procedure, nor with the preceding statement in the second half of line 2. It is possible that if the second half of line 2 had been correctly copied from the original, it would have stated that:

i-na uš 10 *aḫ-ru-uṣ* — From the length, 10 I broke off.

In other words, it is possible that the whole question in IM 54559, if correctly copied from the original, would have asked for the sides of a rectangle when it is known that

a) The front is two-thirds of the length.
b) The length shortened by 10 and the front broadened by 10 together make a rectangle with the area 20.

In quasi-modern symbolic notations, that problem can be expressed as follows (if u is still equal to 40):

a) $s = 2/3\ u$, b) $(u - 10) \cdot (s + 10) =$ ……… (**)

The problem (**) above should be compared with the problem attacked in the solution procedure of IM 54559, which is of the form

a) $s = 2/3\ u$, b) $u \cdot s + 10\ u - 10\ s = 20\ (00)$ (**)

It is difficult to explain this difference between the (tentatively corrected) question (**) in IM 54559 and the ensuing solution procedure. One possible explanation may be that the one who originally designed the problem mistakenly calculated as follows

$(u - 10) \cdot (s + 10) = u \cdot s + 10\ u - 10\ s.$

Fig. 5.4.3 below is a geometric interpretation of the rectangular-linear problem (**) above and its solution procedure. The steps of the solution procedure are illustrated in Fig. 5.4.1 b-d.

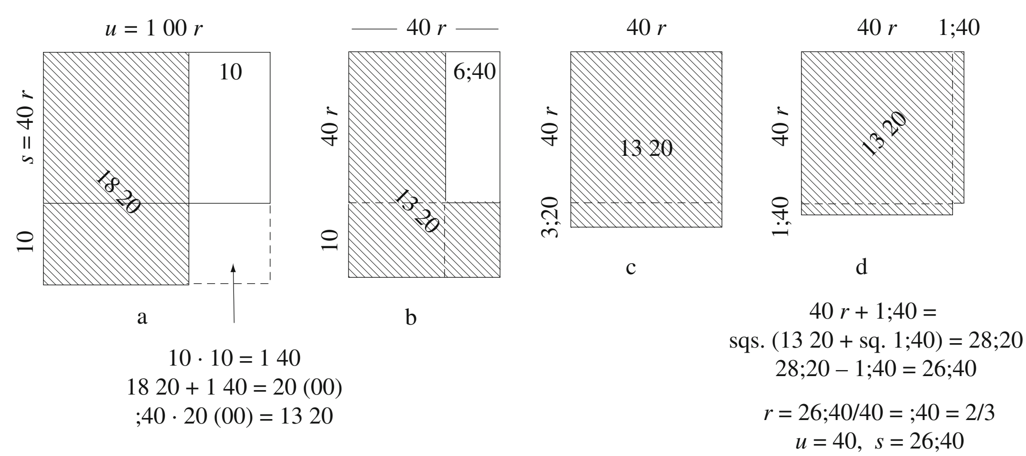


Fig. 5.4.3. IM 54559(**): $s = 2/3\ u$, $(u - 10) \cdot (s + 10) = 18\ 20$.

The revealed multitude of errors in IM 54559, both in the formulation of the question and in the answer, is without parallel in previously published mathematical cuneiform texts.

5.4.2 The Vocabulary of IM 54559

a.šag$_4$	field, area
íb.si.e	equal(side) (= square root)
sag.ki	front (= short side)
ša.na.bi	two-thirds
uš	length (= long side)
at-ta i-na e-pé-ši-ka	you, in your procedure
šum-ma ki-a-am i-ša-al	If so he asked
um-ma šu-ú	saying this
(*a*) *e-li* (*b*) *mi-na-am wa-ta-ar*	(*a*) – (*b*) = ?

šu-ta-ki-il	< *akālu* Št to make eat each other (= to multiply (two sides))
ta-mar	< *amāru* to see (= to find a result)
[*aḫ-ru-uṣ*], ⌜*ḫu-ru-uṣ*⌝	< *ḫarāṣu* to break off (= to subtract)
ḫe-pé	< *ḫepūm* to break (= to halve)
[*i-di*]	< *nadūm* to lay down (= to draw?)
i-ši	< *našūm* to carry (= to multiply (by a number))
na-ás-ḫi-ir	< *saḫāru* N to turn back (= begin again)
a-tu-úr	< *tāru* to return (= start again)
ú-ṣí-[*im-ma*], *ṣí-im-ma, ṣí-ib*	< *waṣābu* to add on
wa-ta-ar	< *watāru* to exceed
iš-te-en	*ištēn* one (of two)
mé-eḫ-ra-am	*meḫru* copy, duplicate
mi-nu-um, mi-na-am	*minu* what?
ši-ni-ip	*šinipu* two-thirds

5.4.3 IM 54559. A Hand Copy of the Tablet

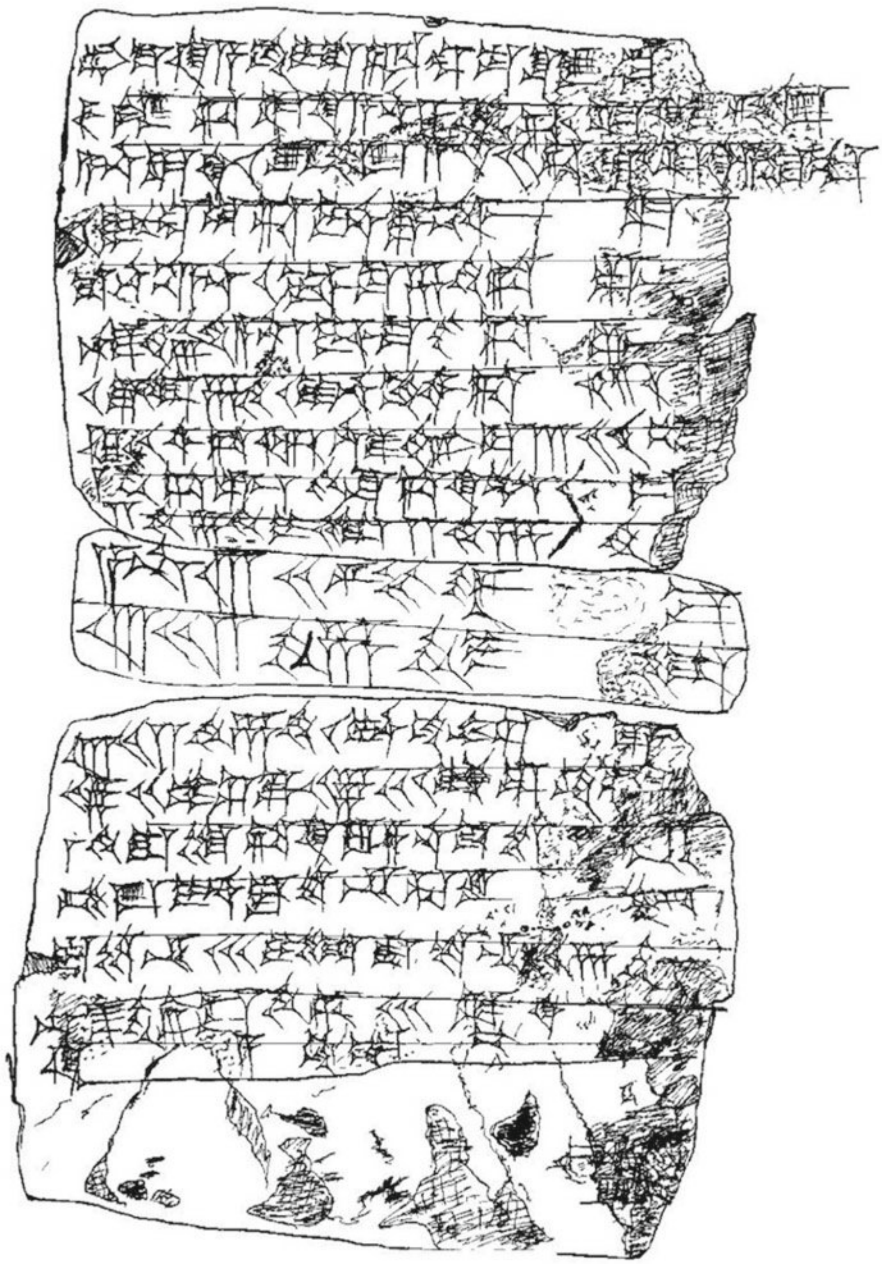


Fig. 5.4.3. IM 54559. Hand copy.

5.5 IM 53963. A Single Problem Text from Old Babylonian Shaduppûm with a Metric Algebra Problem for a Right Triangle

This single problem text was inadequately published by Bruins in *Sumer* 9 (1953), 249, on the same page as IM 31247. No photo or hand copy accompanied Bruins' transliteration of the text.

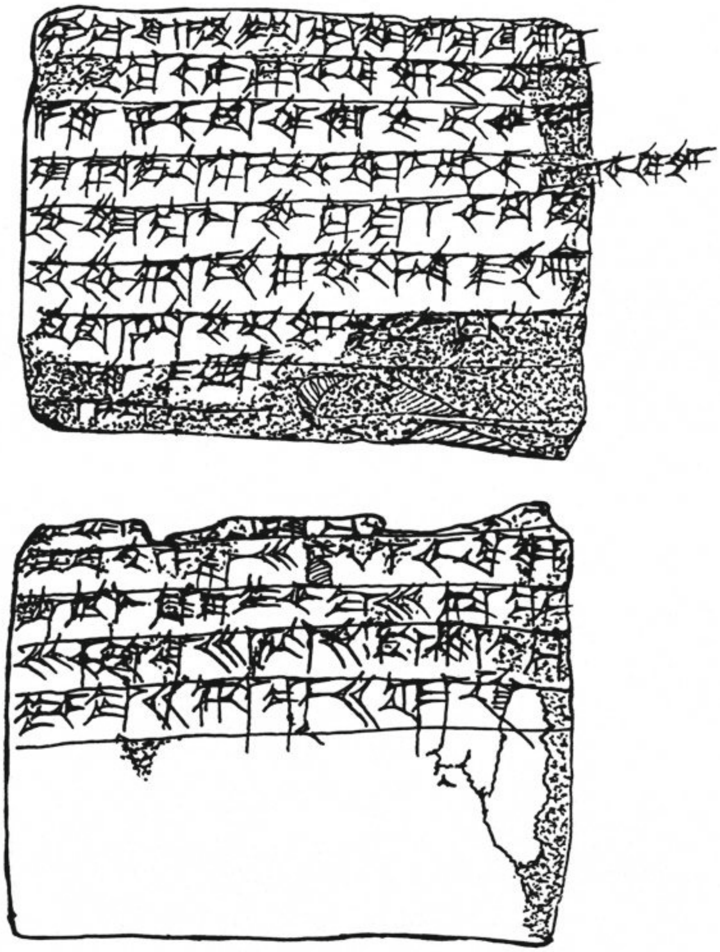


Fig. 5.5.1. IM 53963. Hand copy.

5.5.1 A Form and Magnitude Problem for a Right Triangle

Lines 7-10 of this text are badly damaged and difficult to read, so the transliteration and translation below is only tentative.

		IM 53963
obv.	1	*šum-ma ki-a-am i-ša-al-ka um-ma šu-ú-ma* /
	2-3	[*sa*]-*ta*!-*ku ši-ni-ip ši-di-im pu-tu-u*[*m*] / a.šag$_4$ 5
		ši-id-di ù pu-ti mi-[*nu-um*] /
	4	*at-ta i-na e-pé-ši-ka*
	5	*aš-šum ši-n*[*i-i*]*p ši-di-im* / *pu-tu-um qa-bu-ku-um*
	6	1 *ši-da-am* [*ù*] / 40 *pu-ta-am lu-pu-ut*
		na-ás-ḫi-ir /
	7-8	40 *ša a-na pu-ti-im* [*ta-al-pu-tu a-na*] / [30 x x] *i-ši-ma*! [20 *ta-mar*]
	9	[*pa-ni* 20] / ⸢*pu-ṭú*⸣-[*úr-ma*]
rev.	10	[*a-na* 5 a.šag$_4$ *i-ši-ma*] / [15 *ta-mar*]
	11	[íb.sá 15 *šu-li*] / íb.sá 40 ⸢*i-li*⸣
	12	30 *a-na* 1 *ši-di-ka* / *ša ta-ki-lu i-ši-ma* 30 *ta-mar* /
	13	30 *ši-id-di*
	14	30 *a-na* 40 *ša pu*!-*ti-im* / *i-ši-ma* 20 *ta-mar*
		20 sag.ki

1	If so he asked, saying this: /
2-3	A triangle. Two-thirds of the length, the front. / The field, 5
	My length and my front, wh[at?] /
4	You, in your procedure:
5	Because two-thirds of the length / the front it was said to you:
6	1, the length, [and] / 40, the front, touch.
	Turn back! /
7-8	40 that for the front you touched / [to 30 the x x] carry, then [20 you will see].
9	[The reciprocal of 20 / re[solve, then
10	to 5 carry, then] / [15 you will see].
11	[The equalside of 15 let come up] / ⌜the equalside⌝, 40 (sic!) ⌜will come up⌝.
12	30 to 1, your length, that you held$^?$, carry, then 30 you will see. /
13	30, my length.
14	30 to 40 of the front / carry, then 20 you will see.
	20, the front.

The triangle figuring in this exercise is silently assumed to be a “right triangle”, meaning one half of a rectangle, so that its area is 1/2 times the product of the length *u* and the front *s*. Therefore, the stated problem can be reformulated as follows, in quasi-modern symbolic notations:

$s = 2/3\ u, \qquad A = 1/2\ u \cdot s = 5\ (00).$

It is, obviously, a slightly modified parallel to the form and magnitude problem for a rectangle in IM 121613 # 1 (Sec. 5.1.1 above). In lines 5-6 of the text, *u* and *s* are given the values 1 (00) and 40 (presumably times a reed of unknown length *r*). In lines 7-8, the area of the triangle is computed as

$A = 1/2\ u \cdot s = 30 \cdot 40 = 20\ (00)$ (presumably times sq. *r*).

The value of *r* is then computed in lines 8-9, as follows:

rec. ‘20’ · ‘5’ = ‘15’, sqs. ‘15’ = ‘30’.

With this value of *r*, the values of *u* and *s* are then computed in the usual way, and are found to be, as in most similar cases, 30 and 20.

5.5.2 The Vocabulary of IM 53963

a.šag$_4$	field, area
sag.ki	front (= short side)
at-ta i-na e-pé-ši-ka	you, in your procedure
šum-ma ki-a-am i-ša-al-ka	If so he asked you
um-ma šu-ú	saying this
ta-mar	< *amārum* to see (= to find a result)
lu-pu-ut	< *lapātum* to touch (= to point at?)
i-ši	< *našüm* to carry (= to multiply (by a number))
qa-bu-ku-um	< *qabûm* to say
na-ás-ḫi-ir	< *saḫārum* N to turn back (= begin again)
šu-ta-ki-il	< *akālum* Št to make eat each other (= to multiply (two sides))
aš-šum	because
mi-[*nu-um*]	*minum* what?
pu-tu-um, pu-ti, pu-ta-am	*pūtum* front (= short side)
sa-ta-ku	*santakkum, sattakkum* wedge, triangle
ši-di-im, ši-id-du, ši-id-da-am	*šiddum* side, length (= long side)
ši-ni-ip	*šinipum* two-thirds

6. Further Mathematical Texts from Old Babylonian Mê-Turran (Tell Haddad)

6.1 IM 95771. A Fragment of a Large Mathematical Recombination Text, with Three Partly Preserved Illustrated Geometric Exercises and a Badly Preserved Table of Constants

6.1.1 # 1′. A Few Lines of an Exercise about a City Wall

Only the last three lines of the solution algorithm in this exercise are (partly) preserved. Without the missing context, the object of the exercise cannot be decided, not even conjectured. Without further context, the translation of line 4′ is very uncertain.

IM 95771 # 1′

1′ [*x x x x x*] uš 10 *ki*-[*x x x x x*] /
2′ [*i-n*]*a du-ri-im ka-li-šu ḫu-ru-uṣ-ma* 25 íb.tak$_4$ /
3′ [*x x x x*] *x ša du-ri-im tu-úr-da-am* 25 *ḫa-mi-iš-na-ma-tim* /
4′ [*x x x t*]*a-ak-su-sa-am-ma pu-ur-sa-am te-eb-ḫi*
ki-a-am ne-pé-šu

1′ [x x x x x] the length. 10 x [x x x x x] /
2′ from the whole city wall break off, then 25 is the remainder. /
3′ [x x x x] x that of the city wall you went down, 25, five-in-cubits, /
4′ [the x x x y]ou gnawed off, then the section you surrounded.
Such is the procedure.

An interesting possibility is that this sadly destroyed exercise dealing with a broken city wall may be related to five better preserved exercises in § 1 of MS 3052 (Friberg, *MSCT 1* (2007), Sec. 10.2 a), dealing with repairs of breaches in mud walls.

6.1.2 # 2′. A Trapezoidal Water Reservoir Divided into Five Sections

Only three partly preserved lines of the question and three partly preserved lines of the solution procedure remain of this badly damaged exercise. Fortunately, the diagram associated with the exercise is fairly well preserved. It can be restored in its entirety as follows:

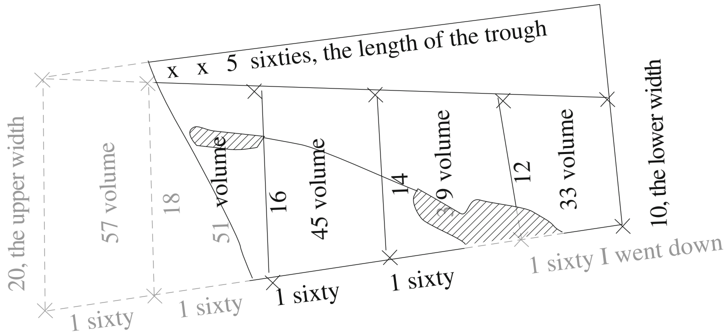


Fig. 6.1.1. IM 95771. The geometric diagram in exercise # 2′.

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7_6

The (reconstructed) diagram shows, in a view from above, a trapezoidal 'trough' (here interpreted as a water reservoir) divided into five parallel sub-sections. The upper and lower widths are explicitly given as 20 and 10, respectively. The upper and lower widths together with the also explicitly indicated four transversals form a steadily decreasing sequence of numbers (in modern terms, a decreasing arithmetic progression) [20], [18], 16, 14, 12, 10. The indicated volumes of the five sub-sections also form a steadily decreasing sequence of numbers, [57], [51], 45, [3]9, 33.

An unusual feature of the diagram is that the sub-divided trapezoid is completed by a (symmetric) triangle with the same 'length' as the trapezoid, and with a width equal to the difference between the upper and the lower width of the trapezoid. This was possibly done for a didactic purpose, since the visually obvious "growth rate" of the triangle (the increase of the width divided by the length) is equal to the less visually obvious growth rate in the opposite direction of the trapezoid.

Another unusual feature of the diagram is that all intersections between lengths and widths are marked by crossed wedges. This is done also in the other two preserved diagrams on this fragment.

Here is what is preserved of the text of this exercise:

IM 95771 # 2´

1 [A.SUG (*buginnu*) *ru-up-šum e-lu-um er*]*-bi-na-ma-tim ru-up-*[*šum šu-up-lum*] /
2 [*ši-ni-na-ma-tim x x x x*]*-a-ni-ka* saḫar *šu-up-lum* [*x*] *šu-ši* [*x x x x*] /
3 [*x x x x x x x x x x x x x x x x*]*-x a-na* 5 *šu-ši* uš
4 *ta-lu-ia ù* saḫar*-ia* / [*mi-nu*]
[*i-na e-pé-ši-ka* 20 *ru-u*]*p-ša-am e-la-am ù* 10 *ru-up-ša-am* /
5 [*ša-ap-la-am ku-mur ḫe-pé*]*-ma* 15 *i-lí* 15 *a-na* ⸢5 uš⸣ [*i-ši-ma* 1 15] /
6 [*i-lí* igi 1 15 *a-na* 3 45 *i-ši-m*]*a* 3 ⸢*šu-up*⸣-[*la-am ta-mar*] /
... [*x x x x x x x x x x x x x x x x x x x x x x x x x*]

1 [A trough. The upper width is f]our cubits. The [lower] wid[th] /
2 [is two cubits. x x x x] x x your [x x] the lower volume [x] sixties [x x x x] /
3 [x x x x x]-x for 5 sixties length.
4 My transversals and my volumes / [are what?]
In your procedure: 20] the upper [wid]th and 10 the [lower] width /
5 [add together, break], then 15 will come up. 15 to ⸢5, the length⸣, [carry, then 1 15] /
6 [will come up. The reciprocal of 1 15 to 3 45 carry], then 3, the ⸢depth⸣ [you will see.] /
... [x x x x x x x x x x x x x x x x x x x x x x x x x x]

In the first line of the question, something has the length four cubits. The word *erbinammatim* 'four cubits' occurs also in exercise # 3 of Haddad 104 (Al-Rawi and Roaf, *Sumer* 43 (1984)), another Old Babylonian mathematical recombination text from Mê-Turran, where it is explained by the authors as a contraction of *erbe ina ammatim*. Note the similar occurrence of the word *hamišnammatim* 'five cubits' in the text of exercise IM 95771 # 1´above. Since 4 cubits = 4 · ;04 rods = ;20 rods, it is likely that '[20] the upper [wid]th', which is mentioned in line 4 of the exercise, is identical with the 'four cubits' given in line 1.

A 'lower volume' seems to be mentioned in line 2 of the question, but what that may mean is not clear. However, apparently a total length of 5 sixties seems to be given in line 2 and is mentioned again in line 3. The question ends by asking for the (lengths of the four) transversals and the volumes of the (five) sub-sections of the reservoir.

There is so much missing of the question that the stated problem cannot be reconstructed. However, at least the meaning of the first, fairly well preserved part of the solution procedure is quite clear. The sum of the (apparently) given upper and lower widths of the whole reservoir is computed and cut in half. The resulting average width (20 + 10)/2 = 15 is multiplied by the (apparently) given length 5 sixties of the whole reservoir. Then the (apparently) given volume 3 45 (00) of the whole reservoir is divided by the product 15 · 5 (00) = 1 15 (00). The result is 3 45 / 1 15 = 3, which can be understood as the common depth of all the sub-sections of the reservoir.

At the end of the solution procedure, the answer to the question in exercise # 2′ is lost. However, this answer was given, prematurely, in the diagram preceding the exercise, where the lengths of the transversals and the volumes of the sub-sections are recorded.

6.1.3 # 3′. A Problem for a Divided Symmetric Triangular Water Reservoir

The text of exercise # 3′ on IM 95771 is well preserved. In addition, it is accompanied by a diagram which, although damaged, still shows almost all that it was supposed to show. The text contains several unusual technical terms. The translation below of lines 2 and 8 is problematical, and there is no obvious explanation of the repeatedly occurring term KI, even if its mathematical meaning is clear.

IM 95771 # 3′

1 A.SUG (*buginnu*)
3 *šu-ši* uš *i-na pí-ša* 20 *ir-pí-iš i-na zi-ba-ti-ša* 10 *ir-pí-iš* /
2 *a-na* ba.si-*ša ki ma-ṣí* uš *ú-ri-id-ma* ba.si *in-ne-mi-id* /
3 *i-na e-pé-ši-ka*
4 20 *ru-up-šum e-lu-um e-li* 10 *ru-up-ši-im* <*ša-ap-li-im*> *mi-nam i-ter* / 10 *i-ter*
10 KI *zi-ba-ti* A.SUG *ka-li-šu i-li*
5 *pa-ni* 3 *šu-ši* uš *pu-ṭur-ma* / 20 *i-lí*
20 *a-na* 10 KI *zi-ba-ti* A.SUG *i-ši-ma* 3 20 *i-lí*
6 3 20 KI *zi -ba-ti-im* / *a-na* 1 *šu-ši* uš
pa-ni 3 20 *pu-ṭú-ur-ma* 18 *i-lí*
7 18 *a-na* 10 *ru-up-ši-im* / *ša-ap-li-im i-ši-ma* 3 *i-lí*
3 *šu-ši* uš *ša-ap-la-nu-um* 10 *ru-up-ši-im* /
8 [10] *tu-ri-id-ma* ba.si-*ka in-ne-mi-id*
ki-a-am ne-pé-šum
============
24 *du-gi-i im-ta-ḫa-ar*

1 A trough (a water reservoir(?)).
3 sixties was the length, at its mouth it was 20 wide, at its tail it was 10 wide. /
2 At its equalsides, how much length did I go down, then the equalsides were made to touch. /
3 In your procedure:
4 20, the upper width, over 10, the <lower> width, what is it beyond? / 10 it is beyond.
10, the KI of the tail of the whole trough will come up.
5 The reciprocal of 3 sixties, the length, release, / 20 will come up.
20 to 10, the KI of the tail of the trough carry, then 3 20 will come up.
6 3 20 is the KI of the tail / for 1 sixty of length.
The reciprocal of 3 20 release, then 18 will come up.
7 18 to 10, the lower width / carry, then 3 will come up.
3 sixties is the lower length of the width 10. /
8 [10] you went down, your equalsides were made to touch.
Such is the procedure.
============
24 x x equalsided.

The diagram accompanying this exercise shows a symmetric ‘trough’ divided into six sub-sections of equal lengths. See Fig. 6.1.2 below. However, only a triangle divided into two stripes by a line parallel to the front is considered in the text of the exercise. As a matter of fact, the exercise begins, in line 1, by mentioning just the first of the two stripes, a trapezoid with the length $u' = 3\ 00$ (‘3 sixties’), with the ‘mouth’ $m = 20$, and the ‘tail’ $t = 10$. In the diagram, this trapezoid corresponds to the first three stripes of the divided triangle, while the ‘mouth’ and the ’tail’ are called the ‘upper width’ and the ‘lower width’, respectively. These alternative names for the two parallel fronts reappear in lines 3 and 6 of the exercise, although the term ‘tail’, too, is used repeatedly.

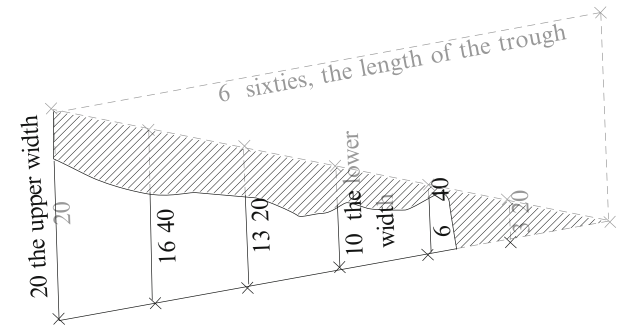


Fig. 6.1.2. IM 95771. The geometric diagram in exercise # 3´.

The solution procedure begins, in lines 3-5, by computing the decrease $m - t$ of the width of the trapezoid (the 'trough'), referred to by the obscure phrase 'the KI of the tail of the whole trough', and dividing it by the length u. The result is the growth rate (f) of the trapezoid, computed as

$f = (m - t)/u' = (20 - 10)$ / 3 00 = 10 / 3 00 = 3 20 / 1 00 = '3 20'.

In lines 5-6, '3 20' is then explained, as 'the KI of the tail for 1 sixty of length', which is correct as long as it means 'the decrease of the width over sixty (nindan) of the length'.

In the second half of the solution procedure, in lines 6-7, the length of the second half of the triangle is considered, corresponding to stripes 4-6 of the divided triangle in the diagram. This length, referred to as 'the lower length of the width 10', is correctly computed as follows:

$u'' =$ rec. $f \cdot t =$ rec. 3 20 · 10 = '3'.

The result is explicitly explained as '3 sixties' (= 3 00).

The problematic passages in lines 2 and 8 of the exercise can, with considerable hesitation, be interpreted as follows: In order to explain that the sides of a (symmetric) trapezoid are extended until they meet and a symmetric triangle is formed, the text talks about 'equalsides' continued until one of them is made to 'touch' the other one. This is the first time in a published mathematical cuneiform text that the term ba.si (literal meaning something like 'it is equal') is used for the equal sides of a symmetric trapezoid or a symmetric triangle. Normally the term (with this or related spellings) refers to the equal sides of a square.

The word *in-ne-mi-id* is an inflected form of the N-stem of *emēdu* 'to lean on', which has the meaning 'to get together', 'to converge'. Among previously published mathematical cuneiform texts, forms of *emēdu* appear in only two known cases. One is the geometric catalog text BM 15285 (Robson, *MMTC* (1999), App. 2; Friberg, *AT* (2007), 126-133), where, for instance, diagram # 7 shows a small square touching a larger square at the four midpoints of its sides. The accompanying text states that

íb.sá *ša ad-du-ú mi-it-ḫa-ar-tam ki-di-tam i-mi-id*	The equalside that I drew touched the outer equalside.

The other case of a known appearance of the term *emēdu* is in the catalog text Str. 364 (Neugebauer, *MKT I* (1935), 248; Friberg, *AT* (2007), Sec. 11.2), in the badly preserved question # 6, concerning a striped triangle, where the text says

[x] *al-li-ik-ma* e *e-mi-id*	I walked [x], and I reached a levee.

It may seem strange that the diagram in exercise # 3´ shows a 6-striped triangle and mentions 'the length of the trough', while the text of the same exercise mentions only a 2-striped triangle composed of a trapezoid and an attached triangle. There is no hint in the text that the 'trough' must actually be a three-dimensional object. A likely explanation for this strange mismatch is that exercise # 3´ and exercise # 2´ in the recombination text IM 95771 both were borrowed from a theme text (similar to Str. 364), in which a large number of exercises illustrated by diagrams were concerned with sub-divided triangular or trapezoidal 'troughs' (water reservoirs) or other objects. The one who borrowed exercise # 3´ from that theme text apparently inadvertently borrowed a diagram and an exercise text that did not belong together.

After exercise # 3´ follows a double-drawn line and the problematic phrase 24 *du-gi-i im-ta-ḫa-ar.* What it means is not at all clear. It may originally have been a catch-line at the end of a theme text, or a subscript like the ones in MS 3049 (Friberg, *MSCT 1* (2007), 299), which was copied without reason along with the text of exercise # 3´. It is similar in appearance to the equally problematic phrase 20 *qù-ut-ti im-ta-ḫa-ar*, at the beginning of the text of the exercise IM 97551 # 5´ (see below).

6.1.4 # 4´. A Few Lines of an Exercise about a Work Norm for the Spreading of Plaster

On the reverse of the fragment IM 95771, only insignificant parts of the end of the solution procedure of exercise # 4´ are preserved. Nevertheless, that is enough for a qualified guess about what the exercise was dealing with. Indeed, what is left of the last few lines of the exercise is the following:

IM 95771 # 4´

[*x x x x x*] *x-ka* 12 *x* [*x x x x*]
[*x x x x*] *x x si-ri-im* 1 lú *ka-la* u_4*-mi-im* [*i-se-er*] /
[*ki-a-am*] *ne-pé-*[*šum*]

[x x x x x] your x 12 x [x x x x]
[x x x x] x x of plaster 1 man for a whole day will spread.
[Such] is the proce[dure].

What is mentioned here appears to be a work norm for the spreading of plaster. The only other known mathematical cuneiform text dealing with this issue is IM 95858 = Haddad 104, exercises ## 7-8 (Al-Rawi and Roaf, *Sumer* 43 (1984)). For the readers' convenience, the texts of the exercises are reproduced below:

Haddad 104 # 7

1 *iš-ka-ar se-ri-im*
2 1 *ši-in-ni qa-ni-e mi-it-ḫa-ar-tam* / *ù* 10 *ú-ba-an mu-ba-am i-ša-ka-an-ma*
3 *ka-la* u_4*-mi-im* / [*i-se-er*]
4 *š*[*um*]*-ma* 1 *ši-in-ni qa-ni mi-it-ḫa-ar-tam* / *ù* 10 *ú-ba-an ú-ub-bi ti-di mi-nu-um*
5 *i-na e-pé-ši-* / *ka* /
6 10 *mu-bi si-ri-im šu-pí-il-ma* 2 *i-lí*
tu-úr-ma /
7-8 1 *mi-it-ḫa-ar-ti si-ri-im* [*š*]*u-ta-k-il-ma-a* / 1 a.šag$_4$ *i-lí*
a-na 2 *šu-up-lim i-ši-ma* 2 saḫar *i-lí* /
9 2 *a-na* 5 *mi-id-di-im i-ši-ma* 10 *i-lí*
10 2 gur *ti-da-am* / 1 lú *ka-la* u_4*-mi-im i-se-er*
ki-a-am ne-pé-šum

1 Work norm for spreading.
2 An equalside of 1, two reeds, / and a thickness of 10, a finger, he will set, then
3 a whole day / he will spread.
4 If a square of 1, two reeds, / and 10, a finger, it was thick, my plaster was what?
5 In your proceure: /
6 10, the thickness of the plaster, make deep, then 2 will come up.
Return, then /
7-8 1, the equalside of the plaster, let eat itself, then / 1, the field, will come up.
To 2, the depth, carry it, then 2, the volume, will come up. /
9 2 to 5, the *middum*, carry, then 10 will come up.
10 2 gur of plaster / 1 man will spread a whole day.
Such is the procedure.

A terminological difficulty in this text is the word *middum*, of unclear meaning. Mathematically, however, it is clear that the term is a mathematical constant with the value '5'. It stands for the number of "cubic sìla-measures” in a volume sar. Indeed, a “cubic sìla” is a convenient expression for a measuring vessel in the form of a cube with the side 6 fingers = ;01 rod. (See Friberg, *MSCT 1* (2007), 101.) Since a volume sar is equal to 1 square rod · 1 cubit, and 6 fingers = 1/5 cubit, it follows that

1 volume sar = 1 sq. rod · 1 cubit = 1 (00 00) · sq. (6 fingers) · 5 · (6 fingers) = 5 (00 00) cubic sìlas.

In the exercise above, the work norm for spreading plaster is given in the form

1 sq. rod · 1 finger of plaster spread per man-day.

(Here “man-day” is a convenient rephrasing of the exercise’s 1 lú *ka-la* u_4*-mi-im* ‘1 man a whole day’.)

The ‘making deep’ in line 6 of the exercise, where the thickness of the plaster is given first as ‘10’, then as ‘2’, can be explained as the following small calculation, where a nindan multiple (normally expressing a horizontal distance) is replaced by an equivalent cubit multiple (normally expressing a *vertical* distance):

1 finger = ;00 10 rods = ;02 cubit.

The ensuing calculation in lines 7-8 leads to the following alternative expression for the given work norm:

1 sq. rods · 1 finger per man-day = ;02 volume sar per man-day.

Through multiplication by the *middum* coefficient ‘5’, this expression for the work norm is replaced, in lines 9-10 of the exercise, by yet another alternative expression, namely

;02 volume sar per man-day = ;02 · 5 (00 00) cubic sìlas per man-day = 10 (00) sìlas per man-day = 2 gur per man-day.

(Maybe the word *middum* was derived from the verb *mâdum* ‘to be numerous’?)

The last few lines of Haddad 104 # 7 are obviously closely parallel to what is preserved of the last few lines of the badly damaged exercise IM 95771 # 4′. It is likely that if the latter exercise had been better preserved, it would have turned out to be a computation of alternative work norms for plastering, just like Haddad 104 # 7, or perhaps a related exercise, like Haddad 104 # 8 (below).

Indeed, Haddad 104 # 8 is a simple variation of the preceding exercise. Here is the text:

Haddad 104 # 8

1-2 *šum-ma* 40 *sa-ma-ni-na-ma-tim mu-li si-ri-im*10 *ú-ba-an* / *mu-bi-e si-ri-ia*
uš *e-se-ru ù ti-di mi-nu-um* /
3 *i-na e-pé-ši-ka*
4 1 sar *ti-dam ša ka-la* u_4*-mi-im* /1 lú *i-se-ru lu-pu-ut*
pa-ni 40 *mu-li pu-ṭur-ma* 1 30 *i-li* /
5 1 30 *a-na* 1 sar *ti-di-im i-ši-ma* 1 30 *i-lí*
6 1 30 *ša-*[*la-aš*] *qa-ni-e* / uš *ka-la* u_4*-mi te-se-er*
7 1 30 uš *a-na* 40 *mu-li* [*si-ri*]*-im* / *i-ši-ma* 1 sar a.šag$_4$ *i-lí*
8 1 *a-na* 2 *mu-bi si-ri-im i-ši-ma* / 2 saḫa*r i-lí*
2 *a-na* 5 *mi-id-di-im i-ši-ma* 10 *i-lí* /
9 10 gur *ti-da-am* 1 lú *ka-la* u_4*-mi-im i-se-er*

1-2 If 40, of 8 cubits, is the height of the spreading, 10, a finger, / the thickness of my spreading,
the length I spread and my plaster are what? /
3 In your procedure:
4 1 sar, the clay that the whole day / 1 man spread, touch.
The reciprocal of 40, the height, release, then 1 30 will come up. /
5 1 30 to 1 sar of plaster carry, then 1 30 will come up. /
6 1 30, of th[re]e reeds, / is the length you spread a whole day.
7 1 30, the length, to 40, the height of the [sprea]ding / carry, then 1 sar of field will come up.
8 1 to 2, the thickness of the spreading, carry, then / 2, the volume, will come up.
2 to 5, the *middum*, carry, then 10 will come up. /
9 10, gur, is the plaster 1 man spreads the whole day.

This exercise is an example of the application of the mentioned work norm for spreading plaster, apparently on the wall of a building, since the 'height' of the plastering is mentioned. It is (silently) assumed that a man is spreading plaster for a whole day on a rectangular wall. The height of the wall is ;40 rods = 8 cubits, expressed in the form '40 *samaninammatim*', and the thickness of the plaster is ;10 rods = 1 finger. The solution procedure begins, in lines 3-4 by mentioning, somewhat obliquely, that the work norm for spreading plaster is 1 area sar per man-day (for the given thickness of 1 finger). Then the length of the rectangle covered by plaster is computed as 1 area sar / ;40 (rods) = 1 sq. rod / ;40 (rods) = 1;30 (rods). This result is explained as 'three reeds', which is correct, since 1 rod = 2 reeds.

In the second half of the solution procedure, in lines 6-9, the amount of plaster spread on the wall is computed, to begin with as the volume 1;30 (rods) · ;40 (rods) · ;02 cubit = ;02 volume sar. Next, as in the preceding exercise, this volume is multiplied by the *middum* constant '5', which gives the result '10', in relative place value notation. In order to specify that '10' here means 10 *sixties*, the result is presented in the form '10 gur', which means " '10' in the region of gur-measures" and is not to be interpreted as 10 gur but as 2 gur! (In the preceding exercises, the result was explicitly given in the form '2 gur'.) As is well known, this somewhat strange and ambiguous way of specifying the order of magnitude of a number written in relative sexagesimal place value notation is not uncommon in Old Babylonian mathematical texts.

6.1.5 # 5′. A Figure Composed of a Rectangle and a Symmetric Triangle

The partly preserved text of exercise # 4′ on the fragment IM 95771 is preceded by a diagram which appears to show a geometric figure composed of a rectangle with the sides 20, 10 and a symmetric triangle with the sides 20, 13 20, 13 20.

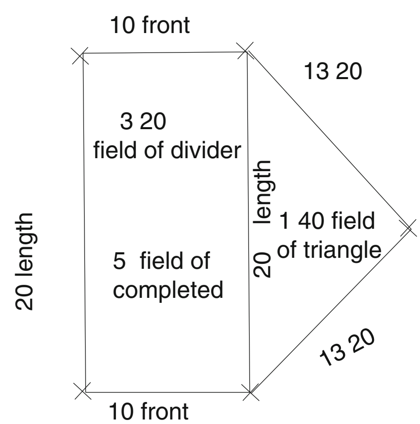


Fig. 6.1.3. IM 95771. The geometric diagram in exercise # 4′.

Inside the rectangle are inscribed two area notations. One is 3 20 a.šag$_4$ bar 'field' (area) of 'divider' (meaning 'diagonal', in a *pars pro toto* construction standing for 'rectangle'). Anyway, since 3 20 = 20 · 10, 3 20 is what would normally be the area of the rectangle. The other area notation is 5 a.šag$_4$ *qù-ut-tim* '5 is the field (area) of the completed (figure) ', obviously referring to the figure composed of a rectangle and a diagonal.

Inside the triangle there is a single area notation 1 40 a.šag$_4$ sag.kak '1 40 is the field (area) of the peg-head (triangle)'. This is difficult to explain. Indeed, if 1 40 is the area of a symmetric triangle with the 'front' 20, then the 'length' (actually, the height) of the triangle must be 10, and the length of the sloping side of the triangle should be

10 · sqs. 2 = appr. 10 · 1;25 = 14;10.

In this small computation the fairly accurate approximation sqs. 2 = 1;25 (= 17/12) is used. However, the author of this exercise seems to have counted instead with the very bad approximation sqs. 2 = appr. 1;20 (= 16/12). Indeed,

10 · 1;20 = 13;20.

As will be shown below, the mistake appears to be repeated in the text of the exercise.

If the area of the rectangle is 3 20 and the area of the triangle is 1 40, as indicated in the diagram on the clay tablet, then the total area of the composite figure is 3 20 + 1 40 = 5. Therefore the phrase 5 a.šag$_4$ *qù-ut-tim* can clearly be explained as meaning '5 is the area of the completed (= whole) figure'.

Here is what is preserved of the text of this exercise. In spite of the appearance, it is possible that not much is missing. Indeed, after the partly preserved fifth line of the text there may have been only one additional line of the exercise.

IM 95771 # 5´

1 20 *qù-ut-ti im-ta-ḫa-ar*
a.šag$_4$ *mi-nu-um*
i-na e-pé-[*ši-ka*]
2 [*x x x*] / 20 uš *a-na* 10 sag *i-ši-ma* 3 20 a.šag$_4$ *na-al-ba-ni-*[*im*]
3 [a.šag$_4$ sag.kak] / *šu-li*
20 sag.ki sag.kak an.na $^1/_2$ *ḫe-pé-ma* 10 *i-lí*
4 [*pa-ni* 1 20 x x x] / *pu-ṭur-ma* 45 *i-li*
45 *a-na* 13 20 uš sag.kak k[i.ta *i-ši-ma* 10 *i-li*] /
5 10 *a-na* 10 *i-ši-ma* 1 40 a.šag$_4$ sag.kak *x x i-lí*
[*x x x x x x x x x x*] /
... [*x x x x x x x x x x x x x x x x x x x x*]

1 20 my completed$^?$ (figure) was equalsided.
The field (area) was what?
In your proc[edure]:
2 [x x x] / 20, the length, to 10, the front, carry, then 3 20, the field of the brick mold (rectangle).
3 [The field of the peg head (triangle)] / let come up.
20, the upper front of the peg head, 1/2 break, 10 will come up.
4 [The reciprocal of 1 20 x x x] / release, then 45 will come up.
45 to 13 20, the lo[wer] length of the peg head [carry, then 10 will come up]. /
5 10 to 10 carry, then 1 40, the field of the peg head x x will come up.
[x x x x x x x x x x] /
... [x x x x x x x x x x x x x x x x x x x x]

According to the statement in the first line of the exercise, the *quttu* is 'equalsided' with 20 as the length of the equalside. Apparently, this specification should be interpreted as saying that the length of the rectangle and the height of the *quttum* should both be equal to 20. This, in its turn, implies that the front of the rectangle and that the height of the triangle should both be equal to 10. (It happens occasionally in Old Babylonian mathematical texts that that the values of "unknown" parameters are mentioned prematurely.)

The solution procedure starts in line 2 by calculating

the area of the 'brick mold' = the length · the front = 20 · 10 = 3 20.

Therefore, *nalbanu* 'brick mold' is a previously unknown Akkadian word for 'rectangle'! Sure enough, most Babylonian brick molds were rectangular, and a rectangle would be a fitting image of a brick mold. (For other likely attestations of 'brick mold' as a word for 'rectangle', see below Sec. 6.2.4 (MS 3049), Sec. 8.5.10 (BM 85196 and BM 85210), and Sec. 10.1.3 (IM 52916).)

Next, the area of the triangle, as usual called called 'peg head', should be found. According to the diagram, the common length of the two sloping sides of the triangle is 13 20. By mistake, this parameter was not specified in the statement of the problem. Anyway, the requested area is calculated as

the area of the peg head = 20 · 1/2 rec. 1 20 · 13 20 = 10 · 45 · 13 20 = 10 · 10 = 1 40.

Clearly, this implies that 1 20 was used here as a (very bad) approximation to sqs. 2. (A much better approximation would have been 1 25. See the discussion of various known Babylonian square-side approximations in Friberg, *BaM* 28 (1997), Sec. 8.) With this approximation to sqs. 2 the height of the triangle was correctly calculated as the length of the sloping side, divided by sqs. 2.

It is likely that what was calculated in the missing last line of the text of exercise # 5´ was

the area of the *quttum* = the area of the brick mold + the area of the peg head = 3 20 + 1 40 = 5.

This result is explicitly indicated in the diagram preceding the exercise.

6.1.6 # 6´. A Broken Table of Constants

The badly broken col. *iv* of IM95771 seems to have been inscribed with at least 20 constants (*igigubbû*). Less than half that number are relatively well preserved and are listed below:

IM 95771 # 6´

1 07 30	*i-gi-gu-bé-e*	[*x-x-x*]*-im* /
[x x]	*i-gi-gu-bé-e*	*na-ak-ki-im*
6 40	*i-gi-gu-bé-e*	*ka-ar mi-*[*it*]*-ḫa-ar-tim* /
[x x]	*i-gi-gu-bé-e*	*ka-ar ki-pa-tim*
6 40	*i-gi-gu-bé-e* /	[*x x x x*] *x x*
4 48	*i-gi-gu-bé-e*	*ka-ar ki-pa-tim* /
[x x]	[*i-gi-gu-bé-e*]	[*x x x x x x*]
1 12	*i-gi-gu-bé-e*	*wa-ri-im* /
[x x]	[*i-gi-gu-bé-e*]	[*x x x x x x*]
[x x]	[*i-gi-gu-bé-e*]	*a-na-ki-im* /

The missing numbers in this brief table of constants, together with a total lack of context, makes it difficult to make much sense of it. The words *na-ak-ki-im*, *wa-ri-im*, and *a-na-ki-im* could not be found in the Akkadian dictionary. However, the phrases *i-gi-gu-bé-e ka-ar mi-it-ḫa-ar-tim* and *i-gi-gu-bé-e ka-ar ki-pa-tim* can be translated as 'the constant for a square grain-store' and 'the constant for a circular grain-store'.

Another approach to this broken table of constants may be to compare the numbers appearing in it with numbers appearing in other, earlier published Old Babylonian mathematical tables of constants. This approach leads to the following observation: In the mathematical table of constants NSd = YBC 5022 (Neugebauer and Sachs, *MCT* (1945), text Ud), two entries with the constants 6 40 and 4 48 refer to the same 'sealed storage house for grain':

6 40	é.kišib še	NSd 30
4 48	é.kišib še	NSd 61

So, maybe also the constants listed in col. *iv* of IM 95771 are in some way associated with storage houses, both square and circular.

6.1.7 The Vocabulary of IM 95771

an.na	upper
A.SUG (*buginnu*)	trough, water reservoir
bar	divider (rectangle, rectangle diagonal)
ba.si	equalside(?)
íb.tak$_4$	remainder
KI	?
sag.kak	peg head? , triangle
sag.ki	forehead, front
saḫar	dirt, mud, volume

1 lú *ka-la* u_4*-mi-im*	1 man for a whole day, man-day
i-na e-pé-ši-ka	in your procedure (*epēšu* action)
ki ma-ṣí	*kī maṣi* how much? (correspondingly)
ki-a-am ne-pé-šu	such is the procedure (*nēpešu* procedure)
pa-ni (*n*) *pu-ṭú-ur*	compute rec.*n* (*pānum* front)
a-na-ki-im	?
du-ri-im	*dūrum* (city) wall, rampart
e-li	*eli* over, above
e-lu-um	*elû* upper
[*er*]*-bi-na-ma-tim*	*erbinammatim* four-in-cubits
qù-ut-ti	completed[?] (figure)
ḫa-mi-iš-na-ma-tim	*ḫamišnammatim* five-in-cubits
ka-li-šu	*kališu* all of it
ka-ar	*karû* grain-store
ki-pa-tim	*kippatu* circle
mi-nam, mi-nu-um	*mīnu* what?
mi-[*it*]*-ḫa-ar-tim*	*mitḫartu* equalside, square
na-ak-ki-im	?
na-al-ba-ni-[im]	*nalbanu* brick mold, molding number, rectangle
pí-ša	*pû* mouth
ru-up-šum, ru-up-ši-im	*rupšu* width
si-ri-im	*sīru* plaster
ša-ap-li-im	*šaplû* lower
ša-ap-la-nu-um	*šaplānu* beneath, under
šu-up-lum	*šuplu* lower
šu-ši	*šūši* sixty
ta-<al>-lu-ia	*tallu* beam, transversal
u_4*-mi-im*	*ūmu* day
wa-ri-im	?
zi-ba-ti-ša	*zibbatu* tail, tail-end
te-eb-ḫi	< *ebēḫu* to gird, to surround
i-lí, i-li, šu-li	< *elû* to come up, to rise (= to result)
in-ne-mi-id	< *emēdu N* get together, combine, touch
ḫu-ru-uṣ	< *ḫarāṣu* to break off
ḫe-pé	< *ḫepû* to break
ta-ak-su-sa-am	< *kasāsu* to chew, to gnaw, to encroach on
im-ta-ḫa-ar	< *maḫāru* Gt become equal to each other, become equalsided
i-ši	< *našû* to carry (= to multiply (by a number))
pu-ur-sa-am	< *parāsu* to cut off, to section
pu-ṭú-ur	< *paṭāru* to loosen, to release
tu-úr-da-am	< *tāru* to return (= start again)
ú-ri-id, tu-ri-id	< *warādu* to go down
i-ter	< *watāru* to exceed

6.1.8 IM 95771. Hand Copies and Conform Transliterations of the Fragment

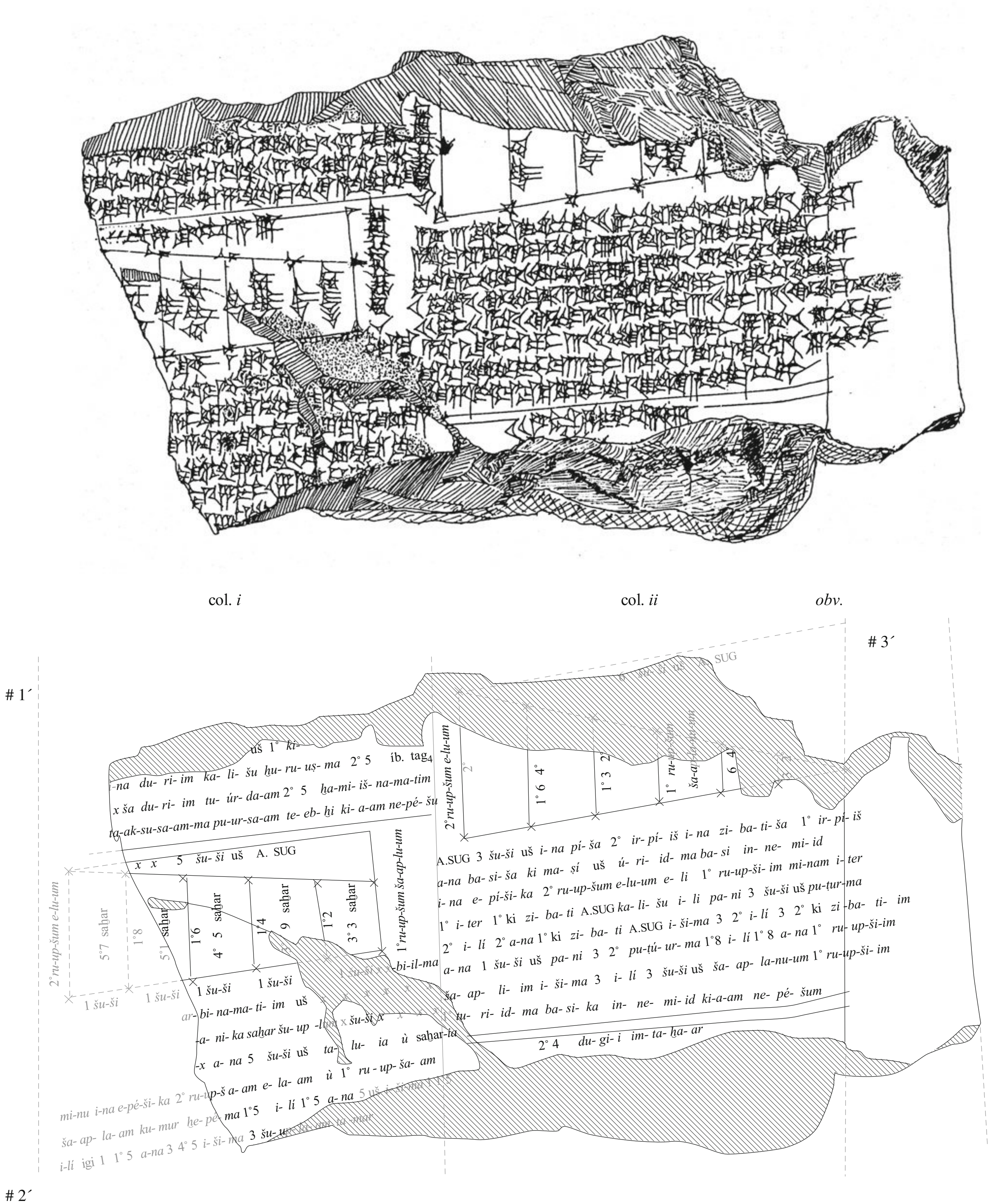


Fig. 6.1.4. IM 95771, *obv.* A hand copy and a conform transliteration.

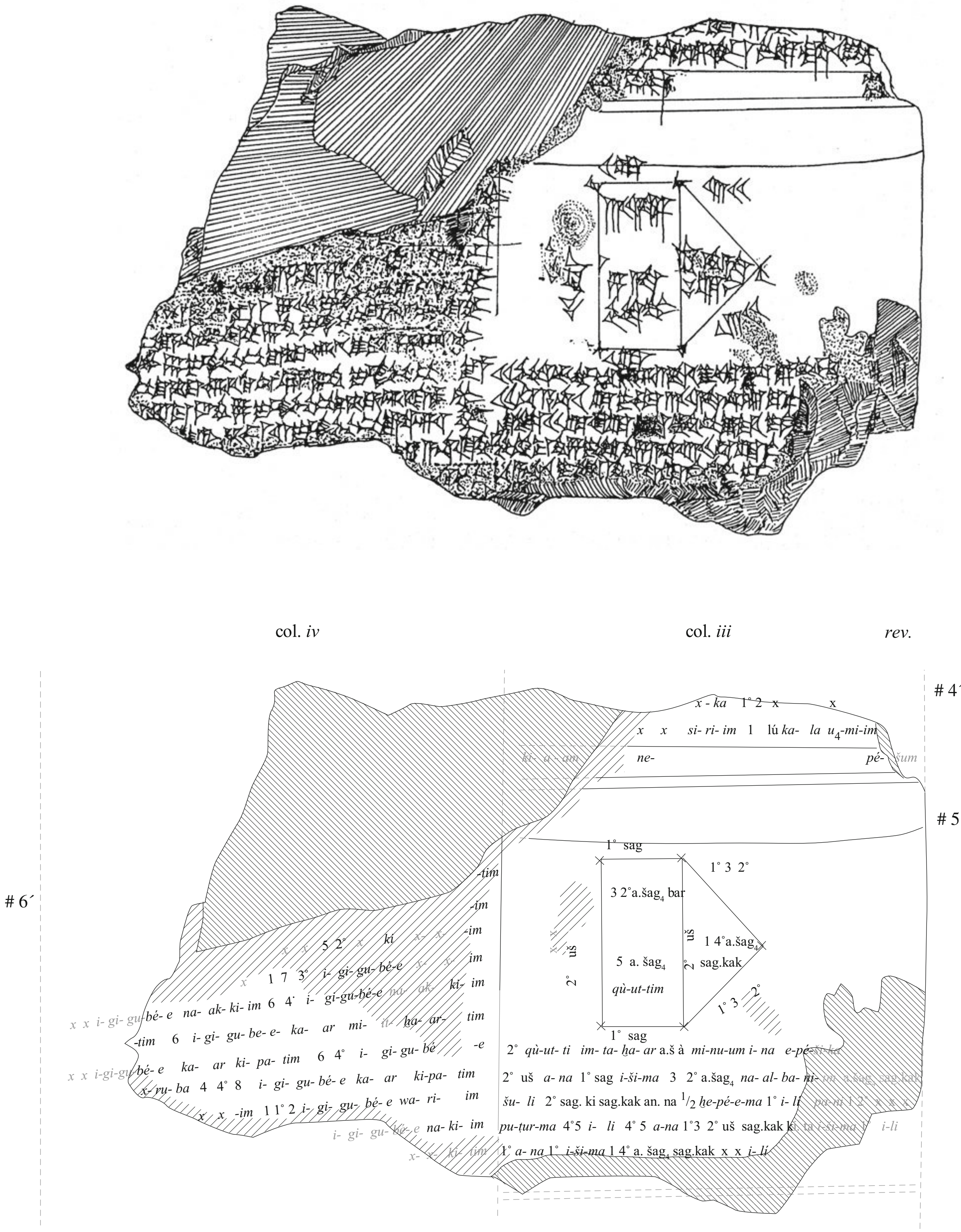


Fig. 6.1.5. IM 95771, *rev.* A hand copy and a conform transliteration.

6.2 IM 121565. A Large Recombination Text with Metric Algebra Problems for Square and Quadrilateral Fields

6.2.1 §§ 1 a-b. Two Quadratic Problems of the Basic Types B4a and B4b

IM 121565 § 1 a (col. *i,* 1-5)

1 *a-na* a.šag$_4$ lagab-*ti-ia* 1 uš daḫ-*ma* 1 50 a.šag$_4$ *ab-ni* /
2 lagab-*ti ki ma-ṣi*
3 za.e 1 *le-qé ḫe-pé šu-ta-ki-il-ma* / 15 *ta-mar*
4 15 *a-na* šag$_4$ 1 50 a.šag$_4$ daḫ-*ma* 1 50 15 / *ta-mar*
ba.sá.e 1 50 15 *šu-li-ma* 10 30 *ta-m*[*ar*] /
5-6 *i-na* šag$_4$ 10 30 30 *ta-ki-il-ta-ka* zi / 10 *ta-mar*
10 lagab

1 To the field (area) of my equalside 1 length I added on, then 1 50 was the field I built. /
2 My equalside was how much?
3 You: 1 take, break, let eat each other, then / 15 you will see.
4 15 onto 1 50 the field add on, then 1 50 15 you will see.
Let the equalside come up, then 10 30 you will [see].
5-6 Out of 10 30 30, your holder, tear off, / 10 you will see.
10 is the equalside.

The stated problem in this exercise can be expressed, in quasi-modern symbolic notations, as a quadratic equation of the following kind:

sq. $s + 1 \cdot s = 1\ 50$.

This is a quadratic equation of the basic type B4a for the unknown *s.* (See Friberg, *AT* (2007), 6). Earlier known examples of quadratic problems of this type are BM 13901 § 1 a (*op. cit.*, 36), and MS 5112 § 1 (Friberg, *MSCT 1* (2007), 309). See also the discussion in Ch. 5 above of various “form and magnitude problems” in texts from Old Babylonian Mê-Turran. Problems of this kind could be solved using a geometric completion of the square. In quasi-modern terms, the solution procedure can be explained as follows:

If sq. $s + q \cdot s = A$, then $A +$ sq. $q/2 =$ sq. $(s + q/2)$, so that $s =$ sqs. $(A +$ sq. $q/2) - q/2$.

(Here, as in MS 121613 # 2, in Sec. 5.1.2 above, $q/2$ is called *takiltum* ‘holder (of your head)’.) With $q = 1$ and $A = 1\ 50$, as in this exercise, these calculations take the following form:

$A +$ sq. $q/2 =$ 1 50;15, $s =$ sqs. 1 50;15 – ;30 = 10;30 – ;30 = 10.

IM 121565 § 1 b (col. *i,* 7-12)

1-2 *i-na* a.šag$_4$ lagab-*ti-ia* 1 uš zi.zi-*ma* / 1 30 a.šag$_4$ *ab-ni*
lagab-*ti ki ma-*⸢*ṣi*⸣
3 za.⸢e⸣ 1 *le-qé* / ⸢*ḫe-pé*⸣ *šu-ta-ki-il-ma* 15 *ta-mar*
4 [15 *a-n*]*a* šag$_4$ / 1 30 a.šag$_4$ daḫ-*ma* 1 30 15 *ta-mar*
5 ba.sá.[e 1] 30 15 / *šu-li-ma* 9 30 *ta-mar*
6 *a-na* šag$_4$ [9 30] 30 / *ta*⸢*ki-il*⸣*-ta-ka* daḫ 10 *ta-mar*
10 lagab

1-2 Out of the field of my equalside 1 length 1 tore off, then / 1 30 was the field I built.
My equalside was how much?
3 You: 1 take, / break, let eat each other, then 15 you will see.
4 [15 ont]o / 1 30, the field, add on, then 1 30 15 you will see.
5 Let the square-si[de of 1] 30 15 / come up, then 9 30 you will see.
6 Onto [9 30] 30 / your holder add on, 10 you will see.
10 is the square-side.

The stated problem in this exercise can be expressed, in quasi-modern terms, as a quadratic equation of the following kind:

sq. $s - 1 \cdot s = 1\ 50$.

This is a quadratic equation of the basic type B4b for the unknown *s*. (See Friberg, *AT* (2007), 6). An earlier known example of a quadratic problem of this type is BM 13901 § 1 b (*op. cit.*, 38). Problems of this kind, too, could be solved using a geometric completion of the square. In quasi-modern symbolic notations, the solution procedure can be explained as follows:

If sq. $s - q \cdot s = A$, then $A +$ sq. $q/2 =$ sq. $(s - q/2)$, so that $s =$ sqs. $(A +$ sq. $q/2) + q/2$.

With $q = 1$ and $A = 1\ 30$, as in this exercise, these calculations take the following form:

$A +$ sq. $q/2 =$ 1 30;15, $s =$ sqs. 1 30;15 + ;30 = 9;30 + ;30 = 10.

6.2.2 *§§ 2 a-b. Two Severely Damaged Quadratic Problems*

IM 121565 § 2 a (col. *i,* 12-22)

1 [*x x x x x x x x x x*]
2 [*x x x x x x x*]-*ma* daḫ-*ma* /
3 [*x x x x x x x x*] *x* lagab /
4 [*x x x x x x šu-ta*]-*ki-il-ma* /
5 [*x x x x x x x x x x*] *x* /
6 1 *x* [*x x x x x x x x*] *x* /
7 1 [*x x x* ba.sá.e 1]45 03 45 /
8 *šu*-[*li-ma* 10 15 *ta-mar*] *i-na* šag$_4$ 10 15 /
9 15 [*ta-ki-il*]-*ta-ka ḫu-ru-iṣ-ma* 10 *ta-mar* /

The text of this exercise is so severely damaged that it has not been possible to reconstruct more than the last few lines of text. However, that is enough to venture the conjecture that the quadratic problem stated in this exercise, expressed in terms of quasi-modern symbolic notations, was the following quadratic equation:

sq. $s + 1/2\ s = 1\ 45$.

Indeed, this equation would have a solution that could be computed as follows:

1 45 + sq. (1/2 · 1/2) = 1 45 + sq. ;15 = 1 45;03 45,
$s =$ sqs. 1 45;03 45 – ;15 = 10;15 – ;15 = 10.

However, a difficulty with this reconstruction of the meaning of the damaged exercise is that it does not explain why this exercise seems to contain 9 lines of text while the two preceding exercises contain only 6 lines each. (Maybe some pieces of the fractured tablet have not been put back in the right places?)

IM 121565 § 2 b (col. *i,* 23-30)

1 [*i-na* a.šag$_4$] lagab-*ti-ia* 1 *x x* uš-*ia* zi.zi /
2 [*x x x x*] lagab-*ti ki-ia* lagab za.e 1 uš *le*-⸢*qé*⸣ /
3 [*ḫe-pé šu-ta-ki*]-*il-ma* 15 *ta-mar* 40 15 /
4 [*x x x x*] 3 15 *ta-mar* 3 *ù* 15 /
5 [*x x x x x x*] *x x* 3 45 *ta-mar* /
6 *x x* [*x x x x x x*]-*ma* ⸢9 45⸣ 15 /
7 *ta-mar* [*a-na* šag$_4$ 9 45 15 *ta*]-*ki-il*<-*ta*>-*ka* /
8 daḫ 10 *ta-mar* 10 lagab

The text of this exercise, too, is severely damaged. No reconstruction of the text seems to be possible.

6.2.3 *§§ 3 a-b. Two Quadratic Problems for Two Equalsides (Squares)*

IM 121565 § 3 a (col. *i,* 31-38)

1 *uš-tam-ḫi*-[*ir*]-*ma – ma-la uš-tam*-⸢*ḫi-ru ul i-de*⸣ /
2-3 [*a-tu-ú*]*r* ⸢1/2⸣ uš *eḫ*!-*pé uš-ta-ki-il a-na* šag$_4$ / [a.šag$_4$ da]ḫ-*ma* 2 05 a.šag$_4$ *ab-ni*
lagab-*ti ki-ia* lagab /
4-5 [za.e] 1 *ù* 1 *lu-pu-ut* 1 uš ki.ta *ḫe-pé-e* / [*šu-ta-k*]*i-il-ma* 15 *ta-mar*
6 15 *a-na* šag$_4$ 1 uš / [an.t]a daḫ-*ma* 1 15 *ta-mar*
igi 1 15 duḫ 48 *ta-mar* /
7 *tu-úr* 48 *a-na* 2 05 *i-ta-aš-ši-ma* 1 40 *ta-mar* /
8 ba.sá.e 1 40 *šu-li-ma* 10 *ta-mar*
10 lagab

1 I made equalsided, but – whatever I made equalsided I did not know. /
2-3 [I retur]ned. ⸢1/2⸣ the length I broke, I let eat itself, onto / [the field I add]ed on, then 2 05 was the field I built.
My equalside, how much was it equalsided? /
4-5 [You]: 1 and 1 touch, 1 the lower length break / [let e]at itself, then 15 you will see.
6 15 onto 1 the / [upp]er length add on, then 1 15 you will see.
The reciprocal of 1 15 release, 48 you will see. /
7 Return. 48 to 2 05 always carry, then 1 40 you will see.
8 The equalside of 1 40 let come up, then 10 you will see.
10 is the equalside.

In this exercise, a square of unknown side length is combined with a second ('lower') square of half that unknown side length. The result is a figure with the total area 2 05. What was then the side length of the first ('upper') square? In quasi-modern symbolic notations, the problem can be presented as the following *homogeneous* quadratic equation, that is a quadratic equation without a linear term:

sq. s + sq. (1/2 s) = 2 05.

The solution is obtained through a routine application of the Old Babylonian "rule of false value", which can be used to solve all kinds of homogenous equations. If s is given the false value 1 (00), times an unknown reed r, then 1/2 s gets the corresponding false value 30, times the same unknown reed, and the combined false area becomes

sq. 1 (00) r + sq. 30 r = 1 15 (00) sq. r.

Therefore, in relative place value notation,

1 15 sq. r = 2 05, and sq. r = rec. 1 15 · 2 05 = 48 · 2 05 = 1 40, so that r = 10.

Therefore,

s = 1 (00) · r = 10.

It is easy to check that 10 is the correct answer also in absolute place value notation, since

sq. 10 + sq. (1/2 · 10) = 1 40 + 25 = 2 05.

An earlier published Old Babylonian exercise dealing with precisely the same problem (in a somewhat different mathematical terminology) is the Larsa text MS 3299 # 3 in Friberg and George, *PGS* (2010), 130. There the question is phrased in the following way:

[*mi-it-ḫar-ti*]? *a-na* 2 *eḫ-pe uš-ta-ki-il a-na* šag$_4$ a.šag$_4$ daḫ-*ma* 2 05 /
[*mi-it-ḫar-ti k*]*i-ia-a im-ta-ḫar*

[My equalside]? in 2 I broke, I let eat itself, onto the field I added on, then 2 05.
[My equalside h]ow much was it equalsided?

Note how the curious phrase lagab-*ti ki-ia* lagab in IM 121565 § 2 a corresponds to the more understandable phrase [*mi-it-ḫar-ti*] *ki-ia-a im-ta-ḫar* in MS 3299 # 3.

Another Old Babylonian exercise of a similar kind is BM 13901, obv. *ii*, 36-43, another text from Larsa (Neugebauer, *MKT 3* (1937), 3). There the question is phrased in the following, less obscure way:

a.šag$_4$ *ši-ta mi-it-ḫa-ra-ti-ia ak-mur-ma* 28 20 /
mi-it-ḫar-tum ra-bi-a-at mi-it-ḫa-ar-tim

The fields of my two equalsides I heaped, then 28 20.
equalside was a fourth of equalside.

IM 121565 § 3 b (col. *i*, 39-47)

1 *uš-tam-ḫi-ir-ma* — *ma-la uš-tam-ḫi-ru ul i-*⸢*de*⸣ /
2-3 *a-tu-úr* $^1/_2$ uš *ḫe-pé-e uš-ta-ki-il-*⸢*ma i-na*⸣ / šag$_4$ a.šag$_4$-*ia* zi.zi-*ma* 1 15 a.šag$_4$ *ab-ni*
4 lagab *ki-ia* / lagab
5 za.e 1 *ù* 1 *lu-pu-ut* 1 uš *le-qé* 1/2.ta / *ḫe-pé šu-ta-ki-il-ma* 15 *ta-mar*
6 15 *i-na* / uš an.ta *ḫu-ru-iṣ-ma* 45 *ta-mar*
7 igi 45 duḫ / 1 20 *ta-mar*

8 *tu-úr-ma* 1 20 *a-na* 1 15 *i-ta-aš-ši-ma* / 1 40 *ta-mar*
9 ba.sá.e 1 40 *šu-li-ma* / 10 *ta-mar*
10 lagab

1 I made equalsided, but — whatever I made equalsided I did not know. /.
2-3 I returned. 1/2 the length broken I let eat each other, then out of / the field I tore off, then 1 15 was the field I built.
4 My equalside, how much was it equalsided?
You: 1 and 1 touch, 1, the length take, 1/2 of it / break, let eat each other, then 15 you will see.
6 15 from / the upper length break off, then 45 you see.
7 The reciprocal of 45 release, / 1 20 you will see.
8 Return, then 1 20 to 1 15 always carry, then / 1 40 you will see.
9 The equalside of 1 40 let come up, then / 10 you will see.

In quasi-modern symbolic notations, this problem can be presented as the following homogeneous quadratic equation:

sq. s – sq. (1/2 s) = 1 15.

The solution s = 10 is again obtained through a routine application of the “rule of false value”.

6.2.4 A Summary

Below the 6 exercises in the first column of IM 121565 there is an intended summary of the contents of that column, in the form of the following subscript:

mu.bi	6 uš zi.zi /	Their names:	6 lengths torn off /
	uš daḫ		lengths added on.

The odd position of this subscript, at the end of the first column rather than, as one would have expected, at the end of the last column, can be explained as follows. IM 121565 is clearly a mathematical recombination text, by which is meant a more or less systematically arranged text containing copies of selected exercises from a number of older mathematical texts. Therefore, it is likely that all the six exercises in the first column of IM 121565 were copies of some part of an older mathematical cuneiform text containing precisely these six exercises, followed by the subscript.

The subscript is interesting because it seems to refer to names used in Old Babylonian schools for quadratic equations of the two types

sq. $s + q \cdot s = A$ uš daḫ
sq. $s - q \cdot s = A$ uš zi.zi

However, the subscript seems to be partly mistaken, since only the two exercises in §§ 1 a-b, and possibly also the damaged exercises in §§ 1 c-d, are of these types. The exercises in §§ 2 a-b are not of the same kind.

Similar summaries in subscripts can be found in (at least) the following four Old Babylonian mathematical texts:

MS 3049 (Friberg, *MSCT* 1 (2007), 299):

mu.bi	6 gúr.meš /	Their names:	6 arcs (problems for circles) /
mu.bi	5 lagab-*ša* /	Their names:	6 equalsides (problems for squares) /
	1 sag.kak /		1 peg head (problem for a triangle) /
	3 *na-al-ba-tum* /		3 brick molds (problems for rectangles) /
	1 šag$_4$.bar ká		1 inner divider (problem for a diagonal in a rectangular gate in a wall)

MS 3052 (*op. cit.*, 278):

[x x]	[5] im.dù.a	[x x]:	5 mud walls (problems for triangular or trapezoidal walls)
	1 *ṣí-li-ip-tum*		1 divider (a problem for a rectangle with a diagonal)
	1 ki.lá		1 excavation (a problem for a rectangular block)
	1 íb.sá		1 equalside (a problem for a square)

IM 121613 (Sec. 5.1.20 above):

[x x]	[x 2/3] uš sag.ki	[x x]:	[x] [2/3] of the length the front (problems for rectangles of a given form)

***TMS* V** (Sec. 8.9 below):

4 22 mu.bi lagab.meš	4 22 their names:	equalsides (problems for squares)

6.2.5 *§§ 4 a-g. Seven Quadratic Problems for Two Concentric Equalsides (Squares)*

IM 121565 § 4 a (col. *ii*, 1-6)

1 [*x x x x x x x x x x*] *x x x* /
2 [*x x x x x x x x x x x x x*] /
3 [*x x x x x x x x x x x x x*] /
4 [*x x x x x ḫu-ru*]*-iṣ-ma* 15 *ta-mar* /
5 15 *x x* [*x x*] *ta-mar* /

So much is lost of the text of this exercise that no reconstruction of the text is possible.

IM 121565 § 4 b (col. *ii*, 7-13)

1 lagab *i-na* šag$_4$ lagab *a*[*d-di-ma* 5.ta].àm *at-ta-sa-am* /
2 6 40 a.šag$_4$ lagab *ki-*[*di-tim*]
3 lagab *qer-bi-tum* / *mi-nu*
4 za.e ba.sá.e [6 40 a.šag$_4$ *šu-li*]*-ma* / 20 [*ta-mar*]
5 *x x x x x x x* [*x x x*] / *x x* [*x x x*]
6 *tu-úr-ma* 5 *ša* [*ta*]*-ta-sa-am* / *a-na* 2 *e-ṣi-ma*
a-na ki-di-tim x qer-bi-tum /
7 x x x lagab x x x x *-tum*

1 An equalside into an equalside I in[scribed, then 5 eac]h way I went away everywhere. /
2 6 40 was the field of the outer equalside.
3 The inner equalside /was what?
4 You: The equalside of [6 40, the field, let come up], then / 20 [you will see].
5 [*x x x x x x*] *x x x* / *x x* [*x x x*]
6 Return, then 5 that you went away everywhere / to 2 repeat, then
to the outer <equalside> x the inner
7 x x x equalside x x x x x

In this exercise, one square is drawn inside another, so that the sides of the two squares are parallel and there is a constant distance between the parallel sides. One may call the configuration, somewhat imprecisely, a pair of concentric squares. The phrase [5.ta].àm *at-ta-sa-am* '5 each way I went out' means that the constant given distance between the parallel sides is 5 (rods) = 30 meters, or perhaps ;05 (rods) = 1 cubit = 1/2 meter. In addition, it is given that the area of the outer square is 6 40, or perhaps ;06 40 (sq. rods).

The stated problem is exceedingly simple. The solution procedure begins in lines 3-4 by computing the side of the outer square as sqs. 6 40 = 20. What then happens is not clear, since the text is broken. However, in lines 5-6 the given constant distance 5 is doubled, probably in order to compute the side length of the inner square as the side length 20 of the outer square, minus 10. Unfortunately, the text is broken again, so it is again not quite clear what happens in the final part of the solution algorithm. Anyway, it is clear that the side of the inner square is 10.

IM 121565 § 4 c (col. *ii*, 14-21)

1 *x x x x x x x x* [*x x x*]*-ia* /
2 *x x x x x x x x x* [*x x*] *x* /
3 *x x* [*x x x*] *x x x* [*ta*]*-mar* /
4 *tu-úr* 8 20 *a-na* 6 *i-ši-ma* 50 *ta-mar* /
5 50 *ḫe-pé-ma* 25 *ta-mar* 25 [*re-eš-ka li-ki-il*] /
6 10 *wa-tu-ra-am ḫe-pé-ma* 5 *ta-mar x* [*x x*] /
7-8 *ta-ki-il-ta-ka a-na* 1 [daḫ*-ma i-na ša-ni-im*] / [*ḫu-ru*]-⸢*iṣ*⸣*-ma*
⸢30⸣ lagab *ki-di-tum*⸢20⸣ lagab ⸢*qer*⸣-[*bi-tum*]

1 *x x x x x x x x* [*x x x*] my /
2 *x x x x x x x x x* [*x x*] *x* /
3 *x x* [*x x x*] *x x x* [you will] see /
4 Return. 8 20 to 6 carry, then 50 you will see. /
5 50 break, then 25 you will see. 25 [let hold your head!] /
6 10, the excess, break, then 5 you will see *x* [*x x*] /
7-8 Your holder to 1 [add on, from the second] / [break off], then
⸢30⸣ is the outer equalside, ⸢20⸣ the in[ner] equalside.

Although nothing remains of the question in this exercise, it is clear this is another problem for a pair of concentric squares, since most of the solution procedure is preserved. Presumably, the solution procedure begins, in the lost line 3, with the computation of the reciprocal of the *watrum* ‘excess’ 10, which almost surely is the amount by which the outer square-side exceeds the inner square-side. In line 4, the number 8 20, apparently the difference between the areas of the two concentric squares, is multiplied by this reciprocal, rec. 10 = 6. The result is 8 20 · 6 = 50. This number, in its turn, is halved, and the new result, 1/2 · 50 = 25 is committed to memory. The Old Babylonian phrase for this action is is *rēška likil* ‘let it hold your head’.

Next, in line 6, the excess 10 is halved, 1/2 · 10 = 5, and then, in lines 7-8, this half excess is either added to or subtracted from the remembered number 25, called *takiltum* ‘that which holds (your head)’, here simply translated as the ‘holder’. The resulting two numbers, a = 30 and b = 20, are the side-lengths of the outer and inner squares.

What is going on here is that the stated, but lost, question can be reconstructed and expressed in quasi-modern symbolic notations as the following pair of equations for the two square-sides a and b:

sq. a – sq. b = 8 20, $a - b$ = 10.

This is a “subtractive quadratic-linear system of equations” of the basic type B3b. See Friberg, *AT* (2007), 6. A quadratic-linear system of equation of the basic type B3b can by solved easily through an application of the “conjugate rule”.

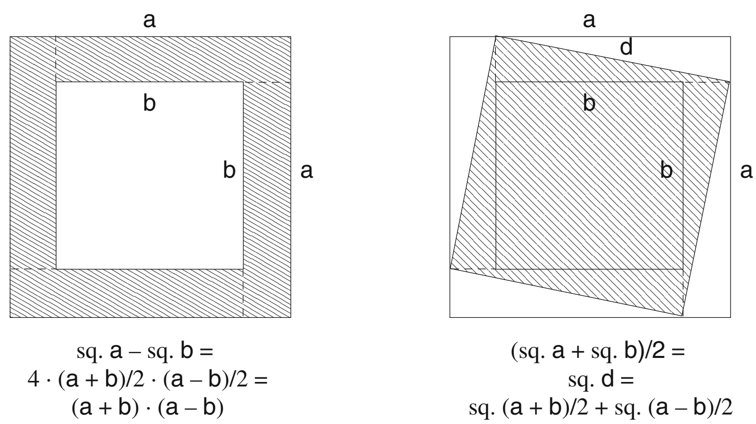


Fig. 6.2.1. IM 121565 § 4 d. Geometric derivations of the conjugate rule and the square sum rule.

The conjugate rule is often used in Old Babylonian mathematics, but it is not known how the rule was found, originally. Anyway, a simple demonstration of the rule can be based on a configuration like the one shown in Fig. 6.2.1, left, above. There, the space between two concentric squares is divided into *a chain of four rectangles*. (Compare with the drawing on the Old Babylonian hand tablet MS 2192 (Friberg, *MSCT* 1 (2007), 203), where the space between two “concentric” equilateral triangles is divided into a chain of three trapezoids.) Therefore, the difference between the areas of the two squares is equal to the combined area of the four rectangles. Now, if the two squares have the sides a and b, then it is easy to see that the four rectangles have the sides $(a + b)/2$ and $(a - b)/2$. Consequently,

sq. a – sq. b = $4 \cdot (a + b)/2 \cdot (a - b)/2 = (a + b) \cdot (a - b)$.

A solution procedure to the problem in IM 121565 § 4 c based on an application of this geometrically demonstrated conjugate rule would proceed as follows:

1/2 $(a + b)$ = 1/2 (sq. a – sq. b)/$(a - b)$ = 1/2 · 8 20 · rec. 10 = 1/2 · 50 = 25.
1/2 $(a - b)$ = 1/2 $(a - b)$ = 1/2 · 10 = 5,
a = 1/2 $(a + b)$ + 1/2 $(a - b)$ = 25 + 5 = 30, $b = (a + b)/2 - (a - b)/ 2$ = 25 – 5 = 20.

This agrees precisely with the preserved part of the solution procedure in IM 121565 § 4 c.

IM 121565 § 4 d (col. *ii*, 22-31)

1 a.šag$_4$ 2 lagab-*ti-ia ak-*⌜*mur-ma*⌝ 21 40
2 [lagab ugu lagab]$^?$ / 10 dirig
lagab *ki-di-tum ù qer-*<*bi*>*-tum mi-*[*nu*] /

3 za.e 21 40 *ḫe-pé-ma* 10 50 *ta-mar x* [*x x*] /
4 10 *wa-tu-ra-am ḫe-pé* 5 *ta-mar*
5 *x* [*x x x*] / *wa-tu-ra-am* [*le-qé*] *šu-ta-ki-il-ma* 2[5 *ta-mar*] /
6 25 *i-na* 10 50 *ḫu-ru-uṣ-ma* 10 [25 *ta-mar*] /
7 ba.sá.e 10 25 *šu-li-ma* 2[5 *ta-mar*] /
8-9 *me-eḫ-ra-am i-di* 5 *ta-ki-i*[*l-ta-ka*] / [*a-na* 1 da]*ḫ-ma i-na ša-ni-im* [zi.zi-*ma*]
10 [30] / lagab *ki*-[*di-tum*] 20 lagab *qer-bi-tum*

1 The fields of my two equalsides I heaped up, then 21 40.
2 [equalside over equalside][?] / 10 exceeded.
The outer and inner equalsides were wh[at]?
3 You: 21 40 break, then 10 50 you will see. *x* [*x x*] /
4 10, the excess, break, 5 you will see.
5 *x* [*x x x*] / the excess [take], let eat itself, then 2[5 you will see]. /
6 25 from 10 50 break of, then 10 [25 you will see]. /
7 The equalside of 10 25 let come up, then 2[5 you will see]. /
8-9 A copy lay down, 5, your hold[er] / [to 1 add] on, from the second [break off, then] /
10 [30] is the ou[ter] equalside, 20 the inner equalside.

In quasi-modern symbolic notations, the question in this exercise can be rephrased as the following pair of equations for the two square-sides a and b:

sq. a + sq. b = 21 40, $a - b = 10$.

This is a "subtractive quadratic-linear system of equations" of the basic type B2b. See Friberg, *AT* (2007), 6. The problem is known also from the Larsa text BM 13901 (Høyrup, *LWS* (2002), 68), where precisely the same question is expressed as follows, in a somewhat different terminology:

1 a.šag$_4$ *ši-ta mi-it-ḫa-ra-ti-ia ak-mur-ma* 21 4[0]
2 *mi-it-ḫar-tum* ugu *mi-it-ḫar-tim* 10 *i-te-er*

The successive steps of the solution procedure in IM 121565 § 4 d can be expressed as follows, in quasi-modern symbolic notations:

(sq. a + sq. b)/2 = 21 40 /2 = 10 50, $(a - b)/2 = 10/2 = 5$, sq. $(a - b)/2$ = sq. 5 = 25,
(sq. a + sq. b)/2 – sq. $(a - b)/2$ = 10 50 – 25 = 10 25, sqs. 10 25 = 25, $a = 25 + 5 = 30$, $b = 25 - 5 = 20$.

Clearly, this solution procedure is based on a "square sum rule" of the following kind:

(sq. a + sq. b)/2 = sq. $(a + b)/2$ + sq. $(a - b)/2$.

Indeed, according to this rule, if both (sq. a + sq. b)/2 and $(a - b)/2$ are known, then $(a + b)/2$ can be computed, and when both $(a + b)/2$ and $(a - b)/2$ are known, then it is easy to compute a and b.

An easy demonstration of the square sum rule is shown in Fig. 6.2.1, right, above. Namely, if the space between the two concentric squares with the sides a and b is divided again into a chain of four rectangles, then the half-sum of the areas of the the two squares is equal to the area of the smaller square plus half the difference between the two squares. That half difference is equal to the combined area of the chain of four triangles (half-rectangles) surrounding the smaller square. Consequently, *the half-sum of the areas of the two squares is equal to the area of the oblique square formed as the sum of the smaller square and the chain of four triangles.* That oblique square has the side d, where d is the length of the diagonal of each one of the four triangles. Since each triangle has the sides $(a + b)/2$ and $(a - b)/2$, it follows that, according to the Old Babylonian "diagonal rule",

(sq. a + sq. b)/2 = sq. d = sq. $(a + b)/2$ + sq. $(a - b)/2$.

This is, of course, the mentioned square sum rule.

IM 121565 § 4 e (col. *ii*, 32-40)

1 lagab *i-na* [šag$_4$ lagab *ad-di* 2/3 lagab *ki-di-tim* lagab *qer-bi-tum*] /
2 1 15 a.šá [*dal*]-*ba-nim-ia*
3 lagab *ki-d*[*i-tum*] / *ù qe*[*r-b*]*i-tum mi-nu*
za.e 1 [*šu-ta-ki-il-ma* 1] /
4-5 *t*[*u-ú*]*r* 40 *lu-pu-ut šu-ta-ki-il* [*i-na* 1 zi.zi] / 33 20 *ta-mar*
igi 33 20 duḫ 1 48 *ta-mar* /

6 1 48 *a-na* 1 15 *i-ta-aš-ši-ma* 2 15 *ta-mar* /
7 ba.sá.e 2 15 *šu-li-ma* 1 30 *ta-mar*
8 1 30 / *a-na* 1 *i-ši-ma* 1 30 lagab *ki-di-tam ta-mar* /
9-10 *tu-úr* 1 30 *a-na* 40 *i-ši-ma* 1 lagab *qer-bi-tam* / *ta-mar*

1 An equalside inside an equalside I inscribed, 2/3 of the outer equalside was the inner equalside] /
2 1 15 was the field of my [int]ervening space
3 The outer / and inner equalsides are what?
4-5 Re[tu]rn! 40 touch, let eat itself, [from 1 tear off] / 33 20 you will see.
The reciprocal of 33 20 release 1 48 you will see.
6 1 48 to 1 15 always carry, then 2 15 you will see. /
7 The equalside of 2 15 let come up 1 30 you will see.
8 1 30 / to 1 carry, then 1 30 the outer equalside you will see.
9-10 Return. 1 30 to 40 carry, then 1, the inner equalside, / you will see.

In this exercise, a crucial part of the question is destroyed. Luckily, the solution procedure is preserved intact, which makes it easy to reconstruct the missing part of the question. Indeed, in quasi-modern symbolic notations, the question can be rephrased as follows:

2/3 $a = b$, sq. a - sq. b = 1 15 (00).

A routine application of the method of false value leads to the solution of the problem, which is

a = 1;30 · 1 (00) = 1 30, b = 1;30 · 40 = 1 (00).

IM 121565 § 4 f (col. *ii*, 32-40 and *iii*, 1-7)

1-2 lagab *i-na* šag$_4$ lagab *ad-di-ma* 5 ta.àm / *at-ta-sa-am*
13 20 a.šag$_4$ *dal-ba-nim-ia* /
3 lagab *ki-di-tum ù qer-bi-tum mi-nu* /
4 za.e igi 5 *ša ta-ta-sa-am* duḫ 12 *ta-mar* /
5-6 12 *a-na* 13 20 a.šag$_4$ *dal-ba-nim-ka* / *i-ta-aš-ši-ma* 2 40 *ta-mar*
7 *a-na* 4 / *x x x x x x x x x x x x* / *x x x*
8 *x x x* igi 4 duḫ 15 *ta-mar*
9 15 *a-na* / 2 40 *i-ta-aš-ši-*[*ma*] 40 *ta-mar* 40 *me-eḫ-ra-*⸢*am i-di*⸣ /
10-11 5 *ša ta-ta-sa-am a-na* 1 daḫ*-ma* / *i-na ša-ni-im ḫu-ru-iṣ*
12-13 45 *lagab ki-di-tum* / 35 lagab *qer-bi-tum*

1-2 An equalside inside an equalside I inscribed, 5 each way / I went away.
13 20 was the field of my intervening space.
3 The outer and inner equalsides were what? /
4 You: The reciprocal of 5 that you everywhere went out release, 12 you will see. /
5-6 12 to 13 20, the field of your intervening space / always carry, then 2 40 you will see.
7 To 4 / *x x x x x x x x x x x x* / *x x x.*
8 *x x x.* The reciprocal of 4 release, 15 you will see.
9 15 to / 2 40 always carry, [then] 40 you will see. 40, a copy, inscribe.
10-11 5 that you went out everywhere to 1 add on, then / from the second break off.
12-13 45 is the outer equalside / 35 the inner equalside.

The problem in this exercise is closely related to the problem in § 4 c above, and the solution procedure is similar in both cases. In quasi-modern symbolic notations, the problem in this case can be rephrased as the following quadratic-linear system of equations:

sq. a – sq. b = 13 20, $(a – b)/2 = 5$.

In § 7a´ of the Old Babylonian catalog text *TMS* V from Susa (Sec. 8.9 below), a problem of the same kind is formulated in the following way:

1	[8 20] a.šag$_4$ *dal-ba-ni* 5 *me-šé-tum* /	[8 20] is my intervening space, 5 is the distance.
2	lagab *ki-di-tum ù qer-bi-tum mi-nu*	The outer and inner equalsides are what?

No answer is given, but it is easy to see that it would have been a = 30, b = 20.

The solution procedure in IM 121565 § 4 f is based on an application of the conjugate rule. The first step of the solution algorithm is to compute

rec. $(a – b)/2$ · (sq. a – sq. b) = ;12 · 13 20 = 2 40.

Unfortunately, it is not clear what happens next in the damaged text, but in view of the conjugate rule

rec. $(a-b)/2 \cdot$ (sq. a – sq. b) = $4 \cdot (a+b)/2$.

This is probably what is said, in some way, in the damaged line 7 of the text. Accordingly, lines 8-9 contain the following computation:

$(a+b)/2$ = rec. 4 · 2 40 = 40.

Next to this number 40, another number 40 is inscribed (somewhere), and in lines 10-11 it is said that the distance 5 shall be added to one of these numbers and subtracted from the other. In this way, the sides of the outer and inner squares are computed, as

$a = (a+b)/2 + (a-b)/2 = 40 + 5 = 45, \quad b = (a+b)/2 - (a-b)/2 = 40 - 5 = 35.$

IM 121565 § 4 g (col. *iii*, 8-[21])

1 lagab *i-na* šag$_4$ lagab *ad-di-ma* 4 12 a.šag$_4$ *re-*[*ḫu*] /
<3 igi.4.gál lagab *ki-di-tim* lagab *qer-bi-tum*>
2 lagab *ki-di-tum ù qer-bi-tum mi-nu* /
3 za.e 48 *ù* 36 *lu-pu-ut*
4-5 *tu-*[*úr*] / 48 *šu-ta-ki-il-ma* 38 24 *ta-mar* / 36 *šu-ta-ki-il-ma* 21 36 *ta-*[*mar*] /
6-7 21 36 *i-na* 38 <24> *hu-ru-iṣ-*[*ma*] / 16 48 *ta-mar*
8 *tu-úr mi-*[*nam a-na*] / [16 48 *lu-uš-ku-un x x x x x x x*] /
9-10 15 *šu-ku-un* 15 *a-*⸢*na* 16 48 a.šag$_4$⸣ [*x x*] / *i-ta-aš-ši-ma* 4 1[2 *ta-mar x x x*] /
11-12 *tu-úr-ma x* [*x x x x x* ba.sá.e 15] / *šuli-ma* [30 *ta-mar x x x x x*] /
[*x x x x x x x x x x x x*]

1 An equalside inside an equalside I inscribed, then 4 12 was the remaining field. /
<3/4 of the outer equalside was the inner equalside.>
2 The outer and inner equalsides were what? /
3 You: 48 and 36 touch.
4-5 Return. / 48 let eat itself, 38 24 you will see. / 36 let eat itself, 21 36 you will [see]. /
6-7 21 36 from 38 <24> tear off, [then] / 16 48 you will see.
8 Return. [What to] / [16 48 shall I set *x x x x x x x*] /
9-10 15 set. 15 ⸢to 16 48, the field⸣ [*x x*] / always carry, then 4 1[2 you will see *x x x*] /
11-12 Return, then *x* [*x x x x x*. The equalside of 15] / carry, then [30 you will see, *x x x x x*] /
[*x x x x x x x x x x x x*]

The text of this exercise is damaged, and an essential part of the question was omitted by the scribe. Nevertheless, it is possible to reconstruct a large part of the missing text, including the missing part of the question. In quasi-modern symbolic notations, the question can be rephrased in the following way:

sq. a – sq. b = 4 12, 3/4 $a = b$.

This is a homogeneous quadratic-linear system of equations for the equalsides a and b. Therefore, the solution can be obtained through an application of the rule of false value. In the text, it is assumed that the two square-sides are 48 and 36, so that the linear condition that 3/4 $a = b$ is satisfied. However, the quadratic condition is not satisfied, because

sq. 48 – sq. 36 = 38 24 – 21 36 = 16 48.

In order to proceed from there, it must be decided how much bigger the given square difference 4 12 is than the false square difference 16 48. However, since 16 48 = 7 · 1 12 is not a regular sexagesimal number, the reciprocal of 16 48 does not exist, and 4 12 / 16 48 cannot be computed as rec. 16 48 · 4 12. Therefore the text must ask instead (in lines 7-8) which number times 16 48 equals 4 12. The answer is ;15, which is the square of the scaling factor ;30 (the square of the unknown reed). Consequently, the (reconstructed) answer to the (reconstructed) stated question is that

a = 48 · ;30 = 24, b = 36 · ;30 = 18.

Indeed, sq. 24 – sq. 18 = 9 36 – 5 24 = 4 12.

6.2.6 §§ 5 a-g. Seven Problems Solved by Use of the Inexact Quadrilateral Area Rule

IM 121565 § 5 a (col. *iii*, [22]-33)

[eleven lines are lost]
12 [*x x x x x x x x x*] a.šag$_4$ *ta-mar*

The text of this exercise is lost, with the exception only of the last two words. However, these two words indicate that the object of the exercise was to compute the area of a field of some kind. Now, all the remaining, well preserved exercises in this last paragraph of IM 121565 are concerned with quadrilateral fields (fields with four straight lines as sides). Therefore, it is clear that the purpose of this initial exercise in a section about quadrilateral fields was simply to give information about *how to compute the area of a quadrilateral field*.

As is well known, in "metro-mathematical" cuneiform texts from the fourth and third millennia BC a certain inexact "quadrilateral area rule" is used in order to compute areas of quadrilateral fields. Examples are the proto-cuneiform text W 19408,76 (Nissen/Damerow/Englund, *ABK* (1993), 58), and the Old Akkadian text *DPA* 34 (Friberg, *MSCT 1* (2007), 401). According to the quadrilateral area rule, the area of a quadrilateral field can be computed as the half-sum of the two lengths times the half-sum of the two fronts. This area rule is fairly exact only for almost rectangular quadrilateral fields, but can be grossly inexact otherwise. (See the further discussion in Sec. 6.2.7 below.)

In all the exercises in §§ 5 b-g below, the areas of quadrilaterals are computed by use of the mentioned inexact quadrilateral area rule.

IM 121565 § 5 b (col. *iii*, 34-49)

1 4 [*x x x*]
2 [$^{2}/_{3}$ uš an].ta ⸢*ki*⸣*-ma* uš ki.ta / [$^{1}/_{2}$ uš ki.ta sag.ki an.ta]
3 [$^{1}/_{3}$] sag.[ki] an.ta sag.ki ki.ta / ⸢a.šag$_4$⸣ 2 46 40
uš-*ia ù* sag.ki-*ia mi-nu* /
4-5 za.e 1 uš an.ta *ù* 40 uš ki.ta *ku-mur* / *ḫe-pé-ma* 50 *ta-mar*
6-7 *tu-*[*úr*] 20 sag.ki / an.ta *ù* 6 40 sag.ki ki.ta *ku-mur* / *ḫe-pé-ma* 13 20 *ta-mar*
8 *tu-úr* / 50 *a-na* 13 20 *i-ta-aš-ši-ma* 11 06 40 *ta-mar* /
9 igi 11 06 40 duḫ 5 24 *ta-mar*
10-11 *tu-úr* / 5 24 *a-na* 2 46 40 *i-ta-aš-ši-ma* / 15 *ta-mar*
12 [ba.sá].e 15 *šu-li-ma* / 30 *ta-mar*
13 *tu-úr* 30 *a-na* 1 *i-ši-ma* / ⸢30 uš⸣ [an.t]a *ta-mar*
14 *tu-*<*úr*> 40 *a-na* 30 *i-ši-ma* / [20 uš k]i.ta *ta-mar*
15 *tu-úr* 30 *a-na* 20 / [*i-ši-ma*] 15 sag.ki an.<ta> *ta-mar*
16 6 40 *a-na* / 30 [*i-š*]*i-ma* 3 20 sag ki.<ta> *ta-mar*

1 4 [*x x x*]
2 [2/3 of the up]per [length] as the lower length, / [1/2 the lower length the upper front],
3 [1/3] the upper front the lower front, / the field 2 46 40.
My lengths and my fronts were what? /
4-5 You: 1, the upper length, and 40, the lower length, heap, / break, then 50 you will see.
6-7 Re[turn]. 20, the upper front / and 6 40, the lower front, heap, / break, then 13 20 you will see.
8 Return. / 50 to 13 20 always carry, then 11 06 40 you will see. /
9 The reciprocal of 11 06 40 release, 5 24 you will see.
10-11 Return. / 5 24 to 2 46 40 always carry, then / 15 you will see.
12 [The equal-sid]e of 15 let come up, then 30 you will see.
13 Return. 30 to 1 carry, then / / ⸢30, the⸣ [upp]er ⸢length⸣ you will see.
14 Re<turn>. 40 to 30 carry, then / [20, the upp]er [length] you will see.
15 Return. 30 to 20 / [carry, then]15, the upp<er> front you will see.
16 6 40 to / 30 [rai]se, then 3 20, the low<er> front, you will see.

The first word in the first line of this exercise is broken but seems to mention the number 4. If this was a previously undocumented word for a 'quadrilateral field' it is quite unfortunate that it is not better preserved.

In quasi-modern symbolic terms, the question in this exercise can be rephrased as the following rectangular-linear system of one rectangular and three linear equations for the four unknowns u, u', the upper and lower lengths, and s, s', the upper and lower fronts:

$$2/3\ u = u', \quad 1/2\ u' = s, \quad 1/3\ s = s', \quad (u + u')/2 \cdot (s + s')/2 = 2\ 46\ 40.$$

All these equations are homogeneous, so the problem can by solved through an application of the rule of false value. The solution procedure proceeds as follows. (An Old Babylonian school boy would have to try to keep the absolute sizes of the numbers in his head, if he could.) First, with false values,

$u = 1, \quad u' = 2/3 \cdot 1 = 40, \quad (u + u')/2 = (1 + 40) = 50$ (lines 4-5)
$s = 1/2 \cdot 40 = 20, \quad s' = 1/3 \cdot 20 = 6\ 40, \quad (s + s')/2 = (20 + 6\ 40) = 13\ 20$ (lines 6-7)
$(u + u')/2 \cdot (s + s')/2 = 50 \cdot 13\ 20 = 11\ 06\ 40$ (line 8)

This means that if the four unknowns are given the indicated false values, which satisfy the three linear equations in the stated question, then the corresponding false area, given by the inexact quadrilateral area rule is 11 06 40. However, the prescribed area was 2 46 40. Therefore, the square of the needed scaling factor or unknown reed *r* can be computed as 2 46 40 divided by 11 06 40, as in the following computation:

sq. r = rec. $11\ 06\ 40 \cdot 2\ 46\ 40 = 5\ 24 \cdot 2\ 46\ 40 = 15$. (lines 10-11)

Consequently,

r = sqs. $15 = 30$ (line12)
$u = 30 \cdot 1 = 30, \quad u' = 30 \cdot 40 = 20, \quad s = 30 \cdot 20 = 10, \quad s' = 30 \cdot 6\ 40 = 3\ 20$ (lines 13-16)

Remark: The Old Babylonian school boy who tried to keep in mind the absolute values of the sexagesimal numbers he was counting with, using relative place value notation, had to his disposition the following helpful pragmatic rule: *The great majority of all side lengths appearing in Old Babylonian mathematical exercises are counted in tens or small numbers of sixties* (of a rods). Therefore, for instance, the false length $u = 1$ must be thought of as 1 (00) = 60, and so on. Similarly, the computed answer in lines 13-16 of the exercise had to be thought of as

$u = 30, \quad u' = 20, \quad s = 10,$ and $s' = 3;20 = 3\ 1/3$.

It is also interesting to reflect over how the same imagined Old Babylonian school boy could perform somewhat complicated computations such as those in lines 10-11 of the exercise above. First, he could (if he wished) do the computation of the reciprocal of 11 06 40 in a number of easy steps, as follows:

$11\ 06\ 40 \cdot 3 = 33\ 20, \quad 33\ 20 \cdot 3 = 1\ 40, \quad 1\ 40 \cdot 36 = 1.$ Therefore,
$11\ 06\ 40 \cdot 3 \cdot 3 \cdot 36 = 1,$ and rec. $11\ 06\ 40 = 3 \cdot 3 \cdot 36 = 3 \cdot 1\ 48 = 5\ 24$.

(Cf. Ch. 2 above.) He could also perform the multiplication $5\ 24 \cdot 2\ 46\ 40$ in a similar number of easy steps:

$5\ 24 \cdot 2\ 46\ 40 = 1\ 48 \cdot 8\ 20 = 36 \cdot 25 = 6 \cdot 2\ 30 = 15.$

This way of counting in a number of easy steps is, of course, possible only because the numbers involved in the computations are regular sexagesimal numbers with obvious factorizations.

IM 121565 § 5 c (col. *iv*, 1-10)

1 1 45 uš ⸢an.ta⸣ 45 a.šag$_4$ uš ki.⸢ta⸣ [*ul i-de*] /
2-3 *ma-la* uš [an].ta [ugu uš ki.ta dirig] / lu sag.ki-*ka*
x [*x x x* uš ki.ta *ù* sag.ki *mi-nu*]
4-5 za.e 1 45 uš [an.ta *šu-ta-ki-il-ma*] / 3 03 45 *ta-mar*
6 a.šag$_4$ ⸢*a-na*⸣ [2 *e-ṣi-ma*] / 1 30 *ta-mar*
7 1 30 *i-na* 3 03 45 [zi.zi]-*ma* / 1 33 45 *ta-mar*
8 ba.sá.e 1 3[3 45] / *šu-li-ma* 1 15 uš ki.ta *ta*-[*mar*]
9-10 [*tu-úr*] / 1 45 uš an.ta ugu 1 15$^{!}$ uš ki.[ta *mi-nam* dirig] / 30 dirig
10 30 sag.ki-*ka ki*-⸢*am*⸣$^{?}$ [*né-pé-šum*] /

1 1 45 was the ⸢upper⸣ length, 45 the field. The low⸢er⸣ length [I did not know]. /
2-3 Whatever the [upp]er length [over the lower length exceeded] / should be my front.
x [*x x x*. The lower length and the front were what?] /
4-5 You: 1 45, the [upper] length [let eat itself, then] / 3 03 45 you will see.
6 The field ⸢to⸣ [2 repeat, then] / 1 30 you will see.
7 1 30 from 3 03 45 [tear off], then 1 33 45 you will see.
8 The equalside of 1 3[3 45] / let come up, then 1 15, the lower length. you will [see]
9-10 [Return.] / 1 45, the upper length, over 1 15$^{!}$, the low[er] length [by what exceeds]? / By 30 it exceeds.
10 30 is your front. Such ⸢is⸣ [the procedure].

This exercise is extensively damaged, yet a meaningful reconstruction as above seems to be possible. If the proposed reconstruction is correct, then in line 1 it is stated that the upper length u = 1 45, and that the area A = 45, but that the lower length u' is unknown. Next, it is said, in lines 2-3, that the difference between the upper and lower lengths is equal to the (single) front s. Apparently the object considered is a *triangle*. To be computed are the lower length and the front.

In lines 4-6, the square of 1 45 is computed and is found to be 3 03 45. Then the given area A = 45 is doubled and is found to be 1 30. Next, in lines 6-8, the difference 3 03 45 – 1 30 = 1 33 45 is computed and is found to be the square of 1 15, which is then called the lower length u'. Finally, in lines 9-10, the single front s is computed as the difference $u - u'$ = 1 45 – 1 15 = 30. The result seems to agree with the initial statement that the area A = 45, since with the computed values of the lengths and the front

$$A = (u + u')/2 \cdot s = (1\ 45 + 1\ 15)/2 \cdot 30 = 1\ 30 \cdot 30 = 45\ (00).$$

In this exercise, the solution procedure is based on the conjugate rule. Indeed, in quasi-modern symbolic notations the stated question can be rephrased as the following equations, where the area of the triangle is computed by use of a simple analog to the inexact quadrilateral area rule:

$$u = 1\ 45, \qquad A = (u + u')/2 \cdot s = 45, \qquad u - u' = s.$$

In the second equation above, s can be replaced by the difference $u - u'$. In view of the conjugate rule, the result is the new equation

$$A = (u + u')/2 \cdot (u - u') = (\text{sq. } u - \text{sq. } u')/2 = 45.$$

This means that the lower length can be computed, as in the text, as the solution to the equation

$$\text{sq. } u' = \text{sq. } u - 2 \cdot 45 = 3\ 03\ 45 - 1\ 30 = 1\ 33\ 45.$$

Hence,

$$u' = \text{sqs. } 1\ 33\ 45 = 1\ 15, \qquad s = u - u' = 1\ 45 - 1\ 15 = 30.$$

IM 121565 § 5 d (col. *iv*, 11-19)

1-2 30 sag.ki an.ta 20 sag.ki ki.[ta 18 45 a.šag$_4$] / uš ugu uš 10 dirig
⸢uš an.ta⸣ [*ù* uš ki.ta *mi-nu*] /
3-4 za.e 30 sag.ki an.ta *ù* 20 [sag.ki ki.ta] / [*ku-mu-ur*]? *ḫe-pé* 25 *ta-mar*
igi 25 [duḫ-*ma* 2 24] /
5-6 2 24 *a-na* 18 45 a.šag$_4$ [*i-ta-aš-ši-ma x x*] / 45 [*ta-mar*] 45 *me-eḫ-ra-*[*am i-di*]
7 [10 dirig] / [*ḫe-pé-ma* 5] *ta-mar*
8 5 *a-na* [*x x x x x x*] / *a-na* [*x x x x x x x*]
9 [*x x x x x x x*] / 50 *ta-mar* ⸢uš an⸣.ta
[*x x x x x x x x x*]

1-2 30 was the upper front, 20 the low[er] front, [18 45 the field], / length over length by 10 exceeded.
⸢The upper length⸣ [and the lower length were what?] /
3-4 You: 30, the upper front, and 20, [the lower front] / [heap together], break, 25 you will see.
The reciprocal of 25 release, 2 24. /
5-6 2 24 to 18 45, the field, [always carry, then x x] / 45 you will see. 45, a cop[y inscribe].
7 [10, the excess] / [break, then 5] you will see.
8 5 to [*x x x x x x*] / to [*x x x x x x x*]
9 [*x x x x x x x*] / 50 you will see, the ⸢upper length⸣.
[*x x x x x x x x x*]

This exercise, too, is extensively damaged, yet again a meaningful reconstruction as above seems to be possible. If the proposed reconstruction is correct, then in lines 1-2 it is stated that the upper front s is 30, the lower front s' is 20, the area A is 18 45, and the difference $u - u'$ between the upper and the lower length is 10. In quasi-modern symbolic notations,

$$s = 30, \qquad s' = 20, \qquad A = (u + u')/2 \cdot (s + s')/2 = 18\ 45, \qquad u - u' = 10.$$

This means that the two lengths u and u' can be computed as the solutions to the pair of linear equations

$$(u + u')/2 \cdot 25 = 18\ 45, \qquad u - u' = 10.$$

In lines 5-7 of the text, 18 45 is multiplied by the reciprocal of 25, and 10 is halved. That corresponds to replacing the pair of equations above for *u* and *u*´with the following, equivalent equations:

$(u + u')/2 = \text{rec. } 25 \cdot 18\ 45 = ;02\ 24 \cdot 18\ 45 = 45, \quad (u - u')/2 = 10/2 = 5.$

The remainder of the text of the exercise is severely damaged, but it is clear anyway that what now takes place is that *u* and *u*´ are computed as

$u = (u + u')/2 + (u - u')/2 = 45 + 5 = 50, \quad u' = (u + u')/2 - (u - u')/2 = 45 - 5 = 40.$

It is easy to check that, as required, (50 + 40)/2 · (30 + 20)/2 = 45 · 25 = 18 45.

IM 121565 § 5 e (col. *iv*, 20-27)

1 7 *šu-ši* [*x x x x x*] /

2 za.e 7 ˹uš˺ [*x x x x x*] /

3 3 30 *me-eḫ-ra-am i-*[*di x x x x*] /

4 [*x x x x x*] *ta-mar tu-*[*úr x x x*] /

5 1 uš *x* [*x x x x x*] /

6 ba.sá.e 4 [*x x x x x*] /

7 *i-na* šag$_4$ 8 33 20 [*x x x x x*] /

8 *ḫu-ru-iṣ* 5 *ta-mar* x [*x x x x x*] /

The text of this exercise is damaged so much that no reconstruction is possible. The only meaningful data preserved are that one length is 7 sixties, and that one other side of the quadrilateral is 1 sixty.

IM 121565 § 5 f (col. *iv*, 28-44)

1 ˹27 uš˺ an.ta 1 48 a.šag$_4$ [*x x x x x x x*] /

2-3 *ma-la* uš ki.ta *x* sag.ki [*x x x x x*] / *lu* sag.ki ki.ta

4 za.e [1 *ù* 20] / ˹*ku-mur*˺ *ḫe-pé-šu* 40 *ta-mar*

5 40 *ki-ma* [*x x x x*] / *lu-pu-ut*

tu-úr 1 uš *x x* [*x x x x x*] /

6 *tu-úr* 40 *a-na* 30 *i-ši-ma* 20 [*ta-mar*]

7 [20 *a-na*] / 1 48 a.šag$_4$ *i-ta-aš-ši-ma* ˹36˺ [*ta-mar*]

8 [27] / *a-na* 30 *i-ši-ma* 13 30 *ta-mar*

9 [*tu-úr*] / 40 *a-na* 13 30 *i-ta-aš-ši-ma* 9 [*ta-mar*] /

10 *tu-úr* 30 *a-na* 9 *i-ši-ma* 4 ˹30˺ [*ta-mar*] /

11-12 4 30 *me-eḫ-ra-am i-di šu-ta-ki-il-*[*ma*] / 20 15 *ta-mar*

13 20 15 *a-na* [36 daḫ-*ma*] / 56 15 *ta-mar*

14 ba.sá.e 56 15 [*šu-li-ma*] / 7 30 *ta-mar*

15 *i-na* šag$_4$ 7 30 4 30 [*ša-ni-im*] / *ḫu-ru-*˹*iṣ*˺ 3 *ta-mar*

16 [*tu-úr*] / 3 *šu-ta-ki-il* 9 *ta-m*[*ar*] /

17 9 uš ki.ta 9 sag.ki an.ta ˹3˺ sag.[ki] / ki.ta

1 ˹27˺ is the upper ˹length˺, 1 48 the field, [*x x x x x x x*] /

2-3 Whatever the lower length x the [*x x*] front [*x x x x*] /

4 You: [1 and 20] / ˹heap together˺, break it, 40 you will see.

5 40 as [*x x x x*] / touch.

Return. 1, the [*x x*] length [*x x x x x*]. /

6 Return. 40 to 30 carry, then 20 [you will see].

7 [20 to] / 1 48, the field, always carry, then ˹36˺ [you will see].

8 [27] / to 30 carry, then 13 30 you will see.

9 [Return.] / 40 to 13 30 always carry, then 9 [you will see]. /

10 Return. 30 to 9 carry, then 4 ˹30˺ [you will see].

11-12 4 30, a copy inscribe, let eat each other, [then] / 20 15 you will see.

13 20 15 to [36 add on, then] / 56 15 you will see.

14 The equalside of 56 15 [let come up, then / 7 30 you will see.

15 Out of 7 30, 4 30, [the second] / break off, 3 you will see.

16 [Return.] / 3 let eat itself, 9 you will s[ee.]

17 9 is the lower length, 9 the upper front, ˹3˺ the lower fro[nt].

In this exercise, the solution procedure is fairly well preserved, while essential parts of the question are destroyed. Fortunately, working backwards from the data appearing in the solution procedure, one can reconstruct the missing parts of the question. So, apparently, the stated problem in this exercise was:

$u = 27, \quad A = 1\ 48, \quad [s = u'], \quad [s' = 1/3\ s].$

It follows, in view of the quadrilateral area rule, that the lower length u' can be computed as the solution to the following quadratic equation:

$$A = (27 + u')/2 \cdot (u' + 1/3\ u')/2 = 1\ 48.$$

The solution procedure begins, in line 4, by computing (1 + 20)/2 = 40. Then, in line 6, 40 is multiplied by 30, which gives 20. This means that the initial quadratic equation has been transformed in the following way:

$$(27 + u')/2 \cdot (u' + 1/3\ u')/2 = (27 + u')/2 \cdot 1\ 20\ u'/2 = (27 + u') \cdot \quad 30 \cdot 40\ u' = 20 \quad \text{sq. } u' \quad + 40 \cdot 30 \cdot 27\ u' \quad = 1\ 48.$$

It is not clear what happens in the broken line 5. Anyway, next both sides of the quadratic equation

$$20 \quad \text{sq. } u' \quad + 40 \cdot 30 \cdot 27\ u' \quad = 1\ 48.$$

are multiplied by 20. In particular, in lines 6-7, 1 48 is multiplied by 20. The result is that 20 · 1 48 = 36. The quadratic equation now has the form

$$\text{sq. } 20 \quad \cdot \text{sq. } u' \quad + 40 \cdot 30 \cdot 27 \cdot 20\ u' \quad = 36.$$

At this point in the solution procedure a new unknown is silently introduced, it will here be called $v = 1/3\ u'$. The result is the following quadratic equation for the unknown v:

$$\text{sq. } v + 40 \cdot 30 \cdot 27\ v \quad = 36.$$

Now, belatedly, in lines 7-9, the coefficient in the linear term is simplified to 40 · 30 · 27 = 40 · 13 30 = 9. The quadratic equation for v then takes its final form,

$$\text{sq. } v + 9\ v = 36.$$

This is a quadratic equation of the basic type B4a (Friberg, *AT* (2007), 6), which is solved in the usual way. The solution is found in line 15, namely $v = 3$. Consequently, $u' = 3\ v = 9$. (Somewhat misleadingly, u' is computed in line 16 as sq. 3 = 9.) Finally, when u' is known, it simple to find also the values of the other two initial unknowns, $s = \quad u' \quad = 9$, and $s' = 1/3 \cdot s = 9$.

Remark: The explanation above, in terms of transformations of equations is, of course, totally anachronistic. The author of the exercise IM 121565 § 5 f *certainly did not think in terms of equations*. Instead he tried, with great skill, to hold in his head various manipulations with numbers multiplying the unknown lower length and its square, or 1/3 of that lower length and the square of that 1/3. Here lies the explanation for the belated transformation of the coefficient 40 · 30 · 27.

IM 121565 § 5 g (col. *v*, 1-22)

1-2 10 *šu-ši* uš an.ta 8 *šu-ši* sag.ki an.ta / 42 a.šag$_4$
3 uš ki.ta ⸢*ul i-de*⸣ *ma-la* uš / ki.ta lu sag ki.ta
4-5 za.e uš ki.ta / *ù* sag.ki ki.ta *ḫe-pé-ma* [*x x*] *i-ši-ma* / 15 *ta-mar*
15 *x x* [*x x x x x*] *lu-pu-ut* /
6 *tu-úr-ma* 15 *a-na* [*x x x x x*] a.šag$_4$ /
7 *i-ta-aš-ši-ma* [*x x x x x*]
… ………………………………………………………
19 *a-na* 3 *i-ši-ma* [*x x x x x x*] *x* /
20 8 sag.ki an.ta [*x x x x x* 4 *ta-mar*]
21 4 uš / ki.ta [4 sag] ⸢ki.ta⸣ /
22 *ki-a-am né-pé-šum*

1 10 sixties was the upper length, 8 sixties the upper front, / 42 the field.
3 The lower length I did not know. Whatever the lower length, the lower front.
4-5 You: The lower length / and the lower front break, then [*x x*] carry, then / 15 you will see.
15 x x [x x x x x], touch. /
6 Return, then 15 to [x x x x x] the field /
7 always carry, then [x x x x x]
… ………………………………………………………
19 To 3 carry, then [*x x x x x x*] *x*
20 8, the upper front [*x x x x x* 4 you will see].
21 4 is the / lower length, [4 the lower front].
22 Such is the procedure.

In this final exercise on IM 121565, the question is perfectly preserved, while most of the text of the solution procedure is lost. Given is a quadrilateral with the upper length '10', the upper front '8', and the area '42'. The lower length is not known, and the lower front is equal to the lower length. In quasi-modern symbolic notations, the question can be rephrased as the following two equations for the two unknown sides of the quadrilateral:

$(10 + u')/2 \cdot (8 + s')/2 = 42, \quad s' = u'.$

Apparently, the first step of the solution procedure was (the Old Babylonian equivalent of) imaging these equations rewritten in the form

$(1/2\, u' + 5) \cdot (1/2\, s' + 4) = 42, \quad s' = u'.$

This simple rephrasing of the problem is possibly what is referred to in lines 3-5, where it is said 'The lower length and the lower front break'. The damaged continuation, '*x x* carry, then 15 you will see' probably means that 1/2 · 1/2 = 1/4 = 15, and that the two equations above for the two unknown sides of the quadrilateral now are thought of as being replaced by the following quadratic equation for the unknown lower length:

$1/2 \cdot 1/2$ sq. $u' + 4\;30\;\; u' + 20 = 42$, or, equivalently, 15 sq. $u' + 4\;30\;\; u' + 20 = 42$.

Line 6 is also damaged but seems to say that the field $A = 42$ shall be multiplied by 15. What this implies is that both sides of the equation 15 sq. $u' + 4\;30\; u' + 20 = 42$ are multiplied by 15, which leads to the following new form of the quadratic equation

sq. $(15\, u') + 4\;30 \cdot (15\;\; u') + 5 = 10\;30$.

Most of the remainder of the solution procedure is lost, but it is clear, anyway, how the solution procedure would continue from here, following the usual Old Babylonian routine. First, the unknown u' would (silently) be replaced by the new unknown 15 u', which we may call v. In terms of this new unknown, the quadratic equation takes the simpler form

sq. $v + 4\;30\; v = 5\;30$.

This is a quadratic equation for v of the Old Babylonian basic form B4a (Friberg, *AT* (2007), 6). It was probably solved in the usual way in the lost part of the solution procedure, but it would hardly be worthwhile to try to find the exact role played by the remaining scattered fragments of the procedure in lines 7-18.

Anyway, the solution to the equation for v is $v = 1$. Consequently,

$u' = 4\, v = 4\,(00)$, and $s' = u' = 4\,(00)$.

It is easy to check the result by calculating

$A = (10 + u')/2 \cdot (8 + s')/2 = (10 + 4)/2 \cdot (8 + 4)/2 = 7 \cdot 6 = 42.$

6.2.7 More About the Quadrilateral Area Rule in Old Babylonian Mathematical Texts

The quadrilateral area rule is used also in some other Old Babylonian mathematical texts, although it must have been known that the rule is inexact. An interesting example is the text *MCT* B = YBC 4675 (Neugebauer and Sachs, *MCT* (1945), 44), where a quadrilateral field with the lengths 5 10 and 4 50 (rods) and the fronts 17 and 7 is said to have the area 2(bùr) = 1 00 00 sq. rods. Indeed, according to the inexact quadrilateral area rule (5 10 + 4 50)/2 · (17 + 7)/2 = 5 00 · 12 = 1 00 00.

The text YBC 4675 is mentioned by Høyrup in his discussion of what he calls the "surveyor's formula" (*LWS* (2002), 229-231. Høyrup concludes his discussion with the following relevant comment:

> "In Old Babylonian teaching, the formula was sometimes used as a pretext for the formulation of complex mathematical problems without any care for its plausibility. …… But actual surveying documents demonstrate that this lack of interest in the precision of the formula was a consequence of its function in the specific context, not of ignorance."

Another interesting commentary, in Neugebauer's discussion of YBC 4675 in *MCT,* 46 is that

> "The scribe scarcely realized, however, that the numbers given are incompatible with the basic assumption of his calculations that b_1, b_2 and d (the fronts and the transversal) are parallel lines."

This is because if 5 10 and 4 50 are the lengths and 17 and 7 the parallel fronts of a parallel trapezoid, then 5 10 and 4 50 are the lengths and 17 – 7 = 10 the front of a triangle, which is impossible since the sum of the lengths of any two sides of a triangle should always be more than the length of the remaining side.

A third commentary to the same text appears in an important discussion of the Babylonian quadrilateral area rule in Bruins and Rutten, *TMS* (1961), 4-8. There it is first shown, by use of modern trigonometry, that *areas computed by use of the quadrilateral area rule are always strictly larger than the correct area*, except in the case when the quadrilateral is a rectangle. In the case of YBC 4675, where, as mentioned, the computed area is equal to the given area, this observation shows that there exists no quadrilateral with the given sides and the given area. Otherwise, the quadrilateral in YBC 4675 would be a rectangle, which it is not!

In addition to in IM 121565 § 5 and YBC 4675, the quadrilateral rule appears also in exercise # 1 in IM 52301, a mathematical cuneiform text from Shaduppûm (Tell Harmal) published by Baqir in *Sumer* 6 (1950). Since Baqir's explanation of the exercise was inadequate, the text will be reexamined below.

IM 52301 # 1 (Baqir, *Sumer* 6 (1950), 130-148)

1-2 *šum-ma* 1 40 uš *e-lu-um me-ḫe-er-šu ḫa-li-iq* sag.ki *e-li-tum* / *e-li* sag.ki *ša-ap-li-tim* 20 *e-te-er* 40 a.šag$_4$
3 *mi-nu-um* uš-*ia-ma* / za.e tuk.zú.dè
4 1 30 *šu-ku-un-ma ḫe-pé šu-ta-ki-il-ma* / 45 *ta-mar*
igi 45 duḫ.ḫa-*ma* 1 20 *ta-m*[*ar*]
5 [1 20 *a*]-*na* 40 a.šag$_4$ *i-ši-ma* / 53 20 *ta-mar*
6 53 20 *e-ṣí-ma* 1 46 40 [*ta*]-*mar* 146 40 *re-eš*$_{15}$-*ka* / *li-ki-il*
7 *tu-ur-ma* 1 40 uš *a-li-a-am ù* 20 *ša* sag.ki *e-li-tum* / *e-li* sag.ki *ša-ap-li-tim i-te-ru ku-mu-ur-ma* 2 *ta-mar* /
8 2 *ḫe-pé-ma šu-ta-ki-il-ma* 1 *ta-mar*
9 1 *a-na* 1 46 40 *ṣí-ib-ma* / 2 46 40 *ta-mar*
ba-se-e 2 46 40 *šu-li-*[*ma*] 1 40 *ta-mar* /
10-11 *a-na* 1 40 *ba-se-ka* 1 *ša tu-uš-ta-ki-lu a-na* 1 40 *ṣí-*[*ib*]-*ma* / 2 40 *ta-mar*
12 *i-na* 2 40 *ša ta-mu-ru* 1 40 uš *a-li-am ḫu-ru-uṣ*$_4$ / 1 *ši-ta-tum* uš *ḫa-al-qú* 1 *ḫe-pé-ma* 30 *ta-mar*
13-14 30 *me-eḫ-ra-am* / *i-di-ma* 20 *ša* sag.ki *e-li* sag.ki *i-te-ru ḫe-pé-ma* / 10 *ta-mar*
15 10 *a-na* 30 *iš-ten ṣí-ib-ma* 40 *ta-mar i-na* 30 *ša-ni-im*/ *ḫu-ru-uṣ*$_4$ 20 *ta-mar* 20 sag.ki *ša-ap-li-tum*
ki-a-am ne-pé-šum

1-2 If 1 40 the upper length, its opposite (length) lost, the upper front / over the lower front 20 I went beyond, 40 the field,
3 what was my length? / You, in your getting(?) it:
4 1 30 set, then break {let eat itself}, then / 45 you will see.
The reciprocal of 45 release, then 1 20 you will s[ee].
5 [1 20 t]o 40 the field carry, then / 53 20 you will see.
6 53 20 double, then 1 46 40 [you] will see, 1 46 40 your head / let it hold.
7 Return, then 1 40 the upper length and 20 that the upper front / over the lower front was beyond heap, then 2 you will see. /
8 2 break, then let eat itself, then 1 you will see.
9 1 to 1 46 40 add on, then / 2 46 40 you will see.
The equalside of 2 46 40 let come up, [then] 1 40 you will see. /
10-11 To 1 40, your equalside, 1 that you let eat itself {to 1 40} ad[d o]n, then / 2 40 you will see.
12 From 2 40 that you saw 1 40 the upper length break off / 1 the remainder is the lost length. 1 break, then 30 you will see.
13-14 30 a copy / inscribe, then 20 that front over front went beyond break, then / 10 you will see.
15 10 to one 30 add on, then 40 you will see, from a second 30 / cut off, 20 you will see, 20 the lower front.
Such is the procedure.

In quasi-modern symbolic notations, the problem in the question of this exercise can be explained as follows: Given is a quadrilateral with the lengths u and u' and the fronts s and s'. In terms of sexagesimal numbers in relative place value notation, these four parameters are restricted by the following three requirements:

$$u = 1\ 40, \quad s - s' = 20, \quad A = (u + u')/2 \cdot (s + s')/2 = 40.$$

Evidently, a fourth requirement is needed before the values of all the four parameters can be found. This fourth requirement was lost somehow when the text of exercise # 1 was produced. Fortunately, the missing requirement can be reconstructed with departure from the solution procedure in the exercise by use of some deft reverse engineering. As it turns out, the lost fourth requirement was, essentially, that

$$s + s' = 1\ 30 \cdot (u + u' - 2).$$

If these various requirements are combined, the question can be rephrased in the following way as an equation for the new unknown $v = u + u'$:

$$v/2 \cdot 1\ 30 / 2 \cdot (v - 2)/2 = 40.$$

This is a quadratic equation for the unknown v. In the solution procedure of the exercise the unknowns are computed in (essentially) the following way:

$v/2 \cdot (v - 2) = \text{rec. } (1\ 30 / 2) \cdot 40 = \text{rec. } 45 \cdot 40 = 1\ 20 \cdot 40 = 53\ 20$ (lines 3-5)
$v \cdot (v - 2) = 2 \cdot 53\ 20 = 1\ 46\ 40$ (line 5)
$s + s' = (u + (s - s'))/2 = (1\ 40 + 20)/2 = 2/2 = 1$ (lines 6-8)
$v - 1 = \text{sqs. } (1\ 46\ 40 + 1) = \text{sqs. } 2\ 46\ 40 = 1\ 40, \quad v = 1\ 40 + 1 = 2\ 40$ (lines 8-11)
$u' = v - u = 2\ 40 - 1\ 40 = 1$ (lines 11-12)
$(s + s')/2 = 1/2 = 30, \quad (s - s')/2 = 20/2 = 10, \quad s = 30 + 10 = 40, \quad s' = 30 - 10 = 20$ (lines 12-15)

Evidently, this solution procedure is partly corrupt. The computation of $v = 1$ as a solution to a quadratic equation proceeds correctly but is interrupted in lines 6-8 by a misplaced computation of $s + s'$. Actually, the computation of $s + s'$ should not be possible before the computation of $v = u + u'$. Therefore, it should not come as a surprise that the computation in lines 6-8 is completely nonsensical, although it produces a correct result, probably known beforehand. The correct computation of $s + s'$ would have been as follows:

$$s + s' = 1\ 30 \cdot (v - 2) = 1\ 30 \cdot 40 = 1.$$

The question in exercise # 2 in IM 52301 is phrased in the following way:

1-2 *šum-ma a-na ši-ni-ip ku-mu-ri* sag *e-li-tim* / *ù ša-ap-li-tim* 10 *a-na qa-ti-ia* daḫ-*ma* 20 uš *ab-ni*
3-4 sag / {*e-li*} *e-li-tum e-li ša-ap-li-tim* 5 *i-te-er* / a.šag$_4$ 2.30
mi-nu-um uš-*ia*

1-2 If to two-thirds of the heap of the upper front / and lower 10 I added to my hand, then 20 the length I built,
3-4 the upper / front over the lower 5 was beyond, / the field 2 30.
What was my length?

In quasi-modern symbolic notation this question can be reformulated as

$$2/3\ (s + s') + 10 = u, \quad s - s' = 5, \quad (s + s')/2 \cdot u = 2\ 30, \quad u = ?$$

(The mention of 20 as the length in line 2 is premature.) Clearly the quadrilateral in this exercise is of a special case since only one length is mentioned. Probably the intended object was a rectangular parallel trapezoid.

The interested reader can find the full text of exercise # 2 and an explanation of the solution procedure in Høyrup, *LWS* (2002), 213-217.

After exercise # 2 follows a brief table of constants with 7 entries, and on the left edge there is the following inscription (rotated here for the readers' convenience), for which Baqir could offer no explanation:

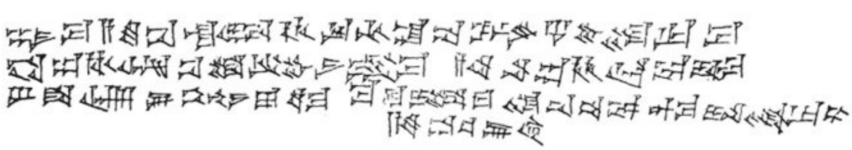


Fig. 6.2.2. IM 52301, *left edge*. Baqir's hand copy.

1 *šum-ma* a.šag$_4$ uš *la mi-it-ḫa-ru-ti*
at-ta
i-gi 4 *pu-ṭú-ur-ma* /
2 *na-ap-ḫa-ar* uš *li-iq-bu-ni-kum-ma a-na na-ap-ḫa-ar* sag *i-ši-ma* /

3 4 *ša-ar er-bé-tim lu-*<*pu*>*-ut-ma*
4 *ma-la i-li-ku tu-uš-ta-ka-an-ma i-na li-ib-bi* / a.šag$_4$ *ta-tam-ša*$_{10}$*-aḫ*

1 If a field, the lengths are not equalsided.
You:
The reciprocal of 4 release, then /
2 the heap of the lengths may they say to you, to the heap of the fronts carry it, /
3 4 of the four winds inscribe, then
4 what comes up for you you will set, inside / the field you will measure.

Apparently this is one of very few known cases when a cuneiform mathematical text contains *a generally worded computation rule* rather than numerically explicit examples. Unfortunately, the understanding of the text is hindered by an awkward formulation of the computation rule. In addition, the text is corrupt. The phrase '4 of the four winds inscribe' should obviously *precede* the phrase 'The reciprocal of 4 release'.

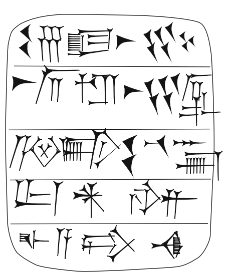
What the author of this computation rule wanted to say but could not express clearly was probably:

Make a diagram of a general quadrilateral field, that is a field where the four sides sides are not equal.
Write down the measures of the four sides, along the two lengths and the two fronts.
Compute rec. 4 and multiply it with the sum of the lengths and with the sum of the two fronts.
Write down the result of the computation inside your diagram. It is the measure of your field.

6.2.8 The Quadrilateral Area Rule in an Old Akkadian Metro-Mathematical(?) Text

It is not difficult to find examples of the application of the quadrilateral area rule in cuneiform texts from the third millennium BC. One particularly interesting example is the small Old Akkadian text Limet (1937), *DPA* 34. The text may be metro-mathematical since it is also a nice example of the application of the "proto-literate field extension procedure". See Friberg, *MSCT 1* (2007), 401.

obv.



26 mir	1$_{\text{géš}}$ 50 kur	26 north	1(60) 50 east
1$_{\text{géš}}$ lá.2 u$_5$	1$_{\text{géš}}$ 50 lá.4$^?$ mar	1(60) – 2 south	1(60) 50 – 4$^?$ west
a.šà.bi 2$_{\text{bùr}}$ 1$_{\text{èše}}$ 3$_{\text{iku}}$ ašag$_5$		its field 2(bùr) 1(èše) 3(iku)	
ur.$^{\text{d}}$iškur		PN	
sig$_7$ a.geštin.na			

A = (26 + 58)/2 · (1(60) 50 + 1(60) 46)/2 = 42 · 1(60) 48 = 1(60 · 60) 15(60) 36,
1(60 · 60) 15(60) 36 sq. nindan = 2(bùr) 1(èše) 3(iku) (+ 36 sar)

Fig. 6.2.3. *DPA* 34. Reproduction of Limet's hand copy.

In *DPA* 34, the sides of a quadrilateral field are given as

26 north, 1(60) 50 east, 1(60) – 2 south, 1(60) 50 – 4 west.

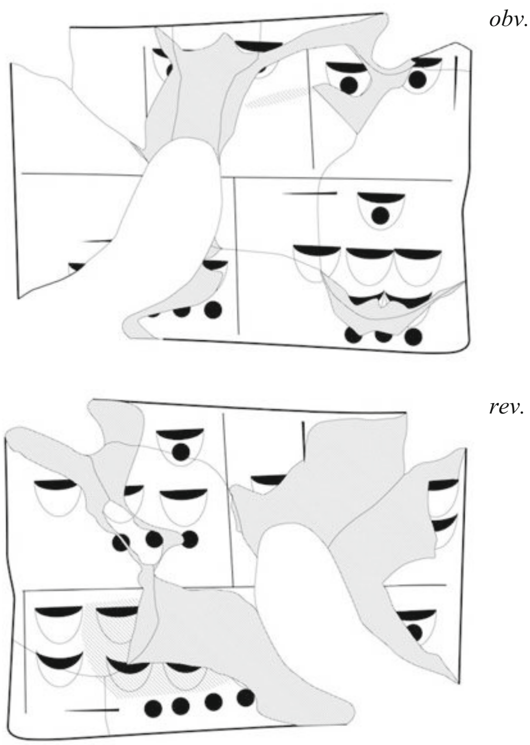
The area is given, without explicit computations, as 2(bùr) 1(èše) 3 iku. This result is essentially correct, as shown in Fig. 6.2.3 above. Note that the exact area is a "nearly round number", since

A = (approximately) 2(bùr) 1 (èše) 3 iku = 2 1/2 bùr = 45 iku.

6.2.9 The Quadrilateral Area Rule in Some Proto-Cuneiform Texts from Uruk IV

The badly broken text W 19408,76+ (cdli.ucla.edu/P003118) was presented and correctly analyzed in Nissen, Damerow, and Englund, *ABK* (1993), Fig. 50. It is from the Uruk IV period (ca. 3350-3200 BC). On the obverse and reverse of this text are given the length measures of two quadrilateral fields, in terms of large sexagesimal numbers, presumably multiples of the basic length unit which in Sumerian and Babylonian texts is called nindan. The two sides “along” are indicated by vertical strokes, and the two sides “across” by horizontal strokes.

It is, of course, tempting to try to calculate the areas of the two quadrilateral fields by use of the (inexact) quadrilateral area rule. This was done in *ABK* mathematically correctly, but anachronistically, in terms of multiplications of decimal numbers. In Fig. 6.2.4 below, the needed multiplications are expressed instead, less anachronistically, in terms of multiplications of sexagesimal numbers.



$A = 20(60) \cdot (15\,(60)\,30 + 14(60)\,30)/2$
$\quad = 20(60) \cdot 15(60) = 5(60 \cdot 60 \cdot 60)$

$5(60 \cdot 60 \cdot 60)$ sq. nindan
$= 10 \cdot 60$ bùr $= 10$(šár) $= 1$(šar’u)

$A = (16(60)\,20 + 23(60)\,30)/2 \cdot (21\,(60)\,20 + 8(60)\,40)/2$
$\quad = 20(60) \cdot 15(60) = 5(60 \cdot 60 \cdot 60)$

$5(60 \cdot 60 \cdot 60)$ sq. nindan
$= 10 \cdot 60$ bùr $= 10$(šár) $= 1$(šar’u)

Fig. 6.2.4. W 19408,76. A metro-mathematical field-sides text from the Uruk IV period with large numbers.

In this connection, it may be a good idea to take a look at what is known about the organizations of the proto-cuneiform systems of length and area measures and of their (Old) Sumerian descendants:

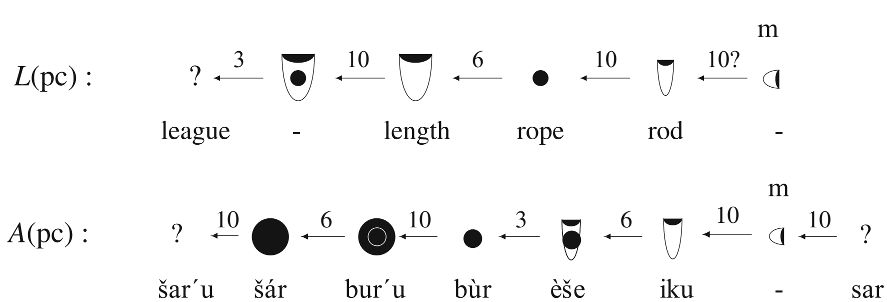


Fig. 6.2.5. The proto-cuneiform systems of length and area units and the corresponding (Old) Sumerian names for the units.

The proto-cuneiform/Sumerian systems of length measures and area measures can be thought of as being linked together by the following kind of "multiplication table":

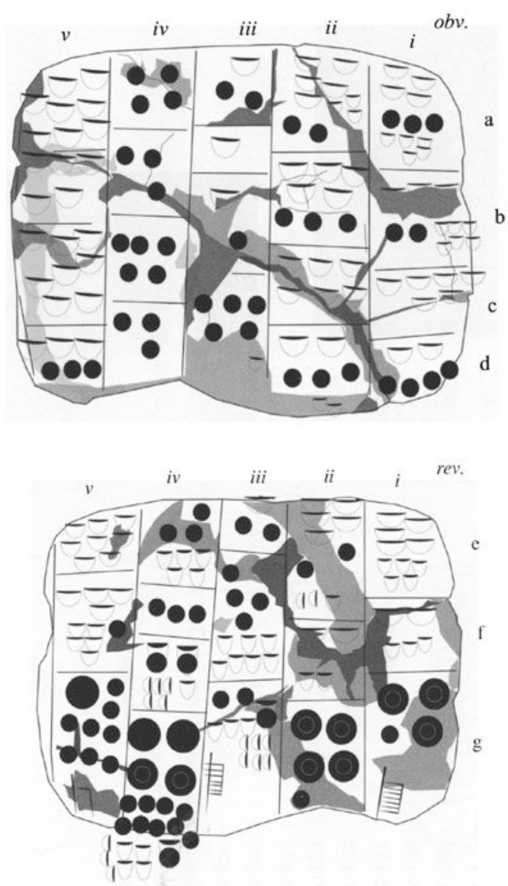
rod · rod	=	1 sar						
rope · rod	=	1m	rope · rope	=	1 iku			
length · rod	=	6m	length · rope	=	1 èše	length · length	=	2 bùr
league · rod	=	1 iku 8m	league · rope	=	1 bùr	league · length	=	6 bùr

Correspondingly, in the case of W 19408,76 in Fig. 6.2.4 above, the computation of the area measures of the two quadrilaterals can be thought of as being carried out in the following simple way, without an abundance of sexagesimal arithmetics:

20 lengths · 15 lengths = 5(60) lengths · lengths = 5(60) · 2 bùr = 10 šár = 1 šar´u.

It is, of course, not known if this is the way the computation of the area measure would have been carried out at the time when W 19408,76 was written, but at least the assumption that this is how it was done is not unreasonable and not anachronistic.

In this connection, it is interesting to reconsider the proto-cuneiform text CUNES 52-06-031, recently published by Monaco in *CUSAS 21* (2012), 38 and correctly (but anachronistically) explained by him in *SEL* 28 (2011). To the left below is shown Monaco's hand copy of the text, to the right a suitable transliteration of the entries in the five columns of the text, continued from the obverse to the reverse.



i obv.	*rev.*
4(60) 35	5(60) 5
⸢3(60)⸣ 26	3(60) 3
6(60)	3(burú) 1(bùr) [1m 5 sar] ašag
2(60) 40	
ii obv.	*rev.*
5(60) 2⸢4 4m⸣	5(60) ⸢42⸣ 2m
5(60) 30	⸢4(60)⸣ 2
⸢6(60)⸣	4(burú) [6(bùr) … ašag]
2(60) 3⸢4⸣	
iii obv.	*rev.*
1(60) 20	1(60) 20
1(60)	5(60) 7
1(60) [20]	2(bùr) 1(èše) 3(iku) 6m ašag
5[4]	
iv obv.	*rev.*
40	⸢4⸣5
30	30
50	2(èše) 1(iku) 5m
30	2(šár) 2(bur´u) 9(bùr) 1(èše) 4(iku) 3m
v obv.	*rev.*
8(60)	8(60)
5(60)	4(60) 15
8(60)	1(šár) 8(bùr) ⸢ašag⸣
3(60) 30	

Fig. 6.2.6. The proto-cuneiform text CUNES 52-06-031.The side lengths of five quadrilateral fields, and explicit computations of their areas by use of the quadrilateral area rule. A summation of the five areas.

In this text, the side lengths *a, b, c, d* of five quadrilateral fields are registered in the five columns on the obverse. In the continuations of the five columns on the reverse, the half sums $e = (a+c)/2$ and $f = (b+d)/2$ are registered, as well as the five corresponding areas, computed by use of the quadrilateral area rule.

In terms of the (Sumerian) length units rod, rope, and length, the given side lengths in column *ii*, for instance, are

a, b, c, d = 5 lengths 2 ropes ⸢4 rods 4m⸣, 5 lengths 3 ropes, ⸢6 lengths⸣, 2 lengths 3 ropes ⸢4 rods⸣.

The corresponding half sums are

$(a + c)/2$ = 5 lengths 4 ropes ⌜2 rods⌝ 2m, $(b + d)/2$ = 4 lengths 2 rods.

These are the length numbers recorded on the reverse of the tablet, in text boxes *ii*:e-f. *Apparently m (the proto-cuneiform sign N08), a small, rotated version of the sign for 1 rod, is a previously undocumented notation for 1/10 rod!*

According to the (inexact) quadrilateral area rule, the area of the quadrilateral with the sides registered in column *ii* on the obverse must then be

5 lengths 4 ropes ⌜2 rods⌝ 2m · 4 lengths 2 rods =
20 lengths · lengths + 16 lengths · ropes + 18 lengths · rods + 8 ropes · rods + 4 rods · rods + 8 lengths · 1m + 4 rods · m =
20 · 2 bùr + 16 èše +18 · 1(60) sar + 8m + 4 sar + 48 sar + 4 sar/10 =
4 bur´u + 5 bùr 1 èše+ 1 èše 4 iku 8m + 8m + 4 sar + 4m 8 sar + 4 sar/10 =
4 bur´u 6 bùr 1m 2 4/10 sar.

Here, apparently, m is a previously unknown notation for§ 1/10 iku! In text box *ii*:g on the reverse is recorded 4(bur´u) 1[..](bùr) […].

Similarly, in column *iii*, the two half-sums of the side lengths recorded on the obverse are

$e = (a + c)/2$ = 1 length 2 ropes, $f = (b + d)/2$ = 5 ropes 7 rods.

According to the quadrilateral area rule, the corresponding area is

1 length 2 ropes · 5 ropes 7 rods = 5 lengths · ropes + 10 ropes · ropes + 7 lengths · rods + 14 ropes · rods =
5 èše + 10 iku + 7(60) sar + 14m = 1 bùr 2 èše + 1 èše 4 iku + 4 iku 2m + 1 iku 4m = 2 bùr 1 èše 3 iku 6m.

This is precisely the area number recorded in text box *iii*:g on the reverse of the tablet.

In column *iv*, the two half-sums of the side lengths recorded on the obverse are

$e = (a + c)/2$ = 4 ropes 5 rods, $f = (b + d)/2$ = 3 ropes.

The corresponding area, according to the quadrilateral area rule, is

4 ropes 5 rods · 3 ropes = 12 ropes · ropes + 15 ropes · rods = 12 iku + 15m = 2 èše 1 iku 5m.

This is the area number in text box *iv*:g.

In column *v*, finally, the two half-sums of the side lengths recorded on the obverse are

$e = (a + c)/2$ = 8 lengths, $f = (b + d)/2$ = 4 lengths 1 rope 5 rods.

The corresponding area, according to the quadrilateral area rule, is

8 lengths · 4 lengths 1 rope 5 rods = 32 lengths · lengths + 8 lengths · ropes + 40 lengths · rods =
32 · 2 bùr + 8 èše + 4(60)m = 1 šár 4 bùr + 2 bùr 2 èše + 1 bùr 1 èše = 1 šár 8 bùr.

This is the area number in column *v*:g.

So far, the computations in the text appear to have been correct. In the case of column *i*, on the other hand, it is clear that the author of the text has made a mistake. The half-sums of the side lengths recorded in column *i* on the reverse should have been

$e = (a + c)/2$ = 5 lengths 1 rope 7 1/2 rods, $f = (b + d)/2$ = 3 lengths 3 rods.

Instead, the length numbers occurring in text boxes *i*:e-f on the reverse are

$e = (a + c)/2$ = 5 lengths 5 rods, $f = (b + d)/2$ = 3 lengths 3 rods.

The corresponding area, according to the quadrilateral area rule, is

5 lengths 5 rods · 3 lengths 3 rods =15 lengths · lengths + 15 lengths · rods + 15 lengths · rods + 15 rods · rods =

15 · 2 bùr + 15 · 6m + 15 · 6m + 15 sar = 3 bur´u 1 bùr 1m 5 sar.

If the mistake in text box *i*:e on the reverse had not been made, the computed area would have been somewhat larger. Clearly, since the error in text box *i*:e is

1 rope 7 1/2 rods – 5 rods = 1 rope 2 1/2 rods,

the resulting error in the area number in text box *i*:g is

1 rope 2 1/2 rods · 3 lengths 3 rods = 3 lengths · ropes + 7 1/2 lengths · rods + 3 ropes · rods + 7 1/2 rods · rods =
3 · 1 èše + 7 1/2 · 6m + 3m + 7 1/2 sar = 3 èše + 45m + 3m + 7 1/2 sar = 1 bùr 4 iku 8m 7 1/2 sar.

The sum of the five area numbers recorded in text boxes *i*-*v*:g on the reverse is

3 bur´u 1 bùr 1m 5 sar + 4 bur´u [6 bùr 1m 2 4/10 sar] + 2 bùr 1 èše 3 iku 6m + 2 èše 1 iku 5m + 1 šár 8 bùr =
2 šár 2 burú 8 bùr 5 iku 3m 7 sar 4 · 1/10 sar.

However, without the error in text box *i*:e on the reverse and the resulting error in text box *i*:g, the sum of the five computed areas would have been instead

2 šár 2 bbur´u 8 bùr 5 iku 3m 7 sar 4 · 1/10 sar + 1 bùr 4 iku 8m 7 1/2 sar =
2 šár 2 bbur´u 9 bùr 1 èše 4 iku 1m 7 sar 9 · 1/10 sar =
approximately 2 šár 2 bur´u 9 bùr 1 èše 4 iku, the area number recorded in text box *i*:g on the reverse.

This means that, as noted already by Monaco in *SEL* 28 (2011), the area number in text box *iv*:h is equal precisely to the sum of the areas of the five quadrilaterals with the side lengths specified in the five columns on the obverse of CUNES 52-06-031, in spite of the error in text box *i*:e!

This curious circumstance can possibly be explained by assuming that CUNES 52-06-031 was a somewhat inexact copy of another tablet with all computations correct.

The computation of the area of the quadrilateral in column *ii*, for instance, was explained above as the rather complicated multiplication of 5(60) 42 2m, interpreted as 5 lengths 4 ropes 2 rods 2m with 4(60) 2, interpreted as 4 lengths 2 rods. Another possibility is, of course, that the area was computed directly through multiplication of the pure number 5(60) 42 2m with the pure number 4(60) 2. However, as it turns out, the multiplication is just as complicated in this case as before, and an added difficulty is that the result of the multiplication, which is 23(60 · 60) 12 4/10, must be converted in some way into an area number.

In cdli.ucla.edu/P411619, the text CUNES 52-06-031 is said to be from the Uruk III period (3200-30000 BC), of unknown provenance. Another text, the fragment *ACPTC* 8 (cdli.ucla.edu/P006413), also mentioned by Monaco in *SEL* 28 (2011), which may have been a parallel to CUNES 52-06-031, is likewise said in cdli.ucla.edu/P006413 to be from Uruk III. However, *ACPTC* 8 differs from CUNES 52-06-031 in that it explicitly states in columns *i, ii,* […] that the items a and c are side lengths 'along' while the items b and d are side lengths 'across'. This circumstance seems to indicate that there was a close connection between *ACPTC* 8 (and CUNES 52-06-031) on one hand and several known "field sides texts" attributed to the Uruk IV period (3350-3200 BC) on the other hand, in particular W 19408, 76 in Fig. 6.2.4 above. See Monaco, *op. cit*., footnote 1.

Incidentally, Monaco writes in *SEL* 28 (2011) that

"True mathematical texts are those which provide, in addition to the input data, the full development of the proposed problems, i.e. the calculations (showing the method adopted to solve the problems), as well as the solutions resulting from the calculations. The two texts proposed in this paper are the most archaic true mathematical texts ever published."

An interesting observation in this connection is that the three known "field sides and area texts" *MSVO 1*, 4-6 from Jemdet Nasr (the Uruk III period) mentions the side lengths of five specified rectangular fields, the areas of the five fields, and the sum of those five areas. Monaco mentions the possibility that also the five fields in each one of those three texts may have been non-rectangular quadrilaterals, but that only the half-sums of the sides of the quadrilaterals are recorded. Thus, there appears to be close links between *MSVO 1*, 4-6 and CUNES 52-06-031 (as well as the related fragments *ACPTC* 8 and CUNES 50-08-076 (cdli.ucla.edu/P325157)). In particular, the use of the sign m as a sign for 1/10 iku is shared by CUNES 52-06-031 and *MSVO 1*, 2 (Friberg, *AfO* 44/45 (1997/1998), Sec. 2).

Indeed, in *MSVO 1*, 2 col. *ii*, the area of a rectangle with the side lengths 5 lengths 1 rope 2 rods and 1 length 3 ropes is recorded as 1 bur´u 5 bùr 1 èše 4 iku ⸢8 m⸣. The corresponding computation is

5 lengths 1 rope 2 rods · 1 length 3 ropes =

5 lengths · lengths 16 lengths · ropes 3 ropes · ropes 2 lengths · rod 6 ropes · rods =

1 bur´u + 16 èše +3 iku + 12 m + 6 m = 1 bur´u 5 bùr 1 èše 4 iku 8 m.

6.2.10 The Vocabulary of IM 121565

an.ta	upper
a.šag$_4$	field, area
ba.sá.e	equalside (= square-side, square root)
daḫ	add on
íb.si.e	equalside (= square root)
igi (*n*) duḫ	release the opposite of (*n*), compute the reciprocal
ki.ta	lower
lagab, lagab-*ti*	equalside, square
sag.ki	front (= short side)
.ta.àm	each (way)
ugu	over
uš	length (= long side)
za.e	you
zi.zi	break off (= subtract)
a-na šag$_4$	onto
i-na šag$_4$	out of
ki ma-ṣi	*kī maṣi* how much?
ki-a-am né-pé-šum	*kīam nēpešum* such is the procedure
ul i-de	*ul īde* I did not know; <*idûm* to know
šu-ta-ki-il	<*akālu* Št to make eat each other (= to multiply (two sides))
ta-mar	< *amāru* to see (= to find a result)
ab-ni	<*banû* to create, to build
šu-li	<*elû* Š to come up, to rise (= to result (of a square root))
e-ṣi-ma	<*eṣēpu* to double
ḫu-ru-iṣ	< *ḫarāṣu* to break off (= to subtract)
ḫe-pé	<*ḫepû* to break (= to halve)
ak-mur, ku-mur	< *kamāru* to heap, to pile up (= to add together)
lu-pu-ut	< *lapātu* to touch
le-qé	< *leqû* to take
uš-tam-ḫi-ir, uš-tam-ḫi-ru	< *maḫāru* Št become equal to each other, become equalsided
ad-di, i-di	<*nadû* to lay down, to inscribe, to draw
i-ši, i-ta-aš-ši	<*našû* to carry (= to multiply (by a number))
at-ta-sa-am, ta-ta-sa-am	< *nesû* to be/become distant
a-tu-úr, tu-úr	< *târu* to return (= start again)
dal-ba-nim	*dalbānu* intervening space
ki-ia	*kiyā* how much each?
ki-di-tum	*kidû* outer
ki-ma	*kīma* as, like
lu	*lū* let it be
ma-la	*mala* whatever, as much as
mé-eḫ-ra-am	*meḫru* copy, duplicate
mi-nu, mi-na-am	*mīnu* what?
qer-bi-tum	*qerbu* inner
re-[*ḫu*]	*rēḫu* remainder
ta-ki-il-ta-ka	*takiltu* holder (of your head)
ša-ni-im	*šanû* second
šu-ši	*šūši* sixty
wa-tu-ra-am	*watru* surplus, excess

6.2.11 IM 121565. Hand Copies and Conform Transliterations of the Tablet

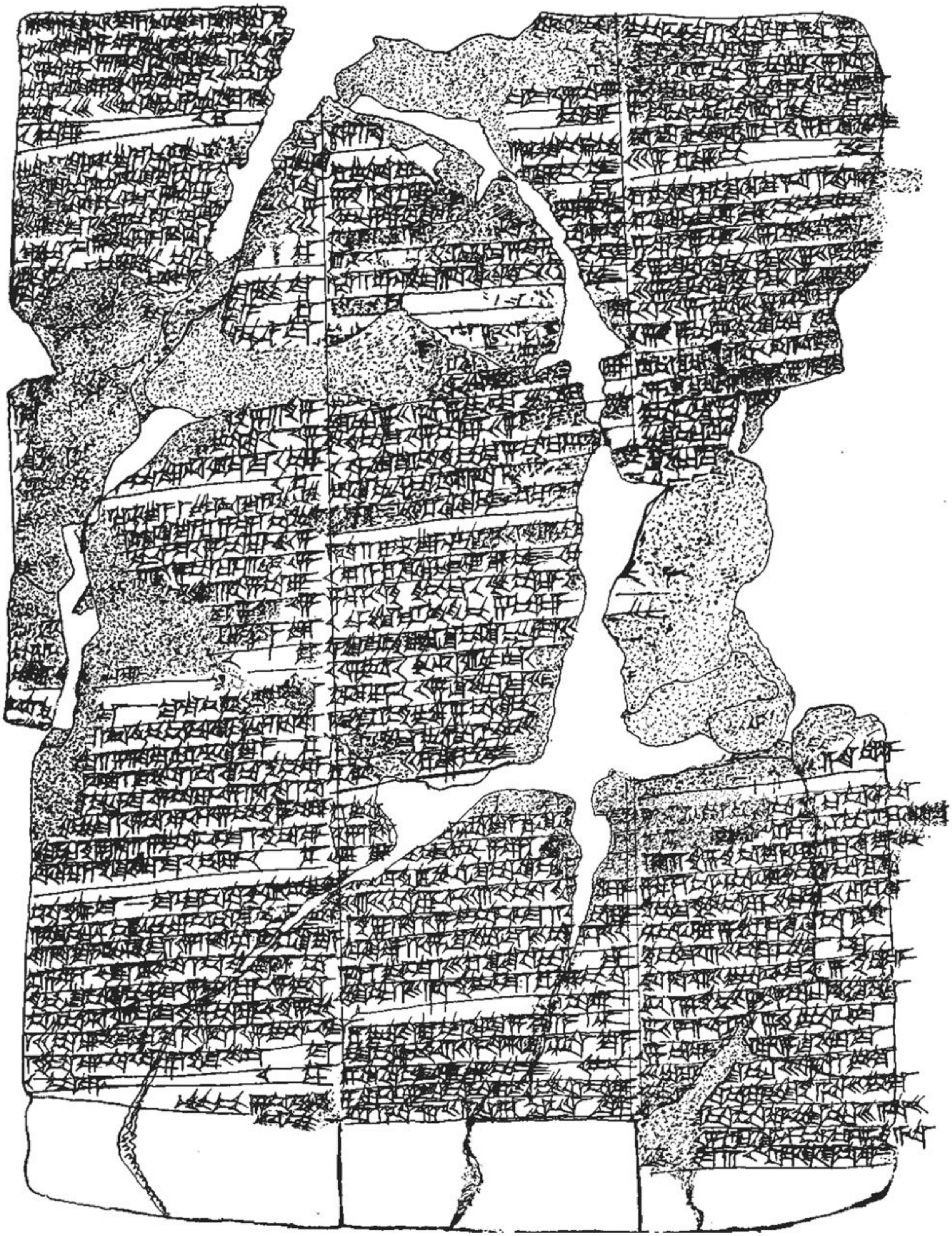


Fig. 6.2.7. IM 121565, *obv*. Hand copy.

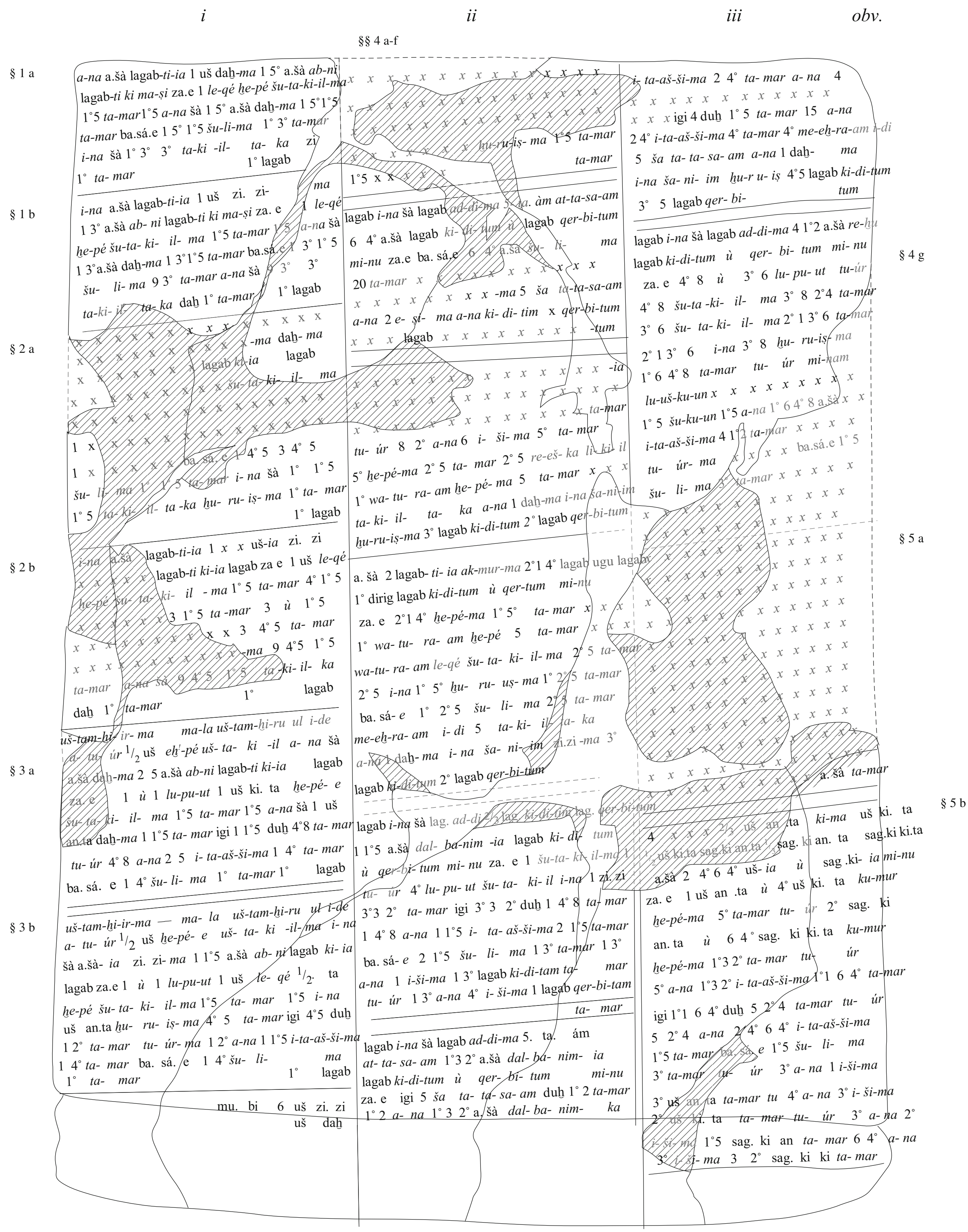


Fig. 6.2.8. IM 121565, *obv*. Conform transliteration.

Fig. 6.2.9. IM 121565, *rev*. Hand copy.

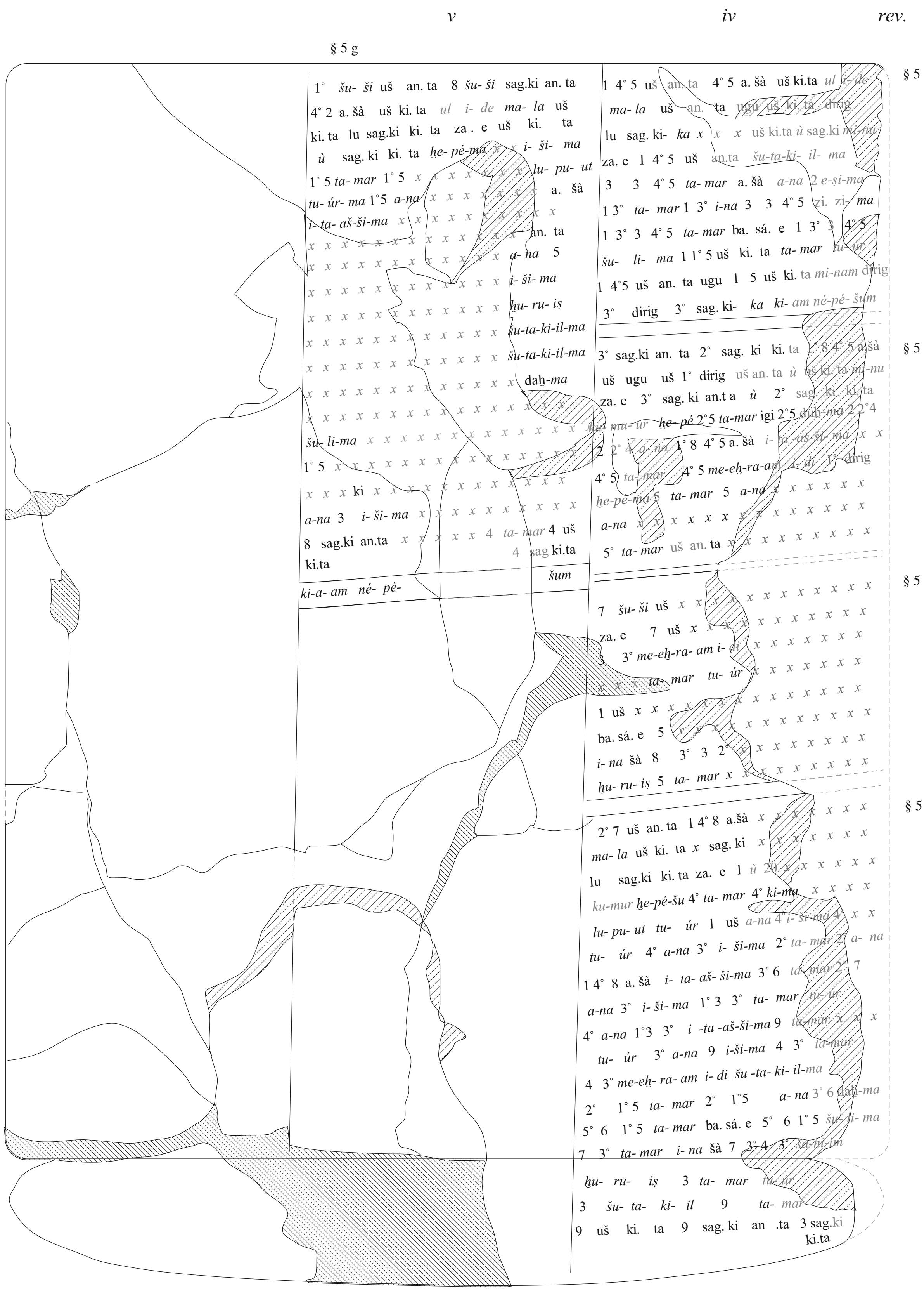


Fig. 6.2.10. IM 121565, *rev*. Conform transliteration.

6.3 IM 121512. A Large Recombination Text with Metric Algebra Problems for Circles and Semicircles

Hand copies of this text, which is moderately well preserved, are shown in Figs. 6.3.3-4 below.

6.3.1 *§§ 1a-j. Metric Algebra Problems for One or Two Circles*

IM 121512 § 1a To find the transversal (diameter) of an arc (circle) when the […] is known.

1 [x x x x x x x x dal *mi-nu*] /
2 [za.e x x x x x x x x x] /
3 [x x x x x] *a-na* šag$_4$ x [x x x x x x] /
4 [x x x *i-ši*]-*ma* 3 20 dal *ta-mar*

1 [x x x x x x x x the transversal (diameter) was what?] /
2 [You: x x x x x x x x x] /
3 [x x x x x] onto x [x x x x x x] /
4 [x x x carry], then 3 20 the transversal you will see.

The text of this exercise is almost completely destroyed. Nevertheless, it is clear from the preserved last few words of the exercise, and from the ensuing paragraphs, that the question was about the length of the transversal (diameter) of a circle, when some other parameter was given.

IM 121512 § 1b To find the string (radius) of a circle when the field (area) is known. Incorrect solution algorithm.

1 ⸢8 20 a.šag$_4$⸣ *ma-at-nu mi-nu* /
2 [za.e 5 igi.g]ub 10 *ma-at-*⸢*na*⸣ *lu-pu-ut* /
3 [igi 5 igi].⸢gub⸣ duḫ-*ma* [12 *ta-mar*] /
4-5 [x x x] 10 *ma-at-na a-na* 12 *i-ši-ma* / [x x x x x x] 1 40 *ta-mar*

1 ⸢8 20 the field⸣. The string was what? /
2 [You: 5 the const]ant, 10 the string touch. /
3 [The opposite of 5 the con]⸢stant⸣ release, then [12 you will see]. /
4-5 [x x x] 10 the string to 12 carry, then / [x x x x x x] 1 40 you will see

The text of this exercise is sufficiently well preserved so that it is clear what is going on. The area of a circle is known, A = 8 20.With '5' as the constant for the area of a circle and with '10' as the constant for the radius r, one has that

A = 5 sq. a, r = 6 a, where a is the (length of the) arc of the circle.

The correct solution procedure with go forward as follows:

sq. a = rec. 5 · A = 12 · 8 20 = 1 40, a = sqs. 1 40 = 10, r = rec. 6 · a = 10 · 10 = 1 40.

The solution procedure in the text begins correctly by computing rec. 5 = 12. However, then it proceeds incorrectly with a nonsense calculation in line 4. In spite of this, the answer in line 5 is correct.

From a linguistic point of view, the use of Akkadian *matnu* 'string' in § 1b as an Old Babylonian word for 'radius' comes as a big surprise. No word for 'radius' has ever appeared before in any known mathematical cuneiform text. (The Akkadian word *perku* 'cross-line' in line 9 of the table of constants BR = *TMS III* is close but is not really a word for 'radius'. The word is derived from *parāku* 'to lie across' and clearly refers to the height of a semicircle, orthogonal to the diameter.) Note that to use 'string' as a word for the radius is quite reasonable, since a taught string fastened at the center of a circle or semicircle can be used, just like the radius, to trace the arc of that circle or semicircle. (The Latin word *radius*, on the other hand, originally meant either 'ray' or, in a more specific sense, 'spoke of a chariot wheel'.)

IM 121512 § 1c To find the field (area) when the arc is known.

1 [10 pat gúr]$^{?}$ a.šag$_4$ *mi-nu*
2 za.e 5 igi.gub gúr / [20 dal gúr]$^{?}$ ⸢*lu-pu-ut*⸣$^{!}$-*ma*
3 10 ⸢pat gúr⸣$^{?}$ ⸢*šu-ta*⸣-*ki-il-ma* / [1 40 *ta-mar*]
4 1 40 *a-na* 5 igi.gub *i-ši-ma* / [8 20 a.šag$_4$] *ta-mar*

1 [10 the border of the arc]$^{?}$. The field was what?
2 You: 5 the constant of the arc, / [20 the border of the arc]$^{?}$ ⸢touch⸣$^{!}$, then
3 10 ⸢the border of the arc⸣$^{?}$ ⸢let⸣ eat itself, then / [1 40 you will see].
3-4 1 40 to 5 the constant carry, then / [8 20 the field] you will see.

In § 1c, the arc is [known] and the area is computed in the usual way, by use of the area constant ‘5’. The constant for the diameter is also (possibly) mentioned, although it will not be needed.

The term pat gúr ‘border of the arc’(?) which is reconstructed in two places in § 1c of IM 121512 reoccurs in lines 1 or 2 of §§ 1e-f, 2a-b and 2e. The normal meaning of the Sumerian word pat, which is ‘to break’ of ‘to find’ does not fit here. Hesitantly, it is suggested that the sign pat may be an abbreviation for the Akkadian word *patu* ‘border’, and that pat gúr in §§ 1e-f simply stands for the length of the arc of the circle, or, in modern terms, for the circumference of the circle. In §§ 2a-b and 2e, the same expression seems to stand for the length of the arc of a semicircle. Note, however, that in several places in § 2 of IM 121512 the phrase used for the length of the arc of a semicircle is instead pat lagab. More about that below.

IM 121512 § 1d to find the field (area) when the […] is known.

1 [x x x x x x x x x x] lagab /
2-4 [za.e x x x *šu*]*-ta-ki-il-ma* / [x x x x x x] x *re-eš-ka* / [*li-ki-il*]
5 [*tu-úr*] 8 20 *a-na* x *i-ši-ma* / [x x *ta-mar*]
[x x] x x 40 e [50]$^{?}$ ⸢a.šag$_4$ *ta-mar*⸣

1 [x x x x x x x x x x] /
2-4 [You: x x x let] eat itself, then / [x x x x x x] x your head / [may hold].
5 [Turn around.] 8 20 to x carry, then / [x x you will see].
[x x] x x 40 [50]$^{?}$ ⸢the field you will see⸣.

The text of this exercise is almost completely destroyed.

IM 121512 § 1e A quadratic equation for the arc of a circle, 5 sq. a + 1 · a = 18 20.

1-2 [*a-na*] ⸢a.šag$_4$⸣ [1 uš daḫ] 18 20 a.⸢šag$_4$⸣ [*mi-nu*] / [*ki*]*-ma* ⸢pat⸣$^{!}$ gúr
za.⸢e⸣ [18 20 a.šag$_4$ 5 igi.gub gúr *lu-pu-ut*] /
3-4 [*t*]*u-úr-ma* 18 20 [a.šag$_4$ *a-na* 5 igi.gub *i-ši-ma*] /[1 3]1 40 *ta-mar re-eš-*[*ka li-ki-il*] /
5-6 [*tu*]*-úr-ma* 1 uš *le-*[*qé ḫe-pé šu-ta-ki-il-ma*] / [1]5 *ta-mar*
7 15 *a-na* šag$_4$ 1 31[40 *ša re-eš-ka*] / [*ú-ki-lu*] daḫ*-ma* 1 46 40 *ta-ma*[*r*]
8 [ba.sá.e] / [1 46 40] *šu-li-ma* 1 20 *ta-*[*mar*]
9-10 [*aš-šum*] / [daḫ *qá-bu-ku*] ⸢*ta-ki-il-ta*⸣*-*[*ka*] / [zi.zi]*-ma* ⸢50 *ta-mar*⸣
11 [*tu-úr-ma* igi 5] / [igi.gub gúr] duḫ*-ma* 12 *ta-mar*
12 [x x x] / ⸢50⸣$^{!}$ *a-*[*na* 12] *i-ši-ma* [10]$^{!}$ uš [*ta-mar*]

1-2 [To] ⸢the field⸣ [1 length I added on] 18 20. The ⸢field⸣ [was what]? / [Li]ke (what) the ⸢ border⸣$^{?}$ of the arc?
Yo⸢u⸣: [18 20 the field, 5 the constant of the arc touch.] /
3-4 [T]urn around, then18 20 [the field to 5 the constant carry, then] /[1 3]1 40 you will see. [Let it hold your] head. /
5-6 [Turn] around, then 1 length ta[ke, break, let eat itself, then] / [1]5 you will see.
7 15 onto 1 31[40 that your head] / [held] add on, then 1 46 40 you will se[e].
8 [The equalside of] / [1 46 40] let come up, 1 20 you [will see].
9-10 [Since] / [I added on it was said to you] [your] ⸢holder⸣/ [tear off], then ⸢50 you will see⸣.
11 [Turn around, then 5] / [the constant of the arc] resolve, then 12 you will see.
12 [x x x] / ⸢50⸣ t[o 12] carry, then [10]$^{!}$ the length [you will see].

In quasi-modern notations, the problem in this exercise can be expressed as the following *quadratic equation for a circle*:

5 sq. a + 1 · a = 18 20.

After multiplication by the area constant ‘5’, as in lines 3-4, this equation is transformed into the following *quadratic equation for a square*:

sq. (5 a) + 1 · 5 a = 5 · 18 20 = 1 31 40.

This is a quadratic equation of the Old Babylonian basic type B4a, just like the equation in IM 121565 § 1a (Sec. 6.2.1 above.) The equation in IM 121512 § 1e is solved routinely in the same way as that equation.

IM 121512 § 1f A quadratic equation for the arc of a circle, 5 sq. $a - 1 \cdot a = 8\ 10$.

1 *i-n*[*a* a.šag$_4$ 1 u]š zi.zi-*ma* 8 10 a.šag$_4$ /
2 [za].⸢e⸣ [8 10 a.šag$_4$⸣ 5 igi.gub gúr *lu-pu-u*[*t*] /
3-4 [x x x 8 10 a].šag$_4$ *a-na* 5 igi.gub *i-ši-*[*ma*] / 40 e <50> *ta-mar re-eš-ka li-ki-*[*il*] /
5-6 *tu-úr* 1 *le-qé ḫe-pé šu-ta-ki-*[*i*]*l-ma* / 15 *ta-mar*
7 15 *a-na* šag$_4$ 40 e 50 *ša re-eš-ka* / *ú-ki-lu* daḫ-*ma* 40 e 50 e 15 *ta-mar* /
8-9 ba.sá.e 40 e 50 e 15 *šu-li-ma* / 49 30*fa-mar*
10 *aš-šum* zi.zi *qá-bu-ku* / [*ta-k*]*i-il-ta-ka* daḫ 50 *ta-mar* /
11 [*tu-úr*]-*ma* igi 5 igi.gub duḫ-*ma* 12 *ta-mar* /
12 [x x x x] 50 *a-na* 12 *i-ši-ma* 10 *ta-mar* pat^{ti} gúr

1 Fro[m the field] x x I tore off, then 8 10 the field. /
2 [You: x x] x 5 the constant of the arc touc[h]. /
3-4 [x x x 8 10 the f]ield to 5 the constant carry, [then] / 40 <50> you will see. Let it hold your head. /
5-6 Turn around. 1 take, break, let eat itself, then / 15 you will see.
7 15 onto 40 50 that your head / held add on, then 40 50 15 you will see. /
8-9 The equalside of 40 50 15 let come up, then / 49 30 you will see.
10 Since "I tore off" was said to you, / [your hol]der add on, 50 you will see. /
11 [Turn around], then the opposite of 5 the constant resolve 12 you will see. /
12 [x x x x] 50 to 12 carry, then 10 you will see, the border$^?$ of the arc.

In quasi-modern notations, the problem in this exercise can be expressed as the quadratic equation

5 sq. $a - 1 \cdot a = 8\ 10$.

After multiplication by the area constant '5', as in lines 3-4, this equation is transformed into

sq. $(5\ a) + 1 \cdot 5\ a = 5 \cdot 18\ 20 = 1\ 31\ 40$.

This is a quadratic equation of the Old Babylonian basic type B4b, just like the equation in IM 121565 § 1b (Sec. 6.2.1 above.) The equation in IM 121512 § 1f is solved routinely in the same way as that equation.

Note that, strictly speaking, the data in the two exercises IM 121512 §§ 1e-f were not computed in the same way by the one who constructed the exercises. Indeed, in terms of sexagesimal numbers in *absolute* place value notation, the right hand side of the quadratic equation in IM 121512 § 1e was computed as follows:

;05 sq. 10 + 1 · 10 = ;05 · 1 40 + 1 · 10 = 8;20 + 10 = 18;20.

If the right hand side of the quadratic equation in IM 121512 § 1f had been computed in a similar way, the result would have been a negative number, against the (silent) conventions of Old Babylonian mathematical texts. Indeed,

;05 sq. 10 – 1 · 10 = ;05 · 1 40 –1 · 10 = 8;20 – 10 = a negative number.

Instead the right hand side of the equation was computed as follows:

;05 sq. 10 – 1 · 10 = ;05 · 1 40 – ;01 · 10 = 8;20 – ;10 = 8;10.

However, for the one who constructed the exercises the difference between the two computations was hardly noticeable, since he worked with sexagesimal numbers in Old Babylonian *relative* place value notation. Actually, the choice he probably was unaware of making in § 1e was between the following two possibilities:

5 sq. 10 + 1 · 10 = 8 20 + 10 = 18 20 or 5 sq. 10 + 1 · 10 = 8 20 + 10 = 8 30.

In § 1f there was only one possible way of making the computation.

Especially noteworthy are the sexagesimal numbers written 40 e 50 and 40 e 50 e 15 in lines 6-8 of IM 121512 § 1f. The numbers in question can be transliterated, more precisely, as 4°e 5°e and 4°e 5°e 1° 5. Then it becomes clear that the e-signs here are a kind of 'zeros' or rather *placeholders*, indicating *vacant places for units*. (Recall that sexagesimal numbers in place value notation are written as a succession of tens followed by units.) Therefore, in modern notation the mentioned numbers can be interpreted as follows:

40 e 50 or 4° e 5° = 40 50, and 40 e 50 e 15 or 4° e 5° e 1° 5 = 40 50 15.

The same kind of notation can be observed below in the following cases:

2 10 e 30 = 2 10 30 in § 2f line 9, and 2 30 e 50 = 2 30 50 in § 2h line 11.

A surprising variant of the same idea can be observed in the following case:

6 e 6 40 = 6 06 40 in § 2i line 7.

Here the sign e indicates a *vacant place for tens*!

IM 121512 § 1g ?

1 [x x x x] x -*ma* x x-*ni* ⸢a.šag$_4$ *mi-nu*⸣ /
2 x x x x [x x x x] x x x x x x x x] /
3 [x x x x x x x x 5]0 gúr
4 za.e 8 [x x x x x x] a.šag$_4$ / *i-na* šag$_4$ [x x x *re-eš-ka li*]-*ki-il*
5 *tu*-[*úr*] / 1 uš *le*-[*qé ḫe.pé*] *šu-ta-ki-il-ma* 15 *ta-ma*[*r*] /
6 15 *a-na* [x x x x x] 5 15 *ta-mar*
7-8 *mi-nam a-na* [x x x x x x] ⸢11 06⸣ 45 a.šag$_4$ / *li-ki-i*[*l-ma* x x x x x x]
9 [x x x x x x] / 5 15 *i-ta-a*[*š-ši-ma* x x x x x x x x] /
10 *tu-úr-ma* [x x x x x x x]

The text of this exercise is badly preserved, beyond repair.

IM 121512 § 1h

The text of this exercise is almost completely destroyed.

IM 121512 § 1i

1 *a-na* š[ag$_4$ a.šag$_4$ x x x x x x x x x x x -*il*] 3 20 /
2 [x x] [*a-na*] 5 igi.gu[b gúr x x x x x 46 40 /
3 [x x x *a-na*] 46 40 [*šu-ta-ki-il* x x zi.zi *a-na*] /
4 [x x x x 3? *lu-pu*-[*ut* x x x x] *šu*-[*li-ma*] x 20 /
5 50?.e *he-pé* x x x šag$_4$ a.šag$_4$ *a-na* 46 40 *i-ši-ma a-na* /
6 *ta-ki-il-ma* [x x x x x]
7-8 33 [4]5 / *a-na* šag$_4$ 1 56 *š*[*a*] *re-eš-ka ú-ki-lu* / dah-*ma* 2 36 ⸢*ta-mar*⸣
9 ba.sá.e 2 3[6] / *šu-li-ma* 1 30 [x x *lu*]-⸢*pu-ut*⸣
10-11 ⸢1 45 45⸣ / *ta-ki-il-ta*-[*ka* x x 4]6 *a-na ta*-[x-x] *ú-ki-lu* / *re-eš-ka li*-[*ki-il*]
12 *tu-úr-ma* / igi 5 igi.gub [duḫ-*ma* 12 *ta*]-*mar*
13 *tu-úr* / 2 a.šag$_4$ *li*-[*ki-il* x x x x x x x x x [*i-ši-ma*] /
14 [x x x x x x x x x x x x x x x x x x x x x x x] /
15 [x x x x x x x x x x x x x x x x x x x x x x x x]

The text of this exercise is also badly preserved, beyond repair.

IM 121512 § 1j. A problem for two concentric circles

1-2 gúr *i-na* šag$_4$ gúr *ad-di* 2 40 x x [x x] / 1 14 40 a.šag$_4$ dal.ba.an.na
3 gúr *ki-d*[*i-tum*] / *ù qer-bi-tum mi-nu*
4 za.e [x x x igi] / 2 40 duḫ-*ma* [22 30 *ta-mar*]
5 [x x x x] / x x x x [x x x x x x x x x x x] /
6 x x x x [x x x x x x x x x x x x x x x x x] /
7 [x x x x x x x x x x x x x x x x x] / ⸢2⸣8 *ta*-[*mar*]
8 x [x x x x x x x x x x x x x x x x] /
9 [x x x x x x x x x x x x x x x x x x x x x x] /
10-11 [x x x x] x x x 2 40 *mi-si-tum* ⸢*a*⸣-[*na*] / [2 *i-ši-ma*] 5 20 *ta-mar*
12 5 20 *i-na* šag$_4$ 12 *ša re-eš-ka* / [*ú-ki*]-*lu hu-ru-is-ma* 6 40 *ta-mar* /
13 ⸢6⸣ 40 *šu-ul-li-iš-ma* 20 gúr *qer-bi-tum ta-mar*

1-2 An arc into an arc I drew, 2 40 x x [x x], / 1 14 40 the field of the intermediate space.
3 The out[er] arc / and the inner arc were what?
You: [x x x. The opposite of] / 2 40 resolve, then [22 30 you will see].
4 [x x x x] / x x x x [x x x x x x x x x x x] /
5 x x x x [x x x x x x x x x x x x x x x x x] /
6 x x x x [x x x x x x x x x x x x x x x x x] /
7 [x x x x x x x x x x x x x x x x x] / ⸢2⸣8 you [will see].
8 x [x x x x x x x x x x x x x x x x] /
9 [x x x x x x x x x x x x x x x x x x x x x x] /
10-11 [x x x x] x x x 2 40 the distance t[o] / [2 carry, then] 5 20 you will see.
12 5 20 out of 12 that your head / [he]ld break off, 6 40 you will see. /
13 ⸢6⸣ 40 triplicate, then 20 the inner arc you will see.

The text of this problem is badly conserved, with the exception of the first few and the last few lines. Therefore, the nature of the problem considered is clear, even if the details of the solution algorithm remain obscure: One 'arc' (circle) is inscribed (concentrically) into another arc. The 'distance' ([*mi-si-tum*]) between them and the 'field' (area) of the 'intermediate space' (dal.ba.an.na) are both given. To be computed are the 'outer' and 'inner' arcs (gúr *ki-ditum ù qer-bi-tum*). This problem is closely related to the problem in Böhl 1821, to find the outer and inner diameters of a circular band (the extension of a circular town), when the length of the extension and the area of the band are known. See Fig. 9.1.1 below. The problem is also related to the problem in the Susa text *TMS* V, § 7 a´, where a square is inscribed concentrically into another square, and where the distance between the squares and the intermediate area are both known. See Fig. 10.3.3, left.

In quasi-modern symbolic notations, the problem in IM 121512 § 1j can be interpreted as as the following problem for the two unknowns a' and a (the outer and inner arcs of the two circles), when the distance $e = 2\ 40$ and the area $A = 1\ 14\ 40$ are known:

$$a' = 3\,d', \quad a = 3\,d, \quad 3\,(d' + d)/2 \cdot e = A, \quad (d' - d)/2 = e.$$

The situation is illustrated in Fig. 6.3.1 below.

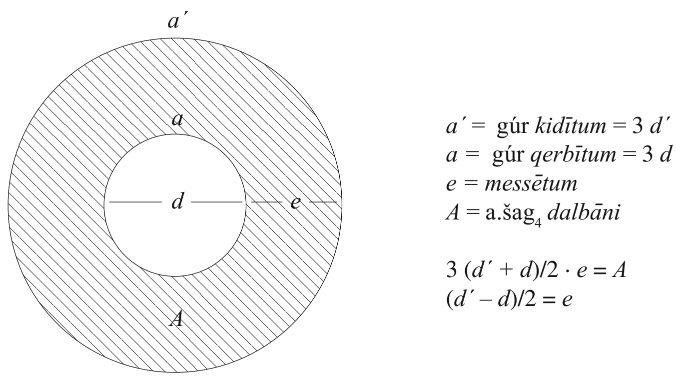


Fig. 6.3.1. IM 121512 § 1j. A metric algebra problem for two concentric circles.

Very little is preserved of the solution procedure in this exercise. Nevertheless, it is clear that towards the end of the solution procedure the diameter d' of the outer circle had been found to be equal to 12, so that the length of the outer arc is $a' = 3\,d' = 36$. Knowing this, the diameter d of the inner circle could be calculated as $d' - 2\,e = 12 - 5\ 20 = 6\ 40$ (lines 10-12). Consequently, the length of the inner arc was found to be $a = 3\,d = 20$ (line 13).

It is likely that in the destroyed part of the solution procedure the diameter d´ of the outer circle was computed in something like the following series of steps:

rec. e · A = 22 30 · 1 14 40 = 28, rec. 3 · 28 = 9 20 = (d´+ d)/2, 9 20 + e = 9 20 + 2 40 = 12.

6.3.2 *Subscript to IM 121512 § 1*

mu.bi.im gúr / 10

Its cases, arcs / 10.

This subscript refers, of course, to the preceding 10 problems for one or two circles.

6.3.3 *§§ 2a-i. Metric Algebra Problems for Semicircles*

(ašag$_4$.)u$_4$.sakar or *uskāru* 'crescent(-field)' is the name used in Old Babylonian mathematical texts for semicircles. Constants for semicircles appear in, for instance, the table of constants BR = *TMS III* (Bruins and Rutten (1961)), where one finds the following three consecutive constants:

15 igi.gub	*šà ús-ka$_4$-ri*	15, the constant	of a crescent (semicircle)	BR 7
40 dal	*šà ús-ka$_4$-ri*	40, the transversal (diameter)	of a crescent	BR 8
20 *pe-er-ku*	*šà ús-ka$_4$-ri*	20, the cross-line (radius)	of a crescent	BR 9

Here, as always, the 'constant' for a geometric object means its area, in the normalized case when its defining parameter is equal to '1'. In the case of circles and semicircles, the defining parameter is the length of the arc. Now, if the arc, the 'transversal' (diameter), and the 'cross-line' (radius) are called *a*, *d*, and *r*, respectively, and if *A* stands for the area, as usual, then the meaning of the three lines BR 7-9 is that

$A = ;15 \cdot a \cdot d, \quad d = ;40 \cdot a, \quad r = ;20 \cdot a.$

It is not difficult to see that these equations are correct if 3 is used as a convenient approximation to π. (In quasi-modern notations,

$a = \pi\, \mathrm{r} = \pi/2\ d, \quad A = \pi/2\ \mathrm{sq.}\ r = 1/2\ a \cdot r = 1/4\ a \cdot d.$)

In the mathematical table of constants NSd = YBC 5022 (Neugebauer and Sachs, *MCT* (1945), text Ud), the entry in line 21 mentions the following pair of constants for a semicircle:

⌜1⌝5 *ù* 10 *ša* ašag$_4$ u$_4$.sakar ⌜1⌝5 and 10 of a crescent-field NSd 21

Here '⌜1⌝5' is probably the same area constant as in BR 7 (see above). The '10' is probably an alternative area constant, standing for ;10 = 1/6, and can be explained by the equation

$A = ;15 \cdot a \cdot d = ;15 \cdot ;40 \cdot \mathrm{sq.}\ a = ;10 \cdot \mathrm{sq.}\ a.$

In § 2a of IM 121512, the constant '40 dal u$_4$.sakar' for the diameter of a semicircle is used. In § 2b, the reciprocal of this constant appears as '1 30 u$_4$.sakar gúr'. In §§ 2c-d, a constant for the area of a semicircle is '10 igi.gub u$_4$.sakar', while in § 2e the constant for the area of a semicircle is instead '15 igi.gub u$_4$.sakar'. The area constant '10 igi.gub' reappears in § 2f line 3, § 2g line 12, § 2h [line 14], and § 2i line 3. In § 2i lines 1 and 6 is mentioned also a constant for the 'string' (radius) of a semicircle as '20 *ma-at-ni/na*'.

IM 121512 § 2a

1 u$_4$.sakar 10 pat gúr dal *mi-nu* /
2 za.e 40 ⌜dal u$_4$.sakar⌝ *lu-pu-ut*
3 40 dal *a-na*[!] / 10 u$_4$.sak[ar *i-š*]*i-ma* 6 40 dal *ta-mar*

1 A crescent (semicircle), 10 the border(?) of the arc. The transversal (diameter) was what? /
2 You: 40 ⌜of the transversal of the crescent⌝ touch.
3 40 of the transversal to / 10 the cresce[nt car]ry, then 6.40 the transversal you will see.

In this exercise, the arc is given as *a* = '10', and the diameter is computed as follows:

$\mathrm{c} = 40, \quad d = \mathrm{c} \cdot a = 40 \cdot 10 = 6\ 40.$

Somewhat surprisingly, the phrase used for the (length of the) arc of the semicircle is pat gúr, and not pat u$_4$.sakar.

IM 121512 § 2b

1 u$_4$.sakar 6 40 dal *ki-ma* ⌜pat⌝ [gúr][?] /
2 za.e 1 30 u$_4$.sakar gúr *lu-pu-ut*
3 1 30 *a-na* 6 40 / dal u$_4$.sakar *i-ta-ši-ma* 10 *ta-mar*
10 pat gúr *ta-mar* /

1 A crescent, 6 40 the transversal. Like (what) the [border of the arc]? /
2 You: 1 30 of the crescent arc touch.
3 1.30 to 6.40 / the transversal of the crescent always carry, then 10 you will see.
10 the border of the arc you will see.

In this exercise, it is instead the transversal (diameter) *d* of the semicircle that is given, as *d* = 6 40, and the arc *a* is computed as follows:

$\mathrm{c} = (\mathrm{rec.}\ 40 =)\ 1\ 30, \quad a = \mathrm{c} \cdot d = 1\ 30 \cdot 6\ 40 = 10.$

IM 121512 § 2c

1 u$_4$.sakar 10 pat lagab a.šag$_4$ *mi-nu* /
2 za.e 10 u$_4$.sakar *šu-ta-ki-il-ma* 1 40 *ta-mar* /
3 1 40 *a-na* 10 igi.gub u$_4$.sakar [*i-ši-ma* 16 40 *ta-mar*] /
4 16 40 a.šag$_4$ ⌜u$_4$.sakar⌝ *ta-*[*ma*]*r*

1 A crescent. 10 the border of the figure(?). The field was what? /
2 You: 10 the <border of the arc> of the crescent let eat itself, 1 40 you will see. /
3 1 40 to 10, the constant of the crescent carry, then 16 40 you will see]. /
4 16 40 the field of the ⸢crescent⸣ you will see.

Here, the arc of a semicircle is again given as a = 10, and the area A is computed as follows:

c = 10, A = c · sq. a = 10 · sq. 10 = 10 · 1 40 = 16 40.

Surprisingly, in this exercise the arc of the semicircle is called pat lagab instead of pat gúr as in §§ 2a-b above. Normally in Old Babylonian mathematical texts, lagab stands for ‘equalside’ or, simply, ‘square’. Indeed, the cuneiform sign itself is the image of a square. The same curious phrase pat lagab returns in the question of § 2d of IM 121512 and in the answers of §§ 2f-i. It always means the (length of the) arc of the semicircle. In the questions of §§ 2f-i the (length of the) arc of the semicircle is simply called uš ‘length’, possibly because uš was routinely used as the name for the unknown in Old Babylonian quadratic equations.

IM 121512 § 2d

1 *šum-ma* 16 40 a.šag$_4$ *ki-ia* <pat> lagab /
2 za.e igi ⸢10⸣ igi.gub u$_4$.sakar ⸢duḫ⸣-*ma* [6] /
3-4 6 *a-na* 16 40 a.šag$_4$ *i-ši-ma* / 1 40 *ta-mar*
5 ba.<sá>.e 1 40 *šu-li-ma* / 10 *ta-mar*
10 pat lagab (written over erasure)

1 If 16 40 the field, how much <the border(?) of> the figure(?) /
2 You: The opposite of ⸢10⸣ the constant of the crescent release, then [6]. /
3-4 6 to 16 40 the field carry, then / 1 40 you will see.
5 The equalside of 1 40 let come up,] / 10 you will see.
10 the border(?) of the figure(?) (written over erasure)

In § 2d the area of a semicircle is given as A = 16 40 and the arc a of the semicircle is computed as follows:

c = 10, sq. a = rec. c · A = 6 · 16 40 = 1 40, a = sqs. 1 40 = 10.

IM 121512 § 2e

1 u$_4$.sakar 10 pat gúr 6 40 dal a.šag$_4$ *mi-nu* /
2-3 za.e 6 40 dal *a-na* 10 uš u$_4$.sakar *i-ši-*[*ma*] / 1 06 40 *ta-mar re-eš-ka li-ki-il* /
4-5 *tu-úr* 15 igi.gub u$_4$.sakar ⸢*lu-pu-ut*⸣ $^?$ /
6-7 15 *a-na* ⸢u$_4$.sakar⸣$^?$ 1 06 40 *ša re-eš-ka* / *ú-ki-lu i-ta-ši-ma* 16 40 a.šag$_4$ *ta-mar*

1 A crescent. 10 the border of the arc, 6 40 the transversal. The field was what? /
2-3 You: 6 40 the transversal to 10 the length of the crescent carry, then / 1 06 40 you will see. Let it hold your head. /
4-5 Turn around. 15 the constant of the crescent ⸢touch⸣. /
6-7 15 to the ⸢crescent⸣ $^?$ 1 06 40 that your head / head always carry, then 16 40 the field you will see.

Here, the values of two parameters are given, the arc a = 10 and the transversal d = 6 40 are given. (The arc is referred to simply as uš ‘the length’.) The area A is then computed as follows:

c = 15, $A = c \cdot d \cdot a$ = 15 · 6 40 · 10 = 15 · 1 06 40 = 16 40.

IM 121512 § 2f

1 *a-na* šag$_4$ a.šag$_4$ u$_4$.sakar 1 uš daḫ-*ma* 26 40 /
2 a.šag$_4$ u$_4$.sakar *ki-ia* lagab <<za.e>> /
3-4 za.e 26 40 a.šag$_4$ *a-na* 10 igi.gub *i-ta-aš-ši-ma* / 4 26 40 *ta-mar re-eš-ka li-ki-il* /
5-6 *tu-úr-ma* 1 uš *le-qé he-pé šu-ta-ki-il-ma* / 15 *ta-mar*
7 15 *a-na* šag$_4$ 4 26 40 *ša re-eš-ka* / *ú-ki-lu* dah-*ma* 4 41 40 *ta-mar* /
8 ba.sá.e 4 41 40 š*u-li-ma* 2 10 *ta-mar* /
9-11 *i-na* šag$_4$ 2 10 e 30 *ta-ki-il-ta-ka* / *hu-ru-iṣ-ma* 1 40 *ta-mar* / *re-eš-ka li-ki-il*
12 *tu-úr-ma* / igi 10 igi.gub duḫ 6 *ta-mar*
13-14 *tu-úr* / 1 40 *ša re-eš-ka ú-ki-lu a-na* 6 *i-ši-ma* / 10 *ta-mar*
10 pat lagab
1 Onto the field of a crescent 1 length I added on, then 26 40. /
2 The field of the crescent how much was it square? /
3-4 You: 26 40 the field to 10 the constant always carry, then / 4 26 40 you will see. Let it hold your head. /

5-6 Turn around. 1 the length take, break and let eat itself, / 15 you will see.
7 15 onto 4 26 40 that your head / held add on, then 4 41 40 you will see. /
8 The equalside of 4 41 40 let come up, then 2 10 you will see. /
9-11 Out of this 2 10 30 the holder / break off, then 1 40 you will see. / Let it hold your head.
12 Turn around, then/ the opposite of 10 the constant resolve, 6 you will see.
13-14 Turn around. / 1 40 that held your head to 6 carry, then / 10 you will see.
10 the border of the figure.

The following quadratic problem for the unknown arc *a* is considered in § 2f:

$A + 1 \cdot a = 26\ 40$, where A is thought of in the form $A = c \cdot \text{sq. } a$, $c = 10$.

These are the successive steps of the customary solution algorithm:

$c \cdot A = 10 \cdot 26\ 40 = 4\ 26\ 40$, $c \cdot A + \text{sq. } 1/2 = 4\ 26\ 40 + \text{sq. } 30 = 4\ 26\ 40 + 15 = 4\ 41\ 40$, $\text{sqs. } 4\ 41\ 40 = 2\ 10$,
$2\ 10 - 30 = 1\ 40\ (= c \cdot a)$, $\text{rec. } 10 \cdot 1\ 40 = 10 = a$.

IM 121512 § 2g

1-2 *i-na* šag$_4$ a.šag$_4$ u$_4$.sakar 1 uš zi.zi-*ma* / 6 40
a.šag$_4$ u$_4$.sakar *ki-ia* lagab /
3-4 za.e 6 40 a.šag$_4$ *a-na* 10 igi.gub *i-ši-ma* / 1 06 40 *ta-mar re-eš-ka li-ki-il* /
5-6 *tu-úr-ma* 1 uš *le-qe he-pé* / *šu-ta-ki-il-ma* 15 *ta-mar*
7-8 15 *a-na* .šag$_4$ / 1 06 40 *ša re-eš-ka ú-ki-lu* dah-*ma* / 1 21 40 *ta-mar*
9 ba.sá.e 1 21 40 / *šu-li-ma* 1 10 *ta-mar*
10-11 *aš-šum* zi.zi / *qá-bu-ku* 30 *ta-ki-il-ka* dah-*ma* / ⸢1 40 *ta*⸣-*mar re-eš-ka li-ki-il* /
12-13 *tu-ûr-ma* igi 10 igi.gub duh-*ma* / 6 *ta-mar*
14-15 *tu-úr* 1 40 *ša re-eš-ka* / *ú-ki-lu a-na* 6 *i-ši-ma* / 10 *ta-mar*
10 pat lagab

1-2 Out of the field of a crescent 1 length I broke, then / 6 40.
The field of the crescent how much was it square? /
3-4 You: 6 40 the field to 10 the constant carry, then / 1 06 40 you will see. Let it hold your head. /
5-6 Turn around. 1 of the length take, break and / let eat itself, then 15 you will see.
7-8 15 onto / 1 06 40 that held your head add on, then / 1 21 40 you will see.
9 The equalside of 1 21 40 / let come up, 1 10 you will see.
10-11 Since break off / it was said to you, 30 the holder add on, then / ⸢1 40 you will⸣ see. Let it hold your head. /
12-13 Turn around. The opposite of 10 the constant resolve, then / 6 you will see.
14-15 Turn around. 1 40 that your head / held to 6 carry, then / 10 you will see.
10 is the border of the figure.

The following variation of the preceding quadratic problem is considered in § 2g:

$A - 1 \cdot a = 6\ 40$, where again A is thought of in the form $A = c \cdot \text{sq. } a$, $c = 10$.

In this case, the successive steps of the customary solution algorithm are:

$c \cdot A = 10 \cdot 6\ 40 = 1\ 06\ 40$, $c \cdot A + \text{sq. } 1/2 = 1\ 06\ 40 + \text{sq. } 30 = 1\ 06\ 40 + 15 = 1\ 21\ 40$, $\text{sqs. } 1\ 21\ 40 = 1\ 10$,
$1\ 10 + 30 = 1\ 40\ (= c \cdot a)$, $\text{rec. } 10 \cdot 1\ 40 = 10 = a$.

IM 121512 § 2h

1-2 *a-na* šag$_4$ u$_4$.sakar uš *ù* dal dah-*ma* / 33 20
a.šag$_4$ u$_4$.sakar *ki-ia* [lagab] /
3-5 za.e 33.20 a.šag$_4$ *a*-[*na* 10 igi.gub] / *i-ta-aš-ši-ma* 5 3[3] 20 *ta-m*[*a*]*r* / *re-eš-ka li-ki-i*[*l*]
6 *tu-úr-ma* / 1 uš 40 dal *ku-mur he-pé-ma* 50 *ta-mar* /
7-8 50 *me-eh-ra-am i-di šu-ta-ki-il-ma* / 41 40 *ta-mar*
9-10 41 40 *e-li* 5 33 20 / *ša re-eš-ka ú-ki-lu* dah-*ma* 6 15 / *ta-mar*
11 ba.sá.e 6 1[5] *šu-li-ma* / 2 30 *ta-mar*
12-13 *i-n*[*a* š]ag$_4$ 2 30 e 50 / *ta-ki-il-ta-k*[*a hu*]-*ru-iṣ-ma* / 1 40 *ta-mar re*-[*eš-k*]*a li-ki-il* /
14-15 *tu-úr-m*a igi 10 igi.gub duh-*ma* / 6 *ta-mar*
16 6 *a-na* 1 40 *ša re-eš-k*[*a*] / *ú-ki-lu i-ši-ma* 10 *ta*-[*mar*]
10 pat lagab

1-2 Onto the crescent the length and the transversal I added on, / 33 20.
The crescent-field how much was it square? /

3-5 You: 33 20 the field to 1[0 the constant] / always carry, then 5 3[3] 20 you will see. / Let it hold your head.
6 Turn around, then / 1 of the length 40 of the transversal heap, 50 you will see. /
7-8 50 a copy lay down let eat itself, / 41 40 you will see.
9-10 41 40 over 5 33 20 / that held your head add on, 6 15 / you will see.
11 The square-side of 6 1[5] let come up, then / 2 30 you will see.
12-13 Out of this 2 30 50 / [your] holder [br]eak off, then / 1 40 you will see. Let it hold [your] head. /
14-15 Turn around. The opposite of 10 the constant resolve, then / 6 you will see.
16 6 to 1 40 that your head / held carry, then 10 you will [see].
10 the border of the figure.

In § 2g, the quadratic problem for the unknown arc *a* considered in § 2f is made more complicated through the addition of a second linear term in the equation, which now has the following form:

$A + a + d = 33\ 20$, where $A = c \cdot \text{sq. } a$, $c = 10$.

Since 40 is the constant for the transversal of the crescent, the new equation can be rewritten as follows:

$A + (1 + 40) \cdot a = 33\ 20$, where $A = c \cdot \text{sq. } a$, $c = 10$.

The corresponding solution algorithm contains the following steps:

$c \cdot A = 10 \cdot 33\ 20 = 5\ 33\ 20$, $c \cdot A + \text{sq. } (1+40)/2 = 5\ 33\ 20 + \text{sq. } 50 = 5\ 33\ 20 + 41\ 40 = 6\ 15$, $\text{sqs. } 6\ 15 = 2\ 30$,
$2\ 30 - 50 = 1\ 40\ (= c \cdot a)$, $\text{rec. } 10 \cdot 1\ 40 = 10 = a$.

IM 121512 § 2i

1-2 *a-na* šag$_4$ a.šag$_4$ u$_4$.sakar uš [dal *ù* 20 *ma-a*]*ṭ-ni* / dah-*ma* 36 4[0]
a.š[ag$_4$ u$_4$.sakar *ki-ia*] lagab /
3-5 za.e 36 40 *a-na* 10 igi.gub *i-ta-aš-ši-ma* / 6 e 6 40 *ta-mar re-eš-ka* / *li-ki-il*
6 *tu-úr-ma* 1 uš 40 dal / 20 *ma-aṭ-na ku-mur ḫe-pé-ma* 1 *ta-mar* /
7-8 1.e <*a*>*-na* 6 e 6 40 *ša re-eš-ka* / *ú-ki-lu* dah-*m*a 7 06 40 *ta-mar* /
9 ba.sá.e 7 06 40 š*u-li-ma* 2 40 *ta-mar* /
10-12 *i-na* šag$_4$ 2 40 1 *ta-ki-il-ta-ka* / *hu-ru-is-ma* 1 40 *ta-mar re-eš-ka* / *li-ki-il*
13 *tu-úr-ma* igi 10 igi.gub / dah-*ma* 6 [*t*]*a-mar*
14-15 *tu-úr* 1 40 *ša* / *re-eš-ka ú-ki-lu a-na* 6 *i-ši-ma* / 10 *ta-mar*
16 10 pat lagab

1-2 Onto the field of a crescent the length, [the transversal and 20 of the str]ing / I added on, then 36 4[0].
The fie[ld of the crescent how much] was it square? /
3-5 You: 36.40 to 10 the constant always carry, / 6 0$^{!}$6 40 you will see. Your head / may it hold.
6 Turn around. 1 the length, 40 the transversal, / 20 the string heap, break, then 1 you will see. /
7-8 This 1 to 6 0$^{!}$6 40 that your head / held add on, then 7 06 40 you will see. /
9 The equalside of 7 06 40 let come up, then 2 40 you will see. /
10-12 Out of 2 40 1 your holder / break off, then 1 40 you will see. Your head / may it hold.
13 Turn around. The opposite of 10 the constant / add on, then 6 [you] will see.
14-15 Turn around. 1 40 that / held your head to 6 carry, then / 10 you will see.
16 10 the border of the figure.

In § 2i, finally the quadratic problem for the unknown arc *a* considered in § 2f is made more even complicated through the addition of a third linear term in the equation, which now has the following form:

$A + a + d + r = 36\ 40$, where $A = c \cdot \text{sq. } a$, $c = 10$.

Since 20 is the constant for the 'string' of the crescent, the new equation can be rewritten as follows:

$A + (1 + 40 + 20) \cdot a = 36\ 40$, where $A = c \cdot \text{sq. } a$, $c = 10$.

The solution algorithm then has the following form:

$c \cdot A = 10 \cdot 36\ 40 = 6\ 06\ 40$, $c \cdot A + \text{sq. } (1 + 40 + 20)/2 = 6\ 06\ 40 + \text{sq. } 1 = 6\ 06\ 40 + 1 = 7\ 06\ 40$, $\text{sqs. } 7\ 06\ 40 = 2\ 40$,
$2\ 40 - 1 = 1\ 40\ (= c \cdot a)$, $\text{rec. } 10 \cdot 1\ 40 = 10 = a$.

Evidently IM 121512 § 2 is a fine example of *a strictly organized mathematical theme text,* starting with very simple defining exercises (in this case giving information about various mathematical constants associated with a semicirle), and proceding through a series of increasingly complicated exercises.

The quadratic equations in IM 121512 §§ 2h-i

$A + a + d = 33\ 20$ and

$A + a + d + r = 36\ 40$, where A is the area, a the arc, d the diameter, and r the radius of a *semicircle,* $A = ;10 \cdot$ sq. a

have interesting parallels in other Old Babylonian mathematical texts. (See Friberg, *MSCCT 1*, 330-331.) For instance, in BM 80209 § 7a (Friberg, *JCS* 33 (1981)) a similar quadratic equation is

$A + d + a = 8\ 33\ 20$, where A is the area, a the arc, and d the diameter of a *circle*, $A = ;05 \cdot$ sq. a.

In *TMS* 20 § 1 (Bruins and Rutten (1961)) a third kind of quadratic equation of the same type is

$A + a + d = 1\ 16\ 40$, where A is the area, a the arc, and d the diameter of a *concave square*, $A = ;26\ 40 \cdot$ sq. a.

Finally, in *TMS* 9 (*op. cit.*) a fourth kind of quadratic equation of the same type is

$A + u + s = 1$, where A is the area, u the length, and s the front of a *rectangle of a certain kind*, $A = u \cdot s$.

6.3.4 MLC 1354. A Badly Conceived Quadratic Problem for the Arc of a Semicircle

MLC 1354, published in Neugebauer and Sachs, *MCT* (1945) as text Eb, is the only previously known example of a mathematical cuneiform text concerned with a problem for a semicircle. According to Goetze's orthographic analysis (see *MCT*, p. xx), MLC 1354 belongs to text group 6, which means that it is probably a text from Old Babylonian Sippar. The writing on the obverse of the tablet is well preserved, but only the first two lines of the writing on the reverse are readable. See the hand copy from *MCT* in Fig. 6.3.2 below.

MLC 1354

drawing
1 ⸢ašag.u$_4$.sakar $^2/_3$⸣ [uš sag] /
2-3 *i-na* sag 5 nindan *a-sú-*⸢*uḫ*⸣-[*ma*] / [1(iku)] ašag. 12 $^1/_2$ sar a.šag$_4$ <<zu.dè>> /
4 uš sag *mi-nu*
za.e ak.ta.<zu.dè> /
5 igi 40 *pu-ṭur a-na* 1 52 30 a.šag$_4$ *i-ši-ma* 2 48 45 *t*[*a-mar*] /
6 igi 15 igi.gub.ba *pu-ṭur-ma* 4 *ta-mar*
7 4 *a-na* 2 4[8 45] / *i-ši-ma* 11 15 *ta-mar*
$^1/_2$.⸢bi⸣ gar.ra /
8-9 igi 40 *pu-ṭur a-na* 5 *ša* zi.zi *i-ši-ma* / 7 30 *ta-mar*
10 *ba-ma-at* 7 30 *ḫe-pé šu-tam-ḫír-ma* / 14 03 45 *ta-mar*
11 14 03 45 / *a-na* šag$_4$ 11 [1]5 [gar.gar-*ma*] 11 29 03 45 / [*ta-mar*]
… ……………………………

drawing (see below)
1 ⸢A crescent field. $^2/_3$⸣ [of the length is the front.] /
2-3 From the front 5 nindan I tore off, [then] / 1(iku) 12 $^1/_2$ sar the field x x x /
4 The length and front were what?
You in your doing it: /
5 The opposite of 40 release, to 1 52 30 the field carry it, 2 48 45 you will see. /
6 The opposite of 15 the constant release, then 4 you will see.
7 4 to 2 4[8 45] / carry, then 11 15 you will see.
$^1/_2$ of it set. /
8-9 The opposite of 40 release, to 5 that you tore off carry, then / 7 30 you will see.
10 The half-part of 7 30 break and make equalsided, / 14 03 45 you will see.
11 14 03 45 / onto 11 [1]5 [add together, then] 11 29 03 45 / [you will see].
… ……………………………

This exercise begins with a line drawing of a semicircle, in the text explicitly mentioned as a "crescent field". Two parameters for the semicircle are called the "length" and the "front", and it is specified that the front is 2/3 of the length. Reasonably, this means that the "length" is the arc of the semicircle, and that the "front" is the diameter. In the line drawing, the constant 2/3 is represented by the sexagesimal number '40'.

It is further specified that the front is decreased by 5 nindan, and that the area (of the semicircle) then becomes equal to 1(iku) 12 1/2 sar. The corresponding sexagesimal multiple of a sar is mentioned in line 5 of the exercise as 1 52 30 (1 52;30). Clearly, the problem which the author of the exercise wanted to set up can be described, in quasi-modern terms, as the following quadratic equation:

$$A = 15 \quad u \cdot s = 15 \cdot u \cdot (40\, u - 5) = 1\ 52\ 30.$$

The area constant '15' for a semicircle is mentioned in the line drawing, next to the constant '40'.

In the solution procedure, multiplications of the area by rec. 40 (= 1 30) and by rec. 15 = 4 and of the subtracted length 5 by rec. 40 (= 1 30) reduces this quadratic equation to

$$u \cdot (u - 1\ 30 \cdot 5) = 4 \cdot 1\ 30 \cdot 1\ 52\ 30 \quad \text{or} \quad u \cdot (u - 7\ 30) = 4 \cdot 2\ 48\ 45 = 11\ 15.$$

The meaning of the instruction '1/2 of it set' in line 7 is not completely clear, but in line 10 the solution algorithm for the quadratic equation begins in the usual way by halving the coefficient 7 30.

Some part of the solution procedure is lost, but it is clear that it proceeded as follows:

$$1/2 \cdot 7\ 30 = 3\ 45, \quad \text{sq. } 3\ 45 = 14\ 03\ 45, \quad 14\ 03\ 45 + 11\ 15 = 11\ 29\ 03\ 45,$$
$$[\text{sqs. } 11\ 29\ 03\ 45 = 26\ 15, \quad 26\ 15 + 3\ 45 = 30].$$

Therefore, the solution to the quadratic problem is that the arc (the "length" *u*) is 30 (nindan). Correspondingly, the modified "front" is $2/3 \cdot 30 - 5 = 15$ (nindan).

Note, however, that the problem in this exercise was badly conceived. If the arc of the semicircle is 30, then the diameter is (according to Old Babylonian conventions) 2/3 of 30 = 20, not 15. So the modified "front" $40\, u - 5 = 15$ cannot be equal to the diameter of the semicircle. Yet the use of the constant '15' in the computation of the area $A = 15 \cdot u \cdot (40\, u - 5)$ shows that *u* and $40\, u - 5$ were thought of as the arc and the diameter of the semicircle.

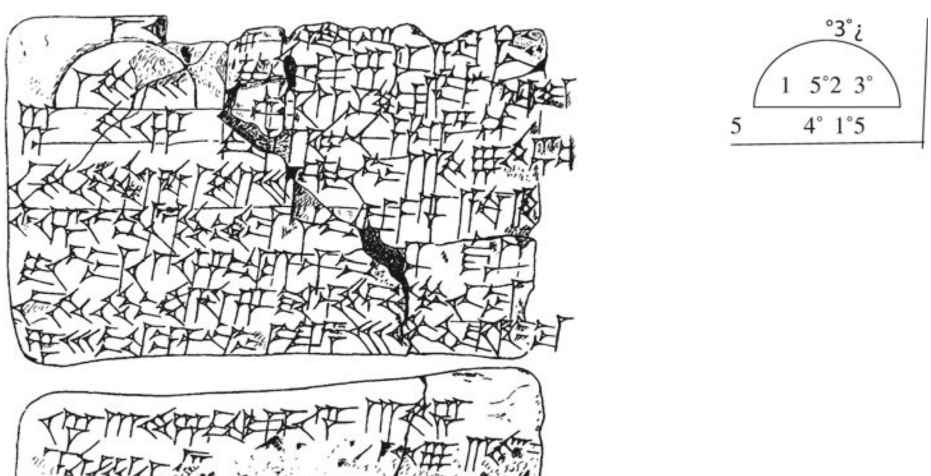


Fig. 6.3.2. MLC 1354. A badly conceived quadratic problem for the arc of a semicircle.

6.3.5 *The Vocabulary of IM 121512*

The vocabulary of the text IM 121512 is nearly identical with the vocabulary of the preceding text IM 121565, except for the addition of the following words and phrases:

dal	transversal (= diameter)
dal.ba.an.na	intermediate space
gúr	(circular) arc
igi.gub	constant
mu.bi.im	cases
pat	border(???)
u_4.sakar	crescent (= semicircle)
i-ta-ši	< *našû* to carry (= to multiply (by a number))
šu-ul-li-iš	< *šalāšu* to triplicate
aš-šum ... *qá-bu-ku*	since it was said to you that .....
re-eš-ka li-ki-il	let it hold your head (= keep this in mind)
ša re-eš-ka ú-ki-lu	that held your head (= that you kept in mind)
ma-at-nu	string (radius)
mi-si-tum	*messētu* distance
qer-bi-tum	*qerbu* inner
šum-ma	if

6.3.6 IM 121512. Hand Copies of the Tablet

Fig. 6.3.3. IM 121512, *obv*. Hand copy.

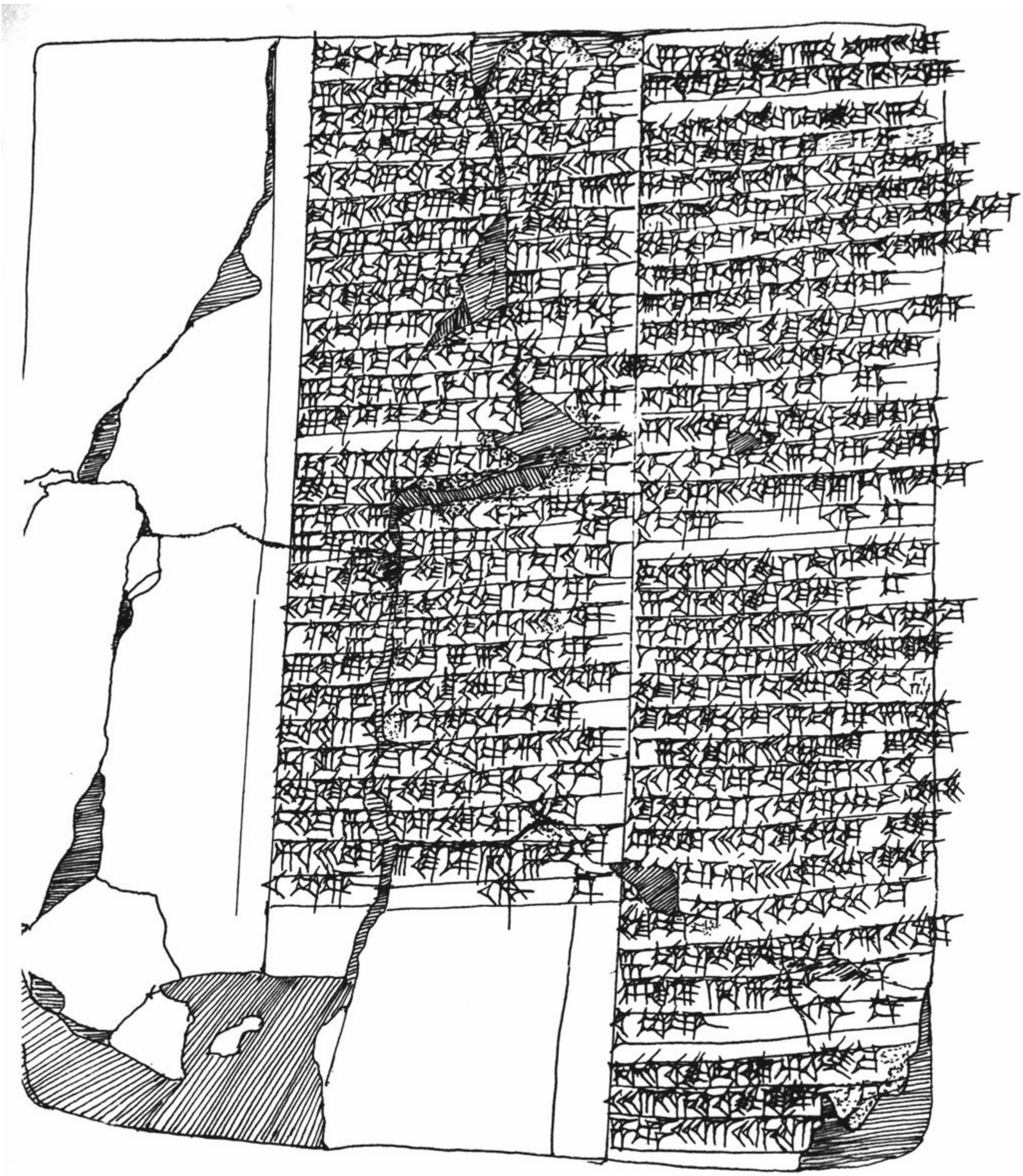


Fig. 6.3.4. IM 121512, *rev*. Hand copy.

6.4 Some Atypically Brief Single Multiplication Tables from Old Babylonian Mê-Turran

Eight atypical single multiplication tables found at Old Babylonian Mê-Turran (Tell Haddad) are surprisingly brief and incomplete, essentially only excerpts from the ordinary kind of Old Babylonian single multiplication tables, where as a rule the head number is multiplied in 23 lines with all integers from 1 to 20, and with 30, 40, 50. In Fig. 6.4.1 below, hand copies of the eight atypical multiplication tables are compared with hand copies of two (previously unpublished) Old Babylonian arithmetical table texts, the single multiplication table IM 18323 and the table of cube sides IM 18236.

Note that the subscript on IM 18232 contains a name and a date, which is an ordinary feature of a subscript. However, the edge inscription is of a previously unknown kind (on multiplication tables). It says

1 mu.bi 23.kam   1, its lines are 23.

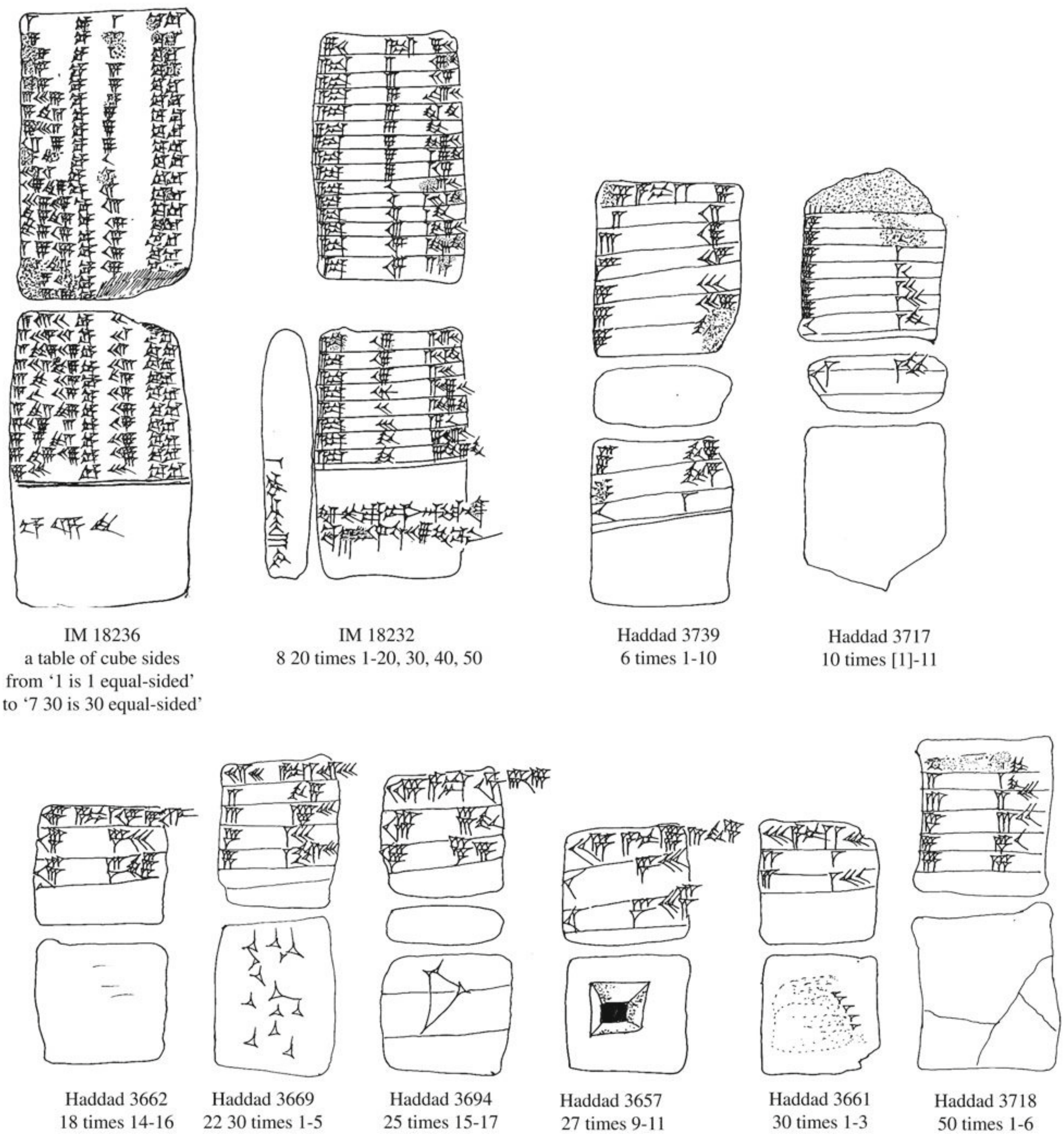


Fig. 6.4.1. Eight multiplication tables from Mê-Turran compared with two ordinary Old Babylonian arithmetical table texts.

6.5 The Mathematical Texts from Old Babylonian Mê-Turran (Tell Haddad)

Below is a list of mathematical cuneiform texts excavated at Mê-Turran (Tell Haddad):

IM 95771	a fragment, geometry and table of constants	6.1
IM 121512	a large recombination text, equations for circles and semicircles	6.3
IM 121565	a large recombination text, equations for square and quadrilateral fields	6.2
IM 121613	a large recombination text, equations for rectangles	5.1
IM 95858	a large recombination text of mixed content: bricks, logs, plaster	6.1.4
	= Haddad 104, Al-Rawi and Roaf (1984) *Sumer* 43	
Haddad x	a badly broken large recombination text, problems for diagonals	
Haddad 350	numbers around the edge of a round tablet	CDLI/P430101
Haddad unp 1		fragments of a large mathematical
cuneiform text	CDLI/P430998	
Haddad unp 2	fragments of a large mathematical cuneiform text	CDLI/P430999
Haddad unp 3	fragments of a large mathematical cuneiform text	CDLI/P431000
Haddad unp 4	fragment of a large mathematical cuneiform text	CDLI/P431001
Haddad unp 5	fragments of a large mathematical cuneiform text	CDLI/P431002
Haddad 3657	× 27	6.4
Haddad 3661	× 30	6.4
Haddad 3662	× 18	6.4
Haddad 3669	× 22 30	6.4
Haddad 3694	× 25	6.4
Haddad 3717	× 10	6.4
Haddad 3718	× 50	6.4
Haddad 3739	× 6	6.4

The mathematical cuneiform texts from the Iraq Museum discussed in this volume have either been confiscated by the authorities in Iraq or are from known sites such as Mê-Turran, Shaduppum, or the Library of Sippar. Most numerous of these are the mathematical texts from Tell Haddad, which now is known to be one part of the remains of the ancient city Mê-Turran, 'the waters of the river Turnat' (modern Diyala Valley).

The remains of Mê-Turran comprise three mounds, the highest of them Baradan, the one in the middle Haddad, and the lowest es-Seeb. Baradan produced some Kassite and New Assyrian texts, with no mathematical texts among them. However, the excavators did not dig deep enough to reach the Old Babylonian strata. The top stratum of Haddad produced mainly Neo-Assyrian texts from the time of Assurbanipal. Most of those texts are literary, but there is also the so called Text of the Water Clock, published by Al-Rawi and George in *AfO* 36.

At the slope of Tell Haddad towards Tell es-Seeb there is an ancient street along which have been found many private houses, official and religious buildings and probably a school. From the Old Babylonian period there are three strata. Most of the tablets were found in the third stratum. The main part of the Sumerian texts, the Akkadian literary texts, and the mathematical texts were found in Room 10 of "Area 2" (see Cavigneaux, "A scholar's library in Meturan" (1999)), while most of the well preserved administrative texts were found in room 25, together with some multiplication tables.

The tablets from Tell Haddad are mostly unbaked but may have been accidentally baked when the buildings where they were kept were destroyed by fire. Many of the texts were badly broken, but in several cases it was possible to painstakingly put them together again.

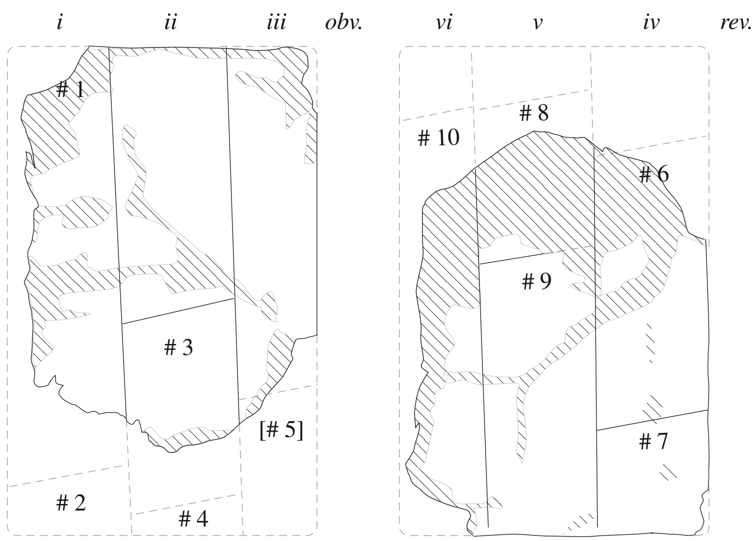
Six arithmetical tablets from Tell es-Seeb were published recently by Isma'el and Robson in an article contained in *Your Praise is Sweet, Memorial Volume for Jeremy Black* (2010). The article ends with a useful survey of all published mathematical tablets from the Eshnuna Region in the Diyala Valley (tablets from the sites Tell Haddad, Tell es-Seeb, Tell edh-Dhiba'i, Ishchali, Tell Harmal, and some unprovenanced tablets sharing the same internal characteristics).

Tell es-Seeb was excavated by the late Hazim Abdulhamid Alnajafi, who also started the excavation at Tell Haddad. He was followed by Dr. Nail Hannoon, and after him the work was taken over by Burhan Shaker. The results of the excavations were assessed by Rasmiah Rashied and Jabbar Alkhawas. Farouk Al-Rawi was invited by all of them to study the texts and was kindly given the permission to publish them from the Board of Archaelogy of Iraq.

7. A Recombination Text from Old Babylonian Shaduppûm Concerned with Economic Transactions

7.1 IM 31210. Economic Transactions

This text was first published by Bruins in *Sumer* 10 (1954), 57-61, in an inadequate transliteration without hand copies or photos, and with in most cases unsatisfactory mathematical interpretations.



1: Six partners. Badly preserved
2: Seven partners. Steadily decreasing shares
3: Three partners. Steadily decreasing shares
4: Explicit computation of a combined market rate
5: Lost
6: Thirteen(?) brothers. Unequal shares
7: Ten partners. Steadily decreasing shares
8: Steadily decreasing shares. Badly preserved
9: Four partners. A system of linear equations
10: Partners. Badly preserved

IM 31210, *obv.* Outline of the clay tablet. Table of contents.

The text is inscribed on a large clay tablet with three columns on the obverse and three on the reverse. The lower part of the tablet is lost, and so is part of the tablet near the left edge, with much of columns *i* and *vi*. Yet, substantial parts of three individual problems remain in columns *ii-iii* on the obverse, as well as substantial parts of three more problems in columns *iv-v*. The common theme of these six problems is *economic transactions* or, more precisely, seven or more exercises concerned with division of property between partners or brothers, and one exercise with an explicit computation of a combined market rate, the only one known, so far.

7.1.1 # 1. Six Partners. A Very Badly Preserved Exercise

	IM 31210 # 1 (col. *i*)
—	[seven lines lost]
1	[*at-ta i-na e-pé*]-*ši-ka* /
2	[*x x x x x x*] *x-ri-da* /
3	[*x x x x x x x x x x x*] *x x-ma* /
4	[*x x x x x x*] *x* 1 *pu-ṭur* /
5	[*x x x x x x*]-*ma x x* /
6	[*x x x*] *x x x aš-šum x x* /
7	[*x x x*]-*ma tu-úr-ma* /
8	[*x x x*] *x x - ma* /

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7_7

9	[*x x x x x x x x x x x x x x*] *i-di* /	
10	[*x x x x x x x x x x x x x x x x x*] /	
11	[*x x x-ma* 16 *i-li* 16 *x x x*] /	
12	[*me-eḫ*]*-ra-am* [*x x x*] *i-di* /	
13	[*x x x*]*-ma* 14 *i-li* 14 *me-eḫ-* / *ra-am i- di* /	
14	[*x x x*]*-ma* 26 *i-li tu-úr* /	
15	[*x x x x x x x x x x x x x x*] *i-li* /	
16	[*x x x x x x x x x x x x x x*] *i-li* /	
17	[*x x x x x x x x x x x x x x x x*] /	
18	[*x x*] 18 [*x*]*-ma x-x-ma* 18 /	
19	[*x x x ri-iš-k*]*a li-*[*ki*]*-il* /	
20	[*x x x*]*-ma pa-ni* 6 *at-ḫi-ka* /	your 6 partners
21	[*pu-ṭur-ma*] 10 *i-li* 10 *a-na* 8 *ša* /	
22	[*x x ša*] *ri-iš-ka ú-ka-lu* /	
23	[*x x x* 3] *i-li* 3 *a-na ši-na* /	
24	*ša ri-iš-ka* [*x*]*-x-ú* /	
25	[*x x x*] 8 *i-li* <8> *me-eḫ-ra-am* /	
26	[*x x x* 10] *i-li* 10 *me-eḫ-ra-am* /	
27	[*x x x* 17] *i-li* 17 *me-eḫ-ra-am* /	
28	[*x x x x x x x x x x x x x x*] *i-li* /	
29	[*x x x x x x x x x x x x x x*]*-ši-ma* /	
—	[several lines lost]	

7.1.2 # 2. Seven Partners. Steadily Decreasing Shares

In the exercise IM 31210 # 2, the question and an unknown number of lines at the beginning of the solution procedure are lost. On the other hand, the last part of the solution procedure and the answer are both well preserved. The explanation below of what is going on in this exercise, is essentially the same as the explanation proposed by Bruins in the first publication of IM 31210 in *Sumer 10* (1954).

	IM 31210 # 2 (col. *ii*)
—	[Several lines missing.]
[1´]	[*i-na* 1 7 30 *ú-su-uḫ-ma* 52 30 *i-li*] /
2´-3´	52 30 *me-eḫ-ra-am i-di* 7 30 / *ú-su-uḫ-ma* 45 *i-li*
4´	*i-na* 45 / 7 30 *ú-su-uḫ-ma* [3]7 30 *i-li* /
5´-6´	*i-na* 37 30 7 30 *ú-su-uḫ-ma* / 30 *i-li*
7´	*i-na* 32[sic] 7 30 *ú-su-uḫ-ma* / 22 30 *i-li*
8´	*i-na* 22 30 *ša i-lu-kum* / 7 30 *ú-su-uḫ-ma* 15 *i-li* /
9´	[*na-ás*]*-ḫi-ir*
10´	*ku-mur-šu-nu-ti-ma* 4 22 30 / *i-li*
11´	*mi-nam a-na* 4 22 30 *lu-*<*uš-*>*ku-un* / *ša* [1 10] kù.babbar *i-ba-an-ni-a-am*
12´	16 / *šu-ku-*[*u*]*n-ma i-ba-ni-kum* /
13´	*na-ás-ḫ*[*i*]*-ir-ma*
14´	16 *a-na* 1 *i-ši-ma* / 16 [x] *i-li* 16 *zi*<*-ti*> *ra-bi-im* /
15´-16´	16 [*a-na*] 52 30 *i-ši-ma* 14 *i-li* / 1[4 x] *ša-*[*nu*]*-um* šu.ba.an.[ti] /
17´-18´	16 [*a-na x* 4]5 *i-ši-ma* 1[2] *i-li* / 12 [*ša-al-šu-um*] šu.ba.an.[ti] /
19´-20´	16 *a-na* 37 30 [*i*]*-ši-ma* [10] *i-li* / 10 *re-bu* šu.ba.a[n.ti]
21´-22´	16 *a-na* 30 / *i-ši-ma* 8 *i-li* [8 *x ḫa*]*-am-šu* / [šu.ba.an.ti]
23´	[16 *a-na* 22 3]0 *i-ši-ma* / 6 [*i-li* 6 *ši-iš-šu* šu.ba.an.ti] /
24´-25´	16 *a-na* 15 [*i-ši-ma* 4 *i-li*] / 4 *se-bu-um* šu.ba.an.ti *ki-a-am* [*né*]- / *pé-šum*

—	[Several lines missing.]
[1´]	[From 1, 7 30 tear off, then 52 30 will come up]. /
2´-3´	52 30, a copy lay down, 7 30 / tear off, then 45 will come up.
4´	From 45, / 7 30 tear off, then [3]7 30 will come up. /
5´-6´	From 37 30, 7 30 tear off, then / 30 will come up.
7´	From 30[!], 7 30 tear off, then / 22 30 will come up.
8´	From 22 30, / 7 30 tear off, then 15 will come up. /
9´	[Turn] back, then
10´	heap them, then 4 22 30 / will come up.
11´	What to 4 22 30 should I set / that will construct for me 1 10 of silver?

12′ 16 / set, then it will construct for you. /
13′ Turn back, then
14′ 16 to 1 carry, then / 16 [x] will come up, 16 the great share. /
15′-16′ 16 [to] 52 30 carry, then 14 will come up, / 1[4 x] the second will rec[eive]. /
17′-18′ 16 [to x 4]5 carry, then 1[2] will come up, / 12 [the third] will rec[eive]. /
19′-20′ 16 to 37 30 carry, then [10] will come up, / 10 the fourth will rec[eive].
21′-22′ 16 to 30 / carry, then 8 will come up, [8 the] fifth / [will receive],
23′ [16 to 22 3]0 carry, then / 6 [will come up, 6 the sixth will receive]. /
24′-25′ 16 to 15 [carry, then 4 will come up] / 4 the seventh will receive.
Such is the procedure.

It is obvious from what is left of the solution procedure and the answer in this exercise that the question (now lost) was phrased, more or less, in the following way:

> 1 mina 10 shekels of silver was divided between 6 partners, so that x x x x x.
> The shares were steadily decreasing. How much were the 6 shares?

Clearly, the extra condition [x x x x x], whatever it was, allowed the computation of the difference between successive shares, which was found to be 7 30, if the big partner's share was 1. Therefore, it is possible that the lost extra condition was simply that

> The difference between each partner's share and the next partner's share was an eighth of the big partner's share.

The solution procedure is a very simple application of the "method of false value", namely as follows: First it is assumed that the big partner's share has the false value 1, and the corresponding false values of the other 6 partner's shares are computed through repeated subtraction of 7 30. After that, the sum of the 7 false shares is computed.

[1 – 7 30 = 52 30]	(line [1′])
52 30 – 7 30 = 45	(lines 2′-3′)
45 – 7 30 = 37 30	(lines 3′-4′)
37 30 – 7 30 = 30	(lines 5′-6′)
30 – 7 30 = 22 30	(lines 6′-7′)
22 30 – 7 30 = 15	(lines 7′-8′)
1 + 52 30 + 45 + 37 30 + 30 + 22 30 + 15 = 4 22 30	(lines 9′-10′)

In modern notations, if the false value of the big partner's share is 1, then the corresponding false values of the other shares are ;52 30, ;45, etc., and the sum of the false values of all the 7 shares is 4;22 30. This value is then compared with the given value of the sum of the 7 shares, apparently [1 mina 10 shekels], and it is found that:

16 · 4 22 30 = 1 10	(lines 10′-12′)

In modern notations, again, 16 · 4;22 30 = 1 10 (shekels), which means that the needed correction factor is 16 (shekels). Consequently, the true values of the 7 shares are as follows:

16 · 1 =	16 (shekels)	(lines 13′-14′)
16 · 52 30 =	14 (shekels)	(lines 15′-16′)
16 · 45 =	12 (shekels)	(lines 17′-18′)
16 · 37 30 =	10 (shekels)	(lines 19′-20′)
16 · 30 =	8 (shekels)	(lines 20′-22′)
16 · 22 30 =	6 (shekels)	(lines 22′-23′)
16 · 15 =	4 (shekels)	(lines 25′-26′)

It is easy to check that 16 + 14 + 12 + 10 + 8 + 6 + 4 = 1 10, as it should be.

7.1.3 # 3. Three Partners. Steadily Decreasing Shares?

It is not quite clear what this exercise is about, since a crucial term in the first line of the question is badly preserved, and since another term repeatedly occurring in the text of the exercise has an unknown meaning. The latter term is *kišīru*, occurring in several lines of the text, but given in CDA without any suggested translation. It is no help to know that the verb *kašāru* means 'to restore, to be successful'. In *Sumer* 10 (1954), Bruins translates *kišīru* as 'purse' without giving any justification for his translation. It is more likely that *kišīru* was a beer vessel of some kind.

IM 31210 # 3 (col. *ii*)

1	*šum-ma i-ša-lu-ka* 1(bán)$^?$ ⸢kaš⸣
2-3	2 *at-ḫu-ú ki-ši-ir-šu ši-na* / *ki-ši-ri-ia*
4	*se-bi-*⸢*it*⸣*-ti* / *x* / *ša iš-te-en ki-ši-ri*
5	*ša-la-*⸢*te*⸣*-šu* / [*m*]*i-nu*
	at-ta i-na e-pé-ši-ka /
6-7	*aš-šum ki-ši-ir-šu ši-na ki-ši-ri-ka* / *se-bi-*⸢*it*⸣*-tu qa-bu-kum*
8-9	7 *a-na* / *ši-na* [*e-ṣ*]*í-ip-ma* 14 *i-li* / [1]4 *ri-iš-ka li-ki-il*
10-11	*tu-úr* / *x x x x x x x x x x* / *x x x x*
	[*na*]*-ás-ḫi-ir-ma* /
—	(several lines lost)

1	If he asked you (about) 1 bán ⸢of beer⸣. /
2-3	2 partners. Their *kišīrus* are twice / my *kišīru*.
4	Seven times the x / x / of one is my *kišīru*.
5	Their three (*kišīrus*) / are what?
	You, in your procedure: /
6-7	Since their *kišīru*s are twice your *kišīru* / (and) seven was said to you,
8-9	7 to two repeat, then 14 will come up. / Let 14 keep your head.
10 -11	Return. / x x x x x x x x x x / x x x x
	Turn back, then
—	(several lines lost)

The problem in this exercise may be about vessels called *kišīru*, of three different sizes, possibly (the sign at the end of the first line is poorly preserved) with a combined capacity measure of 1 bán = 10 sìla. If the three types of vessel are called *a, b, c,* in this order, where *b* is 'my' vessel, then the stated problem can be understood as the following system of three linear equations, in quasi-modern symbolic notations:

$$a + b + c = 1 \text{ bán (?)} \qquad \text{(line 1)}$$
$$a + c = 2\, b \qquad \text{(lines 2-3a)}$$
$$b = 7\, c \qquad \text{(lines 3a-4)}$$

The first of these equations seems to have been inadvertently omitted. The second equation confirms that the three sizes are steadily decreasing in the sense that $a - b = b - c$.

The problem appears to have been solved by use of the rule of false values. Thus, it is silently assumed that

$$c' = 1, \quad \text{so that} \quad b' = 7.$$

Then

$$a' + c' = 2\, b' = 2 \cdot 7 = 14 \qquad \text{(lines 7-8)}$$

Consequently, in the lost part of the solution procedure,

$$a' = 2\, b' - c' = 14 - 1 = 13.$$

The value of c' (the smallest share) is then determined by the equation

$$a + b + c = (13 + 7 + 1) \cdot c' = 21\, c' = 1 \text{ bán (?)}$$

Clearly, something is wrong here because 21 is not a regular sexagesimal number. It is possible that the reading 1(bán) kaš in line 1 is incorrect, but in that case a mention of the sum of the three shares was inadvertently omitted. It ought to have been some multiple of 21 (or 7).

7.1.4 # 4. An Explicit Computation of a Combined Market Rate

In this exercise, the question is almost completely lost, while a substantial initial part of the solution procedure is well preserved. The loss of the question is doubly unfortunate since the exercise is of a new type with no previously published direct parallels, and since the meaning of a crucial term in the solution procedure is unknown. It is possible, however, that the exercise is in some way related to the well known category of Old Babylonian "combined market rate exercises" (Friberg, *MSCT 1* (2007), Sec. 7.2). The explanation of the exercise proposed below differs from Bruins' explanation in *Sumer 10* (1954).

IM 31210 # 4 (col. *iii*)

[*šum-ma ši-na* x *a-na* 1 bán] /
[*iš-te-en* x *a-na* 2 bán] /
[*ša-la-ša-at* x *a-na* 4 bán]
1 [*ra-bi-it*] / [x *a-na* 3 bán]
2 [*še-um*] / *li-li* ⸢*ù*⸣ [*li-ri-da*] /
3 ⸢ganba⸣ [*li-im-ta-ḫar*] /
4 *i*-[*x x x x x x x x*] /
5 *ri*-[*x x x x x x x x*] /
6 *at-ta i*-[*na e-pé-ši-ka*] /
7-8 *pa-ni ši-na š*[*a a-na* 1 bán] / *pu-ṭur-ma* 30 *i*-[*li*] /
9 30 *a-na* 1 [x] *i-ši*-[*ma* 30 *i-l*[*i*] /
10-11 *pa-ni iš-te-en* [*ša*] *a*-[*na* 2 bán] / *pu-ṭur-ma* 1 *i*-[*li*] /
12 1 *a-na* 2 *i-ši-ma* 2 *i-li* /
13-14 *pa-ni ša-la-ša-at ša a-na* / 4$^{!}$ bán *pu-ṭur-ma* 20 *i-li* /
15-16 20 *a-na* 4 bán *i-ši-ma* / 1 20 *i-li*
17-18 *pa-ni ra-bi-it* [*ša*] / *a-na* 3 bán *pu-ṭur-ma* / [1]5 *i-li*
19 15 *a-na* 3$^{!}$ bán / *i-ši-ma* 4$^{!}$5 *i-li* /
20-22 *na-ás-ḫi-ir-ma* / 30$^{!}$ 2 1 20 *ù* 45 *ku-mur* / 4 35 *i-li*
23-24 *mi-nam* / *a-na* 4 35 *lu-uš-ku-un* / [*š*]*a* 1 31 40 *še-x-am i-ba-ni-kum* /
25 [*x*] 20 *šu*-[*ku-un*] *i-ba-ni-kum* /
— <20 *a-na* 30 *i-ši-ma* 10 *i-li*>
26 20 *a-n*[*a* 2 *i-š*]*i-ma* 40 *i-li* /
27 20 *a-na* [1 20 *i-ši*]*-ma* 26 40 *i-li*/
28-29 20 *a-na* [45 *i*]*-ši-ma* / 15 *i-li*
30 [*x x*] *ku-mur-ka* / *i-ši-ma ni*-[*x*]
31 *tu-úr-ma* / *pa-ni* 20$^{?}$ [*x x x*] *x x x* /
32 [*x*] *a-na* [*x x x x x x x x*] /
33 *pa-ni* [*x x x x x x x x*] /
34 1 31 [40 *x x x x x x x x*] /
35 *pa-ni* [*x x x x x x x x*] /
36 *šum*-[*x x x x x x x x x*] /
[several lines missing]

[If two x for 1 bán] /
[one x for 2 bán,] /
[three x for 4 bán,]
1 [four] / [x for 3 bán.]
2 [The grain] / may rise ⸢or⸣ [may fall], /
3 ⸢the market rates⸣ [may be equal.] /
4 *x* [*x x x x x x x x*] /
5 *x* [*x x x x x x x x*] /
6 You, [in your procedure:]
7-8 The reciprocal of two th[at for 1 bán] / release, then 30 will [come up]. /
9 30 to 1 [x] carry, then 30 will come [up]. /
10-11 The reciprocal of one [that] t[o 2 bán] / release, then 1 will come up. /
12 1 to 2 carry, then 2 will come up. /

13-14 The reciprocal of three that for / 4! bán release, then 20 will come up. /
15-16 20 to 4 bán carry, then / 1 20 will come up. /
17-18 The reciprocal of four [that] / for 3 bán release, then / [15] will come up.
19 15 to 3 bán / carry, then 45 will come up. /
20 Turn back, then
21-22 30, 2, 1 20, and 45 heap / 4 35 will come up.
23-24 What / to 4 35 should I set / that 1 31 40, the grain, will construct for you? /
25 [x] 20 s[et], it will construct for you. /
— <20 to 30 carry, then 10 will come up.>
26 20 to [2 li]ft, then 40 will come up. /
27 20 to [1 20 carry], then 26 40 will come up. /
28-29 20 to [45] carry, then / 15 will come up.
30 [x x] your heap / carry, then x.
31 Return, then / the reciprocal of 20 [*x x*] *x x x* /
32 [*x*] to [x x x x x x x] /
33 The reciprocal of [*x x x x x x x x*] /
34 1 31 [40 *x x x x x x x*] /
35 The reciprocal of [*x x x x x x x x*] /
36 *x* [*x x x x x x x x x*]
[several lines missing]

In this exercise, nothing remains of the question. The preserved initial part of the solution procedure is quite clearly structured but without any detailed information about what is really going on. It begins in lines 7-19 with a series of four computations of a common form, where in each case information from the question is cited:

a. There are 'two for 1 bán' rec. 2 · 1 = 30 · 1 = 30 (lines 7-9)
b. There are 'one for 2 bán' rec. 1 · 2 = 1 · 2 = 2 (lines 10-12)
c. There are 'three for 4 bán' rec. 3 · 4 = 20 · 4 = 1 20 (lines 13-16)
d. There are 'four for 3 bán' rec. 4 · 3 = 15 · 3 = 45 (lines 17-19)

Next, the sum of the four computed numbers, 30, 2, 1 20, and 45 is compared with 1 31 40, clearly a number given in the lost question. The result is as follows:

30 + 2 + 1 20 + 45 = 4 35 (lines 21-22)
20 · 4 35 = 1 31 40 (lines 23-25)

The computed number 20 = 1 31 40 / 4 35 is then used as a multiplier for three of the four previously computed numbers (the omission of one of those numbers is an obvious mistake):

<20 · 30 = 10>, 20 · 2 = 40, 20 · 1 20 = 26 40, 20 · 45 = 15 (lines 26-29)

Only frustratingly uninformative traces remain of the rest of the solution procedure.

However, as mentioned above, there are obvious similarities between what is preserved of the solution procedure in IM 31210 # 4 and what is known about the well known category of Old Babylonian "combined market rate exercises" (Friberg, *MSCT 1* (2007), Sec. 7.2; see also Sec. 9.1.3 below). Consider, for instance the following combined market rate exercise, in the form of a numerical array (*op. cit*., p. 160):

MS 2299

			16 53 20
3	20	5 55 33 20	17 46 40
4	15	4 26 40	17 46 40
5	12	3 33 20	17 46 40
6	(10)	2 57 46 40	17 46 40

In this numerical array, the numbers in the first column can be interpreted as four different "market rates". The market rate of a given commodity is *the number of "units" of that particular commodity that can be purchased for 1 shekel of silver*. The nature of a unit depends, of course, on the commodity considered. For *grain*, *etc*., it could be the gur (a capacity unit of 5 00 sìla, each equal to about 1 liter), for *metals* it could be the mina (a weight unit equal to about 500 grams), for *fish* it could be a basket of sixty fishes, and so on. Note that in a market economy *before the invention of money*, it was much more convenient to operate with a market rate *r* for a given commodity than with its reciprocal, the "unit price" *p*, expressing *how much silver was needed to buy 1 unit of the commodity.*

The numbers in the second column of the numerical array above are, of course, the four unit prices corresponding to the market rates in the first column. The sum of these four unit prices can be called the "combined unit price" *P*. In the given example it is easy to check that the combined unit price is

> *P* = 57 (shekels), the silver needed to buy a combination of one unit of each of the four commodities.

This combined unit price is not explicitly mentioned in the numerical array. However, it is clear that it was compared with the "available silver" 16 53 20, inscribed at the top of the array. Indeed, counting with relative sexagesimal numbers, one easily sees that

> 16 53 20 = 20 · 50 40, 50 40 = 40 · 1 16, 1 16 = 4 · 19, so that 16 53 20 = 20 · 40 · 4 · 19.

On the other hand,

> 57 = 3 · 19, so that 16 53 20 = 20 · 40 · 4 · 20 · 57 = 17 46 40 · 57.

Note that the "multiplier" 17 46 40 is inscribed four times in the fourth column of the numerical array. Note also that

> 17 46 40 · 20 = 5 55 33 20, 17 46 40 · 15 = 4 26 40, 17 46 40 · 12 = 3 33 20, and 17 46 40 · 10 = 2 57 46 40.

In view of all these numerical relations, it is now clear that the numerical array above can be interpreted as the answer to a question of the following kind:

> How many times a combination of 1 unit each of four different commodities can be bought for an available amount of 16;53 20 shekels of silver if the market rates for the four commodities are, respectively, 3, 4, 5, 6 units per shekel of silver?

The detailed answer given by the numerical array is that

> A combination of 17;46 40 units of each one of the four commodities can be bought for the following combined price:
> 17;46 40 · (;20 + ;15 + ;12 + ;10) = (5;55 33 20 + 4;26 40 + 3;33 20 + 2;57 46 40) = 16;53 20 shekels of silver.

Remark: In the absence of any specific indications in the text of MS 2299 how the recorded numbers should be interpreted, there is more than one conceivable explanation of the problem and its solution. Instead of commodities purchased, one may think of wares produced, or work finished, *etc*., in a given period of time. (Cf. Friberg, *RlA 7,* Sec. 5.6 h.) Thus, instead of a combined market rate problem of the kind described above, the problem behind the numerical array on MS 2299 may have been, for instance, a "combined work norm problem" of the following kind:

> Given four kinds of wares produced or work finished *in equal quantities* at four different work rates, namely 3, 4, 5, and 6 units per man-day, and given a total of 16;53 20 man-days (understood as 16 and 2/3 and 1/3 of 2/3 man-days). Then the combined cost in labor is ;57 man-days for 1 unit of each kind, and 17;46 40 units of each kind can be produced or finished in the given 16;53 20 man-days. The cost in labor for 17;46 40 units of the first kind is 17;46 40 · ;20 man-days = 5;55 33 20 man-days, *etc.*

After this thorough discussion of a typical example of a combined market rate problem, the meaning of (what is left of) the curious solution procedure in IM 31210 # 4 can be better understood. Actually, the explanation which is proposed here is that the stated problem in exercise # 4 was a combined market rate problem. In that case, the lost question in exercise # 4 may have been formulated in the following way:

> How many times a combination of 1 unit each of four different commodities can be bought for an available amount of, say, 1 31;40 bán of grain, if the market rates for the four commodities are, respectively, 2 for 1 bán, 1 for 2 bán, 3 for 4 bán, and 4 for 3 bán?

The answer to this question seems to have been obtained in the following way:

a	The price of 1 unit with the market rate 2 for 1 bán is	rec. 2 · 1 = ;30 · 1 =	;30 bán	(lines 7-9)
b	The price of 1 unit with the market rate 1 for 2 bán is	rec. 1 · 2 = 1 · 2 =	2 bán	(lines 10-12)
c	The price of 1 unit with the market rate 3 for 4 bán is	rec. 3 · 4 = ;20 · 4 =	1;20 bán	(lines 13-16)
d	The price of 1 unit with the market rate 4 for 3 bán is	rec. 4 · 3 = ;15 · 3 =	;45 bán	(lines 17-19)

Therefore,

The combined price for 1 unit of each kind is ;30 + 2 + 1;20 + ;45 = 4;35 bán (lines 21-22)

Now suppose that the given number '1 31 40' stood for, say, 1 31;40 bán of grain. In that case, the observation that 20 · 4 35 = 1 31 40 can be interpreted as meaning that

20 times the combined price is the given amount of grain (lines 23-25)

Then what remained to be computed was the following:

a´	The price of 20 units of the commodity with the market rate 2 for 1 bán is	20 · ;30 =	10 bán	
b´	The price of 20 units of the commodity with the market rate 1for 2 bán is	20 · 2 =	40 bán	(line 26)
c´	The price of 20 units of the commodity with the market rate 3 for 4 bán is	20 · 1;20 =	26;40 bán	(line 27)
d´	The price of 20 units of the commodity with the market rate 4 for 3 bán is	20 · ;45 =	15 bán	(lines 28-29)

It is easy to check that the sum of these four prices is, indeed, the given amount of grain. Indeed,

10 + 40 + 26;40 + 15 bán = 1 31;40 bán (lines 29-30)?

What happens in the poorly preserved continuation of the exercise, with start in line 31, is not clear. Anyway, the importance of this exercise, # 4 in IM 31210, is that it contains *the only explicit solution procedure for a combined market rate problem in the whole corpus of published mathematical cuneiform texts!*

Remark: It is also interesting to note *the counting in exercise # 4 with something like our common fractions*. Indeed, the unit prices seem to be computed here by use of the following computation rule:

rec. $(n \cdot$ rec. $m)$ = rec. $n \cdot m$.

This is a Babylonian counterpart to the modern computation rule $1/(n/m) = m/n$.

A puzzling observation in this exercise is that the capacity measures in the given market rates

'two for 1 bán', 'one for 2 bán', 'three for 4 bán', 'four for 3 bán'

are not written in the usual way with the special cuneiform number signs for 2(bán), 3(bán), 4(bán)! This phenomenon can possibly be explained as follows: The exercise was almost certainly excerpted from a theme text with many similar exercises. In those other exercises, some of the given market rates may have been given in the form

'two for 1 sìla', 'one for 2 sìla', 'three for 4 sìla', 'four for 3 sìla', etc.

or

'two for 1 shekel', 'one for 2 shekel', 'three for 4 shekel', 'four for 3 shekel', etc.

The measure notations in such expressions would have been perfectly legitimate, since grain and silver could be counted as multiples of the basic units sìla and shekel. Whoever worked for a while with such expressions would develop a feeling for 'two for 1', 'one for 2', 'three for 4', 'four for 3', etc., not very different from our own feeling for common fractions like 2/1, 1/2, 3/4, 4/3, etc. Then writing, for instance, 'four for 3 bán' would not be very much different from writing '4/3 per bán'.

7.1.5 # 5. Lost

7.1.6 # 6. Thirteen Brothers(?), Unequal Shares

IM 31210 # 6 (col. *iv*)

[several lines missing]

1´-4´ [*x*] *x x* [*x x*] *x x-ma* / *x x x* [*x x*] *x x-ma* / *x x x* [*x x*] *x x x* / *i-li*

5´-7´ *a-na* [*x x*] *x-ma* / *x x x x-ma* / [*x x x*] *x x x-ma* / [*x x x*]

8´ [6] 40 *a-na* 1(bán) kaš(?) / [*i-ši-m*]*a* 6 40 x *i-li*

9´ 6 40 *a-na* / [2] *i-ši-ma* 13 20 *i-li* /

10´ 6 40 *a-na* 2 *i-ši-*[*m*]*a* 13 20 *i-li* /

11´-13´ 13 20 *a-na* 2 30 *x x x x* / *zi-ti* [*ša*]-*la*-[*ša-at*] *i-ši-ma* / 33 20 *i-li*

14´-16´ *tu-úr-ma* / 13 20 *ra-bu-um* šu.<ba.an.ti> / 13 20 *x x x x* šu.<ba.an.ti> / 33 20 *ša-al-šum* /

17´ *zi-<it-ti>-šu ku-mur-ma* [1 bán].bi *i-li* /
18´ *ki-a-am né-pé-šum*

11´-4´ [x] x x [x x] x x, then / x x x [x x] x x, then / x x x [x x] x x x / will come up
5´-7´ to [x x] x, then / x x x x, then / [x x x] x x x, then / [x x x]
8´ [6] 40 to 1 bán of beer(?) / [carry], then 6 40 x will come up.
9´ 6 40 to / [2] carry, then 13 20 will come up. /
10´ 6 40 to 2 carry, then 13 20 will come up. /
11´-13´ 13 20 to 2 30 x x x x / the thi[rd] share carry, then / 33 20 will come up.
14´-16´ Return, then / 13 20 the big (brother) will re<ceive> / 13 20 x x x x will re<ceive> / 33 20 the third. /
17´ Their shares heap up, then its [1 bán] will come up.
18´ Such is the procedure.

Most of the text of this exercise is destroyed or illegible. In addition it is not clear what the expression 1 bán.bi stands for. (It occurs also in exercise # 3.) Does it refer to 1 bán (= 10 sìla) of some (somewhat expensive) commodity? Or should it be read as 1(bán) kaš ‘1 bán of beer’, since one of the ways to read the cuneiform sign bi is as kaš ‘beer’? The same question arose in exercise # 3 above.

In spite of the mentioned difficulties, so much is preserved of the solution procedure that it is possible to suggest at least a tentative interpretation of what is going on in exercise # 6 of IM 31210, namely that the question was formulated, essentially, in the following way:

1 bán of some commodity was divided between 13 brothers in such a way that
brothers 2 and 3 got equal shares,
the oldest brother got as much as brothers 2 and 3 together,
the remaining ten brothers also got equal shares and together 2 1/2 times as much as brothers 2 and 3 together.

This problem can easily be solved by use of the rule of false values. Indeed,

If brothers 2 and 3 got 1 each, then the oldest brother got 2, and the remaining 10 brothers got together 2 30 · 2,
so that the sum of all the shares would be 2 + 2 + 2 30 · 2 = 9. However, the sum of all the shares should be 1 bán.
Therefore, the oldest brother got rec. 9 · 2 = 6 40 · 2 = 13 20 (bán) (lines 7´-9´)
brothers 2 and 3 got together also 13 20 (bán) (line 10´)
and the remaining 10 brothers got together 2 30 · 13 20 = 33 20 (bán) (lines 11´-13´)
Indeed, 13 20 + 13 20 + 33 20 = ‘1’ = 1 bán (line 17´)

In other words, the individual shares would be

1 brother: ;13 20 bán, 2 brothers: ;06 40 bán, 10 brothers: ;03 20 bán.

7.1.7 # 7. Ten Partners. Steadily Decreasing Shares

In exercise # 7 of IM 31210, the question is well preserved, although it is not very clearly formulated. A substantial part of the solution procedure is also preserved, but the end of the solution procedure and the answer are both missing.

IM 31210 # 7 (col. *iv-v*)
1 *šum-ma* $1\,^{2}/_{3}{}^{!}$ ma.n[a k]ù.babbar 10 *at-ḫu-*[*ú*] /
2-3 *zi-it-ti ša-la-ša-at ak-mur-ma* / 46 48
ki ma-ṣ[*í*] *ur-ta-ba-bu* /
4 *at-ta i-na e-pé-ši-ka* /
5-6 *pa-ni* 10 *at-ḫi-ka pu-ṭur-ma* / 6 *i-li*
7 6 *a-na* [1 40] kù.babbar [*i-ši-ma*] / 10 *i-li*
8 10 *a-na* 3 *at-ḫi-ka* / *i-ši-ma* 30 *i-li*
9 30 *i-na* 46 48 / *ku-mur-ri-ka ú-su-uḫ-ma* 16 4[8] /
10 16 48 *re-eš-ka li-ki-*[*il*] /
— [several lines missing]

1 If 1 2/3 mina of silver, 10 partner[s], /
2-3 the shares of three I heaped, then / 46 48,
how much were they respectively weaker? /

4 You, in your procedure: /
5-6 The reciprocal of 10 of your partners release, then / 6 will come up.
7 6 to [1 40], the silver, [carry, then] / 10 will come up.
8 10 to 3 of your brothers / carry, then 30 will come up.
9 30 from 46 48, / your heap, tear off, then 16 4[8]. /
10 16 48 let keep your head. /
— [several lines missing]

As observed already by Bruins in the original publication of the text, in *Sumer* 10 (1954), the vaguely presented problem here can be interpreted as follows:

> 1 2/3 mina of silver was divided between 10 partners, so that the first three got ;46 48 mina.
> The shares were steadily decreasing. How much less than each partner's share was the next partner's share?

In modern terms, the shares form a decreasing arithmetical progression. It is likely that this is what is implied by the term *ur-ta-ba-bu* 'they were respectively weaker'.

The solution procedure starts with the following computation:

$\text{igi}\ 10 \cdot 1\ 40 = 6 \cdot 1\ 40 = 10$ (lines 5-7)

It should probably be understood as the computation of the average share, namely the given amount of silver divided by the number of partners, hence as $a = 1/10 \cdot 1;40$ mina = ;10 mina = 10 shekels.

Consequently, if *the constant difference between successive shares* is called d, then the 10 shares are

$a + 9\ d/2,\ a + 7\ d/2,\ a + 5\ d/2,\ a + 3\ d/2,\ a + d/2,\ a - d/2,\ a - 3\ d/2,\ a - 5\ d/2,\ a - 7\ d/2,\ a - 9\ d/2.$

Even if this is not stated explicitly in the solution procedure, it is likely that this observation is what the ensuing part of the solution procedure was based upon. In particular, the combined shares of the 3 first partners can then be expressed as follows:

$46;48 \text{ shekels} = (a + 9\ d/2) + (a + 7\ d/2) + (a + 5\ d/2) = 3\ a + 21\ d/2.$

Therefore, the constant difference d can be computed as follows:

$21\ d/2 = 46\ 48 - 3 \cdot 10 = 46\ 48 - 30 = 16\ 48$ (= 16;48 shekels) (lines 7-10)

The remainder of the solution procedure is lost, but it is obvious that the next step must have been to ask 'what times 21 is 16 48?'. This question has the answer '48 times 21 is 16 48'. Consequently,

$d/2 = 48$ (= ;48 shekels), so that $d = 1\ 36$ (= 1;36 shekel),

and the greatest share, for instance, is

$a + 9\ d/2 = 10 + 9 \cdot 48 = 10 + 7\ 12 = 17\ 12$ (= 17;12 shekels).

However, in the question only the constant difference ($d = 2 \cdot d/2$) was asked for, not the individual shares. Therefore, the answer to the question may have been formulated just as

1 36 *ur-ta-ba-bu* '1 36 they were respectively weaker'.

7.1.8 # 8. Steadily Decreasing Shares. A Very Badly Preserved Exercise

IM 31210 # 8 (col. *v*)

1′-2′ [2 *a-na*] ⌜15⌝ *ša ri*-[*iš-ka*] / [*ú-ka-lu*] *i-ši-ma* ⌜30⌝ *i-li*
3′ [30] / [*ur*]-*ta-ba-b*[*u*]
[*ki-a-am ne-pé-šum*]

1′-2′ [2 to] 15 that [held your] head / carry, then 30 will come up.
3′ [30] / [they were] respectively weaker.
[Such is the procedure.]

Only the last few lines of exercise # 8 are (imperfectly) preserved. It is clear, however, that this exercise is closely related to the preceding exercise, # 7. Apparently, the last computation of the exercise, in lines 1-2, is the computation

$d = 2 \cdot d/2 = 2 \cdot 15 = 30.$

What this probably means is that in this exercise the shares of a certain number of partners were steadily decreasing, with a constant difference between successive shares of 1/2 shekel.

7.1.9 Str. 362 # 1. A Text from Uruk of a Similar Kind, with Steadily Increasing Shares

This text was published and explained by Neugebauer in *MKT I* (1935), 239 ff.

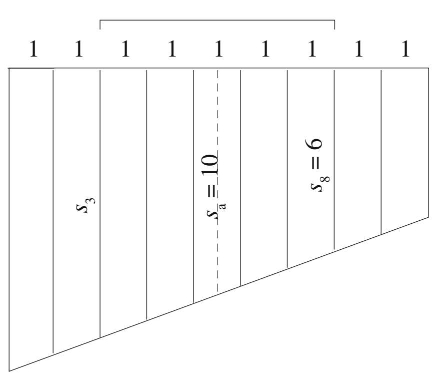
Str. 362 # 1

1-2 10 šeš.e.ne 1 2/3 ma.na kù.babbar / šeš ugu šeš *ú-te-li-li-ma* /
3-4 *ma-la ú-te-li-lu-ú ú-ul i-di* / ḫa.la ki.8 6 gín
5 šeš ugu šeš / *ki ma-ṣi ú-te-li-li* /
6 za.e ak.da.zu.dè /
7-8 igi 10 erin duḫ-*ma* 6 in.šúm 6 *a-na* 1 2/3 ma.na kù.babbar / *ta-na-aš-ši-i-ma* 10 in.šúm
9 10 *a-na ši-na* e.tab-*ma* / 20 in.šúm
10 6 *zi-it-ti sa-am-ni-im* / *a-na ši-na* e.tab-*ma* 12 in.šúm
11 12 *i-na* 20 zi-*ma* / 8 in.šúm 8 *ri-eš-ka li-ki-il* /
12-13 1 *ù* 1 *ša-ap-li-a-am* gar.gar-*ma* 2 in.šúm / 2 *a-na ši-na* e.tab-*ma* 4 in.šúm /
14-15 1 *a-na* 4 *tu-ṣa-am-ma* 5 in.šúm / 5 *i-na* 10 erin zi-*ma* 5 in.šúm /
16-17 igi 5 gál.bi duḫ-*ma* 12 in.šúm / 12 *a-na* 8 nim-*ma* 1 36 in.šúm /
18 1 36 *ša* šeš ugu šeš *ú-te-li-lu-ú*

1-2 10 brothers, 1 2/3 mina of silver / brother over brother repeatedly was above, then /
3-4 how much he was repeatedly above I did not know. / The 8th share was 6 shekels.
5 Brother over brother / how much was he repeatedly above? /
6 You in your doing it:
7-8 The reciprocal of 10 the men release, then 6 will come up. 6 to 1 2/3 mina of silver / you will carry, then 10 it will give.
9 10 to two repeat, then / 20 it will give.
10 6 of the eigth share / to two repeat, then 12 it will give. /
11 12 from 20 tear off, then / 8 it will give. 8 let hold your head./
12-13 1 and 1 the lower heap together, then 2 it will give / 2 to two repeat , then 4 it will give. /
14-15 1 to 4 you will add on, then 5 it will give. / 5 from 10 men tear off , then 5 it will give. /
16-17 The reciprocal of 5 release, then 12 it will give. / 12 to 8 lift, then 1 36 it will give. /
18 1 36 that brother over brother repeatedly is above.

In this exercise, just like in the exercise IM 31210 # 7 in Sec. 7.1.7 above, 10 men divide 1 2/3 mina of silver in unequal shares, with a constant difference between consecutive shares. The object of the exercise is to find the size of that constant difference, given that the 8th share is 6 shekels.

The explicit solution procedure in Str. 362 # 1 was perfectly explained by Neugebauer in *MKT I*, with the exception that his explanation was illustrated by use of a modern bar diagram. If the author of the exercise did think in terms of a diagram, which of course is far from certain, it is much more likely that that diagram looked something like the diagram in Fig. 7.1.1 below. Indeed, striped trapezoids of a similar kind were quite popular in Old Babylonian mathematical texts. See Friberg *AT* (2007), Secs. 11.4-11.10.



$s_1 + s_2 + \ldots\ldots\ldots + s_{10} = 1\ 40$

$s_a = \text{rec. } 10 \cdot 1\ 40 = 10$

$s_3 - s_8 = 2 \cdot (s_a - s_8) = 2 \cdot (10 - 6) = 2 \cdot 4 = 8$

$m = 10 - 1 - 2 \cdot (1 + 1) = 10 - 5 = 5$

$d = (s_3 - s_8)/m = \text{rec. } 5 \cdot 8 = 12 \cdot 8 = 1\ 36$

Fig. 7.1.1. Str. 362 # 1. Explanation of the solution procedure.

The solution procedure in Str. 362 # 1 starts, like the solution procedure inIM 31210 # 7, with the computation of the average share, as rec. 10 · 1 40 = 10. In the diagram above, the average share is called s_a and is shown as the length of the dashed mid-line in the trapezoid. In the same diagram, the 10 shares are shown as the lengths of two parallel fronts and eight parallel transversals, all at a constant distance 1 from each other. In particular, the known eighth share is called s_8 and is drawn as a fat line. The symmetrically placed transversal representing the third share is also indicated and is called s_3. The diagram makes it geometrically obvious that

$s_3 - s_a = s_a - s_8$, so that $s_3 - s_8 = 2 \cdot (s_a - s_8) = 2 \cdot 10 - 2 \cdot 6 = 20 - 12 = 8$ (lines 7-11)

Next, the distance between the third and eighth transversals is computed in a somewhat roundabout way, by noting that s_8 and s_8 both are a distance 2 away from the upper and lower fronts, respectively, which in their turn are at a distance of 10 – 1 from each other. (There are, in general, n – 1 intermediate spaces between n parallel lines.) Therefore, the distance m from s_3 to s_8 is

$m = 10 - (1 + 2 \cdot 1 + 1)) = 10 - 5 = 5$ (lines 12-15)

Then, financially, the constant difference d between consecutive shares can be computed as follows:

$5\,d = s_3 - s_8 = 8$, so that d = rec. 5 · 8 = 12 · 8 = 1 36 (lines 16-17)

7.1.10 # 9. Four Partners. A System of Linear Equations

In exercise # 9 of IM 31210, the question and a large part of the solution procedure are well preserved, but both the end of the solution procedure and the answer are missing.

The explanation of the solution procedure in this exercise suggested by Bruins in *Sumer* 10 (1954) would have been essentially correct if Bruins had not incorrectly read 10 *shik-lu* (which he supposed to mean ‘10 shekels’) instead of 10 5/6 gín (‘10 5/6 shekels’) in lines 4 and 23!

IM 31210 # 9 (col. *v*)

1 *šum-ma i-ša-lu-*[*ka um-ma šu-ú-ma*] /
2 1 $^1/_2$ ma.na kù.babbar 4 *at-*[*ḫu-ú*] /
3-4 *ra-bu-um e-li te-ni* [*x x*] / 10 $^5/_6$ gín kù.babbar *li-il-*[*qi*] /
5-6 *se-bi-at zi-it-ti te-n*[*i x*] / *ša-al-šum*
7-8 *iš-te-en-š*[*e-re-et*] / *zi-it-*[*ti*] *ra-bi ṣé-ḫe-rum* / *li-il-qi* /
9 *at-ta i-na e-pé-ši-*[*ka*] /
10-11 11 *ù* 7 *šu-ta-ki-*[*il*]*-ma* / 1 17 *i-li*
12-13 1 1[7 *a*]*-na ši-na* / *e-ṣí-*[*ip-m*]*a* 2 34 *i-li* / 2 34 [*ri*]*-iš-ka li-ki-il* /
14-15 *tu-úr-*[*ma*] 11 *ù* 7 *ku-mur-ma* / 18 *i-li*
16-18 18 *a-na* 2 34 / *ša ri-iš-ka ú-ka-lu ru-di-ma* / 2 52 *i-*[*li*] 2 52 *ri-iš-ka* / *li-ki-i*[*l*]
19 *tu-úr-ma* 1 30 kù.babbar / *a-na* 11 *i-*[*ši*]*-ma* 16 30 *i-li* / 16 30 *ri-iš-ka li-ki-il* /
20-21 *tu-úr-ma* 1 *ša ra-ma-ni-ka* / *a-na* 11 *ṣí-ib-*[*ma*] 12 *i-li* /
22-23 [12] *a-na* 10 $^5/_6$ gín *i-ši-ma* / 2 10 *i-li*
24-25 2 10 *i-na* 16 30 / *ša ri-iš-ka ú-*<*ka*>*-lu ḫu-ru-iṣ-ma* / [14 20 *i-li*]
— [several lines lost]

1 If he asks [you, saying this:] /
2 1 1/2 mina of silver, 4 part[ners]. /
3-4 The big (partner) over the next(?) [x x] / 10 5/6 shekels of silver may ta[ke]. /
5-6 The seventh of the share of the next(?) / (is) the third.
7-8 The elev[enth] / of the share of the big (partner) the small / may take.
9 You, in [your] procedure: /
10-11 11 and 7 let eat each other, then / 1 17 will come up.
12-13 1 1[7 t]o two / repe[at, th]en 2 34 will come up. / 2 34 may hold your [he]ad. /
14-15 Return, then 11 and 7 heap, then / 18 will come up.
16-18 18 to 2 34 / that held your head add, then / 2 52 will [come up]. 2 52 may hold / your head.
19 Return, then 1 30 of silver / to 11 carry, then 16 30 will come up. / 16 30 may hold your head. /
20-21 Return, then 1 of your own / to 11 add on, [then] 12 will come up. /
22-23 [12] to 10 5/6 shekels carry, then / 2 10 will come up.

24-[25] 2 10 from 16 30 / that [kept] your head break off, then / [14 20 will come up].
— [several lines lost]

The stated problem in this exercise is quite clearly expressed. With a slight reformulation it is as follows:

4 partners share 1 1/2 minas of silver.
The share of the first partner is 10 5/6 shekels more than the share of the second partner.
The share of the third partner is 1/7 of the share of the second partner.
The share of the fourth partner is 1/11 of the share of the first partner.

In modern symbolic notations, if the four shares are called *a, b, c, d,* in decreasing order, then

$a + b + c + d = 1\ 30$ (shekels)
$a = b + 10\ 5/6$ (shekels)
$c = b/7$
$d = a/11$.

This *system of four linear equations for the four unknowns a, b, c, and d* can be solved in a number of different ways. However, what is left of the solution procedure seems to suggest that the the author of exercise IM 31210 # 8 proceeded as follows:

Since the last two equations show that $a = 11\ d$ and $b = 7\ c$, the first two equations can be reformulated as the following *system of two linear equations for the two unknowns c and d*:

$(11 + 1) \cdot d + (7 + 1) \cdot c = 1\ 30$
$11\ d = 7\ c + 10\ 5/6$.

A modern way of solving this system of two linear equations for two unknowns would be to eliminate one of the unknowns, thereby reducing the problem to a single linear equation for one of the unknowns. That is also what the author of this exercise seems to have done, although without telling the reader all the details of what is going on. Apparently he started by observing that in the second equation 11 *d* is given in terms of the unknown *c*. Therefore he multiplied, in his mind, all the numerical coefficients in the first equation by 11, so that he got the following new equation

$(11 + 1) \cdot 11\ d + (7 + 1) \cdot 11\ c = 11 \cdot 1\ 30$.

By use of the equation for 11 *d* in terms of *c* he could then, still in his mind only, reformulate this equation as

$(11 + 1) \cdot (7\ c + 10\ 5/6) + (7 + 1) \cdot 11\ c = 11 \cdot 1\ 30$ (*)

Next, he could count the number of times that the unknown *c* appears in equation (*), as follows:

$(11 + 1) \cdot 7 + (7 + 1) \cdot 11 = 2 \cdot 11 \cdot 7 + (11 + 7) = 2 \cdot 1\ 17 + 18 = 2\ 34 + 18 = 2\ 52$ (lines 10-17)

Further, he could compute the given numbers appearing in equation (*), as follows:

$11 \cdot 1\ 30 = 16\ 30$ and $(11 + 1) \cdot 10\ 5/6 = 12 \cdot 10\ 5/6 = 2\ 10$ (lines 18-23)

Consequently, he could (still in his mind) rewrite equation (*) in the following simpler form

$2\ 52\ c + 2\ 10 = 16\ 30$.

Finally, he could simplify the equation further by collecting together the given terms, with the result that

$2\ 52\ c = 16\ 30 - 2\ 10 = [14\ 20]$ (lines 23-[25])

The remainder of the solution procedure is lost, but it is obvious that it proceeded, essentially, as follows:

What to 2 52 should I set that will construct for me 14 20 of silver? 5 set, then it will construct for you.

In other words, $c = 14\ 20 / 2\ 52 = 5$ (shekels). With *c* known, the remaining shares could be computed directly, as

$b = 7\ c = 35$ (shekels), $a = b + 10\ 5/6 = 45\ 5/6$ (shekels), $d = a/11 = 4\ 1/6$ (shekels).

It is easy to check that then, as it should be,

$a + b + c + d = 45\ 5/6 + 35 + 5 + 4\ 1/6$ (shekels) $= 1\ 30$ (shekels).

It is clear that the lost part of the solution procedure in IM 31210 exercise # 8, with simple computations of the kind shown above, can have occupied at most about 10 lines at the top of column *vi*. No traces remain of that part of column *vi*, which is completely broken off.

Here are some final remarks about surprising features of the text of exercise # 8. First, it is surprising that in line 22, in the course of the solution procedure, the weight number 10 5/6 shekels appears instead of the corresponding abstract number in sexagesimal place value notation, 10 50.

Secondly, the computation in lines 10-17 of the number of times that *c* appears in equation (*) seems to be unnecessarily complicated. A likely reason for this complication of the computation may be that the author of the text wanted the computation to be as detailed as possibly, so that he could keep track of what was happening, and so that it would be easy for him to use the text as a model in another exercise of the same kind, with different given coefficients in the equations.

A problem similar to IM 31210 # 9 is **VAT 8522 # 2**. See Fig. 11.3.5 below. The reconstruction of the damaged part of the question in this exercise is due to Muroi, in *HS* 34 (1988). Muroi was also the first one to attempt to give an explanation of the solution procedure. As is obvious from the conform transliteration of the exercise in Fig. 11.3.5, the chaotically organized solution procedure which follows the question in this exercise is not very helpful.

According to Muroi, the (reconstructed) question in VAT 8522 # 2 is the following:

a. Five brothers share 5/6 mina and 9 shekels of silver.
b. The ‘small’ brother gets 1/2 the share of the ‘big’ brother.
c. The five brothers share in equal parts 45 shekels (the text says, mistakenly, 4 30 shekels).
d. One third of the difference between the shares of the ‘big’ brother and the ‘next’ is equal to the following differences.

How big were the shares?

Since the solution procedure presented in the text is not of much use as it is, it may be a good idea to start by presenting an independent solution procedure, for instance in the following form (this is essentially the solution procedure suggested by Muroi):

Start by dividing in unequal shares what remains after each brother has received in equal shares his part of 45 shekels.
Using the method of “false values”, assume that the ‘small’ brother gets 1.
Then condition d is satisfied if the false shares are **7, 4, 3, 2, 1**, but condition b is not satisfied.
Conditions d and b are both satisfied if an extra amount p is added to each share, where $7 + p = 2 \cdot (1 + p)$.
Clearly, then, $p = 5$, so that the new, improved false values are **12, 9, 8, 7, 6**. The corresponding “false sum” is 42.
Conditions a and c together require that the true sum should be equal to 59 – 45 = **14 shekels**.
Since 42 = 3 · 14, the needed “correction factor” is rec. 3 = ;20 and the the corrected shares are **4, 3, 2;40, 2;20, 2 (shekels)**.
Now, the five equal shares of 45 shekles are all equal to rec. 5 · 45 = ;12 · 45 = **9 (shekels)**.
Therefore, if 9 shekels are added to each one of 4, 3, 2;40, 2;20, 2 (shekels), the result is **13, 12, 11;40, 11;20, 11 (shekels)**.

Observe that all the numbers written in boldface above can be found at various places in the scribbled solution procedure of VAT 8522 # 2 (Fig. 11.3.5 below). Therefore, this is clearly the solution procedure in the mind of the author of that exercise.

However, something is wrong here, since 11 shekels is not 1/2 of 13 shekels, as required by condition b above. Fortunately, the solution procedure described above can be kept intact and will produce a correct solution, if the conditions b and c in the question are modified as follows:

b′. The five brothers share in equal parts 45 shekels.
c′. *The remaining 14 shekels* are divided in unequal parts so that he ‘small’ brother gets 1/2 the share of the ‘big’ brother.

7.1.11 # 10. Partners. A Very Badly Preserved Exercise

IM 31210 # 8 (col. *vi*)
[several lines lost or hard to read]
1 [*x x x x x x x x*] *pu-ṭur-ma* /

2 [*x x x x x x x x*]*-ma* 6 *i-li* /
3 [*x x x x x x x x x*] *ú-x-ka* /
4 [*x x x x x x tu*]*-úr-ma* /
5 [*x x x x x x x x*] *ú-x-ka* /
6 [*x x x x x x x x x x x x x*] /
7 [*x x x x x x x x*] 12 *i-li* /
8 [*x x x x x x x x*] *x x x x* /
9 [*x x x x x x x x*] *x-ma* 9 x *i-li* 12 *i-li* /
10 [*x x x x x x x x*] *x x x x x* /
11 [*x x x x x x x x*] *tu-x-x-ma* /
12 [*x x x x x x x x*] *x x x x x* /
13 [*x x x x x x x x*] *x x x-ma* /
14 [*x x x x x x x x*] *ù* 9 *x ša* /
15 [*x x x x x x*] *a-na x* šu.ba.an.ti /
16 [*x x x x x x x x*] *x* šu.ba.an.ti /
17 [*x x x x x x x x*] *x* /
[possibly 1 or 2 lines missing]

7.1.12 The Vocabulary of IM 31210

kaš	beer(?)	3, 6
kù.babbar	silver	2, 7, 9
šu.ba.an.ti, šu.<ba.an.ti>	he receives	2, 6, 10
aš-šum … qa-bu-kum	since it was said to you that	3
at-ta i-na e-pé-ši-ka	you, in your procedure	3, 4, 7
ki-a-am né-pé-šum	such is the procedure	2, 6
mi-nam a-na (*a*) *lu-uš-ku-un*	what to (*a*) should I set	2, 4
ša (*b*) *i-ba-an-ni-a-am*	that will construct (*b*) for me	2
ša (*b*) *i-ba-ni-kum*	that will construct (*b*) for you	2, 4
pa-ni (*a*) *pu-ṭur*	the opposite (reciprocal) of (*a*) release (compute)	1, 4, 7
re-eš-ka li-ki-il	let it hold your head (= keep this in mind)	7
ri-iš-ka li-ki-il	let it hold your head	1, 3, 9
ša ri-[*iš-ka ú-ka-lu*]	that held your head	1, 7
šum-ma i-ša-lu-ka	if he asked you	3, 9
i-ba-an-ni-a-am	< *banû* build (construct)	2
i-ba-ni-kum	< *banû* build (construct)	2, 4
i-li	< *elû* to come up, to rise (= to result)	2, 3, 4, 6, 7, 9
[*e-ṣ*]*í-ip*	< *eṣēpu* to double	3
ku-mur, ak-mur	< *kamāru* to heap, to pile up (= to add together)	2, 4, 6, 7
li-il-qi	< *leqû* to take	9
ú-su-uḫ	< *nasāḫu* to tear out (= to subtract)	2, 7
i-ši	< *našû* to carry (= to multiply (by a number))	2, 4, 6
ur-ta-ba-bu	< *rabābu* Gt to be weak	7, 8
na-ás-ḫi-ir	< *saḫāru* N to turn back	2, 3, 4
lu-<uš->ku-un	< *šakānu* to put down, to set	2, 4
šu-ku-un	< *šakānu* to put down, to set	2, 4
tu-úr	< *tāru* to return (= start again)	3, 4, 6

at-ḫu-ú, at-ḫi-ka	*athû* pl. partners	3, 7
e-li	*eli* above	9
ḫa-am-šu	*ḫamšu* fifth	2
iš-te-en	*ištēn* (number) one	3, 4
iš-te-en-š[*e-re-et*]	*ištēnšerü* eleventh	9
ki ma-ṣi	*kī maṣi* how much?	7
ki-ši-ir-šu	*kišīru* meaning not known	3
ki-ši-ri-ia, ki-ši-ri-ka, ki-ši-ri	*kišīru*	3
ku-mur-ka, ku-mur-ri-ka	*kumurrü* heap, piling up (= sum)	4, 67
me-eḫ-ra-am	*meḫru* copy, duplicate	2
mi-nam, mi-nu	*mīnu* what?	2, 3, 4
ra-bi, ra-bi-im, ra-bu-um	*rabü* big, old	2, 5, 9
ra-ma-ni-ka	*ramānu* self	9
re-bu, ra-bi-it	*rebû,* f. *rabūtu* fourth, one-fourth	2, 4
se-bu-um	*sebü* seventh	2
se-bi-at, se-bi-x-x	*sebītu* one-seventh	3, 9
[*ši-iš-šu*]	*šiššu* sixth	[2]
ša-al-šum	*šalšu* third	5, 9
ša-la-aš, ša-la-ša-at	*šalāš*, f. *šalāšat* three	3, 4, 7
ša-nu-um	*šanü* second, next	2
ši-na	*šina* two	3, 4
šum-ma	*šumma* if	3, 7, 9
te-ni	*tenû* substitute, replacement	9
zi-ti, zi-it-ti	*zittu* share	2, 5, 7, 9

7.1.13 IM 31210. Hand Copies and Conform Transliterations

Fig. 7.1.2. IM 31210, *obv.* Hand copy.

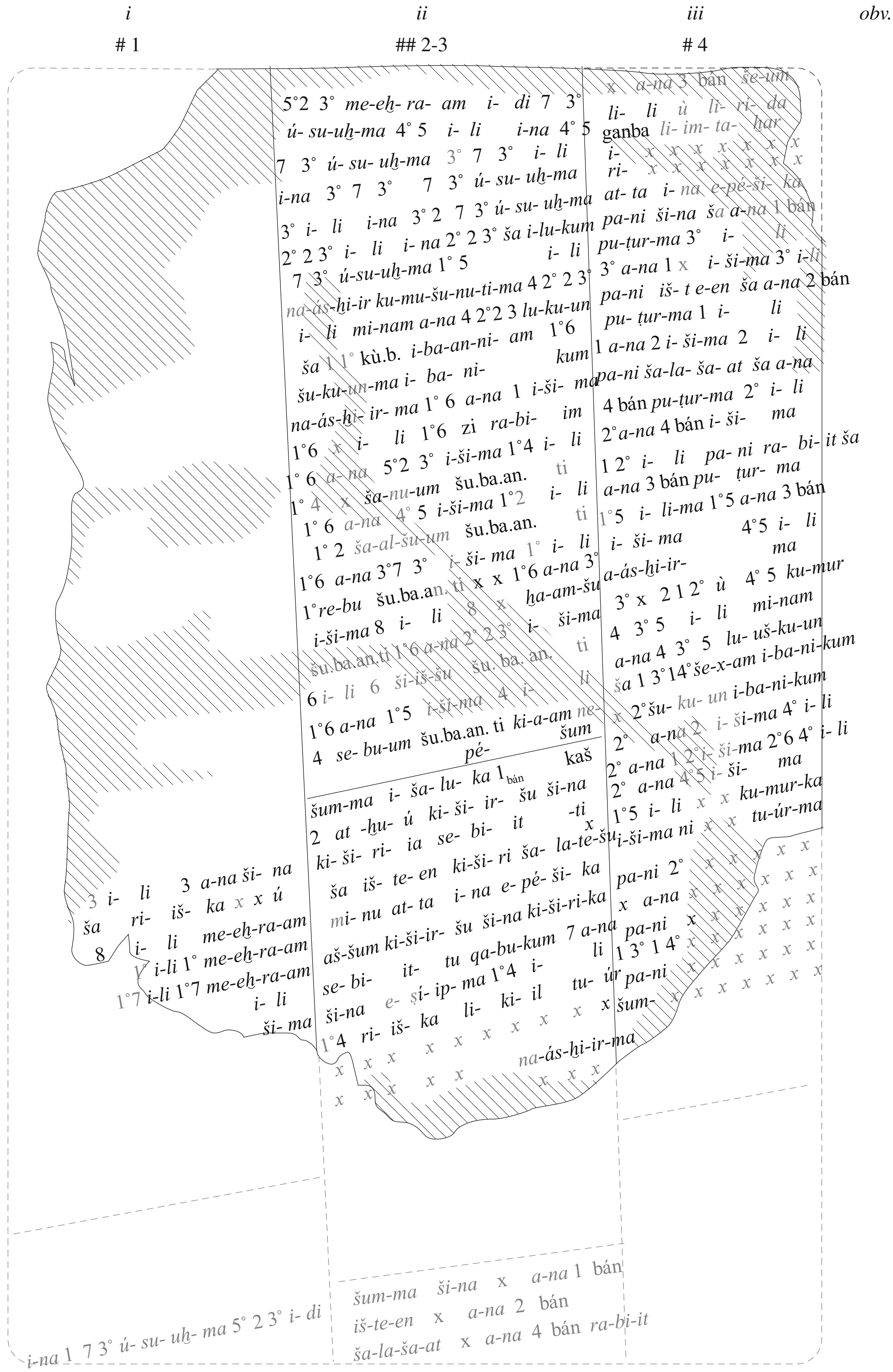


Fig. 7.1.3. IM 31210, *obv.* Conform transliteration.

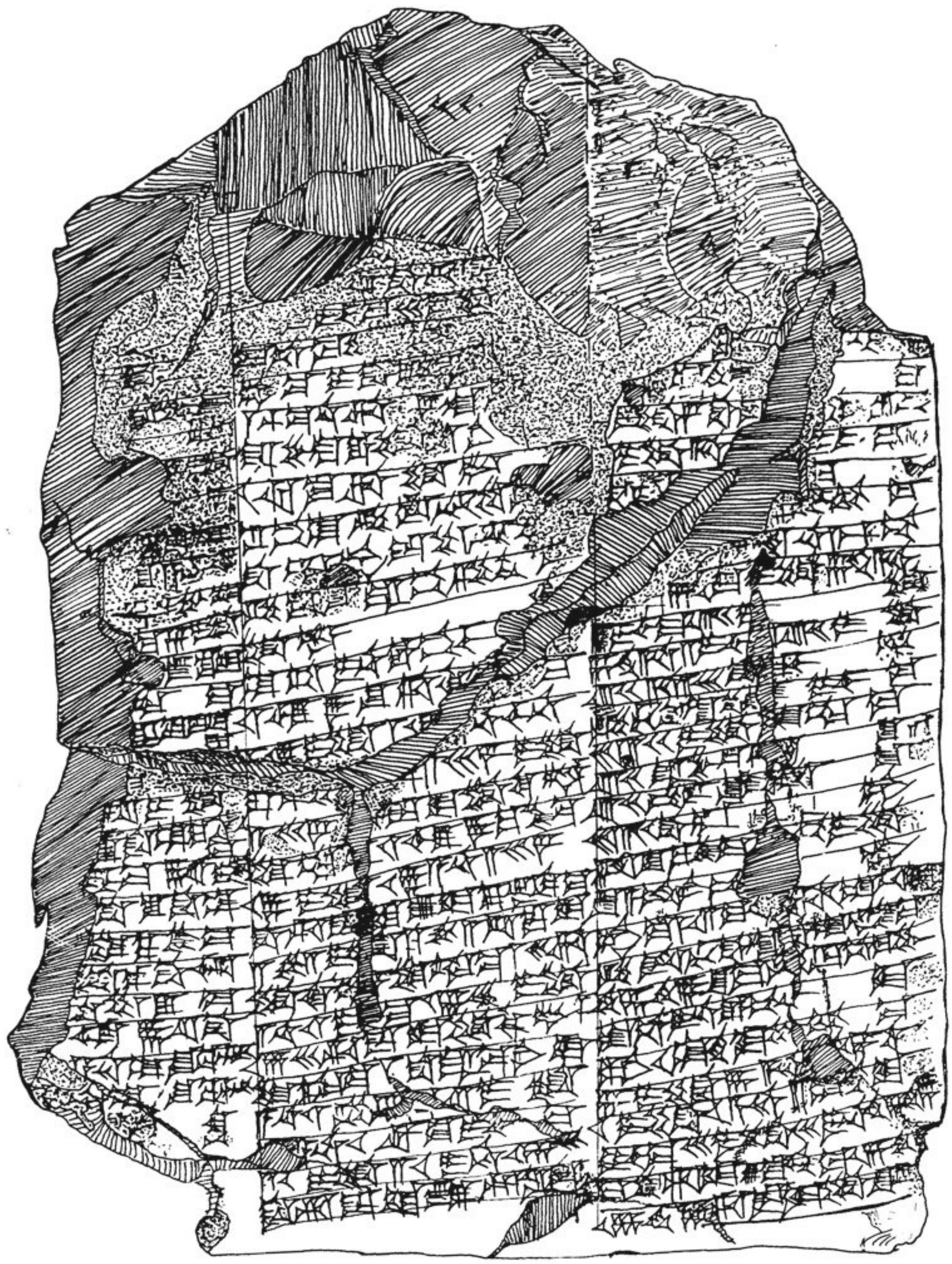


Fig. 7.1.4. IM 31210, *rev.* Hand copy.

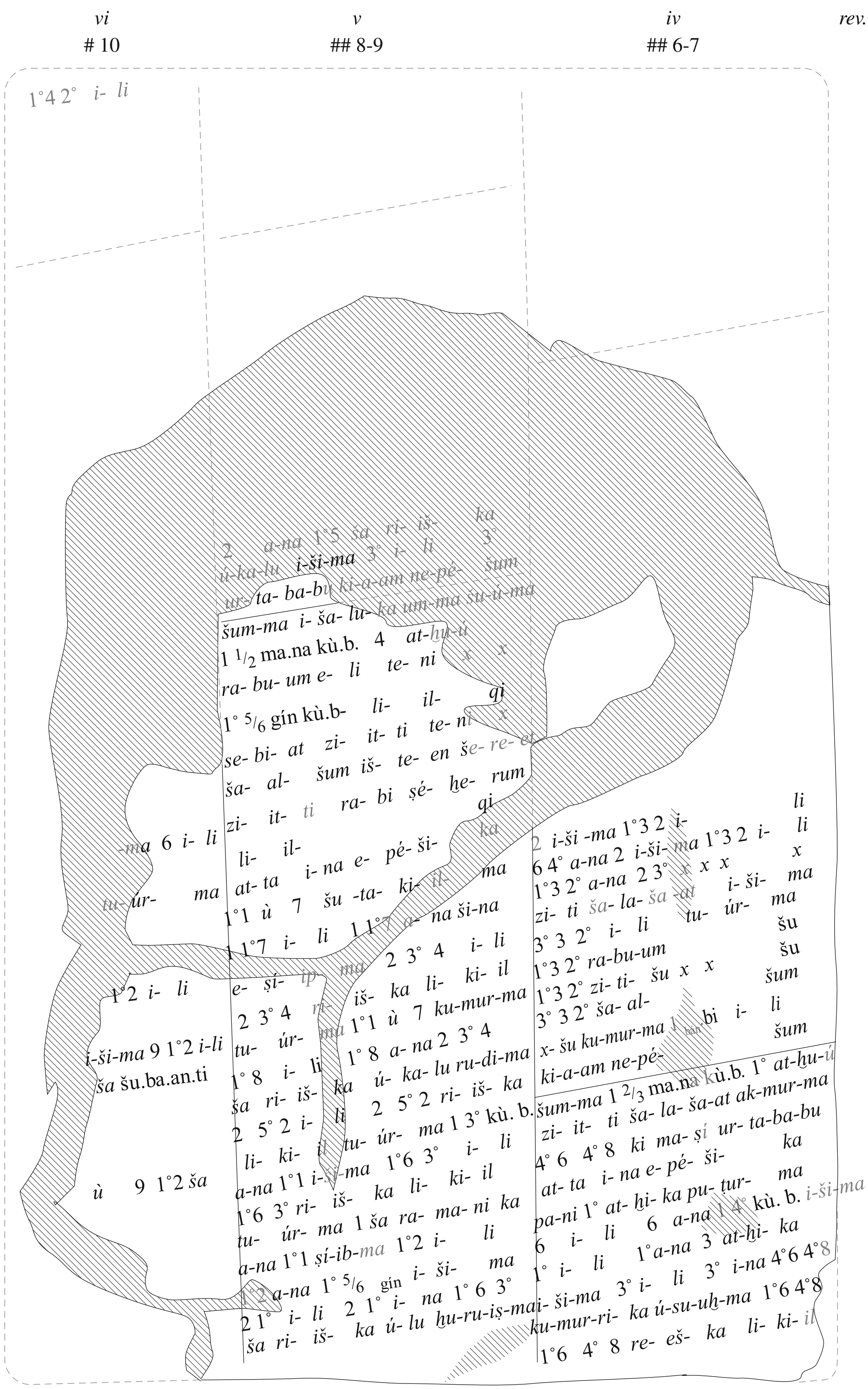
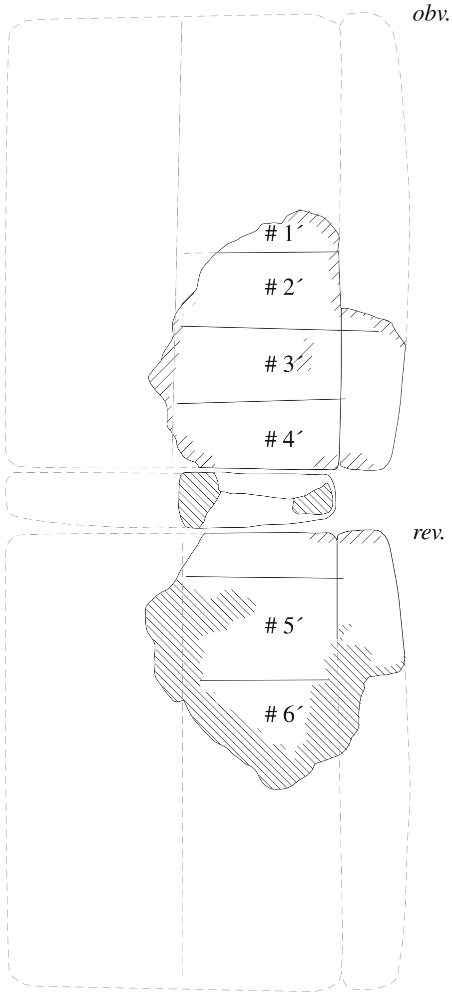


Fig. 7.1.5. IM 31210, *rev.* Conform transliteration.

8. Six Fragments of Problem Texts of Group 6, from Late Old Babylonian Sippar

8.1 BM 80078. A Large Fragment of a Recombination Text with Problems for Bricks

BM 80078 is a relatively large fragment originally forming the lower right corner of an Old Babylonian clay tablet inscribed in two columns on the obverse and two on the reverse with a mathematical recombination text, apparently with problems for bricks as a common theme. See the the suggested reconstruction of the outline of the whole clay tablet in Fig. 8.1.1 below and the hand copy in Fig. 8.1.3.



1´: Bricks of type R3nv(?). A quadratic equation

2´: Bricks of type R3nv. Rectangular-linear equations

3´: Bricks of type R3nv. A cubic equation

4´: Bricks of type R1/2c. Moist and dry. Loading number

5´: A brick of type R1/2c. Small copies. A cubic equation

6´: Bricks of type R1/2c. A badly preserved problem

Fig. 8.1.1. The fragment BM 80078. A suggested outline of the whole tablet.

The fragment contains the relatively well preserved text of 4 mathematical exercises, as well as the less well preserved text of 2 additional exercises. A reasonable estimate seems to be that the intact clay tablet contained about 24 exercises, excerpted from several older theme texts with problems for bricks. There is no way of knowing if the text of the intact clay tablet contained a colophon.

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7_8

Various properties of the mathematical vocabulary of BM 80078, exhibited in Sec. 8.1.7 below, reveal that BM 80078 is a “northern” mathematical text, probably from Sippar. This fact was announced already in Friberg, *RA* 94 (2000), 169.

8.1.1 # 1´. Rectangular Bricks of the Variant Type R3nv(?). A Badly Preserved Problem

BM 80078 # 1´ (*obv.* 1´-3´)

[*x x x x x x x x x x x x x x x x*] /
1´ [*x x x x x x*]-⸢*li*⸣-*iš* [2] 30 *ta-mar*
2 ⸢30⸣ [*x x x x x x*] /
2´-3´ [*x x* 30 *a*]-⸢*na*⸣ 2 30 *iš-te-en* daḫ.ḫa 3 ⸢*ta*⸣-[*mar* 3 uš] / [sig$_4$-*ka*]
[30 *i*]-⸢*na*⸣ 2 30 *ša-ni-*<*im*> ba.zi
2 *ta-mar* 2 ⸢sag⸣ [sig$_4$-*ka*]

[x x x x x x x x x x x x x x x x] /
1´ [x x x] x x [2] 30 you will see.
2 ⸢30⸣ [*x x x x x x*] /
2´-3´ [*x x* 30 t]o 2 30, one, add on, 3 you will [see. 3 is the length of your brick].
[30 fr]om 2 30, the second, tear off,
2 you will see. 2 is the front [of your brick].

In line 1´ of this exercise, the number 2 30 is computed, probably as a square-side (square root), and then (as normally in the solution of a quadratic problem) a second copy of this number is produced. In lines 2´-3´, the number 30 is first added to one 2 30, then subtracted from the other 2 30. The result is 3, the length of a brick, and 2 the front (short side) of the same brick. (This interpretation of the few traces remaining of the end of the text of exercise # 1 is based on the assumption that exercises ## 1´-2´ are intimately related.)

Undoubtedly, this reconstructed end of the solution exercise in exercise # 1´ shows that the length and the front of a brick were computed in exercise 1´ as the solution to a rectangular-linear system of equations, in the same way as the length and front of a brick were computed in the ensuing exercise # 2´. (See below.) In both cases the length and front were found to be 3 and 2, respectively.

8.1.2 # 2´. Bricks of the Variant Type R3nv. A Rectangular-Linear System of Equations

BM 80078 # 2´ (*obv.* 4´-11´)

1 [x x x x 50]$^?$ lú.ḫun.gá *a-gur-ma* 3 28 20 sig$_4$ *il-bi-*⸢*in*⸣ /
2 [*x x x*] ⸢ul⸣.gar uš *ù* sag 5 12 sukud
uš sag *mi-nu*
za.⸢e⸣
3 [igi 3 28 20] / [duḫ.a] ⸢17⸣ 16 48 *ta-mar*
4 *a-na* 4 10 igi.gub.ba túm.[a 1 12] / [*ta-mar*]
igi 12 sukud duḫ.a *a-na* 1 12 túm.a 6 *ta-mar*
5 ⸢sag⸣ / [ḫa.za]
nigín.na $^1/_2$ 5 *ḫe-pé* 2 30 *ta-mar*
šu-tam-ḫir 6 1[5] / [*ta-mar*]
6 ⸢*i*⸣-*na* 6 15 6 *ša* sag ḫa.za 15 íb.tak$_4$
7 ⸢*mi*⸣-[*nam* íb.sá 30] / [íb.sá]
30 *a-na* 2$^!$ 30 *iš-te-en* daḫ.ḫa 3 uš sig$_4$-⸢*ka*⸣ / [*ta-mar*]
8 30 *i-na* 2 30 *ša-ni-im* ba.zi 2 sag sig$_4$-*ka ta*-[*mar*]

1 [For 50 days]$^?$ a man I hired, 3 28 20 bricks [he] molded. /
2 [x x x x] The sum of length and front 5, 12 the height.
Length and front, what?
You:
3 [The reciprocal of 3 28 20] / [release], 17 16 48 you will see.
4 To 4 10 the constant carry, [1 12] / you will see.
The reciprocal of 12 the height release, to 1 12 carry, 6 you will see. [Hold] the head. /

5 Turn around.
1/2 of 5 break, 2 30 you will see.
Make equalsided, 6 1[5 you will see]. /
6 From 6 15, 6 that held the head, 15 the remainder.
7 Wh[at is the equalside? 30] / the equalside.
30 to 2 30, one, add on, 3 the length of ⌜your⌝ brick [you will see], /
8 [x x] 30 from 2 30, the second, tear off, 2 the front of your brick you will see.

Exercise # 2′ is the first exercise on the fragment BM 80078 with a well preserved text. According to the suggested outline of the whole tablet in Fig. 8.1.1 above, the text of exercise # 2′ was inscribed in the middle of the second column of the tablet, preceded by the texts of about 8 other exercises. Presumably, it would have been easier to understand exercise # 2′ if the texts of those preceding exercises had not been lost.

However, relevant information about other known mathematical cuneiform texts dealing with bricks is available in previously published works, such as Friberg, *ChV* (2001) and Friberg, *MSCT 1* (2007), Sec. 7.3. There it is mentioned that an Old Babylonian brick could be either square (S), half-square (H), or rectangular (R), in the latter case always with the sides in the proportions 1 : 2/3. The long side of a brick can be either 1/3, 1/2, 2/3, and 1 cubit, or ;03 and ;04 rods, where the cubit is about 1/2 meter, while ;01 rods is about 1/10 meter. More specifically, the formats of bricks attested in various mathematical cuneiform texts can be named as follows:

S1/3c, S1/2c, S2/3c, S1c	in the case of *square* "tile bricks" of normal thickness (namely 5 fingers = 1/6 cubit),
H2/3c, H4n	in the case of *half-square* "cow bricks" of normal thickness,
R1/2c, R3n	in the case of *rectangular* "bricks" of normal thickness.

For bricks of "variant" thickness (6 fingers = 1/5 cubit), these notations may be augmented by a tag v (v for *variant*), so that, for instance,

A brick of type R3nv has the length ;03 rods, the front ;02 rods, and the thickness ;12 cubit.

Various "constants" are typically associated with the mentioned brick types in cuneiform mathematical texts. One such constant is the "molding number" *L* (Akk. *nalbanum*), which can be defined as follows:

Bricks of a given type have the molding number *L* if the volume of *L* brick sar of such bricks is 1 volume sar.
Here 1 brick sar = 12 00 bricks, and 1 volume sar = 1 square rod · 1 cubit (= 1 n. · 1 n. · 1 c.).

Thus, for instance,

The volume of a brick of type R3nv is ;03 n. · ;02 n. · ;12 c. = 1/20 · 1/30 · 1/5 volume sar = 1/50 · 1/60 volume sar.

Consequently,

1 brick sar of bricks of type R3nv occupy a space of 12 · 1/50 volume sar, and, conversely,
1 volume sar is the volume of 50 · 1/12 brick sar of type R3nv.

Therefore,

The molding number for bricks of type R3nv is *L* = 50 · 1/12 = 50 · ;05 = 4;10.

Interestingly, there is another, perhaps more natural, way of looking at the molding number for bricks of a given type. (See Friberg, *MSCT 1* (2007), 177, and Friberg, *ChV* (2001), Sec. 6.1.) In the Mê-Turran exercise Haddad 104 # 9, a combined work norm for brick making is computed, derived from given work norms for 'crushing', 'molding', and 'mixing'. The result of the computation in the mentioned exercise is that 2 15 bricks of type S2/3cv, with a combined volume of 5 volume shekels (= ;05 volume sar), can be produced in one day's work. This result can be compared with the easily established fact that the molding number (in the sense mentioned above) for bricks of type S2/3cv is 2;15.

Now suppose that it was regularly assumed by Old Babylonian brick makers that

5 volume shekels of bricks *of any given type* could be produced in one day's work of crushing, molding, and mixing.

Then it would follow that also, generally,

L · 60 bricks *of any given type* could be produced in one day's work of crushing, molding, and mixing.

This is because, for any given type of bricks,

$L \cdot 60$ bricks = $L \cdot 1/12$ brick sar = 1/12 volume sar of bricks = 5 volume shekels of bricks.

Consequently, in particular,

4 10 bricks of type R3nv could be produced in one day's work of crushing, molding, and mixing.

In the Babylonian relative place value notation it was, of course, impossible to distinguish between the numbers L and $L \cdot 60$. Therefore, the constant '4 10' mentioned explicitly in line 3 of the exercise BM 80067 # 2´ can be interpreted simultaneously as the number of brick sar of type R3nv contained in 1 volume sar, and as the number of bricks of type R3nv that could be produced in 1 day's work of crushing, molding, and mixing! Never mind that, as will be shown below, this mention of '4 10' as a 'constant' was a mistake made by the author of exercise # 2´ in BM 80078.

The question in BM 80078 # 2´ is almost entirely preserved. It is formulated as follows, in lines 1-2:

a [For x days] a hired man made 3 28 20 bricks (of a certain type).
b The sum of the length and the front (width) of a brick of this type is '5'.
c The height (thickness) of a brick of this type is '12'.
What are the length and the front?

The first half of the solution procedure, in lines 3-4 is not perfectly preserved, although it is clear that the result of initial computation is the number '6', which the reader of the exercise is asked to remember. This number '6' reappears in the second half of the solution procedure, which clearly aims at finding the solution to the following rectangular-linear system of equation of type B1a (see Friberg, *AT* (2007), 6) for the length (u) and the front (s) of a brick of the unspecified type:

$u \cdot s = 6, \quad u + s = 5.$

The solution to this rectangular-linear system of equations is obtained in lines 5-8 of the exercise through standard balancing and completion of the square. See how it is done in Fig. 5.1.19 above. In quasi-modern symbolic notations, but still with Babylonian relative place value notation, the computation in lines 5-8 can be explained as follows:

sq. $(u - s)/2$ = sq. $(u + s)/2 - u \cdot s$ = sq. 2 30 – 6 = 6 15 – 6 = 15, $(u - s)/2$ = sqs. 15 = 30,
$u = (u + s)/2 + (u - s)/2$ = 2 30 + 30 = 3, $s = (u + s)/2 - (u - s)/2$ = 2 30 – 30 = 2.

This result of the computation, that the length is '3' and the front is '2', while the height was said to be '12', clearly means that the bricks figuring in this exercise are bricks of the variant type R3nv, with the length ;03 rods, the width ;02 rods, and the thickness ;12 cubit = 6 fingers.

The volume of a brick of type R3nv is, as mentioned above,

;03 n. · ;02 n. · ;12 c. = 1/20 · 1/30 · 1/5 volume sar = 1/50 · 1/60 volume sar = ;00 01 12 volume sar.

Evidently, the number '1 12' appearing in line 4 of the exercise stands precisely for the volume of one brick of type R3nv. In lines 2-4 of the text, this value for the volume V of one brick is computed as follows (*in relative place value notation*):

V = 4 10 · rec. 3 28 20 = 4 10 · 17 16 48 = 1 12 (sq. rods · cubit).

Here 3 28 20 is the given number of bricks, while '4 10' is called the 'constant' (igi.gub.ba). It is clear that the author of the text mistakenly thought of the molding number '4 10' for bricks of type R3nv. However, instead, the '4 10' appearing in the text of the exercise can be explained as follows: 4 10 = 3 28 20 · V.

However, since 3 28 20 is a pure number and V a volume, '4 10' here must stand for *a given volume*, and not for a pure number like the molding number. How can this observation be reconciled with the fact that in the question it was given only that [x] *hired men were making bricks*?

The likely answer to this question is that the author of BM 80078 # 2´ counted with the Old Babylonian work norm for molding bricks mentioned above, deduced from the exercise Haddad 104 # 9, which was

5 volume shekels of bricks of any given type produced per man and day.

This means that if the given, but lost, number of days a hired man made bricks is assumed to be [50], then the number '4 10 in line 3 of the exercise can be explained as follows

A hired man for 50 days · 5 volume shekels of bricks produced per man and day = 4;10 volume sar of bricks.

If this explanation is correct, it was purely by coincidence that the (implicitly) given volume of bricks produced in 50 man-days happened to be equal to ‘4 10’, the molding number for bricks of type R3nv!

To sum it all up, the given number of men making bricks, together with the known work norm for making bricks gave a computed volume of produced bricks. This computed volume of produced bricks together with the given number of bricks allowed the computation of the volume of one brick. The computed volume of one brick divided by the given thickness of a brick allowed the computation of the product of the length and the width of a brick (of a certain type). Then, finally, the computed product of the length and the width, together with the given sum of the length and the width, made it possible to compute both the length and the width. The end result, although not explicitly mentioned in the text, was that the bricks considered in the exercise were bricks of type R3nv, rectangular bricks of length ;03 rods, width ;02 rods and the variant thickness 1/5 cubit.

Note that this is *the first attestation of bricks of type R3nv in a cuneiform mathematical problem text*.

8.1.3 # 3´. Bricks of the Variant Type R3nv. A Cubic Problem

BM 80078 # 3´ (*obv.* 12´-18´)

1 [x x 50]$^{?}$ lú.ḫun.gá *a-gur-ma* 3 28 20$^{!}$ sig$_4$ ⸢*il*⸣-[*bi*]-⸢*in*⸣
2 *wa-ar-*⸢*ka* sag⸣ *ù* [sukud] / ⸢*aḫ*⸣*-ru-úṣ-ma*
$^2/_3$ uš sag $^1/_2$ sag sukud.bi [uš] sag *ù* sukud-*ša mi-nu* /
3 za.e igi 3 28 20 duḫ.a 17 16 [48] *ta-mar*
4 *a-na* 4 10 igi.gub.⸢ba⸣ túm.a / [1] 12 *ta-mar*
igi 4 sukud duḫ.a *a-na* 1 12 [túm].⸢a⸣ 18 *ta-mar* sag ḫa.za /
5 *ù* 40 *a-na* 1 túm.a 40$^{!}$ *ta-mar*
6 igi 40 [duḫ.a] ⸢*a*⸣*-na* 18 *ša* sag ḫa.za <túm.a> / 27$^{!}$ *ta-mar*
mi$^{!}$*-nam* íb.⸢sá⸣ 3 íb.sá
3 *a-na* [1 túm].a 3 uš sig$_4^{!}$-ka *ta-mar* /
7 40 *a-na* 3 túm.a 2 *ta-mar* 2 sag sig$_4$-*ka* /
8 3 *a-na* 4 sukud túm.a 12 sukud *ta-mar*

1 [For 50 days]$^{?}$ a man I hired, 3 28 20 bricks he molded.
2 Thereafter, the front and the [height] / I specified, then
2/3 of the length is the front, 1/2 of the front is its height.
[The length], the front and its height are what? /
3 You:
The reciprocal of 3 28 20 release, 17 16 [48] you will see.
4 To 4 10 the constant carry, / [1] 12 you will see.
The reciprocal of 4 the height release, to 1 12 [carry], 18 you will see. Hold the head. /
5 And 40 to 1 carry, 40 you will see.
6 The reciprocal of 40 [release], to 18 that held the head <carry>, / 27 you will see.
What is it equalsided? 3 it is equalsided.
3 to [1 carry], 3 the length of your brick you will see. /
7 40 to the 3 carry, 2 you will see. 2 is the front of your brick.
3 to 4 the height carry, 12 the height you will see.

The question in BM 80078 # 3´ is almost entirely preserved. It is formulated as follows, in lines 1-2:

a [For x days] a hired man made 3 28 20 bricks (of a certain type).
b The front (width) of a brick of this type is 2/3 of the length of the brick.
c The height (thickness) of a brick of this type is 1/2 of the front of the brick.
What are the length, the front, and the height?

The solution procedure in this exercise begins, in lines 3-4, in precisely the same way as the solution procedure in exercise # 2´, namely with the computation of the volume V of one brick of the given (but undisclosed) type. The volume is again found to be (in relative place value notation)

V = 4 10 · rec. 3 28 20 = 4 10 · 17 16 48 = 1 12 (sq. rods · cubit).

What follows next can be explained either in terms of “false values” or in terms of a tentative “reference volume” or “reference brick” (terms of a kind introduced by Proust in *TMN* (2007), Ch. 7). A reference brick is considered with the standardized reference length u' = 1 (00) (rods) and the corresponding reference front s' and reference height h', both supposed to satisfy conditions b-c in the question of exercise # 3´. Then (in relative place value notation)

$s' = 2/3\ u' = 2/3 \cdot 1$ (rod) = 40 (rods) and $h' = 1/2\ s'$ = 20 (rods) = 4 (cubits).

Remember that in Old Babylonian metrology lengths and fronts were measured in rods, while heights were measured in cubits, with 1 rod = 12 cubits. It follows that the volume of the reference brick is (again in relative place value notation)

V' = 40 (sq. rods) · 4 (cubits).

In line 4 of the exercise, the computed volume ‘1 12’ is divided by the reference height ‘4’. The result is ‘18’. Next, in lines 5-6 of the exercise, this number ‘18’ is divided by ‘40’, computed as the product of the reference front ‘40’ and the reference length ‘1’. The result is the number ‘27’. What this means is that the volume of a brick of the (unknown) type considered in this exercise is 27 times the volume of the chosen reference brick, which is constructed so that it is of the same type, that is, with length, front, and height in the same proportions as in the brick actually considered in the exercise. In modern terms, think of the actual length of a brick of the considered type as *f* times the reference length, where *f* is some “proportionality factor”. Then the actual front of the brick is also *f* times the reference front, and the actual height of the brick is *f* times the reference height. Consequently, the volume of the actual brick is the cube of *f* times the volume of the reference brick. Since it has also been shown to be 27 times the volume of the reference brick, it follows that the proportionality factor *f* = the cube root of 27 = 3.

This is the meaning of the computation in the first half of line 6 of the exercise. Therefore, finally, as in lines 6-7 of the exercise,

$u = 3 \cdot u'$ = 3 · ‘1’ = ‘3’, $s = 2/3\ u$ = ‘40’ · ‘3’ = ‘2’, and $h = 3 \cdot h'$ = 3 · ‘4’ = ‘12’.

This, in its turn, naturally means that the sides of the brick considered in exercise # 3´ are

u = ‘3’ = ;03 rods, s = ‘2’ = ;02 rods, and h = ‘12’ = ;12 cubit = 1/5 cubit = 6 fingers.

Therefore, the brick in question is of type R3nv.

8.1.4 # 4´. Moist and Dry Bricks of Type R1/2c. A Loading Number

BM 80078 # 4´ (*obv.* 19´- *rev.* 5)

1 $^1/_2$ kùš uš $^1/_3$ kùš sag 5 šu.si sukud-*ša*
i-na ra-aṭ-bu-ti-ša ki ma-ṣi ik-bi-it /
2 *i-na ša-bu-lu-ti-ša ki ma-ṣi iq-li-il* /
3 *ù a-na bi-la-at* 1.še *ki ma-ṣi* šid$^?$ *mi-*⸢*nu-ša*⸣
4 za.e 2 30 uš / ⸢*a*⸣*-na* 1 40 sag túm.a 4 10 *ta-mar*
5 *a-na* 10 sukud túm.a / [41 40] *ta-mar*
nigín.na
6 18 *ṭì-da-am* / [*ù x x*] *x* ba.zi gar.ra
7 18 *ṭì-*⸢*da-am*⸣ [*a-na* 4]1 40 túm.a / ⸢12⸣ 30 *ta-mar*
i-na ra-aṭ-bu-ti-ša ⸢*ki*⸣ [*ma-ṣi ik-bi-it*] /
8 ⸢12⸣ 30 *a-na* 12 *qé-er-ba-nim* túm.a 2 30 *ta-mar*
⸢2 30⸣ [*i-na*] 12 30 ba.zi /
9 [10] *mi-nu* gar 10 ma.na *i-na ša ša-bu-lu-ti-ša iq-li-il* /
10-11 igi 10 *ša i-na ša-bu-lu-ti-ša iq-li-lu* duḫ.a / [*a*]*-na* 1.še túm.a 6 40 *ta-mar* sig$_4$ *ki-ma bi-la-at* 1.še

1 1/2 c. the length, 1/3 c. the front, 5 fingers its height.
In its wet state, how much was it heavy, /
2 in its dry state, how much was it light, /
3 and for the load of 1, how much? The count$^?$ was what of it?
You:

4 2 30 the length / to 1 40 the front carry, 4 10 you will see,
5 to 10 the height carry, / [41 40] you will see.
Turn around.
6 18 <the constant of> the clay / [and the x x] x to tear off set.
7 18 the clay [to 4]1 40 carry, / 12 30 you will see,
in its wet state how [much it is heavy.] /
8 12 30 to 12 of the inside carry, 2 30 you will see.
9 2 30 [from] 12 30 tear off, / [10].
What (shall it be) set? 10 minas that! in! its dry state it is light. /
10-11 The reciprocal of 10 that in its dry state it is light release, / to 1 carry, 6 40 you will see, bricks as the load of 1.

This exercise, the first of its kind to be published, deals with wet, newly made bricks and sun-dried bricks. A clue about what is going on here is given by BM 36776, Fig. 8.1.2 below, a small but very interesting fragment of a table of constants from Late Babylonian Babylon (Robson, *MMTC* (1999), 206; Friberg *MSCT 1* (2007), 171). On that fragment there are two well preserved four-line paragraphs of constants for bricks and traces of two more such paragraphs. The damaged paragraphs can be reconstructed without much hesitation. The result is a partially reconstructed table of constants, with the following entries:

§ 1´ the *molding numbers* for bricks of the four types R1/2c, H2/3c, S2/3c, and R3n
§ 2´a the weights of *(newly made) wet* bricks of the same four types;
§ 2´b the weights of *sun-dried* bricks of the same four types
§ 2´c the weights of *baked* bricks of the same four types

In connection with the present discussion of the exercise BM 80078 # 4´, only the entries for standard bricks (sig$_4$) are of interest. (However, what is said below about relations between the various entries for standard bricks is equally true in the case of entries for the three remaining types of bricks.)

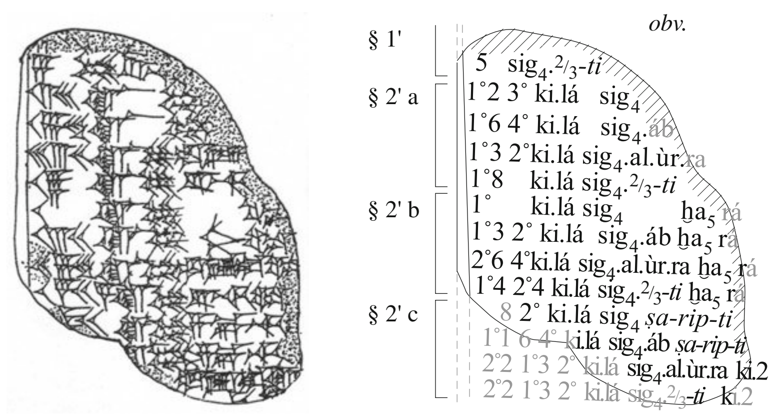


Fig. 8.1.2. BM 36776. A small fragment of a table of constants with four paragraphs of constants for bricks.

BM 36776

§ 1´	[7 12	sig$_4$]			(molding number for) bricks		R1/2c
	[5 24	sig$_4$.áb]			(molding number for) cow bricks		H2/3c
	[2 42	sig$_4$.al.ùr.ra]			(molding number for) tile bricks		S2/3c
	5	[sig$_4$.$^2/_3$-*ti*]			(molding number for) 2/3-bricks		R3n
§ 2´ a	12 30	ki.lá	sig$_4$		weight of bricks		R1/2c
	16 40	ki.lá	sig$_4$[.áb]		weight of cow bricks		H2/3c
	13 20	ki.lá	sig$_4$.al.ùr.ra		weight of tile bricks		S2/3c
	18	ki.lá	sig$_4$.		weight of 2/3-bricks		R3n
§ 2´ b	10	ki.lá	sig$_4$	ḫa$_5$.rá	weight of bricks	dried	R1/2c
	13 20	ki.lá	sig$_4$.áb	ḫa$_5$.rá	weight of cow bricks	dried	H2/3c
	26 40	ki.lá	sig$_4$.al.ùr.[ra]	ḫa$_5$.rá	weight of tile bricks	dried	S2/3c
	14 24	ki.lá	sig$_4$.$^2/_3$-*ti*	ḫa$_5$.rá	weight of 2/3-bricks	dried	R3n

§ 2´ c	⸢8⸣ 20	ki.lá	sig_4	*ṣa-rip-ti*	weight of bricks	baked	R1/2c
	[11 06 40	ki.lá	sig_4.]áb	*ṣa-rip-ti*	weight of cow bricks	baked	H2/3c
	[22 13 20	ki.lá	sig_4.]al.ùr.ra	ki.2	weight of tile bricks	ditto	S2/3c
	[12	ki.lá	sig_4.$^2/_3$]-*ti*	ki.2	weight of 2/3-bricks	ditto	R3n

A first observation is that in this text the weight 10 of (sun-)dried rectangular standard bricks in § 2´b is 4/5 of the weight 12 30 of (presumably) wet, newly made rectangular standard bricks in § 2´a. In the same way, the weight 13 20 of dried "cow bricks" (half square bricks) is 4/5 of the weight 16 40 of wet cow bricks. And so on.

In a similar way, the weight ⸢8⸣ 20 of baked rectangular standard bricks in § 2´c seems to be 5/6 of the weight 10 of a dried rectangular standard brick in § 2´b. Therefore, it is likely that the non-preserved weight of a baked cow brick can be reconstructed as 5/6 of the weight 13 20 of a dried cow brick, that is as 5/6 · 13 20 = [11 06 40], as indicated. Then also the weight of a baked "tile brick" can be reconstructed as 5/6 · 26 40 = [22 13 20], and the weight of a baked "2/3-brick" can be reconstructed as 5/6 · 14 24 = [12].

A final observation is that the (reconstructed) weight [12] of baked 2/3-bricks in § 2´c is the reciprocal of the constant 5 for 2/3-bricks in § 1, which can be explained as the molding number of 2/3-bricks (more precisely, bricks of the type R3n). See Friberg, *ChV* (2001), Table 4.2. And so on.

The mentioned relations between the various items in the paragraphs of the table of constants on the fragment BM 36776 can be summarized as follows:

In the process of being sun-dried, a wet brick loses 1/5 of its weight.
In the process of being baked, a sun-dried brick loses 1/6 of its weight.
The weight of a single baked brick of any given type is the reciprocal of the molding number for bricks of that type.

The weight of a brick is mentioned explicitly in the text *MCT* Oa = YBC 7284, a round hand tablet with the following inscriptions (plus an unrelated number):

obv.	41 40	'41 40'
	8 20	'8 20'
	igi.gub.ba.bi 12	'12'
rev.	1 sig_4	1 brick,
	ki.lá.bi en.nam	its weight is what?
	ki.lá.bi 8 $^1/_3$ ma.na	Its weight is 8 1/3 minas.

On the reverse of the hand tablet, it is asked what the weight is of 1 brick. The answer is given directly, as 8 1/3 minas. Evidently this number corresponds exactly to the number ⸢8⸣ 20 in the first line of BM 36776 # 2´c above, indicating the weight of a *baked* standard brick! On the obverse of the tablet it is explained how this number was computed, with numbers in relative place value notation:

'12' · '41 40' = '8 20'.

According to Neugebauer and Sachs, this computation can be understood as saying, expressed in quasi-modern place value notation, that

12 (00 00) (minas/volume sar) · ;00 00 41 40 (volume sar/brick) = 8;20 (minas/brick).

Therefore, the constant '12' in this text stands for the simple rule that

12 (00) talents is the weight of 1 volume sar of *baked* bricks (obviously regardless of type), with 1 talent = 1 (00) minas.

An interesting observation is that the text on YBC 7284 is divided into two parts, physically separated from each other: On one hand there is a simple metro-mathematical question and its answer, with a frame drawn around the question+answer, and on the other hand there is the easy computation needed to obtain that answer. This kind of pedagogical format for a metro-mathematical exercise is reminiscent of the format used in the case of six Old Babylonian square hand tablets from Nippur (Proust, *TMN* (2007), Sec. 6.4.1), all with computations of areas of squares with given sides! Is this an indication that also the round hand tablet YBC 7284 stems from Nippur?

Now, recall that if *L* is the molding number for bricks of any given type, then the volume of *L* · 12 (00) bricks of this type (= *L* brick-sar) is 1 volume sar. (See above, Sec. 8.1.2.) With the weight of 1 volume sar of baked bricks, regardless of type, being equal to 12 (00) talents, it follows that

The weight of *L baked* bricks with any given molding number *L* is 1 talent.

This agrees with the observation above that in the table of constants BM 36776 the entries in § 2´c are the reciprocals of the entries in § 1´.

In view of the mentioned rules for the proportional loss of weight when wet bricks are dried and when dried bricks are baked, the simple rule above for the relation between weight and volume of baked bricks can be complemented as follows (as observed already by Robson in *MMTC* (1999), 63):

12 (00) talents is the weight of 1 volume sar of *baked* bricks (regardless of type),
6/5 · 12 (00) = 14 24 talents is the weight of 1 volume sar of *sun-dried* bricks (regardless of type),
5/4 · 14 24 = 18 (00) talents is the weight of 1 volume sar of *wet* bricks (regardless of type).

Now return to the text of the exercise BM 80078 # 4´. In lines 1-2 of that text it is stated that the length of the bricks considered is 2/3 cubit, the front is 1/2 cubit, and the height is 5 fingers. This means that the bricks are of type R1/2c, so called standard bricks. It is asked what the weight of a brick of this type is when the brick is wet, and when it is dry, respectively. Something is also asked about a related load, but the terminology used in this question is new and obscure.

The solution procedure begins, in lines 3-5, with the computation of the volume of one standard brick, in (essentially) the following way:

!/2 cubit · 1/3 cubit · 5 fingers = ;02 30 nindan · ;01 40 nindan · ;10 cubit = ;00 04 10 sq. nindan · ;10 cubit = ;00 00 41 40 sar.

Then, in lines 5-6, the text sets down that the (constant of) 'clay' is '18' and that [one-fifth] should be subtracted. In view of what was shown above, *this 'clay' constant can be understood as 18 (00) talents per volume sar of wet bricks, and 1/5 is the relative weight loss due to sun-drying*. Accordingly, in lines 6-7 of the solution procedure, multiplication of the volume '41 40' with the constant '18' gives the result that the weight of one standard brick 'in its wet state' is '12 30'. This result agrees with the entry in the first line of § 2´a of the table of constants BM 36776.

Next, in lines 8-9, the weight '12 30' is multiplied by another constant, called '12 of the inside', and the result, '2 30', is subtracted from the weight, so that only '10' remains. The number 10 is explained as 10 minas, the weight of a standard brick 'in its dry state'. What is going on here can be understood in view of the rule mentioned above that in the process of being dried a brick loses 1/5 of its weight. Therefore, the somewhat strangely named '12 of the inside' stands for 1/5 (= ;12) which is lost when the brick dries out. The result that 10 minas is the weight of a dry standard brick agrees with the entry in the first line of § 2´b of the table of constants BM 36776.

In lines 10-11, the reciprocal of the computed 10 minas is multiplied into 1.$e, whatever that means, and the result is said to be '6 40'. This appears to be a miscalculation, since rec. 10 is '6' (while it is rec. 9 that is equal to '6 40'). In addition, the result '6 40' is specified as sig$_4$ *ki-ma bi-la-at* 1.$e which, maybe, can be translated as 'bricks as the load for 1'. Anyway, this is clearly the answer to the obscurely formulated question in line 3 : *a-na bi-la-at* 1.$e *ki-ma-%i* $id[?] *mi-°nu-$a¿* 'for the load of 1 how much? The count[?] of it was what?'.

A question that would make sense in this connection would be something like 'How many dry bricks is 1 man's load?'. The correct answer to this question would be 6 dry bricks. See the discussion of "loading numbers" in Friberg, *MSCT 1* (2007), 173, where it was demonstrated that '6' was the Old Babylonian loading number for standard bricks. This loading number is mentioned explicitly in, for instance, the brief table of constants *UET 5*, 881 (see the hand copy in Friberg, *ChV* (2001), 71), where it is stated that

6 sig$_4$.si.sá gun lú.1.e	6 regular bricks is the load of 1 man.

A final remark: As was mentioned above, the constant '18', corresponding to the weight per volume unit of wet bricks or wet clay, is attested for the first time in line 6 of the exercise BM 80078 # 4´, in the phrase

18 *ṭì-da-am* gar.ra — set '18', the clay.

A search for the constant '18' in earlier published Babylonian tables of constants (see Friberg, *ChV* (2001), 64-67; see also Robson, *MMCT* (1999), 63) yields an item in the table of constants NSd = YBC 5022, sandwiched between a "walking number" and a "loading number", as shown below:

45	*mu-ut-ta-al-li-ik-tum ša* sig_4	walking number for bricks	NSd 39
18	*im-lu-ú-um*	weight number for clay (or wet bricks)	NSd 40
6	*ma-aš-šu-ú-um*	carrying tray, loading number for dry bricks	NSd 41

In this brief list of three consecutive items in an Old Babylonian table of constants, the constant '6' is the "loading number" mentioned above, meaning that a man can carry 6 (sun-dried) standard bricks (with a total weight of 1 talent). The constant '45' is the related "walking number", meaning that a man can walk 45 (00) rods (about 16 kilometers) in a day's work, carrying such a load of 6 standard bricks. These two constants are explicitly mentioned together in the small Old Babylonian table text MS 2221 (Friberg, *MSCT 1* (2007), 170).

The third constant, 18 *im-lu-ú-um*, has never been adequately explained before, although it is stated in Neugebauer and Sachs, *MCT* (1945), 135, that

> "In all probability, this corresponds to line 14 of (the table of constants) Ue, which gives the coefficient 18 for im.lá. Akkadian '*imlūm* would accordingly be a loanword from Sumerian im-lá. The coefficient 18 is mentioned in mathematical texts dealing with im-lá ....".

The explanation of the constant 18 *im-lu-ú-um* as a "weight number for clay or wet bricks" is natural in view of the explanation above of the obviously related constant 18 *ṭì-du-um*.

The mathematical cuneiform texts dealing with im.lá and mentioned by Neugebauer and Sachs are exercises ## 10-13 in BM 85194 (Neugebauer, *MKT I* (1935), 145-146), an Old Babylonian mathematical recombination text from Sippar, and the entry '18 im.lá' in the table of constants YBC 7243, where it appears in a meaningless context together with other constants for entities like im.dù.a, im.PAR.PAR, im.KI.a, and im.dub. The exercises BM 85194 ## 10-13 are very brief and impossible to understand clearly without their original context in a large theme text. In addition, the one who wrote (or copied) the four exercises had obvious difficulties keeping apart the two terms ki.lá (weight) and im.lá (weight per volume unit). For these reasons, Neugebauer did not try to give any explanation for what is going on in the exercises, but with the newly won insight into the meaning of the constant 18 *im-lu-ú-um* = 18 im.lá, the following explanation can be suggested.

In the four exercises are considered relations between the "weight number" 18 im.lá (which we may refer to as (c), the weight ki.lá (w), the depth gam (d), and the two numbers 4 and 3, which in the text of the exercises are called bal 'fractional parts'. The equation governing these relations is

$$c \cdot 1/(4 \cdot 3) \cdot d = w.$$

In the exercises ## 10-11 and ## 12-13, respectively, the numerical parameters for these parameters are (in Babylonian relative place value notation):

$$18 \cdot 1/(4 \cdot 3) \cdot 40 = 1 \quad \text{and} \quad 18 \cdot 1/(4 \cdot 3) \cdot 20 = 30.$$

A possible (although very shaky) interpretation is that what is considered in, for instance, exercises ## 10-11 is the weight of the clay in a rectangular pit with the sides 1/3 rod = 4 cubits and 1/4 rod = 3 cubits and a depth of ;40 cubit = 2/3 cubit. In modern notations, this weight would be

w = 18 (00) (talents/volume sar) · 1/(4 · 3) · ;40 (volume sar) = 18/12 · 40 talents = 1 (00) talents.

8.1.5 # 5´. Small Copies of Bricks of Type R1/2c

BM 80078 # 5´ (*rev.*: 6-18)

1	⌜1/2⌝ kùš uš 1/3 ⌜kùš⌝ sag 5 šu.si sukud	*aḫ-pí-ši ab-lu-ul-ši-ma* /
2	⌜2/3 uš sag 1/2⌝ sag sukud	
3	2 05 *al-bi-in* sig_4 *ši-i* / [uš sag sukud en.nam]	

4 za.e íb.sá 2 05 ⸢igi.gub⸣.ba / [en.nam 5 íb.sá]
⸢6⸣ *a-na* 5 túm.a 30 *ta-mar* sag ḫa.za /
5 [*aš-šum* $^2/_3$ uš sag] $^1/_2$ sag sukud *qá-bu-kum* 1 40 *ù* 4 gar.ra /
6-7 [30 *ša* sag ḫa.za] *a-na* ⸢1 túm.a⸣ 30 *ta-mar* 3 šu.si uš sig$_4$ / [*ša ta-al-bi-nu*]
8 40 30 *a-na* 40 túm.a 20 *ta-mar* 2 šu.si ⸢sag sig$_4$⸣ / [*ša ta*]*-al-bi-nu*
9 30 *a-na* 4 túm.<a> 2 *ta-mar* 1 šu.si [sukud] ⸢sig$_4$⸣ / [*ša ta-al*]*-bi-nu*
nigín.na igi 30 uš sig$_4$ *ša ta-al*$^!$*-bi-*[*nu* duḫ.a] /
10 [*a-na* 2 30] uš sig$_4$ túm.a 5 *ta-mar*
11 igi 20 sag sig$_4$ [*ša ta-al-bi-nu*] /
[*a-na*] ⸢sag⸣ sig$_4$ túm.a 5 *ta-mar*
12 igi 2 sukud$^!$ sig$_4$ [*a-na* 10] / [sukud sig$_4$] túm.a 5 *ta-mar*
13 5 *a-na* 5 túm.<a> 25 [*ta-mar* 25] / [*a-na*] 5 túm.a 2 05 *ta-mar*
2 05 *ša* [sig$_4$ *ši-i ta-al-bi-nu*]

1 ⸢1/2⸣ cubit is the length, 1/3 ⸢cubit⸣ the front, 5 fingers the height. I broke it, I mixed it, then /
2 ⸢2/3 the length was the front, 1/2⸣ the front the height,
3 2 05 I molded of this brick. / [Length, front, height what?]
4 You: The equalside of 2 05 ⸢the constant⸣ / is what? [5 is the equalside.]
⸢6⸣ to 5 carry, 30 you will see. Hold the head. /
5 [Since 2/3 of the length was the front], 1/2 of the front the height was said to you, 1, 40, and 4 set. /
6-7 [30 that held the head] to ⸢1 carry⸣, 30 you will see. 3 fingers is the length of the brick / [that you molded]
8 40. 30 to 40 carry, 20 you will see. 2 fingers is ⸢the front of the brick⸣ / [that you] molded.
9 30 to 4 carry, 2 you will see. 1 finger is the height of ⸢the brick⸣ / that you molded.
Turn around. The reciprocal of 30 the length of the brick that you molded [release], /
10 [to 2 30] the length of the brick carry, 5 you will see.
11 The reciprocal of 20 the front of the brick / that you molded [release],
[to] ⸢the front⸣ of the brick carry, 5 you will see.
12 The reciprocal of 2 the height [to 10] / [the height of the brick] carry, 5 you will see.
5 to 5 carry, 25 [you will see. 25] / [to] 5 carry, 2 05 you will see.
2 05 of [this brick that you molded].

This exercise contains both a question, a solution procedure with the answer to the question, and a verification of the answer. The solution procedure contains a blatant example of cheating.

The question begins by stating the dimensions of a standard brick (type R1/2c). This brick is then said to have been crushed and remolded, in order to fabricate 2 05 (= 125) new, much smaller bricks, all of the same form as the original brick. The remainder of the question, in line 3, is not preserved, but it is obvious that what is asked for is the sides of the new, smaller bricks.

The solution procedure begins by computing the cube root of the given number 2 05, incorrectly called 'the constant'. In line 4 the computed cube root 5 is multiplied by the number 6, *which appears out of the blue*. The result, 30, is saved for further use. Next, since it was said that the new bricks should be of the same form as the original brick, with 2/3 of the length equal to the front and 1/2 of the front equal to the height, in line 5 of the exercise the sides are tentatively set to be '1' (rod), '40' (rods), and '4' (cubits). Indeed,

if the length is '1', then the front is 2/3 of '1' = '40', and the height is 1/2 of '40' = '20' (nindan) = '4' (cubits).

In lines 6-8, these tentative side lengths are multiplied by the saved number 30. The result is that the length of one of the small bricks is '30', interpreted as 3 fingers (;00 30 rods), the front is '20', interpreted as 2 fingers, and the height is '2', interpreted as 1 finger (;02 cubit).

The verification starts in lines 9-10, where the length of the original standard brick, '2 30' (;02 30 rods = 1/2 cubit), is divided by '30', the length of the new bricks. The result is 5, which shows that the original brick was 5 times longer than the new bricks. Then, in lines 10-12, it is shown in a similar way that the front of the original brick was 5 times the front of the new bricks, and that the height of the original brick was 5 times the height of the new bricks. Consequently, as shown in lines 12-13, the original brick was 5 · 5 · 5 = 2 05 times larger than the newly made, smaller bricks.

It is obvious that the student who wrote the text of this exercise understood that since 2 05 is the cube of 5, the sides of the small bricks were 1/5 of the sides '2 30', '1 40', and '10' of the original bricks and therefore equal to '30', '20', and '2'. This means that the solution procedure could have been very brief, only a few lines of text. Possibly, the reason for the more lengthy solution procedure in the text is that the student had been instructed by his teacher to solve the stated problem by use of the tentative side lengths '1', '40', and '4'. Not understanding how he could do this, the student instead faked a solution method in which he appeared to make use of the tentative side lengths.

A correct solution method, which the student could not find, would have proceeded as follows: As was probably well known to any Old Babylonian advanced student of mathematics, a standard brick (of type R1/2c) has the volume '41 40'. Obviously, the volume of each one of the 2 05 new bricks is 2 05 times smaller, that is, (in Babylonian relative place value notation)

41 40 / 2 05 = 8 20 / 25 = 1 40 / 5 = '20'.

If the sides of the small bricks are tentatively assumed to be '1', '40', and '4', then the corresponding tentative volume is

1 · 40 · 4 = '2 40'.

The true volume '20' can be be compared with this tentative volume '2 40'. The result is that

20 / 2 40 = '7 30'.

On the other hand, if the true sides of the small bricks are assumed to be *f* times the tentative sides '1', '40', and '4', where *f* is a "linear correction factor", then the true volume of one of the small bricks is *F* times the tentative volume, with *F* equal to the cube of *f*. It is difficult to imagine precisely how an Old Babylonian teacher of mathematics would formulate this argument. Anyway, this shows that the cube of *f* must be equal to '20' / '2 40' = '7 30' (1/8). Consequently, *f* itself must be equal to the cube root of '7 30' (1/8), and it follows that the value of the correction factor is *f* = '30' (1/2). From there on, the solution procedure can continue just as in the text of exercise # 5′, lines 5-9.

8.1.6 # 6′. Bricks of Type R1/2c. A Badly Preserved Exercise

BM 80078 # 6′ (*rev.*: 19-25)

1 [1/2 kùš uš] ⸢1/3⸣ kùš sag 5 šu.si sukud 6 šu.⸢si⸣ [*aḫ-pí-ši-ma*] /

2 [*ab-lu*]-⸢*ul*⸣-*ši-ma al-bi-in-ma* ⸢sig$_4$⸣ [*x x x x x*] /

3 [*x x x x x qé-er*]-*ba-nim ki ma-ṣi ma*-⸢*mi*⸣-[*ta*]

4-5 [za.e] / [*x x x x x* gar].ra *uš-tam-ḫir* ⸢sig$_4$⸣ [*x x x x x*] / [*x x x x x* 2]-*at-tam mi-nu* [*x x x x x*] /

6-7 [*x x x x x*]x *ša* ⸢sukud⸣ [*x x x x x*] / [*x x x x x*]-*ma* [*x x x x x*] /

[*x x x x x x x x x x x x x x x x x x*]

1 [1/2 cubit the length], 1/3 cubit the front, 5 fingers the height. 6 fingers [x x x x x] /

2 [I broke it, then I mi]xed it, then I molded it, then bricks [x x x x x] /

3 [x x x of the ins]ide? How much is the number(?) ?

[You]: /

4-5 [x x x x s]et and make equalsided. Bricks [x x x x x] / [x x x x x 2] x x what [x x x x x] /

6-7 [x x x x x] x of the height [x x x x x] / [x x x x x], then [x x x x x]

… [x x x x x x x x x x x x x x x x x x]

8.1.7 The Vocabulary of BM 80078

ba.zi	tear off (subtract)
daḫ.ḫa	add on, attach
gar.ra	set
íb.sá	equal(-side) (square root)

íb.tak$_4$	remainder
igi (*n*) duḫ.a	release the opposite of (*n*) (compute rec. *n*)
igi.gub.ba	constant, fixed number
kùš	cubit (1/6 of a rod)
lú.ḫun.gá	hired worker
nigín.na	turn around (resume)
sag	front (short side)
sag ha.za	hold the head (remember, keep ready)
sig$_4$	brick
šu.si	finger (1/30 of a cubit)
sukud	height
.še	of
túm.a	carry (multiply)
ul.gar	sum
uš	length (long side)
za.e	you
aš-šum ..... qá-bu-kum	since it was said to you that .....
$^{1}/_{2}$ (*a*) *ḫe-pé*	break (*a*) in half
a-gur	< *agārum* to hire
qá-bu	< *qabûm* to say, to command
ta-mar	< *amārum* to see (= to find a result)
ab-lu-ul	< *balālum* to mix (together)
ḫe-pé, aḫ-pí	< *ḫepûm* to break
aḫ-ru-uṣ	< *ḫarāṣum* to break off, to specify
ik-bi-it	< *kabātum* to be heavy
il-bi-in, al-bi-in, ta-al-bi-nu	< *labānum* to make (mold) bricks (lit. to spread out)
iq-li-il	< *qalālum* to be light
šu-tam-ḫir	< *maḫārum* Št become equal to each other, become equalsided
bi-la-at	*biltum* load
iš-te-en	*ištēn* one
ki ma-ṣi	*kī maṣi* how much?
ki-ma	*kīma* like, when, as
ma-mi-ta	*?* number(?)
mi-nu, mi-nam	*minum* what?
qé-er-ba-nim	*qerbēnum* inside
ra-aṭ-bu-ti	*raṭbum* moist, fresh
ša-bu-lu-ti	*šābulum* dry, dried out
ša-ni-im	*šanûm* second
ši-i	*šī* she, this
-ši	*-ši* her (ack.)
ṭì-da-am	*ṭīdum* clay, mud
wa-ar-ka	*warka* afterwards, later

At the beginning of Ch. 9 below is inserted "a survey of known texts from Group 6 and their Sumerian terminology". There it is shown how the members of Group 6 of unprovenanced Old Babylonian mathematical texts are distinctly characterized by the Sumerian part of their vocabulary, notably the presence of the term ul.gar.

8.1.8 BM 80078. A Hand Copy of the Fragment

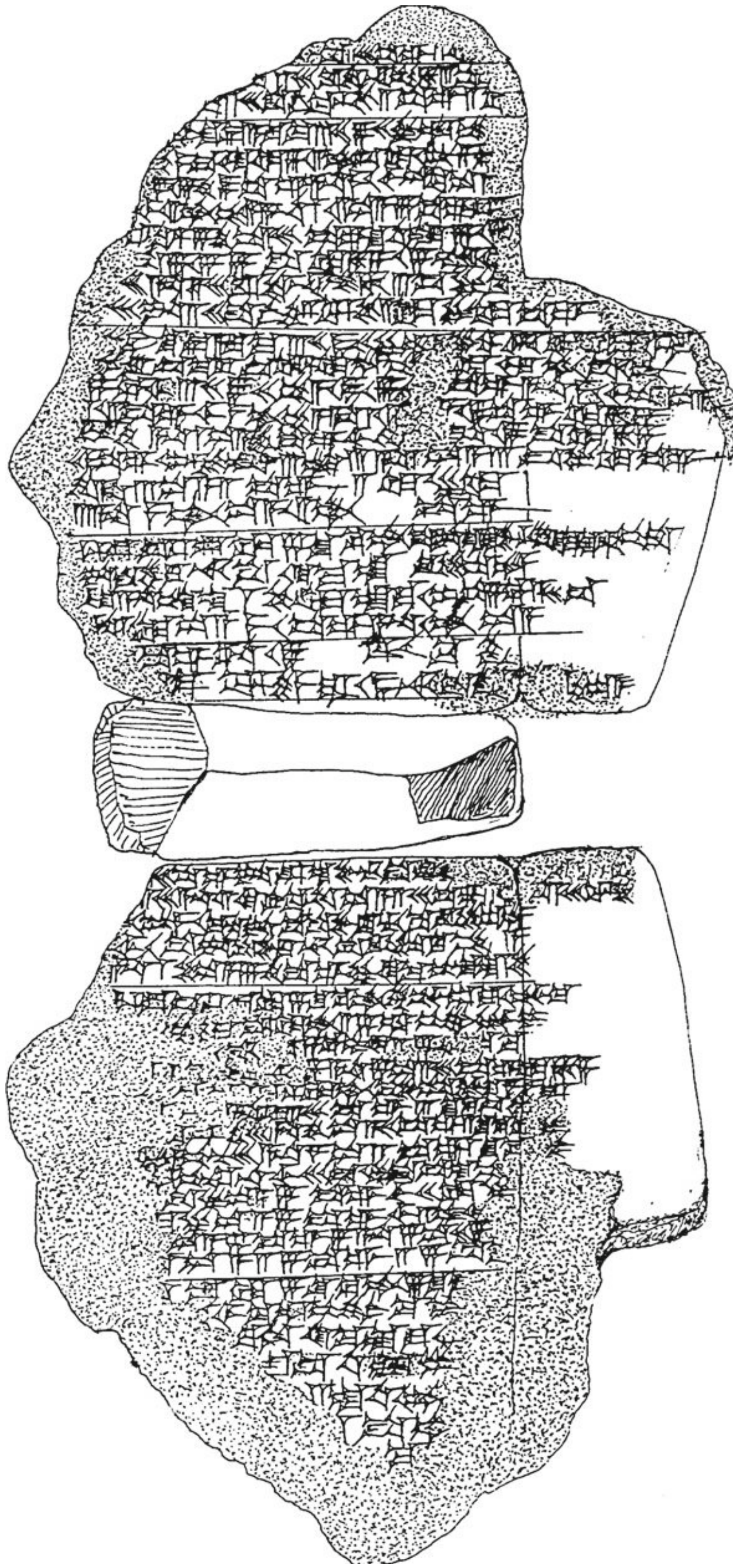


Fig. 8.1.3. BM 80078, hand copy. A small fragment of a large recombination text with problems for bricks.

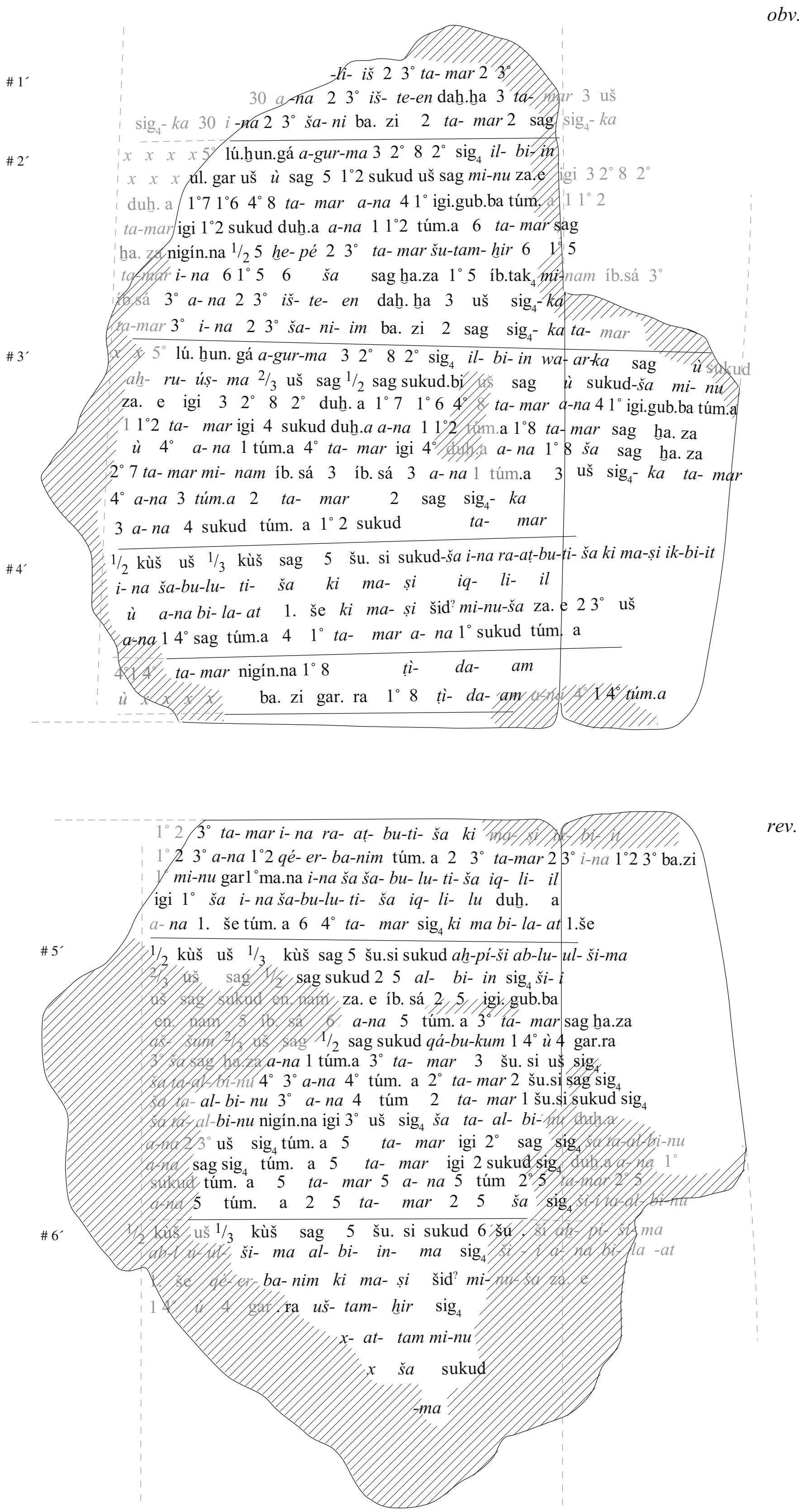


Fig. 8.1.4. BM 80078, conform transliteration.

8.2 BM 54779. A Small Fragment With Homogeneous Quadratic Problems for Squares

See the hand copy and conform transliteration in Fig. 8.2.1 below.

8.2.1 # 2´. A Homogeneous Quadratic Problem for Three Squares

BM 54779 # 2´ (col. *i,* 2´ - 11´)

1	1	30	25
2	40	20	
3	[20]	10	

1+ a.šag$_4$ [3 íb.sá.ḫá ul.gar] /
2+, 3+ *ù ma-la* íb.sá [1 ugu íb.sá 2 dirig] / *uš-tam-ḫir*
4 *a-na* šag$_4$ a.šag$_4$ a.na íb.sá.ḫá / [ul.gar]] 2$_{\text{èše}}$ 3$_{\text{iku}}$ ašag
5 *ši-ni-ip* íb.sá.1 íb.sá.2 / [$^1\!/_2$ íb.sá].2 íb.sá.3
íb.sá.ḫá en.nam /
6 [íb.sá.1 *ù* 2 a.na 1 *ù* 4]0 gar.ra *šu-tam-ḫir* 1 26 40 /
7 [íb.sá.3 a.na 20] gar.ra
nigín.na
8 1 ugu 40 *mi-nam* / [dirig 20 dirig]
20 *šu-tam-ḫir* 6 40 *ta-mar* /
9, 10 [*a-n*]*a* 1 26 40 6 40 *ù* 6 40 *ku-*[*mur*] / [1] 40 *ta-mar*
igi 1 40 duḫ.a 36 *ta-mar* /
… …………………………

1	1	30	25
2	40	20	
3	[20]	10	

1+ The fields [of 3 equalsides I gathered,] /
2+, 3+ and whatever equalside [1 over equalside 2 was beyond] / I let equal itself,
4 onto the fields of as much as were the equalsides / [I gathered], (it is) 2 èše 3 iku of field.
5 Two-thirds of equalside 1 was equalside 2, / [1/2 of equalside] 2 was equalside 3.
The equalsides were what? /
6 [equalsides 1 and 2 as much as 1 and 4]0 set and let equal themselves, 1 26 40. /
7 [equalside 3 as much as 20] set.
Turn around.
8 1 over 40 what / [is it beyond? 20 beyond].
20 let equal itself, 6 40 you will see. /
9, 10 [To] 1 26 40, 6 40 and 6 40 heap, / [1] 40 you will see.
The reciprocal of 1 40 release, 36 you will see. /
… …………………………

The question in this exercise is partly damaged, but what is missing can be restored, thanks to information yielded by the solution procedure. The solution procedure itself is also damaged, with the last part of it missing, but that, too, can be restored without difficulty.

The question is preceded by a brief table, gathering together some numerical data. The question itself, in the first five lines of the exercise, can be reformulated as follows:

Three squares have the unknown sides s, s', and s''.
sq. $(s - s')$ + sq. s + sq. s' + sq. s'' = 2 èše 3 iku, $2/3\ s = s'$, $1/2\ s' = s''$.
s, s', s'' = ?

The solution procedure begins in line 6 by tentatively setting s and s' equal to ‘1’ and ‘40’ = 2/3 of ‘1’ (presumably times a reed of unknown length r). The sum of their squares is then (in Babylonian relative place value notation)

sq. s + sq. s' = sq. 1 + sq. 40 = 1 + 26 40 = 1 26 40 (sq. r) (line 6)

Then, in line 7, s'' is set equal to '20' = 1/2 of '40', and the calculation continues as follows

$s - s' = 1 - 40 = 20\ (r)$, sq. $(s - s')$ = sq. 20 = 6 40 (sq. r) (lines 7-8)

Altogether, then, again in relative place value notation,

sq. s + sq. s' + sq. s'' + sq. $(s - s')$ = 1 26 40 (sq. r) + 6 40 (sq. r) + 6 40 (sq. r) = 1 40 (sq. r) (lines 9-10)

In the final part of what is preserved of the solution procedure of exercise # 2′, in line 10, the reciprocal 36 of 1 40 is computed. The purpose of this operation is, obviously, to compare the given area 2 èše 3 iku with the computed, tentative, area 1 40 (sq. r). Now,

2 èše 3 iku = (2 · 10 (00) + 3 · 1 40) sar = 25 (00) sar = 25 (00) sq. rods.

Consequently, in relative place value notation,

1 40 (sq. r) = 25 (sq. rods), and it follows that sq. r = 25 (sq. rods) / 1 40 = 15 (sq. rods).

Therefore,

r = sqs. 15 = 30 (rods).

Finally, then

s = 1 (00) r = 30 (rods), s' = 40 r = 20 (rods), s'' = 20 r = 10 (rods).

As shown above, the stated problem in this exercise was solved by first setting tentative values for the three unknowns. In this way, the stated problem was (essentially) transformed into a simple *homogeneous quadratic equation* for the unknown length r, which could be solved directly through extraction of a square root. The discussion above also revealed the meaning of the brief numerical table at the beginning of the exercise: The three numbers 1, 40, 20 in the first column of the table are the chosen tentative values for the three square-sides. The three numbers 30, 20, 10 in the second column are the computed values of the three square-sides, and the single number 25 in the third column is the given area, expressed in sq. rods.

8.2.2 # 3′. A Homogeneous Quadratic Problem for Two Squares

BM 54779 # 3′ (col. *ii,* 1′ - 5′)

1′ [*x x x x x x x x x x x x x*] / 3 45 *ta-mar*
2′ [*x x x x x x x x x x*] / 15 íb.sá
3′ 15 *a-na* 1 túm.a [15 *ta-mar*] / 15 íb.sá.1
4′ 15 *a-na* 3 túm.a 45 [*ta-mar*] / 45 íb.sá.2

1′ [x x x x x x x x x x x x x] / 3 45 you will see.
2′ [*x x x x x x x x x x*] / 15 is the equalside.
3′ 15 to 1 carry, [15 you will see,] / 15 is equalside 1.
4′ 15 to 3 carry, 45 [you will see.] / 45 is equalside 2.

Only the last few lines are preserved of the text of this exercise. However, what is left is enough to show that the exercise was about a problem for two squares. It is even possible to make an informed conjecture about how the problem was stated originally.

In lines 1′-2′, the number '3 45' is obtained as the result of some calculation, and it is noted that the square root of '3 45' is '15'. Clearly, '15' is a calculated correction factor in an application of the usual method of false values. In lines 2′-4′, the tentative values '1' and '3' are multiplied by the computed correction factor '15'. The result is that the sides of the two squares are shown to be '15' and '45', meaning, probably, 15 and 45 rods.

Now, if the intended values for the sides of the two squares were 15 and 45 rods, then the sum of the areas of the two squares would be sq. 15 + sq. 45 = 10 · sq. 15 = 10 · 3 45 = 37 30 (sq. rods) = 1 bùr 4 1/2 iku. Therefore, it is likely that the question in this exercise was formulated as follows:

Two squares have the unknown sides s and s'.

sq. s + sq. s' = 1 bùr 2 1/2 iku, $3\,s = s'$, $s, s' = ?$

In the lost part of the solution procedure, the sides of the two squares would have been given the tentative values 1 and 3. Correspondingly, the sum of the areas of the two squares would have the tentative value 1 + 9 = 10. The given value 37 30 for the sum of the areas divided by this tentative value would then be 37 30 / 10 = 3 45. And so on, as in the preserved lines 1´- 4´ of the exercise.

8.2.3 # 4´. Another Homogeneous Quadratic Problem for Three Squares

BM 54779 # 4´ (col. *iii,* 1-4)

1	7	3 30	21 30
2	6	3	
3	1	30	

1+, 2+ a.šag$_4$ 3 íb.sá.ḫá / ul.gar]-*ma* 2$_{\text{èše}}$ ašag 1 30 sar /
3+, 4+ íb.sá.1 / a.na íb.sá.2 *se-bi-a-tim li-im-ṭì* /
.... ………………………………

1	7	3 30	21 30
2	6	3	
3	1	30	

1+, 2+ The fields of 3 equalsides / I heaped, then 2 èše 1 30 sar. /
3+, 4+ equalside 1 / as much as equalside 2 a seventh should be less. /
.... ………………………………

As in the case of exercise 4´, the question in this exercise is preceded by a brief table, gathering together some numerical data. The question itself, partially preserved in the first four lines of the exercise, can be reformulated in quasi-modern symbolic notations as follows:

Three squares have the unknown sides $s, s´,$ and $s´´$.
sq. s + sq. $s´$+ sq. $s´´$ = 2 èše 1 30 sar, $s – 1/7\ s = s´$, $1/6\ s´ = s´´$, $s, s´, s´´ = ?$

The meaning of the brief numerical table preceding the text of the exercise is clear, in view of what was said above in the discussion of the parallel exercise # 2´. Thus, the solution procedure begins by choosing the tentative values 7, 6, 1 for the sides of the three squares. In this way, the two first parts of the question will be satisfied. The corresponding tentative sum of the areas of the three squares will then be

sq. 7 + sq. 6 + sq. 1 = 49 + 36 + 1 = 1 26.

On the other hand, the given sum of the three areas is

2 èše 1 30 sar = (20 (00) + 1 30) sq. rods = 21 30 sq. rods.

Note that the number 21 30 appears in column 3 of the initial numerical table. The given sum divided by the tentative sum is then, in relative place value notation,

21 30 / 1 26 = 15.

This is the square of the needed correction factor. Consequently, the correction factor is ‘30’, which can be interpreted as ;30 = 1/2 (rod). Therefore, the last part of the solution procedure would have been the computation of the sides of the three squares as

7 · ;30 (rods) = 3;30 (rods), 6 · ;30 (rods) = 3 (rods), 1 · ;30 (rods) = ;30 (rods).

This answer to the question in the question in this exercise explains the numbers ‘3 30’, ‘3’, and ‘1’ inscribed in column 2 of the initial numerical table.

8.2.4 The Vocabulary of BM 54779 (and BM 54320)

In addition to terms already listed in the vocabulary of BM 80078 (Sec. 8.1.7 above), the vocabulary of BM 54779 contains the following terms:

a.na	whatever, as much as
ašag,a.šag$_4$	field, area-measure
en.nam	what?
.ḫá	various, several
mi-nam (*a*) ugu (*b*) dirig	what (*a*) over (*b*) is beyond? (*a*–*b* = ?)
igi (*n*) duḫ.a	release the opposite of (*n*) (compute rec. *n*)
a-na šag$_4$	into (šag$_4$ = the interior)
(*a*) *a-na* (*b*) (*c*) *li-im-ṭì*	(*a*) should be less than (*b*) by (*c*)
ku-mur	< *kamārum* to heap, to pile up (= to add together)
li-im-ṭì	< *maṭüm* to become less
ma-la	*mala* as much as, whatever
se-bi-a-tim	*sebītum* one-seventh
ši-ni-ip	*šinipum* two-thirds

Note that this vocabulary works also for the smaller fragment BM 54320 in Sec. 4.3 below.

8.2.5 *BM 54779. A Hand Copy and a Conform Transliteration of the Fragment*

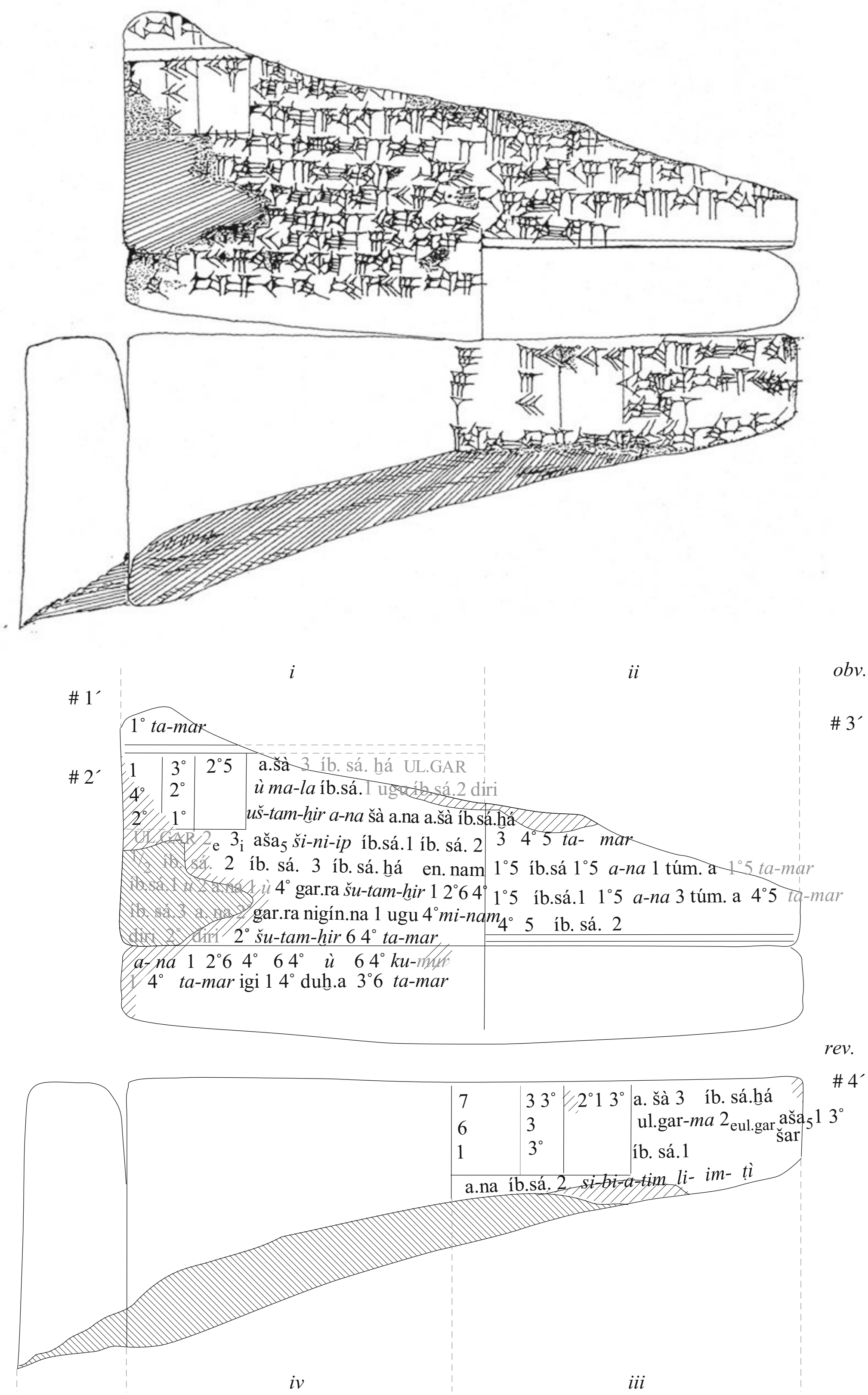


Fig. 8.2.1. BM 54779. Hand copy and conform transliteration.

8.3 BM 54320. A Small Fragment with a Homogeneous Quadratic Problem for Squares

8.3.1 A Homogeneous Quadratic Problem for Three Squares

The fragment BM 54320, shown in Fig. 8.3.1 below, is the lower right corner of a tablet originally inscribed with two columns on the obverse and two on the reverse. On the fragment only part of the solution procedure of a mathematical exercise is preserved. Incidentally, this exercise appears to be identical with the exercise BM 54779 # 4´ (Sec. 8.2.3 above), and also the mathematical terminology appears to be the same. This is clear although there is no overlapping between what is preserved of the texts of the two exercises.

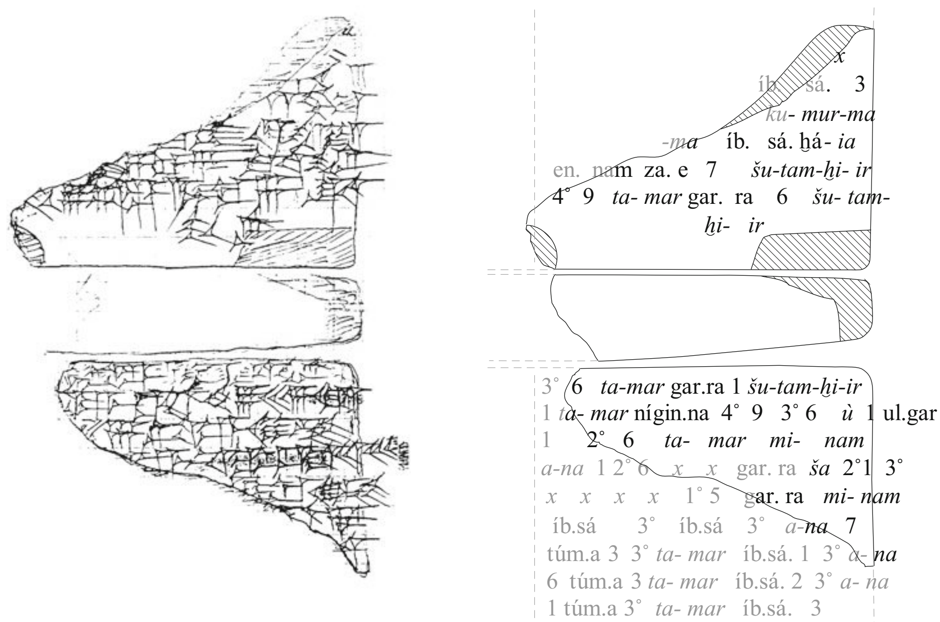


Fig. 8.3.1. BM 54320. A conform transliteration.

BM 54320 (col. *ii,* 1´ - col. *iii*, 7)

obv. 1´ [*x x x x x x x x*] *x* /
2´ [*x x x x* íb].⌜sá⌝.3 /
3´ [*x x x x ku*]*-mur-ma* /
4´ [*x x x x-m*]*a*
6´ za.e 7 *šu-tam-ḫi-ir* / 49 *ta-mar* gar.ra
7´- *rev.* 1 6 *šu-tam-* / *ḫi-ir* / [36] *ta-mar* gar.ra
2 1 *šu-tam-ḫi-ir* / [1] *ta-mar*
nigín.na
3 49 36 *ù* 1 / ul.gar / [1] 26 *ta-mar*
4 *mi-nam* / [*a-na* 1 26 *x x* gar.ra] *ša* 21 30 /
5 [*x x x x* 15 g]ar.ra
6 *mi-nam* / [íb.sá 30 íb.sá]
7 [30 *a*]*-na* 7 / [túm.a 3 30 *ta-mar* íb.sá.1]
[8] [30 *a*]*-na* / [6 túm.a 3 *ta-mar* íb.sá.2]
[9] [30 *a*]*-na* / [1 túm.a 30 *ta-mar* íb.sá.3]

obv. 1´ [x x x x x x x x] x /
2´ [x x x equal]-side 3 /
3´ [x x x x h]eap, then /
4´ [x x x], then
5´ The equalsides / [are] what?

You:
6′ 7 let eat itself, / 49 you will see. Set it.
7′- *rev.* 1 6 let eat itself,/ [36] you will see. Set it.
2 1 let eat itself, / [1] you will see.
Turn around.
3 49, 31, and 1 / heap, / [1] 26 you will see.
4 What / [for 1 26 x x (should be) set] that 21 30 /
5 [x x x x 15 s[et.
6 What / [is the equalside? 30 is the equalside.]
7 [30 t]o 7 / [carry, 3 30 you will see, equalside 1.]
[8] [30 t]o / [6 carry, 3 you will see, equalside 2.]
[9] [30 t]o / [1 carry, 30 you will see, equals-side 3.]

What is preserved of the solution procedure in this badly preserved text evidently coincides precisely with the (reconstructed) solution procedure in the case of the exercise BM 54779 # 4′ in Sec. 8.2.3 above.

8.4 VAT 6505. A Combined Factorization and Doubling and Halving Algorithm

The fragment VAT 6505 was published by Neugebauer in *MKT I* (1935). Since the text of the fragment is severely damaged, Neugebauer managed only to conjecture, correctly, that it has to do with the computation of reciprocals. He also observed that the subscript šu.nigin 12 *ki-ib-*su_5 indicates that the text originally could be divided into 12 separate exercises.

obv.

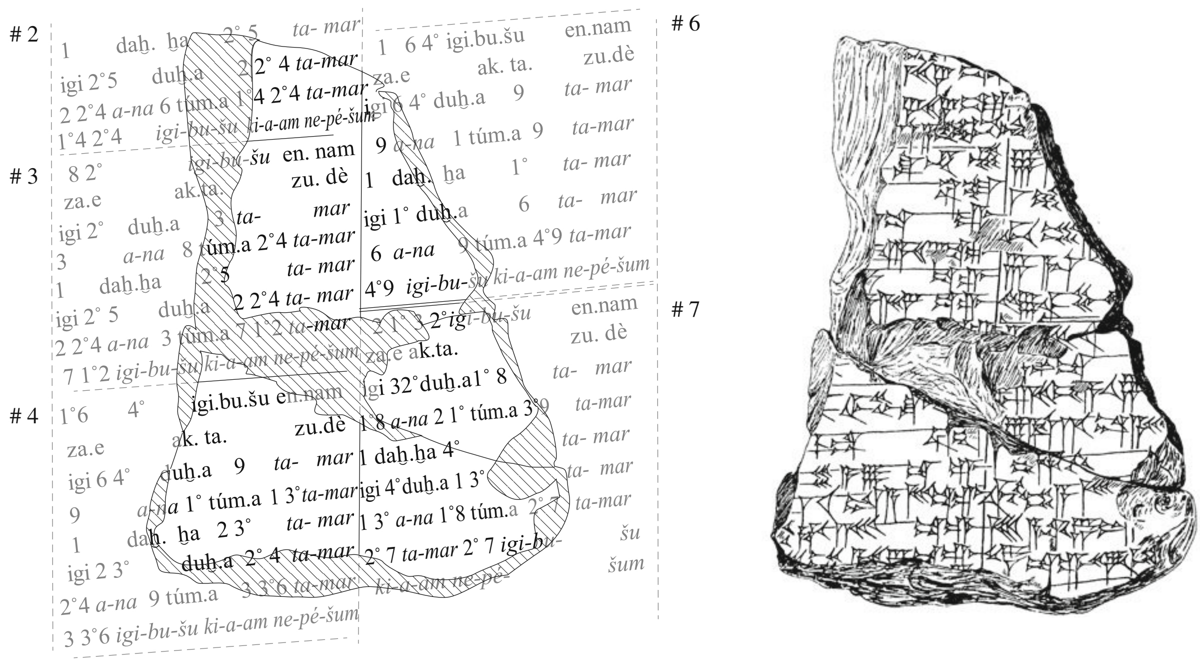


Fig. 8.4.1. VAT 6505. A combined factorization and doubling and halving algorithm.
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VAT 6505 (Neugebauer *MKT I* (1935), 270)

2 [4 10 *igi-bu-šu* en.nam] / [za.e ak.ta.zu.dè] /
[igi 10 duḫ.a 6 *ta-mar*] / [6 *a-na* 4 túm.a 24 *ta-mar*] /
[1 daḫ.ḫa 25 *ta-mar*] /
[igi 25 duḫ.a 2] 24 *ta-mar* / [2 24 *a-na* 6 túm.a 1]4 24 *ta-mar* /
[14 24 *igi-bu-šu k*]*i-a-am ne-pé-šum* /

3 [8 20 *igi-bu*]*-šu* en.nam / [za.e ak.ta].zu.dè /
[igi 20 duḫ.a 3] *ta-mar* / [3 *a-na* 8] túm.a 24 *ta-mar* /
[1 daḫ.ḫa 2]5 *ta-mar* /
[igi 25 duḫ.a] 2 24 *ta-mar* / [2 24 *a-na* 3 túm.a 7 12 *ta*]*-mar* /
[7 12 *igi-bu-šu ki-a-am ne-pé-šum*] /

4 [16 40] *igi-bu-šu* en.nam / [za.e a]k.ta.zu.dè /
[igi 6 40] duḫ.a 9 *ta-mar* / [9 *a*]*-na* 10 túm.a 1 30 *ta-mar* /
[1 da]ḫ.ḫa 2 30 *ta-mar* /
[igi 2 30] duḫ.a 24 *ta-mar* / [24 *a-na* 9 túm.a 3 36 *ta*]*-mar* /
[3 36 *igi-bu-šu ki-a-am ne-pé-šum*] /

… ……………………………………………

6 [1 06 40 *igi-bu-šu* en.nam] / [za.e ak.ta.zu.dè] /
i[gi 6 40 duḫ.a 9 *ta-mar*] / 9 [*a-na* 1 túm.a 9 *ta-mar*] /
1 daḫ.[ḫa 10 *ta-mar*] /
igi 10 duḫ.[a 6 *ta-mar*] / 6 *a-na* [9 túm.a 49 *ta-mar*] /
49 *igi-bu-*[*šu ki-a-am ne-pé-šum*] /

7 [2 13] 20 *igi-b*[*u-šu* en.nam] / [za.e a]k.ta.[zu.dè] /
[igi] 32 duḫ.a 18 [*ta-mar*] / 18 *a-na* 2 10 túm.a 39 [*ta-mar*] /
1 daḫ.ḫa 40 [*ta-mar*] /
igi 40 duḫ.a 1 30 [*ta-mar*] / 1 30 *a-na* 18 túm.[a 27 *ta-mar*] /
<27 *ta-mar*> 27 *igi-bu-*[*šu*] / [*ki-a-am ne-pé-šum*] /

… ……………………………………………

2 [4 10, its reciprocal is what?] / [You, in your doing it:] /
[The reciprocal of 10 release, 6 you will see.] / [6 to 4 carry, 24 you will see.] /
[1 add on, 25 you will see.] /
[The reciprocal of 25 release, 2] 24 you will see. / [2 24 to 6 carry, 1]4 24 you will see. /
[14 24 its reciprocal. S]uch is the procedure. /

3 [8 20] its [reciprocal] is what? / [You in your do]ing it: /
[The reciprocal of 20 release, 3] you will see. / [3 to 8] carry, 24 you will see. /
[1 add on, 2]5 you will see. /
[The reciprocal of 25 release,] 2 24 you will see. / [2 24 to 3 carry, 7 12 you] will see. /
[7 12 its reciprocal. Such is the procedure.] /

4 [16 40] its reciprocal is what? / [You, in your do]ing it: /
[The reciprocal of 6 40] release, 9 you will see. / [9 t]o 10 carry, 1 30 you will see. /
[1 add] on, 2 30 you will see. /
[The reciprocal of 2 30] release, 24 you will see. / [24 to 9 carry, 3 36 you] will see. /
[3 36 its reciprocal. Such is the procedure.] /

… ……………………………………………

6 [1 06 40], its reciprocal is what?] / [You, in your doing it]: /
The [reciprocal of 6 40 release, 9 you will see.] / 9 [to 1 carry, 9 you will see.] /
1 add [on, 10 you will see.] /
The reciprocal of 10 resolv[e, 6 you will see.] / 6 to [9 carry, 54(!) you will see.] /
54(!) [its] reciprocal. [Such is the procedure.] /

7 [2 13] 20 [its] [reciprocal is what?] / [You,] in your do[ing it]: /
[The reciprocal of] 32 release, 18 [you will see.] / 18 to 2 10 carry, 39 [you will see.] /
1 add on, 40 [you will see.] /
The reciprocal of 40 release, 1 30 [you will see.] / 1 30 to 18 carry, [27 you will see.] /
<27 you will see.> 27 [its] reciprocal. / [Such is the procedure.] /

… ……………………………………………

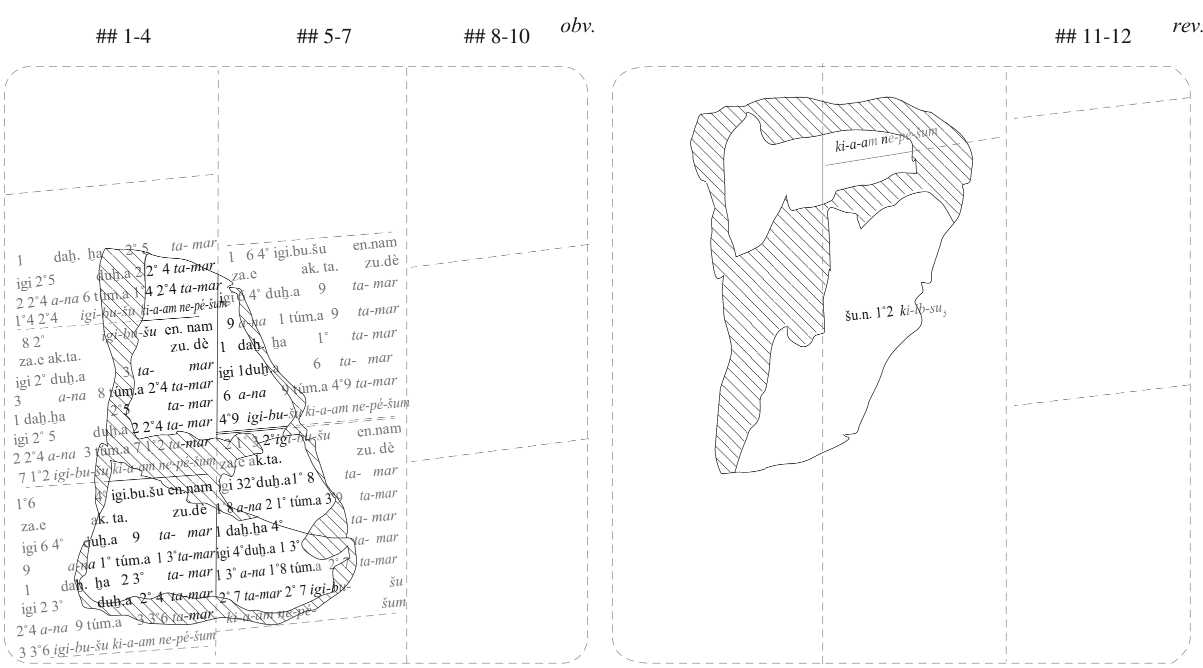


Fig. 8.4.2. VAT 6505. An outline of a suggested reconstruction of the whole text.

On the reverse of VAT 6505, only two lines of text are (partly) preserved. One is the last line of the last exercise, which is, of course, *ki-a-am ne-*[*pé-šum*] 'Such is the pr[ocedure]'. The other is a subscript, which, as Neugebauer observed, says that the whole text contained 12 exercises. Knowing this, it is not difficult to figure out how those 12 exercises were placed on the obverse and reverse of the clay tablet. This is shown in Fig. 8.4.2 above.

Neugebauer did not observe the obvious connection between what remained of the text on VAT 6505 and the perfectly preserved text on the small Ur III/early Old Babylonian algorithm table CBS 10201 (Fig. 13.4.2 below). That algorithm table had been published by Hilprecht in his book about the tablets in "the Temple Library at Nippur". (Actually, Hilprecht's publication in that book of a large number of mathematical and metrological table texts can be said to have been the starting point of the serious study of Mesopotamian mathematics and metrology.) As a matter of fact, Neugebauer published a copy of the text of CBS 10201 among a number of ordinary Old Babylonian tables of reciprocals at the beginning of his *MKT I* (1935). If he had seen the connection between CBS 10201 and VAT 6505, he would have understood both texts much better! On the other hand, Neugebauer did see the connection between the algorithm text CBS 10201 and the list of 13 reciprocal pairs inserted at the end of the combined multiplication table BM 80150 (*MKT I,* 23). That list begins with the pair (2 05, 28 48) and ends with the pair (2 22 13 20, 25 18 45).

Here is the text of CBS 10201:

CBS 10201 (Hilprecht *MMCTTN* (1906); Neugebauer *MKT I* (1935), 24; Fig. 13.4.2 below)

# 1	2 05 12 / igi.gál.bi 28 48 /	2 05 12 / its reciprocal is 28 48 /
# 2	4 10 6 / igi.gál.bi 14 24 /	4 10 6 / its reciprocal is 14 24 /
# 3	8 20 3 / igi.gál.bi 7 12 /	8 20 3 / its reciprocal is 7 12 /
# 4	16 40 1 30 / igi.gál.bi 3 36 /	16 40 1 30 / its reciprocal is 3 36 /
# 5	33 20 18 / igi.gál.bi 1 48 /	33 20 18 / its reciprocal is 1 48 /
# 6	1 06 40 9 / igi.gál.bi 54 /	1 06 40 9 / its reciprocal is 54 /
# 7	2 13 20 18 / igi.gál.bi 27 /	2 13 20 18 / its reciprocal is 27 /
# 8	4 26 40 9 / igi.gál.bi 13 30	4 26 40 / its reciprocal is 13.30.

In exercise # 2 of VAT 6505, it is shown that rec. 4 10 = 14 24. The first step of the algorithmic computation is to compute rec. 10 = 6 as the reciprocal of the "last (double-)place" of 4 10. In line # 2 of CBS 10201, which also mentions that rec. 4 10 = 14 24, this number 6 is placed after 4 10 as kind of hint how to start the computation of the reciprocal.

Similarly in exercise # 3 of VAT 6505, it is shown that rec. 8 20 = 7 12, and the first step of the algorithmic computation of the reciprocal is the computation of rec. 20 = 3, where 20 is the last place of 8 20. In # 3 of CBS 10201, the same number 3 is placed right after 8 20, again as a hint of how to start the computation.

In # 6 of both texts, the number 9 plays a similar role, and in # 7 of both text it is the number 18 that joins the two texts together. However, in # 4 of CBS 10201, the number 1 30 which is placed after 16 40 is the reciprocal of the last place 40, as expected, but in # 4 of VAT 6505, the computation of the reciprocal of 16 40 starts with the computation of rec. 6 40 = 9. In this case, 6 40 is not the single "last place" of 16 40 but instead a kind of generalized "last place", which may be called a "trailing part" of 16 40. This example shows that the actual implementation of the idea of using the last place of a given regular sexagesimal number to start the computation of the reciprocal of that number can vary slightly from case to case.

Both VAT 6505 and CBS 10201 are, of course, closely related to the Old Babylonian algorithm text CBS 1215 which was discussed in Sec. 2.2 above. Recall that the whole Chapter 2 was devoted to a detailed discussion of "direct and inverse factorization algorithms for many-place regular sexagesimal numbers" in both Neo-Babylonian and Old Babylonian algorithm texts.

8.5 BM 96954 + BM 102366 + SÉ 93. A Large Recombination Text Concerned with Pyramids and Cones

The preserved part of this large text consists of three fragment, two from the collection of cuneiform texts in the British Museum and one from the collection of the Couvent Saint-Étienne in Jerusalem. See the hand copies by F. N. Al-Rawi in Figs. 8.5.7-8 below. The two fragments from the British Museum were first copied by F. N. Al-Rawi and were then briefly discussed in Friberg, *PCHM* 6 (1996). The fragment from Jerusalem was published by Jursa and Radner in *AfO* 42-43 (1995-6), 103-105. A complete edition of the three fragments together, unfortunately with a totally inadequate mathematical commentary, was printed in Robson, *MMTC* (1999), App. 3. Much more carefully considered mathematical commentaries to the most interesting exercises in the combined text BM 96954+ were presented in Friberg, *UL* (2005), Sec. 4.8 g and Friberg, *AT* (2007), Sec. 9.3. Those commentaries will be repeated and expanded below.

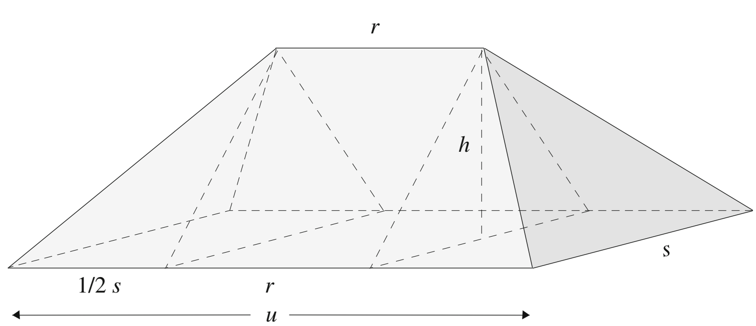
The mathematical terminology used in this combined text clearly shows that, although the text is unprovenanced, it belongs to Group 6 (Friberg, *RA* 94 (2000), 180), which implies that it is is from Late Old Babylonian Sippar. See the survey of texts belonging to Group 6 at the beginning of Ch. 9 below. The table of contents in the outline below (in Fig. 8.5.6) of the combined text clearly shows that BM 96954+ is a mathematical recombination text. It originally comprised about 26 individual exercises, all concerned with three-dimensional solids, in most cases pyramids or cones.

8.5.1 § 1. Metric Algebra Problems for Ridge Pyramids

The solid considered in § 1 of BM 96954+ is what may be called a "ridge pyramid" such as the one depicted in Fig. 8.5.1 below. Expressed in terms of quasi-modern symbolic notations, the following computation rule is consistently used in § 1 for the computation of the volume V of a ridge pyramid with the length u, the front s, the "ridge" r and the height h:

$$V = (u + 1/2\ r) \cdot s \cdot 1/3\ h.$$

It will be explained in Sec. 8.5.10 where this computation rule comes from. Actually, the volume is never explicitly mentioned in the text of § 1. Instead the size of the ridge pyramid is expressed in terms of its *seed measure*, which is obtained from the volume through multiplication by a constant c, which in the text is called 1 30 igi.gub.ba '1 30, the constant'. The ridge pyramid itself is simply called guru$_7$ 'grain heap' or 'granary'. This is understandable, since a large quantity of dry grain carefully poured onto a flat ground will by itself form a heap with all its sides inclined to the same degree.



u = 10 nindan, s = 6 nindan, r = 4 nindan
h = 48 cubits, c = 1 30 gur/volume-sar
C = c · (u + 1/2 r) · s · 1/3 h = 28 48 00 gur

Fig. 8.5.1. BM 96954+ § 1. The computation rule for the seed measure of a 'grain heap' (a ridge pyramid).

The computation rule used in the text for the computation of the seed measure of a 'grain heap' is, then,

$$C = c \cdot (u + 1/2\ r) \cdot s \cdot 1/3\ h.$$

It is likely that five exercises belonging to § 1 were originally contained in the lost first column on the obverse of BM 96954+, and that the whole § 1 was copied from a carefully organized "theme text" concerned with the mentioned ridge pyramid and containing 13 exercises. If so, it is almost certain that the first exercise was the computation of seed measure of the ridge pyramid, in order to set the stage for the following exercises. It is also likely that in the next four exercises each one of the four parameters *u, r, s, h* was computed, one by one, when the values of the remaining parameters were known.

The values of the four parameters are the same in all the exercises of § 1, namely

$u = 10,\ r = 4,\ s = 6,\ h = 48.$

The corresponding values of the volume *V* and the seed measure *C* are

$V = (10 + 1/2 \cdot 4) \cdot 6 \cdot 1/3 \cdot 48 = 12 \cdot 6 \cdot 16 = 19\ 12,$ and $C = 1\ 30 \cdot V = 19\ 12\ 00 + 9\ 36\ 00 = 28\ 48\ 00.$

If *u*, *r*, and *s* were thought of as multiples of 1 nindan (= 6 meters), then the dimensions of the grain pile would be considerable, indeed. The alternative interpretation is that *u*, *r,* and *s* were thought of as multiples of ;01 nindan. Then the grain pile would be diminutive, which is not very likely. Consequently, it is safe to assume that the intended values were *u* = 10 nindan (= 60 meters), *s* = 6 nindan (36 meters), and *h* = 48 cubits (= 24 meters). The volume would then be 19 12 volume-sar (= more than 20,000 cubic meters!), and the seed measure 28 48 00 gur. Note, by the way that the conversion rate from volume-sar to seed measure can be understood as

1 30 gur/volume-sar = 1 30 · 5 00 sìla / volume-sar = 7 30 00 sìla per volume-sar.

This means that the sìla figuring in § 1 of BM 96954+ is the sìla with the *našpaku* 'storing number' 7 30. This particular kind of sìla can be thought of as the content of a rectangular box with both sides of the bottom equal to ;01 nindan = 6 fingers (= 1 decimeter) and with the height equal to ;00 40 nindan = 4 fingers. (See Friberg, *BaM* 28 (1997), 312).

The first (partly) preserved exercise on BM 96954+ is § 1 f below.

BM 96954+ § 1 f (*obv. ii*: 1´-8´)

1 [guru$_7$ 6 sag 48 sukud 28 48 še gur]
2 uš sagšu / [ul.gar 14 uš sagšu] ⸢en.nam⸣
3 za.e igi 48 / [duḫ.a 1 15 *ta*]-*mar*
4 1 15 *a-na* 28 48 še / [*i-ši* 36] *ta-mar*
5 igi 1 30 igi.gub.ba duḫ.a / [40 *a*]-⸢*na*⸣ 36 *i-ši* 24 *ta-mar*
6 igi 6 sag / [duḫ].⸢a⸣ 10 *ta-mar* 24 *a-na* 10 *i-ši* 4 *ta-mar* /
7 [x] 4 *i-na* ul.<gar> ba.zi 10 *ta-mar* 10 uš 4 sagšu /
8 [*ki*]-*a-am ne-pé-šum*

1 [A grain heap. 6 the front, 48 the height, 28 48 gur the grain].
2 The length and the ridge / [I heaped, 14. The length and the ridge] ⸢were what?⸣
3 You: The reciprocal of 48 / [release, 1 15 you] will see.
4 1 15 to 28 48, the grain / [carry, 36] you will see.
5 The reciprocal of 1 30, the constant, release, / [40]. ⸢To⸣ 36 carry, 24 you will see.
6 The reciprocal of 6, the front / [release] 10 you will see. 24 to 10 carry, 4 you will see.
7 [x] 4 from the heap tear off, 10 you will see, 10 the length, 4 the ridge.
8 [Su]ch is the procedure.

In quasi-modern symbolic notations, the question in this paragraph can be rephrased as

$s = 6,\quad h = 48,\quad C = c \cdot (u + 1/2\ r) \cdot s \cdot 1/3\ h = 28\ 48,\quad u + r = 14,\quad u, r = ?$

The solution procedure begins, in lines 2-6 by letting the seed measure C be divided by the known values of the height *h*, the grain constant *c*, and the front *s*. What then remains is

$1/3\ (u + 1/2\ r) = \text{igi}\ s \cdot \text{igi}\ c \cdot \text{igi}\ h \cdot C = 10 \cdot 40 \cdot 1\ 15 \cdot 28\ 48 = 4.$

The remainder of the solution procedure should have proceeded in the following easy steps:

$2\ u + r = 6 \cdot 4 = 24,\quad u = (2\ u + r\ ) - (u + r) = 24 - 14 = 10,\quad r = (u + r) - u = 14 - 10 = 4.$

However, the student who wrote the text of § 1 f did not remember how the teacher had told him to proceed from there, so he chose to bluff and simply stated, *for no other reason than that he knew the answer*, that

$(u + r) - 4 = 14 - 4 = 10 = u,$ and $r = 4.$

BM 96954+ § 1 g (*obv. ii*: 9′-20′)

1 [gur]u_7 6 ⸢sagšu⸣ 48 sukud 28 [4]8 še gur /
2 [$^1/_2$ sagš]u *ki-ma* igi.5.gál uš uš sag⸢šu en.nam⸣ /
3 5 bala uš 2 bala sagšu gar.ra
4-5 nigín.na igi 48 / duḫ.a 1 15 *ta-mar* 1 15 *a-na* 28 48 / *i-ši* 36 *ta-mar*
6 igi 1 30 igi.gub duḫ.a / 40 *ta-mar* 40 *a-na* 36 *i-ši* 24 *ta-mar* /
7-8 igi 6 sag duḫ.a 10 *ta-mar* 10 *a-na* 24 *i-ši* / 4 *ta-mar*
$^1/_2$ 4 *ḫe-pé* 2 *ta-mar* gar.ra /
9-10 $^1/_2$ 2 bala sagšu *ḫe-pé* 1 *ta-mar a-na* 5 bala uš / daḫ.ḫa 6 *ta-mar*
igi.3.gál ba.zi 2 *ta-mar* /
11-12 5 *a-na* 2 *i-ši* 10 *ta-mar* 2 *a-na* 2 bala sagšu / *i-ši* 4 *ta-mar* sagšu
ne-pé-šum

1 |A grain] heap, 6 the ⸢front⸣, 48 the height, 28 [4]8 gur the grain /
2 [1/2 the rid]ge like the 5th part of the front.
3 5 the fraction of the length, 2 the fraction of the ridge, set down.
4-5 Turn around. The reciprocal of 48 / release, 1 15 you will see. 1 15 to 28 48 / carry, 36 you will see.
6 The reciprocal of 1 30, the constant, release / 40 you will see. 40 to 36 carry, 24 you will see. /
7-8 The reciprocal of 6, the ridge, release, 10 you will see. 10 to 24 rise, / 4 you will see.
1/2 of 4 break, 2 you will see. Set it down. /
9-10 1/2 of 2 the fraction of the ridge break, 1 you will see. To 5, the fraction the length / add it on, 6 you will see.
Its 3rd part tear off (sic!), 2 you will see. /
11-12 5 to 2 carry, 10 you will see. 2 to 2, the fraction of the ridge, / carry, 4 you will see, the ridge.
The procedure.

The question in § 1 g can be formulated as follows:

$s = 6, \quad h = 48, \quad C = c \cdot (u + 1/2\, r) \cdot s \cdot 1/3\, h = 28\ 48, \quad 1/2\, r = 1/5\, u, \quad u, r = ?$

The solution procedure starts by setting down 5 and 2, the 'turn-overs' of the length and the ridge. Then it computes, just as in the preceding exercise,

$1/3\ (u + 1/2\, r) = \text{igi}\ s \cdot \text{igi}\ c \cdot \text{igi}\ h \cdot C = 10 \cdot 40 \cdot 1\ 15 \cdot 28\ 48 = 4.$

In the second half of line 8, 4 is divided by 2. This step is misplaced, it should come later.

Indeed, the proper next step would be to use the "method of false values", introducing the false values $r^* = 2$ (bala sagšu), $u^* = 5$ (bala uš), chosen so that they satisfy the equation $1/2 r^* = 1/5\, u^*$. With these values

$1/3\ (u^* + 1/2\, r^*) = 1/3\ (5 + 1) = 1/3 \text{ of } 6 = 2$ (lines 9-10)

Now is the right time to observe that

$1/2 \text{ of } 4 = 2$ (second half of line 8)

This means that for the equation $1/3\ (u + 1/2\ r) = 4$ to be satisfied, u^* and r^* must be multiplied by the "correction factor" 4 : 2 = 2. Consequently

$u = u^* \cdot 2 = 5 \cdot 2 = 10 \quad \text{and} \quad r = r^* \cdot 2 = 2 \cdot 2 = 4$ (lines 11-12)

Remark. Errors like the ones in this exercise where a line in the solution procedure is misplaced, and like the one in the preceding exercise, where the writer of the text had forgotten what the next step should be in the solution procedure but managed to find the right answer anyway, are easy to explain. Indeed, look at the chaotically organized solution procedures in the text VAT 8522 (Fig. 11.3.5 below), which probably were notes written down hastily by a student listening to a teacher's explanations of the proper solution procedures. If a student who had made such sloppy notes then tried to write down the whole text of the exercise with both question and solution procedure, without thinking too much about he was doing, he could easily produce mistakes of the kind mentioned above.

BM 96954+ § 1 h (*obv. ii*: 21′-…; *obv. iii*: 1-8)

1 [gur]u_7 48 sukud 28 48 <še gur> 10 uš
2 $^2/_3$ sag / [*ki-ma* sagšu sag] ⸢sagšu en.nam⸣
3-4 ⸢za.e⸣ igi 48 / [duḫ.a 1 15 *ta-mar* 1 15 *a-na* 28 48] / [*i-ši* 36 *ta-mar*]
… ……………………………

[……………………….] 10 uš /
[……………………….] *i-ši* /
… ……………………………
1′ ……… 3 20 *ta-mar* 1/2 3 20 *ḫe-pé* 1 40 *ta-mar* /
2′-3′ 1 40 nigin 2 46 40 *ta-mar* 2 46 40 *a-na* 2 40 daḫ.ḫa / 5 26 40 *ta-mar*
en.nam íb.sá 2 20 íb.sá /
4′ 1 40 *i-na* 2 20 ba.zi 40 *ta-mar* 1/2 40 *ḫe-pé* 20 *ta-mar* /
5′ 20 nigin 6 40 *ta-mar* igi 6 40 duḫ.a 9 *ta-mar* /
6′ 40 *a-na* 9 *i-ši* 6 *ta-mar* sag
7′ 40 *a-na* 6 *i-ši* / 4 *ta-mar* sagšu /
8′ *ki-a-am ne-pé-šum*

1 [A grain he]ap. 48 the height, 28 48 <gur the grain>, 10 the length.
2 2/3 of the front / [is like the ridge. The front] ⸢and the ridge, what⸣?
3-4 ⸢You⸣: The reciprocal of 48 / [release, 1 15 you will see. 1 15 to 28 48 / [carry, 36 you will see.]
… ……………………………
[………………………] 10, the length /
[………………………] carry /
… ……………………………
1′ ……… 3 20 you will see. 1/2 3 20 break, 1 40 you will see. /
2′-3′ 1 40 square 2 46 40 you will see. 2 46 40 to 2 40 add on / 5 26 40 you will see.
What equalsided? 2 20 equalsided. /
4′ 1 40 from 2 20 tear off, 40 you will see. 1/2 of 40 break, 20 you will see. /
5′ 20 square, 6 40 you will see. The reciprocal of 6 40 release, 9 you will see. /
6′ 40 to 9 carry, 6 you will see, the front.
7′ 40 to 6 carry, / 4 you will see, the ridge. /
8′ Such is the procedure.

The question in § 1 h can be formulated as follows:

$h = 48, \quad C = \text{c} \cdot (u + 1/2\ r) \cdot s \cdot 1/3\ h = 28\ 48, \quad u = 10, \quad 2/3\ s = r, \quad s, r = ?$

The solution procedure starts by computing

$\text{c} \cdot (u + 1/2\ r) \cdot 1/3\ \ s = \text{igi}\ h \cdot C = 1\ 15 \cdot 28\ 48 = 36.$

The middle part of the solution procedure is destroyed, but it is likely that the next step was to compute

$(u + 1/2\ r) \cdot 1/3\ \ s = ;40 \cdot 36 = 24.$

Since u = 10 was given, the problem to consider would then be the rectangular-linear set of equations

$(10 + 1/2\ r) \cdot 1/3\ s = 24, \quad 2/3\ s = r, \quad s, r = ?$

Presumably, a new unknown was introduced, which will here be called p, such that

$r = 6\ p, \quad s = 9\ p.$

The second of the the equations above will then be satisfied, while the first equation takes the new form

$(3;20 + p) \cdot p = 24/9 = 2;40.$

That means that the rectangular-linear system of equations for s and r has been replaced by a quadratic equation for the new unknown p. This equation is solved in the usual way by computing

1/2 of 3;20 = 1;40, sq. 1;40 = 2;46 40, 2;46 40 + 2;40 = 5;26 40, sqs. 5;26 40 = 2;20 (lines 1′ -3′)
p = sqs. (sq. 1;40 + 2;40) – 1;40 = 2;20 – 1;40 = ;40 (line 4′)

Next, the coefficient 9 in the equation $s = 9\ p$ is explained, for no obvious reason!, as

rec. sq. (1/2 · ;40) = rec. ;06 40 = 9 (lines 4′-5′)

Then, finally,

$s = 9\ p = 9 \cdot :40 = 6$ and $r = 2/3\ s = ;40 \cdot 6 = 4$ (lines 6′-7′)

BM 96954+ § 1 i (*obv. iii*: 1′-14′)
1 [guru$_7$ 4 sagšu 48 sukud 28] 48 še gur /
2 [uš *ù* sag ul.gar 16] uš sag en.nam /

3-4 [za.e igi 48] sukud duḫ.a 1 15 *ta-mar* / [1 15] *a-na* 28 48 *i-ši* 36 *ta-mar* /
5-6 igi 1 30 igi.gub duḫ.a 40 *ta-mar* 40 *a-na* 36 *i-ši* / 24 *ta-mar*
7-8 igi.3.gál 24 ba.zi 8 *ta-mar* / $^1/_2$ 4 sagšu *ḫe-pé* 2 *ta-mar* $^1/_3$ 2 ba.zi / 40 *ta-mar*
9 $^1/_3$ 15 ba.zi 5 *ta-mar* 40 *a-na* 5 20 daḫ.ḫa / 6 *ta-mar*
10 $^1/_2$ 6 *ḫe-pé* 3 *ta-mar* nigin 9 *ta-mar* / 8 *i-na* 9 ba.zi <1> *ta-mar*
11 1 en.nam íb.sá / 1 íb.sá
12 1 *a-na* 3 daḫ.ḫa 4 *ta-mar* 1 *i-na* 3 ba.zi / 2 *ta-mar*
13 4 *šu-ul-li-iš* 12 *ta-mar* / 2 *i-na* 12 ba.zi 10 *ta-mar* uš
14 *i-na* 16 ul.gar <ba.zi> / 6 *ta-mar* 6 sag
ne-pé-šum

1 [A grain heap. 4 the ridge, 48 the height, 28] 48 gur the grain. /
2 The length and the front heaped 16].
3-4 [You: The reciprocal of 48], the height, release 1 15 you will see. / [1 15] to 28 48 carry, 36 you will see. /
5-6 The reciprocal of 1 30, the constant, release, 40 you will see. 40 to 36 carry, / 24 you will see.
7-8 The 3rd part of 24 tear off, 8 you will see. / 1/2 of 4, the ridge, break, 2 you will see. 1/3 of 2 tear off / 40 you will see.
9 $^1/_3$ of 16! tear off, 5 <20> you will see. 40 to 5 20 add on, / 6 you will see.
10 1/2 of 6 break, 3 you will see. Square it, 9 you will see. / 8 from 9 tear off, <1> you will see.
11 1 is what equalsided? / 1 equalsided.
13 1 to 3 add on, 4 you will see. 1 from 3 tear off, / 2 you will see.
14 4 triple, 12 you will see. / 2 from 12 tear off, 10 you will see, the length.
From 16, the sum, <tear off>, / 6 you will see, the front.
The procedure.

The question in § 1 i can be formulated as follows:

$$r = 4, \quad h = 48, \quad C = c \cdot (u + 1/2\ r) \cdot s \cdot 1/3\ h = 28\ 48, \quad [u + s = 16], \quad u, s = ?$$

The solution procedure begins by calculating

$$1/3\ (u + 1/2\ r) \cdot 1/3\ s = C \cdot \text{rec. } 48 \cdot \text{rec } 1\ 30 \cdot 1/3 = 8 \qquad \text{(lines 3-7)}$$

Next, silently,

$$p = 1/3\ (u + 1/2\ r) \quad \text{and} \quad q = 1/3\ s$$

are introduced as new unknowns. The question then takes the new form

$$p \cdot q = 8, \quad p + q = 1/3\ (u + s) + 1/3 \cdot 1/2\ r = 1/3 \cdot 16 + 1/3 \cdot 1/2 \cdot 4 = 5\ 20 + 40 = 6, \qquad p, q = ? \qquad \text{(lines 8-9)}$$

This is a rectangular-linear system of equations of the basic type B1a for p and q (Friberg, *AT* (2007), 6). The solution is calculated in the usual way as follows:

$$p = 1/2 \cdot 6 + \text{sqs. } \{\text{sq. } (1/2 \cdot 6) - 8\} = 3 + 1 = 4, \qquad q = 1/2 \cdot 6 - \text{sqs. } \{\text{sq. } (1/2 \cdot 6) - 8\} = 3 - 1 = 2 \qquad \text{(lines 9-13)}$$

After that, u and s are calculated in the following easy steps:

$$u + r/2 = 4 \cdot 3 = 12, \quad u = 12 - 2 = 10, \quad s = 16 - 10 = 6 \qquad \text{(lines 12-14)}$$

BM 96954+ § 1 j (*obv. iii*: 15´-27´)

1 guru$_7$ 4 sagšu 28 48 <še gur> 48 sukud
2 $^1/_2$ uš *ù* 1 sag / uš sag en.nam
3 za.e igi 48 sukud duḫ.a / 1 15 *ta-mar a-na* 28 48 *i-ši* 36 *ta-mar* /
4-5 igi 1 30 igi.gub duḫ.a 40 *ta-mar* 40 *a-na* 36 *i-ši* {24} / 24 *ta-mar*
6 *aš-šum* $^1/_2$ uš *ù* 1 bala sag dug$_4$.ga / 1 *ù* 30 gar.ra nigín.na
7 $^1/_2$ 4 sagšu *ḫe-pé* 2 *ta-mar* / $^1/_3$ 2 ba.zi 40 *ta-mar* $^1/_3$ 1 ba.zi 20 *ta-mar* /
8-9 20 *a-na* 30 *i-ši* 10 *ta-mar* 24 *a-na* 10 *i-ši* 4 *ta-mar* / 4 en.nam íb.sá 2 íb.sá
10-11 igi 30 duḫ.a 2 *ta-mar* / 2 *a-na* 2 *i-ši* 4 *ta-mar* 4 *šu-ul-li-iš* / 12 *ta-mar*
12 2 *i-na* 12 ba.zi 10 *ta-mar* uš / 4 sagšu *i-na* 10 uš ba.zi 6 *ta-mar* sag /
13 *ki-a-am ne-pé-šum*

1 A grain heap. 4 the ridge, 28 48 <gur the grain>, 48 the height, 1/2 the length and 1, the front. /
2 The length and the front, what?
3 You: The reciprocal of 48, the height, release, / 1 15 you will see. To 28 48 carry it, 36 you will see. /
4-5 The reciprocal of 1 30, the constant, release, 40 you will see. 40 to 36 carry, / 24 you will see.
6 Since 1/2 the length and 1 the fraction of the front it was said, / 1 and 30 set down. Turn around.

7 1/2 of 4, the ridge, break, 2 you will see. / 1/3 of 2 tear off, 40 you will see. 1/3 of 1 tear off, 20 you will see. /
8-9 20 to 30 carry, 10 you will see. 24 to 10 carry, 4 you will see. / 4 is what equalsided? 2 equalsided.
10-11 The reciprocal of 30 release, 2 you will see. / 2 to 2 carry, 4 you will see. 4 triple / 12 you will see.
12 2 from 12 tear off, 10 you will see, the length. / 4, the ridge, from 10, the length, tear off, 6 you will see, the front. /
13 Such is the proccdure.

The question in this exercise can be reformulated as follows:

$r = 4, \quad C = \mathrm{c} \cdot (u + 1/2\ r) \cdot s \cdot 1/3\ h = 28\ 48, \quad h = 48, \quad 1/2\ u + 1 = s, \quad u, s = ?$

The solution procedure begins in the usual way by computing

$\text{rec. c} \cdot \text{rec. } h \cdot C = 1/3\ (u + 1/2\ r) \cdot s = \text{rec. } 1\ 30 \cdot \text{rec. } 48 \cdot 28\ 48 = 40 \cdot 1\ 15 \cdot 28\ 48 = 24.$ (lines 2-5)

In this way, the question is simplified to

$1/3\ (u + 1/2\ r) \cdot s = 24, \quad 1/2\ u + 1 = s, \quad r = 4, \quad u, s = ?$

Then, silently, a new unknown p is introduced, so that

$u + 1/2\ r = 1 \cdot p, \quad s = 30 \cdot p$ (line 5)

Then the equation $1/2\ u + 1 = s$ will be automatically satisfied. It is also observed that

$1/2\ r = 1/2 \text{ of } 4 = 2$ (line 6)

Consequently, the problem now takes the following form:

$1/3\ p \cdot 30\ p = 24, \quad p = u + 2$

This is a quadratic equation for p of the simplest kind, which can be solved simply by computing a square-side. However, the writer of § 1 of BM 96954+, who does not seem to have been a very bright student, made the solution procedure more complicated than necessary, proceeding in the following way: He introduced

$q = 1/3 \text{ of } 30\ p$

as a second new unknown. Then the quadratic equation for p was replaced by the following equation for q:

$\text{sq. } q = 30 \cdot 1/3 \text{ of } 24 = 30 \cdot 20 \cdot 24 = 10 \cdot 24 = 4$ (lines 7-8)

(The computation 1/3 of 2 = 40 in line 7 is unmotivated and is not used in the following lines!) It follows that

$q = \text{sqs. } 4 = 2, \quad \text{so that} \quad 1/3 \text{ of } 30\ p = 2, \quad 1/3\ p = \text{rec. } 30 \cdot 2 = 2 \cdot 2 = 4, \quad \text{and} \quad p = 3 \cdot 4 = 12$ (lines 9-11)

After that, it was easy to compute

$u = p - 2 = 12 - 2 = 10, \quad \text{and} \quad s = u - r = 10 - 4 = 6$ (lines 11-12)

Note that it would have been more straightforward to compute s as $1/2\ p = 1/2 \cdot 12 = 6$. Indeed, the equation $s = u - r$ depends on information not really having anything to do with exercise § 1 j, namely that if the inclination f is the same for all the four sides of the ridge pyramid, then

$f = s/h = (u - r)/h, \quad \text{so that} \quad s = u - r.$

See Fig. 8.5.1 above.

BM 96954+ § 1k (*obv. iii*: 28´-34´; *rev. i*: 1-2)

1 guru$_7$ 4 sagšu 28 48 še gur 48 sukud
2 *i-na* 1 kùš / ˹7 30 kùš˺ šag$_4$.gal uš *ù* sag en.nam
3-4 za.e / [7 30 šag$_4$.gal] *a-na* 48 sukud *i-ši* 6 *ta-mar* / *a-na* 4 sagšu daḫ.ḫa 10 *ta-mar* uš /
5-7 igi 48 duḫ.a 1 15 *ta-mar a-na* 28 48 / *i-ši* 36 <*ta-mar*> 40 *a-na* 36 *i-ši* 24 / *ta-mar*
8 $^{1}/_{2}$ 24 *ḫe-pé* 12 *ta-mar* igi.˹3˺.[gál] <ba.zi> / [4] *ta-mar*
9 igi 4 duḫ.a 15 *ta-mar* / 24 *a-na* 15 *i-ši* 6 *ta-mar* sag /
10 *ki-a-am ne-pé-šum*

1 A grain heap. 4 the ridge, 28 48 gur the grain, 48 the height,
2 In 1 cubit, / ˹7 30, cubits,˺ the food. The length and the front, what?
3-4 You: / [7 30, the food] to 48, the height carry, 6 you will see. / To 4, the ridge add it on, 10 you will see, the length. /
5-7 The reciprocal of 48 release, 1 15 you will see. to 28 48 / carry it, 36 <you will see>. 40 to 36 carry, 24 / you will see.
8 1/2 of 24 break, 12 you will see. The 3rd [part] <tear off> / [4] you will see.
9 The reciprocal of 4 release, 15 you will see. / 24 to 15 carry, 6 you will see, the front.
10 Such is the procedure.

In this exercise, the question can be reformulated, in quasi-modern symbolic notations, as

$r = 4, \quad C = \text{c} \cdot 1/2\ (2\ u + r) \cdot s \cdot 1/3\ h = 28\ 48, \quad h = 48, \quad f = (u - r)/h = 7\ 30.$

Indeed, the šag$_4$.gal 'food' is a term used in Babylonian mathematics for the "rate of growth" or simply "inclination" o-5-5f an oblique part of a three-dimensional object. (Cf. the survey in Neugebauer and Sachs, *MCT*, 81, fn. 191.) It is measured in *so and so many horizontal nindan per 1 cubit of descent or ascent*. In this exercise, the rate of growth '7 30' of the sloping sides of the ridge pyramid has to be interpreted as ;07 30 nindan per cubit = 1 1/2 cubit per cubit.

The solution procedure starts with the multiplication of the 'food' f = 7 30 by the height 48, which gives

$u - r = f \cdot h = 7\ 30 \cdot 48 = 6,$ so that $u = r + f \cdot h = 4 + 6 = 10$ (lines 3-4)

Next is computed, as usual,

$1/2\ (2\ u + r) \cdot 1/3\ s = \text{rec. } c \cdot \text{rec. } h \cdot C = 40 \cdot 1\ 15 \cdot 28\ 48 = 24$(lines 5-6)

Somewhat confusingly, since u = 10 and r = 4, also 2 u + r is equal to 24. Therefore, the next steps of the solution procedure must be understood as meaning that

$24 = 1/2\ (2\ u + r) \cdot 1/3\ s = 1/2 \cdot 24 \cdot 1/3\ \text{s} = 4\ s$ so that $s = \text{rec. } 4 \cdot 24 = 6$ (lines 7-9)

BM 96954+ § 1 l (*rev. i*: 3-15)

1 guru$_7$ 10 uš 48 sukud 28 48 še gur /
2 *ma-la* sagšu *ù* $^1/_3$ sag$^!$ ul.gar 6 sagšu sag /
3-4 za.e igi 48 sukud duḫ.<a> 1 15 *ta-mar* 1 15 *a-na* / 28 48 *i-ši* 36 *ta-mar*
5-6 40 *a-na* / 36 *i-ši* 24 *ta-mar* 1 30 *ù* 30 / bala gar.ra
7 igi.3.gál 10 uš ba.zi 3 20 *ta-mar* / $^1/_3$ 1 30 ba.zi 30 *ta-mar*
8 igi.3.gál 30 ba.zi / 10 *ta-mar* 1 30 *a-na* 10 *i-ši* 15 *ta-mar* /
9-11 1 30 *a-na* 3 20 *i-ši* 5 *ta-mar* 24 / *a-na* 15 *i-ši* 6 *ta-mar* 5 *i-na* 6 ba.zi / 1 *ta-mar*
12 igi 10 duḫ.a 6 *ta-mar a-na* 1 *i-ši* / 6 *ta-mar* sag 40 *a-na* 6 sag *i-ši* 4 *mu-ḫu* /
13 *ki-a-am ne-pé-šum*

1 A grain heap. 10 the length, 48 the height, 28 48 gur the grain. /
2 Whatever the ridge and 1/3 the front added together, 6. The ridge and the front?
3-4 You: The reciprocal of 48, the height, release, 1 15 you will see, 1 15 to / 28 48 carry, 36 you will see.
5-6 40 to / 36 carry, 24 you will see. 1 30 and 30, the fractions, set down.
7 The 3rd part of 10, the length tear off, 3 20 you will see. / 1/3 of 1 30 tear off, 30 you will see.
8 The 3rd part of 30 tear off, / 10 you will see. 1 30 to 10 carry, 15 you will see. /
9-11 1 30 to 3 20 carry, 5 you will see. 24 / to 15 carry, 6 you will see. 5 from 6 tear offf / 1 you will see.
12 The reciprocal of 10 release, 6 you will see, to 1 carry / 6 you will see, the front. 40 to 6, the front, carry, 4 the top.
13 Such is the procedure.

The question in this paragraph can be reformulated in the following form:

$u = 10, \quad h = 48, \quad C = \text{c} \cdot 1/2\ (2\ u + r) \cdot s \cdot 1/3\ h = 28\ 48, \quad r + 1/3\ s = 6, \quad r, s = ?$

The solution procedure begins in the usual way by computing

$\text{rec. c} \cdot \text{rec. } h \cdot C = 1/2\ (2\ u + r) \cdot 1/3\ s = 40 \cdot 1\ 15 \cdot 28\ 48 = 24$ (lines 3-6)

In this way, the question is simplified to the following pair of equations for r and s:

$1/2\ (20 + r) \cdot 1/3\ s = 24, \quad r + 1/3\ s = 6.$

With, for instance, the new unknowns $20 + r = p$ and $1/3\ s = q$, this pair of equations can be simplified to

$p \cdot q = 48, \quad p + q = 26.$

This is *a rectangular-linear system of equations for two unknowns*. However, the solution procedure presented in the text of § 1 l shows no trace of the normal kind of solution procedure for such a system of equations. Even the final calculation of the ridge (here called the 'top') is totally unmotivated. Apparently, the student who wrote this final exercise in § 1 of BM 96954+ must have been sleeping when the teacher demonstrated how the question in § 1 l could be answered. As a matter of fact, of the seven preserved exercises in BM 96954+ § 1, four have solution procedures that are more or less inadequate!

8.5.2 § 2. The Volume of a Truncated Ridge Pyramid

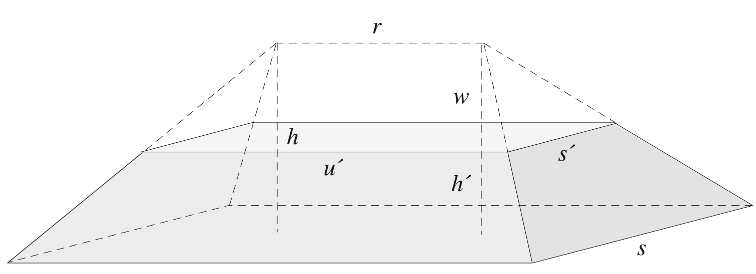
BM 96954+ § 2 (*obv. ii*: 1-8)

1-2 guru$_7$ 10 uš 6 sag 4 sagšu 28 48 <še gur> / 48 sukud
24 *ur-dam* dal *ù še-um* en.nam /
3 [z]a.e igi 48 sukud duḫ.a 1 15 *ta-mar*
4 1 15 *a-na* / [4 dir]ig *ša* uš ugu sag$^!$ *i-ši* 7 30 *ta-mar* /
5 [7 30 *a-na*] 24 *i-ši* 3 *ta-mar*
6 3 *i-na* 10 uš / [ba.zi 7 *ta-mar* 7 d]al
7 3 *i-na* 6 sag ba.zi / [3 *ta-mar* 3 *pé-er-kum*]
[uš *a-na*] sag *i-ši* 1 *ta-mar* /
8-9 [dal *a-na pé-er-kim i-ši*] ⌜2⌝1 / [*ta-mar*]
… ………………

1-2 A grain heap. 10 the length, 6 the front, 4 the ridge, 28 48 <gur the grain>, / 48 the height.
24 I went down. The transversal and the grain were what? /
3 [Y]ou: The reciprocal of 48, the height, release, 1 15 you will see.
4 1 15 to / [4 the ex]cess that the length beyond the front$^!$ carry, 7 30 you will see. /
5 [7 30 to] 24 carry, 3 you will see.
6 3 from 10, the length / [tear off, 7 you will see, 7 the trans]versal.
7 3 from 6 tear off, / [3 you will see, the crossline].
[The length to] the front carry, 1 you will see. /
8-9 [The transversal to the cross-line carry] 21 / [you will see].
… ………………

The question in this paragraph begins by specifying *all* the relevant parameters for a ridge pyramid, and then adds the information that 24 *ur-dam* '24 I went down'. As unknowns to be calculated are mentioned dal 'the transversal' and *šeum* 'the grain'. The situation is known from several known Old Babylonian mathematical texts. Thus, for instance, in MS 3049 § 1 a (Friberg, *MSCT I* (2007), 296 and in BM 85194 ## 21-22 (*op. cit.*, 298), a transversal in a circle is reached by going down a certain distance from the top of the circle (to the left in the diagram). In Str. 364 § 3 (*op. cit.*, 273), a transversal in a triangle is reached by going down a certain distance from the top (that is, the front) of a triangle (again, to the left in the diagram).

Against this background, it is reasonable to assume that in BM 96954+ § 2, the object considered is *a truncated ridge pyramid* as in Fig. 8.5.2 below, where the top of the truncated ridge pyramid is situated 24 (cubits) below the ridge of the ridge pyramid, and where the 'transversal(s)' are the length (u') and front (s') of the rectangular top of the truncated ridge pyramid. The grain which is asked for is, presumably, the seed measure (C') of the truncated ridge pyramid. The height of the truncated pyramid is, of course, $h' = h - w = 48 - 24 = 24$. Note that his means that *the ridge pyramid has been cut off at mid-height!*



$u = 10$ nindan, $s = 6$ nindan, $r = 4$ nindan
$h = 48$ cubits, $w = 24$ cubits
$f = (u - r)/h = ;07\ 30$

$u' = u - f \cdot w = 7$ nindan, $s' = s - f \cdot w = 3$ nindan
$h' = 24$ cubits
$C' = 23\ 24\ 00$ gur

$C' = c \cdot \{(u \cdot s + u' \cdot s') + 1/2\,(u \cdot s' + u' \cdot s)\} \cdot 1/3\ h'$

Fig. 8.5.2. BM 96954+ § 2. The computation rule for the seed measure of a truncated ridge pyramid.

The given parameters for the ridge pyramid are, precisely as in § 1 above, $u = 10$, $s = 6$, $r = 4$, $h = 48$, and consequently, $C = 28\ 48$. The given distance went down or descended is $w = 24$. (w here stands for *wārittu* 'descent'. Both *urdam* and *wārittu* are derived from the verb *warādu* 'to go down'.)

The solution procedure (what is left of it) begins by computing the 'food' f of the ridge pyramid, as

$f = (u - r)/h = (10 - 4)/48 = 1\ 15 \cdot 6 = 7\ 30$ (lines 3-4)

This is, of course, the same value for f as in § 1 k above. Next, f is multiplied by the height of the truncated ridge pyramid, $h' = 48 - 24 = 24$. The product, $f \cdot h' = 7\ 30 \cdot 24 = 3$ will be the rate of growth for both the triangular "front face" and the trapezoidal "length face" of the ridge pyramid, resulting from the rise from the bottom to the top of the ridge pyramid. Consequently,

$$u' = u - f \cdot w = 10 - 3 = 7, \qquad s' = s - f \cdot w = 6 - 3 = 3 \qquad \text{(lines 5-6)}$$

Now all the parameters for the truncated ridge pyramid are known or have been calculated, namely the bottom length u and the bottom front s, the height h', the top 'transversal' u', and the top 'cross-line' s'. (In several known Old Babylonian mathematical texts, including some tables of constants, a *perku* 'cross-line' is a transversal orthogonal to the main transversal, called dal).

It remains to calculate the volume V and the seed measure C' of the truncated ridge pyramid. This calculation begins in the last preserved lines of § 2 with the multiplications

$$u \cdot s = 10 \cdot 6 = 1, \quad \text{and} \quad [u' \cdot s' = 7 \cdot 3] = 21.$$

Now, it can be shown (see Sec. 8.5.10 below) that a correct computation rule for the seed measure of a truncated ridge pyramid like the one in Fig. 8.5.2 above is

$$C' = \text{c} \cdot V', \quad \text{where } V' = \{(u \cdot s + u' \cdot s') + 1/2\,(u \cdot s' + u' \cdot s)\} \cdot 1/3\ h'.$$

It is clear that the computation of the products $u \cdot s$ and $u' \cdot s'$ in the last preserved lines of the solution procedure in § 2 of BM 96954+ were the first two steps in an application of this computation rule. Without doubt, the computation would have continued as follows:

$$u \cdot s' = 10 \cdot 3 = 30, \quad u' \cdot s = 7 \cdot 6 = 42, \quad (u \cdot s + u' \cdot s') + 1/2\,(u \cdot s' + u' \cdot s) = 1\ 21 + 1/2 \cdot 1\ 12 = 1\ 57,$$
$$V' = 1\ 57 \cdot 8 = 15\ 36, \quad C' = 1\ 30 \cdot 15\ 36 = 23\ 24\ (00).$$

8.5.3 *§ 3. The Seed Measure of a Square Pyramid*

BM 96954+ § 3 (*obv. iii*: 9- …)

1 ⌜guru$_7$⌝ [9 kùš.ta].am *im-ta-ḫar* 6 sukud *še-um* en.nam /
2 [za.e igi 12 duḫ.a] 5 *ta-mar* 10 *a-na* 5 *i-ši* 50 *ta-mar* /
3 [45 *šu-tam-ḫir*] ⌜3⌝3 45 *ta-mar* gagar /
4 [*a-na* 6 *i-ši* 3 22] 30 *ta-mar* saḫar.ḫá /
5 [3 22 30 *a-na* 1 30 *i-ši*] ⌜5 03 45⌝ [*ta-mar še-um*]

1 ⌜A grain heap⌝. [9 cubits each] way it was equalsided, 6 the height. The grain was what? /
2 [You: The reciprocal of 12 release,] 5 you will see. 10 to 5 carry, 50 you will see. /
3 [45 make equalsided], ⌜3⌝3 45 you will see, the ground.
4 [To 6 carry it, 3 22] 30 you will see, the mud (volume) /
5 [3 22 30 to 1 30 carry] ⌜5 03 45⌝ [you will see, the grain].

The text of this exercise is badly preserved, but the question seems to ask for the seed measure of some three-dimensional object with a square base of side [x] and the height 6. Presumably, that object is *a square pyramid*. If that is so, then the proper computation rule would be

$$C = \text{c} \cdot \text{sq. } s \cdot h/3, \quad \text{where } s \text{ is the side of the base, and } h \text{ is the height.}$$

The reconstruction above of the lost part of the text is based on the assumption that the 'food' f of the square pyramid, that is its rate of growth, is '7 30', just as in the case of the ridge pyramids §§ 1 and 2 of BM 96954+. More precisely,

$$f = ;07\ 30 \text{ nindan (horizontally) per cubit (vertically)} = 1\ 1/2 \text{ cubit per cubit.}$$

From this assumption, it follows immediately that the side s of the square base of the pyramid must be

$$s = 1\ 1/2 \text{ cubit per cubit} \cdot 6 \text{ cubits} = 9 \text{ cubits} = 9 \cdot ;05 \text{ nindan} = ;45 \text{ nindan.}$$

Then the area A of the base of the square pyramid, and the volume V will be, respectively

$$A = \text{sq. } ;45 \text{ nindan} = ;33\ 45 \text{ sq. nindan} = ;33\ 45 \text{ area-sar}, \quad \text{and} \quad V = 6 \text{ cubits} \cdot ;33\ 45 \text{ area-sar} = 3;22\ 30 \text{ volume-sar.}$$

These results agree with what is left of the text on lines 3-4 of the exercise.

Consequently, the answer to the stated question will be that the seed measure of the square pyramid is

C = 3;22 30 volume-sar · 1 30 gur/volume-sar = 5 03;45 gur.

Note the error in line 2 of the text, where 5 is multiplied by 10 instead of by 9. *In spite of this error, the text continues correctly and apparently arrives at the correct answer!*

An isolated pair of exercises in the large mathematical recombination text BM 96957 + VAT 6598 (Robson, *MMTC*, App. 4; Group 6, Sippar) is also concerned with a square pyramid, although in terms of brick measure, not seed measure.

BM 96957+ # 6 (*obv. ii*: 5-8)

1 5 sar lagab 2 1/2 nindan sukud sig$_4$ en.nam *lu-<ša>-al-bi-in*
2-3 za.e / igi.3.gál 5 sar *le-qé* 1 40 *ta-mar* é.gar8-*tum* 1 40 *a-na* 2 1/2 nindan sukud / *i-ši* 4 10 *ta-mar*
4 2(iku) 1/2(iku) ašag sig$_4$ *tu-ša-al-ba-<an>-ma* 5 sar lagab/ a.na 2 1/2 nindan sukud *ta-am-mar*
ne-pé-šum

1 5 sar square, 2 1/2 nindan the height. The bricks, what should I let be molded?
2-3 You: / The 3rd part of 5 sar take, 1 40 you will see the wall(?). 1 40 to 2 1/2 nindan, the height, / carry, 4 10 you will see.
4 2 1/2 iku sig$_4$ you will let be molded, then 5 sar square, / as much as 2 1/2 nindan height you will see.
The procedure.

BM 96957+ # 7 (*obv. ii*: 9-12)

1 *šum-ma* 2(iku) 1/2(iku) ašag sig$_4$ 2 1/2 nindan sukud lagab en.nam *e-pu-uš*
2 za.e / [igi 2 30 duḫ.a] ˹24˺ *a-na* 4 10 2(iku) 1/2(iku) ašag sig$_4$ *i-ši* 1 40 *ta-mar* /
3 [*ki-il* igi 20 igi.3.gál] duḫ.a 3 *ta-mar* 3 *a-na* / [1 40 *ša tu-ka-lu i-ši* 5 *ta-mar*] .]
4 ˹5 sar lagab˺ / [*ne-pé-šum*|

1 If 2 1/2 iku of bricks, 2 1/2 nindan the height, the square, what did I make?
2 You: / [The reciprocal of 2 30 release] ˹24˺ to 4 10 of 2 1/2 iku of bricks carry, 1 40 you will see. /
3 [Hold. The reciprocal of 20, the 3rd part] release, 3 you will see. 3 to / [140 that you held carry, 5 you will see
4 ˹5 sar square˺. [The procedure.]

In the first of this pair of exercises, a square (base) of area measure *A* = 5 sar is given, together with a height of 2 1/2 nindan. The question is, how many bricks should be made? The solution procedure begins by multiplying the given area by 1/3 and then the result by 2 1/2 nindan, the height. This means that a volume *V* is computed as

V = *A* · 1/3 · *h* = 5 area-sar · 1/3 · 2 1/2 nindan = 1 40 area-sar · 2 1/2 nindan = 4 10 (area-sar · nindan) (lines 2-3)

Obviously, the way in which the volume is computed reveals that the object considered is *a square pyramid*. However, something is not right here, because normally the height would have been converted from a nindan multiple to a cubit multiple. Anyway, the text continues by stating without further computations that

4 10 (area-sar · nindan) = 2 1/2 iku of bricks. (line 3)

Compare this with the expected computation, which would have proceeded as follows:

2 1/2 nindan = 2 1/2 · 12 cubits = 30 cubits,
V = *A* · 1/3 · *h* = 5 area-sar · 1/3 · 30 cubits = 1 40 area-sar · 30 cubits = 50 volume-sar,
B = L · *V* = L · 50 brick-sar, where L is the "molding number" for the kind of bricks considered.

An explanation for the curious "error" mentioned above is that if the silently assumed molding number in this text was L = 5 (brick-sar per volume-sar), then

V = *A* · 1/3 · *h* = 1 40 area-sar · 2 1/2 nindan = 1 40 · 2 30 · 12 volume-sar
= 1 40 · 2 30 · 12 · 5 brick-sar = 4 10 brick-sar = 2 1/2 brick-iku = *B*.

This means that when L = 5 it is possible to skip a pair of steps in the computation, namely the conversion of nindan into cubits and the conversion of volume-sar into brick-sar! The author of the text chose to use this shortcut. (An alternative explanation would be to assume that a couple of lines are missing in the solution procedure, but in view of the second of the mentioned pair of exercises, this explanation is not valid.)

Indeed, in the second exercise, the shortcut equation

$$B = \text{L} \cdot V = \ A \cdot 1/3 \cdot h\ (\cdot 1\ (00))$$

is used to find A when B and h are given. In this way, the simplified solution procedure is as follows:

$$A \ = B/(1/3 \cdot h) = \text{rec. } 20 \cdot \text{rec. } 2\ 30 \cdot 4\ 10 = 3 \cdot 24 \cdot 4\ 10 = 3 \cdot 1\ 40 = 5.$$

Rectangular bricks of type R3n with the molding number 5 appear in two other known Old Babylonian mathematical texts. (See Friberg, *ChV* (2001), 76). One is the brief catalog text *MCT* O = YBC 4607, where rectangular bricks of type R3n occur together with ordinary rectangular bricks of type R1/2c, half-bricks of type H 2/3c, and square bricks of types S 2/3c and S1c. The other is AO 10822 (Friberg, *op. cit.*, 90), a fragment of a very interesting catalog text, where rectangular bricks of type R3n occur together with half-bricks of type H4n, and square bricks of types S3n and S4n.

Note that the square pyramid in BM 96957+ ## 6-7 is much larger than the square pyramid in BM 96954+ § 3. Its base area is 5 area-sar (square nindan). Since 5 is not a square number, the side of the square base can only be given approximately, as 2 nindan 3 cubits (= 27 cubits or 13 1/2 meters). Indeed, sq. 2;15 nindan = 5;03 45 area-sar. The height of the square pyramid is 2 1/2 nindan (= 30 cubits = 15 meters). The number of bricks required to build the pyramid is 4 10 brick-sar = 4 10 · 12 00 = 50 00 00 (= 180,000).

8.5.4 § 4. The Seed Measures of Various Solids

BM 96954+ § 4 a (*rev. i*: 16-20)

1 guru$_7$ sag.kak 30 uš 10 sag 48 sukud / *še-um* en.nam
2 za.e 30 uš *a-na* 10 sag <*i-ši*> {*ta-mar*} / 5 *ta-mar a-na* 48 sukud *i-ši* 4 *ta-mar* /
3 1 30 *a-na* 4 *i-ši* 6 *ta-mar* 6 šár še gur /
4 *ki-a-am ne-pé-šum*

1 A grain heap, peg-head (triangular). 30 the length, 10 the front, 48 the height. / The grain, what?
2 You: 30, the length, to 10, the front, <carry>, / 5 you will see. To 48, the height, carry it, 4 you will see. /
3 1 30 to 4 carry, 6 you will see, 6 šár gur of grain.
4 Such is the procedure.

Forgetting that the 'grain heap' was supposed to be *a prism with a triangular base*, the author of this exercise counts as follows:

$$V = u \cdot s \cdot h = 30 \cdot 10 \cdot 48 = 5\ (00) \cdot 48 = 4\ (00\ 00)\ \text{(volume-sar)}, \quad C = 1\ 30 \cdot 4\ (00\ 00) = 6\ (00\ 00\ 00)\ \text{(gur)}.$$

In the text, the computed number '6' is explained as '6 šár gur of grain'. This is not correct, the correct answer should have been '6 šár.gal gur of grain', where 1 šár.gal = 1 00 šár. Note, by the way, that since 6 šár.(gal) gur is *a very big seed measure,* it is imperative to interpret 30 uš 10 sag 48 sukud as the *large length measures* '30 (nindan) the length, 10 (nindan) the front, 48 (cubits) the height', rather than the diminutive length measures ';30 (nindan) the length, ;10 (nindan) the front, ;48 the height'!

BM 96954+ § 4 b (*rev. i*: 21-27)

1-2 guru$_7$ 30 uš 20 sag an.na 10 sag ki.ta / 48 sukud saḫar.ḫá *ù še-um*
3 za.e sag an.na / *ù* sag ki.ta ul.gar 30 *ta-mar*
4 1/2 30 *ḫe-pé* / 15 *ta-mar* 15 *a-na* 30 *i-ši* 7 30 *ta-mar* /
5 7 30 *a-na* 48 sukud *i-ši* 6 *ta-mar* saḫar.ḫá /
6 [1] 30 *a-n*a 6 *i-ši* 9 *ta-mar še-um* /
7 *ki-a-am ne-pé-šum*

1-2 A grain heap. 30 the length, 20 the upper front, 10 the lower front / 48 the height. The volume and the grain?
3 You: The upper front / and the lower front heap, 30 you will see.
4 1/2 of 30 break / 15 you will see. 15 to 30 carry, 7 30 you will see. /
5 7 30 to 48, the height, carry, 6 you will see, the mud (volume). /
6 [1] 30 to 6 carry, 9 you will see, the grain. /
7 Such is the procedure.

In this exercise is considered *a prism with a trapezoidal base*. The length of the base is u = 30, the upper and lower fronts of the base are s = 20 and s' = 10, respectively, and the height is h = 48. The volume V and the seed measure C can then be calculated as

$V = u \cdot 1/2\,(s + s') \cdot h$ = 30 · 15 · 48 = 6 (00), C = 1 30 · 6 (00) = 9 (00 00).

BM 96954+ § 4 c (*rev. i*: 8-…)

1-2 [guru$_7$ ašag.u$_4$.sakar] 30 gúr 20 dal 48 sukud / [saḫar.ḫá *ù še-um* en.nam]
3 [za].e 30 uš *a-na* 20 dal / [*i-ši* 10 *ta-mar*]
4 [*a-na* 15 igi].gub.ba ašag.u$_4$.sakar / [*i-ši* 2 30 *ta-mar*]
[5] [2 30 *a-na* 48] ⸢sukud⸣ / [*i-ši* 2 *ta-mar* 2 saḫar.ḫá] /
[6] [2 *a-na* 1 30 *i-ši* 3 *ta-mar še-um*]
[*ne-pé-šum*]

1-2 [A grain heap, a crescent-field.] 30 the arc, 20 the transversal, 48 the height. / [The mud (volume) and the grain, what?]
3 [Y]ou: 30, the length, to 20 the transversal, / [carry, 10 you will see].
4 [to 15, the constant for a crescent-field / [carry, 2 30 you will see].
[5] [2 30 to 48], ⸢the height⸣, / [carry, 2 you will see, 2, the mud (volume)]. /
[6] [2 to 1 30 carry, 3 you will see, the grain].
[The procedure]

ašag$_4$.u$_4$.sakar or *uskāru* 'crescent(-field)' is the name used in Old Babylonian mathematical texts for semicircles. Constants for semicircles appear in, for instance, the table of constants BR = *TMS III* (Bruins and Rutten (1961)), where one finds the following three consecutive constants:

15 igi.gub	*šà ús-ka$_4$-ri*	15, the constant	of a crescent	BR 7
40 dal	*šà ús-ka$_4$-ri*	40, the transversal	of a crescent	BR 8
20 *pe-er-ku*	*šà ús-ka$_4$-ri*	20, the cross-line	of a crescent	BR 9

Here, as always, the 'constant' for a geometric object means its area, in the normalized case when its defining parameter is equal to '1'. In the case of circles and semicircles, the defining parameter is the length of the arc. In § 4 c of BM 96954+, the arc of the semicircle is called gúr 'arc'. Now, if the arc, the 'transversal' (diameter), and the 'cross-line' (radius) are called a, d, and p, respectively, and if A stands for the area, as usual, then the meaning of the three lines BR 7-9 is that

$A = {;}15 \cdot a \cdot d, \quad d = {;}40 \cdot a, \quad p = {;}20 \cdot a.$

It is not difficult to see that these equations are correct if 3 is used as an approximation to π. In § 4 c above, a = 30, and therefore d = 20, and

A = 30 · 20 · ;15 = 10 · 15 = 2 30 (lines 2-4)

If this area is understood to be the bottom area of a half circular cylinder with the height h = 48, then the volume and the seed measure of the half-cylinder can be computed as

$V = A \cdot h$ = 2 30 · 48 = 2 (00 00), C = 1 30 · 2 (00 00) = 3 (00 00 00) (lines 4-6)

8.5.5 § 5. The Quadratic Growth Rate of a Circular Cone

BM 96954+ § 5 a (*rev. i*: 1′-5′)

1-2 [*gam-ru-tum* 30 gúr 1 sukud] *i-na* 1 kùš / [šag$_4$.gal en.nam]
3 [za].⸢e⸣ igi 1 sukud duḫ.a 1 *ta-mar* / [1 *a-na* 30 gúr *i*]-*ši* 30 ⸢*ta-mar*⸣
4 1/2 30 *ḫe-pé* / [15 *ta-mar*] 15 *i-na* 1 [kùš] šag$_4$.gal /
5 [*ki-a-a*]*m ne-pé-šum*

1-2 [A whole (cone). 30 the arc, 1 the height], in 1 cubit / [the food was what?]
3 [You): The reciprocal of 1, the height, release, 1 you will see. / [1 to 30, the arc, ra]ise, 30 you ⸢will see⸣.
4 1/2 of 30 break, / [15 you will see]. 15 in 1 [cubit] is the food. /
5 [Suc]h is the procedure.

The *gamr¥tu* 'whole (cone)' in § 5 a has an 'arc' (a circular arc), a height, and a 'food' (probably, as before, a rate of growth). Reasonably, therefore, it must be a circular cone. The arc is a = 30 and the height is h = 1 (00). Asked for is the size of the 'food' f.

Now, in §§ 1-3 above of BM 96954+, the ‘food’ f was always equal to the “rate of growth”, more precisely the increase of a length or a front for each descent of 1 cubit, or decrease for each ascent of 1 cubit. This was equally true in the cases of a ridge pyramid, a truncated ridge pyramid, and a square pyramid. However, here in the case of a cone, the definition of its ‘food’ seems to be different. Indeed, in lines 2-4 of § 5 a the ‘food’ is computed as follows:

f = 1/2 rec. $h \cdot a$ = 1/2 rec. 1 · 30 = ‘15’.

An effort to explain where this strange definition comes from will be postponed until after the examination of also §§ 5 b-d.

BM 96954+ § 5 b (*rev. i*: 6′-9′)

1-2 [*gam*]*-ru-tum* 30 gúr *i-na* 1 kùš 15 šag$_4$.gal / sukud en.nam
za.e 15 šag$_4$.gal tab.ba 30 *ta-mar* /
3-4 igi 30 duḫ 2 *ta-mar* 30 gúr *ana* 2 *i-ši* / 1 *ta-mar* sukud
ne-pé-šum

1-2 [A who]le (cone). 30 the arc, in 1 cubit 15 is the food. / The height, what?
You: 15, the food, double, 30 you will see. /
3-4 The reciprocal of 30 release, 2 you will see. 30, the arc, to 2 carry, / 1 you will see, the height.
The procedure.

The *gamr¥tu* ‘whole (cone)’ figuring in § 5 b is clearly the same as in the preceding exercise. The arc a = 30 and the food f = 15. Asked for is the height h, which is computed as follows:

h = rec. (2 f) · a = rec. (2 · 15) · 30 = ‘1’,

from which it follows that also in § 5 b

f = 1/2 rec. $h \cdot a$.

BM 96954+ § 5 c (*rev. ii*: 1-5)

1 [*gam-ru-tum* 1 sukud *i-na* 1 kùš 15 šag$_4$.ga]l /
2-3 [*ma-ṣa-rum* en.nam za.e 15 šag$_4$.gal] gar.ra / [1 sukud gar.ra]
4 [1 *a-na*] 15 *i-ši* 15 *ta-mar* / [15 íb.sá en.nam 30] íb.sá 30 nindan gúr /
5 [*n*]*e-pé-šum*

1 [A whole (cone). 1 the height, in 1 cubit 15 is the foo]d. /
2-3 [The go-around, what? You: 15, the food] set down, / [1, the height, set down.]
4 [1 to] 15 carry, 15 you will see. / [15 equalsided is what? 30] equalsided, 30 nindan is the arc. /
5 [The p]rocedure.

More than half the text of §5 c is lost, so the reconstruction above of the lost text is only tentative. Anyway, it feels safe to assume that in the same cone as before the height is known, h = 1 (00), and also the food, f = ;15. What is asked for is the ‘go-around’, that is, presumably, the arc a of the circular base of the cone. The arc seems to be computed in the following way:

a = sqs. ($f \cdot h$) = sqs. (15 · 1) = ‘30’.

Consequently, in § 5 c, the food f seems to be related to a and h in the following way:

f = rec. h · sq. a.

BM 96954+ § 5 d (*rev. ii*: 6-12)

1-2 [*gam-ru-tum*] 25 saḫar.ḫá *ma-ṣa-rum ù di-ik-šum* / [en.na]m
za.e 25 *šu-ul-li-iš* 1 15 *ta-mar* /
3-4 igi 5 igi.gub duḫ.a 12 *ta-mar* 12 *a-na* 1 15 *i-ši* / 15 *ta-mar* gar.ra
5 nigín.na igi 15 šag$_4$.gal duḫ.a 4 *ta-mar* / 15 *a-na* 4 *i-ši* 1 *ta-mar*
6 1 en.nam íb.sá 1 íb.sá / sukud
7 15 šag$_4$.gal tab.ba 30 *ta-mar* 30 *a-na* 1 sukud *i-ši* / 30 *ta-mar ma-ṣa-rum*
ne-pé-šum

1-2 [A whole (cone).] 25 the mud (volume). The go-around and the stick-out, / [wh]at?
You: 25 triple, 1 15 you will see. /
3-4 The reciprocal of 5, the constant, release, 12 you see. 12 to 1 15 carry, / 15 you will see. Set it down.

5 Turn around. The reciprocal of 15, the food, release, 4 you will see. / 15 to 4 carry, 1 you will see.
6 1 is what equalsided? 1 equalsided, / the height.
7 15, the food, double, 30 you will see. 30 to 1, the food, carry, / 30 you will see, the go-around.
The procedure.

In § 5 d, the volume of a [‘whole (cone)’] is given, V = ‘25’. In addition, it is silently assumed to be known that the šàg.gal ‘food’ f is equal to ‘15’ as in the preceding paragraphs (line 4). Asked for are the *ma-ṣa-rum* ‘go-around’ and *di-ik-šum* ‘the stick-out’, whatever that may mean.

Now, if the object considered is a circular cone, then

$V = \text{c} \cdot \text{sq. } a \cdot 1/3\, h$, where a is the arc of the circular bottom, h is the height, and c = ‘5’.

Accordingly, the solution procedure begins with the computation of

$\text{sq. } a \cdot h = \text{rec. c} \cdot 3\, V = 12 \cdot 3 \cdot 25 = 15$ (lines 2-4)

Next it is mentioned that f = ‘15’, so that rec. $f = 4$ and

$\text{rec. } f \cdot \text{sq. } a \cdot h = 4 \cdot 15 = 1$ (lines 4-5)

Next it is claimed that the square-side of 1 is the height h. This implies that

$\text{rec. } f \cdot \text{sq. } a \cdot h = 4 \cdot 15 = 1 = \text{sq. } h$ (lines 5-6)

In other words, the computation so far has shown that

h = ‘1’ and $h = \text{rec. } f \cdot \text{sq. } a$ which means that $f = \text{rec. } h \cdot \text{sq. } a$

Therefore, surprisingly, the term šàg.gal ‘food’ does not have the same meaning in BM 96954+ § 5 d as it has in BM 96954+ §§ 1-3! Neither does it seem to have the same meaning as in §§ 5 a-c! Indeed, in § 1 k, the ‘food’ was the rate of increase of *the length of a ridge pyramid* for each descent of 1 cubit from the top of the pyramid (that is, the ridge). In § 5d, the ‘food’ seems to be the increase of *the square of the circumference of a circular cone* for every descent of 1 cubit below the top of the cone!

After the calculation of the height of the cone in lines 2-6, the solution procedure continues as follows:

$m = 2\, f \cdot h = 2 \cdot 15 \cdot 1 = 30$, where m is called *ma-ṣa-rum* ‘the go-around’ (lines 6-7)

This calculation seems pointless, until one recalls that *it is not uncommon in Old Babylonian mathematical exercises that a student has misunderstood the meaning of a computation that a teacher has instructed him to carry out*. In the present case, the computation in lines 6-7 was meant to be a calculation of a as the square-side of sq. $a = f \cdot h = 15$, which is 30. Instead the student explained the calculation as $2 \cdot 15$ is= 30!

This observation also explains why in § 5a, incorrectly,

$f = 1/2 \text{ rec. } h \cdot a$ or, equivalently, $a = 2\, f \cdot h$

and in § 5 b, also incorrectly,

$h = \text{rec. } (2\, f) \cdot a$ or, again, $a = 2\, f \cdot h$

while in § 5 c, correctly,

$a = \text{sqs. } (f \cdot h)$.

Thus, apparently, the definition of the ‘food’ f which was employed, more or less successfully, in §§ 5 a-d of BM 96954+ was that

$f = \text{rec. } h \cdot \text{sq. } a$ = *the increase in the circumference squared of the circular cone for each descent of 1 cubit.*

(Except for the omitted constant c = ;05, this is the same as *the increase in the area of a cross section of the circular cone for each descent of 1 cubit*.) With this kind of definition of f, the ‘food’ in § 5 of BM 96954+ may be called the “quadratic growth rate” of the circular cone.

Anyway, with $h = 1$ and $f = \text{rec. } h \cdot \text{sq. } a = \text{rec. } 1\,(00) \cdot \text{sq. } 30 = 15$, what should have been calculated in lines 6-7 of § 5 d was

$a = \text{sqs. } (f \cdot h) = \text{sqs. } (1\,(00) \cdot 15) = 30$.

It is easy to check that a circular cone with the arc at the bottom equal to 30 nindan (= 180 meters) and with the height equal to 1 (00) cubits (= 50 meters!) is

V = c · sq. a · 1/3 h = ;05 · sq. 30 · 20 = 5 · 15 · 20 = 25 (00) (volume-sar).

Since the result of the computation in § 5 d is called *ma-ṣa-rum* 'the go around', it is now confirmed that 'go-around' was a name for the arc or circumference of the circular base of the cone. It is then also obvious that *di-ik-šum* 'the stick-out' was a similarly descriptive name for the height of the circular cone, so that the question at the beginning of § 5d 'What is the go-around and the stick-out ?' can be understood as meaning 'What is the circumference and the height of the circular cone?'!

8.5.6 *§ 6. The Volume of a Circular Cone Truncated at Mid-Height*

BM 96954+ § 6 (*rev. ii*: 13-23)

1-2 [*x x x*] 30 *ma-ṣa-rum* 1 sukud 25 saḫar.ḫá 2 1/2 nindan / [*ur-dam*] saḫar.ḫá en.nam
3 za.e igi 1 sukud duḫ.a / [1 *ta-mar* 1 *a-na*] 30 *ma-ṣa-rum i-ši* 30 *ta-mar*
4-5 30 *a-na* / [30 sukud ki.ta *i*]-*ši* 15 *ta-mar* 15 *i-na* 30 *ma-ṣa-rum* / [ba.zi 15 *ta-mar*]
[sa]ḫar.ḫá ki.ta <en.nam>
6 30 *ma-ṣa-rum* ki.ta / [nigin 15 *ta-mar a-na* 5 ig]i.gub.ba *i-ši* 1 15 *ta-mar* /
7-8 [15 *ma-ṣa-rum* an.ta nigin 3 4]5 *ta-mar* 3 45 *a-na* / [5 igi.gub.ba *i-ši* 18 45 ta-mar]
9 [1 1]5 *ù* 18 45 / [ul.gar 1 33 45 1/2 1 33 45 *ḫe-pé* 46 5] 2 30 *ta-mar* /
10 [30 *ù* 45 ul.gar 45 *ta-mar* 1/2] 45 *ḫe-pé* / [22 30 *ta-mar* 22 30 nigin 8 26 15 *ta-mar*]
11 [8 2]6 15 / [*a-na* 5 igi.gub.ba *i-ši* 42 11 15 *ta-mar*]
… ………………………………………………

1-2 [A truncared (cone).] 30 the go-around, 1 the height, 25 the mud, 2 1/2 nindan [I went down]. The mud(s), what?
3 You: The reciprocal of 1, the height, release, / [1 you will see. 1 to] 30 the go-around carry, 30 you will see.
4-5 30 to / [30, the lower height, rai]se, 15 you will see. 15 from 30 the go-around / [tear off, 15 you will see.]
[The] lower mud, <what?>
6 30, the lower go-around, / [square, 15 you will see. To 5, the con]stant, carry it, 1 15 you will see. /
7-8 [15, the upper go-around square, 3 4]5 you will see. 3 45 to / [5, the constant, carry, 18 45 you will see.]
9 [1 1]5 and 18 45 / [heap, 1 33 45. 1/2 of 1 33 45 break, 46 5]2 30 you will see. /
10 [30 and 45 heap, 45 you will see. 1/2] of 45 break, / [22 30 you will see. 22 30 square, 8 26 15 you will see.]
11 [8 2]6 15 / [to 5, the constant, carry, 42 11 15 you will see].
… ………………………………………………

In this exercise, something has a *ma-ṣa-rum* 'go-around', probably meaning an arc or circumference a as in § 5 d, a height h, and a volume V. The numerical values of those parameters are the same as in § 5 d, namely a = 30, h = 1, and V = 25 (00), so it seems to be safe to assume that the object considered in this exercise is *the same cone as in § 5 d.* There is one additional given numerical parameter, namely 2 1/2 nindan. As it turns out, the lost word in line 2 of the exercise is [*ur-dam*] 'I went down', and 2 1/2 nindan can be explained as 30 cubits, half the height of the cone. The stated purpose of the exercise is *to compute one or several volumes*. So, the purpose of the exercise seems to be *to compute the volume content of (the two halves of) a cone cut by a plane at mid-height*. See Fig. 8.5.3 below, left. (Compare with § 2 of BM 96954+, where a truncated ridge pyramid was the result of cutting off the original ridge pyramid at mid-height.)

Note, by the way, that the length 2 1/2 nindan is given in standard length measure notation, so the length that went down from the top of the cone *must* be 30 cubits (15 meters), and not ;30 cubits. This confirms the assumption made above that the dimensions of the pyramids and cones considered in BM 96954+ are *huge*, not diminutive! It is likely that the author or authors of the various paragraphs of this text had no thoughts about the plausibility of the numerical parameters he, or they, prescribed for the pyramids and cones.

The solution procedure in § 6, or what remains of it, can be explained as follows: First the upper arc a' is computed, by use of what is in effect a similarity argument, namely as

a' = a – rec. h · a · h' = 30 – 1 · 30 · 30 = 30 – 15 = 15 (lines 2-5)

Note that the curious definition of the 'food' of the cone in § 5 is no longer used in § 6! Anyway, next the areas A and A' of the base and the top, respectively, of the of the truncated cone are calculated:

A = ;05 · sq. a = ;05 · 15 (00) = 1 15, and A' = ;05 · sq. a' = ;05 · 3 45 = 18;45 (lines 5-8)

Then the “average area” A_a is calculated, as the half-sum of A and A':

A_a = 1/2 ($A + A'$) = 1/2 (1 15 + 18;45) = 1/2 · 1 33;45 = 46;52 30 (lines 8-9)

Only traces of the next operation remain, but it seems to have been the calculation of the “middle area” A_m, the area of the section of the truncated cone half-way between the bottom and the top:

A_m = ;05 · sq. {1/2 ($a + a'$)} = ;05 · sq. 22;30 = ;05 · 8 26;15 = 42;11 15 (lines 10-[12])

Now, it can be shown that a correct expression for the volume V' of the truncated cone considered here is

V' = (1/3 A_a + 2/3 A_m) · h' = {1/3 · 46;52 30 + 2/3 · 42;11 15} · 30 = (15;37 30 + 28;07 30) · 30 = 43;45 · 30 = 21 52;30.

See the discussion in Sec. 8.5.11 below. Without doubt, the lost part of the solution procedure in § 6 of BM 96954+ continued in accordance with this correct expression for the volume of a truncated cone.

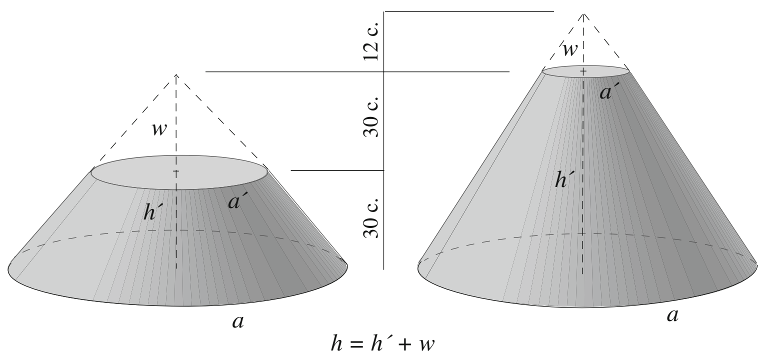


Fig. 8.5.3. BM 96954+ §§ 6-7. Two truncated cones.

The result of the computation described above can be checked in the following way: The difference between the volume V of the whole cone and the volume V' of the truncated cone ought to be equal to the volume V'' of the small cone cut away by the truncation plane. Since the original cone was truncated at mid-height, and since the small cone and the original are *similar solids*, V'' ought to be equal to 1/8 of V. In other words, another way of computing the volume of the truncated cone is to proceed as follows. Since

$V = A$ · 1/3 h = 1 15 sq. nindan · 20 cubits = 25 (00) sar

(as stated in the introduction to the exercise), it follows that

V' = (1 – 1/8) · V = 25 00 sar – 3 07;30 sar = 21 52;30 sar.

The result is, as it should be, the same as above.

8.5.7 § 7. A Circular Cone Truncated Near the Top

BM 96954+ § 7 a (*rev. ii*: … - *iii*… …)

… ……………………………………

1´ ⸢29 51 40 *ta*⸣-[*mar* 29 51 40 x x] /

2´ 29 51 40 *a-na* 1 sukud *i-ši* 29 51 40 *ta-*⸢*mar* saḫar⸣.[ḫá]

… ……………………………………

1´ ⸢29 51 40 you⸣-[will see, 29 51 40 is the x x.] /

2´ 29 51 40 to 1, the height carry, 29 51 40 you ⸢will see⸣.

BM 96954+ § 7 c (*rev. iii*: …)

1-2 [………..] 29 51 40 saḫar.ḫá / [………1/2 5] ⸢*ḫe-pé*⸣ 2 30 / [*ta-mar*) …………………

… ……………………………………

1-2 [………..] 29 51 40 the volume, / [……… 1/2 of 5]⸢break⸣ 2 30 / [you will see] …………………

… ……………………………………

The text of § 7 a is even more damaged than the text of § 6. Only parts of the last couple of lines remain on the lower edge of the tablet. See Fig. 8.5.7 below. Luckily, that is all that is needed for the understanding of what happens in this exercise. The final multiplication of 29 51 40 by the height indicates that the object considered in § 7 a is a solid with the height '1' and the volume '29 51 40'. It is reasonable to assume that here again '1' means 1 (00) cubits = 5 nindan, just as in § 6 above, and that, correspondingly, the volume is V = 29 51;40 sar. It is also reasonable to assume that the solid considered here is a *truncated cone*, as in § 6.

Now, in the comment above to § 6, it was made probable, through an argument based on similarity, that the volume of a cone truncated at mid-height is 1 – 1/8 of the volume of the whole cone. The same kind of argument can be used to show that if a circular cone with the height h is truncated at a distance of $1/n \cdot h$ from the top, where n is an integer, then the volume of the truncated cone is $(1 - 1/n^3)$ of the volume of the whole cone. Now, consider the sexagesimal number 29 51;40. It is close to the round number 30 00. The difference between 30 00 and 29 51;40 is only 8;20, which in its turn is a simple fraction of 30 00. Indeed,

6 · 8;20 = 50, 6 · 50 = 5 00, and 6 · 5 00 = 30 00.

Therefore, the volume mentioned in the last line of § 7 a is

V = 29 51;40 volume-sar = $(1 - 1/6^3)$ · 30 00 volume-sar.

This result implies that the solid considered in § 7 is what remains when a circular cone with the volume 30 00 sar is truncated a distance equal to 1/6 of the height of the cone below the top of the cone. On the other hand, the height h' of the truncated cone is 1 00 cubits. Taken, together, these facts demonstrate that

$(1 - 1/6) \cdot h = h'$ = 1 00 cubits = 5 nindan, so that $h = (1 + 1/5)$ · 1 00 cubits = 1 12 cubits = 6 nindan.

Taking one's cue from § 6, one can now compute the area A and the arc a (that is, the circumference) of the base of the whole cone as

A = 3 rec. $h \cdot V$ = 3 · rec. 1 12 · 30 00 sq. nindan = 1 15 sq. nindan, and
sq. a = rec. ;05 · 1 15 sq. nindan = 15 00 sq. nindan, so that a = 30 nindan.

The arc a' at the top of the truncated cone must then be

$a' = 1/6 \cdot a$ = 5 nindan.

See Fig. 8.5.3, right.

Knowing all this makes it possible to *reconstruct the lost question in exercise § 7 a*, namely as follows:

A cut (cone). 30 the lower go-around, 5 the upper go-around, 1 the height. The mud is what?

In other words, if a truncated circular cone has the bottom circumference a = 30 (nindan), the top circumference a' = 5 (nindan), and the height h' = 1 (00) (cubits), what is then its volume V' ?

The solution procedure probably proceeded as follows. Let V denote the unknown volume of the whole cone, let h be the unknown height of the whole cone, and let $w = h - h'$ be the 'descent'. Then

$w = a'$ · rec. $(a - a') \cdot h'$ = 5 · rec. 25 · 1 (00) = 12, so that h = 1 12 = $6 \cdot w$,
V = ;05 · sq. a · 1/3 h = ;05 · 15 00 · 24 = 30 00,
$V' = V - 1/6 \cdot 1/6 \cdot 1/6 \cdot V$ = 1/6 · 50 = 30 00 – 8;20 = 29 51;40.

Or else, the computation went the following way, using the same computation rule as in § 6:

A_a = ;05 · 1/2 (sq. 30 + sq. 5) = 38;32 30, A_m = ;05 · sq. 1/2 (30 + 5) = 25;31 15, 1/3 A_a + 2/3 A_m = 29;51 40,
$V' = (1/3\ A_a + 2/3\ A_m) \cdot h'$ = 29:51 40 · 1 (00) = 29 51;40.

A third possibility is that the problem was solved by use of the computation rule

V' = ;05 · A_e · 1/3 h', where A_e = sq. $a + a \cdot a'$ + sq. a'.

Where this rule comes form is explained in Sec. 8.5.10 below. Then

V' = ;05 · (15 (00) + 2 30 + 25) · 20 = ;05 · 17 55 · 20 = ;05 · 5 58 20 = 29 51;40.

The text of § 7 b at the top of column *iii* on the reverse of BM 96954+ is totally lost. Of the text of § 7 c remains only a part of the first line of the question, where it is stated that '29 51 40 is the volume'. Also the end of the second line of the paragraph is partly preserved, with the number ⌜3⌝2 30 visible. By a lucky coincidence, this is enough to show that also the (badly preserved) series of exercises §§ 7 b-d were concerned with the same truncated cone as § 7 a, the cone that is shown in Fig. 8.5.3, right, and that §§ 7 a-d were structured in the same way as §§ 1 a-e on the obverse of the tablet, namely *as a systematically organized mathematical theme text*. See the reconstructed outline of the tablet BM 96954+ and the table of contents in Fig. 8.5.5 below.

More precisely, in § 7 a the parameters a, a', and h are given and the volume V is computed. In the next three exercises V and two of the parameters h', a, a' are given, and the third parameter is computed.

The computation of the height h' in § 7 b can have been carried out as follows: If a = 30, a' = 5, then

If a = 30 and a' = 5, then A_e = ;05 · (sq. a + $a \cdot a'$ + sq. a') = 5 · 17 55 = 1 29 35, and 1/3 A_e = 29 51 40.
If, in addition, V' = 29 51 40, then V' = 1/3 A_e · 1, so that h' = 1.

In the same vein, the lower arc a can have been computed as follows in § 7c:

If V' = 29 51 40, a' = 5, and h' = 1, then 5 · (sq. a + 5 a + 25) · 20 = 29 51 40.

Consequently, a can be computed as a solution to the quadratic equation

sq. a + 5 a + 25 = 29 51 40 · 12 · 3 = 5 58 20 · 3 = 17 55,

so that (remember that we are counting with sexagesimal numbers in relative place value notation)

a + 2 30 = sqs. (17 30 + sq. 2 30) = sqs. (17 30 + 6 15) = sqs. 17 36 15 = 32 30, and a = 30.

Note the traces of the phrase ½ 5 *ḫe-pé* 2 30 / *ta-mar* '1/2 of 5 break, 2 30 / you will see' in line 2 of § 7 c.

Similarly, the upper arc a′ can have been computed as follows in § 7 d:

sq. a' + 30 a + 15 = 29 51 40 · 12 · 3 = 5 58 20 · 3 = 17 55,

so that

a' + 15 = sqs. (2 55 + sq. 15) = sqs. (2 55 + 3 45) = sqs. 6 40 = 20, and a' = 5.

8.5.8 Sb 13293. A Problem for a Ridge Pyramid

This problem, published by Bruins and Rutten as *TMS* XIV (1961), remained a mystery until it was explained in Friberg, *PCHM* 6 (1996), and Friberg *AT* (2007), 196.

Sb 13293

1 guru$_7$ a.na 14 24 sahar 3 gi *me-lu-*[*um*] /
2-3 *a-na* 14 24 sahar uš sag *ù qà-aq-qa-da* / *mi-na* gar
4 za.e igi 12 *šu-up-li pu-ṭú-ur* / 5 *ta-mar*
5 5 *a-na* 14 24 saḫar *i-š-íma* / 1 12 *ta-mar*
6 3 gi *me-la-a-am* nigin 9 *ta-mar* / 9 *a-na* 3 *me-le-e te-er-ma* 27 *ta-mar* /
7-9 *i-na* 1 a.rá *ka-a-a-ma-ni* saḫar 20 *šà-lu-uš-ti* / *šà ka-a-a-ma-ni* saḫar zi / 40 *ta-mar*
10 40 *aš-šum* 2 sag guru$_7$ *a-na* 2 tab.ba / 1! 20 *ta-mar*
11 1 20 *a-na* 27 *i-ší-ma* 36 *ta-mar* / 36 *i-na* [1 1]2 zi 36 [*ta-mar*]
12-13 [*tu*]*-úr-ma* / 3 *me-la-a-am* nigin 9 *ta-*[*mar* igi 9 *pu-ṭ*]*ú-úr* / 6 40 *ta-mar*
14 6 40 *a-na* 36 *i-š*[*í-ma*] / 4 *ta-mar* 4 *qà-aq-qa-du*
15-16 3 *me-la-*[*a-am*] / [*aš-š*]*um i-na am-ma-at am-ma-*[*at* šag$_4$.gal] / [*ana* 2] tab.ba 6 *ta-mar* 6 sag
17 6 [*a-na* 4] / [*qà-aq-qa*]*-di* dah 10 *ta-mar* 10 [us] /
… ………………………………………
1′ [12 *a-na*] 3 *me-la-a-am* [*i-ší-ma*] / …… 36 *ta-mar*
2′ 3[6 *a-na* 24 *i-ši-ma*] / [14] 24 *ta-mar* sahar
3′-4′ 14 2[4] saḫar] / *a-na* 8 *na-aš-pa-ak* guru$_7$ [*i-ší-ma*] / 1 55 12 *ta-mar*
5′ 23? [guru$_7$]? / *ù* 2+šu 24$_{gur}$ gur *še-*[*um*]
[*ki-a*]*-am ne-*[*pé-šum*]

1 A grain heap. As much as 14 24 the mud, 3, reeds, the height. /
2-3 For 14 24 of mud, the length, the front, and the head, / what do I set?
4 You: The reciprocal of 12 of the depth, release, / 5 you will see.
5 5 to 14 24 the mud carry it, then / 1 12 you will see.
6 3, reeds, the height, square, 9 you will see. / 9 to 3 of the height lift up, 27 you will see. /
7-9 From 1, the normal step, 20 of mud, a third, / of what <to>? the normal volume you added on, tear off, / 40 you will see.
10 40, since there are 2 fronts of the grain heap, to 2 repeat, / 1! 20 you will see.
11 1 20 to 27 carry, then 36 you will see. / 36 from [1 1]2 tear off, 36 [you will see].

12-13 [Re]turn. / 3, the height, square, 9 you [will see. The reciprocal of 9 re]lease, / 6 40 you will see.
14 6 40 to 36 rai[se, then] / 4 you will see, 4 the head.
15-16 3, the hei[ght], / since in a cubit a cub[it the food], / [to 2] repeat, 6 you will see, 6 the front.
17 6 [to 4] / [the hea]d add on, 10 you will see, 10 the length. /
… ………………………………………
1´ [12 to] 3, the height, [carry, then] / …… 36 you will see.
2´ 3[6 to 24 carry, then] / [14] 24 you will see, the mud.
3´-4´ 14 2[4, the mud] / to 8, the storing number of the grain heap [carry, then] / 1 55 12 you will see.
5´ 23[?] [guru$_7$][?] / and 2 sixties 24 gur the grain.
[Su]ch is the pro[cedure].

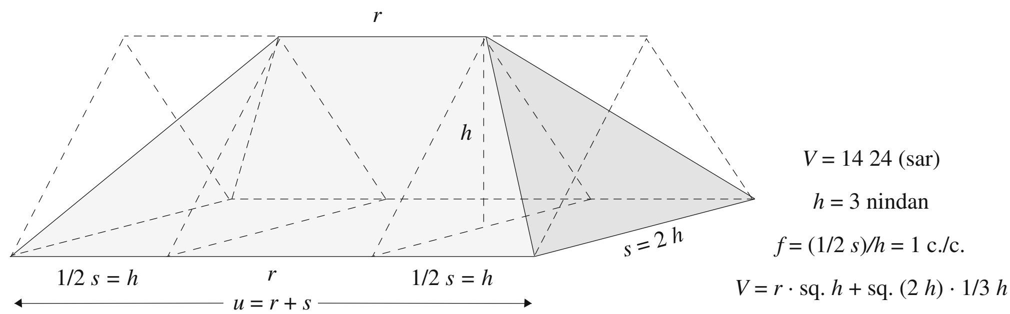


Fig. 8.5.4. Sb 13293. A ridge pyramid.

Clearly, the object considered in Sb 13293 is a ridge pyramid of the same kind as the one in BM 96954+ § 1, although with different numerical values for the parameters. The given volume is V = 14 24, the height is h = 3 nindan, in accordance with Old Babylonian conventions written as '3, (in the range of) reeds', to make it clear that '3' should not be interpreted as ;03 (nindan), which is 60 times smaller. Asked for are the length u, the front s, and *qaqqadu* 'the head', the name used in this text for the ridge r of the ridge pyramid.

Note that the author of the text forgot to mention the extra condition that the 'food' ***f*** should be 1 cubit per cubit, although an explicit reference to this condition is given in line 15.

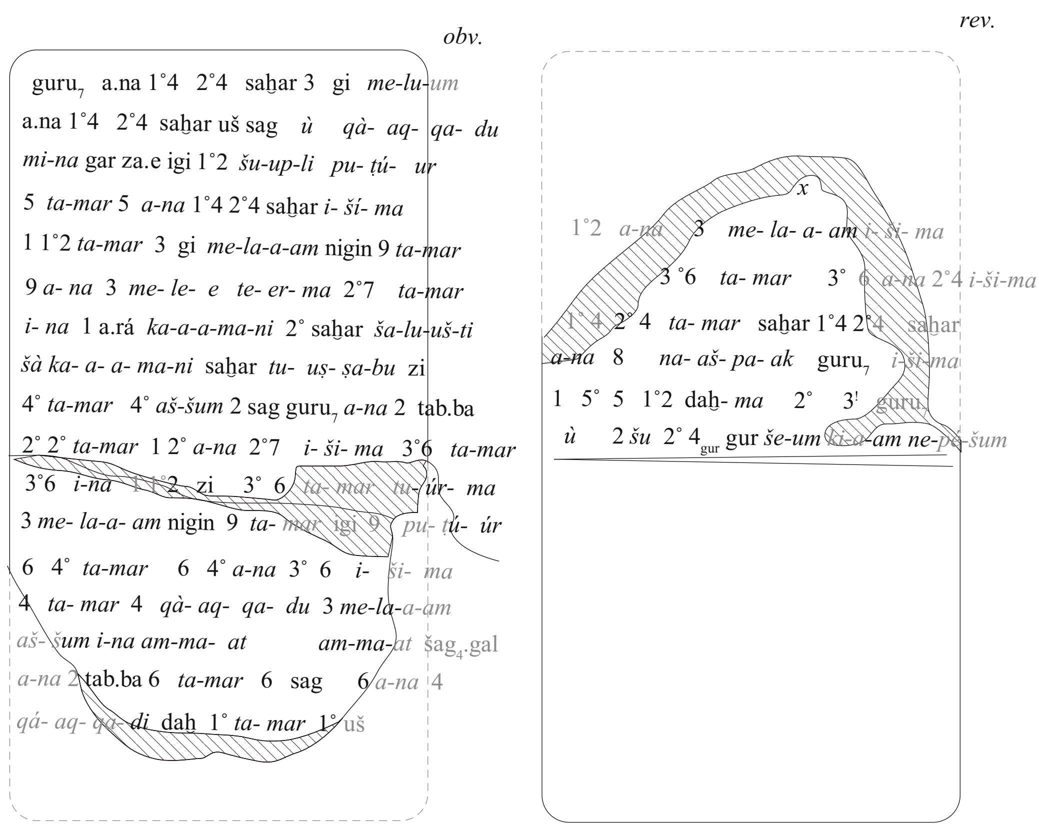


Fig. 8.5.5. Sb 13293. Conform transliteration.

In the first step of the solution procedure, the volume V = 14 24 (sar) = 14 24 (sq. nindan · cubit) is multiplied by the reciprocal of '12 of the depth', by which is meant the conversion factor 12 in the metrological equation 1 nindan = 12 cubits. (Recall that vertical distances were measured in cubits, which explains the reference to the 'depth'.) In this way, the expression for the volume is converted to

V = rec. (12 cubits per nindan) · 14 24 sq. nindan · cubit = 1 12 sq. nindan · nindan (lines 3-5)

What is computed in the second step of the solution procedure is, essentially, the cube of the height, namely

sq. $h \cdot h$ = sq. (3 nindan) · 3 nindan = 27 sq. nindan · nindan (lines 5-6)

This is the volume of a triangular prism with the height h and the sides h and 2 h. There is one such prism indicated in Fig. 8.5.4 on either side of the central prism, which has the height h and the sides r and 2 h, and therefore the volume sq. $h \cdot r$. Note that the pronunciation in line 15 that 'in a cubit a cub[it the food]', seems to imply that the front s of the base of the ridge pyramid shall be equal to *twice* the height h of the pyramid. Somewhat confusingly, the meaning of the 'food' is not the same in Sb 13293 as in BM 96954+ §§ 1-2!

The awkwardly worded passage of the text which then follows can be interpreted as describing the computation of the volume

2 V_{end} = 2 · (1 – 1/3) · sq. $h \cdot h$ = 1;20 · 27 sq. nindan · nindan = 36 sq. nindan · nindan (lines 7-11)

What this means is that the volumes of the pyramids with rectangular bases on either side of the central triangular prism (see again Fig. 8.5.4 above) are computed as two-thirds of the volumes sq. $h \cdot h$ of the two triangular prisms containing them. The sum of those two volumes is 36 sq. nindan · nindan.

In line 11, the sum of the volumes of the two end pyramids is subtracted from the volume V of the whole ridge pyramid. The result is the volume of the central triangular prism:

$V_{central}$ = sq. $h \cdot r$ = V – 2 · (1 – 1/3) · sq. $h \cdot h$ = (1 12 – 36) sq. nindan · nindan = 36 sq. nindan · nindan (line 11)

Next, the ridge r, here called 'the head', is computed in the following way:

r = rec. sq. h · (sq. $h \cdot r$) = rec. sq. $h \cdot V_{central}$ = rec. 9 · 36 nindan = 4 nindan (lines 12-14)

In lines 15-16, the condition that 'in a cubit a cub[it the food]', and the fact that h = 3 leads to the conclusion that s = 6. Finally, therefore, in line 17, $u = r + s$ = 4 + 6 3 = 10. (See again Fig. 8.5.4 above.)

The second part of the solution procedure contains a verification of the result of the preceding computations of r, s, and u. The beginning of the verification is lost, but it seems that first the volume of the ridge pyramid is computed in the following way:

h = 3 nindan = 12 · 3 cubits = 36 cubits, $V = (u + 1/2\ r) \cdot 1/3\ s \cdot h$ = [24] · 36 = 14 24 (lines 1′ -2′)

Next, the computation of the seed measure C of the ridge pyramid is based on the information that

8 *na-aš-pa-ak* guru$_7$ 8 is the storing number for a grain heap (line 3′)

Compare with BM 96954+ § 1 in Sec. 8.5.1 above, where the conversion from volume to seed measure depended on the conversion factor '1 30', which could be explained as

1 30 gur/volume-sar = 1 30 · 5 00 sìla/volume-sar = 7 30 00 sìla per volume-sar.

In the case of Sb 13293, where the storing number is '8', the conversion factor is instead

1 36 gur/volume-sar = 1 36 · 5 00 sìla / volume-sar = 8 00 00 sìla per volume-sar.

Therefore, the seed measure C of the ridge pyramid in Sb 13293, which is computed as

C = c V = 8 · 14 24 = 1 55 12 (lines 2′ -4′)

has to be understood as meaning the following:

C = 14 24 volume-sar · 8 00 00 sìla per volume-sar = 1 55 12 00 00 sìla = 23 02 24 · 5 00 sìla = 23 02 24 gur.

Now, the largest Sumerian/Old Babylonian unit of capacity measure, is called guru$_7$ ('grain heap' or 'granary'), with 1 guru$_7$ = 1 šár gur = 1 00 00 gur. Therefore,

8.5.9 C = 23 02 24 gur = 23 [guru$_7$] 2 sixties 24 gur (lines 4´-5´) BM 96954+. An Outline of the Tablet, a Table of Contents, and Hand Copies

§ 1. The seed measure of a ridge pyramid

C = c · (u +1/2 r) · s · 1/3 h, c = 1 30 gur/volume-sar

[§ 1 a	*u*, *r*, *s*, *h* given	*C* = ?]	computation rule
[§ 1 b	*C*, *r*, *s*, *h* given	*u* = ?]	linear equation
[§ 1 c	*C*, *u*, *s*, *h* given	*r* = ?]	linear equation
[§ 1 d	*C*, *u*, *r*, *h* given	*s* = ?]	linear equation
[§ 1 e	*C*, *u*, *r*, *s* given	*h* = ?]	linear equation
§ 1 f	*C*, *s*, *h* given, *u* + *r* = 14	*r*, *u* = ?	linear equations
§ 1 g	*C*, *s*, *h* given, *r*/2 = 1/5 *u*	*r*, *u* = ?	linear equations
§ 1 h	*C*, *u*, *h* given, 2/3 *s* = *r*	*r*, *s* = ?	quadratic equation
§ 1 i	*C*, *r*, *h* given, [*u* + *s* = 16]	*u*, *s* = ?	rect.-linear eqns
§ 1 k	*C*, *r*, *h* given, (*u* – *r*)/*h* = 7 30	*u*, *s* = ?	linear equations
§ 1 l	*C*, *u*, *h* given, *r* + 1/3 *s* = 6	*r*, *s* = ?	rect.-linear eqns

§ 2. The seed measure of a truncated ridge pyramid

C = c · {(u · s + $u´$ · $s´$) + 1/2 (u · $s´$ + $u´$ · s)} · 1/3 $h´$

§ 3. The seed measure of a square pyramid

C = c · sq. s · 1/3 h

§ 4. The seed measures of various solids

A triangular prism	C = c · u · 1/2 s
A trapezoidal prism	C = c · u · 1/2 (s + $s´$)
A semi-circular cylinder	C = c · a · 1/4 d · 1/3 h

§ 5. The quadratic growth rate of a circular cone

V = sq. s · 1/3 h, (f = 1/2 rec. h · a or) f = rec. h · sq. a

§ 5 a	*a*, *h* given	*f* = ?
§ 5 b	*a*, *f* given	*h* = ?
§ 5 c	*f*, *h* given	*a* = ?
§ 5 d	*f*, *V* given	*a*, *h* = ?

§ 6. The volume of a circular cone truncated at mid-height

$V´ = (1/3\ A_a + 2/3\ A_m) \cdot h´$

§ 7. The volume of a circular cone truncated near the top

$V´ = (1 – 1/6^3) \cdot V$

§ 7 a	[*a*, *a´*, *h´* given]	*V´* = ?
§ 7 b	[*V´*, *a*, *a´* given]	[*h´* = ?]
§ 7 c	*V´*, [*a´*, *h´*] given	[*a* = ?]
§ 7 d	[*V´*, *a*, *h´* given]	[*a´* = ?]

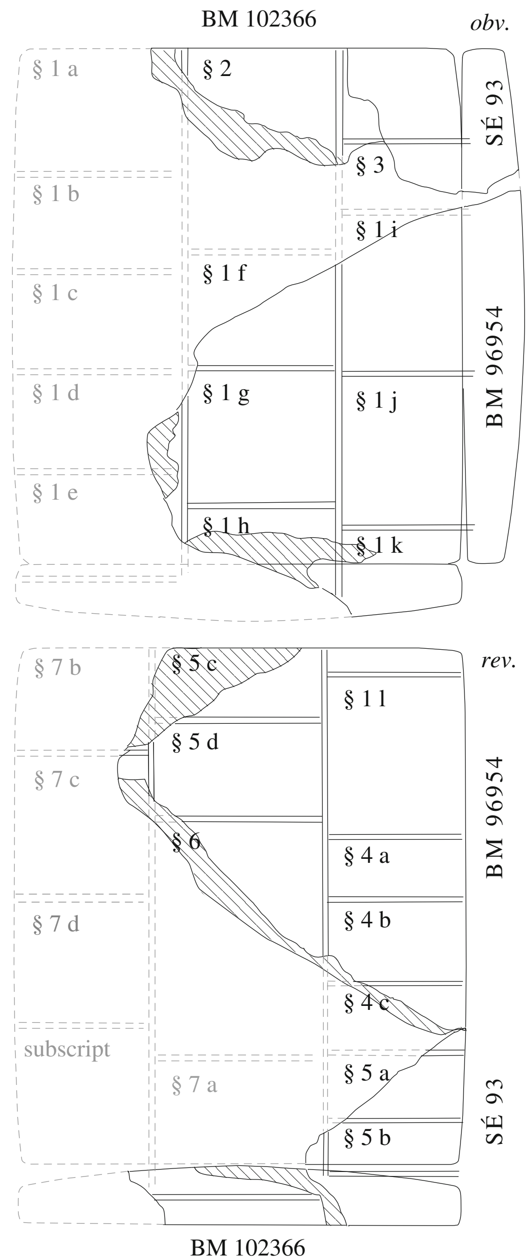


Fig. 8.5.6. BM 96954 + BM 102366 + SÉ 93. A reconstructed outline of the tablet and a table of contents.

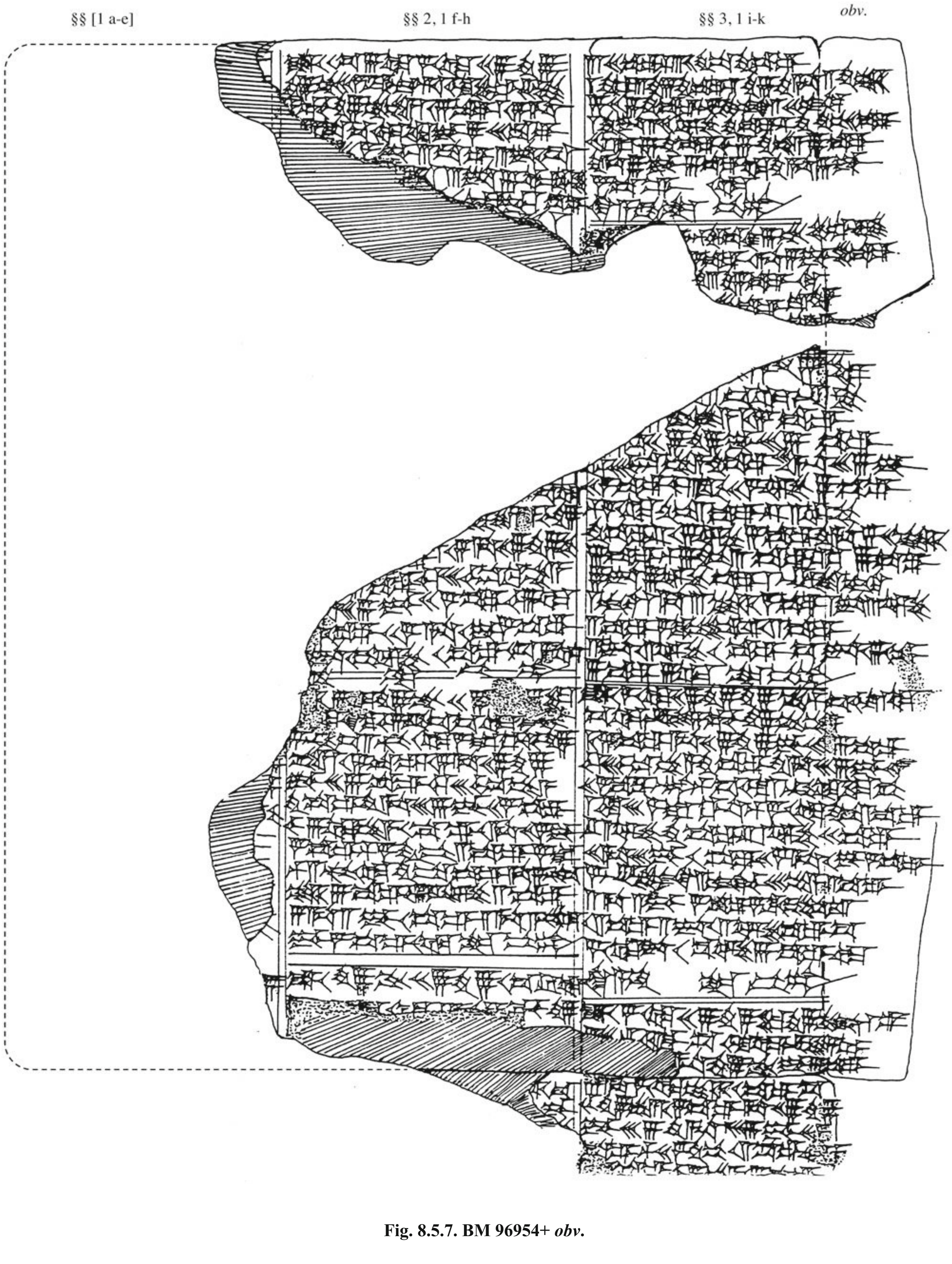


Fig. 8.5.7. BM 96954+ *obv*.

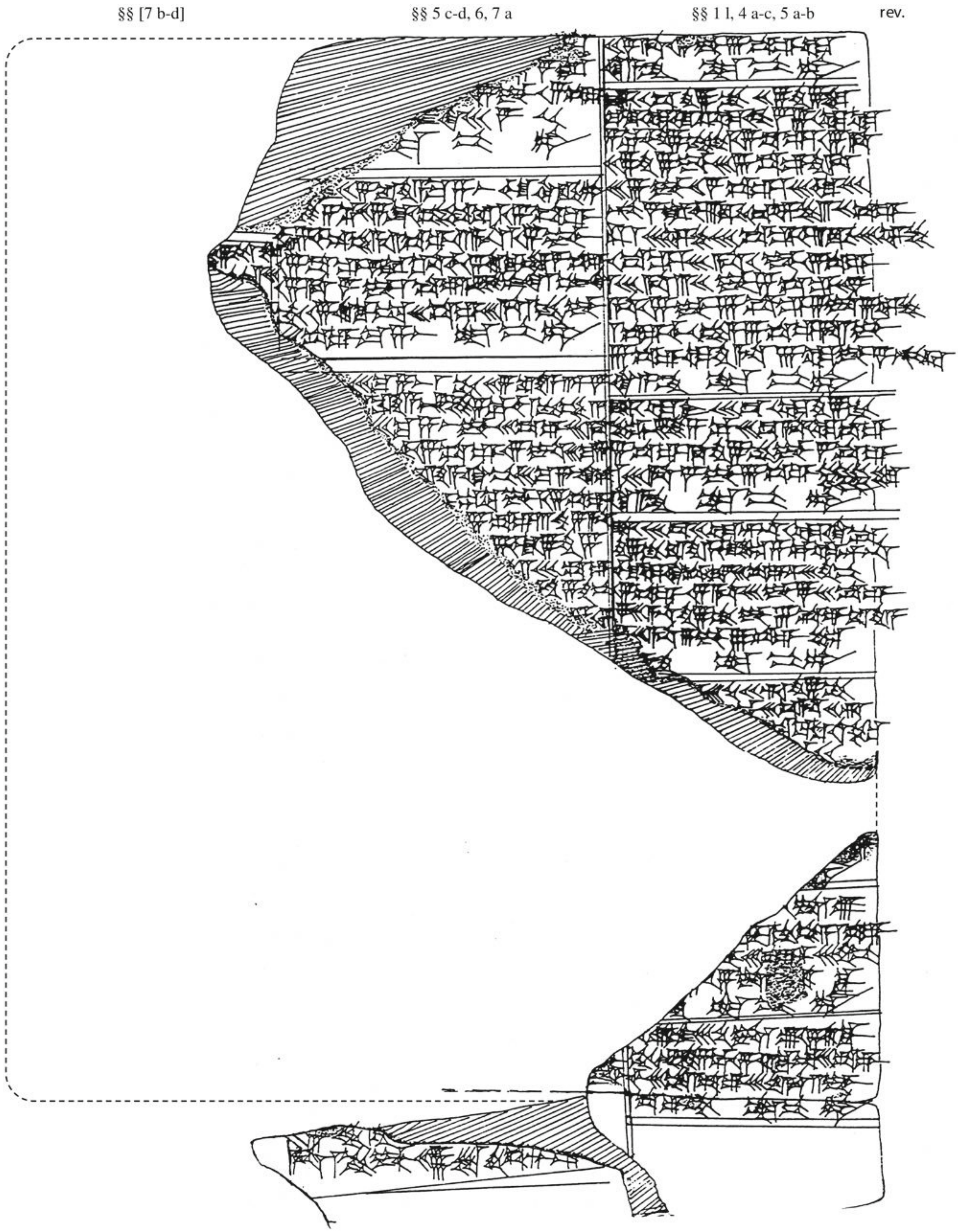


Fig. 8.5.8. BM 96954+ *rev*.

8.5.10 The Vocabulary of BM 96954+

In addition to the terms already listed in the vocabularies above for BM 80078, etc., the following terms appear in the text of BM 96954+:

an.ta	upper
ašag.u$_4$.sakar	crescent field (semicircle)
bala	fraction, ratio
gúr	arc (circumference)
guru$_7$	grain heap, granary
igi.3.gál	the 3rd part
ki.ta	lower
lagab	square
nigin	square
sagšu	ridge, crest
sag.kak	peghead (triangle)
saḫar, saḫar.ḫá	mud (volume)
šag$_4$.gal	food
še-*um*	grain
tab.ba	double

a.rá *ka-a-a-ma-ni*	normal step(?)
aš-šum …… dug$_4$.ga	since it was said that ……
ki-a-am ne-pé-šum	such is the procedure

am-ma-at	*ammatu* cubit
di-ik-šum	*dikšu* swelling, “stick-out” (height of a cone)
[*gam*]-*ru-tum*	*gamrūtu*(?) entirety, whole (cone)
ma-ṣa-rum	*maṣāru* enclosure, *“*go-around” (circumference (of a cone)
me-lu-[*um*], *me-la-a-am, me-le-e*	*mēlû* height, altitude
mu-ḫu	*muḫḫu* top
pé-er-kim	*perku* cross-*line*
qà-aq-qa-da	*qaqqadu* head
šà-lu-uš-ti	*šuluštu*! a third (1/3)
šu-up-li	*šuplu* depth

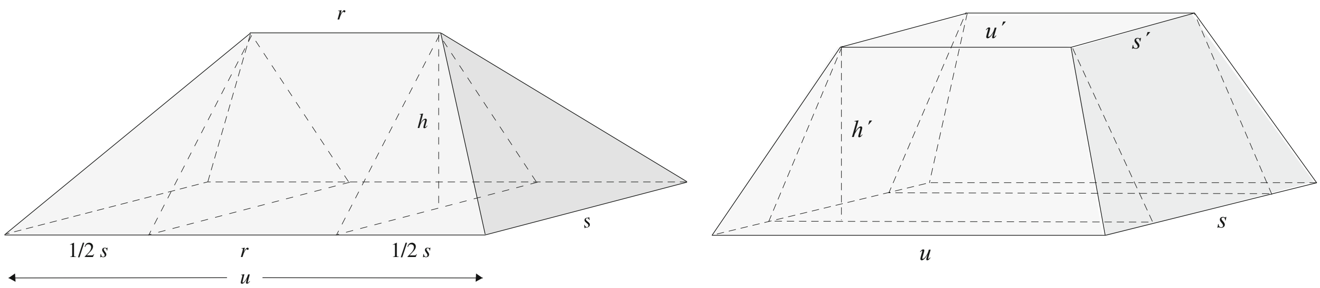
im-ta-ḫar	< *maḫāru* Gt to become equal to each other, to become equalsided (square)
i-ši	< *našû* to carry (= to multiply (by a number))
šu-ul-li-iš	< *šalāšu* D to make threefold
te-er	< *tarû* to lift up
tu-uṣ-ṣa-bu	< *waṣābu* to add on

8.5.11 The Computation Rules for Volumes of Pyramids in BM 96954+ §§ 1-3

If BM 96954+ had been an original mathematical theme text with the theme pyramids and cones, instead of just a mathematical recombination text, its paragraphs would have been arranged in a different order. It is clear from the table of contents in Fig. 8.5.5 above that the systematically arranged §1 of BM 96954+ is an excerpt from such a theme text. Also § 5, as well as the badly preserved § 7, looks like an excerpt from a theme text, although not the same one as in the case of § 1, since §§ 1-4 express the size of ‘grain heaps’ in terms of seed measure, while §§ 6-7 express the size of various objects in terms of their volume.

In a well organized theme text, with exercises of increasing difficulty, the computation of the seed measure of triangular and trapezoidal prisms, as in § 4, would precede the computation of the seed measure of a square pyramid, as in § 3, which in its turn would precede the computation of the seed measure of a ridge pyramid, as in § 1, and of a truncated ridge pyramid, as in § 2, which by the way is inserted by mistake among the exercises of § 1.

Suppose, for the sake of the argument, that a teacher has already instructed his students about computation rules for the volume and seed measure, first, *of triangular and trapezoidal prisms*, and then, *of square pyramids*. How could he go from there to find a computation rule also for the volume and seed measure of ridge pyramids?



A ridge pyramid divided into a triangular prism and two halves of a square pyramid

A truncated ridge pyramid divided into a trapezoidal prism and two halves of a ridge pyramid

Fig. 8.5.9. BM 96954+. How the volumes of a ridge pyramid and of a truncated ridge pyramid can be computed.

In Fig. 8.5.8 above, left, it is shown how *a ridge pyramid* with a common rate of growth for lengths and fronts can be divided into a) *two halves of a square pyramid with the side s and the height h,* and b) *a triangular prism with the length r, the front s, and the height h*. Note that $s = u - r$. The seed measure of the ridge pyramid is then clearly $C = \mathrm{c}\ V$, with c equal to some given storing number and with

$$V = V_{\mathrm{pyramid}} + V_{\mathrm{prism}} = \text{sq. } s \cdot 1/3\ h + r \cdot 1/2\ s \cdot h = (s + 3 \cdot 1/2\ r) \cdot s \cdot 1/3\ h = (u + 1/2\ r) \cdot s \cdot 1/3\ h.$$

This is precisely *the computation rule used in § 1 of BM 96954.*

In Fig. 8.5.8, right, it is shown in a similar way how *a truncated ridge pyramid* with a common rate of growth for lengths and fronts can be divided into a) *two halves of a ridge pyramid with the length u, the ridge r, the front s – s′, and the height h′*, and b) *a trapezoidal prism with the lower length u, the upper length u′, the front s′, and the height h′*. The seed measure of the truncated ridge pyramid is then clearly $C = \mathrm{c}\ V$, with c equal to some given storing number and with

$$V = V_{\mathrm{ridge\ pyramid}} + V_{\mathrm{prism}} = (u + 1/2\ u') \cdot (s - s') \cdot 1/3\ h + 1/2\ (u + u') \cdot s' \cdot h' = \{(u + 1/2\ u') \cdot s + (1/2\ u + u') \cdot s'\} \cdot 1/3\ h'.$$

This is, except for some simple rearrangement, *the computation rule used in § 2 of BM 96954+*, namely

$$V = \{(u \cdot s + u' \cdot s') + 1/2\ (u \cdot s' + u' \cdot s)\} \cdot 1/3\ h'.$$

Note that in the special case when $u = s$, then also $u' = s'$, because it is assumed that the rate of growth is the same all around. This is the case of *a truncated square pyramid*. Then the computation rule above takes the following simpler form:

$$V = \{(\text{sq. } s + \text{sq. } s') + 1/2\ (2 \quad s \cdot s')\} \cdot 1/3\ h'.$$

After a simple rearrangement of the terms, this becomes a computation rule for the volume of a truncated square pyramid with the lower square-side s, the upper square-side s', and the height h':

$$V = (\text{sq. } s + s \cdot s' + \text{sq. } s') \cdot 1/3\ h'.$$

This correct computation rule for the volume of a truncated square pyramid is famously known from exercise # 14 in the Egyptian hieratic text *P. Moscow*. (See, Friberg, *UL* (2005), Sec. 2.2 d.)

In four previously published Old Babylonian mathematical texts, the following *approximate computation rule* is used for the volume of a truncated square pyramid:

$$V_{\mathrm{a}} = A_{\mathrm{a}} \cdot h', \quad \text{where} \quad A_{\mathrm{a}} = 1/2\ (\text{sq. } s + \text{sq. } s') = \text{the “average” square area.}$$

One of those texts is YBC 4708 (*MKT I*, 389), a systematically arranged “tree-like catalog text” with 60 exercises, divided into several sub-catalogs (see Sec. 11.4 below). Exercises ## 53-60 form a mini-catalog concerned with a truncated square pyramid:

YBC 4708 (*MKT I*, 393; a catalog text)

53 1-2 sig$_4$.anše 2 nindan.ta.àm ki.ta íb.sá / 1/2 nindan 2 kùš an.ta íb.sá /
3-4 1/2 nindan 3 kùš sukud.bi / sig$_4$.bi en.nam sig$_4$.bi 1(iku) ašag 44 sar

1-2 A brick pile. 2 nindan each way the lower equalside, / 1/2 nindan 2 cubits the upper equalside, /
3-4 1/2 nindan 3 cubits its height. / Its bricks, what? 1 iku of field 44 sar.

54 1-2 [sig$_4$.anše 1(iku) ašag] 44 sar / $^1/_2$ nindan 2 kùš.ta.à]m an.ta íb.<sá> /
3-5 [$^1/_2$ nindan 3 kùš] sukud.bi / [ki.ta en.t]a.àm íb.sá /[2 nindan.ta].àm ki.ta íb.sá

1-2 [A brick pile. 1 iku of field] 44 sar, / 1/2 ninda 2 cubits each] way the upper equalside, /
3-5 [1/2 nindan 3 cubits] its height. / [Below, what each] way equalsided? / [2 nindan each] way below equalsided.

55 1-2 [2 nindan ta.à] ki.ta íb.sá / [$^1/_2$ ninda]n 3 kùš sukud.bi /
3-4 [an].ta en.ta.àm íb.sá $^1/_2$ nindan an.ta íb.sá

1-2 [2 nindan each] way below equalsided, / [1/2 ninda]n 3 cubit its height. /
3-4 [Abo]ve what each way equalsided? / 1/2 nindan 2 cubits(!) above equalsided.

56 1-2 2 nindan.ta.àm ki.t[a íb.sá] / $^1/_2$ nindan 2 kuš [an].ta í[b.s]á /
3 sukud.bi en.nam

1-2 2 nindan each w[ay equalsided], / 1/2 nindan 2 cubits [ab]ove equ[al]sided.
3 Its height, what?

57 1-2 sig$_4$ sig$_4$.anše 1(iku) ašag 44 sar / $^1/_2$ nindan 3 kùš sukud.bi /
3 íb.sá an.ta *ù* ki.ta / gar.gar-*ma* 2 $^1/_2$ nindan 2 kùš

1-2 Bricks. A brick pile. 1 iku of field 44 sar, 1/2 nindan 3 cubits its height. /
3 The equalsides above and below I added together, then 2 1/2 nindan 2 cubits.

And so on

In exercise # 53 above, a brick pile evidently has the form of a truncated square pyramid. Its lower square-side is s = 2 nindan (= 12 meters), its upper square-side is s' = 1/2 nindan 2 cubits (= ;40 nindan = 4 meters)), and its height is h' = 1/2 nindan 3 cubits (= 9 cubits = 4 1/2 meters). There is no solution procedure, only an answer, but it is clear that the computation proceeded, incorrectly, as follows:

$V_a = A_a \cdot h'$ = 1/2 (sq. s + sq. s') $\cdot$ h = 1/2 (4 + ;26 40) $\cdot$ 9 (sar) = 2;13 20 $\cdot$ 9 = 20 (volume-sar).

However, the answer is given as 1 iku 44 sar = 2 24 sar, clearly to be interpreted as brick-sar. It is easy to see, counting backwards, that the brick number B was computed as follows:

$B = c \cdot V_a$ = 7;12 (brick-sar per volume-sar) = 7;12 $\cdot$ 20 (brick-sar) = 2 24 (brick-sar) = 1 iku 44 sar.

Here, 7 12 is the 'molding number' for standard rectangular bricks (type R1/2c). (See below, Sec. 11.2.2.)

It is interesting to note that the mini-catalog YBC 4708 ## 53-60 is organized in the same way as BM 96954+ § 1 a-l (partly reconstructed). See the table of contents for § 1 in Fig. 8.5.5 above. Indeed, a similar table of contents for YBC 4708 ## 53-60 would have the following form:

# 53	s, s', h' given	B = ?	computation rule
# 54	B, s', h' given	s = ?	linear equation
# 55	B, s, h' given	s' = ?	linear equation
# 56	B, s, s' given	h' = ?	linear equation
# 57	B, h', $s + s'$ given	s, s' = ?	rectangular-linear system of equations

And so on.

YBC 5037 (*MCT* text F) is another catalog text (see again Sec. 11.4 below) with 44 exercises, all labeled with the key phrase ki.lá 'excavations' and dealing with earth-works, man-power, and wages. There are four sub-themes. The last of them, in exercises ## 35-44, is a mini-catalog of the same kind as YBC 4708 ## 53-60, concerned with truncated square pyramids. For obvious reasons, these excavated truncated square pyramids are turned *upside-down*. The same approximate computation rule for the volume of a truncated square pyramid is used in YBC 5037 as in YBC 4708.

YBC 5037 (*MCT* F; a catalog text)

35 1 ki.lá $^1/_2$ ninda[n.ta.à]m an.ta íb.sá 4 kùš.ta.àm ki.<ta> íb.sá $^1/_2$ nindan bùr /
2 saḫar.bi en.nam {1} 1 sar 5 gín saḫar.bi

1 An excavation. 1/2 ninda[n each wa]y above equalsided, 4 cubits each way be<low> equalsided, 1/2 nindan the depth.
2 Its mud (volume) what? 1 sar 5 shekels its mud.

36 1 saḫar ki.lá 1 sar 5 gín <1/2 nindan bùr> 4 kùš ki.ta íb.sá
an.ta en.ta.àm íb.sá 1/2 nindan an.ta

1 The mud of an excavation, 1 sar 5 shekels. <1/2 nindan the depth>, 4 cubits below equalsided.
Above, what each way equalsided? 1/2 nindan above.

37 1 saḫar ki.lá 1 sar 5 gín <1/2 nindan bùr> 1/2 nindan.ta.àm an.na íb.sá
en.ta.à<m> ki.ta 4 kùš

1 The mud of an excavation, 1 sar 5 shekels. <1/2 nindan the depth>, 1/2 nindan each way above equalsided.
What each wa<y> below? 4 cubits.

38 1 saḫar ki.lá 1 sar 5 gín 1/2 nindan.ta.àm an.ta íb.sá
2 4 kùš.ta.àm ki.ta íb.sá / bùr.bi [en.n]am 1/2 nindan bùr.bi

1 The mud of an excavation, 1 sar 5 shekels. 1/2 nindan each way above equalsided,
2 4 cubits each way below equalsided. / Its depth [wh]at? 1/2 nindan its depth.

And so on

In these exercises, just as in YBC 4708 ## 53-60 above, there is no solution procedure, only answers. The parameters for the truncated square pyramid are s = 1/2 ninda, (= 3 meters), s' = 4 cubits = ;20 nindan (= 2 meters), h' = 1/2 nindan = 6 cubits (= 3 meters). The volume V was clearly computed, incorrectly, as

$$V_a = A_a \cdot h' = 1/2\,(\text{sq. } s + \text{sq. } s') \cdot h' = 1/2\,(;15 + ;06\ 40) \cdot 6\ (\text{sar}) = ;10\ 50 \cdot 6 = 1;05\ (\text{volume-sar}).$$

The next example is an isolated exercise in a large mathematical recombination text from Sippar.

BM 85196 # 11 (*MKT II,* 44; Group 6)

1 $^{\text{geš}}$ù.šub 5 nindan.[ta.à]m *im-ta-ḫar* 6 [sukud *i-na* 1 kùš 5 šag$_4$.gal] /
2 sagšu saḫar.ḫà en.na[m z]a.e 5 šag$_4$.ga[l tab.ba 10 *ta-mar*] /
3 10 *a-na* 6 sukud *i-ši* [1] *ta-mar* 1 *i-*[*na* 5 ba.zi 4 *ta-mar* sagšu] /
4 nigin za.zum *ù mu-ḫa-am* [ul.gar 9 *ta-mar* 1/2 9 *ḫe-pé* 4 30 *ta*]*-mar* /
5 4 30 *a-na* 6 sukud *i-ši* 27 [*ta-mar* 27 saḫar.ḫà] /
6 *ki-a-am ne-*[*pé-šum*]

1 A brick mold, 5 nindan e[ach w]ay he made it equalsided, 6 [the height, in 1 cubit 5 the food]. /
2 The ridge (and) the mud, wha[t? Y]ou: 5, the foo[d double, 10 you will see]. /
3 10 to 6, the height, carry, [1] you will see. 1 fr[om 5 tear off, 4 you will see, the ridge.] /
4 The squares of the base and the top [heap, 9(sic!) you will see. 1/2 of 9 break, 4 30 you] will see. /
5 4 30 to 6, the height, carry, 27 [you will see, 27 the mud]. /
6 Such is the pro[cedure].

Some of the terminology used in this exercise is unusual. The word sagšu 'turban, cover of a pot', which in BM 96954+ § 1 is used to denote the 'ridge' of a ridge pyramid, is used in line 2 above to denote the *square top* of a truncated square pyramid. The same square top is called *muḫḫu* 'skull, top' in line 4.

The word $^{\text{geš}}$ù.šub = *nalbanu* 'brick mold' is used to denote the *rectangular base* of the same truncated pyramid. Actually, the base is square since it is 'made equalsided'. The same square base is called za.zum (literal meaning unknown) in line 4. Compare with how the word *nalbanu* is used in IM 95771 (Sec. 6.1.5) to denote a rectangle, and how also the related word *nalbattu* is used (probably) in a subscript to the mathematical recombination text MS 3049 (Sec. 6.2.4 above) and in IM 52916 § 4, a catalog of mathematical problem types (Sec. 10.1.3 below), to denote rectangles.

The (incorrect) computation of the volume of the truncated pyramid proceeds as follows:

$$V_a = A_a \cdot h' = 1/2\,(\text{sq. } s + \text{sq. } s') \cdot h' = 1/2\,(25 + 16) \cdot 6\ (\text{sar}) = 20;30 \cdot 6 = 2\ 03\ (\text{volume-sar}).$$

The next example, too, is an isolated exercise in a large mathematical recombination text from Sippar.

BM 85210 *rev. i*: 23-32 (*MKT I,* 222; Group 6)

1 $^{\text{geš}}$ù.šub 20 *im-ta-ḫar* 6 gam <*i-na*> 5 k[ùš] 5 kùš *a*[*t-t*]*a-ṣí* /
2 $^{\text{geš}}$ù.šub *ša-ni-tam e*[*l-q*]*é-ma lu* igi.3.gál ki igi.6.gál /
3 saḫar en.nam *ù* bar en.nam
4 za.e 20 íb.sá *šu-tam-ḫir* / 6 40 *ta-mar*
i-na 5 kùš 5 kùš šag$_4$.gal ul.gar 10 *ta-mar* /

5 10 *a-na* 6 gam *i-ši* 1 *ta-mar* 1 [*i-na* 2]0 ba.z[i 1]9 [*ta-mar*] /
6-7 19 *šu-tam-ḫir* 6 [01 *ta-mar*] 6 01 *a*-[*na* 6 40 daḫ.ḫa] / 12 41 *ta-mar*
1/2 12 41 [*ḫe-pé* 6 20] 30 [*ta-mar*]
8 [6 20 30] / *a-na* 6 gam *i-ši* 3[8 03 *ta-mar* 38 03 saḫar.ḫà *ta-m*]*ar* /
9 nigín.na saḫar íb.s[á ………] /
10 igi.gub bar *i*-[*ši* x x *ta-ma*]*r* /
… …………………

1 A brick mold, 20 he made it equalsided, 6 the depth, <in> 5, cu[bits], 5, cubits, I went out. /
2 A second brick mold I took, then may the 3rd part be as the 6th part. /
3 The mud, what, and x, what?
4 You: 20, the equalside, make equalsided, / 6 40 you will see.
In 5, cubits, 5, cubits, the food heap, 10 you will see. /
5 10 to 6, the depth, carry, 1 you will see. 1 [from 2]0 tear o[ff 1]9 [you will see]. /
6-7 19 make equalsided, 6 [01 you will see], 6 01 t[o 6 40 add on] / 12 41 you will see.
1/2 of 12 41 [break, 6 20] 30 [you will see].
8 [6 20 30] / to 6, the depth, carry, 3[8 03 you will see, 38 03, the mud, you will s]ee. /
9 Turn around. The mud x x [………] |
10 The constant of x x x […… you will s]ee. /
… …………………

The term $^{\text{geš}}$ù.šub = *nalbanu* 'brick mold' reappears in this exercise with the meaning *rectangular base* of a truncated pyramid. As above, the base is square since it is 'made equalsided'. Interestingly, the *square top* of the same truncated square pyramid is called 'the second brick mold'.

The šag$_4$.gal 'food' has in this text the same meaning as in Sb 13293 (Sec. 8.5.8) and must be doubled (in line 4) in order to give the whole rate of growth of the side of truncated square pyramid for each ascent of 1 cubit. Note that *i-na* 5 kùš 5 kùš šag$_4$.gal '5, cubits, 5, cubits, is the food' means an increase of 1 cubit = ;05 nindan for each descent of 1 cubit = ;05 nindan.

The meaning of the curious phrase 'may the 3rd part be as the 6th part' in line 2 is not clear. Equally obscure is the question 'bar en.nam?' in line 3. This question seems to be answered in the second half of the solution procedure, which begins with the words 'the constant of the bar', but unfortunately the remainder of the solution procedure is lost.

The approximate computation rule for the volume of the truncated square pyramid is used in this text, too, where the volume is computed as follows:

$$V_a = A_a \cdot h' = 1/2\ (\text{sq. } s + \text{sq. } s') \cdot h' = 1/2\ (6\ 40 + 6\ 01) \cdot 6\ (\text{sar}) = 6\ 20;30 \cdot 6 = 38\ 03\ (\text{volume-sar}).$$

A third isolated exercise in a large mathematical recombination text from Sippar is

BM 85194 # 28 (*MKT I,* 150; Group 6)

1 *ḫi-ri-tum* 10.ta.àm *mu-ḫu* 18 sukud *i-na* 1 kùs 1 kùš šag$_4$.gal /
2 za.zum *ù* saḫar.ḫà za.e 5 *ù* 5 ul.gar 10 *ta-mar* /
3-4 [10] *a-na* 18 sukud *i-ši* 3 *ta-mar* 3 *i-na* 10 ba.zi 7 / [*ta-mar*] za.zum
nigín.na za.zum *ù mu-ḫu* 10 ul.gar 17 *ta-mar* /
5-6 [1/2 17 *ḫe-pé*] 8 30 *ta-mar* nigin 1 12 15 *ta-mar* / 1 12 [15 gar].ra
7-8 igi.⸢12⸣.gál 3 dirig *ša mu-ḫu* ugu / za.zum ni[gin x] 45 *a-na* 1 12 15 daḫ.ḫa-*ma* / 1 13 *ta-mar*
9 1⸢8⸣ *a-na* 1 13 *i-ši* 22 30 *ta-mar* / 2(èše) 1(iku) 1/2(iku) ašag saḫar.ḫà
ki-<*a-am*> *ne-pé-šum*

1 An excavation. 10 each way the top, 18 the height, in 1 cubit 1 cubit the food. /
2 The base and the mud? You: 5 and 5 heap, 10 you will see. /
3-4 [10] to 18, the height, carry, 3 you will see. 3 from 10 tear off, 7 / [you will see], the base.
Turn around. The base and the top, 10, heap, 17 you will see. /
5-6 [1/2 of 17 break] 8 30 you will see, square it, 1 12 15 you will see. / 1 12 [15 set] down.
7-8 The ⸢12⸣th part of 3, the excess of the top over / the base squa[red x] 45 to 1 12 15 add on, then / 1 13 you will see.
9 1⸢8⸣ to 1 13 carry, 22 30 you will see, / 2 èše 1 1/2 iku, the mud.
Such is the procedure.

Just as YBC 5037 ## 35-38 above, this exercise is concerned with a truncated square pyramid turned *upside-down*. The square top has the side s = 10, the height (actually, the depth) is h = 18, and the 'food' is 1 cubit per cubit (= ;05 nindan per cubit). Asked for are the (side of the) base and the volume. As in Sb 13293 and BM 85210 *rev. i*: 23-32, the 'food' is counted twice to get the rate of growth (line 2). The side of the base is calculated as

$s' = s - 2f \cdot h$ = 10 – 10 · 18 = 10 – 3 = 7 (lines 3-4)

What follows next is the computation of the volume of the truncated square pyramid by use of *a correct computation rule*, which begins with the computation of the "middle area", the area of a section of the truncated cone at mid-height:

A_m = sq. {1/2 ($s + s'$)} = sq. 8 30 = 1 12 15 (lines 4-5)

The continuation of the solution procedure is damaged and awkwardly worded but appears to say that

V' = {A_m + rec. 12 · sq. ($s - s'$)} · h' = (1 12 15 + 5 · 9) · 18 = 1 13 · 18 = 22 30 (lines 6-8)

This is a slight miscalculation, since 22 30 = 1 15 · 18. Anyway, in the last step of the solution procedure, the computed volume is expressed in standard Old Babylonian notations of volume measure:

V' = 22 30 (volume-sar) = 2 · 10 00 + 1 1/2 · 100 (volume-sar) = 2 èše 1 1/2 iku (line 9)

Where did this correct computation rule come from? To begin with, it is evident that neither $V_a = A_a \cdot h$ = 1/2 (sq. s + sq. s') · h nor $V_m = A_m \cdot h$ = sq. {1/2 ($s + s'$)} · h can be a correct expression for the volume of a truncated square pyramid, for the following reason: In the special case when s' = 0, $h' = h$, that is, in the case of a whole square pyramid, then

$V_a = A_a \cdot h'$ = 1/2 sq. s · h, which is more than the correct expression V = 1/3 sq. $s \cdot h$,

and

$V_m = A_m \cdot h'$ = sq. (1/2 s) · h, which is less than the correct expression V = 1/3 sq. $s \cdot h$.

A correct computation rule ought to give a volume less than V_a and more than V_m, which becomes reduced to V = 1/3 sq. $s \cdot h$ when s' = 0 and $h' = h$. It does not take much effort to find the solution:

V' = 1/3 V_a + 2/3 V_m, which is reduced to (1/3 · ;30 + 2/3 · ;15) · sq. $s \cdot h$ = ;20 sq. $s \cdot h$ when s'= 0 and $h' = h$.

It is easy to check, by the way, that

1/3 V_a + 2/3 V_m = (A_a + 2 A_m) · 1/3 h', with
A_a + 2 A_m = {1/2 (sq. s + sq. s') + 1/2 sq. ($s + s'$)} = sq. $s + s \cdot s'$ + sq. s'.

This is a link between two different but equivalent correct computation rules for the volume of a truncated square pyramid. In addition, it is easy to see that

V' = 1/3 V_a + 2/3 V_m = (1/3 A_a + 2/3 A_m) · h' = {A_m + 1/3 ($A_a - A_m$)} · h', with
1/3 ($A_a - A_m$) = 1/3 {1/2 (sq. s + sq. s') – sq. {1/2 ($s + s'$)}} = 1/3 · sq. {1/2 ($s - s'$)}.

Consequently,

V' = [sq. {1/2 ($s + s'$)} + 1/3 sq. {1/2 ($s - s'$)}] · h'.

This is the correct computation rule applied in BM 85194 # 28 above.

At the beginning of the present section (Sec. 8.5.10) it was shown how to find correct computation rules for the volumes of a ridge pyramid and a truncated ridge pyramid, *provided that it was already known how to compute the volumes of triangular and trapezoidal prisms* (see BM 96954+ § 3 in Sec. 8.5.4 above)*, as well as the volume of a square pyramid*. It is interesting to observe that it was not beyond the powers of talented Old Babylonian teachers of mathematics to find out and to demonstrate, to their own standards, how to compute correctly the volume of a *square pyramid*, or with the method used in Sb 13293, the volume of a *triangular pyramid*. A clue to how they could do that is the computation in BM 96954+ § 2 of the volume of a ridge pyramid *truncated at mid-height*.

Consider Fig. 8.5.9 below where a rectangular pyramid is cut by a horizontal plane at mid-height, and by four vertical planes through the sides of the rectangular base of the cut-off top pyramid.

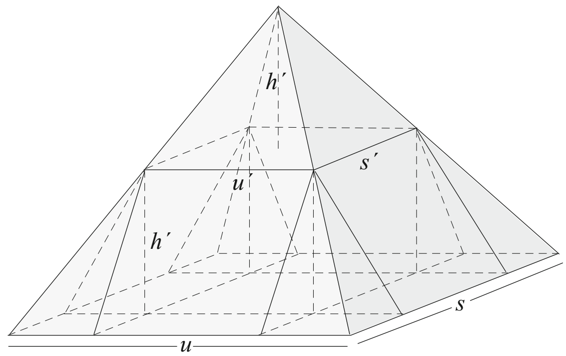


Fig. 8.5.10. A rectangular pyramid truncated at mid-height.

Let *u, s, h* be the length, the front, and the height of the original rectangular pyramid, let *u´, s´* be the length and front of the top of the truncated pyramid, and let *h´* be its height. Let further *V* be the volume of the original pyramid, let *V´* be the volume of the cut-off top pyramid, let V_c be the volume of the central block with the sides *u´*, *s´*, and the height *h´*, let V_u be the volume of each one of the two triangular prisms with the sides *u´*, 1/2 (*s* – *s´*), and the height *h´,* and let V_s be the volume of each one of the two triangular prisms with the sides 1/2 (*u* – *u´*), *s´,* and the height *h´*. Then

$V = V_c + 2\,V_u + 2\,V_s + V' + 4 \cdot 1/4\,V'$, with $V_c = u' \cdot s' \cdot h'$, $V_u = 1/2\,u' \cdot 1/2\,(s - s') \cdot h'$, $V_s = 1/2\,(u - u') \cdot s \cdot h'$, so that
$V = 1/2\ \{2\,u' \cdot s' + u' \cdot (s - s') + (u - u') \cdot s'\} \cdot h' + 2\,V' = 1/2\,(u' \cdot s + u \cdot s') \cdot h' + 2\,V'$.

Now, since the original pyramid was truncated at mid-height, so that *h´*= 1/2 *h*, it follows by use of a similarity argument that also *u´* = 1/2 *u* and *s´*= 1/2 *s*. Therefore, the equation above can be simplified to

$V = 1/4\,u \cdot s \cdot h + 2\,V'$.

Now, it is clear from the form of many geometrical entries in Old Babylonian tables of constants and from the solution procedures in many Old Babylonian geometrical problem texts that Old Babylonian mathematicians knew and exploited the assumption that *the areas of similar plane figures are proportional to the squares of their sides*. It is obvious that Old Babylonian mathematicians were equally aware of and knew how to exploit the assumption *that the volumes of similar solid figures are proportional to the cubes of their edges*. These facts would be self-evident to a people that used bricks as its most important building material!

In particular, since in Fig. 8.5.9 *u´*, *s´*, and *h´* are equal to 1/2 times *u*, *s*, and *h*, it follows that the volume of the cut-off top pyramid is 1/8 of the volume of the whole pyramid, so that 2 *V´* = 1/4 *V*. Therefore,

$V = 1/4\,u \cdot s \cdot h + 1/4\,V$, so that $3/4\,V = 1/4\,u \cdot s \cdot h$, and $V = 1/3\,u \cdot s \cdot h$.

This is *the correct computation rule for the volume of a rectangular pyramid*. Note that the rule is used in § 3 of BM 96954+ (Sec. 8.5.3 above).

One distinguishing feature of Old Babylonian mathematics is that once an interesting problem type and its solution had been discovered, all conceivable variations of that problem type or that solution procedure were also investigated. One particularly clear example are the many known variations of Old Babylonian problems concerned with striped triangles or striped trapezoids. See Friberg, *AT* (2007) Ch. 11. In view of the mentioned tendency in Old Babylonian mathematics, it is natural to ask if Old Babylonian mathematicians may have considered any possible variations of the idea demonstrated in Fig. 8.5.9, where a rectangular pyramid is divided into smaller parts by a plane at mid-height.

A clue is provided by the solution procedure in the exercise Sb 13293 (Sec. 8.5.8), illustrated by Fig. 8.5.4. According to an awkwardly worded passage in that solution procedure, the combined volume of a pair of rectangular pyramids at the two ends of a ridge pyramid is calculated as follows:

$2\,V_{\text{end}} = 2 \cdot (1 - 1/3) \cdot$ sq. $h \cdot h$ = 1;20 · 27 sq. nindan · nindan = 36 sq. nindan · nindan (lines 7-11)

What this means is that the volumes of the rectangular pyramids on either side of a central triangular prism (see Fig. 8.5.4) are computed as the volumes sq. *h* · *h* of the two triangular prisms containing them, minus one third of those volumes. A possible explanation of this strange way of computing volumes of rectangular pyramids is suggested in Fig. 8.5.10. Note that in order to simplify the diagram, *only the left half of a triangular prism* like the ones in Fig. 8.5.4 is considered in Fig. 8.5.10. However, what can be said about that left half can also be said about the right half and about the union of the two.

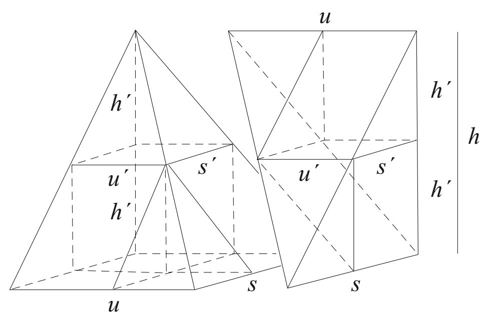


Fig. 8.5.11. A triangular prism divided into a rectangular and a triangular pyramid, both divided at mid-height.

In Fig. 8.5.10 it is shown how a triangular prism with the length *u*, the front *s*, and the height *h* is divided into a rectangular and a triangular pyramid by a slanting plane through one of the fronts and through the opposite end of the ridge at the top of the triangular prism. The rectangular pyramid has the length *u*, the front *s*, and the height *h*, while the triangular pyramid, put on its edge, has the length *h*, the front *s*, and the height *u*.

As shown in Fig. 8.5.10, *the triangular pyramid* can be divided by three planes into two smaller triangular pyramids, both with the length $h' = 1/2\ h$, the front $s' = 1/2\ s$, and the height $h' = 1/2\ h$, and two triangular prisms, both with the length u', the front s', and the height h'. Now let T denote the unknown volume of the original triangular prism and let T' denote the equally unknown volume of each one of the smaller cut-off triangular pyramids. Then, obviously,

$$T = 2\ \ u' \cdot s' \cdot 1/2\ h' + 2\ T'.$$

Now, let it again be assumed that *the volumes of similar solid figures are proportional to the cubes of their edges.* If that is so, then $T' = 1/8\ T$, and it follows that

$$T = 1/8\ \ u \cdot s \cdot h + 1/4\ \ T, \quad \text{so that} \quad 3/4\ T = 1/8\ \ u \cdot s \cdot h, \quad \text{and} \quad T = 1/6\ u \cdot s \cdot h.$$

It is clear that this means that *the volume of the triangular pyramid is one third of the volume of the triangular prism in which it is contained*. The same holds for the triangular pyramid which is the union of the triangular pyramid in Fig. 8.5.10 and its mirror image. This result agrees with what was assumed in the solution procedure of Sb 13293.

The volume of *the rectangular pyramid* to the left in Fig. 8.5.10 can be computed in a similar way, although that is unnecessary, because it is clear anyway that it is equal to 1 – 1/3 = 2/3 of the volume of the triangular prism in which it is contained.

8.5.12 The Computation Rules for Volumes of Cones in BM 96954+ §§ 6-7

In Sec. 8.5.10 above it was shown how clever dissections of pyramids as in Figs. 8.5.8-10 may have been used by talented Old Babylonian teachers of mathematics to find out and to demonstrate, to their own standards, how to compute correctly the volumes of various kinds of *whole or truncated pyramids*. Obviously, no such dissections can be used to find and demonstrate corresponding correct computation rules for the volumes of *whole or truncated cones*. Nevertheless, the use of precisely such computation rules is documented in BM 96954+ § 5 d and §§ 6-7. Where did those computation rules come from?

In BM 96954+ § 3 the seed measure of a square pyramid is computed by use of the correct computation rule $C = \mathrm{c}\ V$, $V = \text{sq. } s \cdot 1/3\ h$. In § 8.5.10 it was shown that, more generally, the volume of a rectangular pyramid can be computed as $V = u \cdot s \cdot 1/3\ h$, and the volume of a triangular pyramid as $V = u \cdot s \cdot 1/6\ h$. Such rules for the computation of the volume of a pyramid can be reformulated as the more elegant rule

The volume of a pyramid is equal to the area of the base of the pyramid times one third of its height.

Obviously, it did not take much imagination to come up with a corresponding rule for the volume of a cone:

The volume of a circular cone is equal to the area of the base of the cone times one third of its height.

In a similar way, a computation rule for the volume of a truncated square pyramid was discussed in Sec. 8.5.10, namely

$V' = (1/3\ A_a + 2/3\ A_m) \cdot h'$, where

$A_a = 1/2$ (sq. s +sq. s') = the "average" square area, and A_m = sq. $\{1/2\ (s + s')\}$ = the "middle square area".

Again, it must have been easy to come up with a corresponding rule for the volume of a truncated cone:

$V' = (1/3\ A_a + 2/3\ A_m) \cdot h'$, where

A_a = ;05 · 1/2 (sq. a +sq. a') = the "average" circle area, A_m = ;05 · sq. $\{1/2\ (a + a')\}$ = the "middle circle area".

This is the computation rule which was, probably, used in the damaged exercise BM 96954+ § 6.

Finally, the computation rule apparently used in § 7, that the volume of a cone of a given form is proportional to the cube of its height is as intuitively obvious as its counterpart for rectangular pyramids.

Examples of exercises using one of the two approximate computation rules $V' = A_m \cdot h'$ and $V' = A_a \cdot h'$ to compute the volume of a truncated cone can be found in four Old Babylonian mathematical texts.

In Haddad 104 # 3 (Al-Rawi and Roaf, *Sumer* 43 (1984)), the seed measure S of an *i-ṣú-um* 'log' with the bottom diameter '5' (nindan), the top diameter '1 40' (nindan), and the length 5' (nindan) is computed as follows:

A_m = 5 · sq. {3 · 1/2 (5 + 1 40)} = 5 · sq. 10 = 8 20, $V' = A_m \cdot h'$ = 8 20 · 1 = 8 20, $S = 6 \cdot V'$ = 50.

Here 6 *na-aš-pa-ki-im* '6 of the storing' is the storing number used in this exercise. (See Sec. 11.3.1 below.)

In VAT 8522 # 1 (Fig. 11.3.5 in Sec. 11.3.8), a [geš]eren 'cedar tree' has the bottom diameter 4, the top diameter 2, the length 5 nindan, while the storing number is 6 40. The seed measure is computed as follows:

A_m = 5 · sq. {3 · 1/2 (4 + 2)} = 5 · sq. 9 = 6 45, $V' = A_m \cdot h'$ = 6 45 · 1 = 6 45, S = 6 40 · V' = 45.

Exercise # 2 in BM 85196 (*MKT II*, 43), a recombination text from Sippar, deals with a [gi]sa 'reed bundle' with a lower arc 4, an upper arc 2, and a height 6 (= 6 cubits, a 'reed'). Its volume is computed as follows:

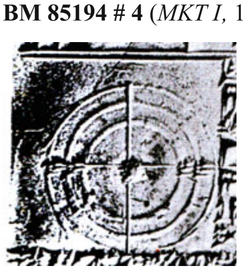
A_a = 1/2 (5 · sq. 4 + 5 sq. 2) = 1/2 (1 20 + 20) = 50, $V' = A_m \cdot h'$ = 50 · 6 = 5.

Exercise # 14 in BM 85194 (*MKT I*, 142), another recombination text from Sippar, is nearly identical with BM 85196 # 2. Also BM 56194 # 14 deals with a reed bundle with the same parameters. However, this time what is computed is the transversal (diameter) at mid-height, which is computed by use of a similarity argument.

8.5.13 BM 85194 # 4. Circular Defense Works with Trapezoidal Cross Sections

An interesting computation rule for the volume of a circular dike with a trapezoidal cross section is the basis for the solution procedure in BM 85194 # 4, an isolated exercise in a large recombination text from Sippar, where also many of the other exercises are concerned with problems for various kinds of solids.

BM 85194 # 4 (*MKT I,* 144; Group 6)



1-3 uru.ki 1+*šu* gúr *ak-pu-up* / 5.ta.àm *it-te-ṣí-ma* / *ḫi-ri-tam ab-ni* 6 gam /

4-6 1 07 30 saḫar.ḫà ba.zi / 5.ta.àm ugu *ḫi-ri-tim* / e *ab-ni*

7 e *šu-ú* / *i-na* 1 kùš 1 kùš šag$_4$.gal /

8 za.zum *mu-ḫu-um ù* sukud en.nam *ù* gúr en.nam /

9 za.e *i-nu-ma* 1+*šu* gúr dal en.nam

10 igi.3.gál 1+*šu* gúr ba.zi / 20 *ta-mar* 20 dal

11 5 *di-ik-ša-am* tab.ba 10 *ta-mar* / 10 *a-na* 20 dal daḫ.ḫa 30 *ta-mar*

12 dal *šu-li-iš* / 1 30 *ta-mar* 1 30 gúr *ša ḫi-ri-tim* /

13-14 nigín.na 1 30 nigin 2 15 *ta-mar* 2 15 *a-na* 5 gúr / *i-ši* 11 15 *ta-mar* gagar

15 11 15 *a-na* 6 gam *i-ši* <1 07 30 *ta-mar*> / igi 10 éš.gàr duḫ.a 6 *ta-mar*

16 6 *a-na* 1 07 30 saḫar.ḫà / *i-ši* 6 45 t*a-mar* 6 45 erín.meš

17-18 igi 6 45 duḫ.a / 8 53 20 *ta-mar* 8 53 20 *a-na* 1 30 gúr *i-ši* / 13 20 *ta-mar*
i-na ḫi-ri-tim uš *ta-pa-la-ak* /

19 nigín.na 10 éš.gàr *a-mur* igi 1 30 gúr duḫ.a 40 *ta-mar*

20 40 *a-na* 13 20 / *i-ši* 8 53 20 *ta-mar*

21 8 53 20 *a-na* 1 07 30 saḫar.ḫà / *i-ši* 10 *ta-mar* éš.gàr

22 nigín.na e *a-mur* 5 šag$_4$.gal tab.ba 10 *ta-mar /* 10 tab.ba 20 *ta-mar*

23 20 *a-na* 1 07 30 *i-ši* 22 30 / *ta-mar*
en.nam *a-na* 22 30 daḫ.ḫa {daḫ.ḫa} íb.sá$^{!}$ *li-pu-*<*ul*> /

24-25 *ù ša* daḫ.ḫa íb.sá *li-pu-ul* 5 03 45 daḫ.ḫa / 27 33 45 *ta-mar*
<27 33 45 en.nam íb.sá> 5 15 íb.sá za.zum /

26 5 03 45 en.nam íb.sá 2 15 íb.sá *mu-ḫu* /

27-28 igi 2+*šu* gúr e duḫ.a 30 *ta-mar* 30 *a-na* 1 07 30 <*i-ši*> / 33 45 *ta-mar*

29 nigín.na za.zum *ù mu-*{*ḫu-*}*ḫa* / ul.gar 7 30 *ta-mar*

30 $^1/_2$ 7 30 *ḫe-pé* 3 45 *ta-*<*mar*> / igi 3 45 duḫ.a 16 *ta-mar*

31 16 *a-na* 3 45 / *i-ši* 9 *ta-mar* 9 sukud *ša i-ki-im* /

32 *ki-a-am ne-pé-šum*

1-3 A town. 1 sixty the arc I curved / 5 each way I went out, then / a ditch I constructed, 6 the depth.

4-6 1 07 30 of mud I tore out / 5 each way beyond the moat / a dike I constructed.

7 This dike / in 1 cubit 1 cubit the food. /

8 The base, the top, and the height what? and the arc what?

9 You: When 1 sixty the arc, the transversal is what?

10 The 3rd part of 1 sixty, the arc, tear off, / 20 you will see, 20 the transversal.

11 5 the stick-out double, 10 you will see. / 10 to 20 the transversal add on, 30 you will see.

12 The transversal triple, / 1 30 you will see, 1 30 the arc of the ditch. /

13-14 Turn around. 1 30 square, 2 15 you will see. 2 15 to 5 of the circle / carry, 11 15 you will see, the ground.

15 11 15 to 6 the depth carry, <1 07 30 you will see.> / The reciprocal of 10 the work norm release, 6 you will see.

16 6 to 1 07 30 the mud / carry, 6 45 you will see, 6 45 the men.

17-18 The reciprocal of 6 45 release, / 8 53 20 you will see. 8 53 20 to 1 30 the arc carry, / 13 20 you will see,
from the excavation the length you mark off. /

19 Turn around. See 10, the work norm. The reciprocal of 1 30 the arc release, 40 you will see.

20 40 to 13 20 / carry, 8 53 20 you will see.

21 8 53 20 to 1 07 30 the mud / carry, 10 you will see, the work norm.

22 Turn around. See the ring wall. 5 the food double, 10 you will see. / 10 double, 20 you will see.

23 20 to 1 07 30 carry, 22 30 / you will see.
What to 22 30 to add on, an equalside it may result in? /

24-25 And what is added on an equalside it may result in? 5 03 45 add on / 27 33 45 you will see.
<27 33 45 what equalsided?> 5 15 equalsided, the base. /

26 5 03 45 what equalsided? 2 15 equalsided, the top. /

27-28 The reciprocal of 2 sixties release, 30 you will see. 30 to 1 07 30 <carry>, / 33 45 you will see.

29 Turn around. The base and the top / heap, 7 30 you will see.

30 1/2 of 7 30 break, 3 45 you will see. / The reciprocal of 3 45 release, 16 you will see.

31 16 to 3 45 / carry, 9 you will see, 9 the height of the dike. /

32 Such is the proceeding.

This exercise is proceeded by a diagram showing three concentric circles, with a distance of 5 (nindan) between each pair of circles. The question is formulated (essentially) as follows:

A circular town has the circumference $a = 1 \cdot 60$ nindan. (line 1)
A new circle is reached by going $e = 5$ (nindan) outwards everywhere. (line 2)
A circular ditch is dug, its depth is $h = 6$ (cubits), its 'food' is $f = 1$ cubit per cubit, its volume $V' = 1\ 07\ 30$ (sar). (lines 3-4)
Another new circle is reached by going $e' = 5$ (nindan) outwards again. (line 5-6)
A circular dike is built. Its inclination is $f = 1$ cubit per cubit. (lines 6-7)
What are the base b', the top t', and the height h' of the dike. What are the new circumferences a' and a''? (line 8)

(The extension of another circular town is the object of an exercise in the Old Babylonian round hand tablet Böhl 1821 (Group 6; from Sippar). See Fig. 9.1.1 below.)

The solution procedure begins by computing *the diameter d of the first circle*

$d = 1/3$ of $a = 20 \cdot 1(60) = 20$. (lines 9-10)

Next the the 'stick-out' (the distance from the first to the second circle) is doubled and added to the diameter d of the first circle. The result is *the diameter d′ of the second circle*:

$2 \cdot e + d = 2 \cdot 5 + 20 = 10 + 20 = 30 = d'$ (lines 10-11)

This second diameter is tripled in order to find *the second circumference a′*:

$3 \cdot d' = 3 \cdot 30 = 1\ 30 = a'$ (lines 11-12)

What happens next is *an effort to explain how the volume V′ = 1 07 30 (volume-sar) was computed*:

$A' = 5 \cdot \text{sq. } a' = 5 \cdot \text{sq. } 1\ 30 = 5 \cdot 2\ 15 = 11\quad 15$, $\quad A' \cdot h = 11\ 15 \cdot 6 = 1\ 07\ 30 = V'$ (lines 13-14)

This means that the given volume 1 07 30 is explained as an excavation in the form of a circular disk with the circumference $a' = 1\ 30$ and the depth $h = 6$. This "explanation" is clearly making no sense, and it is likely that the reference to a depth $h = 6$ was introduced into the statement of the problem by an inept student just in order to make the explanation possible!

In the next few lines answers are given to other questions which were never asked. *The number m of man-days needed to excavate the volume V′ = 1 07 30 (volume-sar)* is computed, under the assumption that the work norm for digging was (the usual) $w = 10$ shekels of mud per man-day:

$w = 10$ (volume-shekels per man-day), rec. $w = 6$ (man-days per volume-sar) (line 15)
$V' \cdot \text{rec. } w = 1\ 07\ 30$ (volume-sar) $\cdot$ 6 (man-days per volume-sar) = 6 45 (00) (man-days) = m (lines 15-16)

Then the length of the second circumference is divided by the number of man-days in order to find out *the progress p along the arc a′ in one man-day of digging*, assuming (unrealistically) that in one man-day a very thin slice of the circular disk with the circumference a' and the depth g was excavated:

$a' \cdot \text{rec. } m = 1\ 30$ (nindan) $\cdot$ rec. 6 45 (00) (man-days) = $1\ 30 \cdot 8\ 53\ 20 = 13\ 20$ ($\cdot\ 60^{-3}$ nindan per man-day) = p (lines 16-18)

The result is verified by counting backwards:

$p \cdot \text{rec. } a' = 13\ 20 \cdot 40 = 8\ 53\ 20 = \text{rec. } m$, $\quad p \cdot \text{rec. } a' \cdot V' = 8\ 53\ 20 \cdot 1\ 07\ 30 = 10 = w$ (lines 19-21)

After all these unmotivated digressions, it is finally time to consider the ring wall (the 'dike') built by use of the excavated mud. The reasoning is silently based on the following c*omputation rule for the volume of a circular ring wall with a trapezoidal cross section, with the base b′, the top t′, and the height h′, when the base of the ring wall is bisected by the arc a′′* (see Fig. 8.5.11 below):

V' = circumference $\cdot$ area of cross section = $a'' \cdot 1/2\ (b' + t') \cdot h'$.

The base b′, the top t′, and the height h′ of the ring wall can therefore be computed as follows:

$2 \cdot 2 \cdot f\ = ;20$ (nindan per cubit) $= 2\ (b' - t') \cdot \text{rec. } h'$, (lines 21-22)
$V' \cdot 2 \cdot 2 \cdot f\ = 1\ 07\ 30 \cdot ;20 = 22\ 30$ (cubic nindan) $= a'' \cdot (b' + t') \cdot (b' - t')$ (line 22)

For the computation to proceed correctly from here, the next step should have been to divide 22 30 by the arc a'' of the outermost circle. The length of which could have been computed as follows:

$d' = 40$, $\quad d'' = d' + 2 \cdot e'\ = 30 + 2 \cdot 5 = 40$, $\quad a'' = 3\ d'' = 2\ (00)$.

Incorrectly, the division by a'' is omitted, and the author of the text continues as if

$V' \cdot 2 \cdot 2 \cdot f$ = 1 07 30 · 20 = 22 30 (cubic nindan) = $(b' + t') \cdot (b' - t')$ = sq. b' – sq. t'.

Note that the equation

sq. b' – sq. t' = 22 30

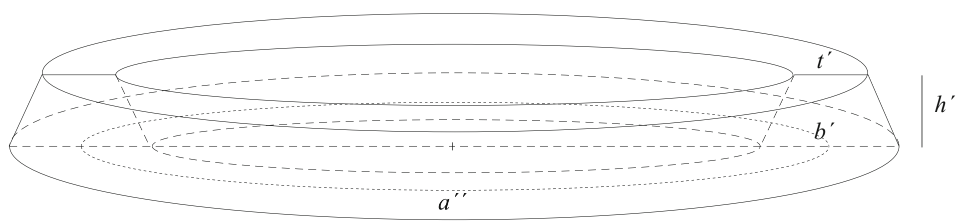
is *an indeterminate quadratic equation*! (Cf. the discussion in Friberg, *AT* (2007), Sec. 13.2 c of Diophantus' *Arithmetica* II.10.) The solution procedure in BM 85194 # 4 solves this indeterminate quadratic equation in the following way, without explaining why:

What added to 22 30 makes a square number and is itself a square number? Answer 5 03 45 (lines 23-24)
22 30 + 5 03 45 = 27 33 45 = sq. 5 15, so that b' = 5 15 and (line 25)
5 03 45 = sq. 2 15 so that t' = 2 15 (line 26)

The last part of the solution procedure shows how to find the height h' of the ring wall:

rec. 1/2 $(b' + t')$ · rec. a'' · V' = rec. 3 45 · rec 2 · 1 07 30 = 16 · 30 · 1 07 30 = 9 = h' (lines 27-31)

(This time the author of the text remembers to take the outer arc a'' into consideration.)



$V' = a'' \cdot 1/2\,(b' + t') \cdot h'$

Fig. 8.5.12. A computation rule for the volume of a ring wall with a trapezoidal cross section.

It was never explained in the solution procedure how the number 5 03 45 was constructed, which defined a solution to the indeterminate quadratic equation sq. b' – sq. t' = 22 30. It is quite possible that the student who wrote the text of the exercise had just made a note of the number instead of listening to this teacher explaining how the number could be found. Actually, it is not particularly difficult to find a solution to an indeterminate quadratic equation of this kind, namely by use of factorization. Indeed, write the equation as

$(b' + t') \cdot (b' - t')$ = 22 30 = 7 30 · 3, for instance. There are many other possible factorizations.

Then it is natural is to choose to set

$b' + t'$ = 7 30, $b' - t'$ = 3.

This is a simple system of linear equations with the obvious solution

b' = 1/2 · (7 30 + 3) = 5 15, t' = 1/2 · (7 30 – 3) = 2 15.

(Actually, if b' and t' are supposed to be of a reasonable order of magnitude, then b' = 5;15 and t' = 2;15, so that $b' + t'$ = 7;30, $b' - t'$ = 3, and $(b' + t') \cdot (b' - t')$ = 22;30.)

If the author of the text had not forgotten to divide 22 30 by a'' = 2 (00) after line 22 he would have had to solve instead the indeterminate quadratic equation $(b' + t') \cdot (b' - t')$ = 11;15. Then, since 11;15 can be factorized as 11;15 = 4;30 · 2;30, for instance, this would lead to the system of equations

$b' + t'$ = 4;30, $b' - t'$ = 2:30.

The corresponding solution to the equation would then be

b' = 1/2 · 7 = 3;30, t' = 1/2 · 2 = 1, and consequently h' = rec. 1/2 $(b' + t')$ · rec. a'' · V' = ;26 40 · 33;45 = 15.

Now, look again at lines 3-4 of the text, where it is said that a moat (a 'ditch') was constructed 5 nindan out from the circular town, that is at the circle with the arc a' = 1 30, and that the volume of the excavated mud was 1 07 30 (sar). (As explained before, the spurious mention of a height h = 6 can be disregarded.) The situation is depicted in Fig. 8.5.12 below.

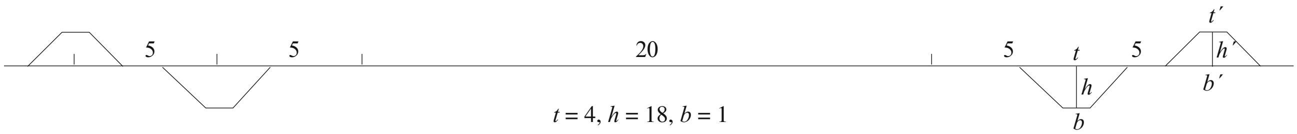


Fig. 8.5.13. Cross sections of a circular moat and a concentric circular ring wall with identical volumes (corrected data).

Reasoning as in the case of the ring wall (see the explanation above of line 22 in the text) one can show that in the case of a moat with the middle arc $a´$, the top t, the base b, the volume 1 07 30, and the 'food' 1 cubit per cubit,

22 30 (cubic nindan) = $a´ \cdot (t + b) \cdot (t - b)$

Since $a´$ = 1 30, it follows that t and b are solutions to the following indeterminate quadratic equation:

$(t + b) \cdot (t - b)$ = 22 30 · rec. 1 30 = 15.

The simplest possible solution to this indeterminate equation is, of course,

$t + b = 5$, $t - b = 3$, so that $t = 4$, $b = 1$, and therefore
h = rec. 1/2 $(t + b)$ · rec. $a´ \cdot V$ = ;24 · ;00 40 · 1 07 30 = 18.

Where did the computation rule for the volume of a circular ring wall come from which is used in BM 85194 # 4, for instance for the computation of the height $h´$ in lines 27-31? It is interesting to observe that the mentioned computation rule is a consequence of the computation rule for the volume of a truncated circular cone, which was (probably) used in the partly preserved exercise BM 96954+ § 7.

That is because *the volume V´ of a circular ring wall with a trapezoidal cross section, like the one depicted in Fig. 8.5.11 above, can be computed as the volume V_o of an outer truncated cone with the lower arc a´´+ 3 b´, the upper arc a´´+ 3 t´, and the height h´, minus the volume V_i of an inner (upside-down) truncated cone with the upper arc a´´ – 3 t´, the lower arc a´´ – 3 b´, and the same height h´.*

Now, recall that it was shown in Sec. 8.5.10 above that the correct Old Babylonian computation rule for the volume of a truncated square pyramid used in BM 85194 # 28 has the form

$V´$ = [sq. {1/2 $(s + s´)$} + 1/3 sq. {1/2 $(s - s´)$}] · $h´$.

A corresponding Old Babylonian computation rule for the volume of a truncated cone with the lower arc a, the upper arc $a´$, and the height $h´$, would (presumably) be

$V´$ = ;05 · [sq. {1/2 $(a + a´)$} + 1/3 sq. {1/2 $(a - a´)$}] · $h´$.

In view of this computation rule, the volumes of the outer and inner truncated cones in Fig. 8.5.11 would be

V_o = ;05 · [sq. {1/2 (2 $a´´$ + 3 $(b´ + t´)$)} + 1/3 sq. {1/2 · 3 $(b´ - t´)$}] · $h´$,
V_i = ;05 · [sq. {1/2 (2 $a´´$ – 3 $(b´ + t´)$)} + 1/3 sq. {1/2 · 3 $(b´ - t´)$}] · $h´$.

Consequently, the volume $V´$ of the circular ring wall in Fig. 8.5.11 would be

$V´ = V_o - V_i$ = ;05 · [sq. {1/2 (2 $a´´$ + 3 $(b´ + t´)$)} – sq. {1/2 (2 $a´´$ – 3 $(b´ + t´)$)}] · $h´$.

By use of the Old Babylonian conjugate rule (see Fig. 6.2.1) this rule can be simplified to

$V´ = V_o - V_i$ = ;05 · [$a´´$ · 2 · 3 $(b´ + t´)$] · $h´$ = $a´´$ · 1/2 $(b´ + t´)$ · $h´$.

(This result can be viewed as the Old Babylonian version of a special case of what in modern mathematics goes under the name "Pappus' centroid theorem for the volume of a solid of revolution". See Heath, *HGM II* (1981), 403. Pappus wrote in the reign of Diocletian, 284-305 AD.)

8.5.14 Pyramids and Cones in Ancient Chinese, Greek and Indian Mathematical Treatises.

BM 96954+ is clearly a mathematical recombination text, although with an easily recognizable theme, pyramids and cones. That it is a recombination text is shown by the table of contents (Fig. 8.5.5 above), which shows that the exercises are not arranged in a logical way, proceeding from the simpler ones to the more complex ones. In a proper theme text, § 4 (the seed measure of various solids) should come first, followed by § 3 (the seed measure of a truncated pyramid), § 1 (the seed measure of a ridge pyramid), and § 2 (the seed measure of a truncated ridge pyramid). As it is now, §§ 2 and 3 are inserted separately among the exercises of § 1. Yet it is also clear that the many exercises of § 1 were excerpted from a well organized theme text.

§§ 5-7 (about whole and truncated circular cones) are probably excerpts from another well organized theme text, because the contents of the pyramids in §§ 1-3 are given in seed measure, while the contents of the cones in §§ 5-7 are given in volume measure.

Interestingly, if one wants to know what an original Old Babylonian theme text dealing with pyramids and cones might have looked like, one can look at Chapter V of *Jiuzhang Suanshu* '*Nine Chapters on the Mathematical Art*'. This is a Chinese mathematical classic, which appears to contain some very ancient material (some of the exercises in the book can be shown to date back to before the Qin dynasty (221-207 BC)), but which probably reached its present form in the first century AD. On one hand *JZSS* Ch. V is concerned with earthworks and the amount of labor required to build them. On the other hand, as is shown in the table of contents in Fig. 8.5.13 below, the chapter begins with simple calculations of the volumes of trapezoidal prisms and then goes on to calculations of the volumes of square prisms, square truncated pyramids, and square whole pyramids, side by side with calculations of the volumes of circular cylinders, circular truncated cones, and circular whole cones.

Next it is shown how a square prism can be cut by a plane section into two triangular prisms, and how a triangular prism can be cut into two pieces, one a triangular and the other a square pyramid. Then there is an exercise where the volume is calculated of an irregular wedge-like solid, followed by exercises dealing with whole and truncated ridge pyramids and with a ring having a trapezoidal cross section. The chapter ends with uninteresting calculations of the volumes of a whole cone, a half-cone, and a quarter-cone. These three exercises may be late additions to the theme text.

All the computation rules for the volumes of various pyramids and cones, etc., in *JZSS* Ch. V are correct, but there are no motivations given for any of those computation rules. In this respect there is nothing except the language, the script, and the measures used, that distinguishes this Chinese theme text from an Old Babylonian theme text with the same content. However, if there ever was a historical connection between ancient Chinese and Old Babylonian mathematics is, of course, impossible to know.

Anyway, in Fig. 8.5.13 below it is explicitly mentioned which Old Babylonian mathematical exercises are known counterparts to exercises ## 2-23 of *JZSS* Ch. V. All the computation rules employed in those Old Babylonian exercises are correct, too, with the possible exception of the computation rule for the volume of an irregular wedge (the *xianchu* in *JZSS* V # 17). Note, however that this is an exceptionally difficult case. See Shen, Crossley, and Lun, *NCMA* (1999), 283, 306, where 12 different varieties of the *xianchu* are enumerated, each requiring a separate derivation of the computation rule for the volume.

A documented attempt to give rigorously correct derivations of the computation rules in *JZSS* Ch. V was first made in the commentaries of Liu Hui (third century AD). See Wagner, *HM* 6 (1979). It is particularly interesting that in his proof of the computation rules for a *yangma* and a *bienao* in *JZSS* V. 15-16, *Liu Hui makes use of the same cutting up of those rectangular and triangular pyramids as in Fig. 8.5.10 above,* possibly used by Old Babylonian mathematicians for the same purpose. However, having cut up the *bienao*, for instance, into two sub-pyramids and two triangular prisms, Liu Hui does not rely, like (probably) the Old Babylonians, on the unproved assumption that the volumes of similar pyramids are proportional to the cubes of their edges. Instead he cuts up the sub-pyramids in the same way as the original *bienao* and continues so, attempting to carry the process to the limit.

As a matter of fact, it is a consequence of a theorem proved by Dehn in *MA* 55 (1902) that any rigorous derivation of a computation rule for the volume of a pyramid must use infinitesimal considerations in one form or another.

In his introduction to *The Method*, Archimedes wrote (see Heath, *HGM I* (1921, 1981), 21):

> "Certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish the actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems that the volumes of a cone and a pyramid are one-third of the volumes of the cylinder and the prism respectively having the same base and the same height, the proofs of which Eudoxus was the first to discover, no small share of the credit should be given to Democritus, who was the first to state the fact, though without proof."

Although Archimedes was (probably) mistaken about the role played by Democritus, it is reasonable to assume, in view of this testimony, that the contents of the stereometric Book XII of Euclid's *Elements* can be ascribed to Eudoxus (the first half of the fourth century BC). Indeed, in the *Elements*, the propositions XII: 1-2 deal with circles, while XII: 3-9, XII: 10-15, and XII: 16-18 are concerned with pyramids, cones, and spheres, respectively.

In the context of the present discussion, the propositions *El*. XII:3-9 and 10-15 are particularly interesting. (See Heath *TBE III* (1956), 378 ff.) Their contents will be outlined below, in intentionally modernized form. (Note, in particular, that Euclid himself avoids speaking about areas and volumes, concepts which are difficult to define rigorously. Instead he speaks about "equal" plane and solid figures.)

XII: 3. A certain dissection of a triangular pyramid.

Every triangular pyramid can be cut (by three planes through the mid-points of the six edges) into two pyramids of equal volumes, similar to each other and to the whole pyramid, and two wedges (triangular prisms) of equal volumes (but not similar to each other).
The combined volume of the two wedges is greater than half the volume of the whole pyramid.

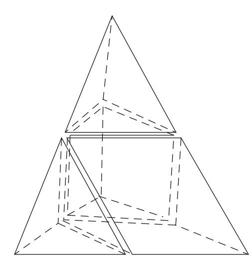
XII: 4. Corresponding dissections of two triangular pyramids of the same height.

Let two triangular pyramids of the same height both be cut into two pyramids and two wedges in the way described in XII:3. Then the combined volumes of the two wedges in each pyramid separately are proportional to the areas of the bases of the pyramids.

XII: 5. The ratio between volumes and areas in triangular pyramids of the same height.

The volumes of two triangular pyramids of the same height are proportional to the areas of their bases.

(This proposition is proved by means of the exhaustion method of Eudoxus. Suppose that the two sub-pyramids produced by the regular dissection described in XII: 3 are in their turn dissected in the same way. The result is two new, smaller wedges and two new, smaller sub-pyramids. The process can be repeated until the combined volumes of all the produced wedges is arbitrarily large compared to the combined volumes of the sub-pyramids. If there are two pyramids of the same height, and if the successive regular dissections are carried out in tandem, then, after each step of the algorithm, the combined volumes of all the produced wedges in each pyramid separately are proportional to the areas of the bases. Therefore, the ratio between the volumes of the pyramids can be neither greater nor smaller than the ratio between the areas of their bases.)

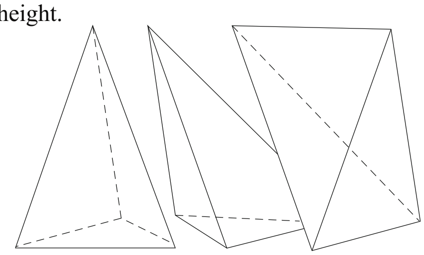


XII: 6. The ratio between volumes and areas in polygonal pyramids of the same height.

The volumes of two polygonal pyramids of the same height are proportional to the areas of their bases.

XII: 7. A dissection of a triangular prism into three triangular pyramids.

Every triangular prism can be cut (by two planes through four of the six vertices) into three triangular pyramids of equal volumes (but not similar to each other). The three sub-pyramids have, two by two, equal heights and bases of equal areas.



Corollary: The volume of a pyramid is one-third of the volume of the prism containing it.

XII: 8. The ratio between the volumes of similar triangular pyramids.

The volumes of similar triangular pyramids are proportional to the volumes of the cubes of their corresponding edges. (This is because, in view of XII:7, the volume of a triangular pyramid is one-third of the volume of the triangular prism containing it.)

XII: 9. The relation between volume, base area, and height of a triangular pyramid.

Two triangular pyramids have equal volumes if and only if the areas of their bases are inversely proportional to their heights. (In other words, the volume of a triangular pyramid is equal to a constant times the product of its height and the area of its base.)

XII: 10. The ratio between the volume of a cone and the volume of the cylinder containing it.

The volume of a cone is one-third of the volume of the cylinder with the same base and the same height.

XII: 11. The relation between volume, base area, and height of a cone.

The volumes of two cones with the same height are to each other as their base areas. (In other words, the volume of a cone is equal to a constant times the product of its height and the area of its base.)

XII: 12. The ratio between the volumes of similar cones.

The volumes of similar cones are proportional to the volumes of the cubes of the diameters of their bases.

Note the statement that similar pyramids are proportional to (the volumes of) the cubes of their edges *because* the volume of a pyramid is one-third of the volume of the prism containing it, and not the other way around as in the (conjectured) Old Babylonian derivation of the computation rule for the volume of a pyramid.

Note also that both Euclid and Liu Hui in their proofs make use of pyramids cut by a plane at mid-height, and compare with Fig. 8.5.2 above where a ridge pyramid is truncated by a plane at mid-height and with Fig. 8.5.3, left, where a cone is truncated by a plane at mid-height.

In Heron of Alexandria's *Metrica, Book II* (written somewhere between 150 and 250 AD), sections II.1-8 contain derivations of correct expressions for the volumes of a pyramid, a truncated triangular pyramid, a cone, a truncated cone, a truncated ridge pyramid, and an "anchor ring" (generated through the rotation of a circle around a point outside the circle). See Heath, *HGM II* (1981), 332.

The first correct expressions for the volumes of pyramids and truncated pyramids, etc. in the preserved corpus of ancient Indian mathematical texts are the ones in Brahmagupta's work. Thus, in *Brāhmasphuṭasiddhānta* V.44 it is stated that (see Colebrook *AAMS* (1973), 312); Sarasvati Amma *GAMI* (1979), 197-208):

"The area of a plane figure, multiplied by the depth, gives the content of the equal excavation, and that, divided by three, is the content of the 'needle'."

An expression for the volume of a truncated pyramid and (possibly) a truncated cone is given in the next two verses, V.45-46:

The area derived from the half-parts of the sums of the sides at the top and the bottom, multiplied by the depth, is the 'practical measure' of the content.
Half the sum of the areas at the top and the bottom, multiplied by the depth, gives the 'gross content'.
Subtract the practical content from the other, divide the difference by three, and add the quotient to the practical content.
The sum is the 'neat content'."

This is clearly the same computation rule as the one discussed in Sec. 8.5.10 above, in connection with the exercise BM 85194 # 28. In the case of a truncated pyramid that computation rule is

$V' = \{A_m + 1/3\,(A_a - A_m)\} \cdot h'$ (the neat content), where

$A_a = 1/2\,(\text{sq. } s + \text{sq. } s')$ (the gross content) and $A_m = \text{sq. } \{1/2\,(s + s')\}$ (the practical content).

V. 2-4. A dike, a wall, a dam	$V = 1/2\,(s + s') \cdot u \cdot h$
V. 5-7. A trench, a moat, a canal OB math *passim*	
V. 8. A square fort	$V = \text{sq. } s \cdot h$
V. 9. A round fort OB math *passim*	$V = \text{sq. } a \cdot 1/12\ h$
V. 10. A square pavilion BM 85194 # 28	$V = (\text{sq. } s + s \cdot s' + \text{sq. } s') \cdot 1/3\ h$
V. 11. A round pavilion BM 96954+ § 6	$V = (\text{sq. } a + a \cdot a' + \text{sq. } a') \cdot 1/36\ h$
V. 12. A square needle BM 96954+ § 3	$V = \text{sq. } s \cdot 1/3\ h$
V. 13. A round needle BM 96954+ § 5 d	$V = \text{sq. } a \cdot 1/36\ h$
V. 14. An embankment *qiandu* BM 85196 # 3	$V = u \cdot s \cdot 1/2\ h$
V. 15. A male horse *yangma*	$V = u \cdot s \cdot 1/3\ h$
V. 16. A turtle's foreleg *bienao* *TMS* XIV	$V = u \cdot s \cdot 1/6\ h$
V. 17. A tunnel entrance *xianchu* BM 85196 + # 4 (??)	$V = (a + b + c) \cdot u \cdot 1/6\ h$
V. 18. A cut grass ridge *chumeng* BM 96954+ § 1	$V = (2\,u + r) \cdot s \cdot 1/6\ h$
V. 19, 21-22. A cut grass overhang *chutong* BM 96954+ § 2	$V = \{(2\,u + u') \cdot s + (u + 2\,u') \cdot s'\} \cdot 1/6\ h$
V. 20. A bent moat *quchi* BM 85194 # 4 ($b = a$)	$V = \{(2\,a + b) \cdot s + (a + 2\,b) \cdot t\} \cdot 1/6\ h$
V. 23. Millet stored on the ground	$V = u \cdot s \cdot 1/3\ h$
V. 24. Soybeans stored against a wall	$V = u \cdot s \cdot 1/3\ h$
V. 25. Rice stored in a corner	$V = u \cdot s \cdot 1/3\ h$

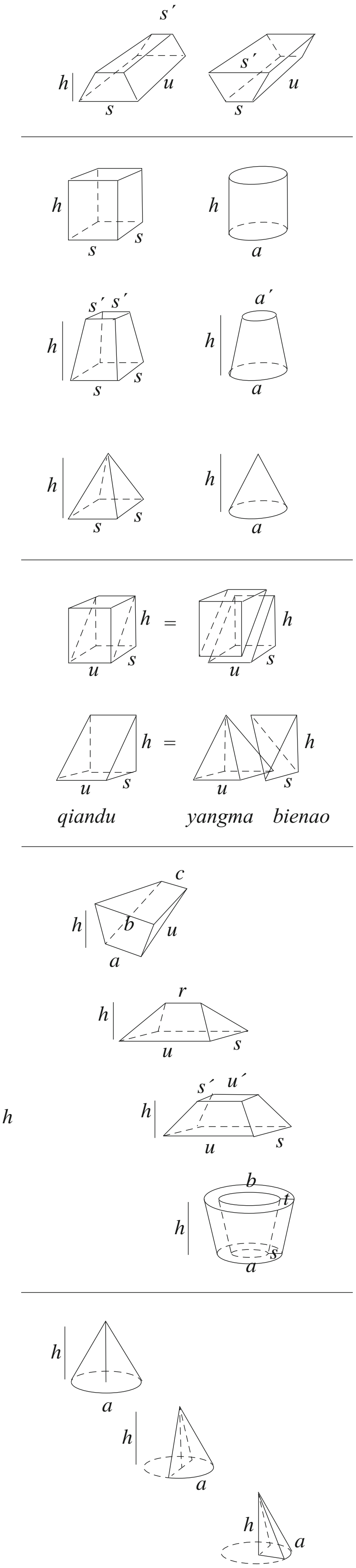


Fig. 8.5.14. Exercises in Ch. V of JZSS, *Nine Chapters on the Mathematical Art*, and their OB counterparts.

8.6 BM 96957+. A Large Recombination Text Concerned with Bricks and Diagonals

This text has an interesting publication history. In its present state it is composed of two large fragments, BM 96957 and VAT 6598. Two of the exercises on VAT 6598, both concerned with rectangles and their diagonals, were published by Weidner in *OLZ* 19 (1916), precisely a century ago. That was the beginning of the study of Old Babylonian mathematical problem texts. Later, a complete edition of VAT 6598 was published by Neugebauer in *MKT I* (1935). The fragment BM 96957 was identified as a join to VAT 6598 by Walker in 1995. The hand copy of the two fragments together shown in Figs. 8.6.3-4 was made by Al-Rawi at that time but has not been published before. It is slightly different from a hand copy in Robson *MMTC* (1999), based on a combination of Neugebauer's copy of VAT 6589 and Walker's copy of BM 96957.

A full edition of BM 96957+VAT 6598 appeared already in Robson (*op. cit*.). The edition below differs from Robson's in a number of important details, such as the explanation of the extra thick bricks of variant types which appear in two of the paragraphs of the text, and the consideration of the square pyramid which presumably appears in a separate paragraph sandwiched in between the two paragraphs dealing with bricks.

8.6.1 § 1. Metric Algebra Problems for a Wall Made of Bricks of the Variant Type R1/2cv

The first (partly) preserved exercise on BM 96957+ is § 1a below.

BM 96957+ § 1a (*obv. i*: 1-4)

1-2 [é.gar$_8$ 2 kùš *ku-bu*]*-ru-ú* 2 1/2 nindan uš 1 1/2 nindan sukud / [sig$_4$ en.nam]
[za].e 2 kùš ⌜sag⌝ *a-na* 2 1/2 nindan uš *i-ši* 25 *qà-qá-rum* *ta-mar* /
3 [25] ⌜*a*⌝*-na* 18 sukud *i-ši* 7 30 7 1/2 sar saḫar-*rum* *ta-mar*
4 7 30 *a-na* 6 / [igi].⌜gub⌝ é.gar$_8$ *i-ši* 45 *ta-mar* 45 sar sig$_4$
ne-pé-šum

1-2 [A wall. 2 cubits the thick]ness, 2 1/2 nindan the length, 1 1/2 nindan the height. / [The bricks were what?]
[Yo]u: 2 cubits the ⌜front⌝ to 2 1/2 nindan the length, raise, 25 the ground you will see.
3 [25 ⌜t⌝o 18 the height raise, 7 30 7 1/2 sar the mud you will see.
4 7 30 to 6 / [the con]⌜stant⌝ of the wall raise, 45 you will see, 45 sar of bricks.
The procedure.

The object considered in this exercise is a (brick) wall, called é.gar$_8$ (Akk. *igāru*), with the thickness s = 2 cubits (about 1 meter), the length u = 2 1/2 nindan (15 meters), and the height h = 1 1/2 nindan (9 meters). The bottom of the wall is then a rectangle with the length 2 1/2 nindan and the front 2 cubits (= ;10 nindan). Hence, the area A of the bottom rectangle (*qaqqaru* "the ground") is ;25 sq. nindan. Since the height of the wall is 1 1/2 nindan (= 18 cubits), it follows that the volume of the wall is

V = ;25 sq. nindan · 18 cubits = 7;30 sar = 7 1/2 sar (line 3)

The kind of bricks used for building the wall is not explicitly mentioned. (The reason for this omission is probably that § 1a of BM 96957+ is an excerpt from an older theme text, in which the kind of bricks used was specified in an exercise near the beginning of the theme text.) Instead it is mentioned, incidentally, that the "molding number" is '6', called '6, the constant of the wall'. As explained in Sec. 8.1 above, this means that the volume of 6 brick sar of bricks of the kind occurring in this exercise is 1 volume sar. Bricks with the molding number L = 6 are *rectangular bricks of the variant type R1/2cv*, more precisely bricks with the length 1/2 cubit, the front 1/3 cubit, and the height 6 fingers = 1/3 cubit. See Friberg, *ChV* (2001), Table 4.2. Compare with the text BM 80078 in Sec. 8.1 above in which occur bricks of the two variant types R3nv and R1/2c.

Therefore, clearly, the number of bricks used to build the wall in § 1a is

B = 6 brick sar/volume sar · 7 1/2 volume sar = 45 brick sar (= 45 · 12 00 bricks) (line 4)

Note the following glosses written in small script below lines 2 and 3:

the ground and *7 1/2 sar, the mud*

The next (partly) preserved exercise is § 1b below.

BM 96957+ § 1b (*obv. i*: 5-10)

1-2 ⸢é⸣.gar$_8$ 2 ½ nindan uš 1 ½ nindan sukud 45 sar sig$_4$ *ku-bu-re-e* / [é].gar$_8$-*ia* en.nam
3 za.e 2 30 uš *a-na* 18 sukud *i-ši-ma* / [45 *ta*]-*mar ki-il* $_{\textit{pa-nu}}$
4 igi 6 igi.gub é.gar$_8$ duḫ.a *a-na* / [45 sar sig$_4$ *i-ši*] 7 30 $_{\text{saḫar-}\textit{rum}}$ *ta-mar ki-il*
5 igi 45 *ša tu-ka-lu* / [duḫ.a 1 20 *ta-mar*]
6 [1] ⸢20 *a-na*⸣ 7 30 $_{\text{saḫar}}$ *ša tu-ka-lu i-ši-ma* / [10 *ta-mar* 2 kúš *ku-bu-re-e* é.gar$_8$]
⸢*ne-pé-šum*⸣

1-2 A [w]all. 2 ½ nindan the length, 1 ½ nindan the height, 45 sar the bricks. The thickness / of my [w]all was what?
3 You: 2 30 the length to 18 the height raise, then / [45 you] will see. Keep! $_{\text{The face.}}$
4 The opposite of 6 the constant of the wall resolve, to / [45 sar of bricks raise) 7 30 $_{\text{the mud}}$ you will see. Keep!
5 The opposite of 45 that you held /[resolve, 1 20 you will see].
6 [1] ⸢20 to⸣ 7 30 $_{\text{the mud}}$ that you held raise, then /[10 you will see. 2 cubits the thickness of the wall].
⸢The procedure.⸣

In this exercise, the same brick wall is considered as before. However, only the the length and the height of the wall are assumed to be known, together with the number of bricks. The object of the exercise is to compute the thickness *s* of the wall. The first step in the solution algorithm is the computation of the area of a face of the wall, explicitly glossed as *pānu* 'face'. (Robson's claim in *MMTC* (1999), 231 that *pānu* is a gloss to igi, which it precedes, is almost certainly a mistake.) That area is, of course,

$A_{face} = u \cdot h = 2;30$ nindan · 18 cubits = 45 nindan · cubits. (line 4)

Consequently, *w* can be computed as follows:

V = rec. (6 brick sar/volume sar) · 45 brick sar = 7;30 volume sar (lines 3-4)

s = rec. $A_{face} \cdot V$ = rec. 45 nindan · cubit) · 7;30 volume sar = ;01 20 · 7;30 nindan = ;10 nindan = 2 cubits

The remaining exercises of § 1 of BM 969957+ are lost, with the exception of the last few lines of the last exercise, below called § 1f. See the suggested reconstruction of § 1 in Fig. 8.6.2 below.

BM 96957+ § 1f (*obv. i*: … - *ii*: 4)

… ……………………
1′ ………… / 1 21 40 *ta-mar*
2′ 25 *ša tu-ka-lu a-na* 1 21 40 / daḫ.ḫa 1 46 40 *ta-mar*
3′ 1 46 40 en.nam íb.<sá> 1 20 íb.sá / *a-di* 2 gar.ra
1 10 *a-na* 1 20 daḫ.ḫa 2 30 *ta-mar* 2 30 ⸢uš é.gar$_8$⸣ /
4′ 1 10 *i-na* 1 20 ba.zi 10 íb.tag$_4$ 2 kùš *ku-bu-re-e* é.gar$_8$
ne-pé-šum

… ……………………
1′ ………… / 1 21 40 you will see.
2′ 25 that you held to 1 21 40 / add on, 1 46 40 you will see.
3′ 1 46 40 is what equ<alsided>? 1 20 equalsided. / Until 2 set.
1 10 to 1 20 add on, 2 30 you will see. 2 30 ⸢the length of the wall⸣. /
4′ 1 10 from 1 20 tear off, 10 the remainder. 2 cubits the thickness of the wall.
The procedure.

It is suggested in Fig. 8.6.2 that in §§ 1c-d the unknowns were the length *u* and the height *h*, respectively, both computed as solutions to simple linear equations. Additionally, in §§ 1e-f, the length *u* and the thickness *s* of the bottom rectangle were computed as the solutions to quadratic equations with $u + s$ and $u - s$ respectively, given, in addition to the height *h* and the number of bricks *B*. This is a common pattern in many Old Babylonian mathematical theme texts. The same pattern can be observed again in the better preserved § 3 of BM 96957+. See again Fig. 8.6.2 below.

It is not difficult to find out that in § 1f the question, reformulated in quasi-modern terms, is

$u - s$ = 2;20 nindan, ($L \cdot u \cdot s \cdot h$ =) B = 45 sar. u, s = ? (L = 6 brick sar/volume sar, h = 18 cubits)

This question can easily be rephrased as the following rectangular-linear problem:

$u \cdot s$ = ;10 · ;03 20 · 45 sq. nindan = ;25 sq. nindan, $u - s$ = 2;20 nindan. u, s = ?

Following the usual solution algorithm, u and w can be computed as follows:

sq. $(u + s)/2$ = sq. $(u - s)/2 + u \cdot s$ = sq. 1;10 + ;25 = 1;21 40 + ;25 = 1;46 40 = sq. 1;20 (lines … - 2′)
u = 1;20 + 1;10 = 2;30 (nindan), s = 1;20 – 1;10 = ;10 (nindan) = 2 cubits (lines 3′ - 4′)

8.6.2 § 2. Problems for a Square Pyramid Made of Bricks of the Type R3n

The first exercise of this paragraph is perfectly preserved. See Fig. 8.6.2 below.

BM 96957+ § 2a (*obv. ii*: 5-8)

1 5 sar lagab *a-na* 2 $^{1}/_{2}$ nindan sukud sig$_4$ en.nam *lu-ša-<al>-bi-in*
2 za.e / igi.3$^{!}$.gál 5 sar *le-qé* 1 40 $_{\text{é.gar8-}tum(?)}$ *ta-mar*
3 1 40 *a-na* 2 $^{1}/_{2}$ nindan sukud / *i-ši* 4 10 *ta-mar*
4 2$_{\text{iku}}$ $^{1}/_{2\text{iku}}$ ašag$_4$ sig$_4$ *tu-ša-al-ba*$^{!}$*-ma* 5 sar lagab / *a-na* 2 $^{1}/_{2}$ nindan sukud *ta-am*$^{!}$*-mar*
ne-pé-šum

1 5 sar the equalside, 2 $^{1}/_{2}$ nindan the height. The bricks what should I mould?
2 You: / The 3$^{!}$rd part of 5 sar take, 1 40 $_{\text{the wall(?)}}$ you will see.
3 1 40 to the 2 $^{1}/_{2}$ nindan the height / raise 4 10 you will see.
4 2 1/2 iku of bricks you will have molded, then 5 sar equalside / at 2 1/2 nindan height you will see.
The procedure.

The gloss é.gar$_8$-*tum* 'the (brick) wall' under line 2 makes no sense whatever. One would have expected to see instead, for instance, the gloss *qà-qá-rum* 'the ground'.

Apparently, the object considered in this exercise is a square pyramid with the bottom area A = 5 area sar (= 5 sq. nindan) and the height h = 2 1/2. (Note that the side of a square with the area 5 sar is sqs. 5 · 1 nindan = appr. 2;14 nindan, so that the height of the pyramid in this exercise, 2;30 nindan, is close to the length of the side of the base.) Since h = 2 1/2 nindan = 2 1/2 · 12 cubits, the volume of such a pyramid is

V = 1/3 $A \cdot h$ = ;20 · 5 · 2 1/2 · 12 volume sar = 1;40 · 30 volume sar = 50 volume sar.

Surprisingly, the value of the product 1/3 $A \cdot h$ is said in line 3 of the text to be 4 10, not 50. However, this unexpected result is immediately reformulated in the remaining part of line 3 as '2 1/2 iku of bricks', which is the same as 4 10 (= 250) sar of bricks. This makes sense if the pyramid was silently assumed to be built of bricks of molding number '5', meaning 5 brick sar per volume sar. This is in Old Babylonian mathematical texts the molding number of *rectangular bricks of type R3n*, more precisely bricks of length ;03 nindan, width ;02 nindan, and the standard thickness of 5 fingers. See Friberg, *ChV* (2001), Table 4.2.

Clearly the author of this text made use of a clever shortcut, making use of the observation that for bricks of molding number 5 the number B of bricks in, for instance, a pyramid of base area A, expressed *in* sq. nindan and height h, expressed in nindan, not cubits, is

B = 1/3 $A \cdot h$, expressed in brick sar.

(The result is, of course, correct only when the computation makes use of sexagesimal numbers in relative place value notation, so that 5 · 12 = '1'.)

In the second exercise of § 2, the number of bricks in the pyramid and the height of the pyramid are given, and what is asked for is the (area of) the base of the pyramid. The computation rule used is the reverse of the computation rule above, namely

A = rec. $h \cdot B$ · rec. 1/3.

BM 96957+ § 2b (*obv. ii*: 9-13)

1 *šum-ma* 2_{iku} $^{1}/_{2\text{iku}}$ ašag$_4$ sig$_4$ 2 $^{1}/_{2}$ nindan sukud lagab en.nam *é-pu-uš*
2-3 za.e / ⸢igi 2 $^{1}/_{2}$ nindan sukud duḫ.a⸣ *a-na* 4 10 2_{iku} $^{1}/_{2\text{iku}}$ ašag$_4$ sig$_4$ *i-ši* 1 40 *ta-mar* / [*ki-il*]
[igi 20 igi.3.gál] duḫ.a 3 *ta-mar*
4 3 *a-na* / [1 40 *ša tu-ka-lu i-ši* 5 *ta-mar*] ⸢5 sar lagab⸣ /
5 [*ne-pé-šum*]

1 If 2 1/2 iku of bricks, 2 1/2 nindan height, the square-side what did I make?
2-3 You: / ⸢The opposite of 2 2/2 nindan height resolve⸣ to 4 10, 2 1/2 iku of bricks raise, 1 40 you will see. / [Keep!]
[The opposite of 20, the 3rd part] resolve, 3 you will see.
4 3 to / [1 40 that you held raise, 5 you will see.] ⸢5 sar the square-side⸣. /
5 [The procedure.]

As explained above, the area of the base of the pyramid is computed here as follows

A = rec. $h \cdot B$ · rec. 1/3 = rec. 2 1/2 · 4 10 · 3 = 1 40 · 3 = ‘5’.

The text of the ensuing § 2c is lost. It is likely that in that exercise the base area and the number of bricks were known and that the height h was computed, by use of another reversal of the shortcut rule above. It is less clear what kind of exercise or exercises came between § 2c and the first exercise of the next paragraph. See again the reconstructed outline of the text of BM 96957+ in Fig. 8.6.2 below.

8.6.3 § 3. Metric Algebra Problems for a Wall Made of Bricks of the Variant Type S2/3cv

The first exercise of § 3 of BM 96957+ is lost, but all the remaining exercises are well preserved. See the reconstructed outline in Fig. 8.6.2 below. Just like in § 1 above, the object considered in this paragraph is a brick wall, é.gar$_8$. However, while the bricks in § 1 were rectangular, called sig$_4$, the bricks in § 3 are square, called sig$_4$.al.ur$_5$.ra. A common feature is that the bricks in both § 1 and § 3 are extra thick, with a thickness of 6 fingers rather than 5. As explained in Sec. 11.2 below this probably means that the bricks in both § 1 and § 3 were of the kind that is reinforced with straw for greater stability.

The first preserved exercise of § 3 of BM 96957+ is

BM 96957+ § 3b (*obv. iii*: 1-6)

1-2 é.gar$_8$ sig$_4$.al.ur$_5$.ra 2 kùš sag 1 nindan sukud 9 sar sig$_4$.al.ur$_5$.ra / uš é.gar$_8$-*ia* en.nam
3 za.e igi 2 15 igi.gub é.gar$_8$ duḫ.a / 26 40 *ta-mar*
26 40 *a-na* 9 sar sig$_4$.al.ur$_5$.<ra> *i-ši* 4 *ta-mar* $_{\text{saḫar-}\textit{rum}}$ ⸢*ki-il*⸣ /
4 igi 12 1 nindan sukud é.gar$_8$ duḫ.a 5 *ta-mar*
5 5 *a-na* 4 *ša tu-ka-lu* / *i-ši* 20 $_{\text{gaba.uš}}$ *ta-mar*
igi 10 sag duḫ.a *a-na* 20 *ša ta-mu-rù i-ši* 2 *ta-mar* 2 nindan uš /
6 *ne-pé-šum*

1-2 A wall of square bricks, 2 cubits the front, 1 nindan the height, 9 sar the square bricks. / The length of my wall was what?
3 You: The opposite of 2 15 the constant of the wall release, / 26 40 you will see.
26 40 to 9 sar of square bricks raise, 4 you will see $_{\text{the mud}}$. ⸢Keep!⸣ /
4 The opposite of 12, 1 nindan, the height of the wall release, 5 you will see.
5 5 to 4 that you kept / raise, 20 $_{\text{the ???}}$ you will see.
The opposite of 10 the front release, to 20 that you saw raise, 2 you will see. 2 nindan the length. /
6 The procedure.

In this exercise, the ‘front’ of the wall (meaning the thickness) is s = 2 cubits, the height is h = 1 nindan, and the number of bricks is B = 9 brick sar. Asked for is the length u of the wall.

The ‘constant of the wall’, meaning the molding number of the square bricks, is given in line 2 as L = 2 15. This means that the bricks are *square bricks of the variant type S2/3cv* with the side 2/3 cubit and the extra thickness 6 fingers. See Friberg, *ChW* (2001), Table 4.2.

The solution algorithm starts with the computation of the volume of the wall as

$V =$ rec. $L \cdot B$ = rec. (2;15 brick sar/volume sar) · 9 brick sar = 4 volume sar (lines 2-3)

Next the area A of the rectangular base of the wall is found through division of the volume by the height h = 1 nindan = 12 cubits:

$A =$ rec. $h \cdot V$ = rec. (12 cubits) · 4 volume sar = ;20 area sar (lines 4-5)

The result is glossed gaba.uš, a term of unknown meaning, possibly standing for *qaqqaru* 'the ground'.

Finally, the length of the wall is found through division of the area of the rectangular base by its 'front' s = 2 cubits = ;10 nindan:

$u =$ rec. $s \cdot A$ = rec. (;10 nindan) · ;20 area sar = 2 nindan (line 5)

In § 3c below, the thickness of the same wall is computed:

BM 96957+ § 3c (*obv. iii*: 7-12)

1 2 nindan uš é.gar$_8$ sig$_4$.al.ur$_5$.ra 1 nindan sukud 9 sar sig$_4$.al.ur$_5$.ra /
2 *ku-bu-re* é.gar$_8$-*ia* en.nam
za.e igi 2 15 igi.gub é.gar$_8$ duḫ.a 26 40 *ta-mar* /
3-4 26 40 *a-na* 9 sar sig$_4$.al.ur$_5$.ra *ša a-na* é.gar$_8$-⸢*ka*⸣ *ta-at-bu-ku i-ši* / 4 *ta-mar* $_{\text{saḫar-}rum}$
igi 1 nindan sukud é.gar$_8$-*ka* duḫ.a *a-na* 4 *ša ta-mu-rù i-ši* 20 *ta-mar* /
5 20 1/3 sar *qà-qá-ar* é.gar$_8$-*ka*
igi 2 nindan uš duḫ.a 30 *ta-mar*
6 30 / ⸢*a*⸣-*na* 20 *i-ši* 10 *ta-mar* 2 kùš *ku-bu-re-e* é.gar$_8$
⸢*ne*⸣-*pé-šum*

1 2 nindan the length of a wall of square bricks, 1 nindan the height, 9 sar the square bricks. /
2 The thickness of my wall was what?
You: The opposite of 2 15 the constant of the wall resolve, 26 40 you will see. /
3-4 26 40 to 9 sar of square bricks that for ⸢your⸣ wall you laid down raise / 4 you will see $_{\text{the mud}}$.
The opposite of 1 nindan the height of your wall to 4 that you saw raise, 20 you will see, /
5 20, 1/3 sar the ground of your wall.
The opposite of 2 nindan the length resolve 30 you will see.
6 30 / to 20 raise, 10 you will see, 2 cubits the thickness of the wall.
The procedure.

Note the difference between the solution algorithms in §§ 1b and 3c. In § 1b, the thickness of the wall is calculated as the volume V of the wall divided by the area $u \cdot h$ of the face of the wall. In § 3c, on the other hand, the width is calculated as the volume V divided first by the height h, then by the length u. This observation, taken together with some differences in the mathematical vocabulary, seems to show that §§ 1 and 3 of BM 96957+ were not excerpted from one and the same original theme text.

Note also the omission of the equation 1 nindan = 12 cubits from the calculation in lines 4-5 of § 3c, where it is said, essentially, that 4 volume sar divided by 1 nindan equals ;20 area sar.

The object of § 3d below is to calculate the height of the same brick wall as before.

BM 96957+ § 3d (*obv. iii*: 13-18)

1-2 2 nindan uš é.gar$_8$ 2 kùš *ku-bu-*<*re*>*-e* é.gar$_8$ 9 sar sig$_4$.al.ur$_5$.ra *a-na* é.gar$_8$ / ⸢*ša*⸣-*ak-na-at*
i-na 9 sar sig$_4$.al.ur$_5$.ra en.nam ⸢*li-li*⸣
3 za.e igi 2 15 igi.gub / [sig$_4$].⸢al⸣.ur$_5$.ra duḫ.⸢a 26⸣ 40 *ta-mar*
4 26 ⸢40 *a-na* 9⸣ [sar sig$_4$.al].⸢ur$_5$.ra <<*i-ši*>>⸣ / [*ša a-na*] ⸢é⸣.gar$_8$ *ša-ak-na-at i-ši* 4 $_{\text{saḫar-[}rum\text{]}}$ *ta-*⸢*mar*⸣ [*ki-il*]
5 [2 nindan uš *a-na* 10] / [*i-ši* 20 *ta-mar* igi 20] ⸢duḫ.a⸣ 3 *ta-mar*
3 *a-na* [4 *ša tu-ka*]-⸢*lu i-ši*⸣ / [12 *ta-mar*]
6 [1 nindan sukud *i*]-⸢*na*⸣ 9 sar sig$_4$.al.ur$_5$.<ra> *tu-ša-qá*
ne-pé-<*šum*>

1-2 2 nindan the length of a wall, 2 cubits the thickness of the wall, 9 sar the square bricks to the wall / placed.
In 9 sar of square bricks what may it be high?
3 You: The opposite of 2 15 the constant / of the square [bricks] releas⸢e 26⸣ 40 you will see.
4 26 ⸢40 to 9⸣ [sar of square [bricks] <<raise>> / [that to] the wall was placed raise, 4 the mud you will ⸢see⸣. [Keep!]
5 [2 nindan the length to 10] / [raise, 20 you will see. the opposite of 20] ⸢release⸣, 3 you will see.
3 to [4 that you ke]⸢pt raise⸣, / [12 you will see].
6 [1 nindan the length] ⸢to⸣ 9 sar of square bricks you will elevate.
The procedure.

The solution algorithm begins by computing the volume of the wall as the given number of brick sar divided by the molding number 2 15. The height is then obtained through division of the volume by the area of the base rectangle.

In §§ 3b, 3c, and 3d, the stated problems had the form of *linear equations* for the unknowns *u*, *s*, and *h*, respectively, essentially simple *division problems*. In the two remaining exercises, in §§ 3e and 3f below, the problems can be reduced to *rectangular-linear problems* for the two unknowns *u* and *s*.

BM 96957+ § 3e (*obv. iii*: 19-28)

1 [é.gar$_8$ sig$_4$.al].ur$_5$.ra 1 nindan sukud é.gar$_8$ 9 sar sig$_4$.al.ur$_5$.ra
2 uš *ù ku-bu-re* / [é.gar$_8$ ul].gar-*ma* 2 10 uš *ù ku-bu-re* é.gar$_8$-*ia* en.nam /
3 [za.e igi 2 15] igi.gub sig$_4$.al.ur$_5$.ra duḫ.a 26 ⸢40 *ta*⸣-*mar*
4 26 40 / [*a-na*] ⸢9 sar⸣ sig$_4$.al.ur$_5$.ra *i-ši* 4 saḫar-*rum ta-mar*
4 saḫar sig$_4$.al.ur$_5$.ra *ki-il* /
5 [igi 12] 1 nindan sukud sig$_4$.al.ur$_5$.ra-*ka* duḫ.a 5 *ta-mar*
6 5 *a-na* 4 *ša tu-ka-lu* / [*i*]-⸢*ši*⸣ 20 *ta-mar* gaba.uš.ta 20 *qà-qá-ra ki-il*
7 1/2 2 10 [ul].⸢gar⸣ uš *ù ku-bu-re* é.gar$_8$ / [*ḫe*]-*pé* 1 05 *ta-mar*
1 05 *šu-tam-ḫir* 1 10 ⸢25⸣ [*ta-mar*]
8 [20 *qà-q*]*á-ra-tum* / [*ša tu*]-⸢*ka-lu*⸣ *i-na* 1 10 25 ba.zi 50 25 [*ta*]-⸢*mar*⸣
9 50 25 en.nam íb.sá / [55 íb.sá]
[55] *a-na* 1 05 *ša* ì.gu$_7$.gu$_7$ daḫ.ḫa 2 *ta-mar* 2 nindan uš /
10 [55 *i-na* 1 05 *ša tu*]-⸢*uš*⸣-*tam-ḫir* ba.zi 10 íb.tag$_4$ 2 kùš *ku-bu-re*
ne-pé-šum

1 [A wall of squ]are bricks. 1 nindan the height of the wall, 9 sar the square bricks.
2 The length and the thickness / [I added tog]ether, then 2 10. The length and the thickness of my wall were what?
3 [You: The opposite of 2 15] the constant of square bricks release, 26 ⸢40 you⸣ will see.
4 26 40 / [to] ⸢9 sar⸣ of square bricks raise 4 the mud you will see.
4 the mud of the square bricks. Keep!
5 [The opposite of 12], 1 nindan the height of your square bricks resolve, 5 you will see.
6 5 to 4 that you held / [ra]ise 20 you will see the x x x. 20 the ground. Keep!
7 1/2 of 2 10, [the]⸢sum⸣ of the length and the thickness of the wall / [br]eak, 1 05 you will see.
1 05 make equalsided 1 10 ⸢25⸣ [you will see].
8 [20 the gr]ound / [that you] ⸢kept⸣ from 1 10 25 tear off, 50 25 [you] ⸢will see⸣.
9 50 25 is what equalsided? / [55 equalsided].
[55] to 1 05 that ate itself add on, 2 you will see. 2 nindan the length.
10 [55 from 1 05 that you] made equalsided tear off, 10 the remainder. 2 cubits the thickness.
The procedure.

Here the wall is the same as before, with the height h = 1 nindan, the number of bricks B = 9 sar, and the sum of the length and the thickness $u + s$ = 2;10 nindan, and with bricks of the same molding number as before, L = 2;15. In quasi-modern terms this means that u and s can be calculated as the solutions to the following rectangular-linear system of equations:

$u + s$ = 2;10 nindan, $u \cdot s$ = rec. $L \cdot B \cdot$ rec. h = 26 40 · 9 · 5 sq. nindan = ;20 sq. nindan (lines 3-6)

(Here again, the area of the rectangular base of the wall is glossed as gaba.uš.) The following solutions are obtained with the help of the usual solution algorithm:

$(u + s)/2$ = 1;05, nindan, $(u - s)/2$ = sqs. (sq. 1;05 – ;20) nindan = sqs. ;50 25 nindan = ;55 nindan (lines 6-9)

u = (1;05 + ;55) nindan = 2 nindan, s = (1;05 – ;55) nindan = ;10 nindan = 2 cubits (lines 9-10)

The next exercise is similar, but with $u - s$ known instead of $u + s$.

BM 96957+ § 3f (*obv. iii*: 29-40)

1-2 [é.gar$_8$ sig$_4$.al.ur$_5$.ra] 9 sa[r sig$_4$.al.ur$_5$.ra] *at-bu-uk* uš *e-li* / [*ku-bu-re* é.gar$_8$ 1 50 dirig]
3 ⸢1 nindan sukud⸣ [uš] é.gar$_8$ *ù ku-bu-re* é.gar$_8$*-ia* / [en.nam]
[za.e igi 2 1]5 igi.gub é.gar$_8$ <sig$_4$>.al.ur$_5$.<ra> duḫ.a 26 40 *ta-mar*
4 26 40 / [*a-na* 9 sar sig$_4$.al].ur$_5$.ra *i-ši* 4 $_{\text{saḫar-}\textit{rum}}$ *ta-mar* saḫar sig$_4$.al.ur$_5$.ra 4 sar /
5 [igi] 12 sukud é.gar$_8$ duḫ.a 5 *ta-mar*
6 5 *a-na* 4 saḫar sig$_4$.al.ur$_5$.ra / *i-ši* 20 *ta-mar* 20 $_{\text{gaba.uš}}$ *qà-qá-ra ki-il*
7 $^1/_2$ 1 ⸢50 *ša*⸣ uš ugu *ku-bu-re* / é.gar$_8$ *i-te-rù ḫe-pé* 55 *ta-mar*
8 [55 *a-di* 2] gar.ra / 55 *šu-tam-ḫir* 50 25 *ta-mar*
9 20 *qà-qá-ra a-na* 50 <2>5 daḫ.ḫa / 1 10 25 *ta-mar*
1 10 25 en.nam íb.sá 1 05 ⸢íb.sá *a-di*⸣ 2 gar.ra /
10 [55 *ša*] *tu-uš-tam-ḫir a-na* 1 05 daḫ.ḫa 2 ⸢*ta*⸣-[*mar* 2 nindan uš] é.gar$_8$ /
11 [55 *i-na*1 05 ba].⸢zi⸣ 10 íb.tag$_4$ 2 kùš *ku-bu-*[*re* é.gar$_8$]
[*ne-pé-šum*]

1-2 [A wall of square bricks.] 9 sar [of square bricks] I laid down, the length over / [the thickness was 1 50 beyond].
3 ⸢1 nindan the height⸣. [The length] of the wall and the thickness of my wall / were what?
[You: The opposite of 2] 15 the constant of square bricks release, 26 40 you will see.
4 26 40 / [to 9 sar of square br]icks raise 4 $_{\text{the mud}}$ you will see. The mud of the square bricks is 4 sar.
5 [The opposite] of 12 the height of the wall resolve, 5 you will see.
6 5 to 4 the mud of the square bricks / ra]se 20 you will see $_{\text{the x x x}}$ the ground. Keep!
7 $^1/_2$ of 1 ⸢50 that⸣ the length over the thickness / of the wallwas above break, 55 you will see.
8 [55 until 2] set. / 55 make equalsided, 50 25 you will see.
9 20 the ground to to 50 <2>5 add on, / 1 10 25 you will see.
9 1 10 25 is what equalsided? 1 05 ⸢equalsided. Until⸣ 2 set./
10 [55 that] you made equalsided to 1 05 add on, 2 ⸢you⸣ [will see. 2 nindan the length] of the wall. /
11 [55 from 1 05 tear] ⸢off⸣, 10 the remainder. 2 cubits the thickn[ess of the wall].
[The procedure.]

(Here in line 6, the area of the rectangular base of the wall is glossed, for the third time, as gaba.uš.)

In § 3g below, the last of the exercises in § 3 of BM 96957, the brick wall has *a trapezoidal cross section*, rather than rectangular as before.

BM 96957+ § 3g (*obv. iv*: 1-18)

1 é.gar$_8$ sig$_4$.al.ur$_5$.ra [2 nindan uš é.gar$_8$]
2-3 [2 kùš] / *a-na ša-ap-li-a-tim ka-ba-ar* [1 kùš *a-na e-le-nim ka-ba-ar*] / 2 nindan sukud
4 *ši-ip-ka-at* é.gar$_8$*-ia* en.nam [*ú-ša-pa-ak ù*] / *i-na* 1 kùš en.nam *i-ku-ul*
5 za.e 2 kùš ki.kal *ki-ta* [*ù* 1 kùš] / *ša a-na* ugu *ip-ḫur* ul.gar 15 *ta-mar*
6 $^1/_2$ 15 *ḫe-pé* / ⸢7⸣ 30 $_{\text{gaba.uš.ta}}$ *ta-mar* 7 30 *a-na* 2 nindan uš *i-ši* 15 *ta-mar*
7 [15 *a*]*-na* 24 sukud (erasure) *i-ši* 6 *ta-mar*
8 6 *a-na* / [2 1]5 igi.gub sig$_4$.al.ur$_5$.<ra> *i-ši* 13 30 *ta-mar* 13 $^1/_2$ sar sig$_4$.al.ur$_5$.ra /
9-10 ⸢$^1/_2$⸣ 13 30 *ḫe-pé* 6 45 *ta-mar* 6 $^2/_3$ <sar> 5 gín / [*it*]*-ti* é.gar$_8$*-ka ši-ip-ka-ti i-ša-pa-ka-ku*
11 nigín.na / ⸢*ša*⸣ *i-na* 1 kùš *i-ku-lu a-mur*
12 2 kùš ki.kal *ki-ta e-li* / ⸢5⸣ *ša a-na e-le-nu* en.nam *i-ter* 5 *i-ter ki-il* /
13-14 [*aš*]-⸢*šum*⸣ *i-na* 1 kùš *ú-ku-*<*la*> en.nam *i-ku-ul* dug$_4$.ga / [5 gar.ra]
15 igi 5 duḫ.a 12 *ta-mar a-na* 2 nindan sukud / [*i-ši* 24 *ta-m*]*ar*
igi 24 duḫ.a 2 30 *ta-mar* <<2 30 *ta-mar*>> /

16-17 [2 30 *a-na*] 5 1 kùš <<x>> *i-ši* 12 30 *ta-mar* / [12 30 *a-na* 6 *i-ši* 1 15] *ta-mar*
18 *i-na* 1 kùš 1 šu.si / ⸢*ù* igi.4.gál 1 šu.si⸣ é.gar$_8$ *ú-ku-la i-ku-ul*
⸢*ne-pé-šum*⸣

1 A wall of square bricks, [2 nindan the length of the wall],
2-3 [2 cubits] / at the lower (end) it is thick, [1 cubit at the upper (end) it is thick] / 2 nindan the height.
4 The heap of my wall, what [did I heap, and] / in 1 cubit what did it eat?
5 You: 2 cubits the lower base [and 1 cubit] / that at the top it gathered add together, 15 you will see.
6 $^1/_2$ of 15 break / ⸢7⸣ 30 $_{\text{the x x x}}$ you will see. 7 30 to 2 nindan the length raise, 15 you will see.
7 [15 to] 24 the height (erasure) raise, 6 you will see.
8 6 to / [2 1]5 the constant of the square bricks raise, 13 30 you will see, 13 $^1/_2$ sar of square bricks. /
9-10 [$^1/_2$] of 13 30 break, 6 45 you will see, 6 $^2/_3$ <sar> 5 gín / with your wall the heap he will heap for you.
11 Turn around. / ⸢That which⸣ in 1 cubit it ate, see.
12 2 cubits the lower base over / ⸢5⸣ that at the upper (end) what is it beyond? 5 it is beyond. Keep! /
13-14 [Sin]ce in 1 cubit (as) feed what did it eat I asked, / [5 set].
15 The opposite of 5 release, 12 you will see. To 2 nindan the height / [raise, 24 you will] see.
The opposite of 24 release, 2 30 you will see. <<2 30 you will see>> /
16-17 [2 30 to] 5, 1 cubit <<x>> raise, 12 30 you will see. / [12 30 to 6 raise 1 15] you will see.
18 In 1 cubit, 1 finger / ⸢and the 4th part of 1 finger⸣ the wall as food ate.
⸢The procedure.⸣

The wall in this exercise is 2 nindan long and has a cross section in the form of a trapezoid with the lower width s = 2 cubits, the upper width s' = 1 cubit, and the height 2 nindan. It is silently understood that the wall is made of square bricks with the same molding number as before, L = 2;15. Asked for are the number of bricks B and the 'food' (inclination) of the wall (see BM 96954+ § 1k in Sec. 8.5 above).

The solution algorithm starts by computing the mean thickness of the wall,
Unexplainably, this *average thickness* is glossed as gaba.uš.ta, a term of unknown meaning. Recall that in §§ 3b, 3e, and 3f above, *the area of the rectangular base* of the wall was glossed as gaba.uš or gaba.uš.ta.

The number of bricks is then computed in a straightforward way, as

$B = L \cdot u \cdot s_m \cdot h$ = 2;15 brick sar/volume sar · 2 nindan · ;07 30 nindan · 24 cubits = 13;30 brick sar (lines 6-8)

In the following two lines, this brick number is halved, without any explication. Presumably, what is computed here is the number of bricks per nindan of length, which turns out to be

B/u = 13;30 brick sar / 2 nindan = 6;45 brick sar per nindan = 6 2/3 sar 5 gín per nindan (lines 9-10)

Now, the 'food' is computed, as

$f = (s - s')/h$ = (;10 – ;05) nindan · rec. 2 nindan = ;05 nindan · rec. 24 cubits = ;05 · ;02 30 nindan per cubit (lines 11-16)

Since

1 nindan = 12 · 30 fingers = 6 00 fingers,

The result of the computation, f = ;00 12 30 nindan per cubit (in the text, of course, expressed simply as '12 30', without any zeros of semicolon), is finally transformed into the more comprehensible

f = ;00 12 30 · 6 00 fingers per cubit = 1;15 fingers per cubit = 1 1/4 finger per cubit (lines 16-18)

8.6.4 § 4. Rectangles and Their Diagonals

§ 4 of BM 96957+ is composed of eight exercises, all apparently concerned with a certain rectangular 'gate' and its diagonal. The text of four of those exercises is well preserved, even if one of the exercises ends prematurely. All the exercises were, presumably, originally illustrated by diagrams, but only five of the diagrams are more or less well preserved. See Figs. 8.6.1-2 below. According to a well known convention in Babylonian mathematical texts, the 'upper' part of the gate is facing to the left in each diagram.

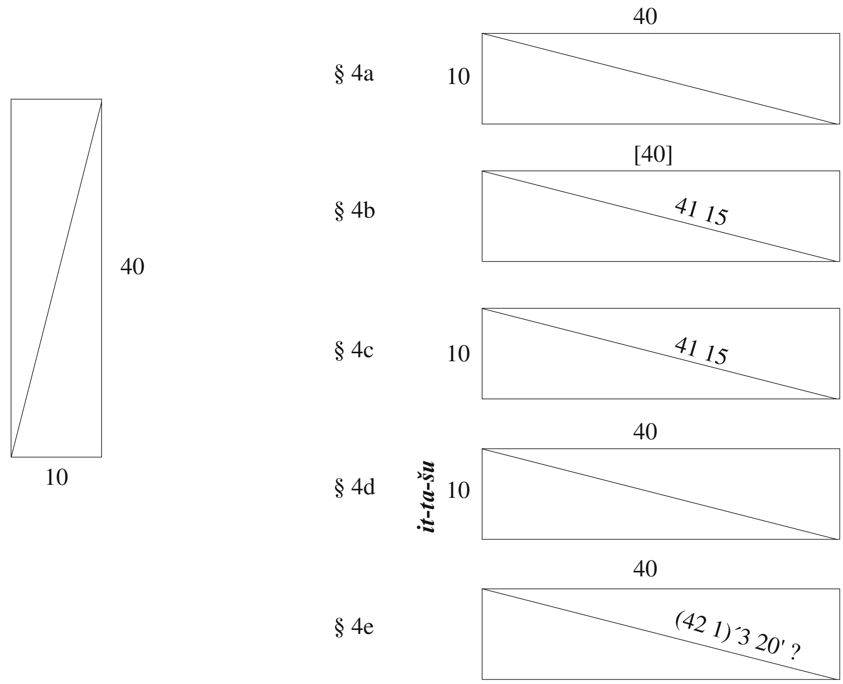


Fig. 8.6.1. BM 96957+VAT 6598. Left: A rectangular gate with given dimensions. Right: The preserved diagrams in the text.

In the first exercise, § 4a, both the height of the gate, h = 1/2 nindan (= 4 meters) and the width (sometimes called the front) of the gate, s = 2 cubits (= 1 meter) are known. To be calculated is the (length of the) diagonal d, called *ṣiliptu* 'the dissector', from *ṣalāpu* 'to cut'. According to the well known "Old Babylonian diagonal rule" (see the survey in Friberg, *MSCCT 1,* Sec. A8.6), the answer should be that

sq. d = sq. h + sq. s = (sq. ;40 + sq.;10) sq. nindan = (;26 40 + ;01 40) sq. nindan = ;28 20 sq. nindan,

d = sqs. ;28 20 nindan

However, the data in this exercise were chosen, probably intentionally, so that ;28 20 is not a square number. (Indeed, sq. d = sq. ;10 · (sq. 4 + sq. 1) sq. nindan = sq. :10 · 17 sq. nindan, so that d = ;10 · sqs. 17 nindan, and 17 is not a square number.) The purpose of the exercise in § 4a seems to be to show how this difficulty can be sidestepped.

BM 96957+ § 4a (*obv. iv*: 19-24)

Fig.
1 ká$^{1}/_{2}$ <nindan> 2 kùš sukud 2 kùš dagal *ṣí-li-*<*ip*>*-ta-šu* en.nam /
2 za.e 10 dagal *šu-tam-ḫir* 1 40 *qà-qá-*⌜*ra ta-mar*⌝ /
3-4 igi 40 kùš sukud duḫ *a-na* 1 40 *qà-qá-ri i-ši* / 2 30 *ta-mar*
$^{1}/_{2}$ 2 30 *ḫe-pé* 1 15 *ta-mar*
5 1 15 / [*a-na* 40 sukud daḫ].ḫa 41 15 *ta-mar*
6 41 15 / [*ṣí-li-ip-tum*]
⌜*ne*⌝*-pé-šum*

1 A gate. 1/2 <nindan > 2 cubits the height, 2 cubits the width. Its dissector (diagonal) what? /
2 You: 10 the width make equalsided, 1 40 the grou⌜nd you will see.⌝ /
3-4 The opposite of 40, cubits the height release, to 1 40 the ground raise, / 2 30 you will see.
1/2 of 2 30 break, 1 15 you will see.
5 1 15 / [to 40 the height add] on, 41 15 you will see.
6 41 15 / the diagonal.

Note that the square of the width of the gate is called *qaqqaru* 'the ground', which seems appropriate if you think of the gate as an opening in a wall. Note also that the height of the gate is

h = 1/2 nindan 2 cubits = 8 cubits = 8 · ;05 nindan = ;40 nindan.

Therefore, '40 cubits' in line 3 has to be understood (in the usual way) as '40 in the range of cubits' = ;40.

It is easy to see that the solution rule made use of in this exercise is of the following form:

d = sq. $s \cdot$ rec. $h \cdot 1/2 + h$ = (;01 40 · 1;30 ·1/2 + ;40) nindan = (;01 15 + ;40) nindan = ;41 15 nindan (lines 2-5)

The solution algorithm above is the result of an application of the well known "Old Babylonian *additive* square side rule" (see Friberg, *BaM* 28, Sec. 8a):

sqs. (sq. $a + R$) = approximatively $a + R/2a$ when R is small compared to a.

Indeed, in the present case, when $a = h$ and R = sq. w, an application of the rule gives that

d = sqs. (sq. h + sq. s) = approximatively h + sq. s / 2 h when w is small compared to h.

The accuracy of the approximation can be checked as follows:

sq. h + sq. s = sq. ;40 + sq. ;10 = ;26 40 + ;01 40 = ;28 20, while sq. ;41 15 = ;28 21 33 45.

The solution rule in lines 2-5 of § 4a can be reversed in order to calculate s when d and h are given. This is done in § 4b below.

BM 96957+ § 4b (*obv. iv*: 25-29)

Fig.

1 *šum-ma* ká 40 kùš sukud 41 15 *ṣí-li-ip-tum* dagal en.nam /

2-3 za.e 40 sukud *i-na* 41 15 *ṣí-li-ip-tum* ba.zi / 1 15 íb.tag$_4$

1 15 ⸢*a-na*⸣ 2 tab.ba 2 30 *ta-mar*

4-5 40 uš / *a-na* 2 30 a.rá ⸢*ša*⸣ *ta-mu-rù i-ši* 1 40 *ta-mar* en.nam íb.sá 10 íb.sá

ne-pé-šum

1 If a gate, 40, cubits, the height, 41 15 the dissector, the width what?

2-3 You: 40 the height from 41 15 the dissector tear off, / 1 15 the remainder.

1 15 ⸢to⸣ 2 double, 2 30 you will see.

4-5 40 the length / to 2 30, the product that you saw raise, 1 40 you will see. What equalsided? 10 equalsided.

The procedure.

The calculation in this exercise is based on the following somewhat simpleminded observation:

If d = (appr.) sq. $s \cdot$ rec. $h \cdot 1/2 + h$, then sq. s = (appr.) $(d - h) \cdot 2 \cdot h$.

Therefore, in the present case,

sq. s = (appr.) $(d - h) \cdot 2 \cdot h$ = ;01 15 · 2 · ;40 sq. nindan = ;01 40 sq. nindan, so that w = ;10 nindan (= 2 cubits).

Alternatively, the solution algorithm used in § 4b can be obtained in the following way. According to the Old Babylonian diagonal rule,

sq. s = sq. d – sq. h = $(d - h) \cdot (d + h)$ = (appr.) $(d - h) \cdot 2\,h$, provided that d is close to h (as in § 4b).

However, in the general case this approximate computation rule involves the messy computation of the square root of a non-square number. It is precisely in order to avoid such computations of the square root of a non-square number that the approximation rule in § 4a can be used.

In the original version of the next exercise, § 4c, the author of the exercises had probably planned to make use of a third approximate solution procedure in order to calculate the height h when d and w were given.

BM 96957+ § 4c (*obv. iv*: 30-32)

Fig.

1 2 kùš dagal 41 15 *ṣí-li-ip-tum* sukud en.nam

za.e

2 nu

3 *iš-te-*[*en-ma*]

1 2 cubits the width, 41 15 the dissector. The height what?

You:

2 Not.

3 [On]e.

Suppose that the author of § 4 of BM 96957+ wanted to make another reversal of the approximate solution algorithm in § 4a in order to calculate h when d and w are given. He could then argue as follows:

If d = (appr.) sq. $s \cdot$ rec. $h \cdot 1/2 + h$, then h is the solution to the quadratic equation $d \cdot h$ – sq. h = (appr.) sq. s.

However, it would hardly make any sense to replace a straightforward computation of h = sqs. (sq. d – sq. s) by the less straightforward computation of the solution of a quadratic equation. Realizing this, the author of § 4 abolished his attempt to find an approximate rule for the computation of h, and wrote that he would 'not' demonstrate a solution procedure in this case. This kind of clearly aborted solution procedure is not known from any other Old Babylonian mathematical text. The uniqueness of the situation my be the reason why the one who copied the text found himself compelled to emphasize that the copy was ‘one’ with the original.

The following approximate solution algorithm can also be used in the case described in § 4c. According to the Old Babylonian diagonal rule in combination with the Old Babylonian *negative* square side rule (see Friberg, *BaM* 28, Sec. 8b)

h = sqs. (sq. d – sq. s) = (appr.) d – sq. s · rec. 2d when w is small compared to d.

With d = ;41 15 and w = ;10, the result is that

h = (appr.) ;41 15 – ;01 40 · rec. 1;22 30 = (appr.) ;41 15 – ;01 13 = ;40 02.

However, 1;22 30 is not a regular sexagesimal number, so that rec. 1;22 30 exists only as an approximation. Indeed. 1;22 30 = 11/8. So, maybe, the solution algorithm in exercise § 4c was abolished because it led to an unpalatable approximate computation of the reciprocal of a non-regular number, no less laborious than the straightforward computation of h = sqs. (sq. d – sq. s).

In the next exercise, in § 4d, the same rectangle and its diagonal is considered, together with a new approximate solution algorithm.

BM 96957+ § 4d (*obv. v*: 1-4)

Fig.
⸢*it-ta-šu*⸣
1 2 kùš dagal 40 kùš sukud *ṣí-li-ip-ta-šu* en.nam
2 za.e 10 sag / *šu-tam-ḫir* 1 40 *ta-mar qà-qá-rum*
3 1 40 *a-na* 40 ⸢kùš⸣ sukud *i-ši-ma* / 1 06 40 *ta-mar a-na* <2> tab.ba 2 13 20 *ta-mar*
4 *a-na* 40 kùš sukud / daḫ.ḫa 42 13 20 *ṣí-li-ip-ta ta-mar*
ne-pé-šum

⸢Its image???.⸣
1 2 cubits the width, 40, cubits the height. Its dissector what?
2 You: 10 the front / make equalsided, 1 40 you will see, the ground.
3 1 40 to 40, ⸢cubits⸣, the height raise, then / 1 06 40 you will see. To <2> double, 2 13 20 you will see.
4 To 40, cubits, the height / add on, 42 13 20 the dissector you will see.
The procedure.

In this exercise, just like in § 4a, the width s = 2 cubits (= ;10 nindan), and the height h = ;40 (nindan) (= 8 cubits) are given, and the dissector d is to be found. The approximate solution procedure employed in this exercise seems to be

d = sqs. (sq. h + sq. s) = (appr.) h + sq. s · 2 · h = (appr.) (:40 + ;01 40 · 2 · ;40) nindan = (appr.) ;42 13 20 nindan.

There is no way to justify or explain the use of this new approximate solution procedure. It may have been just an imperfectly remembered incorrect version of the solution procedure in § 4a! Indeed, what happens here is that sq. s is multiplied instead of divided by 2 h.

Of the next exercise, § 4e, only the upper right part of the diagram is preserved, enough in order to suggest that the height h = 40 and the dissector d = [42 1]3 20 were the given parameters, while the width s was unknown and should be computed. Therefore, possibly, the approximate solution procedure in § 4e may have been just a reversal of the (incorrect) solution procedure in § 4d, in the same way that the approximate solution procedure in § 4b was a reversal of the solution procedure in § 4a.

BM 96957+ § 4e (*obv. v*: …)

Fig.
1′ [………]

Since an attempted second reversal in § 4c of the solution procedure in § 4a was aborted, the author of § 4 apparently did not even attempt to make second reversal of the solution procedure in § 4d in the exercise § 4f (now destroyed). Instead a new series of three related exercises concerned with the same rectangular gate as before begins in § 4f.

BM 96957+ § 4f (*obv. v*: …)
[Fig.]
1′ [……..]

Nothing remains of this exercise. Nevertheless, in view of what comes after it (two partly preserved exercises, apparently concerned with straightforward applications of the Old Babylonian diagonal rule to the same rectangle as before) it is likely that in the width *s* = ;10 nindan and the height *h* = ;40 nindan were given, and that the diagonal *d* was computed, as follows:

sq. *d* = sq. *s* + sq. *h* = sq. ;10 + sq. ;40 = ;01 40 + ;26 40 = ;28 20, so that
d = sqs. ;28 20 = (appr.) ;41 15.

Interestingly, the approximate computation of the square root in this case will have had to make use of the Old Babylonian additive square side rule. For that reason, the computations in § 4f were probably identical with the computations in § 4a, the only difference being that the computations in § 4a were more explicit.

BM 96957+ § 4g (*obv. v*: …)
[Fig.]
1′ [40 kùš sukud 41 15 *ṣí-li-ip*]-*ta-šu* ⌜dagal⌝? en.nam
2′ za.e / [41 15 *ṣí-li-ip-ta šu-tam-ḫir* 28] 20 *ta-mar* /
3′ [40 sukud *šu-tam-ḫir* 26 40 *ta-mar*]
[26 40] ⌜*i-na* 2⌝8 20 / [ba.zi 1 40 *ta-mar*]
[…………]

1′ [4 cubits the height, 41 15] its [dissector]. The ⌜width⌝ what?
2′ You: / [41 15 the dissector make equalsided, 28] 20 you will see. /
3′ [40 the height make equalsided, 26 40 you will see.]
[…………]

In this badly preserved exercise, a reasonable reconstruction seems to show that the height *h* = ,40 nindan and the dissector *d* = (appr.) ;41 15 are known. Then an application of the Old Babylonian diagonal rule shows that

sq. *s* = sq. *d* – sq. *h* = sq. ;41 15 – sq. ;40 = (appr.) ;28 20 – ;26 40 = ;01 40 = sq. ;10 so that *s* = ;10 (lines 3′ - 4′)

BM 96957+ § 4h (*obv. v*: …)
[Fig.]
1′ [2 kùš dagal 41 15 *ṣí-li-ip-ta-šu* sukud en.nam]
2′-3′ za.e / [41 15 *ṣí-li-ip-ta šu*]-⌜*tam-ḫir*⌝ 28 21 30 *tam-ḫir-tim* / [*ta-mar*]
[10 sag *šu-tam-ḫir*] 1 40 *ta-mar*
4′ 1 40 *i-na* 28 20 / [ba.zi 2]⌜6⌝ 40 íb.tag$_4$ en.nam íb.sá 40 íb.sá 40 sukud
ne-pé-<šum>
5′ *iš-te-en-ma*

1′ [2 cubits the width, 41 15 its dissector. The height what?]
2′-3′ You: / [41 15 the dissector make] ⌜equalsided⌝, 28 21 30 the equalsided / [you will see].
[110 the front make equalsided], 1 40 you will see.
4′ 1 40 from 28 20 / [break off, 2]⌜6⌝ 40 the remainder. What equalsided? 40 equalsided. 40 the height.
The proced<ure>.
5′ One.

This exercise, too, is badly preserved. Anyway, in this exercise the width *s* = 2 cubits = ;10 nindan and the diagonal *d* = ;41 15 nindan appear to be given. In lines 2-3, sq. *d* and sq. *s* are then computed as

sq. ;41 15 = ;28 21 33 45 = (appr.) ;28 21 30 and sq. ;10 = ;01 40 (lines 2′ - 3′)

A second approximation is needed in order to facilitate the remainder of the solution procedure. So sq. 41 15 is expressed (just as in the preceding exercise) as (appr.) 28 20, from which it follows that

sq. *h* = sq. *d* – sq. *s* = sq. ;41 15 – sq. ;10 = (appr.) ;28 20 – ;01 40 = ;26 40 = sq. ;40 so that *h* = ;40 (lines 3′ - 4′)

8.6.5 The Colophon

BM 96957+ colophon (*obv. v*: …)
ba-ri sig$_4^{!}$-$^{2}/_{3}$-*ti* /
^{d}iškur.ma.an.sum / dumu ^{m}30-*i-qì-ša-am*

Collated(?). 2/3-bricks / Ishkur-mansum / son of Sin-iqisham

As observed by Robson, *MMTC* (1999), fn. 25, if the reading *ba-ri* 'collated' in the first line of the colophon is correct, this would be "an unprecedentedly early occurrence" of the term. Also the mention of the scribe's name here is exceptionally early in a mathematical cuneiform text.

The reference to sig_4-⅔-*ti* '2/3-bricks' may be a catch-line indicating the contents of "the next tablet" in some series. The term is known from two Late(?) Babylonian tables of constants (Friberg, *ChV* (2001), 65-66). In line 4 of the table of constants RAFb = BM 36776 (Fig. 8.1.2 above), the constant 5 sig_4-⅔-*ti* '5, 2/3 -bricks' clearly refers to the molding number of *bricks of type R3n*, which means bricks of the length ;03 nindan, the width ;02 nindan, and the standard thickness 5 fingers.

In line B4 of another table of constants, Kb = CBS 10996, the strangely named constant 4 10 ^{geš}má.lal sig_4-⅔-*ti* '4 10, cargo-boat 2/3 bricks' clearly refers instead to the molding number of *bricks of the variant type R3nv*, which means bricks of the length ;03 nindan, the width ;02 nindan, and the extra thickness 6 fingers. In both cases, see Friberg, *op. cit.*, Table 4.2. (In Sec. 11.2.2 below it is shown that extra thick bricks of the various variant types may have been made of clay reinforced with straw.)

Interestingly, the '2/3-bricks' appear in both RAFb and Kb in the company of three other types of bricks, namely regular rectangular sig_4 'bricks', half-square sig_4.áb 'cow-bricks', and sig_4.al.ùr.ra 'square bricks'. In particular, in the table of constants Kb, the four types of bricks appear in the following lines:

[6]?	^{geš}má.lal sig_4	[6]?	cargo-boat bricks	(*L* = 6,	type R1/2cv)	Kb B1
⸢4 30⸣	^{geš}má.lal sig_4-*áb*	⸢4 30⸣	cargo-boat cow bricks	(*L* = 4 30,	type H2/3cv)	Kb B2
[2] 15	^{geš}má.lal sig_4.al.ùr.ra	[2] 15	cargo-boat square bricks	(*L* = 2 15,	type S2/3cv)	Kb B3
4 10	^{geš}má.lal sig_4-⅔-*ti*	4 10	cargo-boat 2/3 bricks	(*L* = 4 10,	type R3nv)	Kb B4

In view of the repeated occurrence of this "family of brick types", it is motivated to make the conjecture that § 1 in BM 96957+, concerned with walls made of bricks of the variant type R1/2cv, and § 3 in the mathematical recombination text BM 96957+, concerned with walls made of bricks of type S2/3cv, both were excerpts from a theme text concerned with walls made of bricks of the four variant types R1/2cv, H2/3cv, S2/3cv, and R3nv. In that case, as the catch line of BM 96957+ seems to indicate, two paragraphs on "the next tablet", excerpted from the same older text, may have been concerned with walls made of bricks of the two variant types R3nv and H2/3cv.

8.6.6 BM 96957+. An Outline of the Tablet, a Table of Contents, and Hand Copies

§ 1. A rectangular wall; bricks of type R1/2cv, $L = 6$

$B = L \cdot u \cdot s \cdot h$

§ 1a	u, s, h given	$B = ?$	computation rule
§ 1b	u, h, B given	$w = ?$	linear equation
[§ 1c	w, h, B given	$u = ?$]	linear equation
[§ 1d	u, s, B given	$h = ?$]	linear equation
[§ 1e	$u+s, h, B$ given	$u, s = ?$]	rectangular-linear problem
§ 1f	$u–s, h, B$ given	$u, s = ?$	rectangular-linear problem

§ 2. A square pyramid; bricks of type R3n, $L = 5$

$B = L \cdot h \cdot$ 1/3 sq. s

§ 2a	s, h given	$B = ?$	computation rule
§ 2b	B, h given	$s = ?$	linear equation
[§ 2c	B, s given	$h = ?$	linear equation
[…	…	…]	…

§ 3. A rectangular/trapezoidal wall; bricks of type S2/3cv, $L = 2\ 15$

[§ 3a	$B = L \cdot u \cdot s \cdot h$		
	u, s, h given	$B = ?$]	computation rule
§ 3b	s, h, B given	$u = ?$	linear equation
§ 3c	u, h, B given	$w = ?$	linear equation
§ 3d	u, s, B given	$h = ?$	linear equation
§ 3e	$u+s, h, B$ given	$u, s = ?$	rectangular-linear problem
§ 3f	$u–s, h, B$ given	$u, s = ?$	rectangular-linear problem
§ 3g	$B = L \cdot u \cdot (s + s')/2 \cdot h$, $f = (s - s')/2$		
	u, s, s', h given	$B = ?$, $B/u = ?$, $f = ?$	computation rules

§ 4. A rectangular gate and its diagonal

sq. d = sq. s + sq. h

§ 4a	s, h given	$d = ?$	$d \approx h$ + 1/2 rec. $h \cdot$ sq. s
§ 4b	h, d given	$s = ?$	$s \approx$ sqs. $((d - h) \cdot 2\ h)$
§ 4c	s, d given	$h = ?$	aborted
§ 4d	s, h given	$d = ?$	$d \approx$ sq. $s \cdot 2 \cdot h + h$
§ 4e	[h, d given]	[$s = ?$]	[………]
§ 4f	[s, h given]	[$d = ?$]	[d = sqs. (sq. s + sq. h)]
§ 4g	[h, d given]	$s = ?$	[s = sqs. (sq. d – sq. h)]$^?$
§ 4h	s, d given	$h = ?$	h = sqs. (sq. d – sq. s)

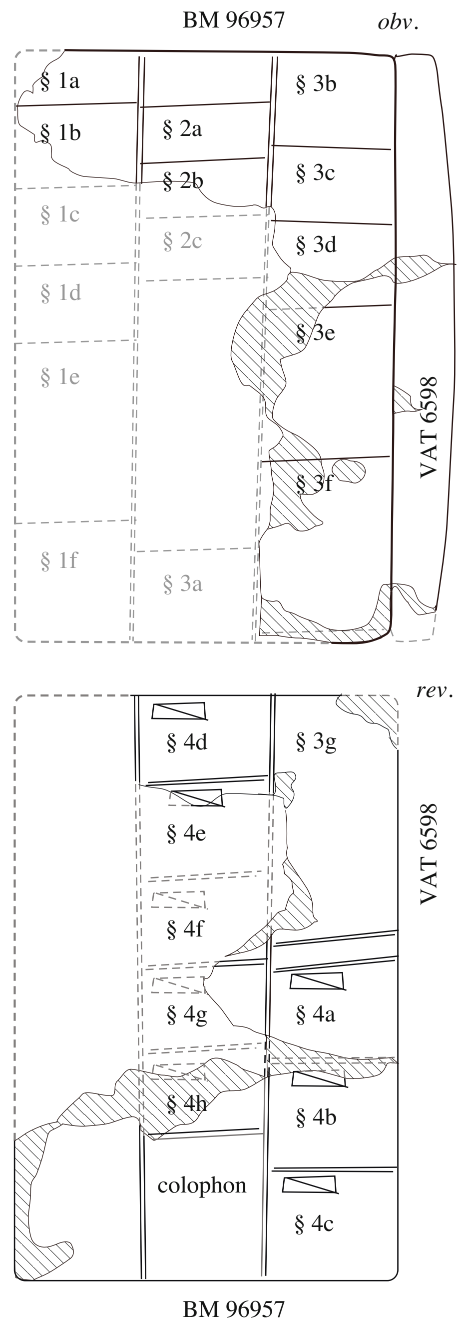


Fig. 8.6.2. BM 96957+VAT 6598. A reconstructed outline of the tablet and a table of contents.

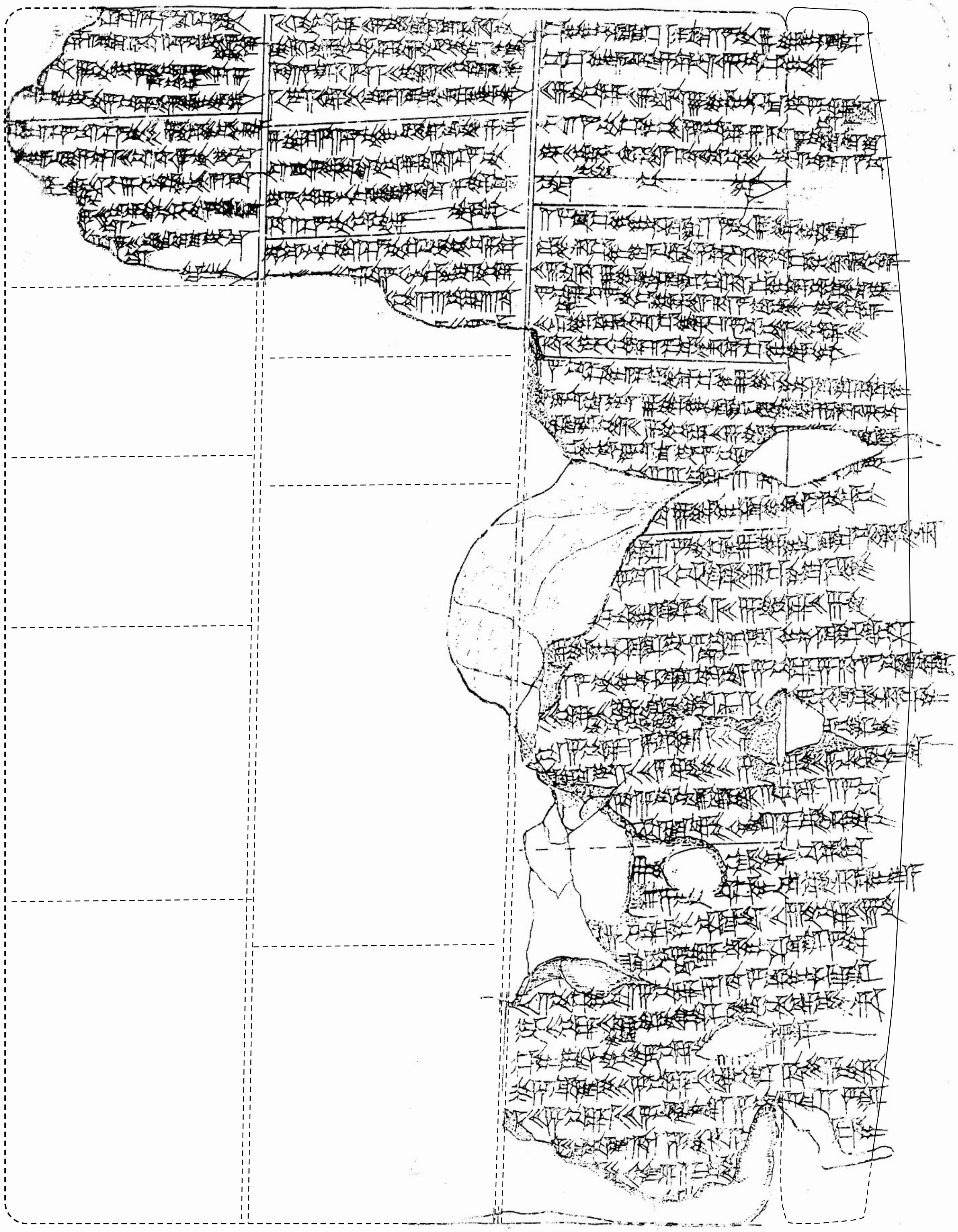


Fig. 8.6.3. BM 96957+VAT 6598, *obv*. Hand copy.

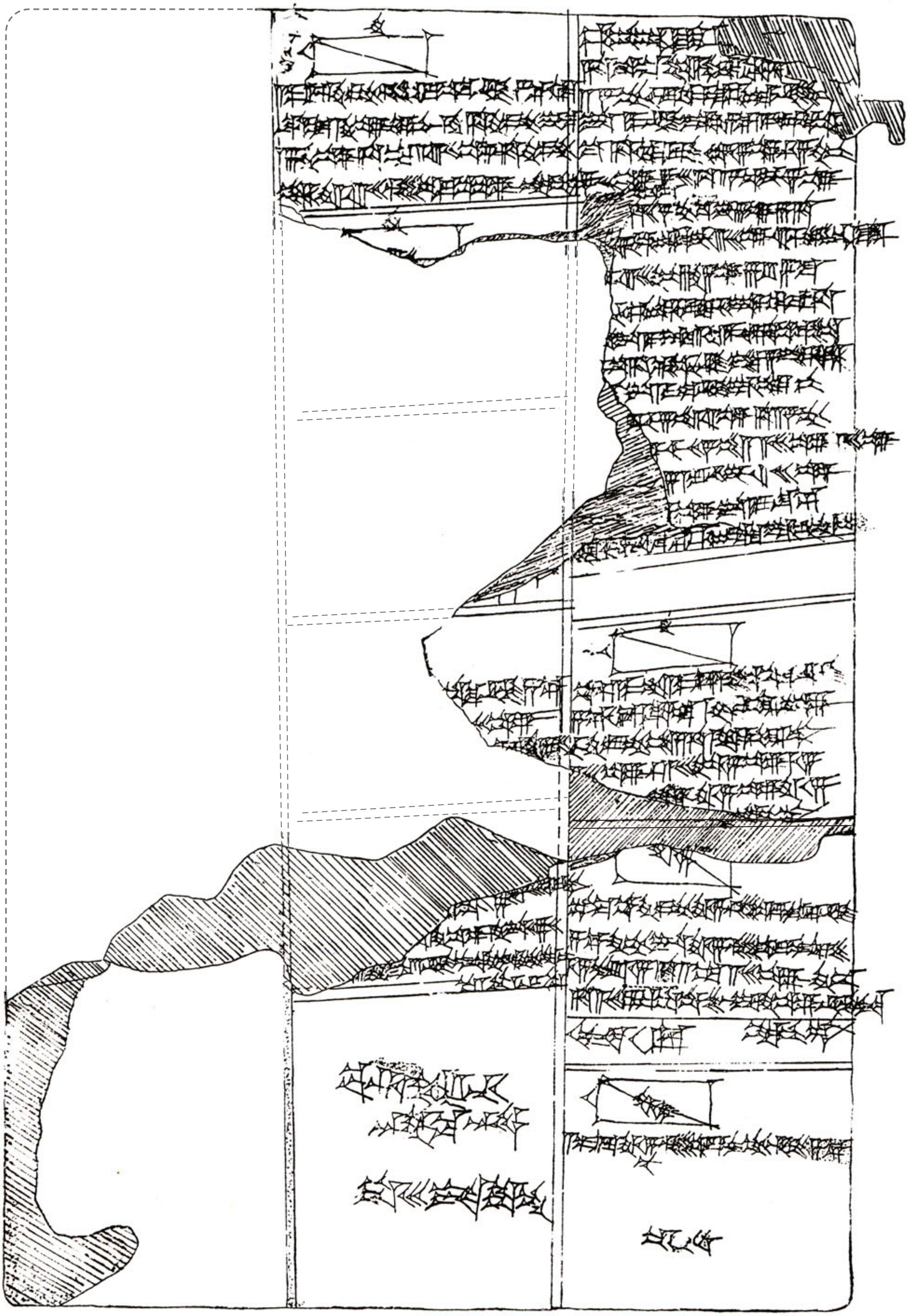


Fig. 8.6.4. BM 96957+VAT 6598, *rev*. Hand copy.

8.6.7 The Vocabulary of BM 96957+

In addition to terms already listed in the vocabularies of BM 80078 (Sec. 8.1.7 above) and BM 54779 (Sec. 8.2.4), the vocabulary of BM 96954+ contains the following terms:

dagal	width
é.gar$_8$	(brick) wall
gaba.uš, gaba.uš.ta	??? (base area?)
íb.sá	equalside (square)
íb.tag$_4$	remainder
ì.gu$_7$.gu$_7$	food (inclination)
ki.kal	base
sag	front
saḫar-*rum* (*eperu*)	mud, earth (volume)
sig$_4$	(rectangular) brick
sig$_4$.al.ur$_5$.ra	square brick

iš-te-en	*ištēn* one
it-ta-šu(?)	*ittu* (?) image?
ku-bu-re-e	*kubru* thickness
ne-pé-šum	*nēpešu* procedure < *epēšu* to make, to build
pa-nu	*pānu* front, face
qà-qá-ar, *qà-q*]*á-ra-tum*	*qaqqaru* ground, earth
ṣí-li-ip-tum	*ṣiliptu* dissection, dissector < *ṣalāpu* to cut open, to dissect
ši-ip-ka-at	*šipkātu* heaping up (number of bricks) < *šapāku* to heap up, to pile up
šum-ma	*šumma* if
tam-ḫir-tim	*tamḫartu* square number < *maḫāru* Gt to make equalsided, to square
ú-ku-la	*ukullû* fodder, feed, food (inclination) < *akālu* to eat

ta-am$^{!}$*-mar, ta-mu-rù*	< *amāru* to see
tu-ka-lu, i-ku-ul, i-ku-lu	< *akālu* to eat
li-li	< *elû* to go up, to be high
é-pu-uš	< *epēšu* to make, to build
i-te-rù	< *watāru* to exceed
ḫe-pé	< *ḫepû* to break
ki-il	< *kullu* D to hold
le-qé	< *leqû* to take
lu-<ša>-al-bi-in, tu-ša-al-ba-<an>	< *labānu* to spread, to make bricks
ip-ḫur	< *paḫāru* to gather, to assemble
i-ša-pa-ka-ku	< *šapāku* to heap up, to pile up
⸢*ša*⸣*-ak-na-at*	< *šakānu* to set, to place
tu-ša-qá	< *šaqû* to be high
ta-at-bu-ku	< *tabāku* to pour out, to lay flat, to lay bricks

a-na e-le-nim	above, at the top
a-na ugu	above, at the top
a-na ša-ap-li-a-tim	below, at the base

9. More Mathematical Cuneiform Texts of Group 6 from Late Old Babylonian Sippar

A Survey of Known Texts from Group 6 and Their Sumerian Terminology

In Ch. IV of Neugebauer and Sachs, *MCT* (1945), A. Goetze divided published Old Babylonian mathematical texts without known provenance into 6 different groups with respect to their Akkadian orthography. Goetze's classification was later refined and extended by J. Høyrup in *LWS* (2002), Ch. IX. Groups 5 and 6 were classified by Goetze as "northern". One of the characteristics of these northern groups is the spelling *ṭì* = TI. This spelling occurs in BM 80078 (Sec. 8.1 above), in the word *ṭì-da-am*. So, according to Goetze's classification, BM 80078 is northern, but nothing more can be said.

Another, more informative, way of dividing Old Babylonian mathematical cuneiform texts into groups (effectively the same groups as the ones discussed by Goetze and Høyrup!) was proposed in Friberg, *RA* 94 (2000), 160-173. There it was shown how an analysis of the Sumerian mathematical terminology, in the form of so called "sumerograms", may provide important clues to the provenance of unprovenanced Old Babylonian mathematical cuneiform texts.

Consider, for instance, the following Old Babylonian mathematical texts:

1	BM 54320	(above, Sec. 8.3)	a small fragment of a 2-column tablet	
2	BM 54779	(above, Sec. 8.2)	a small fragment of a 2-column tablet	
3	BM 80078	(above, Sec. 8.1)	a large fragment of a 2-column recombination text	
4	BM 80088	(below, Sec. 9.3)	a small fragment of a catalog text	
5	BM 80209	(below, Sec. 9.5)	an intact 1-column catalog text	6a
6	BM 85194	(above, Sec. 8.5.12)	a 3-column recombination text	6a
7	BM 85196	(below, Sec. 11.1.3)	a 2-column recombination text	6a
8	BM 85200 + VAT 6599	(*MKT I*, 193)	a 2-column recombination text (fragmentary)	6a
9	BM 85210	(above, Sec. 8.5.10)	a large fragment of at 2-column recombination text	6a
10	BM 96954 + BM 102366 + SÉ 93	(above, Sec. 8.5)	a 3-column recombination text (fragmentary)	6a
11	BM 96957 + VAT 6598	(above, Sec. 8.6)	a 3-column recombination text (fragmentary)	6a
12	Böhl 1821	(below, Sec. 9.1)	a round hand tablet with a single problem	6a
13	CBS 165	(below, Sec. 9.4)	a small fragment of a catalog text	
14	MLC 1354	(*MCT* Eb)	a rectangular hand tablet with a single problem	6b
15	MLC 1950	(*MCT* Ca)	a rectangular hand tablet with a single problem	6b
16	VAT 672	(*MKT I*, 267)	a small fragment	6b
17	VAT 6469	(*MKT I*, 269)	a small fragment	6b
18	VAT 6505	(above, Sec. 8.4)	a small fragment of a three-column tablet	6b
19	VAT 6597	(*MKT I*, 274)	a small fragment	6a

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Below is a brief survey of the Sumerian mathematical terminology used in the enumerated texts, with indications of in which of the texts the respective Sumerian terms appear. Also the use of the Akkadian terms *mi-nu*, *i-ši, ki-a-am ne-pé-šum,* and *ta-mar* is surveyed.

ba.zi	subtract			3		5	6	7	8	9	10	11	12			<u>15</u>				
bala	fraction							7	8	9	10									19
daḫ.ḫa	add on			3		<u>5</u>	6	7	8	9	10	11	12					17	18	19
en.nam or *mi-nu*	what?	1′	2	3′		5′	6	7	8	9	10	11	12		14′	15′		17′	18	
gar.ra	set	1	2	3			6	7		9	10		12		14			17		19
íb.sá	equalside(d)		2	3			6	7	8	9	10	11								
igi (*n*) duḫ.a	compute rec. *n*			3			6	7	8	9	10	11	12			<u>15</u>			18	19
igi.gub.ba	constant			3				7		<u>9</u>	10	<u>11</u>	12		14					
nigín.na	resume	1	2	3			6	7		9	10									19
šag$_4$.gal	food, growth rate						6	7		9	10									
tab.ba	repeat						6	7		9	10	11					16			
túm.a or *i-ši*	multiply		2	3			6′	7′	8′	9′	10′	11′	12′		14′	15′	<u>16</u>	17′	18	19′
ul.gar	add together	1			4	5	6	7	8	9	10	11	12	13		15				19
za.e ak.ta.zu.dè	you in your doing it	<u>1</u>		<u>3</u>			<u>6</u>	<u>7</u>	<u>8</u>	<u>9</u>	<u>10</u>	<u>11</u>	<u>12</u>		14	15	16	17	19	<u>19</u>
ki-a-am ne-pé-šum	such is the procedure						6	7	<u>8</u>	<u>9</u>	10	<u>11</u>	12					17	18	<u>19</u>
ta-mar	is the result	1	2	3			6	7	8	9	10	11	12		14	15			18	19

In the survey above underlined numbers are used to indicate those texts where the abbreviated forms zi, daḫ, duḫ, igi.gub, túm, za.e, *ne-pé-šum* appear instead of the full forms ba.zi, daḫ.ḫa, duḫ.a, igi.gub.ba, túm.a, za.e ak.ta.zu.dè and *ki-a-am ne-pé-šum*. The tagged numbers 1′, 3′, etc. indicate those cases where the Sumerian terms en.nam and túm.a have been replaced by the Akkadian forms *mi-nu* and *i-ši*.

Note that the use of the term ul.gar (clearly in mathematical texts with the meaning 'heap', 'add together') in itself practically guarantees that an unprovenanced mathematical text or fragment belongs to Group 6 of Old Babylonian mathematical cuneiform texts. Indeed, the term ul.gar is documented only in mathematical texts belonging to either Group 6 (probably texts from Sippar) or Group 8 (texts from Susa). In *SBM* 1 (1991), 15, Muroi explains ul.gar as a Sumerian verbal form ul.gar < ù.al.gar.

Similarly the Akkadian term *ta-mar* occurs almost exclusively in texts belonging to Groups 6 (Sippar), 7 (Shaduppûm, Mê-Turran, etc.), and 8 (Susa). (See Høyrup, *LWS* (2002), 360.) One exception is YBC 4662 = *MCT* J, a text belonging to Group 2.

9.1 Böhl 1821. A Metric Algebra Problem for Two Concentric Circular Towns

This text was published by Leemans in *CRRA 2* (1951). Although Leemans' hand copy of the text was accompanied by a photo and a transliteration, as well as by a mathematical commentary written by Bruins, a renewed study of the text is needed. One reason for this is that Leemans' transliteration of the text is completely akkadianized, with all sumerograms replaced by their presumed Akkadian counterparts. As was mentioned above, interesting information about the provenance of a mathematical cuneiform text can be hidden and invisible if the sumerograms in the text are replaced. Besides, the proceedings in which the text was first published are not easily accessible today.

A second reason why a renewed study of the text is needed is that the mathematical commentary offered by Bruins is unsatisfactory. First, explaining the solution procedure in the text in terms of radii of circles, as Bruins does, is anachronistic, since in the solution procedures of Babylonian mathematical problem texts usually only the circumference and the diameter of a circle are considered. The only known exception occurs in the new text IM 121512 (Sec. 6.3 above), where the term *matnu* 'string' stands for 'radius', in most cases of a semicircle. Secondly, Bruins' suggestion that a special unit of length measure was used for the circumference of a circle is unhistoric and unacceptable.

Leemans' French translation is replaced below by an English translation, which is somewhat tentative because of certain awkward and difficult expressions in the question.

		Böhl 1821 (Leemans, *CRRA 2* (1951))
obv.	1-2	uru^{ki} gúr *ak-pu-up-ma* / *ma-la ak-pu-pu ú-ul i-de* /
	3-4	*šu-ub-tum-ia i-ṣa-at-ma* / uru^{ki} *ú-ri-id-di* /
	5	*iš-te-nu-um* uru^{ki} *bi-ri-<ti>-im* 5.ta.àm *et-te-se-e-ma* /
	6	uru^{ki} *ša-ni-a-am* gúr *ak-pu-up* 6 15 a.šag$_4$ *dal-ba-an-na* /
	7	gúr uru gibil *ù* gúr uru libir.ra en.nam
		za.[e] /
	8	5 *da-ki-iš-ta-ka a-di* 3 *a-lik* 15 *t*[*a-mar*] /
	9-10	igi 15 duḫ.a *a-na* 6 15 a.šag$_4$ *dal-ba-an-n*[*a i-ši*] / 25 *ta-mar*
	11	25 *ša ta-mu-ru* / *a-di* 2 gar.ra /
rev.	12-14	5 *ša te-te-es-sú-ú a-na* 1 daḫ.ḫa / *i-na* 2 ba.zi 30 uru gibil 20 uru libir.ra / *ta-mar*
		na-ás-ḫi-ir
	15	a.šag$_4$ gúr uru gibil *ù* gúr uru libir.ra / *a-mu-ur*
		30 *šu-tam-ḫir* 15 *ta-mar*
	16	15 *a-na* 5 igi.gub.ba [gúr] / *i-ši* 1 15 uru gibil *ta-mar*
	17	20 *šu-tam-ḫir a-na* 5 igi.gub.ba gúr / *i-ši* 33 20 uru libir.ra *ta-mar* /
	18	[*ki*]*-a-am ne-pé-šum*

obv.	1-2	A town, an arc (a circle) I curved. / Whatever I curved I did not know. /
	3-4	My settlement grew too small. / A town I added. /
	5	One town, of the distance 5 in each direction, I moved out. /
	6	A second town, an arc I curved, 6 15 was the field of the intermediate space. /
	7	The arc of the new town and the arc of the old town, what?
		You: /
	8	5, your extension, until 3 go, 15 y[ou will see]. /
	9-10	The reciprocal of 15 release, to 6 15 the field of the intermediate space [carry] / 25 you will see.
	11	25 that you saw / until 2 set. /
rev.	12-14	5 that you went away in all directions to 1 add on / from 2 tear off, 30, the new town, 20, the old town, / you will see.
		Turn back.
	15	The fields of the arc of the new town and of the arc of the old town / see.
		30 let equal itself, 15 you will see.
	16	15 to 5, the constant of the [circumference], / carry, 1 15 of the old town you will see.
	17	20 let equal itself, to 5 the constant of the arc /carry, 33 20 of the new town you will see. /
	18	[Su]ch is the procedure.

The question in this exercise can be described as follows, in modern terms :

> Two (concentric) circles are drawn, the second circle at a distance everywhere of 5 (rods) from the first circle.
> The "circular band" bounded by the two concentric circles has the area 6 15 (sq. rods).
> What are the circumferences of the two circles?

The solution procedure is divided into two parts. In quasi-modern symbolic notations, the first part of the solution procedure, in lines 8-13, can be described as follows, with A denoting the area of the circular band and with e denoting its width (= the distance between the two circles, in the text called the 'extension'):

$A / (3\,e) + e = 6\;15 / 15 + 5 = 25 + 5 = 30 = d'$ 'the new town',
$A / (3\,e) - e = 6\;15 / 15 - 5 = 25 - 5 = 20 = d$ 'the old town'.

What is going on here is shown below, in Fig. 9.1.1 right. Evidently, the author of the exercise based his solution procedure on the following *Old Babylonian computation rule for the area of a circular band*:

> The area A of a circular band of width e bounded by two concentric circles with the diameters d and d', respectively, is equal to the median circumference $d_m = (3\,d + 3\,d')/2$ times the width e.

More concisely,

$A = d_m \cdot e = 3\,(d' + d)/2 \cdot e,$ where $e = (d' - d)/2.$

From a modern point of view, the correctness of this rule is obvious, since it is clear that by the use of the conjugate rule, and with 3 as an approximation for π,

$A = A' - A = 3 \text{ sq. } d'/2 - 3 \text{ sq. } d/2 = 3\,(d' + d)/2 \cdot (d' - d)/2 = 3\,(d' + d)/2 \cdot e.$

As a consequence of this simple rule, if A and e are given, as in this exercise, then

$(d' + d)/2 = A / 3\,e,$ and $d' = (d' + d)/2 + (d' - d)/2 = A / 3\,e + e,$ $d = (d' + d)/2 - (d' - d)/2 = A / 3\,e - e.$

Therefore, the computed numbers '30' and '20' in line 13 of the exercise, vaguely called the 'new town' and the 'old town', are really the *diameters* of the two circles. This observation is somewhat surprising, since what was asked for in the stated question in line 7 of the exercise was the *circumferences* of the two circles!

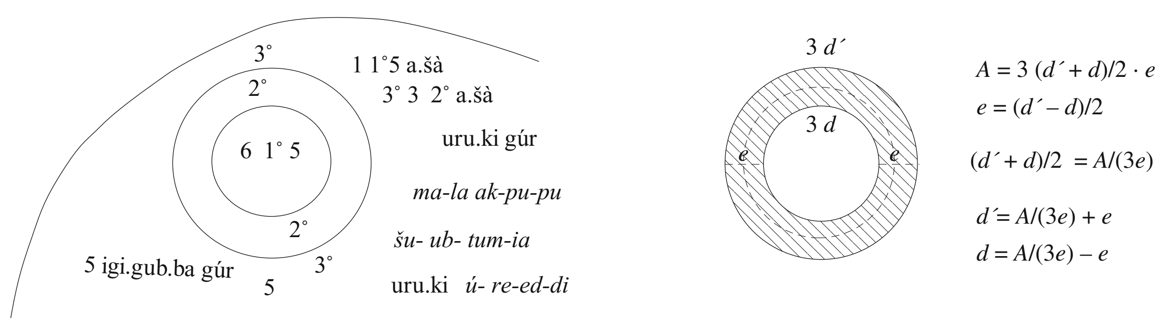


Fig. 9.1.1. Böhl 1821. Finding the outer and inner diameters of a circular ring (the extension of a "town").

In the second part of the exercise, in lines 15-17, there is an attempt to verify the obtained result by computing the areas of the two circles. What is computed is, in relative place value notation,

5 · sq. 30 = 5 · 15 = 1 15 and 5 · sq. 20 = 5 · 6 40 = 33 20.

These values are inscribed to the right of the diagram on the clay tablet. However, here the one who wrote the text made two errors. First, he failed to subtract one of these numbers from the other, in order to verify that the difference between the two areas is equal to the given area of the circular band. As a matter of fact, it is not, because 30 and 20 are the diameters, not the circumferences, of the two circles. What should have been computed, instead, is

5 · sq. (3 · 30) = 5 · sq. 1 30 = 5 · 2 15 = 11 15, 5 · sq. (3 · 20) = 5 · sq. 1 = 5 · 1 = 5 (00), and 11 15 – 5 (00) = 6 15.

Note that further errors were committed in the diagram in the upper left corner of the hand tablet (Fig. 9.1.1, left), where the numbers 30, 20, and 6 15 are placed incorrectly.

Recall that another Old Babylonian mathematical exercise concerned with a circular town and its extension is BM 85194 # 4, Sec. 8.5.13 above.

(An interesting reconstructed image of the ancient city Mari, modern Tell Hariri, in the form of an extended circle with a diameter of over 2 kilometers can be seen below.)

9.1.2. The ancient city of Mari, aerial view.
Balage Balogh GNU Free Documentation License, Wikimedia Commons

9.2 YBC 6492. A Catalog Text with Simple Form and Magnitude Problems

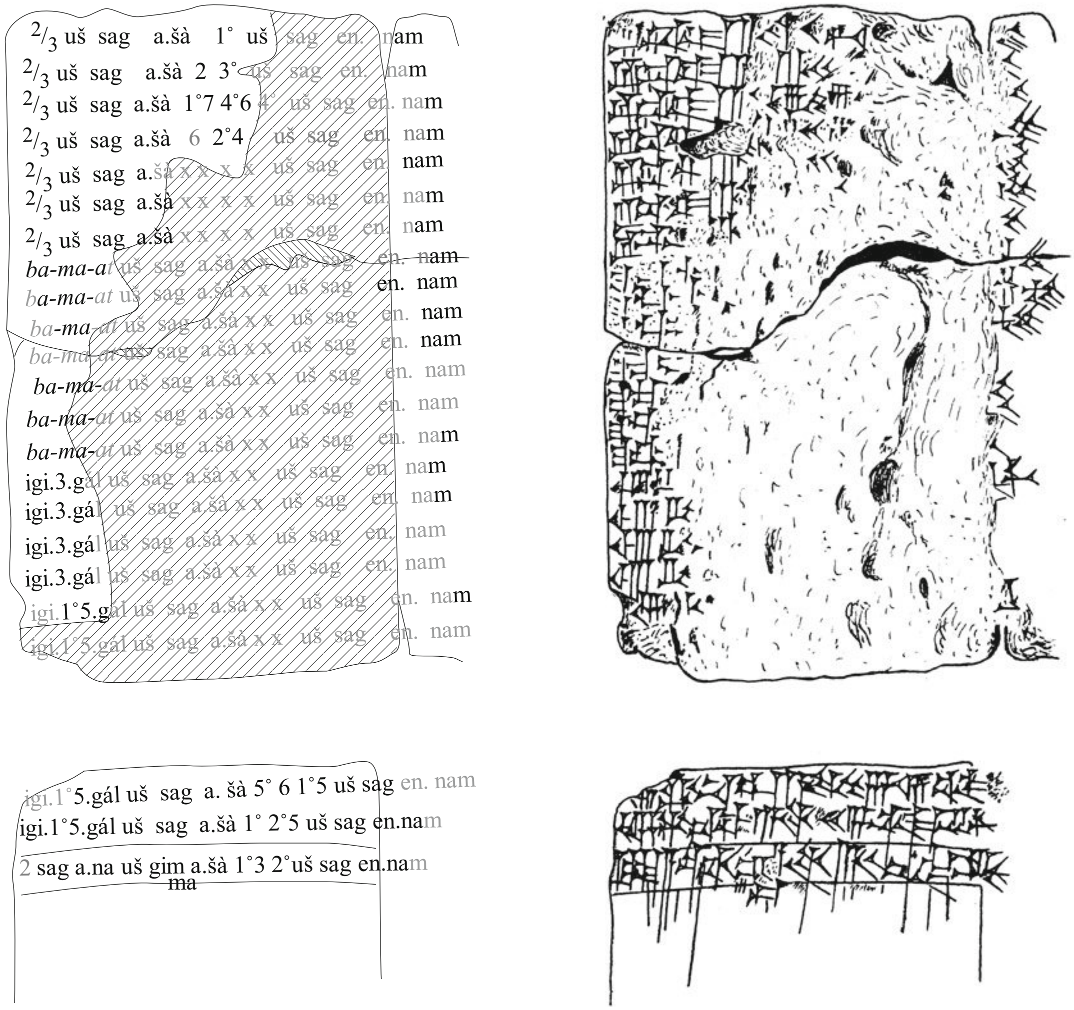


Fig. 9.2.1. YBC 6492. A catalog text for form and magnitude problems with variations of parameters.
The hand copy is reproduced here with the permission of the American Oriental Society.

The catalog text YBC 6492 = *MCT* Sa (Fig. 9.2.1 above) is too uncomplicated to allow it to be assigned to any particular group of Old Babylonian mathematical texts. As a matter of fact, it is concerned, essentially, with *a single rectangular-linear system of equations*, with 23 different *variations of parameters* in the equations. (Compare with what is said below, in Sec. 11.4, about the absence of a notion for general coefficients in Babylonian mathematical texts.) Similar texts are, for instance, the Sippar text BM 80209 (Sec. 9.5 below) and the Susa texts *TMS* V-VI (Secs. 10.3-4 below), which contain large numbers of systematically arranged metric algebra problems for circles in the former text and for one, two, or three squares in the latter, all without solution procedures and answers.

The catalog in YBC 6492 is quite simple, although systematically arranged, as is shown by the following abbreviated transliteration and translation:

2/3 uš sag a.šag$_4$ (A) uš [sag en.n]am	2/3 $u = s$,	A = 10 (00)	(u, s = 30, 20)
2/3 of the length is the front, area A. Length, [front] what?		A = 2 30	(u, s = 15, 10)
		A = 17 46;40	(u, s = 40, 26;40)
		A = [6] 24	(u, s = 24, 18)
	3 lines damaged		
ba-ma-at [uš sag a.šag$_4$ (A) uš sag en.nam]	1/2 $u = s$,	A = ……	(u, s = …, …)
Half [the length is the front, area A. Length, front what?]		7 lines damaged	
igi.3.gál [uš sag a.šag$_4$ (A) uš sag en.nam]	1/3 $u = s$,	A = ……	(u, s = …, …)
A 3rd-part [the length is the front, area A. Length, front what?]		7 lines damaged	
igi.15.gál uš sag a.šag$_4$ (A) uš sag en.na[m]	1/15 $u = s$,	A = ……	(u, s = …, …)
A 15th-part [the length is the front, area A. Length, front what?		2 lines damaged	
		A = 56 15	(u, s = 3 45, 15)
		A = 10 25	(u, s = 12 30, 50)
⸢2⸣ sag a.na uš ⸢gim⸣.ma a.šag$_4$ 13 20 uš [sag en.n]am	⸢2⸣ $s = u$,	A = 13 20	(u, s = 40, 20)
⸢2⸣ fronts as much as the length are ⸢like⸣, area A. Length, [front] what?			

Expressed even more concisely, all the problems in this catalog text are rectangular-linear systems of equations of the following type (except the problem in the last line, which may be a catch line for problems on another tablet):

$$c\,u = s, \quad u \cdot s = A$$

There are *two levels of variations of parameters*. On the primary level, the coefficient c is varied, and on the secondary level the value of A is varied. The answers, which are not given in the text, vary with the parameters. The variation of c is fairly systematic, but the variation of A is not.

Note. All the problems in this catalog text are like the “form and magnitude problem” of exercise # 1 in the Mê-Turran text IM 121613. See Sec. 5.1.1 above. The way in which such problems could be solved is illustrated in Fig. 5.1.1.

9.3 BM 80088. A Catalog Text with Rectangular-Linear Systems of Equations of Types B1a-b

This is a small fragment of a tablet originally inscribed with at least two columns of text. Although the text on the fragment is in a very small script, it is clear that the fragment is only a small piece of what was once a very large tablet.

Just a small part of the text in the first column is preserved. However, the use of the term ul.gar makes it almost certain that the fragment belongs to Group 6 of unprovenanced Old Babylonian mathematical cuneiform texts. (See the introduction above to Ch. 9.)

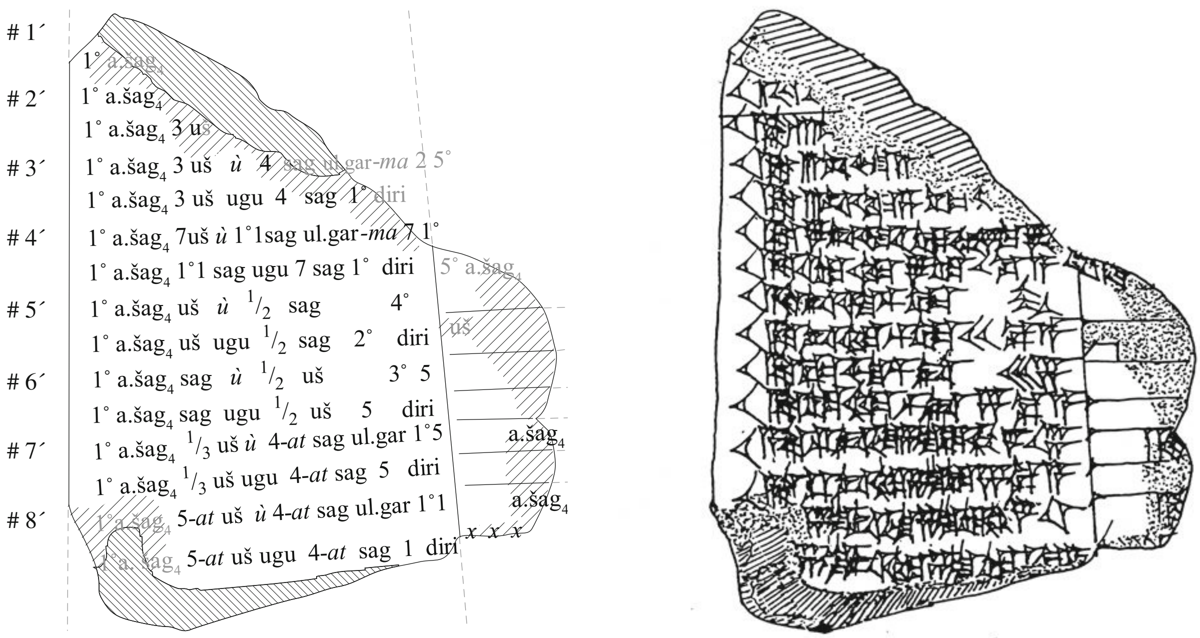


Fig. 9.3.1. BM 80088. A small fragment of a large mathematical catalog text from Sippar with variations of parameters.

The beginning and the end of the text of this paragraph on the fragment are lost, but the preserved part of the text is clearly readable and fairly well organized. It can be divided into a number of sub-sections of two lines each. In the transliteration of the text below, to the left, this division into sub-sections is made visible. Instead of a translation of the usual kind there is, to the right, explanations of the preserved problems in the catalog as rectangular-linear systems of equations, expressed in quasi-modern symbolic notations.

# 1´a	[x x x x	x x x x x x x x x x]	A = 10 (00),	[x x x x x x x x x x]
# 1´b	10 [a.šag$_4$	x x x x x x x x x x]	A = 10 (00),	[x x x x x x x x x x]
# 2´a	10 a.šag$_4$	[x x x x x x x x x x]	A = 10 (00),	[3 u + x x x x x x x x]
# 2´b	10 a.šag$_4$	3 u[š x x x x x x x]	A = 10 (00),	3 u [– x x x x x x x]
# 3´a	10 a.šag$_4$	3 uš *ù* [4 sag ul.gar-*ma* 2 50]	A = 10 (00),	3 u + 4 s = 2 50
# 3´b	10 a.šag$_4$	3 uš ugu 4 sag 10 [dirig]	A = 10 (00),	3 u – 4 s = 10
# 4´a	10 a.šag$_4$	7 uš *ù* 11 sag ul.gar-*ma* 7 10	A = 10 (00),	7 u + 11 s = 7 10
# 4´b	10 a.šag$_4$	11 sag ugu 7 uš$^!$ 10 dirig	A = 10 (00),	11 s – 7 u = 10
# 5´a	10 a.šag$_4$	uš *ù* $^1/_2$ sag 40	A = 10 (00),	u + 1/2 s = 40
# 5´b	10 a.šag$_4$	uš ugu $^1/_2$ sag 20 dirig	A = 10 (00),	u – 1/2 s = 20
# 6´a	10 a.šag$_4$	sag *ù* $^1/_2$ uš 35	A = 10 (00),	s + 1/2 u = 35
# 6´b	10 a.šag$_4$	sag ugu $^1/_2$ uš 5 dirig	A = 10 (00),	s – 1/2 u = 5
# 7´a	10 a.šag$_4$	$^1/_3$ uš *ù* 4-*at* sag ul.gar 15	A = 10 (00),	1/3 u + 1/4 s = 15
# 7´b	10 a.šag$_4$	$^1/_3$ uš ugu 4-*at* sag 5 dirig	A = 10 (00),	1/3 u – 1/4 s = 5
# 8´a	[10 a.šag$_4$	5-*at* uš *ù* 4-*at* sag ul.gar 11	A = 10 (00),	1/5 u + 1/4 s = 11
# 8´b	[10 a.šag$_4$]	5-*at* uš ugu 4-*at* sag 1 dirig	A = 10 (00),	1/5 u – 1/4 s = 1

It is easy to check that all the listed rectangular-linear systems of equations have *the same standard solution*, well known from many previously published cuneiform mathematical texts, namely u = 30, s = 20. Incidentally, conjecturing that all the systems of equations have the same standard solution, a tricky way of finding that solution is to compare to each other the two linear equations in each sub-section. For instance, in ## 4´a and 4´b, 7 u + 11 s = 7 10, and 11 s – 7 u = 10, respectively. Therefore, 2 · 7 u = 7 10 – 10 = 7 (00), so that u = 7 (00) / 2 · 7 = 30, and 2 · 11 s = 7 10 + 10 = 7 20, so that s = 7 20 / 2 · 11 = 20. (The proper way of solving, for instance, exercises # 4´a-b is shown at the beginning of Sec. 9.4.2 below.)

9.4 CBS 165. Another Catalog Text with (Presumably) Rectangular-Linear Systems of Equations of Types B1a-b

9.4.1 CBS 165. A Suggested Partial Reconstruction of the Text

A small fragment of a catalog text similar to the fragment BM 80088 was published by Robson in *Sciamvs* 1 (2000), together with a rudimentary mathematical commentary. Admittedly, the damaged state of the fragment, as well as some unusual kinds of errors in the preserved text makes any detailed reconstruction of the text impossible.

The fragment is not much more than a "surface flake". It was called "unprovenanced" by Robson, but just as in the case of BM 80088 (Sec. 9.3 above), the occurrence of the term ul.gar makes it almost certain that it belongs to Group 6 of Old Babylonian mathematical cuneiform texts, probably from Sippar (just like the three fragments discussed in Ch. 8 above).

The conform transliteration below is based on Robson's hand copy.

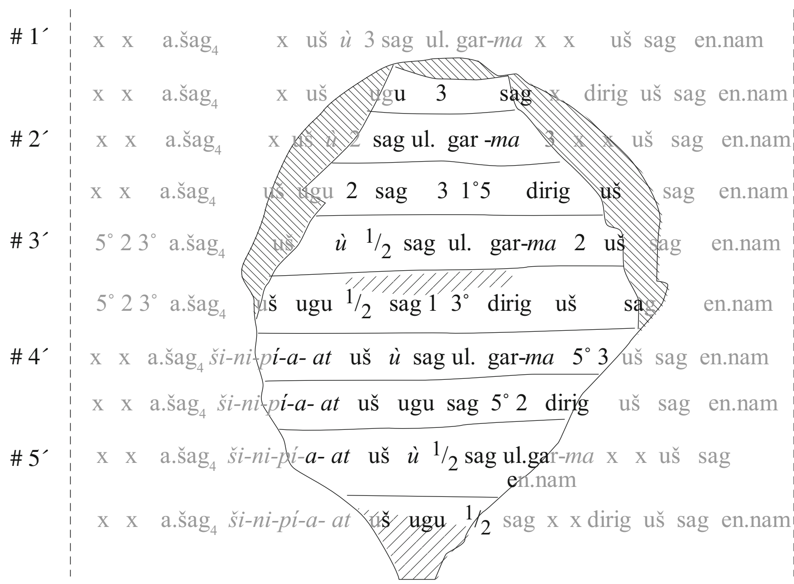


Fig. 9.4.1. CBS 165. A tentative reconstruction and conform transliteration of the text.

The structure of the catalog of metric algebra problems in CBS 165 can be explained as follows:

# 1´a	[x x a.šag$_4$](?)	[x uš *ù* 3 sag ul.gar-*ma* x x]	[uš sag en.nam]	[*A* = x x],	[x *u* + 3 *s* = x x]
# 1´b	[x x a.šag$_4$](?)	[x uš] ˹ugu 3 sag˺ [x x dirig]	[uš sag en.nam]	[*A* = x x],	[x *u*] – 3 *s* = [x x]
# 2 a	[x x a.šag$_4$](?)	[uš *ù* 2] sag ul.gar-*ma* ˹3˺ [x x]	[uš sag en.nam]	[*A* = x x],	[*u*] + 2 *s* = ˹3˺ [x x]
# 2´b	[x x a.šag$_4$](?)	[uš ugu] 2 sag 3 15 dirig	uš ˹sag˺ [en.nam]	[*A* = x x],	[*u*] – 2 *s* = 3 15
# 3´a	[52 30 a.šag$_4$](?)	[uš] ˹*ù* $^1/_2$˺ sag ul.gar-*ma* 2	[uš sag en.nam]	[*A* = 52 30],	[*u*] + 1/2 *s* = 2
# 3´b	[52 30 a.šag$_4$](?)	˹uš˺ ugu $^1/_2$ ˹sag˺ 1 30 dirig	uš ˹sag˺ [en.nam]	[*A* = 52 30],	˹*u*˺ – 1/2 *s* = 1 30
# 4´a	[x x a.šag$_4$](?)	[*ši-ni*]-˹*pí*˺-*a-at* uš *ù* sag 53	[uš sag en.nam]	[*A* = x x],	2/3 *u* + *s* = 53
# 4´b	[x x a.šag$_4$](?)	[*ši-ni*]-˹*pí*˺-*a-at* uš ugu sag 52 dirig	[uš sag en.nam]	[*A* = x x],	2/3 *u* – *s* = 52
# 5´a	[x x a.šag$_4$](?)	[*ši-ni-pí*]-*a-at* ˹uš˺ *ù* $^1/_2$ sag ul.gar-[*ma* x x]	u[š sag / en.nam]	[*A* = x x],	2/3 *u* + 1/2 *s* = [x x]
# 5´b	[x x a.šag$_4$](?)	[*ši-ni-pí-a-at*] ˹uš ugu $^1/_2$˺ [sag x x dirig]	[uš sag en.nam]	[*A* = x x],	[2/3] *u* – 1/2 *s* = [x x]

Like the catalog text BM 80088, the catalog text in CBS 165 is divided into pairs of lines. There are, however, obvious differences between the two texts. One superficial difference is that in CBS 165, but not in

BM 80088, the stated rectangular-linear systems of equations are followed by the question uš sag [en.nam] 'the length and the front [are what]?'. Another, more interesting, difference is that the listed rectangular-linear problems in CBS 165 clearly *do not have the standard solution $u = 30$, $s = 20$*, like the problems in BM 80088.

A first clue admitting a tentative reconstruction of at least one sub-section of the damaged catalog text is given by the pair of equations in sub-section # 3´. Namely,

if $u + 1/2\ s = 2$ and $u - 1/2\ s = 1\ 30$, then $2\ u = 2 + 1\ 30 = 3\ 30$ and $2 \cdot 1/2\ s = 2 - 1\ 30 = 30$, so that $u = 1\ 45$ and $s = 30$.

Next, if it is assumed, for lack of alternatives, that $u = 1\ 45$, $s = 30$ is the fixed answer to the problems in all the five sub-sections, then it follows that in all these cases

$A = u \cdot s = 1\ 45 \cdot 30 = 52\ 30.$

Unfortunately, this tentative reconstruction agrees badly with what is preserved of subsections ## 2´, 4´, 5´.

In # 2´, $u + 2\ s$ is less than $u - 2\ s$, which cannot be correct, since there are no negative numbers in Old Babylonian mathematical texts. Therefore some number in # 2´a or # 2´b must have been miscalculated or miscopied.

In exercise # 4´,

if $2/3\ u + s = 53$ and $2/3\ u - s = 52$, then $2 \cdot 2/3\ u = 53 + 52 = 1\ 45$ and $2\ s = 53 - 52 = 1$, so that $u = 1\ 18\ 45$ and $s = 30$.

In all likelihood, some number or numbers in exercise # 4´, too, must have been miscalculated or miscopied.

Anyway, it seems to be clear that in CBS 165 all the pairs of rectangular-linear problems are of the type

$u \cdot s = A, \quad \text{p}\ u + \text{q}\ s = a, \quad$ and $\quad u \cdot s = A, \quad \text{p}\ u - \text{q}\ s = b,$

with variations of all the parameters *A,* p, q, and also of the solutions *u, s*.

It is not immediately obvious how an Old Babylonian student of mathematics was supposed to find the solutions to rectangular-linear systems of equations like the ones in the catalog texts BM 80088 and CBS 165, discussed above. From a modern point of view, there is an easy solution method that works in every case. Consider, generally, a rectangular-linear system of equations of the type

a) $u \cdot s = A, \quad \text{p}\ u + \text{q}\ s = a \quad$ or $\quad$ b) $u \cdot s = A, \quad \text{p}\ u - \text{q}\ s = b.$

Introducing p u and q s as new unknowns leads to the simplified rectangular-linear system of equations

a) $(\text{p}\ u) \cdot (\text{q}\ s) = \text{p q}\ A, \quad (\text{p}\ u) + (\text{q}\ s) = k \quad$ or $\quad$ b) $(\text{p}\ u) \cdot (\text{q}\ s) = \text{p q}\ A, \quad (\text{p}\ u) - (\text{q}\ s) = k.$

These are *basic* rectangular-linear systems of equations for p u and q s of types B1a and B1b, respectively, (Friberg, *AT* (2007), 6) and can be solved geometrically by use of square bands and square corners as in Fig. 5.1.19 above.

One big difference between the modern approach outlined above and the Old Babylonian approach indicated by the long list of similar problems in the catalogs on BM 80088 and CBS 165, and elsewhere, is that *the Babylonians had no way of indicating general multiples of unknowns*, like p u or q s, with *arbitrary* numerical coefficients p and q. Instead, as in the mentioned catalogs, they listed examples of all kinds of numerical coefficients they could think of, namely *regular numbers* like 3 and 4 in BM 80088 ## 3´a-b above, *non-regular numbers* like 7 and 11 in BM 80088 ## 4´a-b, *basic fractions* like 1/2 and 1/3 in BM 80088 ## 5´a-b, 6´a-b and 7´a-b, *regular unit fractions* like 4-*at* (the '4th') and 5-*at* (the '5th') in BM 80088 ## 7´a-b and 8´a-b, and, possibly, *non-regular unit fractions* like 7-*at* in some lost sub-section of BM 80088. Further examples will be seen in the discussion of the large catalog text *TMS* V (Sec. 10.3 below).

9.4.2 Other Old Babylonian Texts with Rectangular-Linear Problems of Types B1a-b

In the corpus of Old Babylonian mathematical texts, nice examples of rectangular-linear systems of equations like the ones in BM 80088 that can easily be reduced to rectangular-linear systems of the basic types B1a-b are surprisingly rare. However, in the unusually simple catalog text YBC 4612 = *MCT* S, exercises ## 4-5 and ## 9-10 are of precisely these basic types, actually with the following variations of parameters:

4: 1(bur'u)[ašag] a.šag$_4$ uš *ù* sag gar.gar-*ma* / 5 05 uš *ù* sag en.nam / 3 45 nindan uš 1 20 nindan sag
5: 1(bur'u)[ašag] a.šag$_4$ uš ugu sag 2 25 dir<ig> / uš *ù* sag en.nam / 3 45 nindan uš 1 20 nindan sag

9: 2(bùr)[ašag] a.šag$_4$ uš *ù* sag gar.gar-*ma* / 2 54 uš *ù* sag en.nam / 2 30 nindan uš 24 nindan sag
10: 2(bùr)[ašag] a.šag$_4$ uš u<gu> sag 2 06 dir<ig> / uš *ù* sag en.nam / 2 30 nindan uš 24 nindan sag

In terms of quasi-modern symbolic notations, the cited problems can be reformulated as follows:

4: $u \cdot s = 5\ (00\ 00)$, $u + s = 5\ 05$ $u, s = 3\ 45, 1\ 20$
5: $u \cdot s = 5\ (00\ 00)$, $u - s = 2\ 25$ $u, s = 3\ 45, 1\ 20$

9: $u \cdot s = 1\ (00\ \ 00)$, $u + s = 2\ 54$ $u, s = 2\ 30, 24$
10: $u \cdot s = 1\ (00\ 00)$, $u - s = 2\ 06$ $u, s = 2\ 30, 24$

Another example is YBC 9697 = *MCT* Ua (Group 5; Høyrup, *LWS* (2002), 55), where the question is formulated in the following indirect way:

[igi.b]i *e-li* igi 7 *i-ter* / [igi] *ù* igi.bi *mi-nu-um*

Here igi and igi.bi mean two mutually reciprocal sexagesimal numbers, that is two numbers such that

igi · igi.bi = '1'.

The probable Old Babylonian interpretation of this equation in geometric terms would be that igi and igi.bi are *the sides of a rectangle with the area '1'.* Setting u = igi and s = igi.bi, the stated question can be reformulated as the following rectangular-linear system of equations:

$u \cdot s =$ '1', $u - s =$ '7'.

The explicit solution procedure in the text proceeds, essentially, as follows:

sq. $(u - s)/2$ = sq. 3 30 = 12 15, sq. $(u + s)/2 = u \cdot s$ + sq. $(u - s)/2$ = 1 + 12 15 = 1 12 15 = sq. (8 30),

igi = $u = (u + s)/2 + (u - s)/2$ = 8 30 + 3 30 = 12, igi.bi = $s = (u + s)/2 - (u - s)/2$ = 8 30 – 3 30 = 5.

Note that this solution procedure is correct only if $u \cdot s$ = '1' is understood as meaning that $u \cdot s = 60$. In other words,

igi · igi.bi = 1 (00) = 60.

As a matter of fact, this seems to have been the original definition of reciprocal pairs. See Ch. 12 below.

A less explicit occurrence of a simple rectangular-linear system of equations is the exercise BM 80078 # 2′, probably of Group 6a (Sec. 8.1.2 above), where, after a number of preceding calculations, the length u and width s of a brick of a certain type is shown to be such that

$u \cdot s = 6$, $u + s = 5$.

This rectangular-linear system of equations of the basic type B1a is then solved directly in the manner shown in Fig. 5.1.19.

A further example of a mathematical exercise where the stated problem is reduced to a rectangular-linear problem of the basic type B1a after a number of preceding calculations is exercise # 12 in BM 13901 (Group 2; see Høyrup, *LWS* (2002), 71). There the question is:

a.šag$_4$ *ši-ta mi-it-ḫa-*<*ra*>*-ti-ia ak-mur-ma* 21 40	The fields of my two equalsides I added together, then 21 40.
mi-it-ḫa-ra-ti-ia uš-ta-ki-il$_5$-ma 10	The equalsides I let eat each other, then 10.

In quasi-modern symbolic notations:

sq. s_1 + sq. s_2 = 21 40, $s_1 \cdot s_2$ = 10 (00).

In the solution procedure, it is silently assumed that the two squares are new unknowns. Then the given problem is transformed into the following basic rectangular-linear problem for the two squares:

sq. s_1 + sq. s_2 = 21 40, sq. s_1 · sq. s_2 = sq. '10' = '1 40'.

And so on.

Not less interesting is the example of an explicit solution procedure for a rectangular-linear problem of type B1a in *TMS* IX, exercise # 3 (Bruins and Rutten (1961); Høyrup, *LWS* (2002), 89-95). The question in this exercise is expressed as follows:

a.šag$_4$ uš *ù* sag ul.gar 1 a.šag$_4$	The field, the length, and the front I added together, 1 the field.
3 uš 4 sag ul.gar / [17]-*ti-šu a-na* sag daḫ 30	3 lengths, 4 fronts I added together, its 17th to the front I added, 30.

Expressed in quasi-modern symbolic notations, this probably means that the unknown length *u* and front *s* were to be computed as the solutions to the following rectangular-linear system of equations:

$$A + 1\ (00) \cdot u + 1\ (00) \cdot s = 1\ (00\ 00), \qquad 1/17 \cdot (3\,u + 4\,s) + s = 30 \qquad (*)$$

As explained by Høyrup, it was demonstrated in the preceding exercise *TMS* IX # 2, in a surprisingly didactical way, that if the sum of the field, the length, and the front is '1', or, in quasi-modern symbolic notations, if

$$A + 1\ (00) \cdot u + 1\ (00) \cdot s = 1\ (00\ 00),$$

then

$$(u + 1\ (00)) \cdot (s + 1\ (00)) = A + 1\ (00) \cdot u + 1\ (00) \cdot s + \text{sq. } 1\ (00) = 1\ (00\ 00) + 100\ (00) = 2\ (00\ 00).$$

(Høyrup mistakenly calls this a quadratic completion, and therefore pronounces as his opinion that the 'Akkadian way' , *i-na ak-ka-di-i*, which is mentioned in *TMS* IX, exercise # 2, was an Akkadian name for a "quadratic completion". As is clearly shown by Høyrup's own Fig. 19 in *LWS*, what Høyrup calls quadratic completion (making a square complete) should rightly be called "completion by a square".)

The solution procedure of *TMS* IX # 3 begins by transforming the linear equation in (*) above, through multiplication by 17, into the equivalent equation

$$(3\,u + 4\,s) + 17\,s = 17 \cdot 30, \qquad \text{or} \qquad 3\,u + 21\,s = 8\ 30.$$

Therefore, in view of what was explained in the preceding exercise *TMS* IX # 2, the given rectangular-linear system of equations in (*) above can be replaced by the equivalent system of equations

$$(u + 1) \cdot (s + 1) = 2\ (00), \qquad 3\,u + 21\,s = 8\ 30.$$

What follows next of the text is damaged, but apparently these equations for the unknowns *u* and *s* are thought of as replaced by the following equivalent equations for $u' = u + 1$ and $s' = s + 1$:

$$u' \cdot s' = 2\ (00), \qquad 3\,u' + 21\,s' = 8\ 30 + 3 + 21 = 32\ 30 \qquad (**)$$

This is a rectangular-linear system problem of precisely the same kind as some of the problems in the catalog texts BM 80088 and CBS 165. Therefore, the remainder of the solution procedure in *TMS* IX # 3 may be a model for how the problems in the mentioned catalog texts were intended to be solved!

The next step, apparently, is to silently make a new change of unknowns, introducing $u'' = 21\,s'$ and $s'' = 3\,u'$ as the new length and front. (Remember that in Old Babylonian mathematical texts, the length is always assumed to be longer than the front.) The rectangular-linear problem (**) is then replaced by the following equivalent problem, a basic rectangular-linear system of equations of type B1a:

$$u'' \cdot s'' = 3 \cdot 21 \cdot 2\ (00) = 2\ 06\ (00), \qquad u'' + s'' = 32\ 30 \qquad (***)$$

The solution to this system of equations can be obtained as in Fig. 5.1.19 above. It is

$$u''\ (= 21\,s') = 28\ (00), \qquad s''\ (= 3\,u') = 4\ 30.$$

Consequently,

$$u' = 1/3 \cdot 4\ 30 = 1\ 30, \qquad \text{and} \qquad s' = 28\ (00) / 21 = 1\ 20.$$

Therefore, finally,

$$u = u' - 1 = 30, \qquad \text{and} \qquad s = s' - 1 = 20.$$

This is the usual Old Babylonian standard solution to a rectangular-linear system of equations.

Yet another known occurrence of problems of the same kind as those in the catalog texts BM 80088 and CBS 165 can be found in *TMS* VIII, a large fragment from the obverse(?) of a tablet with two fairly well preserved exercises, and traces of a third. The two former exercises contain the (damaged) statements of the following two rectangular-linear problems *as well as the corresponding full solution procedures*:

a) $u \cdot s = 10\ (00), \quad s + 3 \cdot 1/4\, s = u + 5,$ b) $u \cdot s = 10\ (00), \quad s + 1 \cdot 1/4\, s = u - 5.$

Interestingly, after multiplication of both sides of the two linear equations above by 4, the two problems are transformed into the following pair of problems, similar to problems # 3´b and # 4´b in BM 80088:

a´) $u \cdot s = 10\ (00), \quad 7\, s - 4\, u = 20,$ b´) $u \cdot s = 10\ (00), \quad 4\, u - 5\, s = 20.$

Note that in the transliteration below of *TMS* VIII # 1, the restoration of the lost parts of the text in lines 1-2 differs from the restorations suggested by Høyrup in *LWS* (2002) and by Muroi in *SK* 140 (1994). There are also other, minor changes made in the transliteration, compared with what was proposed by Høyrup, and the interpretation of the solution procedure which is proposed below differs significantly from the interpretation proposed by Høyrup.

***TMS* VIII # 1** (Høyrup *LWS* (2002), 188-194; Muroi *SK* 140 (1994))

1-2 [a.šag$_4$ 10 4-*at* sag] *a-na* 3 *a-li-i*[*k a-na* sag daḫ ugu uš] / [5 di]rig
za.e [4 *r*]*e-ba-ti ki-ma* sag gar *re-b*[*a-at* 4 *le-qé* 1 *ta-mar*] /
3 [1 *a-na*] 3 *a-li-ik* 3 *ta-mar*
4 4 *re-ba-at* sag *a-na* 3 d[aḫ 7 *ta-mar*] /
7 *ki-ma* uš gar 5 dirig a.na *na-sí-iḫ* uš gar
5 7 uš *a-na* 4 [sag *i-ší*] / 28 *ta-mar* 28 a.šag$_4$ 28 *a-na* 10 a.šag$_4$ *i-ší* 4 40 *ta-mar* /
6 ⸢5⸣ *na-sí-iḫ* uš *a-na* 4 sag *i-ší* 20 *ta-mar* $^1/_2$ *ḫe-pe* 10 *ta-mar*
7 10 nigin / [1 40] *ta-mar* 1 40 *a-na* 4 40 daḫ 4 41 40 *ta-mar mi-na* íb.si 2 10 *ta-ma*[*r*] /
8 [10 *le*]-*qé a-na* 2 10 daḫ 2 20 *ta-mar*
9 *mi-na a-na* 28 a.šag$_4$ gar *šà* 2 20 *i-na-*[*di-n*]*a* / [5 gar] 5 *a-na* 7 *i-ší* 35 *ta-mar*
10 5 *na-sí-iḫ* uš *i-na* 35 zi / [30 *ta*]-*mar* 30 uš
5 uš *a-na* 4 sag *i-ší* 20 *ta-mar* 20 uš

1-2 [The field is 10. The 4th of the front] to 3 I went, [to the front I added on, over the length] / [it is 5 bey]ond.
You: [4], of the 4th, as the front set, the 4[th of 4 take, 1 you will see]. /
3 [1 to] 3 go, 3 you will see.
4 4, of the 4th of the front, to 3 add [on 7 you will see]. /
7 as the length set. 5, the beyond, as the torn out of the length set.
5 7, the length, to 4, [the front carry] / 28 you will see. 28 is the field. 28 to 10, the field carry, 4 40 you will see. /
6 5, the torn out of the length, to 4, the front, carry, 20 you will see. 1/2 break, 10 you will see.
7 10 square, / [1 40] you will see. 1 40 to 4 40 add on, 4 41 40 you will see. What is it equalsided? 2 10 you will see. /
8 [10 ta]ke, to 2 10 add on, 2 20 you will see.
9 What to 28 should I set that 2 20 will give to me? / [5 set.] 5 to 7 carry, 35 you will see.
10 5, the torn out of the length, from 35 tear off, / [30 you] will see. 30 is the length.
5, the length (sic!), to 4, the front, carry, 20 you will see. 20 is the length (sic!).

The suggested restoration of the severely damaged question in lines 1-2 of this exercise is based on what happens in the ensuing, better preserved, solution procedure. The question can be formulated in quasi-modern symbolic notations as the following rectangular-linear system of equations:

$$A = u \cdot s = 10\ (00), \qquad (s + 3/4\, s) = u + 5.$$

The solution procedure, *which is not of the same kind as the solution procedure in TMS IX above*, begins by (silently) assuming the front to be ‘4’, possibly meaning 4 *r*, where *r* is a reed of unknown size. (A similar interpretation of the basic first step of various solution procedures was made above, in Sec. 5.1, in the discussions of the many exercises in the large recombination text IM 121613.) In terms of this assumed reed, the solution procedure begins by computing, successively,

$$s = 4\, r, \quad 1/4\, s = 1/4 \text{ of } 4\, r = 1\, r, \quad 3 \cdot 1/4\, s = 3/4\, s = 3\, r, \quad s + 3/4\, s = 4\, r + 3\, r = 7\, r, \quad \text{and} \quad u = 7\, r - 5 \quad \text{(lines 2-3)}$$

In line 4 of the exercise, 7 *r* is called the (new) 'length', and the number 5 which is subtracted from 7 *r* is aptly called *nasīḫ* uš 'the torn out of the length'. Now, since $A = u \cdot s = 10\ (00)$, $s = 4\ r$, and $u = 7\ r - 5$, the stated rectangular-linear system of equations has been transformed into the following *quadratic* equation:

$(7\ r - 5) \cdot 4\ r = 10\ (00)$.

The situation, illustrated in Fig. 9.4.2 below, is quite similar to the situation in the case of the exercise IM 121613 # 14, illustrated in Fig. 5.1.13 above. In that exercise, the quadratic equation has the form

$(1\ (00)\ r - 30) \cdot 40\ r = 7\ 30$.

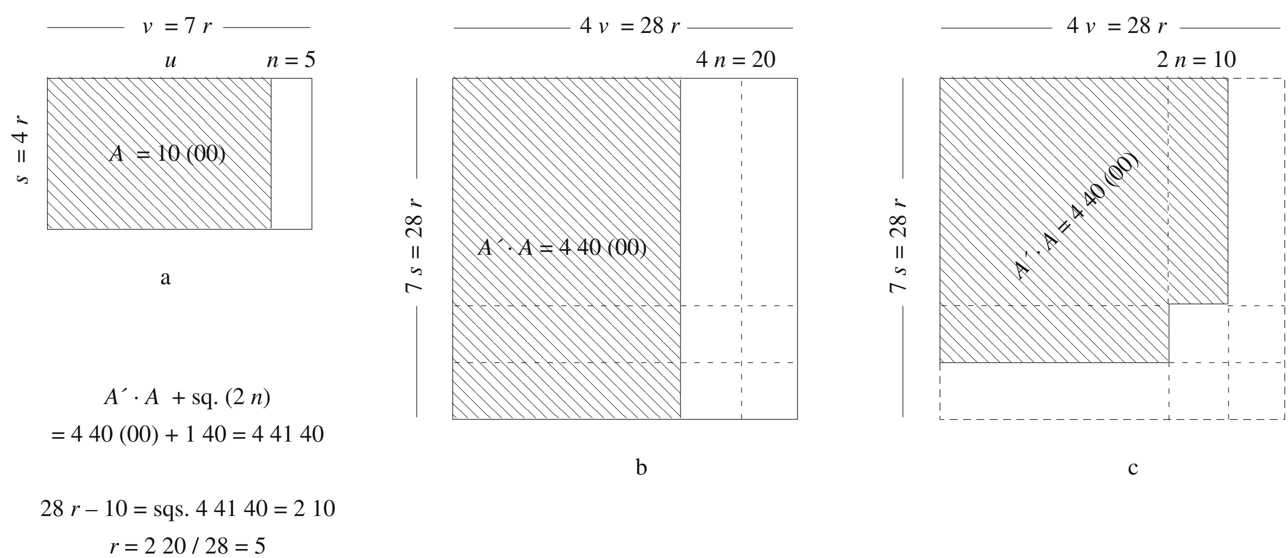


Fig. 9.4.2. *TMS* VIII # 1: $u \cdot s = 10\ (00)$, $(s + 3 \cdot 1/4\ s) = u + 5$.
Note that the diagrams a and b-c, respectively, are not drawn to the same scale.

The next step of the solution procedure is the equalization via crosswise multiplication, whereby one side of the rectangle in Fig. 9.4.2 a is multiplied by 4, the other side by 7. The result is the transformation of the rectangle into a square with the side 28 *r*. At the same time that the sides are scaled up by 4 and 7, respectively, the 'field (area)' *A* is scaled up by the product $A' = 7 \cdot 4 = 28$. Confusingly, also *A´* is called 'the field (area)' in lines 4 right - 5 left. (In a similar solution procedure in the mentioned exercise IM 121613 # 14, the corresponding product is more accurately called 'the false field'.) As shown in Fig. 9.4.2 b, the result is a square with the side $A' \cdot r = 28\ r$, divided into a rectangle with the area $A' \cdot A = 28 \cdot 10\ (00) = 4\ 40\ (00)$, and another rectangle with the sides 28 *r* and $4\ n = 4 \cdot 5 = 20$ (see line 6 left). From a modern point of view, the quadratic equation $(7\ r - 5) \cdot 4\ r = 10\ (00)$ has now been transformed into the simpler quadratic equation

$(28\ r - 20) \cdot 28\ r = 4\ 40$.

The remainder of the solution procedure, in lines 6 right to 7 of the exercise is, as usual, a balancing and a completion of the square. The result is the new equation

$28\ r - 10 = 2\ 10$, from which follows that $28\ r = 2\ 20$, $r = 5$.

Consequently, in lines 9-10,

$u = 5 \cdot 7 - 5 = 30$, and $s = 5 \cdot 4 = 20$.

In quasi-modern symbolic notations, what is going on in the solution procedure of *TMS* VIII # 1 can be summarized as follows: The stated rectangular-linear problem for the sides *u* and *s* of a rectangle is first (silently) transformed into the following quadratic equation for the unknown *s* alone:

$(s + 3 \cdot 1/4\ s - 5) \cdot s = 10\ (00)$.

In order to avoid operations with fractions, *s* is replaced by 4 *r*, where *r* is a silently understood reed of unknown length. The mentioned quadratic equation for *s* is then replaced by the following quadratic equation for *r*:

$(7\,r - 5) \cdot 4\,r = 10\ (00).$

Multiplication of both sides of this equation by the product $4 \cdot 7 = 28$ transforms the equation into

$(28\,r - 20) \cdot 28\,r = 10\ (00).$

This is a quadratic equation for 28 *r*, which can be solved in the usual way by completion of the square. And so on.

The question in the second preserved exercise on the fragment *TMS* VIII was either so badly written by the original author of the text or so badly copied in *TMS* that it now makes very little sense. Nevertheless, what is preserved of the solution procedure is quite comprehensive and clearly proceeds along the same lines as the solution procedure in the first exercise on the fragment, which was discussed in some detail above.

TMS **VIII # 2**

1 [a.šag$_4$ 10] 4-*at* sag *a-na* uš daḫ *a-na* 1 *a-li-ik* :
a.na *šà* uš ugu sag *i-ší-ma* (should be *i-ter*) [5] /
2-3 [za].e 4 *re-ba-ti ki-ma* sag gar *re-ba-at* 4 *le-qé* 1 *ta-mar* 1 *a-na* 1 *a-li-*[*ik*] / [1 *ta-mar*]
4 gaba 4 gar 1 *ta-lu-ka a-na* 4 daḫ 5 *ta-*<*mar*> *ki-ma* uš gar /
4 [5 *ki-ma*] *wa-ṣí-ib* uš gar 5 uš *a-na* 4 sag *i-ší* 20 *ta-mar* 20 a.šag$_4$ /
5 [*a-na* 10 a.šag$_4$] *i-*[*ší*] 3 20 *ta-mar* 5 *wa-ṣí-ib a-na* 4 sag *i-ší* 20 *ta-mar* /
6 [$^1/_2$ *ḫe-pe* 10 *ta-mar*] 10 nigin 1 40 *ta-mar* 1 40 *a-na* 3 20 daḫ 3 21 40 *ta-mar* /
7 [*mi-na* ib.si 1 50 *ta-mar*]
[10] *le-qé i-na* 1 50 zi 1 40 *ta-mar* /
8 [igi 20 a.šag$_4$ *pu-ṭú-úr* 3 *ta-mar* 3] *a-na* 1 40 *i-ší* 5 *ta-m*[*ar*]
9 5 *a-na* 5 uš / [*i-ší* 25 *ta-mar*]
[5 *wa-ṣí-ib* uš *a*]*-na* 25 daḫ 30 *ta-mar* 30 uš
10 [5 *a-na* 4 sag *i-ši* 20 *ta-mar*] 20 sag

1 [The field is 10]. The 4th of the front to the front[!] I added on, to 1 I went,
what the length over the front was beyond[!] [5]. /
2-3 You: 4, of the 4th, as the front set, the 4th of 4 take, 1 you will see. 1 to 1 go, / [1 you will see].
4, a copy of 4 set, 1, the going, to 4 add on, 5 you <will see>, as the length set. /
4 [5 as] the added on of the length set, 5, the length, to 4, the front. carry, 20 you see, 20 the field. /
5 [to 10, the field], carry, 3 20 you will see. 5, the added on, to 4, the front, carry, 20 you will see. /
6 [1/2 break, 10 you will see]. 10 square, 1 40 you will see. 1 40 to 3 20 add on, 3 21 40 you will see. /
7 [What is it equalsided? 1 50 you will see.]
[10] take, from 1 50 tear off, 1 40 you will see. /
8 [The reciprocal of 20, the field, release, 3 you will see. 3] to 1 40 carry, 5 you will see.
9 5 to 5, the length, / carry, 25 you will see.
[5, the added on of the length, t]o 25 add on, 30 you will see. 30, the length.
10 [5 to 4, the front, carry, 20 you will see]. 20, the front.

The question in this exercise can (after needed corrections) be reformulated in quasi-modern symbolic notations as the following rectangular-linear system of equations:

$A = u \cdot s = 10\ (00), \qquad (s + 1/4\ s) + 5 = u.$

Here, if $s = 4\,r$, then $u = 5\,r + 5$, and the problem can be reformulated again as the following equation for *r*:

$(5\,r + 5) \cdot 4\,r = 10\ (00).$

Solving this equation, one finds that $r = 5$. Therefore, $u = 5 \cdot 5 + 5 = 25 + 5 = 30$, and $s = 5 \cdot 4 = 20$.

It is particularly noteworthy that in these two exercises *common fractions* of the type *m*/*n* appear in an embryonic form. Indeed, in exercise #1, the expression

4-*at* sag *a-na* 3 *a-li-ik*

stands for $3 \cdot 1/4$ of the front = 3/4 of the front. Even more obviously, in exercise # 2, the expression

4-*at* sag …… *a-na* 1 *a-li-ik*

stands for $1 \cdot 1/4$ of the front = 1/4 of the front.

9.5 BM 80209. A Catalog Text with Metric Algebra Problems for Squares and Circles

BM 80209 was first published and explained in Friberg, *JCS* 33 (1981). It is a small, relatively well preserved single-column catalog text with variations of parameters. It begins with four very simple metric algebra problems for squares in § 1. The remaining four paragraphs are all about metric algebra problems for circles. In §§ 2, 4 and 5 there are 4 variations of parameters in a single problem, while in § 3 there are 7 variations of parameters in a single pair of problems. The text ends with some scribbled numbers in the left over empty space in the lower part of the reverse.

The use of the term ul.gar 'to add together' shows that the text belongs to Group 6 of unprovenanced Old Babylonian mathematical texts, which means that it probably is from Sippar, like BM 80088 in Sec. 9.3 above, and like the three fragments in Ch. 8 above. (See the survey at the beginning of this chapter.)

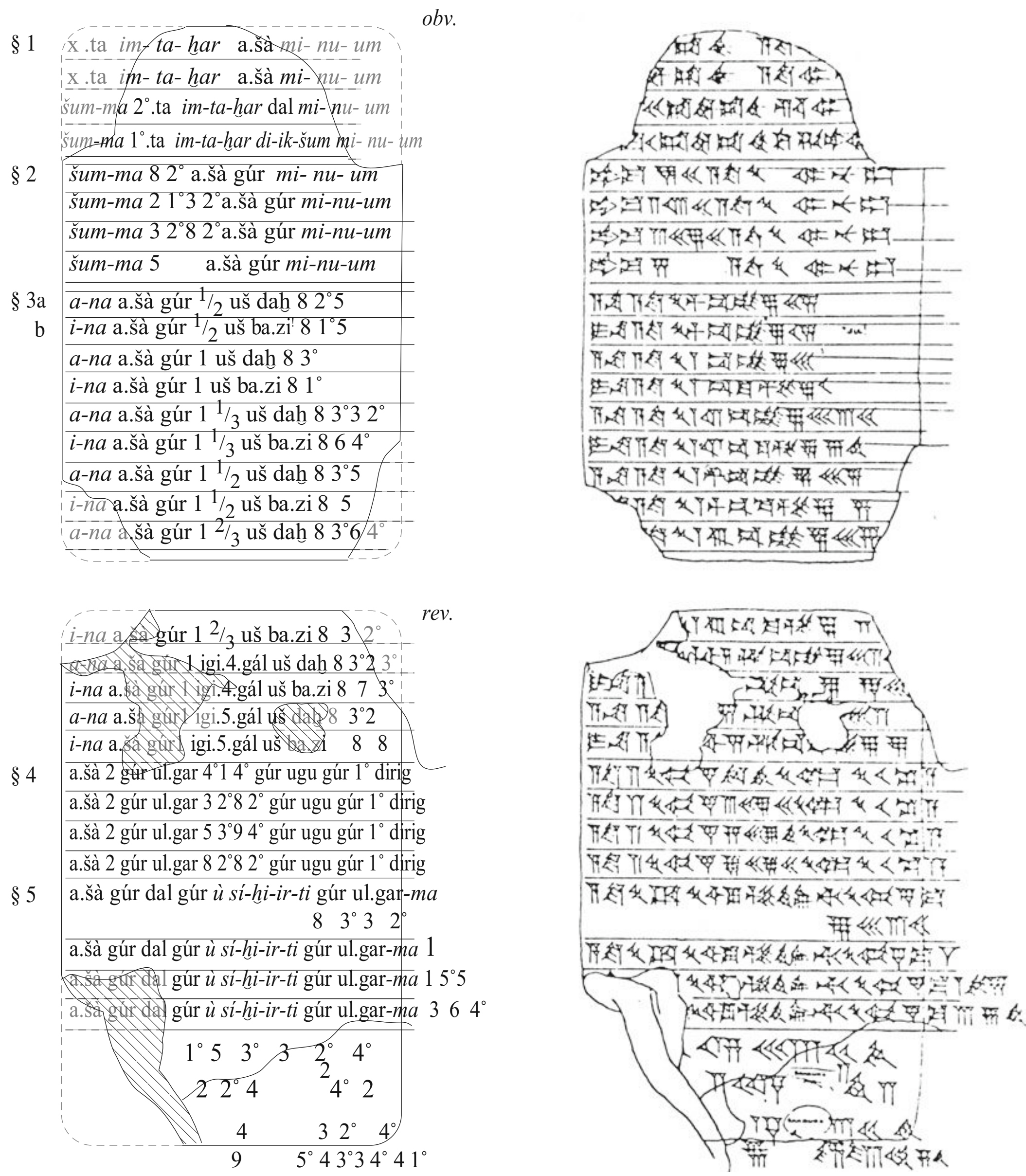


Fig. 9.5.1. BM 80209. An Old Babylonian metric algebra catalog text from Sippar with variations of parameters.

The first two (damaged) lines of § 1 of BM 80209 ask for the area of a square when it side *m* is given:

m = [x], sq. m = ?

In the third line, m = 20 is given, and what is asked for is d = dal, the 'transversal' or 'diagonal'. In view of the well known Old Babylonian "diagonal rule", the question can be reformulated as the equation

m = 20, d = sqs. (2 sq. m) = ?

On one hand, the expected answer can have been that

d = sqs. (2 sq. 20) = sqs. 13 20 = (appr.) 30 – 1 40 · rec. (2 · 30) = (appr.) 30 – 1;40 = (appr.) 28;20.

On the other hand, it can have been that

d = 20 · sqs. 2 = (appr.) 20 · 1;25 = (appr.) 28;20.

In the final line of § 1, the question is:

m = 10, the *dikšum* 'extension' is what?

A similar question is known from BM 131432 = *UET 5*, 864 (Friberg, *RA* 94 (2000)), where the related term *di-ki-iš-ti-im* seems to mean a square extended in four directions in such a way that the area of the extended figure is double the area of the original square. In § 1, line 4, the square with the side 10 would have to be extended by 2;30 in each direction to form a cross-like figure with the area 2 · 1 40 = 3 20.

In §§ 2-5 of BM 80209, the objects considered are circles. As is well known, it is always assumed in Old Babylonian mathematical texts that *the arc a* (that is the circumference), not the *transversal* (the diameter) is the fundamental parameter of a circle and that (approximately)

the area A = ;05 · sq. a, the transversal d = 3 a, so that, conversely, a = ;20 d.

In condensed form, the questions in §§ 2-5 of BM 89209 can be reproduced as follows, in transliteration and translation:

§ 2	*šum-ma* (A) a.šag$_4$ gúr *mi-nu-um* If A was the field (the area), the arc (the circle) was what?	(4 cases)
§ 3a	*a-na* a.šag$_4$ gúr (c) uš daḫ (P) To the field of the arc, c (of) the length I added, P.	(7 cases)
§ 3b	*i-na* a.šag$_4$ gúr (c) uš ba.zi (Q) From the field of the arc, c (of) the length I tore off, Q.	(7 cases)
§ 4	a.šag$_4$ 2 gúr ul.gar (S) gúr ugu gúr 10 dirig The fields of two arcs I added together, S. Arc over arc was 10 beyond.	(4 cases)
§ 5	a.šag$_4$ gúr dal gúr *ù sí-ḫi-ir-ti* gúr ul.gar-*ma* (B) The field of the arc, the transversal of the arc, and the circumference of the arc I added together, then B.	(4 cases)

In terms of quasi-modern symbolic notations, with numbers in relative place value notation, the questions can be reformulated as follows:

§ 2	5 sq. a = P,	a = ?		P = 8 20	(a = 10)
				P = 2 13 20	(a = 40)
				P = 3 28 20	(a = 50)
				P = 5	(a = 1 (00))
§§ 3a-b	5 sq. a + c a = P,	(a = ?)	c = 1/2	P = 8 25	(a = 10)
	5 sq. a – c a = Q,	(a = ?)		Q = 8 15	(a = 10)
			c = 1	P = 8 30	(a = 10)
				Q = 8 10	(a = 10)
			c = 1 1/3	P = 8 33 20	(a = 10)
				Q = 8 8 06 40	(a = 10)

				c = 1 1/2	P = 8 35	(a = 10)
					Q = 8 05	(a = 10)
				c = 1 2/3	P = 8 36 40	(a = 10)
					Q = 8 03 20	(a = 10)
				c = 1 rec. 4	P = 8 33 20	(a = 10)
					Q = 8 07 30	(a = 10)
				c = 1 rec. 5	P = 8 32	(a = 10)
					Q = 8 08	(a = 10)
§ 4	5 sq. a + 5 sq. a' = S,	$a - a'$ = 10		(a, a' = ?)	S = 41 40	(a, a' = 20, 10)
					S = 3 28 20	(a, a' = 40, 30)
					S = 41 40	(a, a' = 50, 40)
					S = 8 38 20	(a, a' = 1 (00), 50)
§ 5	5 sq. a + 20 a + a = B,	(a = ?)			B = 8 33 20	(a = 10)
					B = 1	(a = 20)
					B = 1 55	(a = 30)
					B = 3 06 40	(a = 40)

It is interesting to observe how the data for this catalog text with variations of parameters were constructed by the author of the text. In § 2 *the answers (the values of a) were varied (fairly) systematically*. (It is likely that the present text is a shortened copy of an original catalog text were § 2 contained also lines with the answers a = 20 and a = 30.) The values of P were *computed* with departure from the answers.

In §§ 3a-b, on the other hand, the answer remained the same throughout, but *the value of the coefficient c was varied quite systematically*, and the area of the circle was *alternatingly increased and decreased* by an extra term of the form c a. The values of P and Q were *computed*.

In § 4, *the answers (the values of a and a′) were varied (fairly) systematically*. (It is likely that the present text is a shortened copy of an original catalog text were § 4 contained also a line with the answers a, a' = 30, 20.) The values of S were *computed* with departure from those answers.

In § 5, finally, *the answers (the values of a) were varied quite systematically*, and the values of B were *computed* with departure from the answers.

The solution procedures are not given in a catalog text like this one. Anyway, it is clear that the problems in § 2 can be reduced to the *computation of square-sides* (square roots) simply through multiplication by 12 = rec. 5, which leads to simplified equations of the type

sq. a = 12 P.

Similarly, the problems in §§ 3a-b can be reduced through division by 5 to the following equations:

sq. a + 12 c a = 12 P, and sq. a – 12 c a = 12 Q.

These are *quadratic equations of the basic types B4a and B4b*. (See Friberg, *AT* (2007), 6.) They could be solved by use of standard procedures.

The problems in § 4 can be reduced to the simplified systems of equations

sq. a + sq. a' = 12 S, $a - a'$ = 10.

These are *quadratic-linear systems of equation of the basic types B2a and B2b*. (See again *AT* (2007), 6.)

The problems in § 5, finally, can be simplified to equations of the type

5 sq. a + c a = B, where c = 1 20.

These are problems of the same type as one of the problems in § 3a!

10. Goetze's Compendium from Old Babylonian Shaduppûm and Two Catalog Texts from Old Babylonian Susa

The three tablets IM 52916 (in the present chapter, Sec. 10.1), and IM 52685 + IM 52304 (in Sec. 10.2) were published and correctly interpreted by Goetze in *Sumer* 7 (1951). Goetze called them together "a mathematical compendium from Tell Harmal". The three tablets are very poorly preserved. This unfortunate circumstance was explained by Goetze in the following words:

> "The tablets would probably be in a better condition, had they been found in situ. They had been dug up illicitly and left on the spot as worthless because of their poor state of preservation. They were restored in the (Iraq) Museum laboratory from numerous splinters, work which left a few unattached slivers but yielded three larger pieces."

10.1 IM 52916. A Mathematical Recombination Text from Old Babylonian Shaduppûm

IM 52916 (Figs. 10.1.1-3 below) is a mathematical recombination text. Apparently, the text on the *obverse* of IM 52916 contained only two or three paragraphs, here called

[§ 1]	[………]
§§ 2a-b	a metric algebra catalog with variations of parameters, for quadratic equations of the basic types B4a-b

The latter may have been copied from some catalog of metric algebra problems with variations of parameters, somewhat similar to *TMS* VI in Sec. 10.4 below. The damaged and badly organized text on the *reverse* and the *left edge* seems to contain the following paragraphs, with excerpts from various different original texts:

§§ 3a and 3d:	a table of constants for geometrical figures	(13? lines)
§ 3b:	a table of constants for bricks, mud, and straw	(3 lines)
§ 3c:	a table of constants for [xxx], poorly preserved	(6 lines)
§ 3e:	a table of constants for metals and [xxx], poorly preserved	(16? lines)
§§ 4a-b:	a catalog of problems for figures within figures and for canals	(8 + 1 lines)
§ 4c:	a catalog of geometric(?) problems, poorly preserved	(10 lines)
§ 4d:	a metric algebra problem for a striped triangle(?), with variations of parameters	(7+[x] lines)
§ 4e:	a catalog of problems, poorly preserved	(10+[x] lines)
§ 5:	a computation of a combined work norm, with variations of parameters	(5 lines)

10.1.1 §§ 2a-b. A Catalog of Quadratic Equations of the Basic Types B4a-b

The text on the obverse of IM 52916 is severely damaged. Nothing remains of the text of what may have been § 1, and not a single full line is preserved of the text of §§ 2a-b. The suggested reconstructions of some of those badly preserved lines in Fig. 10.1.1 are only tentative and conjectural. Nevertheless, it is quite clear that, in spite of some similarities between §§ 2a-b of IM 52916 on one hand and §§ 4a-b of *TMS* V (Sec. 10.3) or §§ 1a-b of *TMS* VI (Sec. 10.4) on the other, there are also several conspicuous differences.

Consider, for instance, the following generic lines of §§ 2a-b of IM 52916:

§ 2a	*a-na* a.šag$_4$ lagab (*c*) uš-*ia wa-ṣa-ba-am*	To the field of the square *c* of my length to add on
	a-na a.šag$_4$ lagab (*b*) *ši-di-i ù* (*c*) uš-*ia wa-ṣa-ba-am*	To the field of the square *b* of my side and *c* of my length to add on

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7_10

§ 2b *i-na* a.šag$_4$ lagab (*c*) uš-*ia a-su-uḫ-ma* — From the field of the square *c* of my length I tore off, then
i-na a.šag$_4$ lagab (*b*) *ši-di-i ù* (*c*) uš-*ia a-su-uḫ-ma* — From the field of the square *b* of my side and *c* of my length ……

The first of those generic lines in § 2a, for instance, should possibly be understood as meaning

(I learned) to add *c* of my length to the field (area) of the block (square) (in order to set up quadratic equations).

It is puzzling that the generic line in § 2b is not formulated in the corresponding way!

Compare with the following generic lines in §§ 1a-b of *TMS* VI:

§ 1a *a-na* a.šag$_4$ nigin-*ia* (*c*) uš daḫ-*ma* (*B*) — To the field of my squares *c* (of) the length I added, then *B*.
§ 1b *i-na* a.šag$_4$ nigin-*ia* (*c*) uš zi-*ma* (*B*) — From the field of my squares *c* (of) the length I tore off, then *B*.

There, for each explicitly given value of the coefficient *c*, the corresponding value of *B* is also explicitly given, while in §§ 2a-b of IM 52916 the values of *B* are never mentioned. This means that §§ 1a-b of *TMS* VI could be used as a handy source of assignments to hand out to eagerly waiting students, while § 2a of IM 52916 is only a theoretical enumeration of problem types and could not be used in that way. Another important difference between the otherwise parallel texts is that in *TMS* VI (and *TMS* V) all numerical coefficients are given in the form of cuneiform *number signs* for integers and fractions (and all values of *B* are given as sexagesimal place value numbers), while in §§ 2a-b of IM 52916 all numerical coefficients are given in the form of Akkadian *number words* for integers and fractions!

The Akkadian number words figuring in §§ 2a-b of IM 52916 (and in IM 52685 §§ 1b-c in Fig. 10.2.1 below) seem to be (more or less badly preserved and with various syllabic spellings):

1	m.	*ištēn*
2	m.	*šina*
3	f.	*šalāšat*
4	f.	*erbēt*
5	f.	*ḫamšat*
6	f.	*šiššet*
7	f.	*sebet*
8	f.	*samanat*
9	f.	*tišet*
10	f.	*ešeret*

2/3	*šinepât, patu!*
1/2	*bāmat*
1/3	*šalištu*
1/4	*rabât*
1/5	*ḫamšât*
1/6	*šiššât*
1/7	*sabât*
1/8	*samnât*
1/9	*tišât*
1/10	*išrêt*

10.1.2 § 3. Various Parts of a Table of Constants

The table of constants in §§ 3a and 3d of IM 52916, badly preserved in several places, is concerned with various parameters for the following geometric figures, discussed by Robson in *MMTC* (1999), Secs. 3.1-7:

[*x x x*]	[x x x]			
kippatu	arc, circle	IM 52916	*rev.* 4-5	*MMTC,* Sec. 3.1
sag.kak	peg head, triangle		*rev.* 6-8, 47	Sec. 3.3
apsamikku	concave square		*rev.* 9-11	Sec. 3.7
īn alpim	ox-eye		*rev.* 12	Sec. 3.5
uskāru	crescent, half-circle		*rev.* 13	Sec. 3.2

Another, quite brief table of constants in § 3b of IM 52916 is about carrying bricks, mud, and straw:

45 *i-gi-gu-bu na-az-ba-al* sig$_4$	'45, constant, carrying bricks'	*rev.* 22
1 40 *i-gi-gu-bu na-az-ba-al* saḫar	'1 40, constant, carrying mud'	*rev.* 23
3 20 *i-gi-gu-bu na-az-ba-al* še.in.nu.da	'1 40, constant, carrying straw'	*rev.* 24

What such constants may stand for is discussed in connection with a combined work norm problem for carrying mud, mixing (mud with straw ?), and molding bricks in YBC 6473, § 1e. That problem will be discussed in Sec. 11.2.1 below.

§ 3c (*rev*. 31-36) and § 3d (*rev*. 47) contain further lines of tables of constants, unfortunately so damaged that it is not even clear what they are about.

The four first brief and badly preserved columns of text on the left edge of IM 52916 contain additional lines of tables of constants. The second of these columns contains constants associated in some unknown way with the following sequence of metals (see Robson, *MMCT,* Sec. 8.1):

[urudu]?	[copper]?
[kù.babbar]?	[silver]?
kù.gi	gold
abāru	lead
annaku	tin
alulutu	?

The fourth column on the left edge begins with constants for an *eleppu* 'boat' and for an *išittu* 'store room'. Without any further context, the meaning of the constants is not clear.

10.1.3 § 4. Various Parts of a Unique Catalog of Mathematical Problem Types

Different parts of § 4 of IM 52916 are scattered all over the reverse and left edge of the tablet. It is a new kind of catalog of Old Babylonian mathematical problems, not quite like any other Old Babylonian mathematical catalog. However, it is a continuation of § 2a on the obverse, where the generic line of text was

a-na a.šag$_4$ lagab (*c*) uš-*ia wa-ṣa-ba-am* To add *c* of my length to the field of the square.

As mentioned, this curious construction should possibly be understood as meaning something like

(In school I learned) to add *c* times the length (of the square) to the area of the square (in order to set up a quadratic equation).

The first line of § 4a is phrased as follows:

na-al-ba-tam i-na li-bu na-al-ba-tim e-pé-šam To make a brick mold inside a brick mold.

Here the literal sense of the word *nalbattu*, which is 'brick mold', must be rejected and replaced by a term for some familiar geometrical object. Since brick molds are rectangular, an obvious candidate for a translation of the term is simply 'rectangle'. Recall that in the discussion of IM 95771 # 5 in Sec. 6.1.5 above, the term *nalbanum* 'brick mold' was identified as a word for 'rectangle'. In addition, in the subscript to MS 3049, mentioned in Sec. 6.2.4 above, problems for objects called *nalbattum* 'brick mold' probably means problem for rectangles. Otherwise, rectangles are normally not mentioned by name in mathematical cuneiform texts, only their long and short sides uš (*šiddu*) and sag (*pūtu*). Anyway, with the suggested translation of the word *nalbattu*, the first line of § 4a can be understood as meaning

(In school I learned) to make a rectangle within a rectangle (in order to set up a problem).

A number of Old Babylonian mathematical texts concerned with "figures within figures" were discussed in Friberg, *AT* (2007), Sec. 6.2. One of the examples discussed was the hand tablet YBC 7359 (*op. cit.*, Fig. 6.2.7), which shows on the obverse a teacher's neat drawing of a square inside a square, and on the reverse a student's less elegant reproduction of the same drawing. Numbers inscribed in various places indicate that the mathematical problem illustrated by those drawings was of the following form:

The area between two (concentric and parallel) squares is 1 31.
The distance between the two squares is 3 30. Find the sides of the two squares.

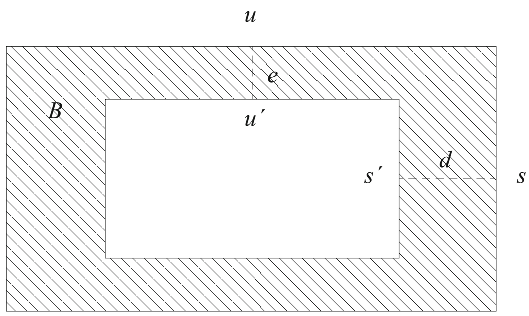
In terms of quasi-modern abstract notations this geometric problem can be rephrased as the following quadratic-linear system of equations of the basic type B3b for the sides of the squares (see Friberg, *AT*, 6):

sq. p – sq. $q = B = 1\ 31$, $p - q = d = 3;30$. $p, q = ?$

It is easy to find the solution, which is $p = 10$, $q = 3$.

If the problem suggested by the first line of § 4a of IM 52916 was of a similar kind, it could have been expressed as follows:

> The area between two (concentric and parallel) rectangles of a given form is given.
> The distances between the sides of the two rectangles are also given. Find the sides of the two rectangles.



$s = c\,u$, $s' = c\,u'$
$u \cdot s - u' \cdot s' = B$
$u - u' = 2\,d$, $s - s' = 2\,e = 2\,c\,d$

sq. u – *sq.* u' = *rec.* $c \cdot B$

Example: c = 2/3, B = 7 30, d = 7;30
u, s = 30, 20 u', s' = 15, 10

Fig. 10.1.1. IM 52916, § 4a, line 1. A surmised metric algebra problem for a rectangle within a rectangle.

Altogether four problems of the type "figures within figures" are mentioned in IM 52916 § 4a. They are

na-al-ba-tam i-na li-bu na-al-ba-tim e-pé-šam	To make a brick mold (a rectangle) inside a brick mold.	§ 4a, 1
ki-pa-tam i-na li-bu ki-pa-tim e-pé-šam	To make an arc (a circle) inside an arc.	§ 4a, 5
i-na li-bu na-al-ba-tim ki-pa-ta-am	Inside a brick mold (rectangle) an arc	§ 4a, 7
i-na li-bu ki-pa-tim na-al-ba-ta-am	Inside an arc a brick mold (rectangle)	§ 4a, 8

Actually, as mentioned above, a metric algebra problem for *a square within a square* is illustrated on YBC 7359. A metric algebra problem for *a circle within a circle* can be found on the Old Babylonian round hand tablet Böhl 1821, discussed above in Sec. 9.1. A metric algebra problem for a *circle within a square* is illustrated in the Old Babylonian square hand tablet MS 2985 (Friberg, *MSCT* 1 (2007), Figs. 8.1.1 and 8.2.12).

Moreover, a problem for *an equilateral triangle within an equilateral triangle* is illustrated on the round hand tablet MS 2192 (Friberg, *op. cit.*, Fig. 8.2.2), and metric algebra problems for *a concave square within a square,* and for *a concave square within a rectangle,* are formulated and solved in the Susa text *TMS* 21 (Muroi, *Sciamvs* 1 (2000); Friberg, *AT* (2007), Fig. 6.2.6). A badly preserved drawing of a circle within a regular 6-front (a hexagon) can be seen on MS 1938/2 (*op. cit.*, Fig. 8.2.14).

In addition to the four mentioned figure within figure problems, the list of mathematical problem types in § 4a of IM 52916 includes the following four items:

na-ra-am na-ar ta-ra-ḫi-i e-pé-šam	To make a canal with banks a canal. (?)	§ 4a, 2
na-ar ša-ka-na-ka-tim na-ra-am šu-zu-ba-am	To rescue a canal (to) a governors' canal. (?)	§ 4a, 3
pa-ta-am e-pé-ša-am	To make a water course.	§ 4a, 4
ki-pa-tam a-na ši-na at-ḫi za-za-am e-pé-ša-am	To allot and make an arc (a circle) to two partners.	§ 4a, 6

It is not easy to know what correct translations would be of such brief sentences taken out of context. The problems mentioned in lines 3, 4 and 6 of § 4a are not known from any published mathematical cuneiform text, and it is not possible even to conjecture what they would be about.

However, the problem type mentioned in line 2 of § 4a is probably known from a couple of Old Babylonian mathematical texts. One of them is the catalog text YBC 7164 = Neugebauer and Sachs, *MCT* L, which is concerned with the digging and maintenance of irrigation canals. In particular, exercise # 8 of that text is phrased in the following way:

pa_5.sig.libir-ra 5 uš uš 1 kùš dagal 1 kùš bùr.bi /	An old little canal. 5 uš the length, 1 cubit the width, 1 cubit its depth. /
te-er-di-is-sà $^1/_2$ kùš *ta-ra-ḫi-ša ú-ša-mi-iṭ*	Its deposits removed 1/2 cubit of its banks.
saḫar.bi en.nam $^1/_3$ sar 5 gín	Its mud (volume) was what? 1/3 sar 5 shekels.

Apparently, an irrigation canal, 5 · 60 nindan long, 1 cubit wide and 1 cubit deep has its banks narrowed by 1/2 cubit of deposited silt (on each side). The silt has to be removed for the maintenance of the canal. The volume of the removed silt can be computed as

5 00 nindan · 2 · 1/2 cubit · 1 cubit = 5 00 · ;05 sq. nindan · 1 cubit = 25 volume-sar.

The answer given in the text is off by a factor of 60, clearly since it was calculated as

5 · 5 = 25, with the result incorrectly interpreted as 25 volume-shekels = 1/3 sar 5 shekels.

The term *taraḫḫu* 'bank of a canal' appears also in exercise # 15 of the mathematical recombination text BM 85196 (*MKT* II, 45), which begins with a drawing of the cross section of a silted canal, with the question formulated as follows:

2 30 1 40	20 15 3 1	2 30 1 40

a-ta-ap ta-ra-aḫ-ḫi pa-na-nu ša / — A banked canal. Formerly of
20 *mu-ḫu* 15 za.zum *ù* 3 bùr — 20 the top, 15 the base, and 3 the depth.
2 30 *e-li-nu-um* / *aš-lu-uṭ* 1 40 ki.ta *aš-lu-uṭ* — 2 30 at the top / I cut off, 1 40 below I cut off,
to the former canal, / 1 cubit of height I added.
a-ta-pa-am pa-ni-[*a-am*] / 1 kùš sukud daḫ.ḫa — The old mud (volume) / the new mud were <what>,
saḫar.ḫá *la-bi-ru-tum* / saḫar.ḫá gibil$_4$ <en.nam> — and mark off the length (to be dug by) 1 man.
ù 1 lú uš *pu-lu-uk*

The rudimentary diagram in this text can be explained as in Fig. 10.1.2 below. After removal of the silt from a silted canal, the top of the canal has been widened by ;02 30 nindan = 1/2 cubit on each side, the bottom has been lowered by 1 cubit and widened by ;01 40 nindan = 1/3 cubit on each side.

A rudimentary diagram of the same kind as in BM 85196 # 15 (above) appears also on the obverse of YBC 9874 = *MCT*, M. In that case, a silted canal with a rectangular cross section has the initial width '5', that is ;05 nindan = 1 cubit, and the depth '1' = 1 cubit. After clearing of the silt, the width of the canal is increased by '1 40' = 1/3 cubit on each side. The number '3 20' to the right of the diagram can be interpreted as the total expansion of the width, namely 2 · '1 40' = '3 20', and a second noted number '3 36' is the length of the canal, 3;36 nindan. Obviously, then, the volume of the silt removed from the sides of the canal, and from the bottom of the canal were

3 36 · 3 20 · 1 40 = 6 · 3 20 = 20 (volume-sar) and 3 36 · 5 · 40 = 2 24 · 5 = 12 (volume-sar).

The numbers '20' and '12' are also recorded to the right of the diagram.

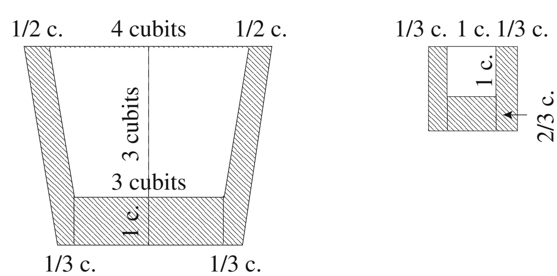


Fig. 10.1.2. BM 85196 # 15 and YBC 9774. Silted canals with a trapezoidal or rectangular cross section.

Note that, in a way, also problems of the kind illustrated in Fig. 10.1.2 are about *figures within figures*! In addition, the problem mentioned in the isolated line called § 4b in Fig. 10.1.9 may be about a figure within a figure. Indeed, the damaged line can tentatively be reconstructed as

[*ki-pa*]*-at-am* [*i-na li-bu i-in*] *al-pí-im e-pé-ša-am* To make an [a]rc (a circle) [inside] an ox-[eye]. § 4b

The *īn alpim* or igi. gu$_4$ is an oval figure bounded by two circular arcs. The term appears in the geometric table of constants in § 3a of IM 52916 in the line

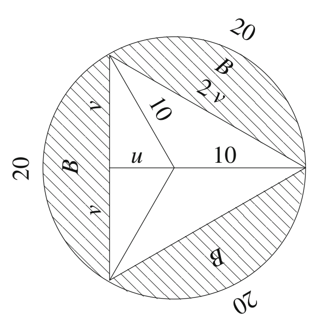
i-in al-pi-im 16 52 30 *i-gi-*[*gu-bu-šu*]	An ox-eye, 16 52 30 its con[stant].	§ 3a, 12

This line in § 3a can be compared with the following lines from the table of constants BR = *TMS* III (Bruins and Rutten (1961)):

1⌜3⌝ 20 igi.gub *šà* a.šag$_4$ še	13 20, the constant of a grain figure	BR 16
5⌜6⌝ 40 dal *šà* a.šag$_4$ še	56 40, the transversal of a grain figure	BR 17
2⌜3⌝ 20 [*pi*]*-ir-ku šà* a.šag$_4$ še	23 20, the cross line of a grain figure	BR 18
1⌜6⌝ 52 30 igi.gub *šà* igi.gu$_4$	16 52 30, the constant of an ox-eye	BR 19
52 30 dal *šà* igi.gu$_4$	53 20, the transversal of an ox-eye	BR 20
30 *pi-ir-ku šà* igi.gu$_4$	30, the cross line of an ox-eye	BR 21

These constants are of interest in the present connection partly because they are closely connected with certain geometric problems for *figures inscribed in figures*. Take, for instance the case of a triangle inscribed in a circle, which is known from two Old Babylonian mathematical texts. One is the Susa text *TMS* I, where a symmetric (i.e. isosceles) triangle is inscribed in a circle. See Friberg, *MSCT* 1, Fig. 8.2.6. The other is MS 3051 (Friberg, *op. cit.*, Figs. 8.1.1 and 8.2.4), where an equilateral triangle is inscribed in a circle. In the latter example, the purpose of the text is to compute the various parts of the circle inside or outside the triangle (although this endeavor was not carried out successfully). How the computation should have been performed is shown by the diagram in Fig. 10.1.3 below, drawn in imitation of the diagram on *TMS* I.

Note that the circle depicted in Fig. 10.1.3 is "normalized" in the sense that its main parameter, which is the circumference, is assumed to be 1 (00). If the circumference is instead some other number, then all linear parameters mentioned in the figures have to be multiplied by this number, and all areas mentioned have to be multiplied by the square of this number.



the radius of the circle = half the diameter = $1/2 \cdot 1\ 00 / 3 = 10$

$u : 10 = v : 2\ v$ and sq. v = sq. 10 – sq. u

so that $u = 5$ and $v = 5 \cdot$ sqs. 3 = appr. $5 \cdot 1;45 = 8;45$

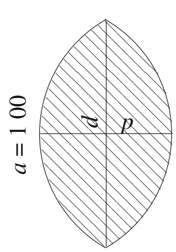
A_{triangle} = appr. $15 \cdot 8;45 = 2\ 11;15$

$B_{\text{segment}} = (A_{\text{circle}} - A_{\text{triangle}}) / 3$ = appr. $(5\ 00 - 2\ 11;15) / 3 = 2\ 48;45 / 3 = 56;15$

the height of each segment = $10 - u = 5$

Fig. 10.1.3. MS 3051. An equilateral triangle inscribed in a circle and certain relevant parameters.

The connection between an equilateral triangle inscribed in a circle and the figure called "ox-eye" is that the ox-eye is the result if you take one of the segments outside the equilateral triangle in the circle and join it to its mirror image, as in Fig. 10.1.4, 1 below.

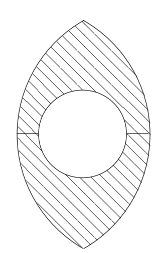


$a = 3 \cdot 20 = 1\ 00$ (normalization)

$d = 3 \cdot 2\ v = 6 \cdot 8;45 = 52;30$

$p = 3 \cdot 2\ u = 6 \cdot 5 = 30$

$A_{\text{ox-eye}} = 9 \cdot 2\ B_{\text{segment}} = 18 \cdot 56;15 = 16\ 52;30$

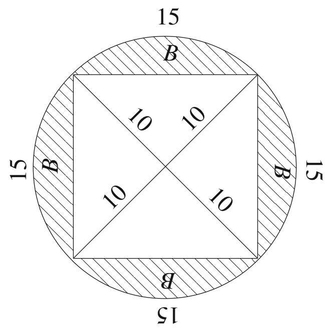


1. 2.

Fig. 10.1.4. 1) Computation of the parameters for an ox-eye listed in BR = *TMS* III, 16-18. 2) A circle within an ox-eye.

The mathematical problem type mentioned in § 4b of Goetze's text IM 52916, "to make a circle inside an ox-eye" can now be illustrated by the drawing in Fig. 10.1.4, 2 above.

The problem of inscribing a square in a circle and computing various relevant parameters is known from the round hand tablet MS 3050 (Friberg, *MSCT* 1, Fig. 8.2.2). It is shown in Fig. 10.1.5 below how those parameters can be computed in the case of a normalized circle with the circumference 1 (00).



the diameter of the circle = appr. 1 00 / 3 = 20

the side of the square = 10 · sqs. 2 = appr. 10 · 1;25 = 14;10

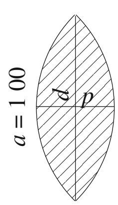
the height of the segment = (20 – 14;10)/2 = 2;55

A_{square} = 2 · sq. 10 = 3 20

B_{segment} = (A_{circle} – A_{square})/4 = appr. (5 00 – 2 · sq. 10)/4 = 1 40 / 4 = 25

Fig. 10.1.5. MS 3050. A square inscribed in a circle and certain relevant parameters.

The connection between a square inscribed in a circle and a "grain figure" is that the latter is the result if you take one of the circle segments outside the square and join it to its mirror image, as in Fig. 10.1.6.



a = 4 · 15 = 1 00 (normalization)

d = 4 · 14;10 = 56;40

p = 4 · 2 · 2;55 = 23;20

$A_{\text{seed figure}}$ = 16 · 2 B_{segment} = 16 · 50 = 13 20

Fig. 10.1.6. Computation of the parameters for a grain figure listed in BR = *TMS* III, 19-21.

The catalog of mathematical problem types continues in IM 52916 § 4c, which unfortunately is very badly preserved. All that can be said is that three of the lines of § 4c seem to deal with the diagonals of one or several geometric figures.

Also the continuation of the catalog of mathematical problem types in § 4 d is badly preserved. However, the first three lines of this paragraph can be read. They are:

ši-ša-at sag.ki an.ta [*i-na* uš *wa-ra-da*]-*am*	To [go down] a sixth of the upper front [from the length].
ḫa-am-ša-[*at* sag.ki an.ta] *i-na* uš *wa*-[*ra*]-*da-am*	To go down a fif[th] of [the upper front] from the length.
[*ra*]-*ba-at* sa[g.ki an.ta]*i-na* uš *wa*-[*ra*]-*da-am*	To go down a [fou]rth of the [upper fr]ont from the length.

This is clearly a catalog of mathematical problems with variations of parameters. However, no other Old Babylonian mathematical texts with formulations of precisely this kind are known. On the other hand, the supremely interesting catalog text Str. 364 with *metric algebra problems for striped triangles* may be a more or less distant relative. (The problems in Str. 364 were thoroughly analyzed in Friberg, *AT* (2007), Sec. 11.2.) Take, for instance, Str. 364 § 3 (see Fig. 10.1.7 below), where the question is formulated as follows:

sag.kak uš *ù* sag an.<ta> nu.zu /	A peg head (triangle). The length and the upper front I did not know.
1(bùr) 2(èše)$^{\text{ašag}}$ a.šag$_4$	1 bùr 2 èše was the field (area).
iš-tu sag an.<ta> / 33 20 *ur-dam-ma* 40 dal /	From the upper front / 33 20 I went down, 40 the transversal.
uš *ù* sag en.nam /	The length and the front were what?

In this problem, just like in IM 52916 § 4c, the two unknowns are the length and the "upper front". (There is no lower front!) Also (possibly), in both cases a transversal, parallel to the front, is reached by "going down" a specified distance along the length of the triangle.

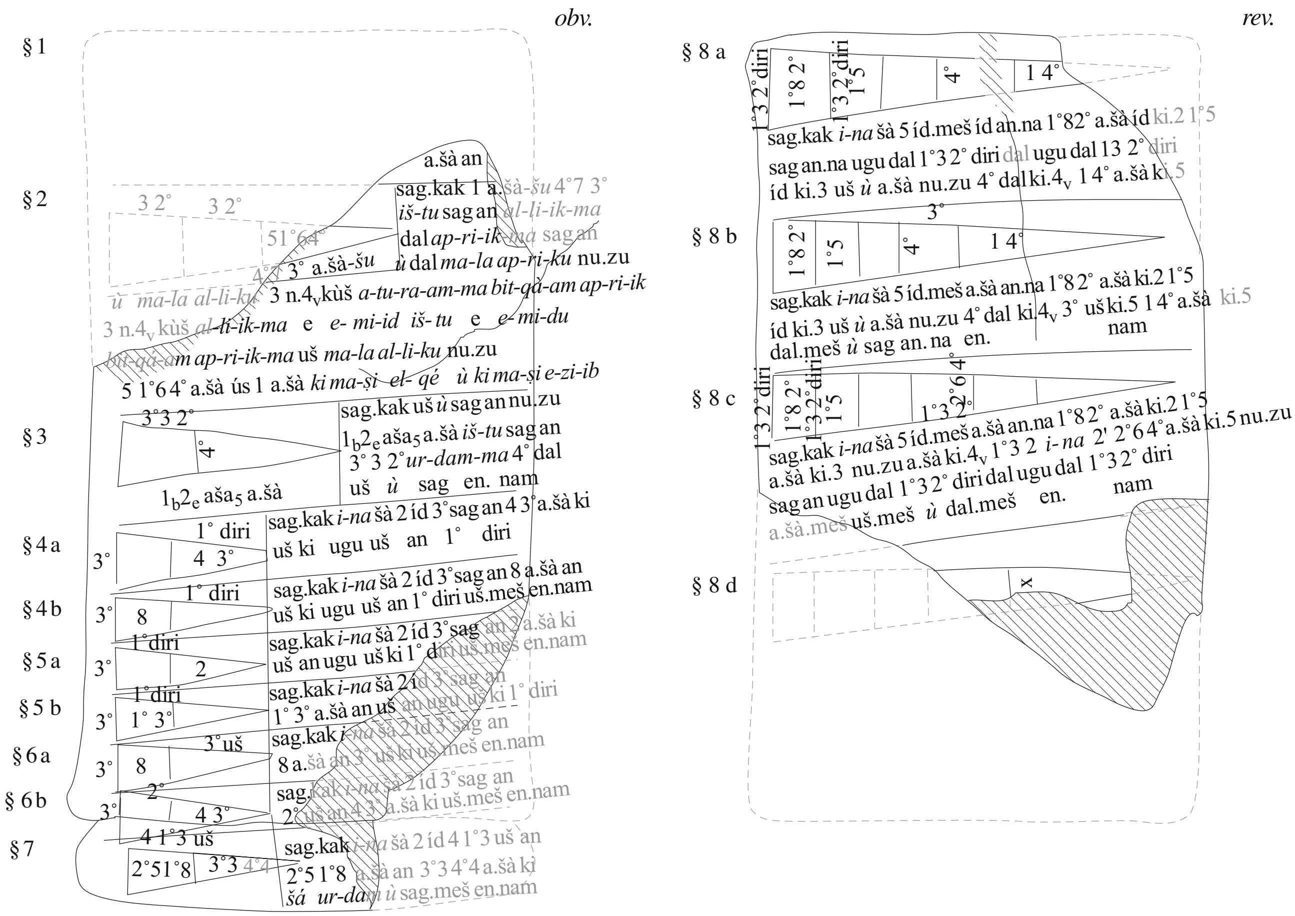


Fig. 10.1.7. Str 364. A catalog of metric algebra problems for striped triangles.

The last part of the catalog of mathematical problem types in § 4 of IM 52916, in § 4c on the left edge, is badly preserved. The few lines of § 4c that are preserved to some degree are the following ones:

še$_{20}$*-e ka-ra-am* [*ša-ka-na*]*-am*	To [pu]t grain in a grain store.
[*iš*]*-ka-ar e-*[*ṣe-di e-pé-ša*]*-am*	To make the [wo]rk norm of ha[rvesting].
[*iš*]*-ka-ar za-ri-i ša še-im e-pé-*[*ša-am*]	To ma[ke] the [wo]rk norm of seeding of grain.
qa-na-am el-qé-e mi-in-da-[*x x x*]	A reed I took, the measu[re x x x].
i-na ša-am-mu-ti-šu 1 *iš-r*[*um x x x*]	Of the plants, 1 ten[th x x x].

Three of these five mathematical problem types are undocumented in the known corpus of Old Babylonian mathematical texts. However, the first mentioned problem type, 'To put grain in a grain store' may very well refer to problems of the type appearing in the large mathematical recombination text BM 96954+ (Friberg, *AT* (2007), Fig. 9.3.2; Robson, *MMTC* (1999), App. 3). In that text, the frequently occurring term guru$_7$ 'granary' refers to a number of differently formed granaries or, possibly, piles of grain.

Also the problem type beginning "a reed I took" looks familiar. Indeed, two of the well known metric algebra "broken reed problems" (see Friberg, *RlA 7* (1990), §§ 5.4b, 5.4e, and *MSCT* 1 (2007), Secs. 10.1 a-c, and Neugebauer, *MKT I* (1935), 311) begin as follows:

gi *el-qé-e-ma mi-in-da-as-sú* nu.zu	A reed I took, its measure I did not know.	Str. 368
qa-na-am el-qé-e-ma / [*mi-in-da-su*] / *ú-ul i-de-e-ma*	A reed I took, [its measure] I did not know.	IM 53965

The broken reed problem in Str. 368 , for instance, is a curiously phrased exercise, possibly inspired by at the time current practices among Old Babylonian architects or surveyors. From a reed of unknown length, a piece is broken off, and the shortened reed is used to measure the length of the rectangle. Then the original reed is reconstituted and used to measure the front. If the area is known, what is then the length of the initial reed? The problem leads to a rectangular-linear system of equations.

10.1.4 § 5. A List of Combined Work Norms, Variations of Parameters

In IM 52916, § 5 on the reverse consists of the following five relatively well conserved lines:

a.na 40	nindan *a-za-bi-il*	*ù*	2 13 20	*al-lu-um*	As much as 40 rods I carried and	2 13 20	was the combined work norm.
a.na 20	nindan *a-za-bi-il*	*ù*	4	*al-lu-um*	As much as 20 rods I carried and	4	was the combined work norm.
a.na 15	nindan *a-za-bi-il*	*ù*	[5]	*al-lu-um*	As much as 15 rods I carried and	[5]	was the combined work norm.
a.na [10]	nindan *a-za-bi-il*	*ù*	6 40	*al-lu-um*	As much as [10] rods I carried and	6 40	was the combined work norm.
a.na 5	nindan *a-za-bi-il*	*ù*	10	*al-lu-um*	As much as 40 rods I carried and	10	was the combined work norm.

The interpretation of the word *allum* 'hoe, pick-axe' as a term for a '(combined) work norm' in connection with the fabrication of mud bricks is discussed below, in Sec. 11.2.1 in the discussion of YBC 4673 § 1e. The term is known from the exercises §§ 9-10 in Haddad 104 (Al-Rawi and Roaf, *Sumer* 43 (1984)).

The five lines appear here out of context, *without specifying the values of the work norms involved*. However, apparently two work norms are involved in some kind of calculation, one for carrying some stuff and another for processing that stuff. Assume that the two work norms are

w units of the stuff carried 1 rod per man-day
w' units of the stuff processed per man-day.

Then also

$w \cdot$ rec. d units of the stuff could be carried a distance d rods per man-day.

Consequently,

1 unit of the stuff could be carried d rods and then processed in ($d \cdot$ rec. w + rec. w') man-days.

Therefore, the combined work norm W for carrying d rods and processing was

W = rec. ($d \cdot$ rec. w + rec. w') units per man-day.

This means that, for instance, the first two lines of § 5, when d = 40 and 20, respectively,

2;13 20 = rec. (40 · rec. w + rec. w') and 4 = rec. (20 · rec. w + rec. w').

Inverting these equations, one finds that

40 · rec. w + rec. w' = rec. 2;13 20 = ;27 and 20 · rec. w + rec. w' = rec. 4 = ;15.

The solution to this system of linear equations for the two unknown work norms is easy to find, since

20 · rec. w = ;27 – ;15 = ;12 so that rec. w = ;03 · ;12 and w = 20 · 5 = 1 40,
rec. w' = 2 · ;15 – ;27 = ;03 so that w' = rec. ;03 = 20.

A likely interpretation of this result is that *the two work norms silently understood in IM 52916 § 5* were

w = 1 40 volume-shekels of mud per man-day carried 1 rod and rec. w = ;00 36 man-days per volume-shekel · rod,
w' = 20 volume-shekels per man-day molded into bricks and rec. w′ = ;03 man-days per volume-shekel.

(Compare with the discussion of YBC 4673 § 1e and Haddad 104 ## 9-10 in Sec. 10.2.1 below.)

The five lines of § 5 can now be explained as *a computation of the combined work norm W for carrying and molding when the mud is carried 40, 20, 15, 10, or 5 rods, respectively.* Indeed,

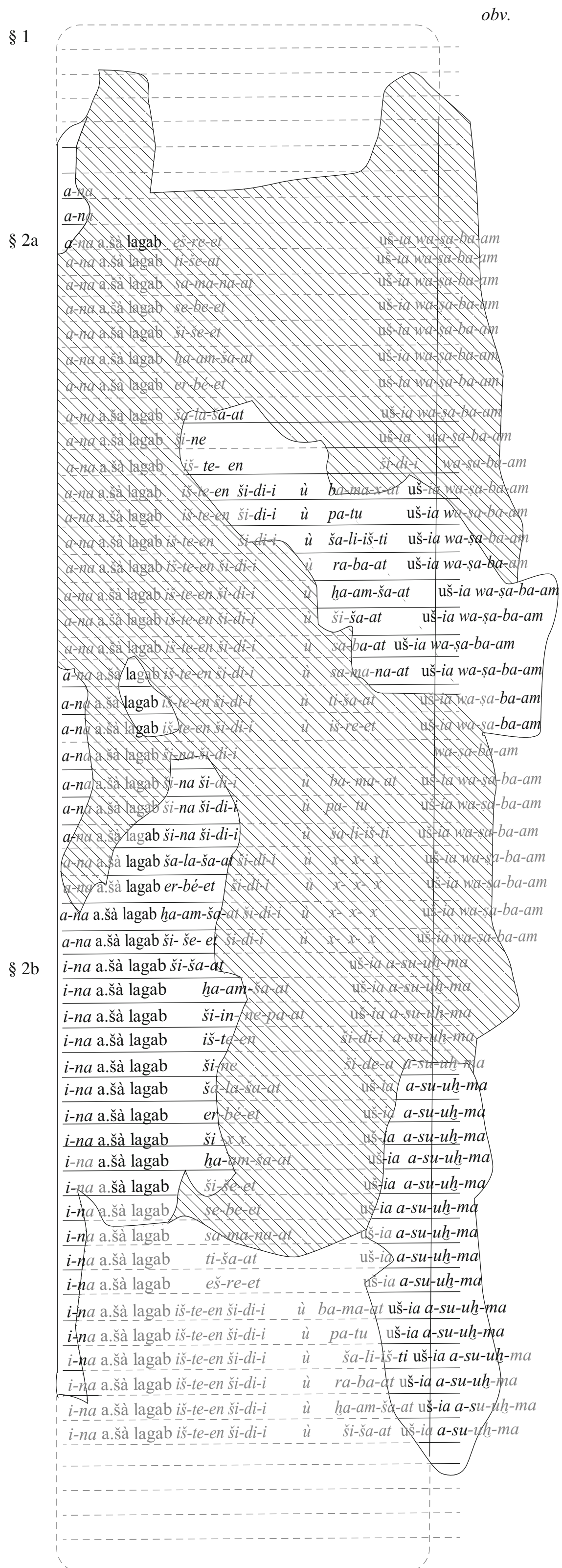
d = 40: W = rec. (40 · ;00 36 + ;03) = rec. (;24 + ;03) = rec. ;27 = 2;13 20 (volume-shekels per man-day),
d = 20: W = rec. (20 · ;00 36 + ;03) = rec. (;12 + ;03) = rec. ;15 = 4 (volume-shekels per man-day),
d = 15: W = rec. (15 · ;00 36 + ;03) = rec. (;09 + ;03) = rec. ;12 = 5 (volume-shekels per man-day),
d = 10: W = rec. (10 · ;00 36 + ;03) = rec. (;06 + ;03) = rec. ;09 = 6;40 (volume-shekels per man-day),
d = 5: W = rec. (5 · ;00 36 + ;03) = rec. (;03 + ;03) = rec. ;06 = 10 (volume-shekels per man-day).

Somewhat surprisingly, the numbers in these computations fit together beautifully in such a way that all numbers considered are *regular* sexagesimal numbers and the reciprocals can be computed exactly. However, this is by design of the author of the paragraph. This becomes clear when the reason for the absence of the case d = 30 is considered. Namely,

d = 30: W = rec. (30 · ;00 36 + ;03) = rec. (;18 + ;03) = rec. ;21 = ?

Thus, in this particular situation, if d = 30 (rods), then the combined work norm W turns out to be the reciprocal of a *non-regular* sexagesimal number, so that its value cannot be computed, at least not easily.

10.1.5 IM 52916. Hand Copies and Conform Transliterations



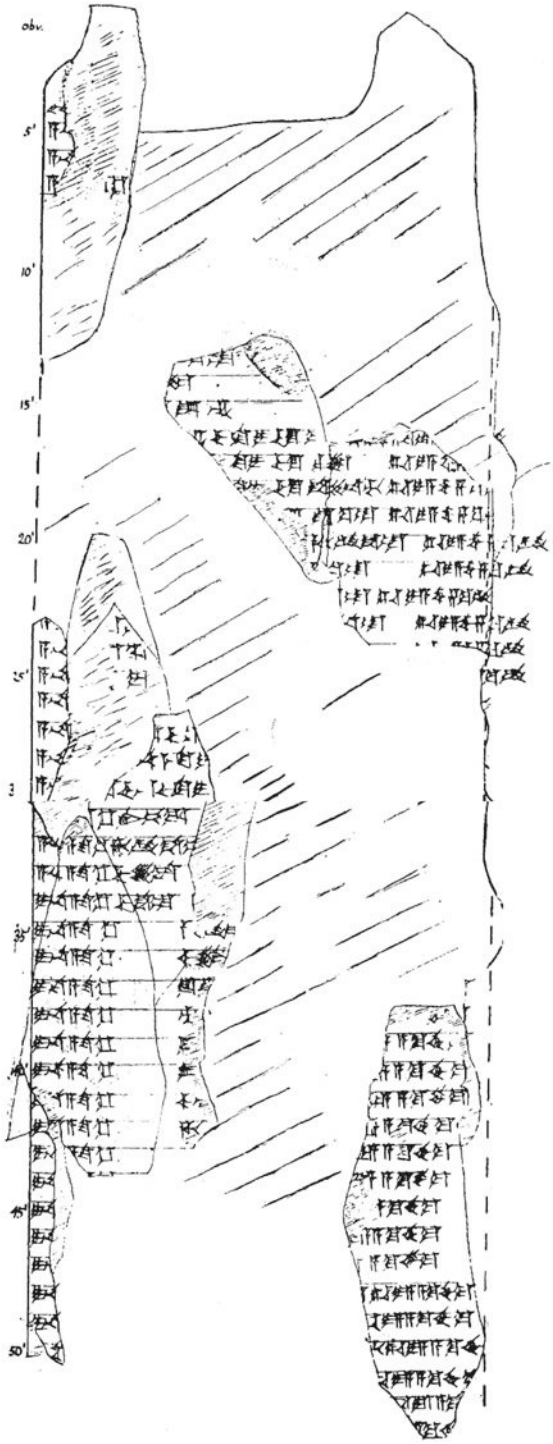


Fig. 10.1.8. IM 52916, *obv.* A catalog of quadratic equations of basic types B4a and B4b, with variations of parameters.

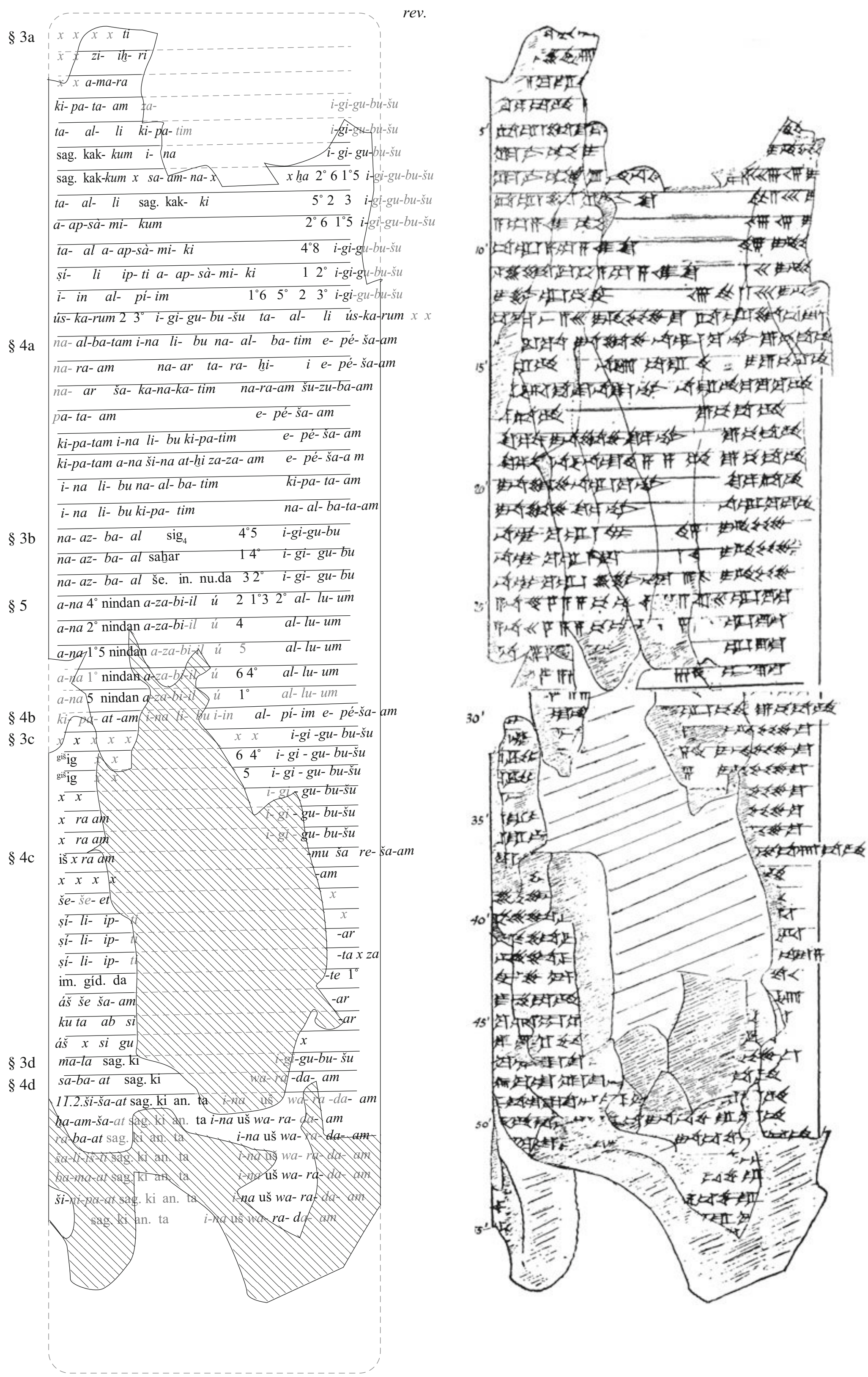


Fig. 10.1.9. IM 52916, *rev.* A mathematical table of constants, and other mathematical catalogs.

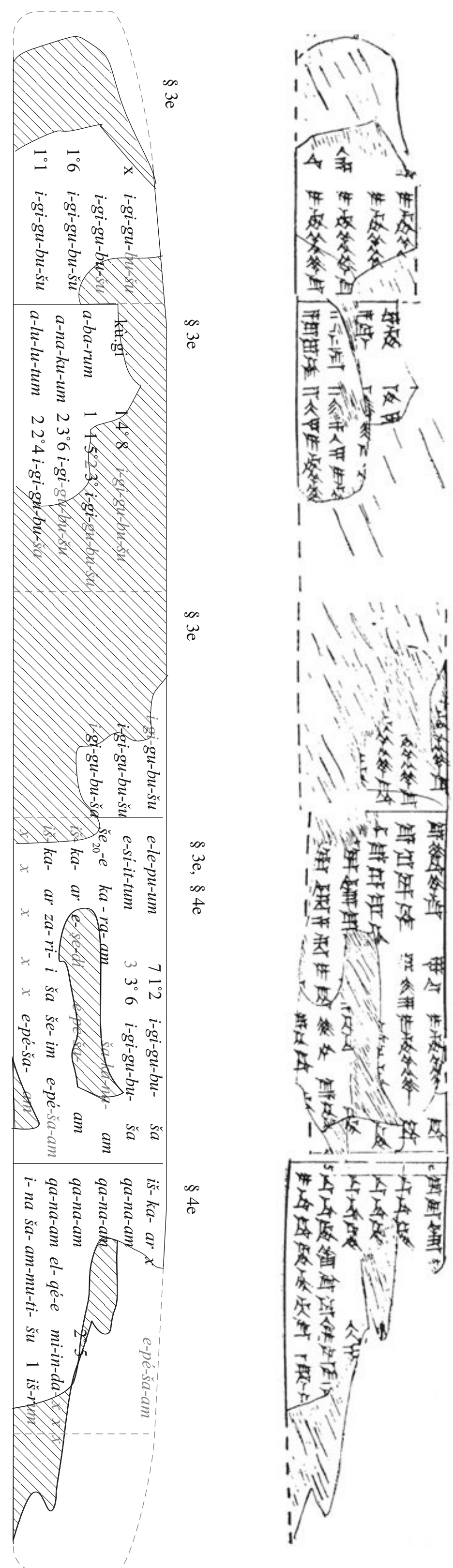


Fig. 10.1.10. IM 52916, *left edge.* A mathematical table of constants, and a catalog of mathematical problems.

10.2 IM 52685 + 52304. A Similar Mathematical Recombination Text from Old Babylonian Shaduppûm

IM 52685 and IM 52304 are probably two physically unconnected fragments of a single mathematical recombination text. The reverse of IM 52304 is not preserved. Several unconnected small fragments are conserved, two of which seem to fit in the places indicated in Fig. 10.2.1 below.

§ 1a on IM 52304 and § 1b on the upper half of the obverse of IM 52685 are poorly preserved, but seem to be similar to (but not identical with) §§ 2a-b of IM 52916. See Sec. 10.1.1 above. The whole combined text can be divided into the following paragraphs:

§§ 1a-b:	a badly preserved catalog with variations of parameters, quadratic problems of types B4a-b	(xx lines)
§ 1c:	a catalog of problems for several squares, variations of parameters	(4 lines)
§ 1d:	metric algebra operations, variations of parameters	(5 lines)
§ 1e:	a rectangle of given form and given area, variations of parameters	(12 lines)
§ 1f:	a linear equation, variations of parameters	(8 lines)
§ 1g:	metric algebra operation, meaning not clear	(2 lines)
§ 1h:	catalog of commercial problems, badly preserved	(13 lines)
§ 2:	table of constants, damaged	(xx lines)
§ 1i:	catalog of problems, damaged (includes the leaning reed problem)	(xx lines)

10.2.1 § 1c. A Catalog of Problems for Several Squares, Variations of Parameters

The four lines of IM 52685 § 2 are relatively well preserved:

a.šag$_4$ *ši-ta mi-it-ḫa-ra-tim ka-m*[*a-ra-am*] /	To add to[gether] the fields (areas) of two equalsides (squares)
a.šag$_4$ *ša-la-aš mi-it-ḫa-*[*ra-tim*] *ka-ma-*[*ra-am*] /	To add to[gether] the fields of three equal-[sides]
a.šag$_4$ *er-bé-e* [*mi-it-ḫa-ra-tim*] *ka-ma-*[*ra-am*] /	To add to[gether] the fields of four [equalsides]
a.šag$_4$ *mi-it-ḫa-ra-tim* [*er*]*-bé-tim i-na* [*x x na-sa-ḫa-am*]	To [tear off] four equalsides from [x x x]

Expressions of the same kind as in the first three lines above are known from the Old Babylonian metric algebra theme text BM 13901 (Group 1c; Neugebauer, *MKT III* (1937), 1; Høyrup, *LWS* (2002), Chs. 3, 5):

a.šag$_4$ *ši-ta mi-it-ḫa-ra-ti-ia ak-mur-ma*	The fields of my two equalsides I added together, then	BM 13901 §§ 8-14
a.šag$_4$ *ša-la-aš mi-it-ḫa-ra-ti-ia ak-mur-ma*	The fields of my three equalsides I added together, then	§§ 17, 18, 24
a.šag$_4$ [*er-bé-e*] *mi-it-ḫa-ra-ti-ia ak-mur-ma*	The fields of my four equalsides I added together, then	§ 15

In the metric algebra series text YBC 4714 (*MKT III*, 4; *LWS*, Ch. 5) similar expressions, in Sumerian, are

a.šag$_4$ íb.sá 2.e / gar.gar-*ma*	The fields of 2 equalsides / I added together, then	YBC 4714, # 29
a.šag$_4$ íb.sá 3.e / gar.gar-*ma*	The fields of 3 equalsides / I added together, then	# 4
a.šag$_4$ íb.sá 4.e / gar.gar-*ma*	The fields of 4 equalsides / I added together, then	# 2
a.šag$_4$ íb.sá 6.e / gar.gar-*ma*	The fields of 6 equalsides / I added together, then	# 3

There are no known counterparts in published Old Babylonian mathematical text to the (damaged) expression in the fourth line of IM 52685 § 2.

10.2.2 § 1d. A Catalog of Metric Algebra Problems, Variations of Parameters

There are five lines, more or less damaged in IM 52685 § 3:

ši-ni-ip uš sag.ki	Two-thirds the length the front,
i-na uš [*na-sa-ḫa*]*-am a-na* [uš *wa-ṣa-ba-am*] /	to [tear off] from the length, [to add] to [the length] /
ši-ni-ip uš sag.ki	Two-thirds the length the front,
a.šag$_4$ uš *ù* sag.ki *ma-la* [*ma-ṣú-ú ka-ma-ra-am*] /	[to add together] the field, the length and the front, as much as [they are]. /
[1/2 uš sag.ki] *ša-li-iš-ti* uš sag.ki	[1/2 the length the front], a third the length the front,
ra-ba-at uš sag.ki / [*ḫa*]*-am-ša-at* uš sag.ki	a fourth the length the front, / [a fi]fth the length the front
ši-ša-at uš sag.ki *sa-ba-at* uš sag.ki /	a sixth the length the front, a seventh the length the front, /
[*sa*]*-am-na-at* uš sag.ki *ti-ša-at* uš sag.ki	an [ei]ghth the length the front, a ninth the length the front,
iš-re-et uš sag.ki /	a tenth the length the front.

All the lines of this paragraph are about metric algebra problems for a rectangle of a given form. In the first two lines, the front of the rectangle is 2/3 of the length of the rectangle. This is precisely the situation that was encountered in Sec. 5.1 above, in exercises ## 1-19 of the Mê-Turran text IM 121613.

In particular, in the *second* line of IM 52685 § 3, the front is 2/3 of the length and the area, the length, and the front of the rectangle are added together. The same situation was encountered in IM 121613 # 5 (Sec. 5.1.5), where the first line of the question begins as follows:

$^2/_3$ uš sag.ki	2/3 of the length (was) the front.
a.šag$_4$ uš *ù* sag.ki *ak-mur-ma* ...	The field, the length and the front I added together, then …

As mentioned, in the *first* line of IM 52685 the front is 2/3 of the length. In addition, there is an instruction to subtract something from the length, or add something to the length. What should be added is not said. However, it is possible that what is meant is that *the field* (the area) should be subtracted from, or added to the length. In that case, a related situation was encountered in IM 121613 # 11 (Sec. 5.1.11 above), where the question begins as follows:

$^2/_3$ uš sag.ki	2/3 of the length (was) the front.
a.šag$_4$ *i-na* uš *ás-su-ḫa-a*[*m-m*]*a* ...	The field from the front I tore off, then …

The remaining lines of IM 52685 § 3 contain *variations of parameters*, where it is mentioned that the coefficient two-thirds figuring in the first two lines can be replaced by either one of the coefficients

[1/2], a third, a fourth, a fifth, a sixth, a seventh, an eighth, a ninth, a tenth.

Note that here in § 3, just like in the two preceding paragraphs, Akkadian number words are used instead of cuneiform number signs, possibly with the exception of [1/2].

10.2.3 § 1e. A Catalog of Form and Magnitude Problems, Variations of Parameters

The generic expression in this paragraph is

uš sag.ki *ma-la e-li-ia ṭà-bu ša-ka-nam*	To set length (and) front as much as over me is good,
A a.šag$_4$ *ba-na-am* /	to build the field *A*.

What this means is, clearly, to set up the elementary linear-rectangular system of equations

$s = \mathrm{c}\, u,\quad s \cdot u = A,$ where the coefficient c can be chosen arbitrarily, and where A is a given area number.

In § 4 the given area numbers *A* are chosen in a somewhat unsystematic way, in the following 12 lines:

[uš] sag.ki *ma-la e-li-ia ṭà-bu ša-ka-nam* 1(èše)$^{\text{ašag}}$ a.šag$_4$ *ba-na-am* /
[uš sag].ki *ma-la e-li-ia ṭà-bu ša-ka-nam* 1(bùr)$^{\text{ašag}}$ [a.šag$_4$] *ba-na-am* /
[uš sag].ki *ma-la e-li-ia ṭà-bu ša-ka-nam* 2(èše)$^{\text{ašag}}$ a.[šag$_4$ *ba-na-am*] /
[uš sag.ki *ma-la*] *e-li-ia ṭà-bu ša-ka-nam* 1(èše) 4(iku)$^{\text{ašag}}$ a.[šag$_4$ *ba-na-am*] /
<uš sag.ki *ma-la e-li-ia ṭà-bu ša-ka-nam*> 1(èše) 3(iku)$^{\text{ašag}}$ a.šag$_4$ *ba-na-am* /
[uš sag.ki *ma-la*] *e-li-ia ṭà-bu* [*ša-ka-nam* 1(èše) 2(iku)$^{\text{ašag}}$ a.šag$_4$ *ba-na-am*] /
[uš sag.ki] *ma-*[*la e-li-ia ṭà-bu ša-ka-nam* 1(èše) 1(iku)$^{\text{ašag}}$ a.šag$_4$ *ba-na-am*] /
[uš sag].ki *ma-l*[*a*] *e-li-*[*ia ṭà-bu ša-ka-nam* 1(iku)$^{\text{ašag}}$ a.šag$_4$ *ba-na-am*] /
[uš sag].ki *ma-la e-li-ia ṭà-bu ša-ka-nam* [2(iku)$^{\text{ašag}}$ a.šag$_4$ *ba-na-am*] /
uš sag.ki *ma-la e-li-ia ṭà-bu ša-ka-nam* 3(iku)$^{\text{ašag}}$ a.šag$_4$ [*ba-na-am*] /
uš sag.ki *ma-la e-li-ia ṭà-bu ša-ka-nam* 4(iku)$^{\text{ašag}}$ a.šag$_4$ *b*[*a-na-am*] /
uš sag.ki *ma-la e-li-ia ṭà-bu ša-ka-nam* 5(iku)$^{\text{ašag}}$ a.šag$_4$ [*ba-na-am*]

The problem type is known from the basic exercise IM 121613 # 1 in Sec. 51.1 above, where $c = 2/3$.

10.2.4 § 1f. A Catalog of Linear Equations, Variations of Parameters

This time the generic expression is

sag.ki a.na uš (*n*) *ma-ṭà-am*	The front to be whatever the length, *n* less.

This generic equation is subjected to a long series of variations of parameters, in the following 8 lines:

sag.ki a.na uš 1 *ma-ṭà-am* sag.ki a.na uš 2 *ma-ṭà-am* /
sag.ki a.na uš 3 *ma-ṭà-am* sag.ki a.na uš 4 *ma-ṭà-am* /
sag.ki a.na uš 5 *ma-ṭà-am* sag.ki a.na uš 6 *ma-ṭà-am* /
sag.ki a.na uš 7 *ma-ṭà-am* sag.ki a.na uš 8 *ma-ṭà-am* /
sag.ki a.na uš 9 *ma-ṭà-am* sag.ki a.na uš [10 *ma-ṭà-am*] /
sag.ki a.na uš 20 *ma-ṭà-am* sag.ki [a.na uš 25 *ma-ṭà-am*] /
sag.ki a.na uš 30 *ma-ṭà-am* sag.ki a.na uš [35 *ma-ṭà*]*-am* /
sag.ki a.na uš 40 *ma-ṭà-am* sag.ki a.na uš [45 *ma-ṭà*]*-am* /

It is likely that it is silently understood in § 5 that the area of the rectangle is given, as in § 4 before it, but the linear relation between the front and the length is of a new type. With this assumption, the problem considered in § 5 of IM 52685 is the following:

$u - s = d, \quad s \cdot u = A,$ where d is a given length number (and A a given area number).

This is a linear-rectangular system of equations of the basic type B1b (Friberg, *AT* (2007), 6). Astonishingly, explicitly stated systems of equations of this type are nonexistent in known Old Babylonian mathematical texts. However, they appear implicitly, for instance in *MCT* Ua = YBC 6967 (Høyrup, *LWS* (2002), 55), where the question is

igi.bi *e-li* igi 7 *i-ter*	Its reciprocal over the reciprocal is 7 beyond.

With igi.bi = u and igi = s, the problem can be reformulated as

$u - s = 7, \quad u \cdot s = 1.$

Another example is the commercial text YBC 4698 § 5a in Sec. 11.1 below, where the market rate problem

1(gur) gur ì.geš / *i-na* sám $1^!$ gín / 2 sìla šuš$_4$ / 7 $^1/_2$ gín kù.dirig

after some manipulation is transformed into a system of equations of the basic type B1b.

10.2.5 § 1g. Geometric Division into Equal Parts(?)

This paragraph consists of a single expression, which may tentatively be translated as follows:

ma-la uš *ma-la* sag.ki a.šag$_4^{li}$ *im-ṣí-ú*	Whichever length, whichever front my field corresponded to,
ši-ma-at mi-it-ḫa-ra-tim e-pé-[*ša-am*]	[to ma]ke the mark of equal parts.

What this expression seems to say, in very vague and general terms, is that a rectilinear geometric figure like a triangle, a rectangle, or a trapezoid should be divided into equal parts. If this interpretation is correct, then the type of Old Babylonian metric algebra problem that immediately comes to mind is that of bisected trapezoids. There are many Old Babylonian mathematical texts with various generalizations of this problem type. See, for instance, Friberg, *AT* (2007), Ch. 11.3. Anyway, the basic problem type with a quadrilateral figure divided by a transversal into two parts of equal area appears in YBC 4675 (Neugebauer and Sachs, *MCT* (1945), text B; Friberg, *op. cit*., 273), where the question begins in the following way:

šum-ma a.šag$_4$ uš.uš.i.ku	If a field of 'length-eats-length',
uš 1.e 10 uš 2.e 4 50 / sag an.ta 17 sag ki.ta 7	1st length 10, 2nd length 4 50, upper front 17, lower front 7,
a.šag$_4$.bi 2(bùr)$^{\text{ašag}}$ /	its field 2 bùr,
1(bùr)$^{\text{ašag}}$.ta.am a.šag$_4^{lam}$ *a-na ši-na* <*a*>*-zu-ú-uz*	1 bùr each, the field in two <I> divided.
ta-al-li qá-ab-lu-ú ki ma-ṣi /	The middle transversal, how much?

Actually, the problem type was known long before the Old Babylonian period. Thus, the Old Akkadian round hand tablet IM 58045 contains a drawing of a bisected trapezoid with indication of relevant parameters (Friberg, *AT* Fig. 11.3.1), and a Neo-Sumerian legal document from Ur III Umma shows numerically how an irregular plot can be equally divided between five partners (Friberg, *CDLJ* 2009:3).

Note: The translation above of *ši-ma-at* as 'mark' corresponds to the gloss *šimtu* '(ownership) mark' in *CAD* (Black et al (1999)). However, in *SBM* 3 (2007), 19, Muroi reads instead *šīmtu* 'what is fixed' and translates as follows:

ma-la uš *ma-la* sag.ki a.šag$_4^{li}$ *im-ṣí-*[*ú*]	To construct whatever length, whatever width my area complied with,
ši-ma-at mi-it-ḫa-r[*a-ti*]*m e-pé-*[*ša*]*-a*[*m*]	by the formula for square roots (= solution to quadratic equations).

Accordingly, he understands this pair of lines as the concluding last lines of § 1f above.

10.2.6 § 1h. A Catalog of Commercial or Financial Problem Types

This subparagraph contains an enumeration of 24 mathematical problem types concerned with either commerce (market rates) or finance (profits, taxes and losses).

ma-ḫi-ra-am e-pé-ša-am	To make a market rate.	§ 1h, line 1
ma-ḫi-ir še-im [*x x x*] /	To [x x x] the market rate of grain.	
[*ma*]*-ḫi-ir* še.ì.geš	Market rate of sesame.	line 2
*ma-ḫi-i*r zu.lum	Market rate of dates.	
ma-ḫi-ir [x x x] /	Market rate of [x x x].	
[*ma-ḫi-ir* ì.geš]	Market rate of common oil.	line 3
[*ma-ḫi*]*-ir* ì.sag.dùg.ga	Market rate of fine oil.	
ma-ḫi-[*ir* ì.šaḫ] /	Market rate of lard.	
[*ma-ḫi-ir* x x x]	Market rate of [x x x].	line 4
ma-ḫi-ir a.bul	Market rate of bitumen.	
ma-ḫi-ir [x x x] /	Market rate of [x x x].	
ma-ḫi-r[*a-am na*]*-sa-ḫa-am ša-ma-am ù k*[*a-ma-ra-am*] /	To tear off, buy, and a[dd together] market ra[tes].	line 5
ne-mé-la-[*am x x*]*-am*	To [x x] profit	line 6
kù.babbar *ša-qa-la-a*[*m*] /	To weigh up silver.	
ì.geš *ù* ì.šaḫ *ša-ma-am*	To buy common oil and lard.	line 7
kù.babbar ì.geš *ù* ì.šaḫ *ka-ma-ra-am* /	To add together silver, common oil, and lard.	
ma-ḫi-ir ì.geš *ù* ì.šaḫ *ka-ma-ra-am* /	To add together market rates for common oil and lard.	line 8
mi-ik-sa-am e-pé-ša-am /	To make tax.	line 9
[*x x x x x x*]*-i-lu-um ba-ba-am*	To [x x x x x] the gate.	line 10
ṣí-it-[*x x x x x*] /	To [x x x] the exit tax.	
[*x x x x x x*] *x ka-ma-ra-am*	To add together [x x x x].	line 11
at-ḫi [*x x x x x*] /	To [x x x] partners.	
[*x x x x x x*] *ma-la ur-ta-ba-bu-ú ḫu-lu-*[*qa-am x x x*] /	To lose [x x] as much as it increased (?)	line 12
[*x x x x x x x x x*]*-ša-am i-na x* [*x x x x x*] /	To [x x x x x] from [x x x x x].	line 13

There is obviously a close connection between the list of topics in lines 1-8 above and the mathematical recombination text YBC 4698 with its many commercial problems. See the detailed discussion of that text in Sec. 11.1 below. However, no mathematical cuneiform texts are known concerned with taxation, as in lines 9-10 above.

The long list of mathematical problem types on IM 52685 ends here. Then follows, apparently, a *table of constants*, of which almost nothing remains.

Note that there is an obvious connection between the recombination text IM 52685 (+ 52304) in the present section, Sec. 10.2, and the recombination text IM 52916 in the previous section, Sec. 10.1. Not only were they *found together*, as told by Goetze, they are also *the only known cuneiform texts to contain (excerpts from) catalogs of mathematical problems, without any numerical parameters*.

10.2.7 IM 52685+. Hand Copies and Conform Transliterations

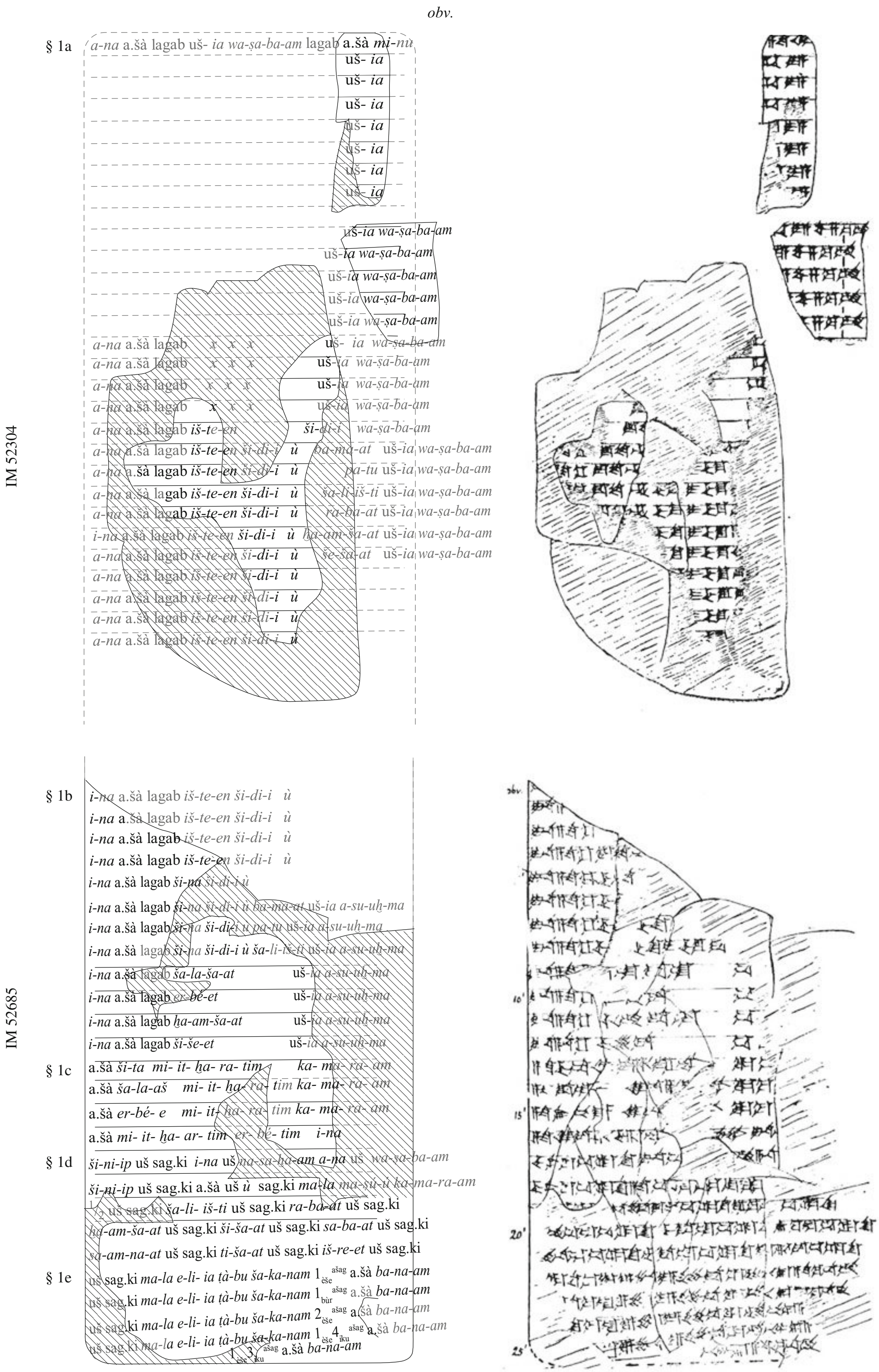


Fig. 10.2.1. IM 52685 + IM 52304, *obv.* An Old Babylonian recombination text with various mathematical catalogs

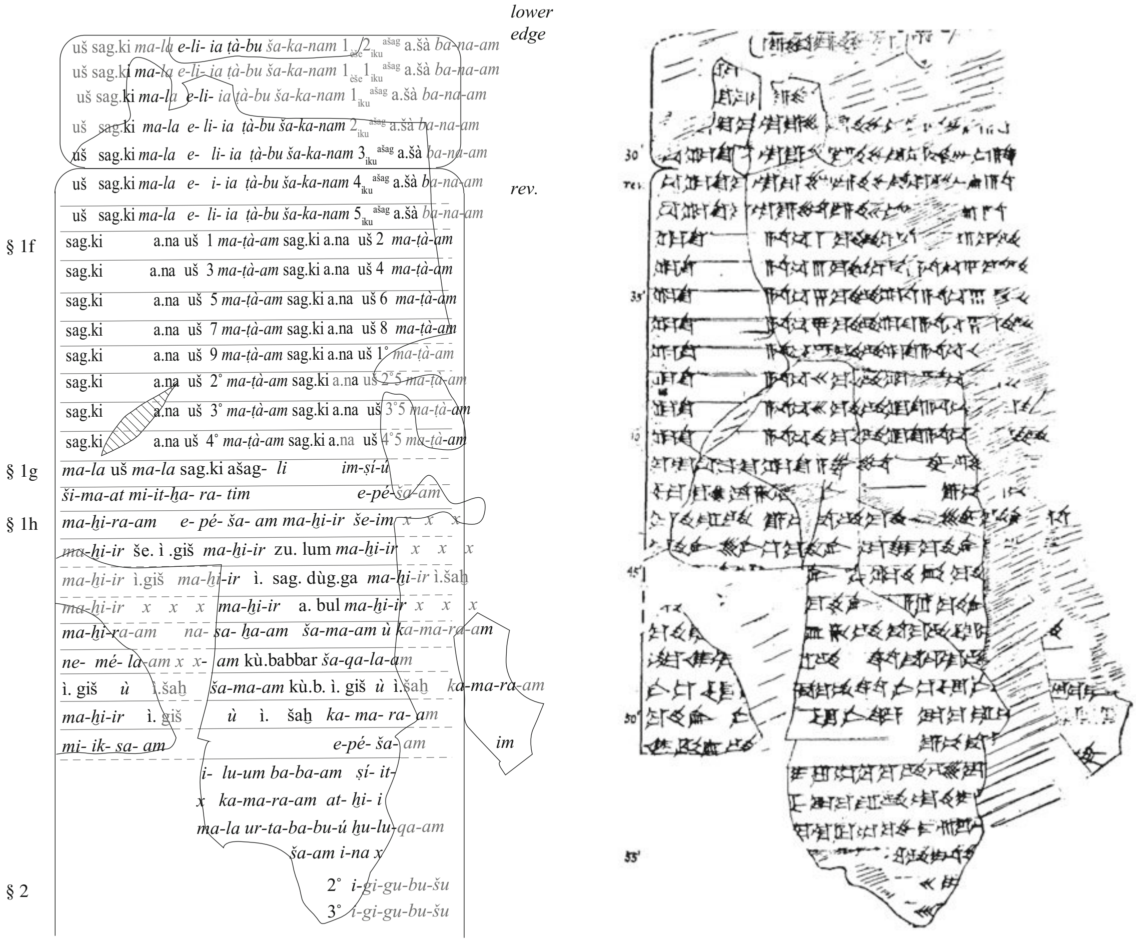


Fig. 10.2.2. IM 52685, *rev.* Additional mathematical catalogs

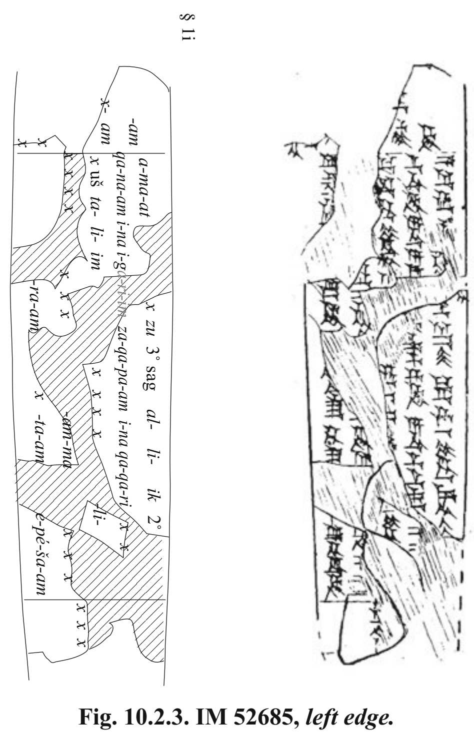


Fig. 10.2.3. IM 52685, *left edge.*

10.3 *TMS* V. A Large Catalog Text with Metric Algebra Problems for Squares

The large catalog text *TMS* V (Figs. 10.3.1-2 below) was published and essentially correctly explained in Bruins and Rutten, *TMS* (1961). The text is fairly well preserved and it is interesting, in particular, because of its unorthodox notations for fractions.

In condensed form, the questions in §§ 1-6 of *TMS* V can be reproduced as follows, in transliteration and literal translation:

§ 1a	(*u*) lagab (*c*) uš-*ia mi-nu*	*u* was the square (equalside), *c* (of) my length was what?
§ 1b	[(*c*) uš-*ia* (*p*) lagab *mi-nu*]	[*c* (of) my length was *p*, the square was what?]
§ 1c	lagab *ù* (*c*) uš-*ia* gar.gar-*ma* (*p*)	The square and *c* (of) my length I added together, then *p*.
§ 1d	lagab ugu (*c*) uš (*q*) dirig	The square over *c* (of) the length was *q* beyond.
§ 2a	(*u*) lagab (*c*) a.šag$_4$ *mi-nu*	*u* was the square, *c* (of) the field was what?
§ 2b	(*A*) a.ša lagab *mi-nu*	*A* was the field, the square was what?
§ 2c	(*c*) a.šag$_4$ (*B*) lagab *mi-nu*	*c* (of) the field was *B*, the square was what?
§ 3a	(*u*) lagab a.šag$_4$ (*c*) uš *mi-nu*	*u* was the square, the field of *c* (of) the length was what?
§ 3b1	a.šag$_4$ (*c*) uš *ù* a.šag$_4$ gar.gar-*ma* (*B*)	The field of *c* (of) the length and the field I added together, then *B*.
§ 3b2	a.šag$_4$ *ù* a.šag$_4$ (*c*) uš gar.gar-*ma* (*B*)	The field and the field of *c* (of) the length I added together, then *B*.
§ 3c1	a.šag$_4$ (*c*) uš ugu a.šag$_4$ (*q*) dirig	The field of *c* (of) the length over the field was *q* beyond.
§ 3c2	a.šag$_4$ ugu a.šag$_4$ (*c*) uš (*q*) dirig	The field over the field of *c* (of) the length was *q* beyond.
§ 4a	(*a-na* a.šag$_4$ lagab-*ia*) (*c*) uš daḫ-*ma* (*B*)	(To the field of my square) *c* (of) the length I added, then *B*.
§ 4b	(*i-na* a.šag$_4$ lagab-*ia*) (*c*) uš zi-*ma* (*B*)	(From the field of my square) *c* (of) the length I tore off, then *B*.
§ 4c	(*c*) lagab ugu a.šag$_4$ (*q*) dirig	*c* (of) the square over the field was *q* beyond.
§ 4d	(*c*) lagab *ki-ma* a.šag$_4$ ⸢gim⸣	*c* (of) the square like the field was equal.
§ 5	lagab.ba a.šag$_4$ *ab-ni mi-nu* ib.[x x] x	The field of a square I built, what [x] x [x]?
§ 6	(*c*) a.šag$_4$ *it-ba-al* íb.tag$_4$ / (*B*) lagab *mi-nu*	*c* (of) the field he took away, remainder / *B*. The square was what?

In this text, the term lagab 'block, square' is used in the same way as the terms íb.sá and *mitḫartum* in other Old Babylonian mathematical texts. The meaning of the two latter terms is literally 'something that is equal' and refers to the sides of a square. A convenient translation is 'equalside'. Note that a 'square', just as an 'equalside', can have both a 'length' (side) and a 'field' (area). Note also that the cuneiform sign lagab has the form of a square, so that a better translation in the case of this text may be, simply, 'square'. This interpretation is supported by the observation that in § 7 of *TMS* V the plural of lagab 'square' is written as nigin, a cuneiform sign in the form of two squares!

In terms of quasi-modern symbolic notations, the questions in §§ 1-6 can be reformulated as follows:

§ 1a	*u* given,	c *u* = ?	(*u* is the length of the square-side, c is a given coefficient.)	20 cases
§ 1b	[c *u* given,	(*u* = ?)]		[20 cases]
§ 1c	*u* + c *u* given,	(*u* = ?)		20 cases
§ 1d	*u* – c *u* given,	(*u* = ?)		17 cases
§ 2a	*u* given,	c sq. *u* = ?		16 cases(?)
§ 2b	c sq. *u* given,	*u* = ?		22 cases(?)
§ 3a	*u* given,	sq. (c *u*) = ?		20 cases
§ 3b1	sq. (c *u*) + sq. *u* given	(*u* = ?)	(Cases when the coefficient c is greater than 1.)	3 cases
§ 3b2	sq. *u* + sq. (c *u*) given,	(*u* = ?)	(Cases when the coefficient c is smaller than 1.)	17 cases(?)
§ 3c1	sq. (c *u*) – sq. *u* given	(*u* = ?)	(Cases when the coefficient c is greater than 1.)	[3 cases(?)]
§ 3c2	sq. *u* – sq. (c *u*) given,	(*u* = ?)	(Cases when the coefficient c is smaller than 1.)	17 cases
§ 4a	sq. *u* + c *u* given,	(*u* = ?)		27 cases
§ 4b	sq. *u* – c *u* given,	(*u* = ?)	(Cases when sq. *u* is greater than c *u*.)	27 cases(?)
§ 4c	c *u* – sq. *u* given,	(*u* = ?)	(Cases when c *u* is greater than sq. *u*.)	3 cases
§ 4d	c *u* = sq. *u*	(*u* = ?)	(The border case.)	1 case
§ 5	???		(Damaged text, the meaning of the problem is not clear.)	1 case
§ 6	sq. *u* – c sq. *u* given,	*u* = ?	(Here c is less than 1.)	5 cases

obv.

§ 1d, §§ 2a-c

§ 1a

§ 1b

§ 1c

§ 3a

§ 3b1

§ 3b2

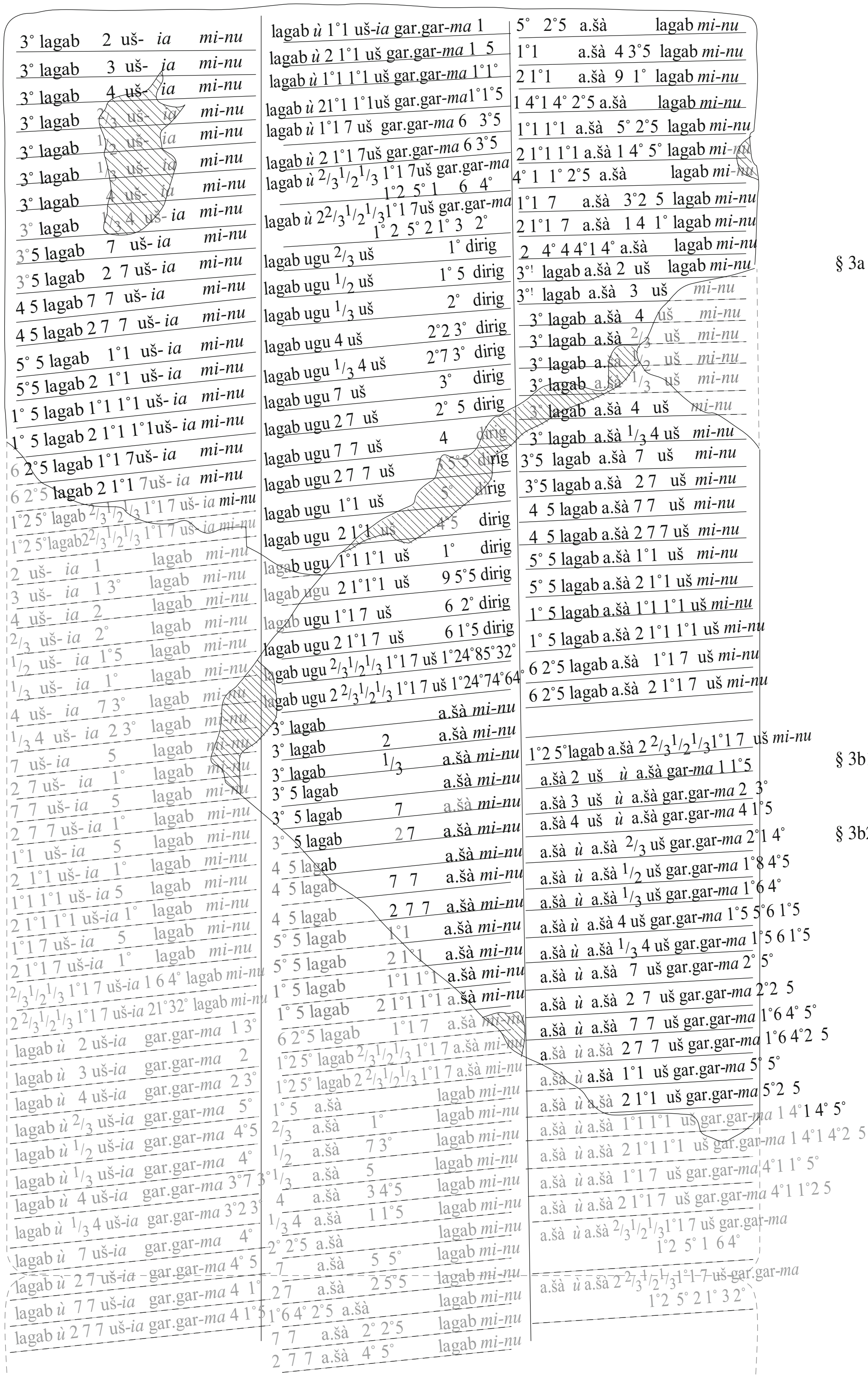


Fig. 10.3.1. *TMS* V, *obv*. An Old Babylonian metric algebra catalog text from Susa with variations of parameters.

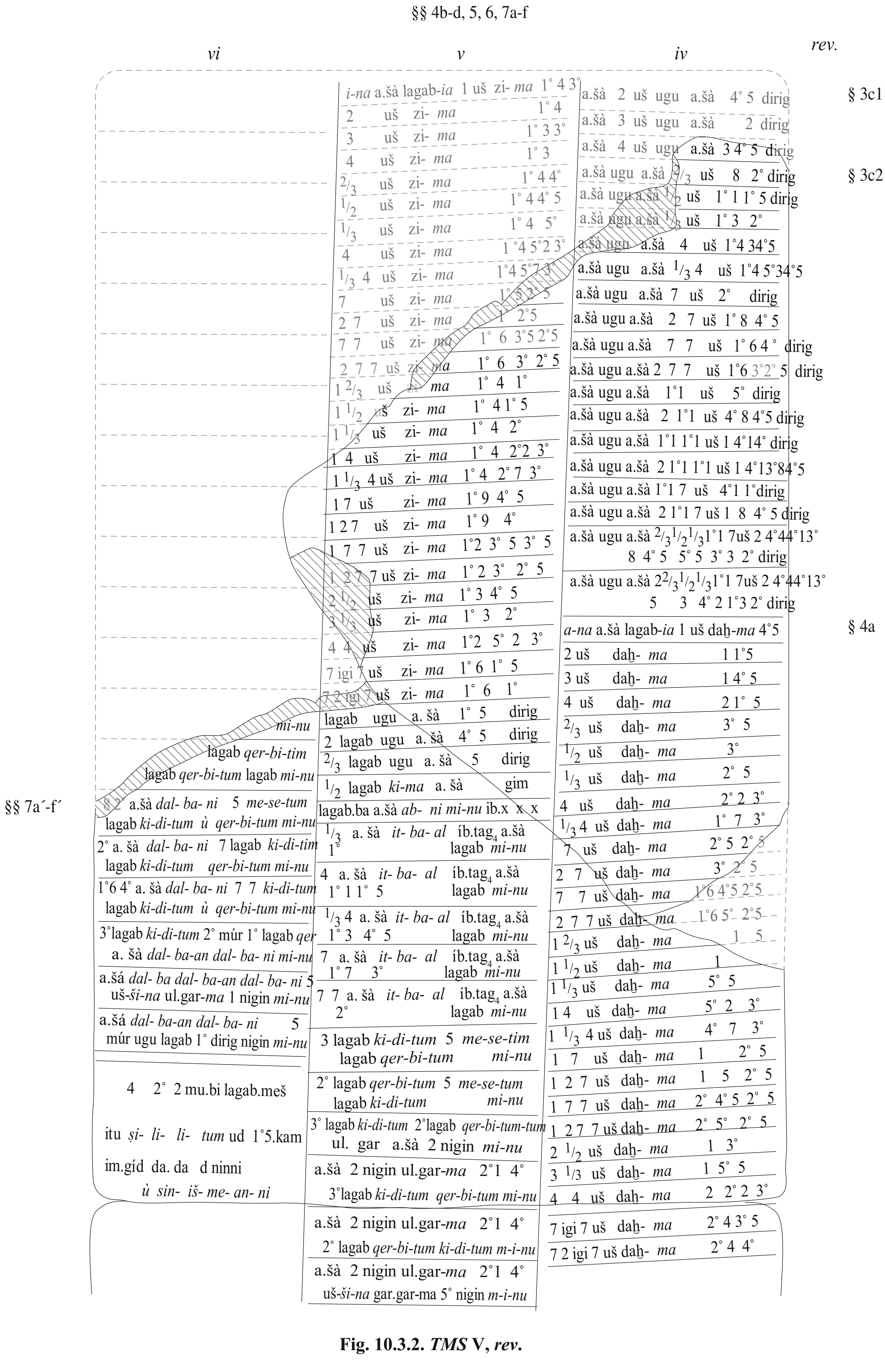


Fig. 10.3.2. ***TMS*** **V,** ***rev.***

These are all, essentially, elementary metric algebra problems, namely in § 1 *linear equations*, in §§ 2-3 and § 6 computations of *squares and square-sides* (square roots), in §§ 4-5 *quadratic equations of the basic types B4a-c*. (See Friberg, *AT* (2007), 6.) As will be shown below, the real difficulty in all these problems is the *counting with complicated coefficients involving compounded fractions*.

Note that great care is taken in this text to make sure that *a greater number is never subtracted from a smaller number*. (Apparently, the concept of negative numbers was not known.) In § 3b similar care is taken to ensure that the biggest number in a sum is mentioned first. Note, finally, the form of the equation in § 4d, which is c *u* = sq. *u* rather than c *u* – sq. *u* = 0. (The concept of 0 as a number was not known.)

As a confusing kind of shorthand, many of the fractions figuring in this text are written in an otherwise undocumented space-saving way, with, for instance,

7 for rec. 7 (= 1/7), 2 7 for 2 rec. 7 (= 2 · 1/7), 7 7 for rec. 7 · rec. 7 (= 1/7 · 1/7), 2 7 7 for 2 rec. 7 · rec. 7 (= 2 · 1/7 · 1/7),
4 for rec. 4 (= 1/4) and 1/3 4 for 1/3 · rec. 4 (= 1/3 · 1/4).

Note, in § 4a, the inconsistent notations

4 4 for 4 rec. 4 (= 4 1/4), but 7 igi 7 for 7 rec. 7 (= 7 1/7), and , 7 2 igi 7 for 7 · 2 rec. 7 (= 7 · 2 · 1/7).

In order to be able to count with such complicated constructions of the coefficient c, the author of the text has modified accordingly the values of the length *u* of the square-sides in the text. In the survey on the next page, it is shown in which paragraphs of the text the various values of c appear, and which the corresponding values are of the length *u*. (Also the squares of *u* are included in the survey.) *The various values of u appearing in the text are mentioned explicitly in § 1a*. They can be explained as follows:

30 = 5 · 6, 35 = 5 · 7, 4 05 = 5 · 7 · 7, 55 = 5 · 11, 10 05 = 5 · 11 · 11, 6 25 = 5 · 11 · 7 and 12 50 = 10 · 11 · 7.

The data in *TMS* V are usually correctly computed, with the interesting exception that the author of the text evidently had *great difficulties keeping track of the absolute values of numbers written in relative place value notation* (floating values). Here are some examples of how the data were computed:

§ 1a:	*u* = 10 05, c = 11 11:	c *u* = 1/11 · 1/11 · 10 05 = 1/11 · 55 = 5
§ 1c:	(*u* = 12 50), c = 2/3 1/2 1/3 11 7:	*u* + c *u* = 12 50 + 2/3 1/2 1/3 · 10 = 12 50 + 1:06 40 = 12 51;06 40
§ 1d:	(*u* = 30), c = 1/3 4:	*u* – c *u* = 30 – 1/3 1/4 · 30 = 30 – 2:30 = 27:30
§ 3a:	*u* = 6 25, c = 11 7:	sq. (c *u*) = sq. (1/11 1/7 · 6 25) = sq. 5 = 25
§ 3b:	(*u* = 4 05), c = 2 7 7:	sq. *u* + sq. (c *u*) = sq. 4 05 + sq. (2 1/7 1/7 · 4 05) = 16 40 25 + 1 40 = 16 42 05
	(*u* = 12 50), c = 2/3 1/2 1/3 11 7:	sq. *u* + sq. (c *u*) = sq. 12 50 + sq. 1;06 40 = 2 44 41 40 + 1;14 04 26 40 = ……
§ 3 c:	(*u* = 4 05), c = 2 7 7:	sq. *u* – sq. (c *u*) = sq. 4 05 – sq. (2 1/7 1/7 · 4 05) = 16 40 25 **+ 1 40** = 16 42 05!
§ 4a:	(*u* = ;30), c = 1 1/2:	sq. *u* + 1 1/2 *u* = ;15 + ;45 = 1
	(*u* = ;35), c = 1 7:	sq. *u* + 1 1/7 *u* = ;20 25 + ;40 = 1;00 25
	(*u* = 35), c = 7 igi 7:	sq. *u* + 7 1/7 *u* = 20 25 + 4 10 = 24 35
§ 4b:	(*u* = 4;05), c = 7 7:	sq. *u* – 1/7 1/7 *u* = 16;40 25 – ;05 = 16;35 25
	(*u* = 30), c = 1 2/3:	sq. *u* – 1 2/3 *u* = 15 00 – 50 = 14 10
	(*u* = 30), c = 2 1/2:	sq. *u* – 2 1/2 *u* = 15 00 – 1 15 = 13 45
§ 4c:	(*u* = ;30), c = 2/3:	2/3 *u* – sq. *u* = ;20 – ;15 = ;05
§ 4d:	(*u* = ;30), c = 1/2:	1/2 *u* = ;15 = sq. *u*
§ 6:	(*u* = 35), c = 7:	sq. *u* – 1/7 sq. *u* = 20 25 – 2 55 = 17 30
	(*u* = 35!), c = 7 7:	sq. *u* – 1/7 1/7 sq. *u* = 20 25 – 25 = 20

The trouble the author of the text had with the relative place value notation is evident from the examples above, where it is shown how the data were computed sometimes as if *u* = 30, 35, 4 05, etc., but other times as if *u* = ;30, ;35, 4;05, etc. Except for that, the author was quite successful with his computations of the data. In spite of the large number of computed numbers and the considerable amount of work obviously involved in the computations, there are very few errors.

One interesting error appears in the example above taken from § 3c, where the author of the text absentmindedly added 1 40 to 16 40 25 instead of subtracting it.

The text of §§ 1-6 is well organized. This is shown, quite convincingly, by the amazing regularity of the repertory of coefficients used in the problems of §§ 1-6, as displayed in the table below.

Table 9.5.1. A survey of the coefficients, square-sides and squares appearing in the text of 239 problems in §§ 1-6 of *TMS* V.

c	*u*	sq. *u*	§ 1a	§ 1b	§ 1c	§ 1d	§ 2a	§ 2b	§ 3a	§ 3b	§ 3c	§ 4a	§ 4b	§§ 4c-d	§ 6
1	30	15	+	[+]	[+]		+	[+]		+		+	[+]	+	
2			+	[+]	[+]		+	[+]	+	+	[+]	+	[+]	+	
3			+	[+]	[+]				+	+	[+]	+	[+]		
4			+	[+]	[+]				+	+	+	+	[+]		
2/3			+	[+]	[+]	+			[+]	+	+	+	[+]	+	
1/2			+	[+]	[+]	+			[+]	+	+	+	[+]	+	
1/3			+	[+]	[+]	+	+	[+]	[+]	+	+	+	[+]		+
4			+	[+]	[+]	+			+	+	+	+	[+]		+
1/3 4			+	[+]	[+]	+			+	+	+	+	[+]		+
7	35	20 25	+	[+]	[+]	+	+	[+]	+	+	+	+	[+]		+
2 7			+	[+]	[+]	+	+	[+]	+	+	+	+	[+]		
7 7	4 05	16 40 25	+	[+]	[+]	+	+	[+]	+	+	+	+	+		+
2 7 7			+	[+]	[+]	+	+	[+]	+	+	+	+	+		
11	55	50 25	+	[+]	+	+	[+]	+	+	+	+				
2 11			+	[+]	+	+	[+]	+	+	+	+				
11 11	10 05	1 41 40 25	+	[+]	+	+	[+]	+	+	[+]	+				
2 11 11			+	[+]	+	+	[+]	+	+	[+]	+				
11 7	6 25	41 10 25	+	[+]	+	+	[+]	+	+	[+]	+				
2 11 7			+	[+]	+	+	[+]	+	+	[+]	+				
2/3 1/2 1/3 11 7	12 50	2 44 41 40	[+]	[+]	+	+	[+]	+	+	[+]	+				
2 2/3 1/2 1/3 11 7			[+]	[+]	+	+	[+]		+	[+]	+				
1 2/3	30	15										+	+		
1 1/2												+	+		
1 1/3												+	+		
1 4												+	+		
1 1/3 4												+	+		
1 7	35	20 25										+	+		
1 2 7												+	+		
1 7 7	4 05	16 40 25										+	+		
1 2 7 7												+	+		
2 1/2	30	15										+	+		
3 1/3												+	+		
4 4												+	+		
7 igi 7	35	20 25										+	+		
7 2 igi 7												+	+		

The final paragraphs of this part of the text, §§ 4c-d and § 6 are surprisingly brief, with only a handful of coefficients used in each one of them. A likely explanation for this conciseness is that the author, or perhaps rather compilator, of *TMS* V wanted to use the remaining space on the tablet for the many problems in § 7. Note by the way, that while §§ 1-6 are a spectacular example of a well organized mathematical catalog text *with variations of parameters*, § 7 is instead an example of an *ordinary* well organized catalog text with variations of the setups of the problems.

The topic of the catalog text in § 7 is *metric algebra problems for two or three "concentric" squares*. Unfortunately, only six problems at the beginning and six in the last part of this paragraph are conserved, as is shown in Fig. 10.3.2. It would be very difficult to attempt to reconstruct all the lost problems, probably as many as fourteen. Below is shown the text of the preserved twelve problems.

§ 7a	30 lagab *ki-di-tum* 5 *me-se-tim* / lagab *qer-bi-tum mi-nu*	30 the outer square, 5 the distance. / The inner square was what?
§ 7b	20 lagab *qer-bi-tum* 5 *me-se-tum* /	20 the inner square, 5 the distance. / m-
	The text of this exercise is too badly preserved for any attempt to reconstruct it to be meaningful.	
	lagab *ki-di-tum mi-nu*	The outer square was what?
§ 7c	30 lagab *ki-di-tum* 20 lagab *qer-bi-tum* / ul.gar a.šag$_4$ 2 nigin *mi-nu*	30 the outer square, 20 the inner square. / The sum of the fields of the 2 squares was what?
§ 7d	a.šag$_4$ 2 nigin ul.gar-*ma* 21 40 / 30 lagab *ki-di-tum qer-bi-tum mi-nu*	The sum of the 2 squares I added together, then 21 40. / 30 the outer square. The inner was what?
§ 7e	a.šag$_4$ 2 nigin ul.gar-*ma* 21 40 / 20 lagab *qer-bi-tum ki-di-tum mi-nu*	The sum of the 2 squares I added together, then 21 40. / 20 the inner square. The outer was what?
§ 7f	a.šag$_4$ 2 nigin ul.gar-*ma* 21 40 / uš-*ši-na* gar.gar-*ma* 50 nigin *mi-nu*	The sum of the 2 squares I added together, then 21 40. / Their lengths I added together, then 50. The squares were what?
	………………	………………
§ 7a´	[8 20] a.šag$_4$ *dal-ba-ni* 5 *me-se-tum* / lagab *ki-di-tum ù qer-bi-tum mi-nu*	[20] the field of the intermediate space, 5 the distance. The outer and inner squares were what?
§ 7b´	20 a.šag$_4$ *dal-ba-ni* 7 lagab *ki-di-tim* / lagab *ki-di-tum qer-bi-tum mi-nu*	20 the field of the intermediate space, 1/7 of the outer square / the inner(!) square. The inner was what?
§ 7c´	16 40 a.šag$_4$ *dal-ba-ni* 7 7 *ki-di-tum* / lagab *ki-di-tum ù qer-bi-tum mi-nu*	20 the field of the intermediate space, 1/7 · 1/7 the outer square / <the inner square.> The outer and inner squares were what?
§ 7d´	30 lagab *ki-di-tum* 20 múr 10 lagab *qer-bi-tum* / a.šag$_4$ *dal-ba-an dal-ba-ni* *mi-nu*	30 the outer square, 20 the middle, 10 the inner square. / The field of the intermediate of the intermediate space was what?
§ 7e´	a.šag$_4$ {*dal-ba*} *dal-ba-an dal-ba-ni* 5 / uš-*ši-na* ul.gar-*ma* 1 nigin *mi-nu*	The field of the intermediate of the intermediate space 5. / Their lengths I added together, then 1. The squares were what?
§ 7f´	a.šag$_4$ *dal-ba-an dal-ba-ni* 5 / múr ugu lagab 10 dirig nigin *mi-nu*	The field of the intermediate of the intermediate space 5. / The middle over the <…> square 10 beyond. The squares were what?

Note that the term for 'add together' is gar.gar in *TMS* V §§ 1c and 3b, just as in the great majority of unprovenanced Old Babylonian mathematical texts. However, in *TMS* V § 7 it is ul.gar, precisely as in unprovenanced Old Babylonian mathematical texts of Group 6, texts from Sippar.

The objects considered in § 7 are pairs or triples of "concentric (parallel) squares" and the "square bands" between them, as in Fig. 10.3.3 below.

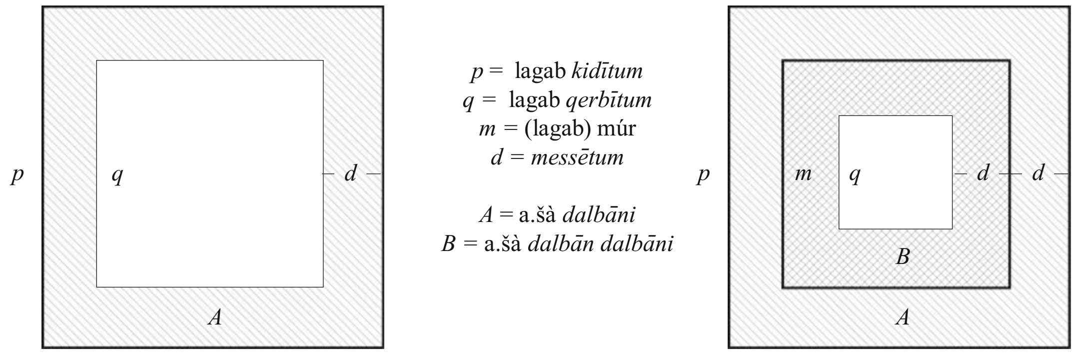


Fig. 10.3.3. *TMS* V, § 7. Concentric squares and square bands.

In terms of quasi-modern symbolic notations, explained in Fig. 10.3.3, the problems in § 7 can be reformulated as follows:

§ 7a	$p = 30$, $d = (p - q)/2 = 5$	$q = ?$	(20)
§ 7b	$q = 20$, $d = (p - q)/2 = 5$	$p = ?$	(30)
§ 7c	$p = 30$, $q = 20$	sq. p + sq. q = ?	(21 40)
§ 7d	sq. p + sq. q = 21 40, $p = 30$	$q = ?$	(20)
§ 7e	sq. p + sq. q = 21 40, $q = 20$	$p = ?$	(30)
§ 7f	sq. p + sq. q = 21 40, $p + q = 50$	$p, q = ?$	(30, 20)
…	…………	……………	
§ 7a´	A = sq. p – sq. q = [8 20], $(p - q)/2 = 5$	$p, q = ?$	(30, 20)
§ 7b´	A = sq. p – sq. q = 20, 1/7 $p = q$	$p, q = ?$	(35, 5)
§ 7c´	A = sq. p – sq. q = 16 40, 1/7 1/7 $p = q$	$p, q = ?$	(4 05, 5)
§ 7d´	$p = 30$, $m = 20$, $q = 10$	sq. m – sq. q = ?	(5 (00))
§ 7e´	B = sq. m – sq. q = 5 (00), $p + m + q$ = 1 (00), ($p = 30$)	$p, m, q = ?$	(30, 20, 10)
§ 7f´	B = sq. m – sq. q = 5 (00), $m - q = 10$	$m, q = ?$	(20, 10)

Here the problems in §§ 7a-c are simple introductory problems, setting the stage for the remaining problems. Also §§ 7d-e are uninteresting problems, solved by simple computations of square-sides (square roots). The problem in § 7f is an example of *an additive quadratic-linear system of equations of the basic type B2a* (Friberg, *AT* (2007), 6). Similarly, the problems in §§ 7a´ and 7f´ are examples of *negative quadratic-linear systems of equations of the basic type B3b* (Friberg, *op. cit.*). The problem in § 7e´ is probably of a similar kind, but it is underdetermined, lacking some crucial piece of information.

The problems in §§ 7b´-c´ were probably intended to be solved as follows:

§ 7b´: $p = 7\ q$, sq. p – sq. q = (49 – 1) · sq. q = A = 20, sq. q = rec. 48 · 20 = 1 15 · 20 = 25, $q = 5$, $p = 7 \cdot 5 = 35$.

§ 7c´: $p = 7 \cdot 7\ q = 49\ q$, sq. p – sq. q = (40 01 – 1) · sq. q = A = 16 40, sq. q = rec. 40 · 16 40 = 1 30 · 16 40 = 25, $q = 5$, p = 4 05.

An interesting observation in this connection is that in the following problem in § 7c´

16 40 a.šag$_4$ *dal-ba-ni* 7 7 *ki-di-tum* / lagab *ki-di-tum ù qer-bi-tum mi-nu*

the same abbreviated notation 7 7 with the meaning '1/7 · 1/7' is used as often in §§ 1-6. This is contradicting the otherwise tempting assumption that §§ 1-6 and § 7 were copied onto *TMS* V from two unrelated original texts, one a catalog text with variations of parameters, the other an ordinary catalog text!

Would it, by the way, have been possible to formulate variations of § 7c´ with 7 7 (= 1/7 · 1/7) replaced by 11 11 (= 1/11 · 1/11) or 11 7 (= 1/11 · 1/7)? The surprising answer is that it would not, for the following reason: Solution procedures for the modified problems imitating the proposed solution procedure for the problem in § 7c´ would begin with the following computations:

p = 11 · 11 q = 2 01 q, sq. p – sq. q = (sq. 2 01 – 1) sq. q.
p = 11 · 7 q = 1 17 q, sq. p – sq. q = (sq. 1 17 – 1) sq. q.

However, unlike sq. 49 – 1 = 40 (00), sq. 2 01 – 1 and sq. 1 17 – 1 are not regular sexagesimal numbers, so the attempted solution procedures cannot be completed as in the case of § 7c´. Indeed, it follows from the conjugate rule that

sq. 49 - 1 = (49 + 1) · (49 – 1) = 50 · 48, and both 50 and 48 are regular sexagesimal numbers.
sq. 2 01 - 1 = (2 01 + 1) · (2 01 – 1) = 2 02 · 2 (00), and 2 02 is not a regular sexagesimal number.
sq. 1 17 - 1 = (1 17 + 1) · (1 17 – 1) = 1 18 · 1 16, and neither 1 18 nor 1 16 is a regular sexagesimal number.

Another interesting observation in this connection is that it is almost certainly anachronistic to say that, for instance, 7 7 is an "abbreviated notation with the meaning '1/7 · 1/7'". Actually, igi *n* with a meaning like 'the *n*th part', where *n* was an integer, was probably *not understood in mathematical cuneiform texts as a number*. It is more likely that igi *n* was understood as *a mathematical operation*. So, for instance, in tables of reciprocals, igi *n* stood, at least initially, for the *n*th part of 1 (· 60). (See Ch. 12 below.) Besides, how could igi 7 be understood as a number, when 7 was sexagesimally non-regular, and when our concept of common fractions had not yet been invented? So, a less anachronistic way of interpreting a notation like 7 7 in *TMS* V could be as a short form of either 'the reciprocal of 7 · 7' or 'the 7th part of the 7th part of (whatever)'.

The text of *TMS* V ends, after § 7, with a colophon, mentioning first the number of problems:

4 22 mu.bi lagab.meš 4 22 (= 262) cases of (problems for) squares (squares).

Then follows a date and a couple of personal names. The whole text is called an im.gíd.da 'long tablet', a term usually, but not always, referring to table texts. (Also the 3-column text IM 121613 in Sec. 5.1, for instance, is called an im.gíd.da in its subscript.) The mentioned number of problems can be compared with the number of reconstructed problems in §§ 1-6 and 7 of *TMS* V (see Figs. 10.3.1-2), which is 239 + 26 = 265.

10.4 *TMS* VI. A Closely Related Catalog Text

The catalog text *TMS* VI was published and essentially correctly explained in Bruins and Rutten, *TMS* (1961). A part of the text is fairly well preserved and the missing parts of the text on both obverse and reverse can easily be reconstructed, as in Figs. 10.4.1-2 below.

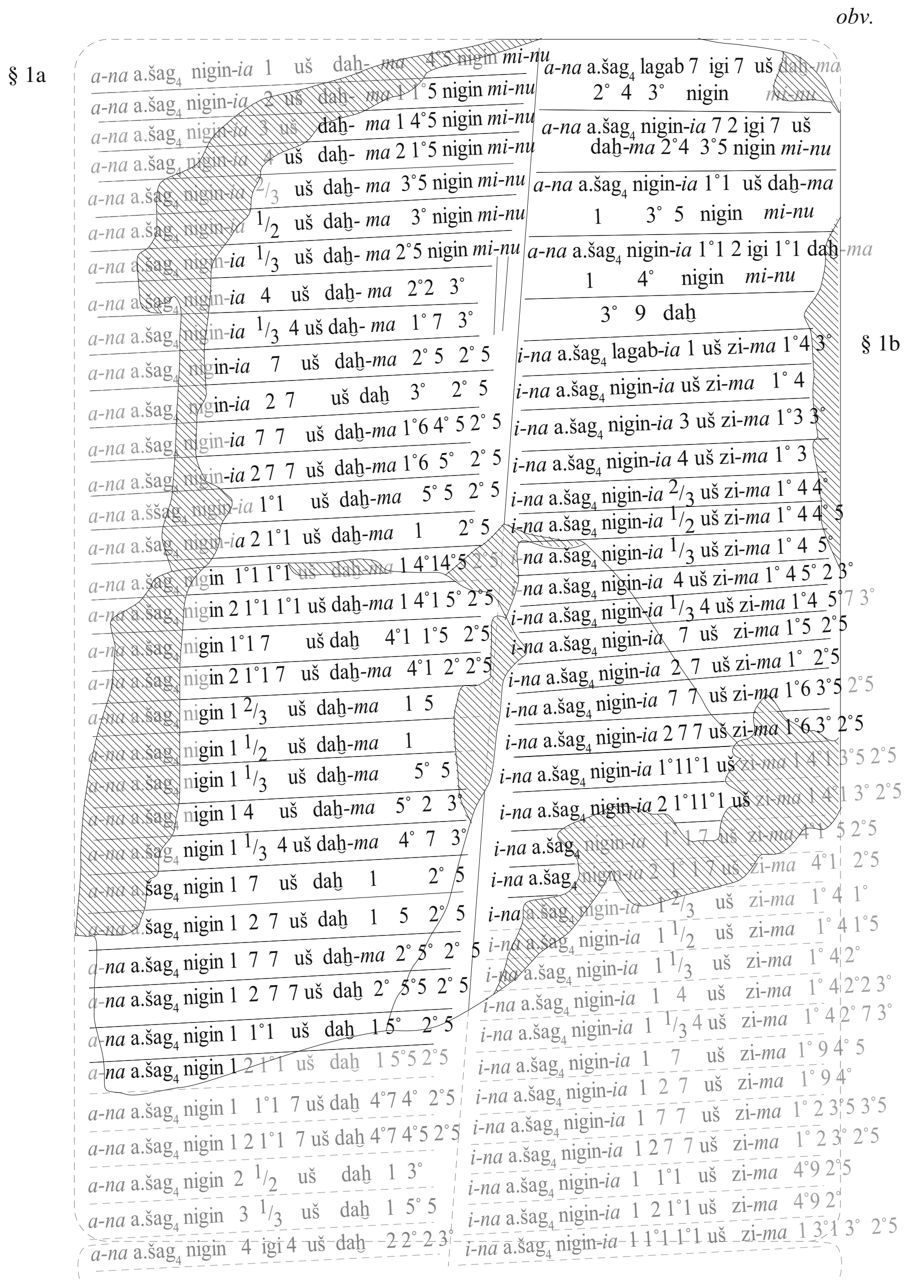


Fig. 10.4.1. *TMS* VI, *obv.* A catalog text for squares, with variations of parameters. A parallel to *TMS* V.

rev.

§ 1b ctd.

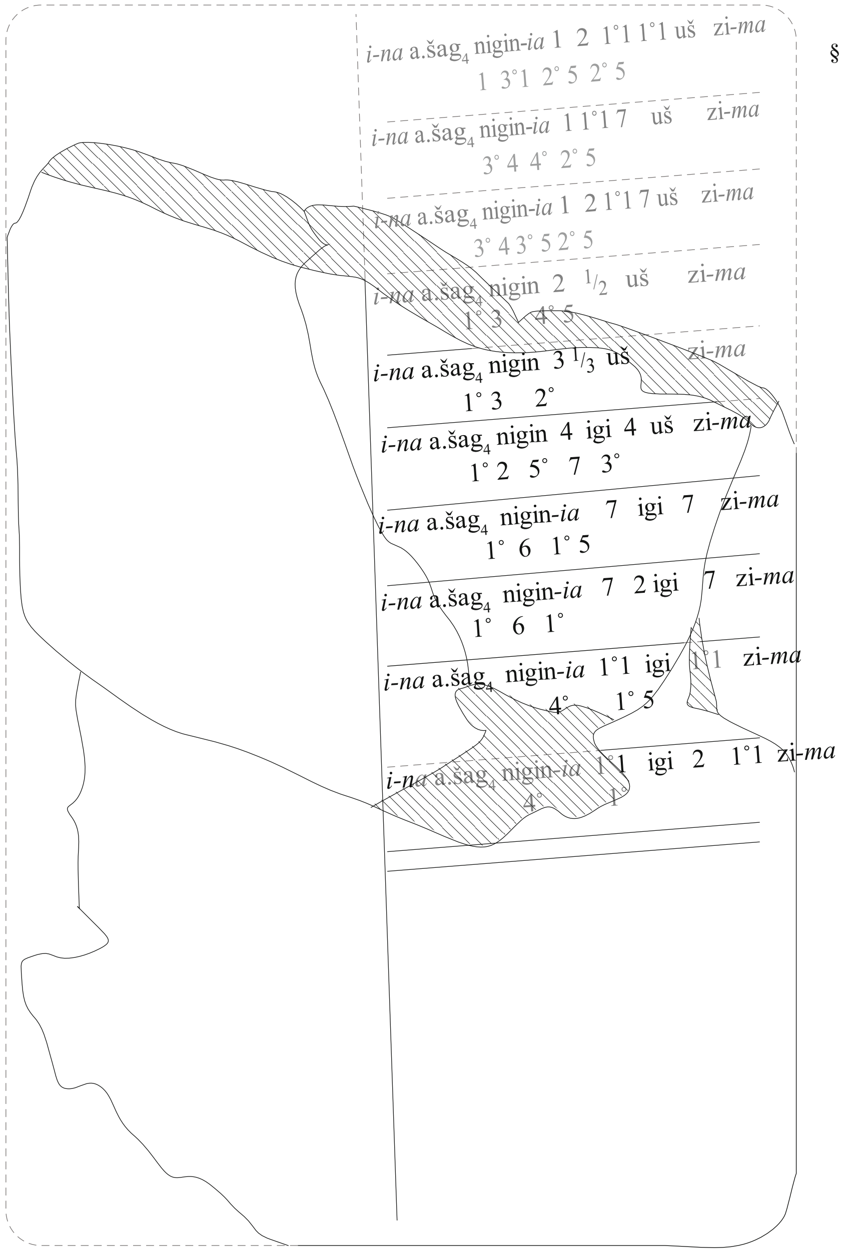


Fig. 10.4.2. *TMS* VI, *rev.*

In condensed form, the questions in §§ 1a-b in *TMS* VI can be reproduced as follows, in transliteration and literal translation:

§ 1a	*a-na* a.šag$_4$ nigin-*ia* (c) uš daḫ-*ma* (B)	To the field of my squares c (of) the length I added on, then B.
§ 1b	*i-na* a.šag$_4$ nigin-*ia* (c) uš zi-*ma* (B)	From the field of my squares c (of) the length I tore off, then B.

Essentially, these two paragraphs in *TMS* VI are the same as §§ 4a-b in *TMS* V. However, there are certain differences, which together clearly show that the text of *TMS* VI is not simply an excerpt from the text of *TMS* V. One difference is that the cuneiform sign nigin (an image of two squares) almost everywhere takes the place of lagab (an image of a single square). Another difference is that in *TMS* VI the phrases

a-na a.šag$_4$ nigin-*ia* (c) uš daḫ-*ma* (B) and *i-na* a.šag$_4$ nigin-*ia* (c) uš zi-*ma* (B)

are always written out in full, not in an abbreviated form as in all lines except the initial ones in § 4a [and 4b] of *TMS* V. A third difference is that the repertory of coefficients in §§ 1a-b of *TMS* VI is an expanded version of that of coefficients in *TMS* V, §§ 4a-b. Indeed, at the end of § 1a in *TMS* VI there is a subscript saying

39 daḫ 	'39 (problems with) add on',

which means that in § 1a there were (according to the author of the text) 39 cases of *additive quadratic equations (of the basic type B4a).* In § 4a of *TMS* V, on the other hand, there were only 27 cases of such equations.

There are various mistakes and omissions in the text of *TMS* VI (all pointed out in the original publication of the text in *TMS* (1961)).

11. Three Old Babylonian Recombination Texts of Mathematical Problems without Solution Procedures, Making up Group 2b

11.1 YBC 4698. An Old Babylonian Numbered Recombination Text with Commercial Problems

11.1.1 An Old Babylonian Mathematical Text with Unusual Sumerian Terminology

YBC 4698 is *a mathematical recombination text,* a mixed bag of loosely related exercises, all concerned with commercial problems. See the hand copy and conform transliteration in Fig. 11.1.1 below. Since exercises belonging more closely together are sometimes separated from each other in this text, the exercises are numbered in two different ways in the conform transliteration, both in the order they are inscribed on the tablet (## 1-17) and according to their problem types (§§ 1-5).

The long history of the decipherment and explanation of YBC 4698 is surprisingly complicated and is well worth narrating in full detail, which will be attempted below.

YBC 4698 was first published by Neugebauer in *MKT III* (1937), 42 ff., more than 75 years ago. Although the text is almost perfectly preserved, only 2 of its 17 mathematical exercises were correctly interpreted by Neugebauer. The reason for his failure is easy to understand. The text was, and still is, the only one of its kind. It is written entirely in Sumerian, except for the occasional *ù* 'and' and *-ma* 'and then', with Sumerian logograms sometimes used in unusual ways to denote various kinds of commercial transactions. Moreover, YBC 4698 is a mathematical recombination text with briefly worded questions, only occasionally with explicit answers, but always without solution procedures. The recombination text includes several only superficially connected types of mathematical exercises with a commercial background, some of them with no or very few parallels in the whole presently known corpus of Old Babylonian mathematical texts.

The only exercises in YBC 4698 understood by Neugebauer were ## 1-2, completely uninteresting introductions to more advanced exercises concerning principal and interest. Thureau-Dangin, Neugebauer's competitor as an interpreter of Babylonian mathematical texts, was more successful. In his review in *RA* 34 (1937) of Neugebauer's *MKT III*, he explained most of the terminology in exercises # 16 (§ 5 a), # 3 (§ 2 a), and ## 6-8 (§§ 4 a-c). In another review of *MKT III,* in *AfO* 12 (1937), Waschow correctly observed a similarity between exercise # 12 (§ 2 b) in YBC 4698 and VAT 7530 (Fig. 11.1.6 below), a brief theme text with six exercises, all without solution procedures and answers. VAT 7530 had been discussed by Thureau-Dangin the year before, in *RA* 33 (1936), 162, with limited success.

In Neugebauer and Sachs, *MCT* (1945), 17-20, a clear and interesting connection was established between the data appearing in the six exercises on VAT 7530 and eight small square hand tablets from the Yale Babylonian Collection (YBC 7234, etc.), all inscribed with numbers ordered into tabular arrays. However, what was really going on in all these texts was far from clear.

In the discussion in Friberg's *Survey* (1982), 57 of Thureau-Dangin's review of *MKT III*, the "profit problem" in exercise # 9 (§ 4 d) was explained. In Friberg's *Lecture Notes* (1984), 108-117, attempts were made to explain all the exercises in YBC 4698, but the unusual terminology still remained partly

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7_11

unexplained. Then, in § 5.2 h of Friberg's article on "Mathematics" in *RlA 7* (1990), the eight Yale tablets with tabular arrays of numbers were explicitly explained as "help tables" for combined market rate problems, a kind of problems which in their turn were mentioned in § 5.6 i of the same article, and which are precisely the kind of problems appearing in VAT 7530 and in §§ 2 a-e of YBC 4698.

The discussion of YBC 4698 was continued in Friberg, *AT* (2007), 67, where a new interpretation was suggested for exercise # 4 (§ 3 a). Finally, in Sec. 7.2 of Friberg *MSCT 1* (2007), the mentioned regular arrays of numbers in YBC 7234 were discussed and compared with six new texts of the same kind (MS 2830, etc., and the Nippur text N 3914, published without any valid explanation in Robson, *SCIAMVS* 1 (2000)). The same section of *MSCT 1* contained a renewed discussion of VAT 7530, explained as a "theme text with combined market rate problems".

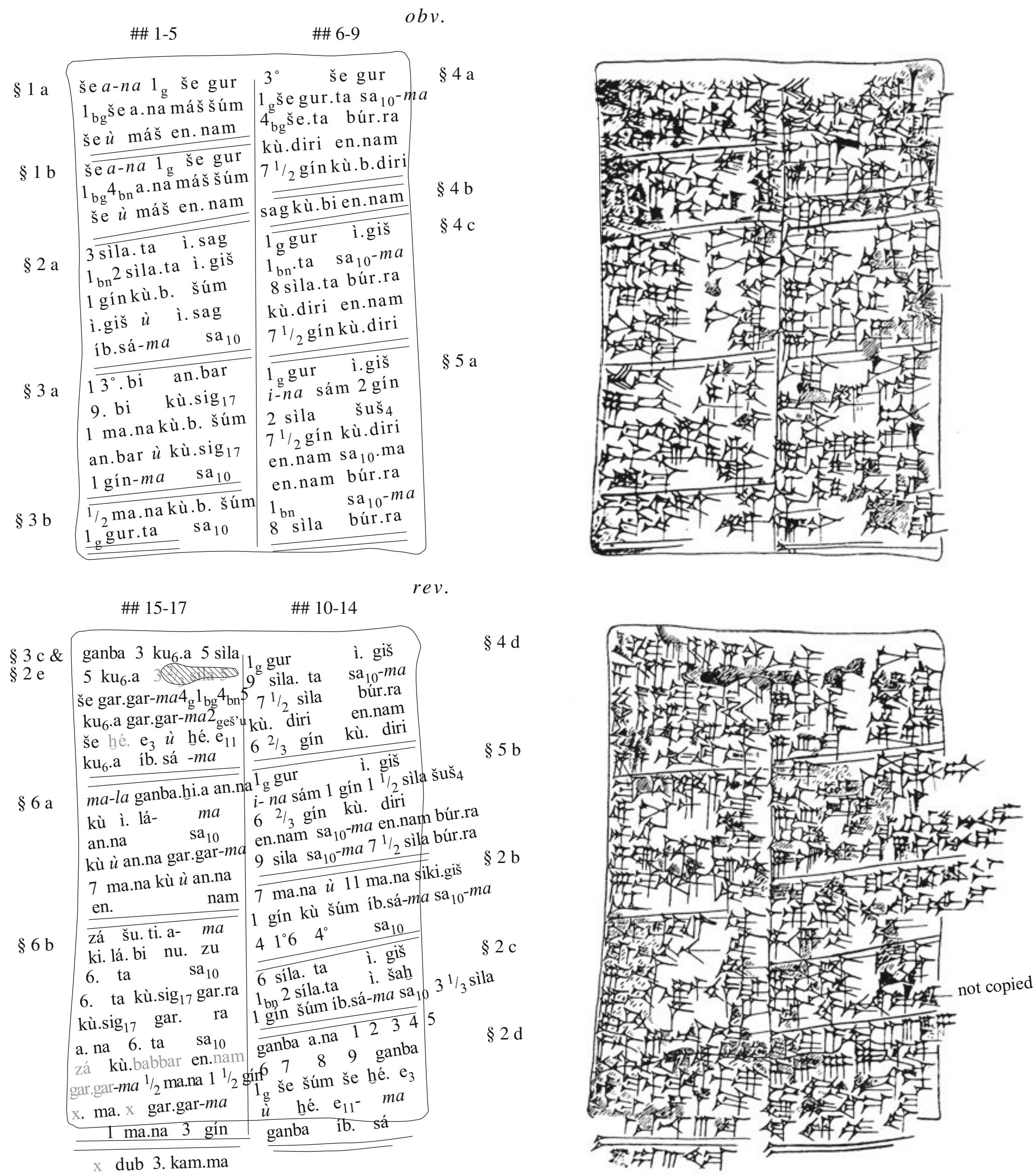


Fig. 11.1.1. YBC 4698. An Old Babylonian numbered recombination text with commercial exercises.

Hand copy by O. Neugebauer, *MKT III*. Reproduced here with the permission of Springer.

(Photo online at cdli.ucla.edu/P255010.)

A big step forward in the attempted interpretation of YBC 4698 was taken recently with Proust and Middeke-Conlin, *CDLJ* 2014:3, an online publication where the remaining undeciphered terms of the text were finally successfully explained, where all the questions of the text were considered together for the first time, and where it was also for the first time observed that a capacity number inscribed on the edge of the clay tablet was missing in Neugebauer's hand copy of the clay tablet. However, since some of the explanations proposed by Proust and Middeke-Conlin for the individual exercises in the text can be contested, it may be justified to make a renewed effort below to explain again together all the seventeen exercises on YBC 4698. Also the "Conclusion" in § 7 of Proust and Middeke-Conlin's paper may be questioned. There it is claimed that

> "This set of 17 statements presents a strong consistency, and most of the statements appear as variations of a basic mathematical sketch. …… Some of the uncertainties of interpretation originate in our lack of pieces of information. This probably results from the fact that this tablet is just one tablet in a series, and that relevant information may have been provided in the first two tablets of the series. …… It must be noted that some of the statements have parallels in the known corpus of mathematical procedure texts and school exercises. This means that the author of the tablet reused old mathematical material. But, unlike catalogue texts, the list of statements noted on YBC 4698 doesn't seem to be a compilation gathering statements from different sources, but rather a systematic elaboration of new material from old material. This is a typical feature of series texts."

Contrary to what is stated in the cited conclusion, it may be argued that YBC 4698 is a typical example of a *mathematical recombination text* (above called a "compilation"). That YBC 4698 is not a "systematic elaboration" of new material from old material is obvious, for instance, from the fact that the questions in ## 1-2 have no relation at all to the subsequent questions. Instead they look precisely as the introductory paragraph to a theme text about a completely different topic, namely principal and interest.

It is also not evidently true that "most of the statements appear as variations of a basic mathematical sketch". This ought to be obvious from the following list of topics in YBC 4698:

§ 1	Two descriptions of commonly used interest rates
§ 2	Five *combined market rate problems (solved by the rule of false value)*
§ 3	Three *systems of linear equations (solved by the rule of double false value)*
§ 4	Three *computations* of profit (solved by simple arithmetic)
§ 5	Two inverse profit problems (leading to *rectangular-linear systems of equations*)
§ 6	Two simple *linear equations* (solved by simple arithmetic)

Answers but *no solution procedures* are given in the text.

The question concerning what is "a typical feature of a series text " will be addressed below, in Sec. 11.4.

11.1.2 § 1. Two Commonly Used Interest Rates

		YBC 4698 § 1 a	(## 1-2, *obv. i*: 1-6)
§ 1 a	1-3	še *a-na* 1(gur) še gur / 1(barig) še a.na máš šúm / še *ù* máš en.nam	
	1-3	Grain. For 1 gur of grain / 1 bariga of grain as interest I paid. / Grain and interest were what?	
§ 1 b	1-2	še *a-na* 1(gur) še gur / 1(barig) 4(bán) an.a máš šúm / še *ù* máš en.nam	
	1-2	Grain. For 1 gur of grain / 1 bariga 4 bán of grain as interest I paid. / Grain and interest were what?	

In lines 1-2 of§ 1 a, it is stated that the (annual) interest charged on a loan of grain with the principal 1 gur is 1 barig. Since in the Old Babylonian period 1 gur = 5 barig, this means that *in § 1 a the (annual) interest on a loan of grain is 1/5 of the principal*.

In lines 1-2 of § 1 b it is stated instead that the interest charged on a loan of grain with the principal 1 gur is 1 bariga 4 bán. Since in both the Old Babylonian and other periods 1 bariga = 6 bán, it follows that 1 gur = 30 bán and 1 bariga 4 bán = 10 bán. Therefore, *in § 1 b the (annual) interest on a loan of grain is 1/3 of the principal.*

This is in agreement with the Code of Hammurabi (S i 4'-12'; t iii 35-40), where it is stated that

“If a merchant gives grain or silver against interest, he shall for 1 gur take 1 bariga 4 bán of grain as interest.” (= 1/3)
“If he gives silver against interest, he shall for 1 shekel of silver take 1/6 (shekel) 6 grains as interest.”(= 1/5)

Indeed, since 1 shekel was equal to 3 00 (180) grains, it follows that

1/6 (shekel) 6 grains = 36 grains = 1/5 of 1 shekel.

The seemingly meaningless question in the mentioned couple of exercises YBV 4698 § 1 a-b can be interpreted as meaning ‘What is the *sum* of principal and interest?’ As a matter of fact, in VAT 8528, discussed immediately below, for instance, the phrase kù.babbar *ù* máš.bi is used repeatedly with the meaning ‘the silver *plus* the interest’.

It is clear that exercises § 1 a-b once were the deliberately simple introductory exercises of some Old Babylonian metro-mathematical theme text with the theme ‘principal and interest’. No such theme text is known, but some relatively advanced isolated exercises from one or more theme texts of that kind are known.

One example is

YBC 4669 # B 11 (*rev. iii*: 18-11; Neugebauer *MKT I*, 516 and *MKT III*, 28)

kù.babbar *a-na* 1 ma.na / 12 gín a.na máš šúm.*ma* /
i-na mu 3.kam.ma / *al-li-ik-ma* / 1 gín kù.babbar *el-qé-a* /
sag kù.babbar en.nam / 34 43 20

Silver. For 1 mina / 12 shekels he paid as interest. /
In the 3rd year / I went, 1 shekel of silver I received. /
The silver principal was what? / 34 43 20.

In this exercise, the annual interest charged on a loan of silver with the principal 1 mina is 12 shekels or 1/5 of the principal. After 3 years an unknown principal plus the compounded interest amounts to 1 shekel. How much was then the principal? The question is easy to answer. Namely, if the principal is called P, then

cube (1;12) · P = 1 shekel, so that P = cube ;50 (shekels) = ;34 43 20 (shekels) = ;34 43 20 (shekels).

If a proper conversion to a weight measure of this numerical answer had been attempted, it would have given (after round-off) the result that

P = ;34 43 20 shekels = 1/2 shekel 14;10 grains = (approximately) 1/2 shekel 14 grains.

Another example is

AO 6770 # 2 (Neugebauer *MKT II,* 37)

1-2 a.ga.na 1(gur) gur *a-na ṣi-ba-at i-di-in-ma* / *i-na ki ma-ṣi ša-na-tim li-im-ta-ḫa-*[*ru-ú*] /

1-2 Now, 1 gur he gave for interest, then / in how many years shall they be equal?

In other words, after how many years will the compounded interest be equal to the principal? (The interest rate is not explicitly mentioned, but in the solution procedure it is silently understood that the annual interest is 1/5 of the principal, as in YBC 4698, § 1 a.)

Even more interesting and complicated examples are the following two:

VAT 8528 (Neugebauer *MKT I*, 353, *MKT II* pl. 56, and *MKT III*, 59; Thureau-Dangin *RA* 33, 65-68)
only the questions and answers are shown below

\# 1 [1 ma.n]a kù.ba[bbar] / [*a-na* 1] ma.na 12 gí[n] / [a.na máš.b]i in.na.šúm-*ma* /
[kù *ù* m]áš.bi 1(gú) gú 4 ma.n[a] / [šu].ba.an.ti
ki ma-ṣi u_4*-mi* / *iš-ša-ki-in* /
…………
mu 30.kam *ša* 1 04 kù *ù* máš.bi

\# 3 10 še.gur *a-na* 1(gur) gur 1(barig) še / an.a ḫar.ra *ad-di-in-ma* /
1(bán) še.ta *i-na* ud 1.kam / mu 1.kam *el-te-qé* /

ša še-am ad-di-nu-šum / še-um ša te-el-te-qú-ú / ṣí-ib-ta-am x x ú-ul i-šu-⌜ú⌝ /
še-am ki ma-ṣi e-li-ia ir-ši /
…………
6 01 *še-am e-li-ia ir-ši*

1 [1 min]a of silver. / [For 1] mina 12 shekels / [as it]s [interest] he paid, then /
[silver plus in]terest 1 talent 4 minas / I received.
For how many days / did he deposit it?
…………
It was the 30th year that 1 04 (minas) was silver plus interest.

 # 3 10 gur of grain. For 1 gur 1 bariga of grain / as interest I gave, then /
1 bán of grain per day / for 1 year I received. /
He whom I gave grain, / (if for) the grain that I(!) received there is no x x interest /
How much grain can he claim from me?
…………
6 01 is the grain that he can claim from me.

The solution procedure in VAT 8528 # 1 is based on the simplifying assumption that interest was calculated and added to the principal only every fifth year, so that the silver owed by the borrower was doubled every fifth year. The question and the solution procedure in VAT 8528 # 3 are more interesting. As shown by Thureau-Dangin (*op. cit.*), the somewhat vaguely worded question can be interpreted as follows:

> 10 gur of grain was borrowed from me, at an annual interest rate of 1 bariga per gur (= 1/5). The grain was repaid to me at a rate of 1 bán per day for a full year (of 12 · 30 = 6 00 days). If no interest was charged for the repaid grain, what was I owing at the end of the year?

The solution procedure starts by computing the principal plus interest of the original loan, which in quasi-modern terms can be expressed as

10 gur · 1;12 = 50 00 sìla · 1;12 = 1 00 00 sìla = 6 00 · 10 sìla = 6 00 · 1 bán.

(Remember that 1 gur = 5 barig, 1 bariga = 6 bán, 1 bán = 10 sìla, and 1 sìla = 60 shekels.) This means that the loan actually can be repaid at a rate of 1 bán per day for a full year. However, the borrower should not be paying interest for grain that he has already returned. Another way of saying this is that interest should be paid by me, the original loan-giver, for the returned grain.

For the first day's bán of grain, a full year's interest should be paid, for the second day's bán interest for 1 year minus 1 day should be paid, and so on, until only 1 day's interest should be paid for the last day's bán. Now, 1 year's interest for 1 bán of grain is 1/5 of 1 bán or 2 sìla, and 1 day's interest for 1 bán is 2 sìla / 6 00 = ;00 20 sìla. Therefore, the total amount of grain owed by the original loan-giver can be expressed as the sum of an arithmetic progression with 6 00 terms, where the first term is ;00 20 (sìla) and the last term 2 (sìla). Indeed, the sum is computed correctly in the solution procedure of VAT 8528 # 3 as follows (here in quasi-modern terms):

(2 + ;00 20)/2 · 6 00 (sìla) = 1;00 10 · 6 00 (sìla) = 6 01 (sìla).

In other words, the excess repayment was 6 01 sìla = 1 gur 1 bariga 1 sìla. However in the text this final result is not computed absolutely correctly. Instead the computed number '6 01' is multiplied, without motivation, by '1', standing for the original principal plus interest. This error does not change the final result which still can be expressed as '6 01' in relative sexagesimal place value notation. Thureau-Dangin's comment to the mentioned false final step is the following (in French):

> "This mistake is certainly his own (the student's), and the only conclusion that one may draw from this is that he had badly understood the lesson he had listened to."

A final example of an advanced problem about principal and interest is the following Old Babylonian mathematical cuneiform text from Susa, published by Bruins and Rutten in *TMS* (1961). It is an unusually puzzling and interesting text, which was inadequately explained by Bruins. Therefore, it is well motivated to take a renewed look at it here.

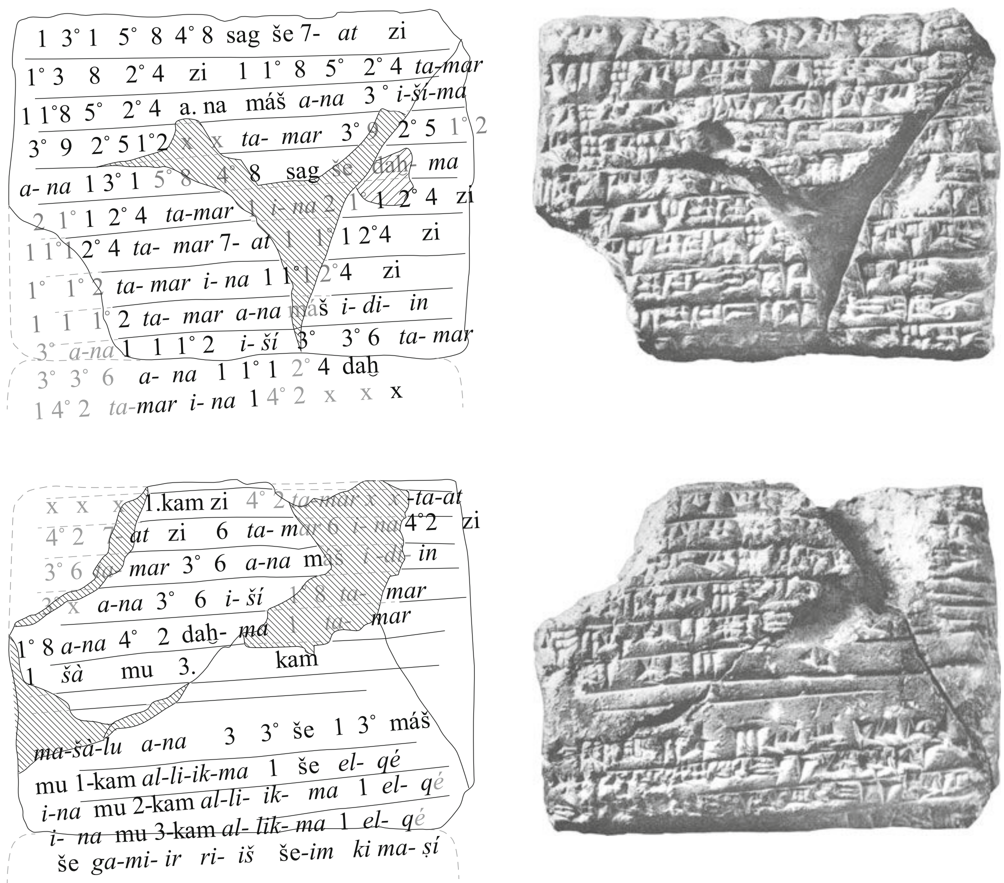


Fig. 11.1.2. ***TMS*** **XXII. An Old Babylonian profit, interest, and annuity exercise from Susa.**

TMS **XXII** (Bruins and Rutten 1961)

obv. 1 1 31 58 48 sag še
2 7-*at* zi / 13 08 24 zi 1 18 50 24 *ta-mar* /
3-4 1 18 50 24 a.na máš *a-na* 30 *i-ši-ma* / 39 25 12 [*x x*] *ta-mar* /
5-6 39 25 12 / *a-na* 1 31 [58 4]8 sag [še daḫ]-*ma* / [2 1]1 24 *ta-mar*
7 [1 *i-na* 2 1]1 24 zi / [1 11] 24 *ta-mar*
8 7-*at* [1 1]1 24 zi / [10 12] *ta-mar*
9 *i-na* 1 1[1 2]4 zi / [1 01 1]2 *ta-mar* a.na [má]š *i-di-in* /
10 [30 *a-na*] 1 01 12 *i-ši* [30 36] *ta-mar* /
11-12 [30 36] *a-na* 1 11 [2]4 daḫ / [1 42 *ta*]-*mar*
rev. 1 *i-na* 1 42 [x x x] / [x x x] 1.kam zi [42 *ta-mar ši*]-*ta-at* /
2-3 [42 7]-*at* zi 6 *ta-m*[*ar* 6 *i-na*] 42 zi / [36 *ta*]-*mar* 36 a.na m[áš] *i-di-in* /
4 30 [x] *a-na* 36 *i-ši* [18 *ta*]-*mar*
5-6 18 *a-na* 42 daḫ-*ma* [1 *ta*]-*mar* / 1 *šà* mu 3.kam

==

7 *ma-ša-lu*
8 *a-na* 3 30 še 1 30 máš / mu 1.kam *al-li-ik-ma* 1(barig) še *el-qé* /
9 *i-na* mu 2.kam *al-li-ik-ma* 1 *el-q*[*é*] /
10-11 *i-na* mu 3.kam *al-lik-ma* 1 *el-q*[*é*] / še *ga-mi-ir*
re-eš še-*im ki ma-ṣí*

obv. 1 1 31 58 48 is the original grain.
2 A 7th tear off / 13 08 24 tear off 1 18 50 24 you will see. /
3-4 1 18 50 24 as interest to 30 carry, then / 39 25 12 you will see. /
5-6 39 25 12 / to 1 31 [58 4]8 the original grain add on, then / [2 1]1 24 you will see.

7 [1 from 2 1]1 24 tear off / [1 11] 24 you will see.
8 A 7th of [1 1]1 24 tear off / [10 12] you will see.
9 From 1 1[1 2]4 tear it off / [1 01 1]2 you will see, as [intere]st he gave. /
10 [30 to] 1 01 12 carry, [30 36] you will see. /
11-12 [30 36] to 1 11 [2]4 add on, / [1 42 you] will see.
rev. 1 From 1 42 [x x x x] / [x x x] the 1st tear off, [42 you will see, the remainder] /
2-3 [42 a 7]th, 6 you will [see. 6 from] 42 tear off, / [36 you]]will see. 36 as inte[rest] he gave. /
4 [x x] 30 to 36 carry, [18 you] will see
5-6 18 to 42 add on, then [1 you] will see, / 1, that of the 3rd year.

7 To make equal.
8 For 3 30 of grain, 1 30 interest / the 1st year I went, then 1(barig) of grain I received. /
9 In the 2nd year I went, then 1(barig) I re[ceived]. /
10-11 In the 3rd year I went, then 1(barig) I recei[ved]. / The grain was completed.
The original grain was how much?

The strange layout of this text is obvious. It begins in the middle of some mathematical exercise and proceeds to the end of that exercise, marked by a double ruled line. After that there is a question, apparently the beginning of another mathematical exercise. However, as it turns out upon a closer examination, it is the beginning of the same exercise!

It is not difficult to come up with a plausible explanation for how a text like this may have been written. Presumably, a student set out to improve his writing skills by making an exact copy of an older mathematical cuneiform text dealing with principal and interest. Unfortunately, the clay tablets of a certain standard format which were available to write the copy on were smaller than the original clay tablet, so two tablets were needed for the copy. When there turned out to be some empty space remaining on the second clay tablet after the copy had been completed, the diligent student simply started from the beginning again, filling all available space on the second tablet with writing.

Now, the first of the two tablets with the copied text has not been preserved, so we are left with only the second clay tablet which, by a lucky coincidence, happens to contain *the whole verification of the solution procedure on the original clay tablet, followed by the whole question on the same tablet*!

The best way of dealing with this text is, of course, to start with the question in the last five lines of the reverse. That question begins with the heading *mašālu* 'to become equal'. (A related, well known mathematical term is *mišlu* 'half'.) As will be shown below, what is to be made equal are the principal plus the compounded interest minus the annuities, that is, the yearly payments. The rate of interest is awkwardly and unconventionally described in the following way:

a-na 3 30 še 1 30 máš 'for 3 30 of grain, 1 30 interest'.

In other words, the annual interest on a principal of, say, 3;30 gur of grain is 1;30 gur. However, this is not the way in which interest is computed in the remaining part of the text. The alternative expression can be explained as follows: 3 30 is equal to 7 times 30 and 1 30 is equal to 3 times 30. Therefore,

3 30 – a 7th of 3 30 = 6 · 30 = 3, and consequently (3 30 – a 7th of 3 30) times 30 = 1 30.

This means that

If a 7th is torn off the principal and if the remainder is halved, the result will be the interest.

In modern terms, the reasoning can be explained, anachronistically, as follows, by use of common fractions:

1;30/3;30 = 3/7 = (1 – 1/7)/2.

The question continues by stating, essentially, that

With the mentioned interest rate, and with a yearly payment of 1 bariga as annuity, the loan was repaid after 3 years.
What was the grain principal?

Since the solution procedure is missing in this text, but not the verification, an effort will be made below to recreate the solution procedure, using the same kind of arguments as in the verification. In quasi-modern terms, let C be the original grain principal at the beginning of the first year, let C' be the principal at the beginning of the second year, after adding one year's interest and subtracting the first annuity payment, and let C'' be the principal at the beginning of the second year. Then C, C', and C'' can be computed as the solutions to the following *chain of linear equations*:

$$C + C \cdot (1 - 1/7) \cdot 30 = 1 + C',$$
$$C' + C' \cdot (1 - 1/7) \cdot 30 = 1 + C'',$$
$$C'' + C'' \cdot (1 - 1/7) \cdot 30 = 1.$$

These equations can all be simplified through multiplication by 7. The result is the new chain of linear equations

$$7\,C + C \cdot 6 \cdot 30 = 7 + 7\,C', \quad \text{or} \quad 10\,C = 7 \cdot (1 + C'),$$
$$7\,C' + C' \cdot 6 \cdot 30 = 7 + 7\,C'', \quad \text{or} \quad 10\,C' = 7 \cdot (1 + C''),$$
$$7\,C'' + C'' \cdot 6 \cdot 30 = 7. \quad \text{or} \quad 10\,C'' = 7.$$

The solutions to this chain of linear equations are easily computed in the following three steps:

$$C'' = 7 \cdot \text{rec. } 10 = 7 \cdot 6 = 42,$$
$$C' = 7 \cdot 6 \cdot (1 + C'') = 7 \cdot 6 \cdot 1\ 42 = 7 \cdot 10\ 12 = 1\ 11\ 24,$$
$$C = 7 \cdot 6 \cdot (1 + C') = 7 \cdot 6 \cdot\ \ 2\ 11\ 24 = 7 \cdot 13\ 08\ 24 = 1\ 31\ 58\ 48.$$

In metrological terms,

C = 1 31;58 48 sìla = (approximately) 1 bariga 3 bán 1 5/6 sìla 26 1/2 grains.
C' = 1;11 24 sìla = 1 bariga 1 bán 1 1/3 sìla 12 grains,
C'' = 42 sìla = 4 bán 2 sìla.

The verification of the exercise, in the preserved part of the text of *TMS* XXII, starts with the computed principal C = 1 31 58 48, and then continues, essentially, as follows:

$$C + C \cdot (1 - 1/7) \cdot 30 = 1\ 31\ 58\ 48 + 1\ 18\ 50\ 24 \cdot 30 = \ \ 1\ 31\ 58\ 48 + 39\ 25\ 12 = 2\ 11\ 24 = 1 + 1\ 11\ 24 = 1 + C',$$
$$C' + C' \cdot (1 - 1/7) \cdot 30 = 1\ 11\ 24 + 1\ 01\ 12 \cdot 30 = 1\ 11\ 24 + 30\ 36 = 1\ 42 = 1 + 42 = 1 + C'',$$
$$C'' + C'' \cdot (1 - 1/7) \cdot 30 = 42 + 36 \cdot 30 = 42 + 18 = 1.$$

11.1.3 § 2. Oil, Wool, Fishes. Combined Market Rate Problems

YBC 4698 § 2 (# 3, *obv. i*: 7-11 and ## 12-15, *rev. i*: 11-*ii*: 6)

§ 2 a	1-3	3 sìla.ta ì.sag / 1(bán) 2 sìla.ta ì.geš / 1 gín kù.babbar šúm /
	4-5	ì.geš *ù* ì.sag / íb.sá-*ma* sa$_{10}$
	1-3	At 3 sìla of fine oil / at 1 bán 2 sìla of common oil / 1 shekel of silver I paid. /
	4-5	Common oil and fine oil I made equal, then I bought.
§ 2 b	1-2	7 ma.na *ù* 11 ma.na siki.geš / 1 gín kù šúm
	3	íb.sá-*ma* sa$_{10}$-*ma* / 4 16 40 sa$_{10}$
	1-2	(At) 7 minas and 11 minas of common(?) wool (I bought) / 1 shekel of silver I paid,
	3	I made equal and bought, then / 4 16 40 I bought.
§ 2 c	1-3	6 sìla.ta ì.geš / 1(bán) 2 sìla.ta ì.šaḫ / 1 gín kù šúm íb.sá-*ma* sa$_{10}$ 3 1/3 sìla
	1-3	At 6 sìla of common oil / at 1 bán 2 sìla of lard / 1 shekel of silver I paid, I made equal, then I bought. 3 1/3 sìla.
§ 2 d	1-3	ganba a.na 1 2 3 4 5 / 6 7 8 9 ganba / 1(gur) še šúm
	4-5	še ḫé.e$_3$ / *ù* ḫé.e$_{11}$-*ma* / ganba íb.sá
	1-3	Market rates. As much as 1 2 3 4 5 / 6 7 8 9 were the market rates / 1 gur of grain I paid.
	4-5	The grain may rise / or fall, but the market rates I made equal.
§ 2 e	1-2	ganba 3 ku$_6$.a 5 sìla / 5 ku$_6$.a [3 2/3 sìla 5]? /
	3-4	še gar.gar-*ma* 4(gur) 1(barig) 4(bán) ⸢5⸣ / (ku$_6$.a gar.gar-*ma* 2(geš'u) /)
	5-6	še [ḫé].e$_3$ *ù* ḫé.e$_{11}$- / *ma* / ku$_6$.a íb.sá
	1-2	Market rates. 3 fishes 5 síla, 5 fishes [3 2/3 sìla 5]? /
	3-4	The grain I added together, then 4 gur 1 bariga 4 bán ⸢5⸣ <sìla>. / (The silver I added together, then 2(10 · 60).)
	5-6	The grain may rise or fall, / but / the fishes I made equal.

All the questions in § 2 of YBC 4698 are examples of combined market rate problems (see Friberg, *RlA 7* (1990), § 5.6 i and Friberg, *MSCT 1* (2007), Sec. 7.2). Old Babylonian texts concerned with combined market rate problems are usually in the form of small hand tablets with tabular arrays of numbers, in which the *market rates* (units per shekel of silver or sìla of grain) for several commodities are listed in col. *i*, the corresponding *unit prices* (shekels or sìla per unit) in col. *ii*, the prices paid for N units of each commodity in col. *iii,* and the number N of units bought, the same for all commodities, in col. *iv*. The number N is computed so that the combined price will be a given amount of silver, or grain.

YBC 4698 § 2 a (# 3)

In the case of YBC 4698 § 2 a, there are two given market rates, 3 and 12 sìla of fine and common oil, respectively, per shekel of silver, and the total amount of silver available 1 shekel. There is no explicit question, no solution procedure, and no answer in the text of this paragraph.

Although YBC 4698 § 2 contains no solution procedure, it can be shown that the solution to this problem is that the price for 1 sìla of fine oil is ;20 shekel, while the price for 1 sìla of plain oil is ;05 shekel, so that *the combined price for 1 sìla of each kind* of oil is ;25 shekel. Now recall that rec. ;25 = 2;24. Therefore *the combined price for 2;24 sìla of each kind* of oil is 2;24 · ;25 shekel = 1 shekel, the price of 2;24 sìla of fine oil is ;48 shekel, and the price for 2;24 sìla of plain oil is ;12 shekel. In other words, 2;24 sìla (2 1/3 sìla 4 shekels) per shekel is the *combined market rate* for the two kinds of oil.

Here is a more detailed way of explaining the solution procedure: Let m = 3 and m' = 12 (sìla per shekel) be the given market rates, let p = rec. m = ;20 (shekels per sìla) and p' = rec. m' = ;05 (shekels per sìla) be the corresponding unit prices, and let S = 1 (shekel) be the given amount of silver available. Then the problem in YBC 4698 § 2 a can be interpreted in quasi-modern terms as a linear equation of the following type for the number N of sìla bought of each kind of oil for the silver available:

$$N \cdot p + N \cdot p' = S.$$

With the data given in the text, the solution to this linear equation can be calculated easily as follows:

$N \cdot (;20 + ;05) = 1$, so that N = rec. ;25 = 2;24.

For N sìla of each kind separately are then paid the following N-fold unit prices

$N \cdot p$ = 2;24 · ;20 = ;48 (shekel) and $N \cdot p'$ = 2;24 · ;05 = ;12(shekel).

The simplest way to visualize the detailed solution procedure outlined above is by means of the following simple *tabular array,* which contains all the numerical data of the solution procedure, expressed in Old Babylonian relative place value notation. Numbers given in the question are written in boldface.

sums:	$(p + p')$	$\boldsymbol{S}$		sums:	(25)	**1 (shekel)**	
$\boldsymbol{m}$	p = rec. m	$N \cdot p$	$N = S \cdot$ rec. $(p + p')$	**3**	20	48	2 24
$\boldsymbol{m'}$	p' = rec. m'	$N \cdot p'$	N	**12**	5	12	2 24

Note that in known Old Babylonian numerical tabular arrays of this kind, no top row with a *sum* like (25) above is ever included, although it would have been helpful. That sum simply had to be kept in the head by the one who made the calculation.

YBC 4698 § 2 b (# 12)

In this example, there are again two given market rates, namely m = 7 and m' = 11 minas per shekel, of two kinds of wool, and the total amount of silver available is again 1 shekel. The problem is deliberately complicated by the fact that both 7 and 11 are non-regular sexagesimal numbers, so that the unit prices p and p' cannot be computed exactly. (*However, it is essential for the solvability of the problem that 7 + 11 is a regular sexagesimal number.*) The difficulty can be avoided, however, by considering the *$m \cdot m'$-fold unit prices*. These are the number of shekels that you have to pay for $m \cdot m'$ = 7 · 11 = 1 17 units of the two kinds of commodities. In the present case, these $m \cdot m'$-fold unit prices are

m' = 11 (shekels) and m = 7 (shekels).

Therefore, as in other similar cases (see Friberg, *MSCT 1*, Sec. 7.2 e), the linear equation which has to be considered in YBC 4698 § 2 b is of the following modified type:

$$N \cdot m' + N \cdot m = m \cdot m' \cdot 1 = 7 \cdot 11 \text{ (shekels)}.$$

The solution is again easily calculated, as follows:

$$N \cdot (m' + m) = m \cdot m', \quad \text{so that} \quad N = m \cdot m' \cdot \text{rec. } (m + m') = 7 \cdot 11 \cdot \text{rec. } 18 = 1\ 17 \cdot ;03\ 20 = 4;16\ 40.$$

Indeed, the price for $N = m \cdot m' \cdot \text{rec. } (m + m') = 7 \cdot 11 \cdot \text{rec. } 18$ units of the two kinds of commodities is

$$m' \cdot \text{rec. } (m + m') = 11 \cdot \text{rec. } 18 = 11 \cdot ;03\ 20 = ;36\ 40 \text{ (shekels)}$$

and

$$m \cdot \text{rec. } (m + m') = 7 \cdot \text{rec. } 18 = 7 \cdot ;03\ 20 = ;23\ 20 \text{ (shekels)},$$

respectively. Consequently, *the combined price for* $N = m \cdot m' \cdot \text{rec. } (m + m') = 7 \cdot 11 \cdot \text{rec. } 18$ *units of each kind of commodity* is equal to

$$m' \cdot \text{rec. } (m + m') + m \cdot \text{rec. } (m + m') = ;36\ 40 + ;23\ 20 = 1 \text{ (shekel)}.$$

The numerical data of this solution procedure can be presented in the form of the following tabular array, where the numbers in col. *ii* are 1 17-*fold unit prices*, the prices for $m \cdot m' = 7 \cdot 11 = 1\ 17$ units of each kind of wool:

sums:	(18)	**1 (shekel)**	
7	11	36 40	4 16 40
11	7	23 20	4 16 40

Note that the combined market rate ‘4 16 40’ is given as an explicit answer in the last line of exercise § 2 b. In metrological notations, the answer could have been be expressed as the capacity number 4 minas 16 2/3 shekels.

Note, by the way, that what is going on here can be explained in modern terms, anachronistically, as the following straightforward counting with common fractions:

$$N = 1/(1/7 + 1/11) = 7 \cdot 11/(11 + 7) = 7 \cdot 11\ /\ 18.$$

The earliest known example of counting *explicitly* with common fractions in this way appears in the Early Roman demotic papyrus BM 10520 § 5 e. See Friberg, *UL* (2005), Sec. 3.3 e.

YBC 4698 § 2 c (# 13)

This example appears to be much like the example in § 2 a above. There are two given market rates, 6 and 12 sìla of plain oil and lard, respectively, per shekel of silver. The total amount of silver available is again 1 shekel. An answer is explicitly given, in the form of the capacity number 3 1/3 sìla inscribed on the edge. See Fig. 10.1.3 below. (Note that the number 3 1/3 sìla appears to be written on top of some other capacity number.)

Fig. 11.1.3. YBC 4698 § 2 c. The capacity number 3 1/3 sìla inscribed on the edge.

The missing solution procedure in this straightforward case can be replaced by the following tabular array:

sums:	(15)	**1 (shekel)**	
6	10	40	4
12	5	20	4

However, this would lead to a combined market rate of 4 sìla, not 3 1/3 sìla as indicated in the text. A possible explanation for the discrepancy is that the one who wrote the text of § 2 c had first produced cols. *i-ii* in a tabular array like the one above, but then misread the number 5 in col. *ii* for a number 8, which is easily done. As a result of this misreading, the sum of the two numbers in col. *ii* became 18 instead of 15, and the calculated combined market rate became rec. 18 = 3 20 instead of rec. 15 = 4. The number 3 20 was then expressed in metrological terms as 3 1/3 sìla.

YBC 4698 § 2 d (# 14)

In this exercise, commodities are paid for in grain. In Old Babylonian mathematical texts, *1 gur of grain is worth as much as 1 shekel of silver*. See, for instance, YBC 4666 # 1 (Neugebauer and Sachs, *MCT* p. 76, text K), where a worker was paid 1 bán of grain each day for his work:

1(bán) še á.bi lú.ḫun.gá	1 bán of grain, the wages of a hired man.

Compare with, for instance, YBC 7164 # 1 (*MCT* p. 81, text L), where a worker was paid 6 grains of silver each day for his work:

6 še á lú.ḫun.gá	6 grains, the wages of a hired man.

The corresponding wages for 1 month's work would then be either 30 bán = 1 gur of grain or 30 · 6 grains = 1 shekel of silver.

In YBC 4698 § 2 d, the market rates for 9 unspecified commodities are 1, 2, …, 9 (units of that commodity for 1 sìla(?) of grain). It is likely that the "units" were themselves metrological. *For equal amounts of all the 9 commodities is paid altogether 1 gur (= 5 00 sìla) of grain.* No question is asked, but presumably the object of the exercise was to calculate the combined market rate, just as in exercises §§ 2 a-c above. For that purpose a tabular array of the following form could be used where, because 7 is a non-regular sexagesimal number, 7-fold unit prices (the number of sìla paid for 7 units of the commodity) are listed in col. *ii*:

sums:	(appr. 20)	(appr. **5**)	
1	7	1 45	1 45
2	3 30	52 30	1 45
3	2 20	35	1 45
4	1 45	26 15	1 45
5	1 24	21	1 45
6	1 10	17 30	1 45
7	1	15	1 45
8	52 30	13 07 30	1 45
9	46 40	11 40	1 45

Note that a round-off has to be used in this computation, where 19 48 10 (or more precisely, 19;48 10 sìla for 7 units of each kind of commodity) has been approximated by 20. In order to get the stipulated combined price of 1 gur = 5 00 sìla, this number has to be multiplied by 15. Therefore, the resulting combined market rate has to be understood as 15 · 7 = 1 45 (units of each kind per gur(!) of grain).

Compare with the tabular array on the "hand tablet" N 3914 (Friberg, *MSCT 1* (2007), 165), where the given market rates are 1, 2, …, 10 and the sum of the corresponding 7-fold unit prices is 20;30 10. However, in that text the need of a round-off is evaded by assuming that the available amount of silver or grain is not 1 or 5, but 3 25 01 40.

YBC 4698 § 2 e (# 15)

In this exercise, two kinds of fish are paid for in grain. One market rate is given as '3 fishes for 5 sìla', which is the same as the somewhat unrealistic market rate ;36 fish per sìla, or a unit price of 1 2/3 sìla per fish. The other market rate is partly destroyed, what remains is '5 fishes for [x sìla]'. The total amount of grain paid for equal quantities of the two kinds of fish is given as 4 gur 1 bariga 4 bán ⸢5⸣ (sìla) which is the same as 21 45 sìla.

Now, 21 45 = 45 · 29, where 29 is a non-regular sexagesimal number. Therefore, if the market rate problem in this particular case was set up so that there would be an *exact* answer to the problem, which is a quite reasonable assumption, the total price, the sum of the two entries in col. *ii* of the corresponding tabular array, should be a regular sexagesimal number times the factor 29. Against this background, it is not difficult to find out that there are two likely values for the lost capacity number [x sìla]. One is [1 1/3 sìla], the other is [3 2/3 sìla 5 (gín)**]**. The reconstructed second market rate would then be either

'5 fishes for [1 1/3 sìla]' which is the same as 3;45 fishes for 1 sìla or 16 sìla for sixty fishes

or

'5 fishes for [3 2/3 sìla 5 (shekels)]' which is the same as 1;20 fishes for 1 sìla or 45 sìla for sixty fishes

Fig. 11.1.4. YBC 4698 § 2 e. Note the destroyed number [3 2/3 sìla 5]$^?$ in the second line of this paragraph.

With these two possible reconstructions of the lost number, the corresponding tabular arrays would be:

sums:	(1 56)	**21 45 (sìla)**			sums:	(2 25)	**21 45 (sìla)**	
36	1 40	18 45	11 15	or	**36**	1 40	15	9
[3 45]	16	3	11 15		**[1 20]**	45	6 45	9

In the former case, the price in grain for 11 15 = 11 sixties 15 fishes of each kind would be

11;15 sixties · 1 40 sìla per sixty + 11;15 sixties · 16 sìla per sixty = (18 45 + 3 00) sìla = 21 45 sìla.

In the latter case, the price in grain for 9 sixties fishes of each kind would be

9 sixties · 1 40 sìla per sixty + 9 sixties · 45 sìla per sixty = (15 00 + 6 45) sìla = 21 45 sìla.

In either case the tabular array above would be of the same form as the tabular arrays found on, for instance, the round hand tablet MS 2268/19 (see Friberg *MSCT 1*, 164). However, only the reconstructed capacity number [3 2/3 sìla 5] seems to agree with the traces of the number in the second line visible in the photo detail above.

The total number of fishes in YBC 4698 § 2 e would then be 2 · 9 00 = 18 00, not 20 00 as erroneously stated in line 4 of the exercise! *It is obvious that that part of the text is corrupt, a line from a related but different exercise has mistakenly been copied to the wrong place.* Or else, this was an attempt to combine two exercises into one. See the renewed discussion of exercise § 2 e below under the new name § 3 c.

The market rates for two kinds of fish figuring in § 2 e are, supposedly, ;36 and [1;20] fishes per sìla of grain, respectively. Since 1 gur = 5 00 sìla, these market rates can also be expressed as

5 00 · ;36 = 3 00 fishes for 1 gur of grain and 5 00 · 1;20 = 6 40 fishes for 1 gur of grain.

In view of the usual value relation that 1 gur of grain is worth as much as 1 shekel of silver, the two market rates can also be interpreted as *3 sixties and 6 sixties 40 fishes for 1 shekel of silver.* That these particular market rates are inside the usual range of such market rates is shown, for instance, by HE 113, an Old Babylonian account from a fish market in Larsa, published by Scheil in *RA* 15 (1918). See Fig. 11.1.5 below.

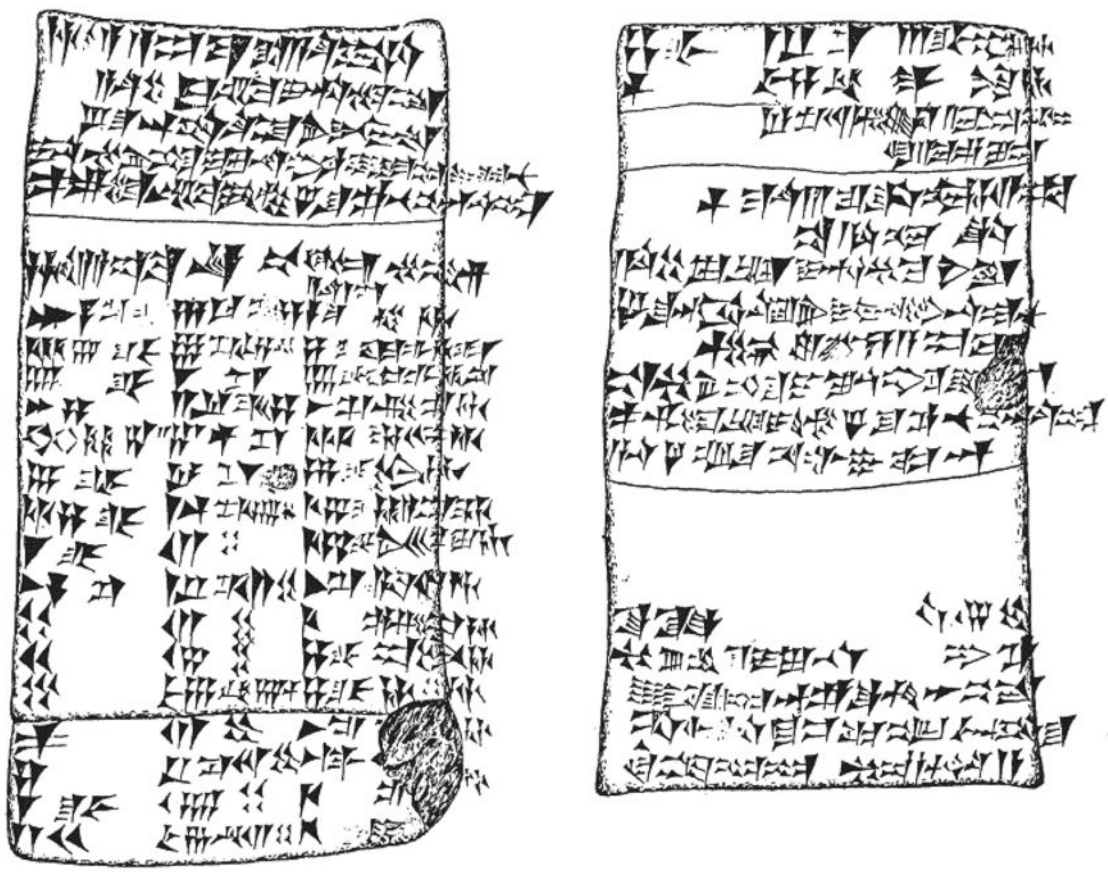


Fig. 11.1.5. HE 113. An Old Babylonian account from the fish market at Larsa, in the reign of Samsu-Iluna.

This text contains among other things a table of four columns (occupying most of the space on the obverse). The four columns have the headings

ku_6.ḫá a.ab.ba	'various kinds of saltwater fish'	kù	'silver'
ganba *a-na* 1 gín	'market rates for 1 shekel'	mu.bi.im	'their names'

Tabular accounts with this kind kind of layout appear exclusively among Old Babylonian cuneiform texts from the ancient city Larsa, as shown by Robson in *HMT* (1999). Known examples of mathematical cuneiform texts with tables of the same kind, and therefore presumably from Larsa, are Plimpton 322 (Friberg, *MSCT 1* (2007), App. 8), and CBS 1766 (Friberg, *AfO* 52 (2013)).

Eighteen varieties of fish are enumerated in the table on HE 113. Some are counted, often in sixties, and others are measured in capacity measure. The smallest number of fishes of any particular type is 20, while the biggest number is 2 šár 4 *nēr* 5 (*šu-ši*) (= 2 45 00). The smallest capacity number is 10 sìla, while the biggest is 3 gur 2 bariga (= 17 00 sìla). The market rates for the counted kinds of fish vary from 3 sixties to 30 sixties per shekel of silver, and the market rates for measured kinds of fish vary from 40 to 6 40 (= 400) sìla per shekel.

HE 113 (Scheil, *RA* 15 (1918))

ku$_6$.ḫá a.ab.ba		kù		ganba *a-na* 1 gín		mu.bi.im
3(gur) 2(barig) gur	(17 00 sìla)	6 ⅔ gín 24 (še)	(6;48 gín)	2(barig) 3(bán)	(2 30 sìla)	nam.ku$_6$
3(*nēr*) 7 *šu-ši*	(37 00)	9 gín igi.4.gál	(9;15 gín)	4 *šu-ši*	(4 00)	ka.mar.ku$_6$ sag
8 *šu-ši*	(8 00)	1 gín	(1 gín)	8 *šu-ši*	(8 00)	ka.mar.ku$_6$ uš
2(gur) 4(barig)	(14 00 sìla)	2 ⅔ gín 24 (še)	(2;48 gín)	1(gur) gur	(5 00 sìla)	*etc.*
2(*šár*) 4(*nēr*) 5 ´´	(2 45 00)	5 ½ gín	(5;30 gín)	3(*nēr*)	(30 00)	
5 *šu-ši*	(5 00)	⅚ gín	(;50 gín)	6 *šu-ši*	(6 00)	
2(*nēr*) 4 *šu-ši*	(24 00)	1 ½ gín 18 še	(1;36 gín)	1(*nēr*) 5 *šu-ši*	(15 00)	
1 *šu-ši*	(1 00)	12 še	(;04 gín)	1(*nēr*) 5 *šu-ši*	(15 00)	
1(gur) 2(barig) gur	(7 00 sìla)	1 ⅓ gín 12 še	(1;24 gín)	1(gur) gur	(5 00 sìla)	
40	(40)	12 še	(;04 gín)	1(*nēr*)	(10 00)	
20	(20)	15 še	(;05 gín)	4 *šu-ši*	(4 00)	
50	(50)	igi.6.gál 7 ½	(;12 30 gín)	4 *šu-ši*	(4 00)	
2(bán)	(20 sìla)	12 še	(;04 gín)	1(gur) gur	(5 00 sìla)	
3(barig)	(3 00 sìla)	⅓ gín 21 še	(;27 gín)	1(gur) 1(barig) 4(bán)	(6 40 sìla)	
1 *šu-ši*	(1 00)	18 še	(;06 gín)	1(*nēr*)	(10 00)	
2 20	(2 20)	igi.6.gál 12 še	(;14 gín)	1(*nēr*)	(10 00)	
4 *šu-ši*	(4 00)	1 ⅓ gín	(1;20 gín)	3 *šu-ši*	(3 00)	
1(bán)	(10 sìla)	igi.4.gál	(;15 gín)	4$_{\text{bán}}$	(40 sìla)	

Proust and Middeke-Conlin in *CDLJ* 2014:3 were the first who correctly read the following two parallel phrases written in Sumerian in the last few lines of the two Old Babylonian combined market rate exercises YBC 4698 §§ 2 d and 2 e:

še ḫé.e$_3$ / *ù* ḫé.e$_{11}$-*ma* / ganba íb.sá 'the grain may have risen / or fallen, but the market rates may have been equal'

and

še [ḫé].e$_3$ *ù* ḫé.e$_{11}$- / *ma* / ku$_6$.a íb.sá 'the grain may may have risen or fallen / but / the fishes I made equal'

Parallel phrases in Akkadian were already known from VAT 7530, a brief catalog text concerned with combined market rates (Neugebauer, *MKT I* (1935), 288; Thureau-Dangin, *RA* 33 (1936), 162). See Fig. 11.1.6 below. Exercises ## 3, 5-6 of VAT 7530 end with the following three variants of the same phrase:

kù.babbar *li-li ù li-ri-da ma-ḫi-ru li-im-ta-ḫar* 'the silver may rise or fall, but the market rates may be equal'
kù.babbar *li-li li-ri-id-ma* ganba *li-im-ta-ḫi-ra*
[kù.babbar] *li-li ù li-ri-id-ma* [ganb]a *li-im-ta-ḫa-ar*

Neither Neugebauer nor Thureau-Dangin managed to understand the exercises on VAT 7530. The understanding was partly hindered by Thureau-Dangin's misreading of a new term repeatedly occurring in VAT 7530 as *ši-za-a-at*, which he interpreted as a variant writing of the known word *šizû* 'one third (cubit)'. The mistaken reading still persisted in Neugebauer and Sachs, *MCT* (1945), 18. An effort there to explain the meaning of the exercises on VAT 7530 ended with the following admission:

> "We are completely in the dark as to the exact application of this set of coincidences to the problems given in VAT 7530, but it seems obvious that there is a significant relation between VAT 7530 and the tables given above (the tabular arrays YBC 11127, *etc.*)."

The meaning of the exercises on VAT 7530 and the related tabular arrays on YBC 11127, *etc.*, was explained in Friberg, *RlA 7* (1990) § 5.2 h. Both the exercises on VAT 7530 and the tabular arrays were then again extensively discussed in Friberg, *MSCT 1* (2007), Sec. 7.2. In particular, the elusive term previously mistakenly read *ši-za-a-at* was explained there as igi.4$^{a\text{-}at}$ = *rebât* 'a 4th-part, a quarter'.

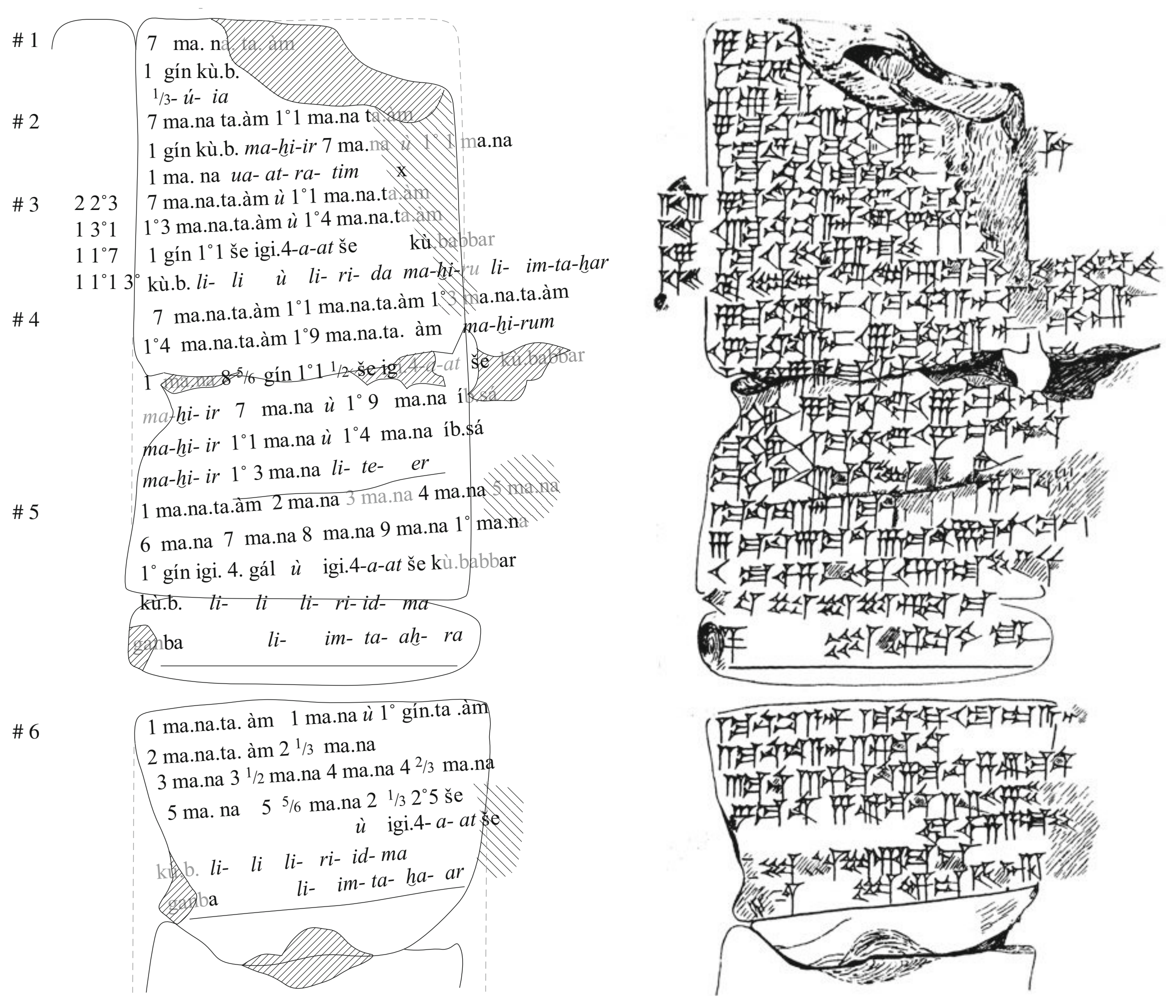


Fig. 11.1.6. VAT 7530. An Old Babylonian catalog text text with combined market rate exercises.
Hand copy by O. Neugebauer, *MKT II*. Reproduced here with the permission of Springer.

Take, for instance, exercise # 3 on VAT 7530. The related tabular array is of the following form:

	(6 22 30)	**1 gín 11 še igi.4**$^{a\text{-}at}$	
7	2 23	23 50	2 46 50
11	1 31	15 10	2 46 50
13	1 17	12 50	2 46 50
14	1 11 30	11 55	2 46 50

Note that the four numbers inscribed on the left edge of the tablet are the numbers in col. *ii* of the tabular array. The could be computed as 7 · 11 · 13 = 16 41 divided by 7, 11, 13, and 14, respectively.

The sum of the entries in the third column is

23 50 + 15 10 + 12 50 + 11 55 = 1 03 45.

This sexagesimal number in place value notation corresponds to the weight number 1 gín 11 še igi.4$^{a\text{-}at}$ inscribed on top of the last two columns of the tabular array. Indeed, it is easy to check that

1;03 45 shekels = 1 shekel + 3 · 3;45 grains = 1 shekel 11;15 grains = 1 shekel 11 grains and a 4th-part (of a grain).

Note, by the way, that the interesting phrase in YBC 4698 § 2 e

še [ḫé].e$_{3}$ *ù* ḫé.e$_{11}$- / *ma* / ku$_{6}$.a íb.sá 'the grain may have risen or fallen, / but / the fishes I made equal'

has a parallel not only in several lines of VAT 7530 but also in the following phrase in the isolated exercise # 18 in BM 85196, an Old Babylonian mathematical recombination text from Sippar:

li-li / *ù li-r*[*i-i*]*d-ma* ḫar kù.babbar *li-im-ta-aḫ-ru* / 'may it go up / and may it go down, but the silver may be equal'

However, the problem in BM 85196 # 18 is not a combined market rate problem as all the other exercises where similar phrases are mentioned, but a problem that can be expressed as a system of two linear equations for two unknowns. Since this particular exercise has never been adequately explained before, it may be justified to take a new look at it below.

BM 85196 # 18 (*MKT II*, 46; rev. *ii*, 18-29)

1 [igi.7.gál] ḫar kù.babbar 1 *ù* igi.11.gál ḫar kù.babbar *ša-ni-im* / [*s*]*é-ru-uḫ-ma*
2 [*lu*]-*ú* 1 gín *ša ta-aḫ-ru-ṣ*[*ú*]
3 *li-li* / *ù li-r*[*i-i*]*d-ma* ḫar kù.babbar *li-im-ta-aḫ-ru* /
4 za.e bal u[l.gar 18] *ta-mar* igi 18 duḫ.a 3 20 *ta-mar* /
5-6 3 20 *a-na* 7 *i-ši* 22 30 *ta-mar* 3 20 *a-na* [11] *i-ši* / 37 30 *ta-mar*
7 37 30 *i-na* 1 ḫar kù.babbar *aḫ-ru-iṣ* / [2]2 30 *i-na* 2 ḫar kù.babbar *aḫ-ru-iṣ*
8-9 22 30 *a-na* / 11 *i-ši* 4 07 30 *ta-mar* 37 30 *a-na* 7 *i-ši* / 4 22 30 *ta-mar*
10-11 22 30 *i-na* 4 07 30 ba.zi / 3 45 *ta-mar* 37 30 *i-na* 4 22 30 ba.zi / 3 45 *ta-mar mi-it-ḫa-ru-ú*
12 *ki-a-am ne-pé-šum*

1 [A 7th-part of] silver ring 1 and an 11th-part of a second silver ring were destroyed, then /
2 may 1 shekel be what you destroyed.
3 May it go up / and may it go down, but the silver may be equal.
4 You: The ratios a[dd together, 18] you will see. The reciprocal of 18 release, 3 20 you will see. /
5-6 3 20 to 7 carry, 22 30 you will see, 3 20 to [11] carry / 37 30 you will see.
7 37 30 from silver ring 1 I broke off / [2]2 30 from silver ring 2 I broke off.
8-9 22 30 to / 11 carry, 4 07 30 you will see, 37 30 to 7 carry / 4 22 30 you will see
10-11 22 30 from 4 07 30 tear off / 3 45 you will see, 37 30 from 4 22 30 tear off / 3 45 you will see. They will be equal.
12 Such is the procedure.

Silver rings (often in the form of spiral coils) or silver sheets were used as one of several means of payment in the Old Babylonian period, long before the invention of money. In BM 85196 # 18, parts of two silver rings of unknown weight are used to pay some expenditure, with pieces cut off from both, and after the operation the two rings are arranged to have the same weight. In quasi-modern symbolic notations, the question in this (contrived) exercise can be expressed as a system of linear equations, with s and s' denoting the unknown initial weights of the two silver rings:

$$1/7 \cdot s + 1/11 \cdot s' = 1 \text{ (shekel)}, \qquad s - 1/7 \cdot s = s' - 1/11 \cdot s'.$$

A particularly clever way of solving this system of linear equations starts by *introducing the weights of the cut-off pieces as new unknowns*, calling them, for instance, c and c'. Then the system of linear equations above is replaced by the following equivalent system of linear equations:

$$c + c' = 1 \text{ (shekel)}, \qquad 6\,c = 10\,c'.$$

A system of linear equations of this kind is easy to solve, simply by multiplying the first equation by 6, alternatively by 10, and then using the second equation. The result is again a new system of linear equations, namely

$$10\,c' + 6\,c' = 6 \text{ (shekels)}, \qquad 10\,c + 6\,c = 10 \text{ (shekels)}.$$

Therefore, clearly,

$$c = 6 \cdot \text{rec.}\,(10 + 6) = 6 \cdot ;03\ 45 = ;22\ 30 \text{ (shekels)}, \qquad c' = 10 \cdot \text{rec.}\,(10 + 6) = 10 \cdot ;03\ 45 = ;37\ 30 \text{ (shekels)}.$$

However, what is calculated in line 4 to (the first part of) line 6 of BM 85196 is instead, incorrectly,

$$7 \cdot \text{rec.}\,(11 + 7) = 7 \cdot ;03\ 20 = ;22\ 30(!) \text{ (shekels)}, \qquad 1\,1 \cdot \text{rec.}\,(11 + 7) = 11 \cdot ;03\ 20 = ;37\ 30(!) \text{ (shekels)}.$$

This means that *in the text someone started with incorrect numbers (11 and 7 instead of 10 and 6) and made a couple of incorrect computations (7 · ;03 20 should have been ;23 20 and 11 · ;03 20 should have been ;36 40), yet ended up with two correct numbers (;22 30 and ;37 30).* From there on the text proceeds correctly. In lines 6-7 it is stated that ;37 30 and ;22 30 is what was broken off from the two silver rings, and after that the initial weights of the two silver rings are computed as 11 · ;22 30 = 4;07 30 and 7 · ;37 30 = 4;22 30 (shekels). Then, the correctness of the computation is checked by showing that if ;22 30 is torn off from 4;07 30 and if ;37 30 is torn off from 4;22 30 the result is, in both cases, 3;45 (shekels). That means that the second of the two initial equations is satisfied. It is also clear that the first equation is satisfied, since ;22 30 + ;37 30 = 1 (shekel).

It is easy to imagine a scenario that can explain this strange text where several mistakes lead to a correct answer. A teacher may have shown his students how to solve the problem formulated in the first three lines of BM 85196 # 18, allowing the students to take notes during his demonstration, and then asked them to write down the details of the solution. A student who had not listened carefully forgot that the solution procedure started with a change of unknowns and started incorrectly but got the result right in the end because he relied on the notes he had made. The same kind of phenomenon is known from the Kassite problem text AO 17264 (Neugebauer, *MKT I* (1935), 126; Friberg, *AT* (2007), Sec. 11.6), where, in the case of an unusually advanced problem, two completely nonsensical first steps of the solution procedure are followed by six correct steps and a correct answer.

Thureau-Dangin who first published BM 85196 in *RA* 32 (1935) made the *ad hoc* assumption that the author of exercise # 18 had, for some reason, silently subtracted ;00 50 from 7 · ;03 20 = ;23 20 in order to get the recorded number ;22 30 and added the same number ;00 50 to 11 · ,03 20 = ;36 40 in order to get the recorded number ;37 30. Then Neugebauer, in *MKT II* (1935), 57-59, tried to mathematically justify Thureau-Dangin's *ad hoc* assumption, which led to a long and unnecessarily complicated computation.

11.1.4 § 3. Iron, Gold, and Fishes. Systems of Linear Equations

YBC 4698 § 3 (## 4-5, *obv. i*: 12-18 and # 15, *rev. ii*: 1-6)

§ 3 a 1-3 1 30.bi an.bar / 9.bi kù.sig$_{17}$ / 1 ma.na kù.babbar šúm /
4-5 an.bar *ù* kù.sig$_{17}$ / 1 gín-*ma* sa$_{10}$

1-2 Its 1 30 iron / its 9 gold. / 1 mina of silver I paid. /
3 Iron and gold / 1 shekel, then I bought.

§ 3 b 1-2 $^1/_2$ ma.na kù.babbar šúm / 1(gur) gur.ta sa_{10}

1-2 1/2 mina of silver I paid / at 1 gur I bought.

§ 3 c 1-2 (ganba) 3 ku_6.a 5 sìla / 5 ku_6.a [3 $^2/_3$ sìla 5] /
3-4 še gar.gar-*ma* 4(gur) 1(barig) 4(bán) ⸢5⸣ / ku_6.a gar.gar-*ma* 2(geš'u) /
5-6 (še [ḫé].e_3 *ù* ḫé.e_{11}- / *ma* / ku_6.a íb.sá)

1-2 (Market rates.) 3 fishes 5 síla, 5 fishes [3 2/3 sìla 5] /
3 The grain I added together, then 4 gur 1 bariga 4 bán ⸢5⸣ <sìla>. / The silver I added together, then 2(10 · 60).
4 (The grain may have risen or fallen, / but / the fishes I made equal.)

YBC 4698 § 3 a (# 4)

In this exercise, it is clear that *1 mina of silver is paid for 1 shekel of iron and gold.* Much less clear is the meaning of the phrase

1 30.bi an.bar / 9.bi kù.sig_{17} / 'its 1 30 iron / its 9 gold'

The price of gold is known to have varied in the Old Babylonian period, and it is not unreasonable to assume that the strange expression 9.bi kù.sig_{17} 'its 9 gold' simply means that gold was assumed to be 9 times more valuable than silver, in other words that the unit price of gold was 9 shekels of silver per shekel of gold. However, if 1 mina = 60 shekels of silver was the combined price of 1 shekel of iron and gold, then not only gold but also iron must have been assumed to be valuable. Iron was, of course, quite rare in the Old Babylonian period, before the advent of the iron age in Mesopotamia towards the end of the second millennium BC.

If the cited phrase means that *the unit price of gold was 9 shekels of silver per shekel of gold,* then it must also mean that the unit price of iron was either 1;30 or 1 30 shekels of silver per shekel of iron. The first alternative must be rejected because in that case the combined price of 1 shekel of iron and gold would be less than 9 shekels, instead of 1 mina (= 60 shekels). Therefore, the only remaining possibility is that *in this exercise iron was assumed to be 1 30 (= 90) times more valuable than silver and 10 times more valuable than gold!* Maybe the author of the exercise had no idea about how valuable iron should be assumed to be, and so he just chose a value that would lead to an exactly solvable mathematical problem.

Now, let i and g (shekels) be the weights of iron and gold, respectively. Then the question in YBC 4698 § 3 a can be rephrased, in quasi-modern terms, as the following system of two linear equations for two unknowns:

$$1\ 30\ i + 9\ g = 1\ 00, \qquad i + g = 1.$$

In the Old Babylonian text VAT 8391 # 2 (Neugebauer, *MKT I*, 320; Friberg, *MSCT 1*, 334-335), a system of linear equations of precisely this type is solved explicitly. If the solution method used in that case is applied in the present case, one finds that the question in YBC 4698 #§ 3 a can be answered as follows. Start with a tentative "false" solution

$$i^* = g^* = ;30.$$

Then the second of the two linear equations above is satisfied. However, the first is not, since

$$1\ 30\ \ i^* + 9\ g^* = 45 + 4;30 = \ 49;30 \qquad \text{which is 1 00 minus a } \textit{deficit} \text{ of } 10;30.$$

In order to eliminate that deficit, the value of i^* must be increased by a certain amount and the value of g^* must be decreased by the same amount, so that the second equation will continue to be satisfied. For instance,

If i^* is increased and g^* decreased by ;01, then 1 30 $i^* + 9\ g^*$ is increased by 1;30 – ;09 = 1;21.

Evidently, the two factors 1 30 = 10 · 9 and 9 were chosen so that both they and their difference 9 · 9 would be regular sexagesimal numbers! Therefore, the solution to the problem can be calculated as follows:

In order to eliminate the whole deficit of 10;30, i^* must be increased and g^* decreased by rec. 1;21 · 10;30.

Since rec. 1;21 · 10;30 = rec. 1 21 · 10 30 = rec. 27 · 3 30 = rec. 9 · 1 10 = ;06 40 · 1 10 = 7; 46 40, it follows that

i = ;30 + 7;46 40 = ;37 46 40 and g = ;30 – 7;46 40 = ;22 13 20.

In metrological terms, the *exact* answer to the question in YBC 4698 § 3 a is

;37 46 40 shekel = 1/2 shekel 23 1/3 grains of iron and ;22 13 20 shekel = 1/3 shekel 6 2/3 grains of gold.

YBC 4698 § 3 b (# 5)

The question in this brief paragraph makes no sense at all. It is likely that what should have been written in the second line of § 3 b was erroneously replaced by a line borrowed from some exercise nearby in the text from which YBC 4698 § 3 a and the first line of § 3 b both were copied.

YBC 4698 § 3 c (# 15)

As explained above, the text of YBC 4689 § 2 e / 3 c seems to be an effort to formulate two problems simultaneously, the combined market rate problem of § 2 e, and the system of linear equations in § 3 c, both problems based on the same pair of market rates for two kinds of fish. With the same assumed reconstruction as before of the destroyed capacity number in line 2 of the exercise, the two *market rates* are ;36 and 1;20 (fishes per sìla). The corresponding *unit prices* are p = 1 40 and p' = 45 (sìla per sixty fishes). Now let there be f and f'' sixties of fishes of the two kinds. Then f and f'' are, in quasi-modern terms, the solutions to the following system of linear equations:

$p \cdot f + p' \cdot f''$ = 1 40 $\cdot f$ + 45 $\cdot f''$ = 21 45 (sìla) *and* $f + f''$ = 2 geš'u = 20 sixties.

This is a system of linear equations of precisely the same kind as in exercise § 3 a (and in VAT 8389 and 8391). Therefore, the same solution method can be used here as in § 3 a.

Start with a tentative "false" solution

$f^* = f''^*$ = 1/2 · 20 sixties = 10 sixties.

Then the second of the two linear equations above is satisfied. However, the first is not, since

1 40 $\cdot f^*$ + 45 $\cdot f''^*$ = (16 40 + 7 30) sìla = 24 10 sìla which is 21 45 sìla plus an *excess* of 2 25 sìla.

In order to eliminate that excess, the value of f^* must be decreased by a certain amount and the value of f''^* must be increased by the same amount, so that the second equation will continue to be satisfied. For instance,

If f^* is decreased and f''^* increased by 1 sixty, then 1 40 $\cdot f^*$ + 45 $\cdot f''^*$ is decreased by (1 40 – 45) sìla = 55 sìla.

Now, observe that

2;38 · 55 = 2 38 – 2;38 · 5 = 2 38 – 13;10 = 2 24;50 = approximately 2 25.

Therefore,

In order to eliminate the whole excess of 2 25 sìla, f^* must be decreased and f''^* increased by (approximately) 2;38 sìla.

Consequently,

f = (appr.) (10 – 2;38) sixties = 7 22 fishes and f'' = (appr.) (10 + 2;38) sixties = 12 38 fishes.

Indeed,

7 22 + 12 38 = 20 sixties, and 1 40 · 7;22 + 45 · 12;38 = 2 16;40 + 9 28;30 = 21 45;10 sìla = (appr.) 21 45 sìla.

The Old Babylonian exercise VAT 8391 # 2 was mentioned above as a parallel to the exercises in YBC 4698 § 3 with their systems of linear equations. For the readers' convenience, the question of that exercise is copied below.

VAT 8391 # 2 (*obv. i*: 21-*obv. ii*: 34) (Neugebauer, *MKT I* (1935), 320; Friberg, *MSCT 1*, 334-335)

1-2 *šum-ma i-na* 1(bùr) ašag 4(gur) še.gur *am-ku-us* / *i-na* 1(bùr) ašag *ša-ni-im* 3(gur) še.gur *am-ku-us* /
3-4 *i-na-an-na* 2 garim a.šag$_4$ *ši-na* gar.gar-*ma* 30 / *ù* še-*a ši-na* gar.gar-*ma* 18 20 /
5 garim-*ú-a* en.nam
……………

1-2 If from 1 bùr of field 4 gur of grain I collected / from a second bùr of field 3 gur of grain I collected. /
3-4 Now, (for) 2 meadows the two fields I added together, then 30, / and my grains I added together, then 18 20.
5 My meadows were what?
……………

What this means is that grain is assumed to be collected as rent from fields at two different given rates, either 4 or 3 gur of grain per bùr of surface area. In the case described, grain is collected from two 'meadows' of unknown areas, at the two mentioned rates. The sum of the two areas and the sum of the two amounts of collected grain are given, and the size of the two areas has to be computed.

In quasi-modern terms, let A and A' be the two unknown areas, let C and C' be the two unknown amounts of collected grain, and let r and r' be the two given rates of rent collection. Then the question in the text of VAT 3891 # 2 can be rephrased as the following system of two linear equations for the two unknowns A and A':

$A + A' = 30$ (· 60 sar), $\quad r \cdot A + r' \cdot A' = 18\ 20$ (sìla).

The *explicit solution procedure in the text* proceeds in a number of steps, as follows. First the measures mentioned in the question are converted into sexagesimal numbers in relative place value notation:

1 bùr = 3 · 6 · 1 40 sar = 30 (· 60 sar), 4 gur = 4 · 5 · 60 sìla = 20 (· 60 sìla), 3 gur = 3 · 5 · 60 sìla = 15 (· 60 sìla).

Then tentative values of the areas A and A' satisfying the first linear equation are found by simply assuming that A and A' are equal:

$A^* = A'^* = 1/2\,(A + A') = 1/2 \cdot 30$ (· 60 sar) = 15 (· 60 sar).

Next, the given rates of rent collection are computed in terms of sexagesimal numbers in place value notation:

r = rec. 30 · 20 = ;40 (sìla/sar), $\quad r'$ = rec. 30 · 15 = ;30 (sìla/sar).

After that it is shown that the tentative values of A and A' do not satisfy the second linear equation:

$r \cdot A^* + r' \cdot A'^* =$ 40 · 15 + 30 · 15 =10 + 7 30 = 17 30 (sìla),
which is less than the prescribed value 18 20 by a *deficit* of 50.

Next, it is (silently) assumed that A^* is increased by '1' and that A'^* is decreased by the same amount. This tentative correction does not affect the first of the two linear equations, but it increases the left hand side of the second linear equation by an amount

$r - r'$ = ;40 – ;30 = ;10.

Consequently, in order to completely eliminate the deficit 50 in the second linear equation, A^* must be increased and A'^* decreased by the amount of

rec. ;10 · 50 = 6 · 50 = 5 (· 60).

Therefore, the following values of A and A' will satisfy also the second linear equation:

$A = A^* + 5 = 15 + 5 = 20$ (· 60), $\quad A' = A'^* - 5 = 15 - 5 = 10$ (· 60).

In terms of Old Babylonian area measure:

A = 2 èše, $\quad A'$ = 1 èše.

Finally,

$C = r \cdot A$ = ;40 (sìla/sar) · 20 (· 60 sar) = 13 20 (sìla), and $C' = r' \cdot A'$ = ;30 (sìla/sar) · 10 (· 60 sar) = 5 00 (sìla).

In terms of Old Babylonian capacity measure:

C = 2 gur 3 bariga 2 bán and C' = 1 gur.

Note: The discussion above of VAT 8391 # 2 can be compared with a similar discussion of VAT 8389 # 1 and VAT 8391 # 3 in Høyrup *LWS* (2002), 77-85.

11.1.5 § 4. Buying, Selling, and Making a Profit.

YBC 4698 § 4 (## 6-8, *obv. ii*: 1-11 and # 10, *rev. i*: 1-5)

§ 4 a	1-3	30 še gur / 1(gur) še gur.ta sa_{10}-*ma* / 4(barig) še.ta búr.ra /
	4-5	kù.dirig en.nam / 7 $^1/_2$ gín kù.babbar dirig
	1-3	30 gur of grain / at 1 gur of grain I bought, then / at 4 bariga of grain I sold.

	4-5	The profit in silver was what? / 7 1/2 shekels of silver was the profit.
§ 4 b	1	sag kù.bi en.nam
	1	The initial silver was what?
§ 4 c	1-3	1(gur) gur ì.geš / 1(bán).ta sa_{10}-*ma* / 8 sìla.ta búr.ra
	4-5	kù.dirig en.nam / 7 $^1/_2$ gín kù.dirig
	1-3	1 gur of common oil / at 1 bán I bought, then / at 8 sìla I sold.
	4-5	The profit in silver was what? / 7 1/2 shekels was the profit in silver.
§ 4 d	1-3	1(gur) gur ì.geš / 9 sìla.ta sa_{10}-*ma* / 7 $^1/_2$ sìla búr.ra /
	4-5	kù.dirig en.nam / 6 $^2/_3$ gín kù.dirig
	1-4	1 gur of common oil / at 9 sìla I bought / at 7 1/2 sìla I sold. /
	5-6	The profit in silver was what? / 6 2/3 shekels was the profit in silver.

YBC 4698 § 4 a-b (## 6-7)

In this exercise, 30 gur of grain were bought at a market rate of 1 gur per shekel of silver and sold at a reduced market rate of 4 bariga of grain per shekel of silver. What was then the profit?

The answer must, of course, be that the profit was the difference between the amount of silver for which the grain was bought and the amount for which it was sold. Let b be the silver needed to buy C = 30 gur of grain at the market rate m = 1 gur per shekel, and let s be the amount of silver obtained when selling the same 30 gur of grain at the reduced market rate m' = 4 bariga of grain per shekel. Recall that

1 gur = 5 · 6 · 10 sìla = 5 00 sìla,   4 bariga = 4 · 6 · 10 sìla = 4 00 sìla,   and   30 gur = 30 · 5 00 sìla = 2 30 00 sìla.

Then, in Babylonian relative sexagesimal numbers, the profit of the business transaction will be

$s - b$ = rec. $m' \cdot C$ – rec. $m \cdot C$ = rec. 4 · 2 30 – rec. 5 · 2 30 = 15 · 2 30 – 12 · 2 30 = 37 30 – 30 = 7 30.

In terms of Old Babylonian units for capacity measure,

30 gur of grain was bought for 30 shekels = 1/2 mina of silver and sold for 37 1/2 shekels = 1/2 mina 7 1/2 shekels.
Therefore, the profit was 7 1/2 shekels (as indicated in the text of § 4 a),
and the originally invested capital was 1/2 mina of silver.

Indeed, the question in § 4 a has the explicitly given answer '7 1/2 shekels is the profit in silver'.

YBC 4698 § 4 c (# 8)

The problem in this exercise is a close parallel to the problem in § 4 a. However, this time 1 gur (= 5 00 sìla) of common oil was bought at a market rate of 1 bán (= 10 sìla) per shekel of silver and sold at a reduced market rate of 8 sìla per shekel. Asked for is as before the profit.

This time, let again m and m' be the two market rates, let C be the original amount of oil. Then, of course, the net profit n of the business transaction will be, as in the preceding exercise,

rec. $m' \cdot C$ – rec. $m \cdot C$ = rec. 8 · 5 – rec. 10 · 5 = 7 30 · 5 – 6 · 5 = 37 30 – 30 = 7 30.

Indeed, the question in § 4 c has the explicitly given answer '7 1/2 shekels is the profit in silver', which agrees with the number 7 30 calculated above.

YBC 4698 § 4 d (# 10)

This problem is essentially the same as the problem in § 4 a, but with different data. In this case, the computation of the profit is as follows:

$s - b$ = rec. $m' \cdot C$ – rec. $m \cdot C$ = rec. 7 30 · 5 – rec. 9 · 5 = 8 · 5 – 6 40 · 5 = 40 – 33 20 = 6 40.

The explicitly given answer is '6 2/3 shekels is the profit in silver', which agrees with the computed profit.

11.1.6 § 5. Buying, Selling, and Making a Profit. The Inverse Problem

YBC 4698 § 5 (# 9, *obv. ii*: 12-19 and # 11, *rev. i*: 6-10)

§ 5 a 1-4 1(gur) gur ì.geš / *i-na* sám $1^!$ gín / 2 sìla šuš_4 / 7 $^1/_2$ gín kù.dirig

5-8 en.nam sa$_{10}$-*ma* / en.nam búr.ra / 1(bán) sa$_{10}$-*ma* / 8 sìla búr.ra

1-4 1 gur of common oil. / From the purchase price for 1! shekel / 2 sìla I cut off. /
5-8 What did I buy, then / what did I sell? / (At) 1 bán I bought / (at) 8 sìla I sold.

§ 5 b 1-3 1(gur) gur ì.geš / *i-na* sám 1 gín $^1/_2$ sìla šuš$_4$ / 6 $^2/_3$ gín kù.dirig /
4-5 en.nam sa$_{10}$-*ma* en.nam búr.ra / 9 sìla sa$_{10}$-*ma* 7 $^1/_2$ sìla búr.ra

1-3 1 gur of common oil. / From the purchase price for 1 shekel 1 1/2 sìla I cut off. /
5-6 (At) what did I buy, (at) what did I sell? / (At) 9 sìla I bought, then (at) 7 1/2 sìla I sold.

YBC 4698 § 5 a (# 9)

It is stated in the question of this exercise (somewhat vaguely) that 1 gur (= 5 00 sìla) of common oil was bought and sold with a market rate difference of 2 sìla and a net profit of 7 1/2 shekels of silver. Asked for are the two market rates at which the oil was bought and sold, respectively. The explicitly given answer is that the two market rates were 1 bán (= 10 sìla) and 8 sìla (per shekel of silver).

Let *m* and *m´* be the two unknown market rates, let *p* and *p´* be the corresponding unit prices (the reciprocals of the market rates), let *C* be the given amount of oil, and let *P* be the given profit in silver. Let also *d* be the given difference between the market rates. Then (cf. the related problem in YBC 4698 § 4 c above) *m* and *m´* can be determined as the solution to the following pair of equations:

$P = C \cdot (p' - p), \qquad m - m' = d,$ where *P, C* and *d* are given.

In order to simplify the problem, the first equation may be multiplied by the product of the market rates. Then the pair of equations above is replaced by the following equivalent pair of equations:

$m \cdot m' \cdot P = C \cdot (m - m') = C \cdot d, \qquad m - m' = d.$

Explicitly, with the given data, *C* = 5 00, *d* = 2, and *n* = 7;30,

$m \cdot m'$ = $C \cdot d \cdot$ rec. P = 5 00 · 2 · rec. 7;30 = 1 20, $\quad m - m' = d = 2.$

This is *a rectangular-linear system of equations of the basic type B1b* for the two unknowns *m* and *m´*. (See Friberg, *AT* (2007), 6.) Rectangular-linear systems of equations of this kind can easily be solved by a geometric method, illustrated in Fig. 5.1.19 above. In the present case, an application of that method shows that the solution to the system of equations above for *m* and *m´* is

m = sqs. (sq. *d*/2 + *C* · *d* · rec. *P*) + *P*/2 = sqs. (1 + 1 20) + 1 = sqs. 1 21 + 1 = 9 + 1 = 10
m´ = sqs. (sq. *d*/2 + *C* · *d* · rec. *P*) – *P*/2 = sqs. (1 + 1 20) – 1 = sqs. 1 21 – 1 = 9 – 1 = 8.

This result agrees with the explicitly given answer in lines 7-8 of § 5 a.

YBC 4698 § 5 b (# 11)

This exercise is just like the exercise in § 5 a, only the data are different. Here *C* = 5 00, *d* = 1;30, and *P* = 6;40. Consequently,

$m \cdot m'$ = $C \cdot d \cdot$ rec. P = 1;30 · 5 00 · rec. 6;40 = 1 07;30, $\quad m - m' = d$ = 1;30.

Therefore, by the solution method shown in Fig. 5.1.19, as in the preceding example,

m = sqs. (sq. *d*/2 + *C* · *d* · rec. *n*) + *n*/2 = sqs. (;33 45 + 1 07;30) + ;45 = sqs. 1 08;03 45 + ;45 = 8;15 + ;45 = 9
m´ = sqs. (sq. *d*/2 + *C* · *d* · rec. *n*) – *n*/2 = sqs. (;33 45 + 1 07;30) ;45 = sqs. 1 08;03 45 – ;45 = 8;15 – ;45 = 7;30.

This result agrees with the explicitly given answer in line 5 of § 5 b. Indeed, the explicitly given answer is that the two market rates were 9 sìla and 7 1/2 sìla (per shekel of silver).

Another example of an Old Babylonian mathematical text with a commercial problem leading to a rectangular-linear system of equations is *TMS* XIII. The question in that exercise is a close parallel to the questions in YBC 4698 §§ 5 a-b, but several of the special commercial terms which are expressed as sumerograms in YBC 4698 §§ 5 a-b are expressed in Akkadian in *TMS* XIII.

***TMS* XIII** (Bruins and Rutten, *TMS* (1961), text 13; Høyrup, *LWS* (2002), 206; Friberg, *UL (2005)*, 218)

1 2(gur) gur 2(barig) bariga 5 bán ì.geš sa$_{10}$ /
2-3 *i-na* sám 1 gín kù.babbar / 4 sìla.ta.àm ì.geš *ak-ši-iṭ-ma* /
4 $^2/_3$ ma.na kù.babbar *ne-me-la a-mu-ur*
5 *ki ma-ṣí* / *aš-šà-am ù ki ma-ṣí ap-šu-ur*
… ………

1 2 gur 2 bariga 5 bán of common oil I bought. /
2-3 From the purchase price for 1 shekel of silver / 4 sìla each time of common oil I cut off, then
4 2/3 mina of silver as profit I saw.
5 (At) how much did I buy and (at) how much did I sell?
… ………

MLC 1842 is yet another example of an Old Babylonian mathematical text with a commercial problem leading to a rectangular-linear system of equations. However, it is not such a close parallel to YBC 4698 §§ 5 a-b as *TMS* XIII. Anyway, this text, too, has an interesting Akkadian terminology, and the question in the text is an interesting variant of the questions in *TMS* XIII and YBC 4698 §§ 5 a-b .

MLC 1842 (Group 5?; Neugebauer and Sachs, *MCT* Sb; Friberg, *UL*, 219))

1-2 ganba.e *i-li-ma* 30 še gur *a-ša-am* / ganba *iš-pí-il-ma* 30 še gur *a-ša-am* /
3-4 *ma-ḫi-ri-ia ak-mu-ur-ma* 9 / kù.babbar *ma-ḫi-ri-ia ak-mu-ur-ma* / 1 ma.na 7 $^1/_2$ gín /
5-6 ganba <*ki-ia*> *a-ša-am ù ki-ia* / *ap-šu-ur*

1-2 The market rate increased, 30 gur of grain I bought, / the market rate decreased, 30 gur of grain I bought.
3-4 My market rates I added together, 9 / the silver of my market rates I added together, 1 mina 7 1/2 shekels.
5-6 The market rates, <at> how much did I buy, and <at> how much / did I sell?

In this exercise, 30 gur of grain is bought at a high market rate (that is, at a low price), and then another 30 gur of grain is bought at a lower market rate (that is, at a higher price). Note the phrase

ganba.e *i-li-ma* …… ganba *iš-pi-il-ma* ……

which is reminiscent of similar phrases occurring in some of the combined market rate exercises, such as YBC 4698 §§ 2 d-e (Sec. 11.1.3 above) and VAT 7530 ## 3, 5-6 (Fig. 11.1.6 above).

In the question in MLC 1842, the capacity of the grain bought is C = 30 gur = 2 30 00 sìla, the sum of the market rates is '9', and the sum of the two amounts of silver paid is S = 1 mina 7 1/2 shekels. The final question in lines 5-6 is awkwardly and incorrectly formulated, but what is wanted is, of course, to find the size of the two market rates.

In quasi-modern terms, the question can be rephrased as a system of linear equations:

$m + m' =$ '9' (sìla per shekel of silver), $C \cdot (p + p') =$ 1 mina 7 1/2 shekels = 1 07;30 shekels = S.

Here, as usual, *m* and *m´* are the two unknown market rates, *p* and *p´* are the corresponding unit prices, and *C* = 30 gur is the amount of grain bought on both occasions. The solution procedure in this case is a variant of the solution procedure in the case of YBC 4698 § 5 a (see above). The idea is again to multiply the second equation by the product of the market rates (and divide by *S*). The result is the following equivalent rectangular-linear system of equations for two unknowns, for simplicity with the numbers *in relative place value notation*:

$$m + m' = 9\,, \qquad m \cdot m' = C \cdot (m + m') \cdot \text{rec. } S = 2\ 30 \cdot 9 \cdot \text{rec. } 1\ 07\ 30 = 2\ 30 \cdot 8 = 20.$$

This is a rectangular-linear system of equations of the basic type B1a. (See again Friberg, *AT* (2007), 6.) The solution is given by the solution method shown in Fig. 5.1.19 as

$$m = 4\ 30 + \text{sqs. (sq. } 9/2 + 20) + 9/2 = \text{sqs. } (20\ 15 - 20) = 4\ 30 + \text{sqs. } 15 = 4\ 30 + 30 = 5$$
$$m' = 4\ 30 - \text{sqs. (sq. } 9/2 - 20) - 9/2 = \text{sqs. } (20\ 15 - 20) = 4\ 30 - \text{sqs. } 15 = 4\ 30 - 30 = 4.$$

Actually, however, the sum of the market rates, which is simply given as ‘9’ in the question, must be 9 00 (sìla per shekel), the product of the market rates must be 20 00 00, and the more realistic answer must be that the two market rates are 5 00 sìla = 1 gur and 4 00 sìla = 4 bariga per shekel of silver!

(Note that in their discussion of *MCT* Sb, Neugebauer and Sachs, who did not understand the conversion of 30 gur into ‘2 30’ sìla, invented an erroneous solution procedure and got a wrong answer to the problem.)

A final example of an Old Babylonian mathematical text with a commercial problem leading to a rectangular-linear system of equations is the single exercise tablet MS 3895 (Friberg and George, *PF 40* (2010), Sec. 3.) Only the question and the answer of that interesting text are reproduced below.

MS 3895

1-1b [*a-na šá*]*m* 1(barig) 3(bán) ì.sag *i-na* é / um.mi.a kù.babbar *el-qé-ma* /
2 *a-ša-am-ma ma-ḫi-ir a-ša-mu ú-ul i-de* /
3 [*a*]*p-š*[*u-u*]*r ma-ḫi-ir ap-šu-ru ú-ul i-de*
……………
23-24 3 4[5 *ma-ḫi-ir a-š*]*a-mu* / 3 [*m*]*a-*[*ḫi-ir*] *ap-šu-ru*

1-1b [To bu]y 1 bariga 3 bán of fine oil I borrowed silver from a lending-house,
2 I bought, but the market rate at which I bought I did not know,
3 I so[ld], but the market rate at which I sold I did not know.
……………
23-24 3 4[5 was the market rate at which I] bought / 3 the market [rate] at which I sold.

11.1.7 § 6. Lead, Silver, and Precious Stones. Linear Equations

YBC 4698 § 6 (## 16-17, *rev. ii*: 7-22)

§ 6 a 1-3 *ma-la* ganba.ḫá an.na / kù ì.lá-*ma* / an.na sa$_{10}$ /
4-5 kù *ù* an.na gar.gar-*ma* / 7 ma.na
6 kù *ù* an.na / en.nam

1-3 As much as the market rates of lead / silver I weighed up, then / lead I bought. /
4-5 Silver and lead I added together, then / 7 minas.
6 Silver and lead / were what?

§ 6 b 1-3 zá šu.ti.a-*ma* / ki.lá.bi nu.zu / 6.ta sa$_{10}$ /
4-6 6.ta kù.sig$_{17}$ gar.ra / kù.sig$_{17}$ gar.ra / a.na 6.ta sa$_{10}$ /
7 ˹zá˺ kù.˹babbar˺ en.˹nam˺ /
8 ˹gar.gar˺-*ma* $^1/_2$ ma.na 1 $^1/_2$ gín /
9 [x] ma [x] gar.gar-*ma* / 1 ma.na 3 gín

1-3 Precious stones I received, but / their weights I did not know. / At 6 I bought.
4-6 At 6 of gold set / the silver set, / as much as at 6 I bought.
7 The stones (and) the silver were what?
8 Add them together, then 1/2 mina 1 1/2 shekel.
9 [x] x [x] add together, then / 1 mina 3 shekels.

YBC 4698 § 6 a (# 16)

In this exceedingly simple exercise, an unknown quantity of lead was paid for in silver, at the going market rate for lead, which is not explicitly mentioned. When the lead was weighed together with the silver paid for it, the result was 7 minas. How much lead was there, and how much silver?

In order to solve this problem, let *s* (shekels) be the weight of the silver, and let *m* be the market rate of lead paid for in silver. Then

$m \cdot s + s = 7\ 00$ (shekels) or $(m + 1) \cdot s = 7\ 00$ (shekels).

The most reasonable assumption in this situation seems to be to assume (tentatively) that in this text lead was 6 times less valuable than silver, so that $m = 6$. Then, clearly, the answer to the question in § 3 d is that

6 minas of lead was bought for 1 mina of silver, so that the combined weight of lead and silver was 7 minas.

YBC 4698 § 6 b (# 17)

In § 3 e, the unit price (not the market rate!) is denoted by the term 6.ta. That means that in this text gold was assumed to be 6 times more valuable than silver, which is quite reasonable. Apparently what is going on here is that (golden) gems of unknown weight were bought for silver and that, just like in the preceding exercise, the gems and the silver paid for them were weighed together. In line 8 of the exercise, which seems to be misplaced (it should have preceded line 7), it is said that the combined weight was 1/2 mina 1 1/2 shekels = 31;30 shekels. What was then the weight of the gems and what was the weight of the silver?

In order to solve this problem, let *g* (shekels) be the weight of the gems. Then

$6\ g + g = 31;30$ (shekels) or $7\ g = 7 \cdot 4;30$ (shekels).

The solution to this simple linear equation is obviously that

$g = 4;30$ (shekels), so that golden gems weighing 4 1/2 shekels were bought for 27 shekels of silver.

Line 9 of exercise § 3 e is damaged and possibly does not belong to this exercise at all. However, if it does, it may just mention an alternative total weight of the golden gems and the corresponding silver. Clearly, if the combined weight of gems and silver was twice as much as before, then also the weight of the gems and the weight of the silver must have been twice as much. Therefore (possibly) the answer to this variant of the question was meant to be that

Golden gems weighing 9 shekels were bought for 54 shekels of silver. Together, gems and silver weighed 1 mina 3 shekels.

11.1.8 ***Subscript***

YBC 4698 subscript

[………] dub 3.kam.ma

[………] the 3rd tablet.

This subscript appears to indicate that YBC 4698 is a so called "series text", like quite a few other Old Babylonian and Kassite(?) numbered texts published by Neugebauer in *MKT I-III* (1935), and by Neugebauer and Sachs in *MCT* (1945). In *ZA* 99 (2009), Proust published two additional series texts. For a recent extensive and thought provoking discussion of all the mentioned texts, now for additional clarity divided into two categories, proper "series texts" and "catalogues", interested readers are referred to Proust, *NTM* 20 (2012). However, since the subject is as difficult to come to terms with as it is interesting, it may not be out of the way to say a few more words about it. This will be done in Sec. 11.4 below.

11.1.9 The Vocabulary of YBC 4698

a.na	*mala*	as much as, whatever
an.bar		iron

an.na		lead, tin
.bi		its(?)
búr.ra	*pašāru* (MS 3895)	to release, to sell
dirig		excess, beyond
e$_3$		to go out, to take away
e$_{11}$ (ed$_3$)	*elû, warādu* (VAT 7530)	to go up, to go down
en.nam		what?
ganba	*maḫīru* (MS 3895)	market rate
gar.gar		to add together
geš'u		10(· 60)
íb.sá		to make equal
ki.lá		weight
kù.babbar		silver
kù.dirig, dirig	*nēmelu* (MS 3895)	profit, excess of silver
kù.sig$_{17}$		gold
ku$_6$.a		fish
máš	*ṣibtu* (AO 6770)	interest
sa$_{10}$	*šâmu* (MS 3895)	to buy
sag	*rēšu* (*TMS* XXII)	principal, capital
sám		purchase price
še		grain
šúm	*nadānu* (AO 6770)	to give, to pay
šu.ti.a	*leqû* (VAT 8528)	to take, to receive
šuš$_4$	*kašāṭu* (*TMS* XIII)	to peel off
zá		gem, precious stone
a.na máš šúm	*a-na ṣi-ba-at i-di-in* (AO 6770)	he paid as interest
ganba íb.sá	*ma-ḫi-ru li-im-ta-ḫar* (VAT 7530)	the market rates I made equal / may be equal
ḫé.e$_3$ *ù* ḫé.e$_{11}$	*li-li ù li-ri-id* (VAT 7530)	it may go up or it may go down
nu.zu	*ú-ul i-de* (MS 3895)	I did not know
(*n*) sìla.ta, gur.ta *etc.*		at a rate of (*n*) sìla or gur *etc.* (per unit of silver or grain)

11.2 YBC 4673. An Old Babylonian Numbered Recombination Text with Problems about Bricks, Mud, and Reeds

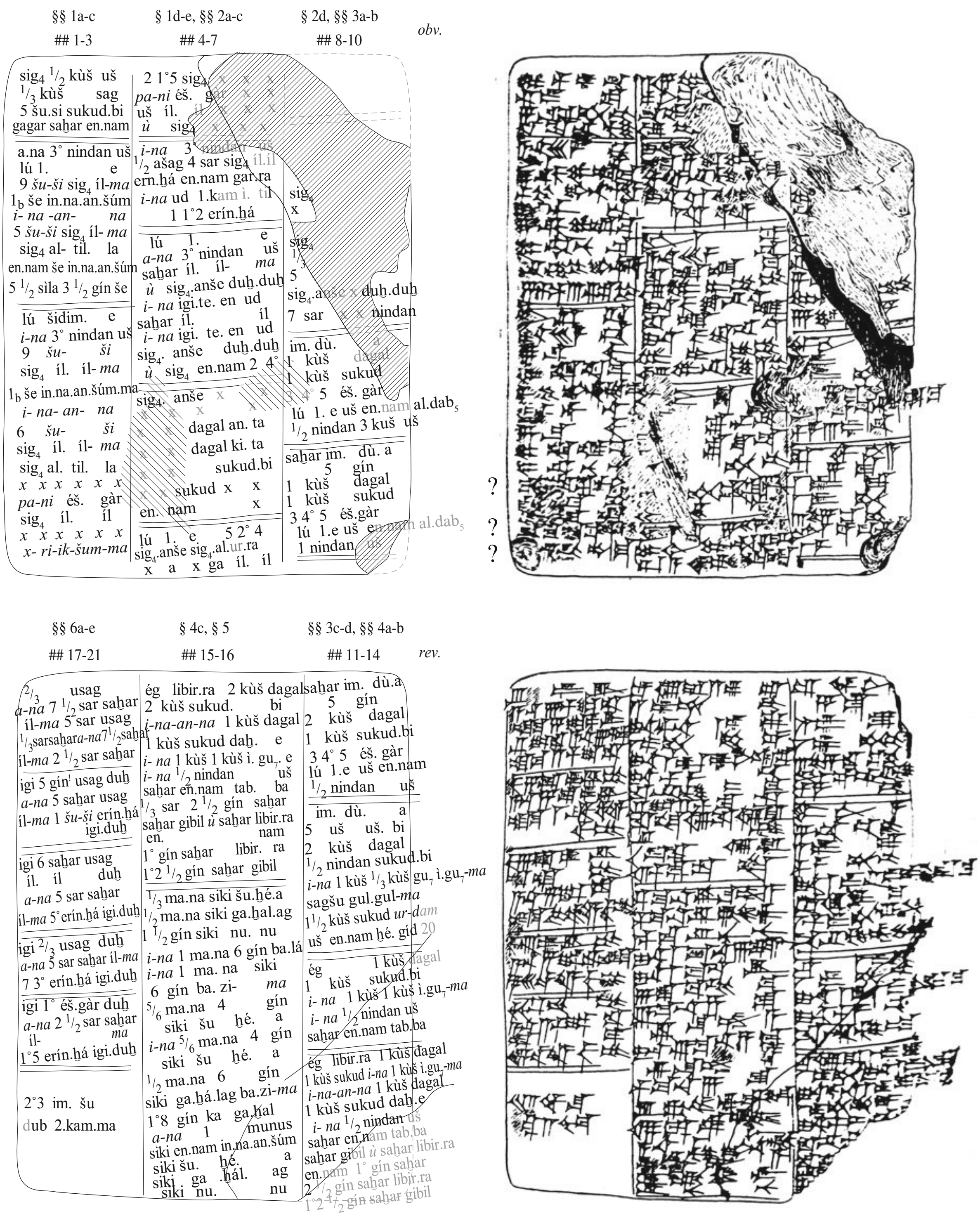


Fig. 11.2.1. YBC 4673. An Old Babylonian numbered recombination text.

Hand copy by O. Neugebauer, *MKT III*. The hand copy is reproduced here with the permission of Springer.

YBC 4673 (Neugebauer, *MKT III* (1937), 29-31) is a fairly well organized Old Babylonian mathematical recombination text in 3 columns. The obverse of the tablet is damaged with the right upper corner missing, but the reverse is well preserved. The text can be divided into 5 consecutive paragraphs, as follows:

§ 1 Four problems for rectangular bricks of type R1/2c
§ 2 Four problems about brick piles
§ 3 Four problems about building and rebuilding a mud wall
§ 4 Three problems about old and new levees
§ 5 One problem about three kinds of wool
§ 6 Five problems about 'reed work' (reinforced underwater bases of dams)

The wool problem does not fit in with the other four kinds of problems, which are all about three-dimensional constructions. Several of the three-dimensional problems were discussed already in Friberg, "Bricks and Mud" in *ChV* (2001), and also, briefly, in Robson *MMTC* (1999).

Answers but *no solution procedures* are given in the text.

11.2.1 § 1. Problems about Bricks of Type R1/2c

YBC 4673 § 1 (## 1-4; *obv. i*: 1-*ii*: 9) Cf. Friberg, *ChV* (2001), Secs. 4.7, 6.4.

§ 1a 1-3 sig$_4$ 1$^1/_2$ kùš uš / $^1/_3$ kùš sag / 5 šu.si sukud.bi /
4 gagar saḫar en.nam

1-3 Bricks. 1/2 cubit the length / 1/3 cubit the front / 5 fingers its height. /
4 Ground (bottom area) and mud (volume) what?

§ 1b 1-4 a.na 30 nindan uš / lú 1.e / 9 *šu-ši* sig$_4$ íl-*ma* / 1(bán) še in.na.an.šúm /
5-8 *i-na-an-na* / 5 *šu-ši* sig$_4$ íl-*ma*/ sig$_4$ al.til.la / en.nam še in.na.an.šúm /
9 5 $^1/_2$ sìla 3 $^1/_3$ gín še

1-4 As much as 30 rods of length / 1 man / 9 sixties of bricks carried, then / 1 bán of grain I gave him. /
5-8 Now / 5 sixties of bricks he carried, then / the bricks were finished. / What of grain did I give him? /
9 5 1/2 sìla 3 1/3 shekels of grain.

§ 1c 1-5 lú.šidim.e / *i-na* 30 nindan uš / 9 *šu-ši* / sig$_4$ íl.íl-*ma* / 1(bán) še in.na.an.šúm.ma /
6-10 *i-na-an-na* / 6 *šu-ši* / sig$_4$ íl.íl-*ma*/ sig$_4$ al.til.la / x x x x /
11-15 *pa-ni* éš.gàr / sig$_4$ íl.íl / x x x x / [*x*]-*ri-ik-šum-ma /* 2 15 sig$_4$ [x x x] /
16-18 *pa-ni* éš.g[àr x x] / uš íl.[íl x x x] / *ù* sig$_4$ [x x x]

1-5 A brick worker / from 30 rods of length / 9 sixties / of bricks carried, then / 1 bán of grain I gave him. /
6-10 Now / 6 sixties of bricks / he carried, then / the bricks were finished. / x x x x /
11-15 The reciprocal of the work norm / for brick carrying / x x x x / x x x x / 2 15 bricks [x x x] /
16-18 The reciprocal of the work no[rm x x x] / the length he carri[ed x x x] / and the bricks [x x x].

§ 1d 1-3 *i-na* 30 [nindan uš] / $^1/_2$(iku) ašag 4 sar sig$_4$ [íl.íl] / erín.ḫá en.nam gar.ra /
4-5 *i-na* ud 1. k[am ì.til] / 1 12 erín.ḫá

1-3 From 30 [rods of length] / 1/2 iku of field 4 sar of bricks he carried. / Of men, what to set /
4-5 (so that) in 1 day they will b[e finishe]d? / 1 12 men.

YBC 4673 § 1a (# 1)

Asked for in this introductory exercise is the *bottom area* (called gagar 'ground') and *volume* (called saḫar 'mud') of a (certain kind of) brick with the length 1/2 cubit, the width 1/3 cubit, and the height 5 fingers. These are the dimensions of *a standard rectangular brick of type R1/2c* (see Friberg, *ChV* (2001), 74). The bottom area and the volume are, of course,

A = 1/2 cubit · 1/3 cubit = ;02 30 rods · ;01 40 rods = (;0)4 10 area-shekels,
V = 1/2 cubit · 1/3 cubit · 5 fingers = 1/6 sq. cubit · 1/6 cubit = ;00 04 10 sq. rods · ;10 cubit = (;00) 41 40 volume-shekels.

YBC 4673 § 1b (# 2)

This exercise starts by mentioning the standard "carrying number" for bricks of type R1/2c. The carrying number (Akk. *nazbalum*) is an important brick parameter, different for different types of bricks. It is the (Old Babylonian) *work norm for a man carrying bricks*, in terms of *number of bricks carried · distance walked*. (See Friberg, *op. cit.*, 72.) In the case of bricks of type R1/2c the work norm is

9 sixties of bricks · 30 rods of length.

It is further mentioned in this exercise that the man carrying the bricks is paid 1 bán of grain for his work. (This is an example of the well known *standard Old Babylonian wages for a hired man*, namely 1 bán of grain or 6 grains of silver per day, corresponding to 1 gur of grain or 1 shekel of silver per month.) The question now is what a hired man would be paid if there were only 5 sixties of bricks (to be carried the standard distance of 30 rods). The answer is, in a simple application of "the rule of three",

5 · 1/9 of 1 bán = 5 · 1/9 · 10 sìla = 1/9 of 50 sìla = 5 1/2 sìla 3 1/3 shekel of grain.

This is also the answer given in the text. However, the computation was probably carried out in the form

1/9 = 6 40, 5 · 6 40 = 33 20.

YBC 4673 § 1c (# 3)

The first 6 lines of § 1c are, essentially, the same as the first 5 lines of the preceding § 1b, mentioning the standard work norm for carrying bricks of type R1/2c, as well as the standard daily wages. The question now seems to be what a hired man would be paid if there were 6 sixties of bricks to be carried. Unfortunately, several lines of text in § 1c are hard to read, but it is clear, anyway, that instead of giving a numerical answer, *in lines 11-14 the general procedure for finding the answer is described verbally, starting with the instruction to compute the reciprocal of the work norm (pa-ni* éš.gàr sig_4 íl.íl 'the reciprocal of the work norm for carrying bricks').

The lines of text in *obv. ii*: 1-4, which are badly damaged, may possibly have been concerned with a variant of the same exercise, with different data, 2 15 bricks to be carried instead of 6 sixties. Note that the ingress of the paragraph, specifying the work norm and standard wages, is not repeated. This is the only instance of a tree-like structure in this text! (Remember that extended tree-like structures are characteristic for an important category of "series texts"! See Proust, *ZA* 99 (2009) and Sec. 11.4 below.)

Note also that not only the two different ways of formulating the answer to the stated question in § 1b and § 1c, respectively, but also a slightly different use of vocabulary, seems to indicate that these two sub-paragraphs of the recombination text YBC 4673 were copied from two different original theme texts!

YBC 4673 § 1d (# 4) (Cf. Friberg, *ChV (2001)*, Sec. 4.7.)

In this exercise, it is silently assumed that the work norm for carrying bricks is the same as in §§ 1b and 1 c. However, this time the available bricks are not counted in sixties as before, but in "brick-sar", where 1 sar of bricks = 12 sixties of bricks. Consequently, the mentioned number of bricks in this sub-paragraph, to be carried the standard length of 30 rods, can be understood as

1/2 iku 4 sar of bricks = 54 sar of bricks = 54 · 12 sixties of bricks.

Again by the rule of three, this is the following multiple of the daily norm:

1/9 · 54 · 12 = 6 · 12 = 1 12.

In the text of § 1d, this computation is not mentioned. However, the answer to the question 'How many men will finish the job in 1 day?' is given, correctly, as 1 12 erín 'men'. The answer can be interpreted as meaning that

1/2 iku 4 sar of bricks can be carried a distance of 30 rods in 1 12 "man-days" (men · days).

11.2.2 § 2. Problems about Carrying Mud and Molding Bricks

YBC 4673 § 2 (## 5-8; *obv. ii*: 10-*iii*: 8´)

§ 2a 1-4 lú.1.e / a.na 30 nindan uš / saḫar íl.íl-*ma* / *ù* sig_4.anše duḫ.duḫ /
5-7 *i-na* igi.te.en ud / saḫar íl.íl / *i-na* igi.te.en ud / sig_4.anše duḫ.duḫ /

8-9 *ù* sig$_4$ en.nam / 2 40

1-4 1 man / as much as 30 rods of length / mud he carried, / and brick piles he molded. /
5-9 In (what) fraction of a day / did he carry mud / in (what) fraction of a day / did he mold brick piles /
8-9 and the bricks were what? / 2 40.

§ 2b 1-5 sig$_4$.anše x x / [x x] x [x] / [x x] dagal an.ta / [x x] dagal ki.ta / [x x] sukud.bi /
6-7 [x x] sukud x x / en.nam [x]

1-5 A brick pile. x x / [x x] x [x] / [x x] the upper width / [x x] the lower width / [x x] its height. /
6-7 [x x] the height x x / what [x] ?

§ 2c 1-3 lú.1.e 5 24 x / x sig$_4$.anše sig$_4$ al.ur$_5$.ra / x x x x íl.íl /
… ………

1-3 1 man 5 24 x / x brick pile square bricks / x x x x carried.
… ………

§ 2d 1´-5´ sig$_4$ [x x x x x] / x [x x x x x] / sig$_4$ [x x x x x] / 1/3 [x x x x x] / 5 [x x x x x] /
6´-7´ sig$_4$.anse duḫ.duḫ / 7 sar [x x] x

1´-5´ Bricks. [x x x x x] / x [x x x x x] / bricks [x x x x x] / 1/3 [x x x x x] / 5 [x x x x x] /
6´-7´ a brick pile he molded / 7 sar [x x] x.

YBC 4673 § 2a (# 5)

Here a *man carries mud over a distance of 30 rods and then uses the mud he carried in order to mold bricks (presumably still standard rectangular bricks of type R1/2c). The question is what part of the day was used for carrying mud, and what part for molding bricks. Asked for is also the number of bricks molded (in one day).* An answer to the exercise is given in the imprecise form ‘2 40’.

An only partly successful attempt to explain this exercise was made in Friberg, *ChV (2001)*, Sec. 6.4. The limited success was caused in part by an unfortunate misreading of a line of the text.

Clearly two important brick and mud parameters were assumed to be known in this exercise, namely *the work norms for carrying mud and for molding bricks.* Such work norms are known from a number of Old Babylonian “tables of constants”. The following list of constants that may be relevant in the present situation is excerpted from the survey in Friberg, *op. cit.*, Sec. 1.

TMS, text III	5 *šà na-al-ba-ni*	‘5 of molding’	BR 45
IM 49949	2 13 20 igi.gub [geš]íl *šu-up-ši-ki-im*	‘2 13 20, constant of a carrying basket’	E 3
	1 40 *na-az-ba-al* saḫar	‘1 40, carrying mud’	E 4
	4 30 *na-az-ba-al* sig$_4$	‘4 30, carrying bricks’	E 5
	7 12 *a-ma-ri-im*	‘7 12 of a brick pile’	E 6
	45 igi.gub *mu-ta-li-ik-tim*	‘45, constant of walking’	E 16
IM 52916	*na-az-ba-al* sig$_4$ 45 *i-gi-gu-bu*	‘45, constant, carrying bricks’	§ 3b, 1
	na-az-ba-al saḫar 1 40 *i-gi-gu-bu*	‘1 40, constant, carrying mud’	§ 3b, 2
	na-az-ba-al še.in.nu.da 3 20 *i-gi-gu-bu*	‘1 40, constant, carrying straw’	§ 3b, 3
A 3553	7 12 igi.gub sig$_4$	‘7 12, constant, bricks’	Ka 24
	45 igi.gub a.rá	‘45, constant, walking’	Ka 28
	4 30 igi.gub íl.íl	‘4 30, constant, carrying’	Ka 29
YBC 5022	4 30 *na-az-ba-al* sig$_4$	‘4 30, carrying bricks’	NSd 2
	1 40 *na-az-ba-al* saḫar	‘1 40, carrying mud’	NSd 8
	7 12 *na-al-ba-an-ša*	‘7 12, molding them’	NSd 3
	7 12 *ša* sig$_4$.anše	‘7 12 of a brick pile’	NSd 16
	45 *mu-ut-ta-al-li-ik-tum*	‘45, walking’	NSd 39
	1 40 *ma-aš-šu-ú-um ša* saḫar	‘1 40, carrying board for mud’	NSd 42
YBC 7243	7 12 sig$_4$.anše	‘7 12, brick pile’	NSe 40
	4 30 *na-az-*[*ba-al*]*-lu-um*	‘4 30, carrying’	NSe 41
	45 *ta-la-ak-tum*	‘45, walking’	NSe 46

The constant '5' of 'molding' is explained by the exercise Haddad 104, # 9 (see Friberg, *op. cit*., Sec. 6.1), where 5 (volume-shekels per man-day) is the combined work norm for *crushing* clay or straw, *mixing* clay with straw, and *molding* bricks. (Regarding combined work norms, see Friberg *RlA 7* (1990), § 5.6h.) The corresponding individual work norms are w = 20, w' = 20, and w'' = 10 (volume-shekels per man-day). The combined work norm is computed as follows:

rec. w + rec. w' + rec. w'' = rec. 20 + rec. 20 + rex. 10 = 3 + 3 + 6 = 12, rec. 12 = 5.

The constant '1 40' for carrying mud is similarly explained by the exercise Haddad 104, # 10 (see Friberg, *ChV (2001)*, Sec. 6.2). Apparently, mud could be carried in baskets with a standard volume of ;02 13 20 volume-shekels (= 1/27 volume-shekel). *In a day's work, a man was expected to walk a distance of 1 uš (= 60 rods) 45 times, carrying a basket full of mud*. The total amount of mud carried in this way was

;02 13 20 volume-shekel · 45 uš = 1 40 volume-shekels · rods (per man-day).

The related constant '4 30' for carrying bricks (of type R1/2c) can, of course, be explained as the work norm

9 sixties of bricks · 30 rods = 4 30 bricks · 1 uš (per man-day).

Finally, *the constant '7 12' for a brick pile, or for molding,* has the following explanation. A standard rectangular brick had the volume ;00 41 40 volume-shekel. (See exercise YBC 4673 § 1a above.) Therefore, 7 12 such bricks had the combined volume 5 volume-shekels. Indeed, in relative place value notation,

7 12 · 41 40 = (1/5 of 36) · (1/36 of 25) = 5.

This means that *the 'molding number' L = 7 12 for bricks of type R1/2c is the number of such bricks that together have a volume of 5 volume-shekels, and 5 volume-shekels per man-day is precisely the work norm for molding bricks*. (Brick types and molding numbers were discussed already in Sec. 8.1 above, in connection with four preserved problems for bricks on the fragment BM 80078.)

Now return to exercise § 1e of YBC 4673, where a man carries mud over a distance of 30 rods and then uses the mud he carried in order to mold bricks. As mentioned, the work norm for carrying mud was

1 40 volume-shekels · rods per man-day which is the same as 3 1/3 volume-shekels · 30 rods per man-day.

In other words, in one man-day 3;20 volume-shekels of mud could be carried the required distance. Conversely,

1 volume-shekel of mud could be carried 30 rods in rec. 3;20 = ;18 man-days.

In a similar way, the work norm for molding bricks was, as mentioned above, that

5 volume-shekels of bricks could be fabricated in 1 man-day,

and conversely,

1 volume-shekel of bricks could be fabricated in rec. 5 = ;12 man-days.

Now, *if it is assumed that the mud bricks were made entirely out of mud* (or clay), it follows that

1 volume-shekel of mud could be carried the required distance and converted to bricks in
rec. 3;20 + rec. 5 = ;18 + ;12 = ;30 man-days.

Conversely, again,

rec. (rec. 3;20 + rec. 5) = rec. ;30 = 2 volume-shekels of mud could be carried 30 rods and converted to bricks in 1 man-day.

This kind of arguing is typical for the computation of a combined work norm. The time required would be

2 · ;18 = ;36 man-days for carrying the clay and 2 · ;12 = ;24 man-days for molding the bricks.

It remains to be calculated how many standard rectangular bricks of type R1/2c could be made out of the 2 volume-shekels of mud. Now, as was mentioned above,

The combined volume of 7 12 bricks of type R1/2c = 5 volume-shekels.

Consequently,

2 volume-shekels of mud was enough for 2 · rec. 5 · 7 12 = 2 52;48 bricks of type R1/2c.

This result is close to but not precisely equal to the answer given in the last line of § 1e, namely 2 40 (bricks).

It is not difficult to see that in order to get the desired answer that the number of fabricated bricks was precisely 2 40, one has to assume that the molding number was not 7 12 but 6 40. (Note that YBC 4673 § 1e is is an exercise in a recombination text *without context*, and therefore without any indication of what value the molding number was supposed to have.) Indeed, if the molding number was 6 40, then

2 volume-shekels of mud would be enough for 2 · 1/5 · 6 40 = 2 40 bricks.

Unfortunately, this attempted explanation of exercise § 1e does not work, since the molding number 6 40 is not attested in any known cuneiform mathematical text. See Friberg, *ChV (2001)*, Table 4.1. (Indeed, 6 40 would have to be the molding number for an unattested half-brick with the long side 2/3 cubit.)

Except for the unpalatable way out to assume that the stated answer in § 1e was erroneous, the only remaining possibility is to assume that the mud bricks in this exercise were *not* entirely made of mud. Assume, for instance, that "reinforced mud bricks" were made of, with regard to volume, *5 parts mud and 1 part straw*. Then

;50 volume-shekels of mud were needed to make 1 volume-shekel of bricks.

With this assumption, an alternative calculation of a combined work norm would proceed as follows:

1 volume-shekel of bricks could be fabricated in rec. 5 = ;12 man-days,
;50 volume-shekels of mud could be carried 30 rods in ;50 · rec. 3;20 = ;50 · ;18 man-days = ;15 man-days.

Therefore,

;50 volume-shekels of mud could be carried 30 rods and used to make 1 volume-shekel of bricks in ;27 man-days.

Conversely,

mud could be carried 30 rods and mixed with straw to make rec. ;27 = 2;13 20 volume-shekels of bricks in 1 man-day.

Now, make the further assumption that the bricks in question were 6 fingers thick rectangular bricks of the variant type R1/2cv. (See Friberg, *ChV* (2001), 75.) Such bricks have the molding number '6', which means that

6 sixties bricks of type R1/2cv have a volume of 5 volume-shekels.

Hence, by the rule of three,

2;13 20 (60/27) volume-shekels of bricks is the volume of 2;13 20 · 1/5 · 6 sixties = 2 40.

The argument shows that *if the bricks considered in YBC 4673 § 1e were 6 fingers thick rectangular bricks of type R1/2cv, then the answer given in the text of the exercise is correct.*

Even more interesting is the surprising discovery that generally

Variant bricks with a thickness of 6 fingers instead of the usual thickness 5 fingers
were bricks made of 5 parts mud and 1 part straw!

This conclusion is strongly supported by the observation that the Old Babylonian mathematical exercises Haddad 104 ## 9-10 (Friberg, *ChV (2001)*, Secs. 6.1-2), which contain the computation of *combined work norms* for crushing (whatever that means), *molding, mixing, and carrying*, in a way reminiscent of the explanation above of YBC 4673 § 1e, deal with variant 6 fingers thick square bricks of type S2/3cv (Friberg, *op. cit.*, 80).

The question in § 1e about what part of the day was used for carrying mud, and what part for molding bricks is now easy to answer. It was shown above that ;50 volume-shekels of mud could be carried 30 rods in ;15 man-days and used with straw to make 1 volume-shekel of bricks in ;12 man-days. Consequently,

mud for 2;13 20 (60/27) volume-shekels of brick could be carried 30 rods in 2;13 20 · ;15 man-days = ;33 20 man-days
and used with straw to make bricks in an additional 2;13 20 · ;12 man-days = ;26 40 man-days.

In other words, *mud was carried for ;33 20 man-days (5/9 of the day) and bricks were molded for ;26 40 man-days (4/9 of the day).*

It deserves to be mentioned here that an alternative interpretation of YBC 4673 § 1e was made by Robson in *MMTC* (1999), 90, with departure from a new, improved reading of the following entry in Baqir's Old Babylonian table of constants:

IM 52301 4 10 *ša* sig_4 '4 10 of bricks' B 3

(Compare with the observation that 5/6 · 5 volume-shekels = 4;10 volume-shekels of mud per man-day would be required for the molding of bricks with mud and straw in the proportions 5 to 1.) According to Robson, ;04 10 volume-sar per man-day can be interpreted as the work norm for piling standard bricks. Then the combined work norm for carrying mud over a distance of 30 rods and making brick piles could be computed as follows:

rec. (rec. ;03 20 + rec. ;04 10) = rec. (18 + 14;24) = rec. 32;24 = ;01 51 06 40 (50/27) volume-sar of standard bricks per man-day.

This would correspond to the probably incorrect result that

;01 51 06 40 volume-sar of bricks · 7;12 · 12 sixties standard bricks per volume-sar = 2 40 standard bricks (of type R1/2c).

YBC 4673 §§ 2b-d (# 6-8)

These three exercises are too badly preserved to allow any meaningful reconstruction or explanation of them. Nevertheless, it is clear that at least one of them deals with carrying [mud] and building sig_4.anše 'brick piles' of sig_4.al.ur.ra 'square bricks'. At least that is what is suggested by the following preserved lines of text in § 2b:

lú.1.e 5 24 / sig_4.anše sig_4.al.⌜ur⌝.ra / x x x x íl.íl / …………

11.2.3 § 3. Problems about Building and Rebuilding a Mud Wall

YBC 4673 § 3 (## 9-12; *obv. iii*: 9´-14´- *rev. i*: 14) Cf. Friberg, *ChV (2001)*, Sec. 5.5.

§ 3a 1-3 im.dù.[a] / 1 kùš [dagal] / [1] kùš sukud / [3 4]5 éš$^!$.gàr /
4-5 lú.1.e uš en.[nam] al.dab_5 / ½ nindan 3 kùš uš

1-3 A clay wa[ll]. / 1 cubit the [width] / [1] cubit the height / [3 4]5 the work norm. /
4-5 1 man, what length did he take? / 1/2 rod 3 cubits.

§ 3b 1-5 saḫar im.dù.a / 5 gín / 1 kùš dagal / 1 kùš sukud / 3 45 éš$^!$.gàr /
6-7 lú.1.e uš e[n.nam al.dab_5] / 1 nindan [uš]

1-5 The mud of a mud wall / 5 shekels. / 1 cubit the width / 1 cubit the height / 3 45 the work norm. /
6-7 1 man, the length was [what]? / 1 rod was the [length].

§ 3c 1-5 saḫar im.dù.a / 5 gín / 2 kùš dagal / 1 kùš sukud.bi / 3 45 éš$^!$.gàr /
6-7 lú.1.e uš en.nam / ½ nindan uš

1-6 The mud of a mud wall / 5 shekels. / 1 cubit the width / 1 cubit the height / 3 45 the work norm. /
7 1 man, the flank was what? / 1/2 rod was the length.

§ 3d 1-5 im.dù.a / 5 uš uš.bi / 2 kùš dagal / ½ nindan sukud.bi / *i-na* 1 kùš ⅓ kùš gu_7 ì.gu_7-*ma* /
6-8 ⌜sagšu⌝ gul.gul-*ma* / 1 ½ kùš sukud *ur-d*[*am*] / uš en.nam ḫé.gíd ⌜20⌝

1-5 A mud wall. / 5 uš (5 · 60 rods) its length / 1 cubit the width / 1/2 rod its height. / From 1 cubit 1/3 cubit of food it ate. /
6-8 Its ⌜top⌝ was destroyed, then / 1 1/2 cubits the height wen[t down]. / The length what shall it be long? ⌜20⌝.

YBC 4673 § 3a (# 9)

In this introductory exercise is considered an im.dù.a 'mud wall' with a given square cross section. Both sides of the cross section are 1 cubit. The work norm '3 45' for building(?) the mud wall is also given, probably meaning *w = 3;45 volume-shekels per man-day*. Asked for is the length of the mud wall with the given cross section that 1 man can build in 1 day. The answer is obvious. Call the area of the cross section *A* and call the unknown length *u*. Then

A = 1 square cubit = ;05 (rods · cubit), w = ;03 45 (sq. rods · cubit),
$u = w / A$ = ;03 45 · rec. ;05 = ;03 45 · 12 = ;45 (rods) = 1/2 rod 3 cubits (c. 7 1/2 meters).

The answer is given in the text.

YBC 4673 § 3b (# 10)

In the second exercise of this paragraph, the same mud wall with a square cross section is considered. The work norm w = '3 45' is also given, but that just helps to confuse the issue, since what is really asked for is the length u of a mud wall with the given cross section and the volume V = 5 (volume-)shekels. The answer is again obvious:

V = ;05 (sq. rods · cubit) = $u \cdot A = u$ · ;05 (rods · cubit), $u = V / A$ = ;05 · rec. ;05 = 1 (rod).

The answer is given in the text.

YBC 4673 § 3c (# 11)

Here the mud wall has a rectangular cross section, with he width twice as large as in the preceding exercise. Since the volume remains the same, the length worked in a day is halved.

YBC 4673 § 3d (# 12)

The mud wall in this exercise has a triangular cross section. Its length is u = 5 uš = 5 · 60 rods , its width s = 2 cubits, and its height h = 1/2 rod (= 6 cubits). Consequently its "decrease rate" called gu_7 $ì.gu_7$ 'the food eaten' is 1/3 cubit per cubit. This quaint expression means that the width decreases by 1/3 cubit for every cubit the height increases. Conversely, of course, the width increases by 1/3 cubit for every cubit the height decreases. Now, the top of the wall was destroyed, and the height decreased by h' = 1 1/2 cubits. Consequently, the triangular cross section was replaced by a trapezoidal cross section and the volume decreased. For some reason the question that follows is incomplete, asking how long the length should be so that ………

Fortunately, a related exercise is known where the question is complete. In MS 3052 § 1 (Friberg, *MSCT 1* (2007), 258) the top of a mud wall is torn down and the mud that was torn out of the wall is used to extend the wall. The question then is

How much can the lowered wall be lengthened by use of the mud torn out from the top ?

In the case of YBC 4673 § 3d it is easy to answer this question. (See Fig. 11.2.2 below.) When the height is decreased by 1 1/2 cubit, the width is increased by 1/3 · 1 1/2 = 1/2 cubit. Consequently the top of the new, trapezoidal cross section has the width d = 1/2 cubit. Therefore

The volume torn down from the top of the wall is $u \cdot d/2 \cdot h' = u \cdot 1/4 \cdot 1\ 1/2$ sq. cubits = u · ;22 30 sq. cubits.

On the other hand,

The volume of the extension of the wall is $e \cdot (s + d)/2 \cdot (h - h') = e \cdot 2\ 1/2 / 2 \cdot 4\ 1/2$ sq. cubits = e · 5;37 30 sq. cubits.

The computations can be executed in various ways, more or less cleverly. The end result, however, will be that

e = rec. 5;07 30 · ;22 30 · u = rec. 15 · u = 20 rods.

This also appears to be the answer given in the text.

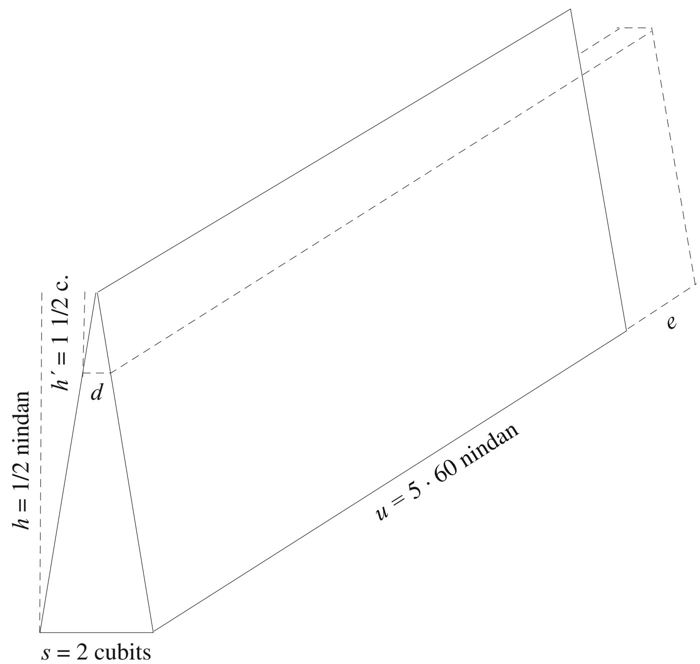


Fig. 11.2.2. YBC 4673 § 3d. A lowered and lengthened wall with an originally triangular cross section.

11.2.4 § 4. Problems about Old and New Levees

YBC 4673 § 4 (## 13-15; *rev. i*: 15-19) Cf. Friberg, *ChV (2001)*, Sec. 5.6; Muroi, *SBM* 2 (1992).

§ 4a 1-4 ég 1 kùš dagal / 1 kùš sukud.bi / *i-na* 1 kùš 1 kùš í.gu$_7$-*ma i-na* $^1/_2$ nindan uš / saḫar en.nam tab.ba

1-4 A levee. 1 cubit width / 1 cubit its height / in 1 cubit 1 cubit it ate, then in 1/2 rod of length / of mud, what, doubled?

§ 4b 1-2 ég libir.ra 1 kùš [dagal] / 1 kùš sukud *i-na* 1 kùš <1 kùš> ì.gu$_7$-ma /
3-6 *i-na-an-na* 1 kùš dagal / 1 kùš sukud daḫ.e / *i-na* $^1/_2$ nindan ⸢uš⸣ / saḫar en.n[am tab.ba] /
7-8 saḫar gi[bil *ù* saḫar libir.ra] / en.[nam]
9-10 [10 gín saḫar] / 2 [$^1/_2$ gín saḫar libir.ra] / [7 $^1/_2$ gín saḫar gibil]

1-2 An old levee. 1 cubit the width / 1 cubit the height, in 1 cubit <1 cubit> it ate, then /
3-6 Now, 1 cubit width / 1 cubit height I added on. / In 1/2 rod of length / of mud, what, doubled?
7-8 N[ew mud and old mud] / were w[hat]?
9-10 [10 shekels the mud] / 2 [1/2 shekels the old mud] / [7 1/2 shekels the new mud]

§ 4c 1-2 ég libir.ra 2 kùš dagal / 2 kùš sukud.bi /
3-5 *i-na-an-na* 1 kùš dagal / 1 kùš sukud daḫ.e / *i-na* 1 kùš 1 kùš ì.gu$_7$.e /
6-8 *i-na* $^1/_2$ nindan uš / saḫar en.nam tab.ba / $^1/_3$ sar 2 $^1/_2$ gín saḫar /
9-12 saḫar gibil *ù* saḫar libir.ra / en.nam / 10 gín saḫar libir.ra / 12 $^1/_2$ gín saḫar gibil

1-2 An old levee. 1 cubit the width / 2 cubits the height. /
3-5 Now, 1 cubit width / 1 cubit height I added on / in 1 cubit 1 cubit it ate /
6-8 In 1/2 rod of length / of mud what, doubled? / 1/3 sar 2 1/2 shekels of mud. /
9-12 New mud and old mud / were what? / 10 shekels old mud 12 1/2 shekels new mud.

YBC 4673 § 4a (# 13)

The object considered in this exercise is a ‘levee’ (Sum. ég) bordering an irrigation canal. The levee has both width and height equal to 1 cubit, and its decrease rate is 1 cubit per cubit. The calculations in §§ 4b-c below show that it must be assumed that *the levee bordered the canal on both sides*, as in Fig. 11.2.3 below.

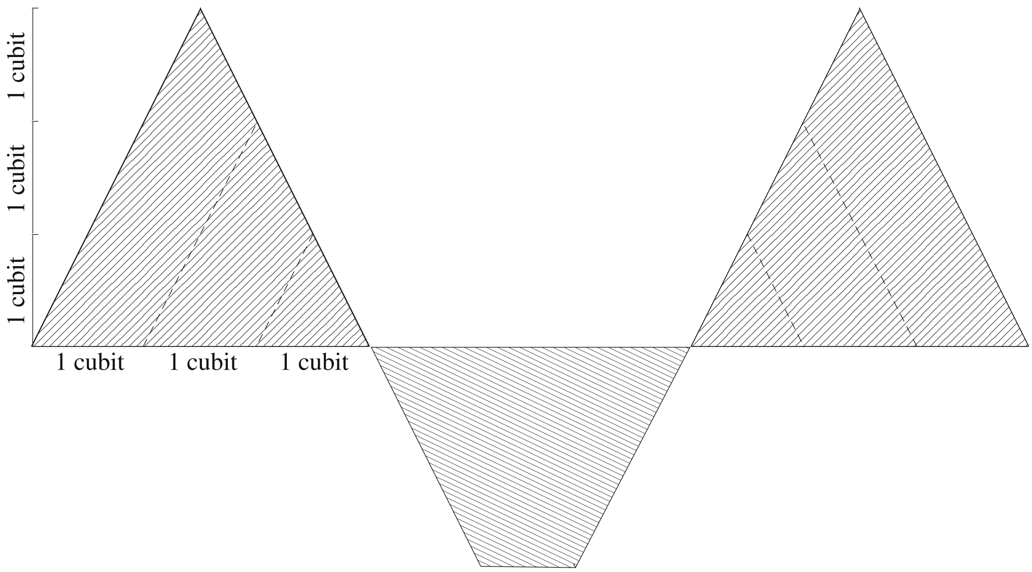


Fig. 11.2.3. YBC 4673 § 4a-c. Successive extensions of a levee bordering a canal on both sides.

The question is what the volume of the levee would be per half-rod (or reed) of length. The easy answer is

$V = 2 \cdot 1/2 \cdot ;05 \cdot 1 \cdot 1/2 = ;02\ 30$ sq. rods · cubit = 2 1/2 volume-shekels.

YBC 4673 § 4b (# 14)

In this exercise, the width and the height of the levee have both been doubled. The volume of the levee per half-rod will then, of course, be 4 times as much as in the preceding exercise, that is 10 volume-shekels. Of these 10 volume-shekels, 2 1/2 come from the "old levee" and the remaining 7 1/2 from the added "new levee". This answer is given in the text.

YBC 4673 § 4c (# 15)

Here the width and height of the levee are three times as much as the width and height in the first exercise. The volume of the levee per half-rod is then 9 times larger, that is 9 · 2 1/2 = 22 1/2 volume-shekels. Of these 22 1/2 volume-shekels, 10 shekels are from the "old levee" in the preceding exercise, and the remaining 12 1/2 volume-shekels are from the "new levee." This answer, too, is given in the text.

11.2.5 § 5. An Obscure Problem about Three Kinds of Wool

YBC 4673 § 5 (# 16; *rev. ii*: 13-30)

1-3 $^1/_3$ ma.na siki šu.ḫé.a / $^1/_2$ ma.na siki ga.ḫal.ag / 1 $^1/_2$ gín siki nu.nu /
4-8 *i-na* 1 ma.na 6 gín ba.lá / *i-na* 1 ma.na siki / 6 gín ba.zi-*ma* / $^5/_6$ ma.na 4 gín / siki šu.ḫé.a /
9-12 *i-na* $^5/_6$ ma.na 4 gín / siki šu.ḫé.a / $^1/_2$ ma.na 6 gín / siki ga.ḫal.ag ba.zi-*ma*/
13 18 gín ka ga.hal.<ag> /
14-18 *a-na* 1 munus / siki en.nam in.na.an.šúm / siki šu.ḫé.a / siki ga.hal.ag / siki nu.nu

1-3 1/3 mina of wool type 1 / 1/2 mina of wool type 2 / 1 1/2 shekels of wool type 3 /
4-8 From 1 mina 6 shekels I weighed. / From 1 mina of wool / 6 shekels I tore off / 5/6 mina 4 shekels / of wool type 1. /
9-12 From 5/6 mina 4 shekels / of wool type 1 / 1/2 mina 6 shekels / of wool type 2 I tore off, then /
13 18 shekels of wool$^!$ type 2. /
14-18 For 1 woman / what of wool did I give her / wool type 1 / wool type 2 / wool type 3 ?

This is a problem about three kinds of wool. It is a completely isolated exercise among the other exercises on YBC 4673, which are all about bricks and mud. As a matter of fact, there are no related texts at all within the whole known corpus of Old Babylonian mathematical texts. Besides, the text of the exercise is without doubt corrupt; it makes almost no sense at all.

In the first three lines of the exercise, three quantities of three kinds of wool are enumerated, 1/3 mina of the first, 1/2 mina of the second, but only 1 1/2 shekel of the third, which must be a mistake. (One would have expected 2/3 mina instead. Note that rec. 2/3 = 1 1/2.) Next it is specified, twice, that 6 shekels should be subtracted from 1 mina (of wool). The result is stated in lines 7-8 as 5/6 mina 4 shekels (= 54 shekels) of wool of type 1. In lines 9-13, 1/2 mina 6 shekels (= 36 shekels) of wool of type 2 are subtracted from the 54 shekels of wool of type 1, and the remainder is said to be 18 shekels of wool of type 2! The text of the exercise ends with an incomplete question about the three kinds of wool for a woman.

Neugebauer (*MKT III*, 33, fn. 18) made the remark that $54 = 3 \cdot 18$ and $36 = 2 \cdot 18$. Other than that, nothing intelligent can be said about this enigmatic exercise.

11.2.6 § 6. Problems about Reinforcing Reed Works

YBC 4673 § 6 (## 17-21; *rev. iii*: 1-20) Cf. Robson, *MMTC* (1999), Sec. 6.4.

§ 6a 1-3 $^2/_3$ usag / *a-na* 7 $^1/_2$ sar saḫar / íl-*ma* 5 sar usag /
4-5 $^1/_3$ ⸢*ša*⸣? saḫar *a-na* 7 $^1/_2$ saḫar / íl-*ma* 2 $^1/_2$ sar saḫar

1-3 2/3 reed work / to 7 1/2 sar of mud / carry, then 5 sar of reed work /
4-5 1/3 mud to 7 1/2 sar mud / carry, then 2 1/2 sar of mud

§ 6b 1-4 igi 5 gín! usag duḫ / *a-na* 5 <sar> saḫar usag / íl-*ma* 1 *šu-ši* erín.ḫa / igi.duḫ

1-4 The reciprocal of 5 shekels! of reed work release / to 5 <sar> the mud of reed work / carry, then 1 sixty men / you will see.

§ 6c 1-4 igi 6 saḫar usag / íl.íl duḫ / *a-na* 5 sar saḫar / íl-*ma* 50 erín.ḫá igi.duḫ

1-4 The reciprocal of 6 of reed work / carrying release / to 5 of mud / carry, then 50 men you will see.

§ 6d 1-3 igi $^2/_3$ usag duḫ / *a-na* 5 sar saḫar íl-*ma* / 7 30 erín.ḫá igi.duḫ

1-3 The reciprocal of 2/3 of reed work release / to 5 sar of saḫar carry, then 7 30 men you will see.

§ 6e 1-3 igi $^2/_3$ usag duḫ / *a-na* 2 $^1/_2$ sar saḫar / íl-*ma* / 15 erín.ḫá igi.duḫ

1-3 The reciprocal of 2/3 of reed work release / to 2 1/2 sar of mud carry, then 15 men you will see.

The five exercises in § 6 of YBC 4673 all deal with something called usag (ú.sag$_{11}$), apparently meaning 'reed work' or more precisely 'reed bundles used for reinforcement of dams or barrages'.

YBC 4673 § 6a (# 17)

The translation of this exercise is slightly complicated by the fact that in cuneiform mathematical texts the word saḫar can mean both 'mud, earth' and 'volume'. Anyway, a given volume(!) of 7 1/2 sar is multiplied first by 2/3 usag apparently to be read as '2/3, the constant of reed work' and then by $^1/_3$ ⸢*ša*⸣? saḫar, apparently correspondingly to be read as '1/3, the constant of mud work'. The result is that the given volume is divided into *5 sar of reed work and 2 1/2 sar of mud work*. Apparently, dams and barrages built in or near water were supposed to be made of, by volume, reed bundles for the lower 2/3 and mud or bricks for the upper 1/3.

A well known example is exercise #1 in the large Old Babylonian mathematical recombination text BM 85196 from Sippar (Neugebauer *MKT II* (1937), 43). There in the construction of a triangular dam called *ap-pu-um* 'nose', reed work and mud work are divided according to the following rule:

i-na ku-ta-li-šu 3 kùš bùr *i-na pa-ni me-e* 6 bùr 'in the rear 3 cubits depth, in the face of the water 6 <cubits> depth'.

Since 6 cubits = 1/2 rod = 1 reed, which was the standard length of reed bundles, it appears that the reed bundles were packed standing up in the under-water part of the construction. In the solution procedure of the same exercise, the volume of the whole construction is calculated and is found to be 11 15 volume-shekels. After that the volumes of mud work and reed work, respectively, are computed as follows:

> $^2/_3$ 11 15 ba.zi 3 45 *ta-mar* saḫar.ḫá / íb.tag$_4$ 7 30 ^{gi}sa kin
> ‘2/3 of 11 15 tear out, 3 45 you will see, the mud work, / the remainder 7 30 is the reed bundle work’.

YBC 4673 § 6b (# 18)

In this exercise, the 5 volume-sar of reed work calculated in the previous exercise are multiplied by the reciprocal of what appears to be a work norm for reed work, namely *5 volume-shekels per man-day* of ???. The result is, of course, 1 sixty of man-days. This answer is given in the text.

Robson, *MMCT* (1999), 88 and 102, mentions the following five known work norms related to reeds:

IM 52916	*na-az-ba-al* še.in.nu.da 3 20 *i-gi-gu-bu*	‘carrying straw, 3 20 is its constant’	G 24
TMS, text III	10 *šà šu-ri ka-sà-mi*	‘10, of cutting reed bundles’	BR 68
	5 *ša šu-ri za-ba-li*	‘5 of carrying reed bundles’	BR 69
YBC 5022	40 *ša* ^{geš}*šu-rum*	‘40 of reed bundles’	NSd 59
YBC 7243	5 20 usag	‘5 20, reed work’	NSe 48

The constant BR 69 for ‘carrying reed bundles’ may be the constant ‘5’ mentioned in § 6b, but carrying reeds is not mentioned in § 6b, only in § 6c!

YBC 4673 § 6c (# 19)

In this exercise, the 5 volume-sar of reed work calculates in the previous exercise are multiplied by the reciprocal of a new work norm related to reeds, namely *6 volume-shekels per man-day of carrying reeds (for some unmentioned distance)*. The result is 50 man-days. This answer, too, is given in the text.

None of the constants mentioned above is the work norm ‘6’ mentioned in § 6c.

YBC 4673 § 6d (# 20)

In this exercise, the computation in § 6a is reversed. The same volume of reed work as before, 5 volume-sar, is divided by the constant 2/3. The result should be 7 30 volume-shekels, but it is by mistake given as 7 30 erín.ḫá ‘7 30 men (man-days)’.

YBC 4673 § 6e (# 21)

In this exercise, the 2 1/2 volume-sar of mud work calculated in § 6a are divided by a work norm called just 10 éš.gàr ‘10, the work norm’ without specification of what it refers to. It may be the following constant:

TMS III	10 *šà* saḫar *za-ba-li* éš.gàr lú.1	‘10 of carrying mud, work norm for 1 man’	BR 46

(It is unclear how this work norm for carrying mud relates to another work norm for carrying mud, namely ‘1 40’, mentioned in the entries E 4, G E23, NSd 8, and NSd 42. See the discussion above of YBC 4673 § 1e.) Anyway, multiplying ‘2 1/2 (volume-sar) of mud’ by the reciprocal of ‘10 (volume-shekels) of carrying mud’ leads to the result ‘15 man-days’, which is the answer to the exercise in the last line of the text.

11.2.7 Subscript

> 23 im.šu / dub 2.kam.ma ‘23 hand tablets / the 2nd tablet’

11.2.8 The Vocabulary of YBC 4673

a.na	as much as
an.ta	upper
dagal	width

ég		ring wall, levee
en.nam		what
erín		man, soldier, man-day
éš.gàr		work norm
gibil		new
gu$_7$ ì.gu$_7$		food he ate, rate of decrease/increase
igi	*pa-ni*	opposite, reciprocal
igi.te.en		fraction, proportion
im.dù.a		mud wall
ki.ta		lower
libir.ra		old
lú.1		1 man, man-day
lú.šidim		brick worker, builder
munus		woman
sagšu		top
saḫar		mud, earth, volume
sig$_4$		brick
sig$_4$ al.ur$_5$.ra		square brick
sig$_4$.anše		brick pile
siki		wool
sukud		height
ud.1.kam		1 day
usag		reed work
dab$_5$		to take
daḫ		to add on
duḫ		to loosen, release, resolve
gar.ra		to set
gíd		to be long
gu$_7$		to eat, food
gul.gul		to destroy (intensively)
íl		to carry
íl.íl		to carry (repeatedly)
igi.duḫ		to see
lá, ba.lá		to subtract
sig$_4$.anše duḫ.duḫ		to mold bricks and make brick piles (intensively)
šúm, in.na.an.šúm.ma		to give
tab.ba		to double
til		to complete, to finish
zi, ba.zi		to tear off, to subtract
i-na-an-na		now
šu-ši		sixty
ur-dam (?)		< *warādu* to go down

11.3 YBC 4669. An Old Babylonian Unnumbered Recombination Text with Mixed Problems

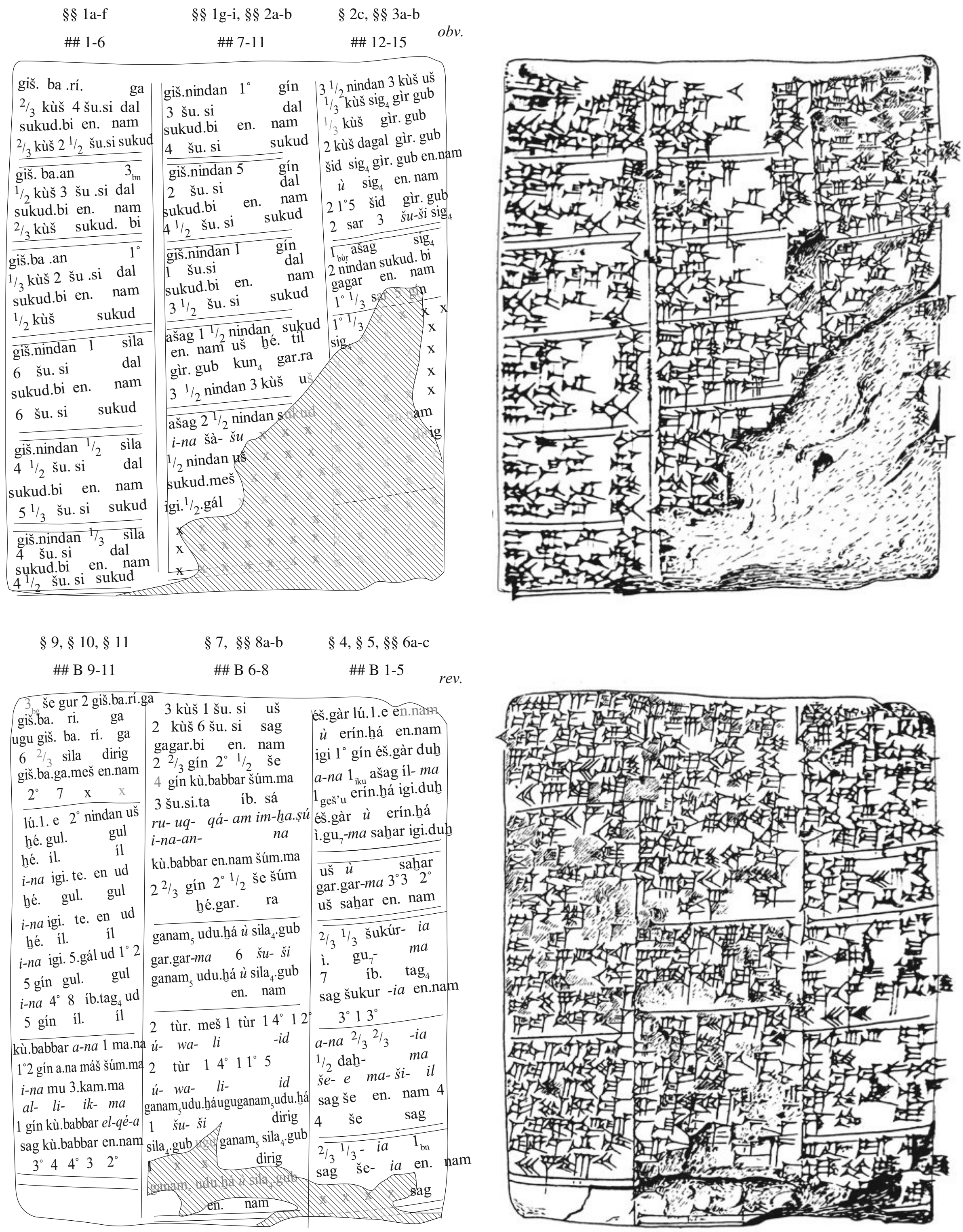


Fig. 11.3.1. YBC 4669. An Old Babylonian unnumbered recombination text with mixed problems.
Hand copy by O. Neugebauer, *MKT III*. Reproduced here with the permission of Springer.
(Photo online at cdli.ucla.edu/. P254987.)

YBC 4669 (Neugebauer, *MKT I* (1935), 514-515; *MKT III* (1937), 26-29) is a badly organized Old Babylonian mathematical recombination text in 3 columns. The obverse of the tablet is damaged with the right lower corner unreadable, but the reverse is fairly well preserved. The text can be divided into 11 consecutive paragraphs, as follows:

§ 1 Nine problems for a decreasing series of cylindrical measuring vessels, measured in cylinder-sìla
§ 2 Three problems for a stair-case of square bricks of type S1/3c
§ 3 Two (or three) problems for standard rectangular bricks of type R1/2c
§ 4 One problem involving a work norm for digging
§ 5 A cubic equation for a rectangular block
§ 6 Three linear equations for grain
§ 7 A problem for a sheet of silver
§ 8 Two systems of linear equations for ewes and sheep
§ 9 A system of linear equations for two containers
§ 10 A combined work norm problem about crushing and carrying bricks
§ 11 A problem about compounded interest

Several of these problems were discussed already in Friberg, *MPIWG* (1996); *ChV (2001)*; *MSCT 1* (2007), and in Robson *MMTC* (1999).

Answers but *no solution procedures* are given in the text.

11.3.1 § 1. Problems for a Decreasing Series of Cylindrical Measuring Vessels

YBC 4669 § 1 (## 1-9; *obv. i*: 1-*ii: 12*) Cf. Friberg, *MSCT 1* (2007), Sec. 4.7.

§ 1a 1-4 $^{\text{geš}}$ba.rí.ga / $^{2}/_{3}$ kùš 4 šu.si dal / sukud.bi en.nam / $^{2}/_{3}$ kùš 2 $^{1}/_{2}$ šu.si sukud
1-4 A barig-vessel / 2/3 cubit 4 fingers the transversal. / Its height is what? / 2/3 cubit 2 1/2 fingers the height.

§ 1b 1-4 $^{\text{geš}}$ba.an 3(bán) / $^{1}/_{2}$ kùš 3 šu.si dal / sukud.bi en.nam / $^{2}/_{3}$ kùš sukud.bi
1-4 A bán-vessel of 3 bán / 1/2 cubit 3 fingers the transversal. / Its height is what? / 2/3 cubit its height.

§ 1c 1-4 $^{\text{geš}}$ba.an 10 <sìla> / $^{1}/_{3}$ kùš 2 šu.si dal / sukud.bi en.nam / $^{1}/_{2}$ kùš sukud
1-4 A bán-vessel of 10 <sìla> / 1/3 cubit 2 fingers the transversal. / Its height is what? / 1/2 cubit the height.

§ 1d 1-4 $^{\text{geš}}$nindan 1 sìla / 6 šu.si dal / sukud.bi en.nam / 6 šu.si sukud
1-4 A nindan-vessel of 1 sìla / 6 fingers the transversal. / Its height is what? / 6 fingers the height.

§ 1e 1-4 $^{\text{geš}}$nindan $^{1}/_{2}$ sìla / 4 $^{1}/_{2}$ šu.si dal / sukud.bi en.nam / 5 $^{1}/_{3}$ šu.si sukud
1-4 A nindan-vessel of 1/2 sìla / 4 1/2 fingers the transversal. / Its height is what? / 5 1/3 fingers the height.

§ 1f 1-4 $^{\text{geš}}$nindan $^{1}/_{3}$ sìla / 4 šu.si dal / sukud.bi en.nam / 4 $^{1}/_{2}$ šu.si sukud
1-4 A nindan-vessel of 1/3 sìla / 4 fingers the transversal. / Its height is what? / 4 1/2 fingers the height.

§ 1g 1-4 $^{\text{geš}}$nindan 10 gín / 3 šu.si dal / sukud.bi en.nam / 4 šu.si sukud
1-4 A nindan-vessel of 10 shekels / 3 fingers the transversal. / Its height is what? / 4 fingers the height.

§ 1h 1-4 $^{\text{geš}}$nindan 5 gín / 2 šu.si dal / sukud.bi en.nam / 4 $^{1}/_{2}$ šu.si sukud
1-4 A nindan-vessel of 5 shekels / 2 fingers the transversal. / Its height is what? / 4 1/2 fingers the height.

§ 1i 1-4 $^{\text{geš}}$nindan 1 gín / 1 šu.si dal / sukud.bi en.nam / 3 $^{1}/_{2}$ šu.si sukud
1-4 A nindan-vessel of 1 shekel / 1 fingers the transversal. / Its height is what? / 3 1/2 fingers the height.

The questions in all the sub-paragraphs of § 1 of YBC 4669 are concerned with the dimensions of possibly standardized dimensions for a decreasing series of measuring vessels for units of capacity measures. As is well known (see, for instance, Friberg *MSCT 1* (2007), Sec. 3.1), the basic units for Old Babylonian capacity measures are the gín (shekel), the sìla, the bán, the barig, and the gur. Each basic unit is a multiple of the preceding unit in this list. More precise information is given by *the factor diagram for the system C of capacity units* in Fig. 11.3.2 below. In the factor diagram are displayed the Sumerian and Akkadian names of the basic capacity units, as well as the series of conversion factors (bundling numbers), all of them sexagesimally regular integers.

		basket				shekel
C(OB):	gur ← 5 —	barig ← 6 —	bán ← 10 —	sìla ← 60 —		gín
	kurru	*parsiktu*	*sūtu*	*qû*		*šiqlu*
	5 00 s.	1 00 s.	10 s.	1 s.		;01 s.

Fig. 11.3.2. Factor diagram for the Old Babylonian System *C* of basic units of capacity numbers.

While the relative sizes of the various capacity units always were fixed as in this factor diagram, the absolute sizes were not. As a matter of fact, there existed a number of different standards for the sìla, in particular. (See Friberg, *BaM* 28 (1997), Sec. 7.) One of the sìla standards was the cubic sìla, equal to the content of a "unit cube", *a cube with all its sides equal to 6 fingers = 1/60 of a rod (about 1 decimeter)*. For convenience, this length unit will below be called 1 n.

The relation between a given Old Babylonian sìla standard and Old Babylonian volume measure could be described by the *našpakum* 'storing number' *S* for that sìla. For the *cubic sìla*, in particular, the storing number was *S* = 5 00, for the reason that

1 volume-shekel = ;01 sq. rod · 1 cubit = 1 00 sq. n. · 5 n. = 5 00 sq. n. · n., so that
1 volume-shekel contained *S* = 5 00 cubic sìla.

However, the capacity numbers figuring in YBC 4669 § 1 are not based on the cubic sìla but on a smaller sìla which may be called a "cylinder sìla", equal to the content of a "unit cylinder", *a cylindrical container with the bottom diameter and the height both equal to 6 fingers = 1 n.* For the cylinder sìla, the storing number was S = 6 40, for the reason that according to a convenient Old Babylonian convention the circumference *a* and the volume *V* of a cylinder with the bottom diameter *d* and the height *h* could be computed as (approximately)

$a = 3\ d$, and $V =\ ;05 \cdot \text{sq. } a \cdot h =\ ;45 \cdot \text{sq. } d \cdot h$.

(Recall that in modern notations $a = \pi\ d$ and $V = \pi\ r^2 \cdot h = \pi/4\ d^2 \cdot h$.) Therefore,

1 volume-shekel = ;01 sq. rod · 1 cubit = 1 00 sq. n. · 5 n. = 5 00 sq. n. · n. = 6 40 · ;45 sq. *n.* · *n.*, so that
1 volume-shekel contained S = 6 40 cylinder sìla.

The usefulness of the cylinder sìla is obvious. Since the volume of a cylinder with the bottom diameter *d* and the height *h* was calculated as ;45 sq. *d* · *h*, and since 1 volume-shekel was equal to 6 40 cylinder sìla, it followed that if the bottom diameter and the height were both counted as multiples of the rod (in relative sexagesimal numbers), then *a cylindrical container with the bottom diameter d and the height h contained precisely sq. d · h cylinder sìla.*

Now, consider the following tabular display of the data in the nine sub-paragraphs of § 1 in YBC 4669:

§	name	capacity *C*	diameter *d*	height *h*	sq. *d* · *h*
1a	$^{\text{geš}}$ba.ri.ga	1 barig	24 f. = 4 n.	22 1/2 f. = 3;45 n.	1 00 sq. n. · n.
1b	$^{\text{geš}}$ba.an 3(bán)	3 bán	18 f = 3 n.	20 f. = 3;20 n.	30 sq. n. · n
1c	$^{\text{geš}}$ba.an 10 <sìla>	1 bán	12 f. = 2 n.	15 f. = 2;30 n.	10 sq. n. · n
1d	$^{\text{geš}}$nindan 1 sìla	1 sìla	6 f. = 1 n.	6 f = 1 n.	1 sq. n. · n
1e	$^{\text{geš}}$nindan 1/2 sìla	1/2 sìla	4 1/2 f. = ;45 n.	5 1/3 f. = ;53 20 n.	;30 sq. n. · n
1f	$^{\text{geš}}$nindan 1/3 sìla	1/3 sìla	4 f. = ;40 n.	4 1/2 f. = ;45 n.	;20 sq. n. · n
1g	$^{\text{geš}}$nindan 10 shekels	10 shekels	3 f. = ;30 n.	4 f. =;40 n.	;10 sq. n. · n
1h	$^{\text{geš}}$nindan 5 shekels	5 shekels	2 f. = ;20 n.	4 1/2 f. = ;45 n.	;05 sq. n. · n
1i	$^{\text{geš}}$nindan 1 shekel	1 shekel	1 f. = ;10 n.	3 1/2 f. = ;35 n.	c. ;01 sq. n. · n

Now it is clear that all the computations of the heights in the nine exercises in § 1 of YBC 4669 were correct, provided that the capacity measures were assumed to be based on the cylinder sìla with S = 6 40.

All the questions in § 1 of YBC 4669 are of the type

The diameter *d* and the capacity *C* of a measuring vessel are known. What is its height *h*?

The form of the solution is obvious. Assuming that the capacity measures in all the exercises are based on the cylinder sìla, one finds that

If $d = p$ n., $h = q$ n., and $C = c$ sìla$^{cyl.}$ then $C = c$ sìla$^{cyl.}$ = ;45 · sq. $p \cdot q$ sq. n. · n. = sq. $p \cdot q$ · :45 sq. n. · n.
Therefore, c = sq. $p \cdot q$ and consequently $q = c$ · rec. (sq. p).

Examples:

§ 1a: C = 1 bariga = 1 00 sìla, d = 2/3 cubit 4 fingers = 24 fingers = 4 n.
Therefore, $h = q$ n., where q = 1 00 · rec. 16 = 1 00 · ;03 45 = 3;45,
so that h = 3;45 n. = 22 1/2 fingers = 2/3 cubits 2 1/2 fingers.

§ 1i: C = 1 shekel = ;01 sìla$^{cyl.}$, d = 1 finger = ;10 n.
Therefore, $h = q$ n., where q = ;01 · rec. ;01 40 = ;01 · 36,
so that h = ;36 n. = 3;36 fingers = (approximatively) 3 1/2 fingers.

Note that in the Old Babylonian tables of constants B and BR, for instance, the storing number for the cylinder sìla is expressed as follows (see Friberg, *BaM* 28 (1997), 307):

IM 52301	6 40 *i-gi-gu-ub-bi-im qu-up-pi-im*	'6 40 of the constant of a container'	B1
TMS III	6 40 *šà na-aš-pa-ki-im*	'6 40 of storage'	BR 50

Among published Old Babylonian mathematical cuneiform texts, there are a few scattered examples of counting with cylinder sìla. They were all inadequately explained before, so it may be a good idea to consider them more closely here. The four examples below occur in various places in BM 85194, a large Old Babylonian mathematical recombination text from Sippar.

BM 85194 (Neugebauer, *MKT I* (1935), 142 ff.)

22 1 ^{geš}ba$^{!}$.rí$^{!}$.ga 4 dal 1(barig) še gam *ù* bùr$^{!}$ en.nam /
2-3 za.e dal nigin 16 *ta-mar* igi 16 duḫ.a / 3 45 *ta-mar*
3 45 bùr *ne-pé-šum*

1 A barig-vessel , 4 the transversal 1 bariga the grain. Circumference and depth$^{!}$ were what? /
2-3 You: Square the transversal, 16 you will see. The reciprocal of 16 release / 3 45 you will see.
3 45 was the depth. The procedure.

(Note: In this exercise bùr 'depth' is written with 1 oblique wedge and gam 'circumference' with two aligned oblique wedges. Apparently the one who wrote the text tended to confuse the two similar signs with each other so that he sometimes wrote gam instead of bùr.)

24 1 ^{geš}bán še.a.am *ma-li-a-at i-na lìb-bi* [1 sìla] šu.ti 2 /
2 bùr en.nam *lu-ri-id-ma lu-ú* [1 sìla]
3 [za].e 2 30 bùr / *a-na* 10 *zu-uz*$_5$ igi 10 duḫ.a 6 *ta-mar*
4 [2] 30 *a-na* 6 *i-ši* / 15 *ta-mar* 10 šu.si 1/2 10 šu.si bùr$^{!}$
5 1 sìla *lu-mur* 2 dal nigin / 4 *ta-mar* 15 bùr *a-na* 4 *i-ši* 1 *ta-mar* 1 sìla še /
6 *ki-a-am ne-pé-šum*

1 A bán-vessel filled with grain. Out of it, I received [1 sìla]. <Transversal> 2. /
2 What depth shall I go down so that <I get> 1 sìla?
3 [Yo]u: 2 30, the depth, in 10 divide. The reciprocal of 10 release, 6 you will see.
4 [2] 30 to 6 carry / 15 you will see, 10, a finger, 1/2 10, a finger, the depth$^{!}$.
5 1 sìla may I see. 2, the transversal square / 4 you will see. 15, the depth to 4 carry, 1 sìla of grain.
6 Such is the procedure.

34 1-2 *pi-sa-nu-um* 3 [kùš gam *ak*]*-pu-up-am* / 3(barig) *ma-aḫ-ru-tum ki-ma* [*x x*]*x-ša-di-im* /
3-4 *wa*$^{?}$*-ar-ku-tum i-na ka-*[*x x ig-g*]*a-am-ru* / sukud *ú-ri-di-im* en.nam
5 za.e [3 nigin 9 *ta*]*-mar* / 9 *a-na* 5 gam *i-ši* 45 *ta-mar*
6 [igi 45 duḫ.ḫa] / 1 20 *ta-mar* igi 6 40 *a-na* 9 *i-ši* [12 *ta-mar*] /
7 12 nindan sukud *ú-ri-*<*di-*>*im ne-pé-šum*

1-2 A vessel. 3, [cubits the circumference I] curved / 3 bariga earlier as x x /
3-4 later, in x [x were] filled. / The height of the descent was what?
5 You: [3 square 9 you] will see. / 9 to 5, the circumference carry, 45 you will see.
6 [The reciprocal of 45 release] / 1 20 you will see. The reciprocal of 6 40 to 9 (sic!) carry [12 you will see.] /
7 12, a rod was the height of the descent. The procedure.

35 1-2 1 *qa-am* 10 šu.si *am-ṣú-ur* uš en.nam / *al-li-ik*
3 za.e 10 nigin 1 40 *ta-mar* 1 40 / *a-na* 5 gam *i-ši* 8 20 *ta-mar*
4 igi 8 20 duḫ.a / 7 12 *ta-mar*
5 igi 6 40 *a-na* 7 12 *i-ši* / 1 04 48 uš *ta-mar ne-pé-šum*

1-2 1 sìla-vessel 10, a finger, I went around. The length what / did I go?
3 You: 10 square, 1 40 you will see. 1 40 / to 5 of the circumference carry, 8 20 you will see.
4 The reciprocal of 8 20 release / 7 12 you will see.
5 The reciprocal of 6 40 to 7 12 carry / 1 04 48 the length you will see. The procedure.

In **BM 85194 # 22**, the author of the text is silently counting with *cylinder sìla.* As a matter of fact, *this exercise is a direct parallel to YBC 4669 § 1a,* except that all length measures are expressed in terms of rods rather than in terms of cubits and fingers. Indeed

C = 1 bariga = 1 00 sìla$^{\text{cyl.}}$, d = '4' = 4 n.
Therefore, $h = b$ n., where b = 1 00 · rec. sq. 16 = 1 00 · ;03 45 = 3;45,
so that h = '3 45' . The result can be interpreted as h = 3;45 n. = 22 1/2 fingers = 2/3 cubits 2 1/2 fingers.

Note that the computation rule clearly used here is

If C is counted in cylinder sìla and both the diameter d and the height h are counted in rods, then $C = h$ · sq. d.

In **BM 85194 # 24**, the author of the text is again silently counting with *cylinder sìla.* It is silently assumed to be known that, *precisely as in YBC 4669§ 1c*, but in terms of rods rather than cubits and fingers, a cylindrical bán-vessel has the diameter 2 n. and the height 2;30 n. From such a cylindrical bán-vessel, full of grain, 1 sìla of grain is removed. Asked for is how much the the grain level is lowered. The calculations proceed as follows:

A cylindrical bán-vessel has the diameter '2' = 2 n. and the height '2 30' = ;2;30 n.
When 1 sìla is removed from the top, this is a 10th of 1 bán. Therefore, the height of the grain is decreased by a tenth of '2 30', which is '6 · 2 30' = '15' = '10' + '5' = ;10 n. + ;05 n. = 1 1/2 fingers.

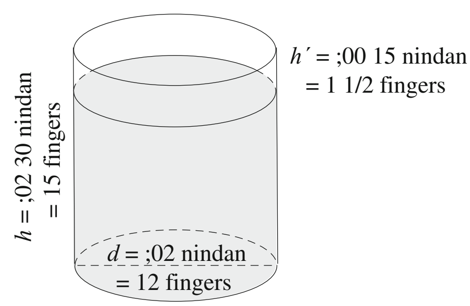


Fig. 11.3.3. BM 85194 # 24. One cylinder sìla of grain removed from the top of a cylinder bán-vessel full of grain.

In **BM 85194 # 34**, the object considered is a *pisannum*, which normally means a container made out of reed. In the present case, it apparently means a cylindrical reed basket, 'formerly' containing 3 bariga of grain. Unfortunately, the text of the exercise is damaged in several crucial places. In addition the solution procedure apparently begins incorrectly and contains a miscalculation. Anyway, the computation rule apparently used in this exercise is the following:

If C is counted in cylinder sìla, the circumference a in rods, and h in cubits, then $C = h$ · ;05 sq. a · S,
where S = 6 40 cylinder sìla per volume-shekel.

In particular, if C = 3 cylinder bariga = 3 00 cylinder sìla, and if a = 3 cubits (= ;15 n.), then it follows from the mentioned computation rule that (in relative place value notation) *h* = *C* · rec. (5 · sq. 15) · rec. 6 40 = 3 · 12 · 16 · 9 = 1 26 24, meaning 1;26 24 cubits = 7;12 n.

In the mentioned basket holding 3 cylinder bariga when full, 1 cylinder bariga would, of course, take up a height of 7;12 n./3 = 2;24 n., 1 cylinder bán a height of 2;24 n./6 = ;24 n., and 5 cylinder sìla a height of ;12 n. With departure from these numbers, one may try to begin understanding what is going on in the damaged and incorrectly solved exercise BM 85194 # 34.

The occurrence of the words *ma-aḫ-ru-tum* and *wa-ar-ku-tum*, probably meaning 'earlier' and 'later' suggests that this exercise, just like BM 85194 # 24 above, is about how much the level of grain goes down if a certain amount of grain is removed from a vessel full of grain, here apparently a 3 bariga basket with the circumference 3 cubits = 15 n. and the height 7;12 n. Unfortunately, since the text is damaged it is not clear how much grain was removed.

One possibility is that the one who wrote the text knew the answer beforehand, but did not know the correct solution procedure. In that case it is possible that he manipulated his computations so that he would obtain the correct result, namely '12 rods ' as in line 7, probably meaning ';12 cubits = 1 n.'. This would mean that the amount of grain removed was 2 bán 5 sìla (= 25 sìla). Another possibility is that the calculations were correct, except that they started with computing the square of 3 (for 3 cubits) instead of the square of 15 (for 15 n.). In that case, the result would be 25 times to large, meaning that the amount of grain removed was just 1 sìla.

In **BM 85194 # 35**, an apparently tube-like object, a thin and long cylinder, has its content expressed as the capacity number 1 *qa* (or *qû*) = 1 sìla. The cylinder has the circumference 10 šu.si '10, a finger', which means 1 finger = ;00 10 rods or ;10 n. Asked for is the length *u*, which is calculated as follows:

First the area *A* of the cross section of the tube is calculated as *A* = 5 · sq. *a* = 5 · 1 40 = 8 20.
Then the length *u* of the tube is found as rec. 8 20 · rec. 6 40 = 7 12 · 9 = 1 04 48.

Here, clearly, '6 40' is the storing number *S* for cylinder sìla. Recall that

There are 6 40 cylinder sìla in a volume-shekel (or 6 40 00 cylinder sìla in a volume-sar).

In order to try to explain the mentioned solution procedure, start by assuming that the length *u* = *p* cubits. Then

A · *u* = ;05 · sq. (;00 10 rods) · *p* cubits = ;00 00 08 20 · *p* volume-shekels.

Consequently,

S · *A* · *u* = the volume of 6 40 · ;00 00 08 20 · *p* cylinder sìla.

Therefore, the length of the cylinder in exercise # 35 is *u* = *p* cubits, where

p = rec. ;06 40 · rec. ;08 20 = 9 · 7;12 = 1 04;48.

This means that the answer to the question in the exercise is formally correct, if it is taken to mean that *u* = 1 04;48 (cubits). However, the data for this exercise seem to have been badly chosen and are quite unrealistic.

A final example of an Old Babylonian mathematical cuneiform text counting with cylinder sìla is the unusually interesting exercise **VAT 8522 # 1** in a carelessly organized and badly written recombination text**.**

VAT 8522 # 1 (Neugebauer, *MKT I* (1935), 368) conform transliteration in Fig. 11.3.5 below.
The exercise has been discussed by Muroi in *JHSJ* II:25 (1986), in Japanese with a summary in English.

1-3 [$^{\text{geš}}$er]en 5 nind[an u]š.bi / ⸢1 04⸣ sìla *i-na iš-di-šu ik-bi-ir* / 8 sìla *i-na ap-pi-šu ik-bi-ir* /
4-6 3 $^{1}/_{3}$ ma.na kù.babbar {*ga*} *ga-am-ru-um* / *ša* $^{1}/_{3}$ ma.na kù.babbar $^{\text{geš}}$eren *ki ma-ṣi* / *li-ik-ki-sú-nim*
(Figure)
7-11 6 *a-na* ⸢$^{1}/_{2}$ íl⸣ / 3 *a-na* 3 [*ša*]-*lu-uš-tim* / 9 ki 9 / 1 ⸢21⸣ [*a-na*] 5 igi.gub ⸢*ki-pa-tim*⸣ / 6 45 a.na [*k*]*a-lu-šu* /
12-14 6 45 *a-na* 6 40 igi.gub ⸢*na-aš-pa-ki-im*⸣ / 45 *ka-lu-šu ik-bi-ir* / 4 30 *i-na ap-pi-šu i-na-ki-sú-nim* / Etc.

1-3 A [ced]ar tree, 5 nind[an] its le[ngth] / ⸢1 04⸣ sìla in its base it was thick / 8 sìla in its nose it was thick. /
4-6 3 1/3 minas of silver was the total / for 1/3 mina of silver how much cedar / should he cut off?
(Figure)

7-11 6 to ⸢1/2 carry⸣ / 3 to 3 of the [th]reefold / 9 times 9 / 1 ⸢21⸣ [to] 5 the constant of the ⸢circle⸣ / 6 45 as all of it / 6 45 to 6 40 the constant of ⸢storing⸣ / 45 all of it was thick / 4 30 in the nose he cut off. / Etc.

The figure in the text shows a longish trapezoid divided at its midpoint by a transversal. The length of the trapezoid is marked as ‘5’, the two ends as ‘1 04’ and ‘8’. There are also brief notes in the figure saying

4 ba.si, 2 ba.si, dal gar.gar ‘4 is the cube root 2 is the cube root the transversals add together’

Obviously, the trapezoid in the figure is a two-dimensional rendering of the cedar tree in the text.

The curiously worded information that the tree is 1 04 sìla thick at the base and 8 sìla thick at the nose (that is, the top) can be understood as follows. A “cubic cylinder”, as long as it is thick, is cut off at either end of the tree. The two cubic cylinders measure 1 04 (= 64) and 8 cylinder sìla. Since the cube roots of 1 04 and 8 are 4 and 2, respectively, this means that the diameters of the tree at the base and the top, respectively, are 4 and 2 n. (equal to 4 and 2 decimeters). Additional information about the cedar tree is that the value of the whole tree was 3 1/3 mina of silver, but that a piece of it worth only 1/3 mina of silver should be cut off.

The solution procedure starts by halving the sum 6 (n.) of the base and top diameters. The result 3 is multiplied by ‘3 of the threefold’, clearly referring to how the circumference of the tree midway between the base and the top is (approximately) 3 times the diameter there. The resulting 9 is then squared and multiplied by the ‘circle constant’ 5, meaning as usual ;05. The result is 1 21 · 5 = 6 45 (meaning 6;45 sq. n.). This is the area of the cross section at the middle of the tree.

Now, if this middle cross section area is multiplied by the length of the tree, namely 5 rods = 1 00 cubits, the result is (approximately) the volume of the whole tree, namely

6;45 sq. n. · 1 00 cubits = 6 45 sq. n. · cubit = 6;45 volume-shekel.

(Here the volume of a truncated cone, which is difficult to compute exactly, is assumed to be approximately equal to the volume of a cylinder with the same cross section as the cross section of the truncated cone half-way between its two ends and with the same length. see the discussion in Sec. 8.5.11 above.) In Babylonian relative place value notation this whole operation amounts to multiplication of 6 45 by ‘1’, which was not deemed to be worth mentioning at all in the text of the exercise. Next, 6 45 is multiplied by the storing number 6 40 for cylinder sìla. The result of this operation is 6 45 · 6 40 = 45, which can be interpreted as follows:

6;45 volume-shekel · 6 40 cylinder sìla per volume-shekel = 45 00 cylinder sìla.

Next, it is (silently) observed that 1/3 mina is a 10th of 3 1/3 mina. Therefore, for 1/3 mina precisely a 10th of the whole tree can be bought, namely 1/10 · 45 00 = 4 30 cylinder sìla, to be cut off at the top of the tree.

What remains of the solution procedure is so badly preserved that it defies interpretation. Presumably, an effort was made to compute the length of the cut off piece of the tree. Hover to do this correctly would mean solving a cubic equation of the type

$(d-2)\cdot \text{sq.}\,(d+2) = 7;12.$

This was certainly far beyond what even a mathematically competent Old Babylonian scribe could do.

The use of cylinder sìla with the storing number S = 6 40 was very suitable for many kinds of computations involving cylinders and cones, as in all the examples discussed above. However, in exercises ## 1-5 of the large Old Babylonian recombination text Haddad 104 (Al-Rawi and Roaf, *Sumer* 43 (1984)) computations of precisely the same kind are carried out using the alternative storing number S = 6 (see Sec. 8.5.11 above). In BM 96954+ § 1, the storing number is 7 30 (Sec. 8.5.1), and in Sb 13293 it is 8 (see Sec. 8.5.8).

11.3.2 § 2. Problems for a Stair-Case of Bricks of type S1/3c

YBC 4669 § 2 (## 10-11; *obv. ii*: 13-21; and # 12; *iii*: 1-8) Cf. Friberg, *ChV* (2001), Sec. 4.3.

§ 2a 1 ašag 2 1/2 nindan sukud /
2-4 en.nam uš ḫé.til / gìr.gub kun$_4$ gar.ra / 3 1/2 nindan 3 kùš uš

1 A field. 2 1/2 rods the height. /
2-4 What length should I finish / to set the step of the staircase? / 3 1/2 rods 3 cubits was the length.

§ 2 b 1-4 ašag 2 ½ nindan su[kud] / *i-na* šag$_4$-*šu* [x x x] / ½ nindan uš [x x x] / sukud.meš [x x x x] /
5-8 igi.½.gál [x x x x] / ……………

1-4 Afield. 2 1/2 rods the hei[ght] / out of it [x x x] / 1/2 rod the length [x x x]. / The heights [x x x x] /
5-8 The reciprocal of 1/2 [x x x x] / …………

§ 2 c 1-4 3 ½ nindan 3 kùš uš / ⅓ kùš sig$_4$ gìr.gub / [⅓] kùš gìr.gub / 2 kùš dagal gìr.gub /
5-6 šid sig$_4$ gìr.gub en.nam / *ù* sig$_4$ en.nam /
7-8 2 15 šid gìr.gub / 2 sar 3 *šu-ši* sig$_4$

1-4 3 1/2 rods, 3 cubits the length / 1/3 cubit the bricks of the steps /
[1/3] cubit the steps / 2 cubits the width of the steps. /
5-6 The count of the brick steps was what / and the bricks were what? /
7-8 2 15 was the count of the steps / 2 sar 3 sixties were the bricks.

These three exercises are almost certainly random excerpts from an extensive theme text, which explains why some crucial information is missing from § 2a. The term ašag 'field' or 'area' with which §§ 2a-b begin is probably a miscopy by the ancient scribe who wrote YBC 4669. It is likely that he misinterpreted a similar-looking sign. One would have expected to see the sign kun$_4$ 'staircase' introducing these two paragraphs.

In § 2a, the total height of a long staircase is given, h = 2 1/2 rods (= 30 cubits), and one is asked to find the corresponding length. This cannot be done without knowing, for instance, the inclination of the staircase. Since the answer says that the length is u = 3 1/2 rods 3 cubits (= 45 cubits), it is easy to see that the never mentioned inclination must have been equal to $u : h = 45 : 30 = 3 : 2$.

In § 2b, the given height is the same as in § 2a, h = 2 1/2 rods , but most of the text of the exercise is destroyed, so no interpretation attempt is possible.

§ 2c, on the other hand, is almost perfectly preserved. It seems to be asking for the number of steps in a staircase, and the number of bricks needed to build those steps, when it is known that the total length of the steps is 3 1/2 rods 3 cubits (= 45 cubits), that 1/3-cubit square bricks (5 fingers thick bricks of type S1/3c) are used for the individual steps, that 1/3 cubit is the height of each step, and that 2 cubits is the width of each step. See Fig. 11.3.4 below.

The solution procedure is omitted in § 2c but can easily be provided. First, it is clear that 2 · 6 = 12 bricks of type S1/3c are needed for each step. Indeed, six bricks side by side are needed in order to obtain the requested width, namely 2 cubits, and two bricks on top of each other are needed in order to obtain the requested height of a step, namely 1/3 cubit = 10 fingers. Moreover, the length of a step being equal to 1/3 cubit, which is the side of one of the square bricks, it follows that *the number of steps* must be equal to 3 · 45 = *2 15.* Therefore, *the total number of bricks* is 2 15 · 12 = 2;15 · 12 00 = 2 1/4 brick sar = *2 brick sar 3 sixties of bricks*. This is precisely the answer given in the text.

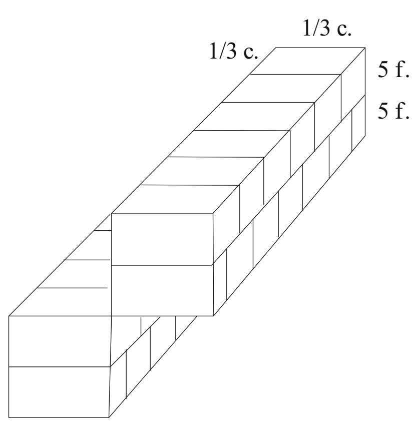


Fig. 11.3.4. YBC 4669 § 2c. Two steps of a staircase built with square bricks of type S1/3c.

11.3.3 § 3. Problems for Bricks of Type R1/2c

YBC 4669 § 3 (## 13-14; *obv. iii*: 9-…)

§ 3a 1-4 1(bùr)[ašag] sig_4 / 2 nindan sukud.bi / gagar en.nam / 10 $^1/_3$ s[ar 5 gí]n

1-4 1 búr of bricks / 2 rods it height. / The ground (bottom area) was what? / 10 1/3 s[ar 5 shek]els.

Solving this problem requires the careful use of certain well known metrological equations. First,

1 bùr of bricks = 3 · 6 · 100 sar of bricks = 30 00 brick-sar, where 1 brick-sar = 12 00 bricks.

Secondly, the molding number *L* for standard rectangular bricks of type R1/2c is 7;12, which means that

7;12 brick-sar of bricks of type R1/2c have the volume 1 volume-sar.

Consequently,

1 brick-sar of bricks of type R1/2c = rec. 7;12 · 1 volume-sar of bricks = ;08 20 volume-sar of bricks, and
1 bùr of bricks of type R1/2c = 30 00 brick-sar = 30 00 · ;08 20 volume-sar of bricks = 4 10 volume-sar of bricks.

Now, it was given that the bricks were piled up in a (presumably rectangular) brick pile of the height 2 rods = 24 cubits. Moreover, 1 volume-sar = 1 square rod · 1 cubit. Therefore, the bottom area of the brick pile would be

rec. 24 · 4 10 sq. rods = ;02 30 · 4 10 sq. rods = 10;25 sq. rods = 10;25 area-sar = 10 1/3 area-sar 5 shekels.

This is precisely the answer given in the text of YBC 4669 § 3a.

The brick pile figuring in this exercise was quite substantial. Indeed,

30 00 brick-sar = 30 00 ·12 00 bricks = 6 00 00 00 bricks = 1,296,000 bricks,

and the bricks could be piled into a brick pile with rectangular bottom and the dimensions

60 cubits · 25 cubits · 24 cubits (= 30 meters · 12,5 meters · 12 meters = 4,500 cubic meters).

The remainder of § 3 is almost completely destroyed.

11.3.4 § 4. A Problem Involving a Work Norm for Digging

YBC 4669 § 4 (# B 1; *rev. i*: 1-7)

… ………
1-2 éš.gàr lú.1.e e[n.nam] / *ù* erín.ḫá en.nam /
3-5 igi 10 gín éš.gàr duḫ / *a-na* 1(iku)[ašag] íl-*ma* / 1(geš'u) erín.ḫá igi.duḫ /
6-7 éš.gàr *ù* erín.ḫá / ì.gu_7-*ma* saḫar igi.duḫ

… ………
1-2 The work norm for 1 man w[as what] / and the man-days were what?
3-5 The reciprocal of 10 shekels, the work norm, release / to 1 iku carry, then /
6-7 the work norm and the man-days / let eat each other (multiply), the mud (volume) you will see.

The first few lines of this exercise are missing. However, since it is mentioned in line 3 of the (preserved part of the) exercise that 10 gín éš.gàr '10 shekels was the work norm', it is likely that the missing lines mentioned work concerned with digging a canal or making some other kind of excavation. See, for instance, the survey in Robson *MMTC* (1999), 100, of mathematical exercises in BM 85196, Erm 15073, and YBC 4657, 4666, 5037, 7164, mentioning this work norm. In line 4 it is mentioned that the volume of the excavated mud was 1 iku.

The last part of the question in this exercise is preserved, as well as, exceptionally in a recombination text of the present kind, *the solution procedure*. Asked for are the number of erín 'soldiers, men, man-days' and (superfluously) the work norm. The solution procedure is exceedingly simple. Indeed, the number of man-days required for the excavation work is obtained by multiplying the given volume 1 iku with the reciprocal of the given work norm 10 (volume-)shekels (per man-day). The result of the multiplication is

1 volume-iku · rec. (10 volume-shekels per man-day) = 100 · 60 · rec. 10 man-days = 10 · 60 man-days.

The result of the calculation is written as 1(geš'u) man-days. (In *MKT III*, 27, the sign 1(geš'u) was misread as 1 10.)

An outline of a verification in general terms is appended to the exercise, mentioning that if the work norm is multiplied by the man-days, the result will be the excavated volume.

11.3.5 § 5. A Cubic Equation without Context

YBC 4669 § 5 (# B 2; *rev. i*: 8-10)

1-3 uš *ù* saḫar / gar.gar-*ma* 33 20 / uš saḫar en.nam

1-3 Length and mud (volume) / I added together, then 33 20. / Length, mud were what?

In the question of this exercise it is stated that the sum of the length and the volume of some three-dimensional object is 33 20. Obviously, this brief exercise had been copied from some theme text where it was preceded by other exercises mentioning precisely what was going on. *Without this context*, and *without an explicit answer* given in the text, it is impossible to know how to answer the question. Besides, it is clear that the question must lead to *a cubic equation*, and cubic equations could be solved in Old Babylonian mathematical exercises only in exceptionally simple cases.

In *MKT II*, 63-64, Neugebauer made the remark that this exercise, together with several of the exercises in YBC 4696 (a "series text" with 52 problems for divided triangles),

> "*quite clearly shows that it is absolutely justified to say that the main stress in Babylonian mathematics was on algebraic, not geometric relations*. The purpose of the geometric dress is just to define the form, the relations between the entities; everything else is then a purely algebraic formalism, as is clearly proved, for instance, by the geometrically nonsensical inhomogeneous problems in (YBC 4696) ## 14-17, 21.22, 32-35" and in YBC 4669 # B 2.

11.3.6 § 6. Linear Equations for Grain. Products of Basic Fractions

YBC 4669 § 6 (## B 3-5; *rev. i*: 11- 23) (Neugebauer, *MKT III*, 27; Neugebauer and Sachs, *MCT*, 103)

§ 6a 1-3 $^2/_3$ $^1/_3$ šukúr-*ia* / ì.gu$_7$-*ma* / 7 íb.tag$_4$ / sag šukúr-*ia* en.nam / 31 30

1-3 2/3 of 1/3 of my ration / I ate, then / 7 remained. / The original ration was what? 31 30

§ 6b 1-4 *a-na* $^2/_3$ $^2/_3$-*ia* / 1(bán) daḫ-*ma* / še-*e ma-ši-il* / sag še en.nam {3(barig)} / 3(barig) še sag

1-4 To 2/3 of my 2/3 / 1 bán I added, then / the grain was halved / The original grain was what? / 3 bariga was the original.

§ 6c 1-3 $^2/_3$ $^1/_3$-*ia* 1(bán) / sag še-*ia* en.nam / [4(bán) 5 sìla] sag

1-4 2/3 of my 1/3 was 1 bán / My original grain was what? / [4 bán 5 sìla] was the original.

YBC 4669 § 6a

The term íb.tag$_4$ 'it remained' is not the appropriate word here. In order to arrive at the proposed the answer 31 30, you must understand the question as asking how much you ate if you consumed 2/3 of 1/3 of your ration.

Apparently all the problems in §§6a-c were, to some part, exercises in how to handle *products of basic fractions* like 1/3, 1/2, and 2/3 (if you know nothing about common fractions). Thus, the question in § 6a, for instance, was probably meant to be answered as follows:

Since 2/3 · 1/3 = ;40 · ;20 = ;13 20, the problem can be rephrased as ;13 20 · *C* = 7 (sìla).
The answer is that *C* = rec. ;13 20 · 7 = 4;30 · 7 = 31;30 (sìla).

YBC 4669 § 6b

Let *C* be the original quantity of grain. In modern terms, the question can be rephrased as the following linear equation:

2/3 · 2/3 · *C* + 10 (sìla) = 1/2 *C*, or ;26 40 *C* + 10 = ;30 *C*.

Consequently,

;03 20 *C* = 10, so that *C* = rec. ;03 20 · 10 (sìla) = 18 · 10 (sìla) = 3 · 60 (sìla) = 3 barig.

This is the answer given in the text.

YBC 4669 § 6c

This exercise is even simpler. The question can be rephrased as

2/3 · 1/3 *C* = 1 bán = 10 sìla, or ;13 20 *C* = 10 (sìla).

Consequently,

C = rec. ;13 20 · 10 (sìla) = 4;30 · 10 (sìla) = 45 (sìla) = 4 bán 5 sìla.

11.3.7 § 7. A Problem for a Foil of Silver. The Rule of Three

YBC 4669 § 7 (# B 6; *rev. ii*: 1-11) (Neugebauer, *MKT III*, 27; Friberg, *MSCT 1* (2007), 348)

1-4 3 kùš 1 šu.si uš / 2 kùš 6 šu.si sag / gagar.bi en.nam / 2 $^{2}/_{3}$ gín 20 $^{1}/_{2}$ še /
5-7 [4] gín kù.babbar šúm.ma / 3 šu.si.ta íb.sá / *ru-uq-qá-am im-ḫa-ṣú* /
8-11 *i-na-an-na* / kù.babbar en.nam šúm.ma / 2 $^{2}/_{3}$ gín 20 $^{1}/_{2}$ še šúm / ḫé.gar.ra

1-4 3 cubits 1 finger the length / 2 cubits 6 fingers the front. / The ground (area) was what? 2 2/3 shekels 20 1/2 grains. /
5-7 [4] shekels of silver were given / 3 fingers equally / the foil they beat. /
8-11 Now / what of silver was given (so that) / 2 2/3 shekels 20 1/2 grains were given / you may set?

This exercise begins(in lines 1-4) by mentioning a rectangle with the sides

3 cubits 1 finger (= ;15 10 n.) and 2 cubits 6 fingers (= ;11 n.)

Then the area of this rectangle is computed. It is

;15 10 n. · ;11 n. = ;02 46 50 sq. n. = 2 2/3 shekels 20 1/2 grains.

This area is correctly mentioned in line 4 of the exercise.

Next it is mentioned that *[4] shekels of silver could be beaten into a square of side 3 fingers*. Since there are 6 00 fingers in a nindan, this means that, proportionally,

4 00 · 4 shekels = 16 minas of silver could be beaten into a square of the size 1 area-shekel.

Now, the question in the exercise (in lines 8-11) is

How much silver can be beaten into a rectangle of area 2 2/3 (area-)shekels 20 1/2 grains (= ;02 46 50 sq. n.)?

The answer to this question is not given in the text, but it can easily be computed as

16 minas per area-shekel · 2;46 50 area-shekels = 44;29 20 minas = 44 1/3 minas 9 1/3 shekels of silver.

The whole exercise is, of course, an application of the rule of three.

11.3.8 § 8. Systems of Linear Equations for Ewes and Lambs

YBC 4669 § 8 (# B 7-8; *rev. ii*: 12-25)
(Neugebauer, *MKT III* (1937), 28; von Soden, *ZDMG* 91 (1937), 203, fn. 1;
(Neugebauer and Sachs, *MCT* (1945), 130; Muroi, *SBM* 3 (2007), 11)

8a 1-2 us$_5$.udu.ḫá *ù* sila$_4$.gub / gar.gar-*ma* 9 šu.si /
3-4 us$_5$.udu.ḫá *ù* sila$_4$.gub / en.nam

1-2 Ewes and lambs / add together, then 9 sixties.
3-4 Ewes and lambs / are what?

8b 1-4 2 tùr.meš 1 tùr 1 40 1 20 / *ú-wa-li-id* / 2 tùr 1 40 1 15 / *ú-wa-li-id* /
5-8 us$_5$.udu.ḫá ugu u[s$_5$.udu.ḫ]á / 1 *šu-ši* dirig / sila$_4$.gub [ug]u sila$_4$.gub / 1 [*šu-ši*] dirig /
7-8 us$_5$.[udu.ḫá *ù* sila$_4$.g]ub / e[n.n]am

1 2 (sheep) folds. Fold 1, 1 40 (to) 1 20 / gave birth, / fold 2, 1 40 (to) 1 15 / gave birth. /
5-8 Ewes over ew[es] / 1 sixty were beyond, / lambs over lambs / 1 [sixty] were beyond. /
7-8 Ewes and lambs / were what?

YBC 4669 § 8a

In § 8a it is clear that some information is missing. Just as in the case of § 5 this is presumably because the exercise had been excerpted from the middle of a theme text, where the needed information was given in the preceding paragraphs.

Luckily, the needed information may be present in the subsequent § 8b, which begins by mentioning two herds of ewes and lambs. As we shall see, the ewes gave birth to lambs at a rate of 1 20 lambs per 1 40 ewes in one of the herds (80 percent!), and at a rate of 1 15 lambs per 1 40 ewes (75 percent!) in the other. Apparently uneducated shepherds made no use of sexagesimal numbers, counting sheep decimally, in hundreds! Counting, nevertheless, with sexagesimal numbers, one can express the two breeding rates as

$b = 1\ 20 \cdot \text{rec. } 1\ 40 = 1\ 20 \cdot ;00\ 36 = ;48$ lambs per ewe, $\quad b' = 1\ 15 \cdot \text{rec. } 1\ 40 = 1\ 15 \cdot ;00\ 36 = ;45$ lambs per ewe.

If it is assumed that the breeding number in § 8a was, for instance, ;48 lambs per ewe, then the stated problem can be reformulated as the following simple system of linear equations for the numbers *e* of ewes and *s* of lambs:

$e + l = 9\ 00, \quad l = ;48 \cdot e.$

The solution is easily found, namely that

$e \cdot (1 + ;48) = 9\ 00$, so that $e = \text{rec. } 1;48 \cdot 9\ 00 = ;33\ 20 \cdot 9\ 00 = 5\ 00$, and $l = ;48 \cdot 5\ 00 = 4\ 00$.

If the breeding number was instead ;45 lambs per ewe, the answer would be harder to find, since 1;45 = ;15 · 7 is not a regular sexagesimal number.

YBC 4669 § 8b

As mentioned above, in this exercise there are two herds of ewes and lambs. In one of the herds the birth rate is b = ;48 lambs per ewe, in the other b´ = ;45 lambs per ewe. Now, it is stated that there were 60 ewes and 1 [00] lambs more in the first herd than in the second. The question is how many ewes and lambs there were in the two herds. Suppose there were *e* ewes and *l* lambs in the first herd, and *e´* ewes and *l´* lambs in the second. Then the stated problem can be transformed, in quasi-modern terms, into the following system of linear equations :

$e - e' = 1\ 00, \quad ;48 \cdot e - ;45 \cdot e' = 1\ [00].$

This is a system of linear equations, of the same kind as the systems of linear equations in YBC 4698 § 3 (discussed above in Sec. 11.1.4). It can easily be solved as follows, for instance:

According to the first equation $e' = e - 1\ 00$. Therefore,
according to the second equation $(;48 - ;45) \cdot e + 45 = 1\ 00$, so that $;03 \cdot e = 15$ and $e = 20 \cdot 15 = 5\ 00$.

Consequently,

$e' = 5\ 00 - 1\ 00 = 4\ 00, \quad l = ;48 \cdot e = 4\ 00, \quad l' = ;45\ e' = 3\ 00.$

A closely related exercise on a roughly made square hand tablet looks like this:

YBC 7326 (Neugebauer and Sachs, *MCT* (1945), 130; photo : Nemet-Nejat, *UOS* (2002), 278-279)

obv. 1 *šum-ma ši-na ta-ar-ba-ṣú-a-mi* /
2-3 *ta-ar-ba-ṣú-ú e-li ta-*<*ar*>*-ba-ṣí* / 8 20 *i-te-er*
4 *li-li-du e-li li-*[*li*]*-di* 10 50 *i-te-er*

rev.

[1] 40	50	8 20
1 20	40	
1 40	41 40	10 50
1 10	29 10	

1 If (he says): my two sheep folds. /
2-3 Fold over fold / 8 20 was beyond, /
4 Lambs over lambs 10 50 were beyond.

The inscription on the reverse shows that in this exercise the two breeding rates are

$$b = 1\ 20 \cdot \text{rec. } 1\ 40 = 8 \cdot \text{rec. } 10 = ;48 \qquad \text{and} \qquad b' = 1\ 10 \cdot \text{rec. } 1\ 40 = 7 \cdot \text{rec. } 10 = ;42.$$

Therefore, the badly formulated question in YBC 7326 can be expressed more correctly as the following system of linear equations for e and e', the unknown numbers of ewes in the two sheep folds:

$$e - e' = 8\ 20, \qquad ;48 \cdot e - ;42 \cdot e' = 10\ 50.$$

Proceeding in the same way as in the suggested solution procedure for YBC 4669 § 8b above, one can solve this system of linear equations as follows:

$$\text{Set} \quad e' = e - 8\ 20. \qquad \text{Then} \quad (;48 - ;42) \cdot e + ;42 \cdot 8\ 20 = 10\ 50, \qquad \text{so that} \quad ;06 \cdot e = 10\ 50 - 5\ 50 = 5\ 00.$$

Consequently,

$$e = 10 \cdot 5\ 00 = 50\ 00, \text{ and } e' = 50\ 00 - 8\ 20 = 41\ 40, \quad \text{and} \quad l = ;48 \cdot 50\ 00 = 40\ 00, \quad l' = ;42 \cdot 41\ 40 = 29\ 10.$$

These are the numbers mentioned in the middle column on the reverse of YBC 7325.

Just for the sake of completeness, a third Old Babylonian mathematical text dealing with sheep must be mentioned here. In the chaotically formatted text, exercise # 3 on the reverse can be transliterated as below, to a great deal thanks to von Soden's and Muroi's improved readings in *ZDMG* 91 and *SMB* 3. However, neither von Soden nor Muroi could find an adequate explanation for what is going on in this puzzling exercise.

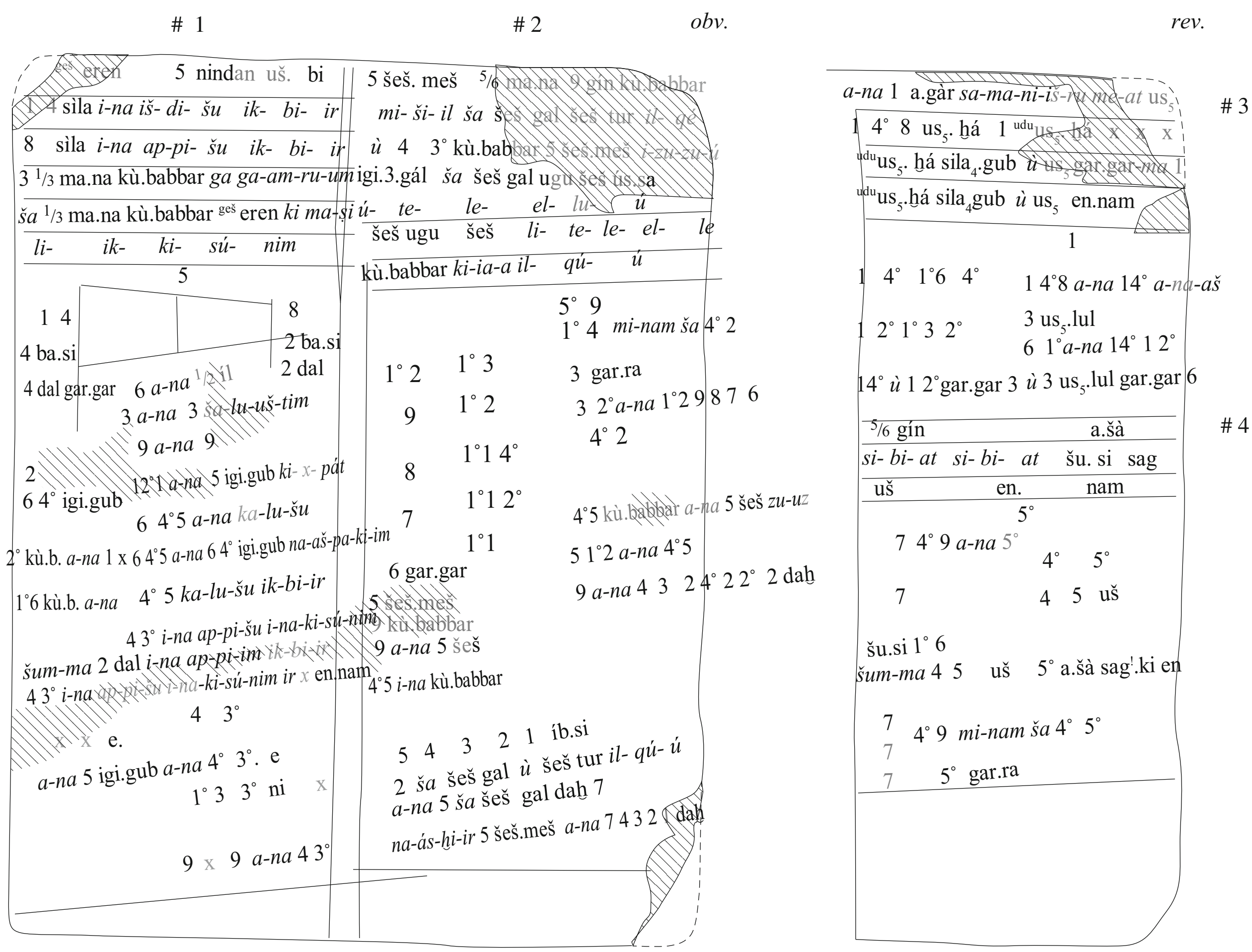


Fig. 11.3.5. VAT 8522. A mathematical recombination text with chaotically organized explicit solution procedures.

For exercises ## 1, 2 and 4, see Secs. 11.3.1 and 7.1.9 above, and Sec. 13.6 below, respectively.

VAT 8522 # 3 (*rev*. 1-4, 1a-c, 2a-b, 3b)
(Neugebauer, *MKT I* (1935), 369; von Soden, *ZDM* 91 (1937), 200; Muroi, *SBM* 3 (2007), 12)

a-na 1 a.gàr *sa-ma-ni-i*[*š-ru me-at* us$_5$] / 1 48 us$_5$.ḫá 1 udu.u[s$_5$.ḫá] / udu.us$_5$.ḫá silà$_4$.gub *ù* [us$_5$ gar.gar-*ma* 1] / udu.us$_5$.ḫá silà$_4$.gub *ù* us$_5$ en.nam /

	1	
1 40	16 40	1 48 *a-na* 1 40 *a-n*[*a*]-*aš*-<*ši*> /
1 20	13 20	3 us$_5$.lul /
	6 10 *a-na* 1 40 1 20 /	

1 40 *ù* 1 20 gar.gar 3 *ù* 3 us$_5$.lul gar.gar 6

To 1 meadow eight[een hundred sheep], / 1 48 sheep 1 ew[e], / ewes, lambs, and [sheep I heaped, then 1]. / Ewes, lambs, and sheep, what?

	1	
1 40	16 40	1 48 to 1 40 I carried, /
1 20	13 20	3 the false sheep, /
	6, 10 to 1 40 1 20 /	

1 40 and 1 20 heap 3 and 3, the false sheep, heap 6.

In this exercise, eighteen hundred sheep are confined to 'meadow 1', while unknown numbers of ewes and lambs are confined to a second meadow, although this is not stated explicitly. Actually, also the number of sheep in meadow 1 should have been unknown, but it happens now and then in Old Babylonian mathematical exercises that numbers that should be unknown by mistake are mentioned prematurely.

Let e, l, and s be the unknown numbers of ewes, lambs, and sheep, respectively, in this exercise. *Assume that the the silently assumed birth rate is the usual 1 20 (80) lambs per 1 40 (1 hundred) ewes, or more concisely b = ;48 lambs per ewe.* Then the vaguely stated question may be rephrased and made more precise as follows, in quasi-modern symbolic notations:

$$s = 1;48\ e, \quad l = ;48 \cdot e, \quad e + l + s = [1\ (00\ 00)].$$

Note that the solution procedure starts by prominently displaying the number '1', which of course stands for the given (but not preserved) value of the sum.

The equations above form an easily solved system of three linear equations for three unknowns. The sketchy and badly organized solution procedure appears to start with the following 'false' values:

$$e^* = 1\ 40, \quad l^* = 1\ 20.$$

Then a corresponding false value for s, called us$_5$.lul 'the false sheep' will be

$$s^* = 1;48 \cdot 1\ 40 = 3\ (00).$$

The corresponding false value for the sum of ewes, lambs, and sheep will be

$$e^* + l^* + s^* = 1\ 40 + 1\ 20 + 3\ (00) = 3\ (00) + 3\ (00) = 6\ (00).$$

Since the sum was required to be [1 00 00)], the needed correction factor is rec. 6 = 10. Consequently, the answer to the stated question is

$$e = 10 \cdot 1\ 40 = 16\ 40\ (10\ \text{hundred}), \quad l = 10 \cdot 1\ 20 = 13\ 20\ (8\ \text{hundred}), \quad s = 10 \cdot 3\ (00) = 30\ (00)\ (18\ \text{hundred}).$$

Evidently all parts of this solution procedure can be found somewhere in the text of the exercise.

Note by the way, that the strangely unorganized form of the solution procedure suggests that *the essential steps of the solution procedure were jotted down by a student as he watched his teacher explain how the problem could be solved.* (The same goes for the solution procedures of the other three problems on VAT 8522.) Unfortunately, VAT 8522 is the only known Old Babylonian mathematical text in which such hastily scribbled notes from the teacher's presentation have been preserved. On the other hand, quite a few Old Babylonian mathematical exercises are known, like BM 80078 # 5′ in Sec. 8.8 above, where a student has started out with a false solution procedure but then ended with the latter half of a correct solution procedure. That can have happened if the student could only understand half of his scribbled notes of how the the teacher had shown that the problem could be solved.

11.3.9 § 9. A System of Linear Equations for Two Baskets

YBC 4669 § 9 (# B 9; *rev. iii*: 1-6) (Neugebauer, *MKT III*, 28)

1-6 ⸢3(barig)⸣ še.gur 2 ^{geš}ba.rí.ga / ^{geš}ba.rí.ga / ugu ^{geš}ba.rí.ga / 6 ⸢2/3⸣ sìla dirig / ^{geš}ba.<rí>.ga.meš en.nam / 27 x x

1-6 ⸢3 bariga⸣ of grain 2 barig-baskets / bariga / over bariga / 6 ⸢2/3⸣ beyond / The barig-vessels what? / 27 x x

The reading ⸢3 (barig)⸣ še.gur and the translation '⸢3 bariga⸣ of grain' is problematic, and so is the reading 6 ⸢2/3⸣ sìla. However, '2 barig-(vessels)', here probably meaning '2 baskets of about the size of 1 barig' should each hold not much more (or less) than 1 barig, and the sexagesimally regular number 6 2/3 = 6;40 is incomparably more common in mathematical cuneiform texts than the non-regular number 6 1/3 = 6;20. Besides, 27 · 6 2/3 sìla = 3 00 sìla = 3 barig, which may help to explain the number 27 x x in the last line of the exercise. (Here x x stands for two well preserved but unreadable cuneiform signs.)

If the suggested improved reading is correct, then it is easy to solve the stated problem, which can be reduced to the following system of linear equations for the unknown contents C and C' of the two baskets:

$C + C' = 3\ 00$ (sìla), $C - C' = 6;40$ (sìla).

The solution would then be, of course,

$2\ C = (3\ 00 + 6;40)$ (sìla),

$C = 1/2 \cdot 3\ 06;40$ (sìla) = 1 33;20 (sìla) = 1 bariga 3 bán 3 1/3 sìla, $C' = 1\ 26;40$ (sìla) = 1 bariga 2 bán 6 2/3 sìla.

(If instead, as in Neugebauer's hand copy, $C + C' = 4\ 00$ sìla and $C - C' = 6;20$ sìla, then it can be shown in the same way that $C = 2$ bariga 3 1/3 sìla anda $C' = 1$ bariga 5 bán 6 2/3 sìla.)

11.3.10 § 10. A Combined Work Norm Problem for Bricks

YBC 4669 § 10 (# B 10; *rev. iii*: 7-17) (Friberg, *ChV (2001)*, Sec. 6.3.)

1-3 lú.1.e 20 nindan uš / ḫé.gul.gul / ḫé.íl.íl /
4-7 *i-na* igi.te.en ud / ḫé.gul.gul / *i-na* igi.te.[en ud] / ḫé.íl.[íl] /
8-11 *i-na* igi.5.gál ud 12 / 5 gín gul.gul / *i-na* 48 íb.tag$_4$ ud / 5 gín íl.íl

1-3 1 man (for) 30! nindan of length / may destroy / (and) may carry. /
4-7 In (what) portion of a day / shall he destroy / in (what) por[tion of a day] / shall he car[ry]?
8-11 In a 5th-part of a day, 12 / 5 shekels he destroyed / in 48, the remainder of the day / 5 shekels he carried.

This is yet another example of an isolated problem in a mathematical recombination text appearing totally out of context. It is not clear what is being destroyed and what is being carried, and the work norms in play in this problem are not explicitly mentioned. It is not clear where the twice mentioned amount '5 shekels' comes from. Obviously, however, the shekels figuring in the question are volume-shekels, not weight-shekels.

Fortunately, the explicitly given answers hints at what is going on. Thus, if 5 shekels are destroyed in a fifth of a day, then 25 shekels of whatever are destroyed in 1 day. This work norm is known from the table of constants BR = *TMS* III, where the following constant is mentioned:

25! *šà* sig$_4$ *šà-ḫa-a-ṭì* 25 of tearing out bricks BR 41

The 5 in 25 is clearly visible in the photo of the tablet (Bruins and Rutten (1961), pl. IV). This means, in other words, that the first work norm is

w = 25 volume-shekels per man-day, the work norm for tearing out bricks (from brick-piles?) BR 41

As for the second work norm implicitly occurring in the exercise, if 5 volume-shekels of bricks are carried 30 rods in ;48 days, then rec. ;48 · 5 = 1;15 · 5 = 6;15 volume-shekels of bricks are carried 30 rods in 1 man-day. (Note that the text erroneously says 20 rods.) Therefore, the second work norm in this exercise is

w' = 6;15 volume-shekels of bricks · 30 rods per man-day is the work norm for carrying bricks.

On the other hand, a well known Old Babylonian work norm for carrying bricks is

9 sixties of bricks (rectangular bricks of type R1/3c) · 30 rods per man-day.

See, for instance, Sec. 11.2.1 above, in the commentary to the exercise W 4673 § 1b. It is also known that

the volume of a rectangular brick of type R1/3c is ;00 41 40 volume-shekels.

See again Sec. 11.2.1 above, in the commentary to the exercise W 4673 § 1a. Now 9 00 · ;00 41 40 = 6;15. Therefore, *in terms of volume measure, the Old Babylonian work norm for carrying bricks (of arbitrary type!) can be expressed as*

6;15 volume-shekels of bricks · 30 rods per man-day (or 3;07 30 volume-shekels · 1 00 rods per man-day).

This explains where the second work norm implicitly occurring in YBC 4669 § 10 came from. Note, however, that expressing work norms for carrying bricks in terms of volume measure is not known from any other Old Babylonian mathematical text. That is, of course, why neither '6 15' nor '3 07 30' appears in any known Old Babylonian table of constants.

The vaguely stated question in YBC 4669 § 10 can now be more precisely formulated, as follows:

1 man may tear out 25 shekels of bricks in 1 day, and he may carry 6 15 shekels of bricks 30 rods in 1 day.
What shall he tear out and carry in 1 day?
In what portion of a day shall he destroy, in what portion of a day shall he carry?

This is *a combined work norm problem* with the following solution:

rec. w = rec. 25 = ;02 24 man-days per volume-shekel, rec. w' = rec. 6;15 = ;09 36 man-days per volume-shekel,
rec. W = rec. w + rec. w' = (;02 24 + ;09 36) man-days per volume-shekel = ;12 man-days per volume-shekel,
W = 5 volume-shekels per man-day,
5 volume-shekels · rec. w = ;12 man-days, 5 volume-shekels · rec. w' = ;48 man-days.

11.3.11 § 11. A Principal and Interest Problem

YBC 4669 § 11 (# B 11; *rev. iii*: 18-24; Neugebauer *MKT I*, 516 and *MKT III*, 28).

1-2 kù.babbar *a-na* 1 ma.na / 12 gín a.na máš šúm.ma /
3-5 *i-na* mu 3.kam.ma / *al-li-ik-ma* / 1 gín kù.babbar *el-qé-a* /
6-7 sag kù.babbar en.nam / 34 43 20

1-2 Silver. For 1 mina / 12 shekels he paid as interest. /
3-5 In the 3rd year I went, 1 shekel of silver I received.
6-7 The silver principal was what? / 34 43 20.

For a discussion of this exercise, see Sec. 11.1.2 above.

11.3.12 ***The Vocabulary of YBC 4669***

a.na	as much as
ašag	field (area)
$^{\text{geš}}$ba.rí.ga	barig-basket
dagal	width
daḫ	add to
dal	transversal (diameter)
dirig	is beyond (is more than)
duḫ	release (compute reciprocal)
en.nam	what?
erín	soldier, man (man-day)
éš.gàr	work norm
gagar	ground (bottom area)
ganam$_5$ udu	ewe
gar.gar	add together
gar.ra, ḫé.gar.ra	to set

gìr.gub		step
gu$_7$, ì.gu$_7$		eat (multiply together)
gul, gul.gul, ḫé.gul.gul	(*šaḫāṭu* BR 41)	to destroy
íb.tag$_4$		remainder
igi.duḫ		to see
igi.te.en		portion
íb.sá		equally, equalside (square)
íl, íl.íl, ḫé.íl.íl		to carry (multiply)
kù.babbar		silver
kun$_4$		staircase
lú.1.e		1 man
máš		interest
sag		head, origin, original
saḫar		mud (volume)
sig$_4$		brick
sila$_4$.gub		lamb
sukud		height
šid		count
šukúr		ration
šúm, šúm.ma		to give
til, ḫé.til		to finish
tùr		sheep fold
ud		day
ugu		over
uš		length (long side)
al-li-ik		< *alāku* to go
el-qé-a		< *leqû* to take, to receive
im-ḫa-ṣú		<*maḫāṣu* to beat
i-na-an-na		now
ma-ši-il		< *mašālu* to be equal, to be halved
ru-uq-qu-um		foil, thin plate
šu-ši		sixty
ú-wa-li-id		< *walādu* to give birth

11.4 On the Form and Purpose of Old Babylonian Mathematical Series Texts

Below is a slightly extended and rearranged version of the list of *series texts* and *catalog texts* discussed by Proust in *NTM* 20 (2012). However, for the sake of greater clarity, Proust's series texts are renamed here as "arborescent" or "tree-like". (Actually, the term *arborescent* appeared first in Proust *ZA* 99 (2009).)

Each line of the list begins with the collection number of the text in question and its place of publication. Next follows information about the "series number" *n* of the text, and about the number of "text sections" it contains, whenever they appear in colophons of the texts. The series numbers are written in the form dub (*n*).kam '*n*th tablet', the number of text sections in the form (*m*) im.šu '*m* hand tablets'. The number of im.šu 'sections' is provided in all tree-like texts except one and in all theme texts except three. In contrast to this, the term im.šu appears nowhere else in the known corpus of Old Babylonian mathematical texts, except in the subscripts of the large single-column recombination text MS 3052 (Friberg, *MSCT 1* (2007), 256, 276), and the large double-column theme text MS 5112 (*op. cit.*, 308, 325), both probably from Uruk.

In the 5th column of the list below is shown the numbering of the series texts and catalogs used by Proust in *NTM* 20, from S1 to S20 for 20 series texts and from C1 to C8 for 8 catalog texts. In the next column are shown the classifications of the texts in the list as belonging to either Groups Sa, Sb or Groups 2a, 2b, according to their Akkadian spellings and mathematical terminology. See Friberg, *RA* 94 (2000), 162-4, 172-3, 180, and also Høyrup *LWS* (2002), 346. Note, in particular, that without exception *all texts* belonging to Groups Sa, Sb or Groups 2a, 2b according to Friberg, *RA* 94, 180 are included in the list. Finally, in a 7th column is noted if the text is a single-column text (S) or a multi-column text with *m* columns on the obverse and *n* on the reverse (M(*m*/*n*)).

The recently published texts AO 9071 and AO 9072 (Proust, *ZA* 99 (2009)) were, of course, not considered in Friberg, *RA* 94. Nevertheless, it is clear that they belong to Group Sa, as indicated below.

A 24194	*MCT* T	no. 10	4 *šu-ši* im.šu	S19	Gr. Sb	M(5/5)	240 systems of equations	tree-like
A 24195	*MCT* U	—	—	S20	Gr. Sb	M(5/5)	177 systems of equations	tree-like
AO 9071	*ZA* 99	no. 7	1 35 im.šu	S13	Gr. Sa	M(3/3)	95 systems of equations	tree-like
AO 9072	*ZA* 99	[…]	[2 42 im.šu]?	S14	Gr. Sa	M(4/5)	162 systems of equations	tree-like
VAT 7537	*MKT I*, 466	[…]	[………]	S7	Gr. Sa	M(3/3)	42+ systems of equations	tree-like
YBC 4668	*MKT I*, 422	no. 3	4 46 im.šu	S4	Gr. Sa	M(5/5)	286 systems of equations	tree-like
YBC 4695	*MKT III*, 34	no. 5	1 37 im.šu	S11	Gr. Sa	M(3/3)	97 systems of equations	tree-like
YBC 4697	*MKT III*, 40	no. 3	25 im.šu	S9	Gr. Sa	M(2/2)	25 systems of equations,	tree-like
YBC 4709	*MKT I*, 412	no. 5	55 im.šu	S3	Gr. Sa	M(3/3)	55 systems of equations	tree-like
YBC 4710	*MKT I*, 402	no. 4	35 im.šu	S2	Gr. Sa	M(3/3)	35 systems of equations	tree-like
YBC 4711	*MKT III*, 45	no. 6	2 11 im.šu	S12	Gr. Sa	M(3/3)	131 systems of equations	tree-like
YBC 4712	*MKT I*, 429	no. 13	48 im.šu	S6	Gr. Sa	M(3/3)	48 systems of equations	tree-like
YBC 4713	*MKT I*, 422	no. 10	37 im.šu	S5	Gr. Sa	M(3/3)	37 systems of equations	tree-like
YBC 4715	*MKT I*, 478	no. [..]	5[0] im.šu	S8	Gr. Sa	M(3/3)	50? systems of equations	tree-like

YBC 4696	*MKT II*, 60	—	52 im.šu	S15	Gr. Sa	M(3/3)	52 problems for divided triangles	tree-l./catalog
YBC 4708	*MKT I*, 389	no. 1	1 *šu-ši* im.šu	S1	Gr. Sa	M(3/3)	60 problems for brick piles	tree-l./catalog
VAT 7528	*MKT I*, 508	no. 3	[24] im.šu	S17	—	M(3/3)	24? problems for small canals	catalog!
YBC 4607	*MCT* O	—	10 im.šu.meš	C3	Gr. 2a	S	10 problems for bricks	catalog
YBC 4612	*MCT* S	—	—	C1	Gr. 2a!	S	15 simple problems for rectangles	catalog
YBC 4652	*MCT* R	—	22 [im.šu]	C4	Gr. 2a	S	22 problems for weight stones	catalog
YBC 4657	*MCT* G	—	31 im.šu	C5	Gr. 2a	S	31 problems for trenches	catalog
YBC 4666	*MCT* K	—	26 im.šu	C7	Gr. 2a	S	23! problems for small canals	catalog
YBC 4714	*MKT I*, 487	no. 4	43 im.šu	S10	Gr. ?	M(5/5)	39! problems for several squares	catalog!
YBC 5037	*MCT* F	—	44 im.šu	C6	Gr. 2a	S	44 problems for trenches	catalog
YBC 6492	*MCT* Sa	—	—	C2	—	S	23 simple equations for rectangles	catalog
YBC 7164	*MCT* L	—	—	C8	Gr. 2a	S	18 problems for small canals	catalog
YBC 4669	*MKT III*, 26	—	—	—	Gr. 2b	M(3/3)	26 mixed problems	recomb. text
YBC 4673	*MKT III*, 29	no. 2	23 im.šu	S16	Gr. 2b	M(3/3)	21! mixed problems	recomb. text!
YBC 4698	*MKT III*, 42	no. 3	[………]	S18	Gr. 2b	M(2/2)	17 diverse commercial problems	recomb. text!

Actually, *all the texts listed above, except the recombination texts YBC 4669 and YBC 4673 with mixed problems, are mathematical theme texts*, in the obvious sense that each one of them is devoted to a number of closely related mathematical problems, according to the theme of the text. *They are all written mainly in terms of Sumerian logograms, and do not include a solution procedure.*

The name "series texts" was originally used by Neugebauer in *MKT I*, 383 ff. to denote Old Babylonian mathematical texts with the subscript dub (*n*).kam '*n*th tablet', with the understanding that all the known mathematical series texts are what has been preserved of several large groups of serially ordered mathematical texts. In the same spirit, Proust in *NTM* 20 (2012), 124, speaks of

> "certain categories of mathematical texts such as catalogues (lists of problem statements) and series texts (lists of problem statements written on numbered tablets)".

In spite of this statement, Proust included into the category of series texts, without a comment, also the unnumbered mathematical texts YBC 4696 and A 24195, the latter obviously because of its apparent affinity with A 24194.

As shown in the list above, there is *a fairly good agreement between Proust's series texts S1-S20 on one hand and Groups Sa-Sb in RA 94 on the other, and similarly between Proust's catalog texts C1-C8 and Group 2a in RA 94*. Note that YBC 4612 was omitted in *RA* 94 from Group 2a, but may very well belong to that group. However, see the remark in Høyrup, *LWS*, 346, fn. 407, that YBC 4612 "is in a much coarser hand than the others".

Almost all the texts classified as series texts in *NTM* 20 are *numbered theme texts with variations of the theme, variations of the variations, and so on*, that is, what Proust calls an arborescent (tree-like) structure. Often the variations of the theme are amazingly contrived and complicated. The texts classified as catalog texts in *NTM* 20 are *unnumbered theme texts with simple variations of the theme*. Several of them mention the number of problems they contain. YBC 7164, for instance, does not, but on the other hand YBC 7164 is a direct continuation of YBC 4666, which does. (See the remark in *MCT*, 81, that problem # 23 in YBC 4666 is a catch-line, referring forward to problem # 1 in YBC 7164.)

The two numbered texts VAT 7528 (S17) and YBC 4714 (S10) are considerably less tree-like than the great majority of the series texts. In that sense, they are much more like catalog texts than series texts. Above they are ordered among the catalog texts.

In *RA* 94, 164, four texts were assigned to Group 2b. One of them, VAT 7528, cannot with certainty be assigned to any group. The three remaining texts of Group 2b, YBC 4698, YBC 4673, and YBC 4669, discussed above in Secs. 11.1-3, are clear examples of *recombination texts*. (The term recombination text refers, as usual, to a more or less well organized compilation of excerpts from several, not necessarily closely related theme texts). Incidentally, *characteristic for Group 2b is the use of the term* gar.ra 'set'*, a term which occurs only in these three texts of all the texts in the list above.* (Elsewhere it occurs in texts of Group 6a, see

Ch. 8 above, and in texts of Group 4a, see Friberg, *RA* 94 (2000), 166.) *The term* šúm (alternatively transcribed as sum, or sì) *'to give' occurs only in the three texts of Group 2b, in VAT 7528, of unclear group, in texts of Group 2a, such as MKT D = YBC 4666, and in texts of Group 3, such as Str. 362, etc.*

Of the three mentioned recombination texts belonging to Group 2b, YBC 4698 is fairly well structured, around the broad theme *commerce*. YBC 4673 is also fairly well structured, around the broad theme *bricks and mud* (except for an odd text about wool), while YBC 4669 is chaotically structured. *Yet even if these three texts are neither tree-like nor ordinary theme texts, they still share some common traits with all the other texts in the long list above. Two of them are numbered, one or two of them count the numbers of problems they contain, all of them are written predominantly with Sumerian logograms, and they are all of the type question plus answer, without solution procedures.*

In this connection, it is interesting to note the role played by the following two texts:

YBC 4662	*MCT* J	—	—	P5c	1 col.	28 problems for digging of trenches	theme text with procedures
YBC 4663	*MCT* H	—	—	P5a	1 col.	8 problems for digging of trenches	theme text with procedures

(Here P is Proust's notation for "procedure text".) As remarked in Neugebauer and Sachs *MCT*, 73,

> "These three texts (YBC G, H, and J) form a closely knit group. G (YBC 4657) contains the statements of 31 problems which were worked out on three other tablets, two of which are preserved: H (YBC 4663), which deals with the problems Nos. 1 to 8; and J (YBC 4662), which treats Nos. 19 to 28, omitting the last three examples of the main text."

In other words, *MCT* H and J contain explicit solution procedures (*written in Akkadian* with only occasional Sumerian logograms) for exercises ## 1-8 and 19-28 of *MCT* G, a catalog text with answers but no solution procedures. This observation suggests that the purpose of the catalog texts listed above was to be together an extremely rich source of mathematical problems to be handed out by teachers and to be solved by advanced students. Compare with the following words in Neugebauer and Sachs, *MCT*, 116:

> "This text (*MCT* T), like the following one (*MCT* U) can best be compared to an extensive collection of problems from a chapter of a textbook. It is obvious that a collection of this sort was used in teaching mathematical methods. They constitute a large reservoir of problems from which individual problems of any required type (say, speaking from a modern point of view, of a certain category of quadratic equations) could be selected. This explains the schematic arrangement of the examples and made possible the employment of a terse style which, if isolated, would be extremely ambiguous."

The comparison with chapters from a modern textbook (in mathematics) certainly makes sense. Remember that in the Old Babylonian period, a long time before the invention of printing and with only clay as a medium for writing, there was no such thing as a "textbook" with mathematical theory and mathematical exercises. The best an ambitious and gifted master teacher could do was to to write his own textbook chapters in the form of *well structured catalog texts*. He could then orally teach his students, showing them how to solve the problems in the catalog texts. Particularly advanced students could perhaps borrow the master's catalog texts in order to make copies of them, and in some cases even in order to complete sections from them with full solution procedures. (Nice examples are the two procedure texts YBC 4662 (*MCT* J) and YBC 4663 (= *MCT* H), mentioned above, which correspond to two sections of the catalog text YBC 4657 = *MCT* G). Less advanced students could copy only one or a few of the exercises and try to complete them with the proper solution procedures. In a later stage, a new teacher who did not posses any suitable catalog texts and who was not able to make his own, could instead write new *pseudo-catalog texts* by copying onto a large clay tablet any collection of such excerpts from the previous teacher's catalog texts that he happened to have in his possession The result would be a typical *mathematical recombination text*.

The assertion in the cited proposition in *MCT* that even tree-like theme texts like *MCT* T and *MCT* U could be used as a source of problems to be handed out to students is probably not correct, however. The individual exercises in tree-like texts were simply too briefly and elliptically formulated to make any sense if they were copied directly by students. It is much more likely that such tree-like texts were the products of over-zealous, nerdy, and obsessive mathematical master teachers with too much time on their hands. Or else, of master teachers who really wanted to explore the outer limits of the mathematical ideas and methods that they were familiar with. It is interesting to note, by the way, that the extreme complexity of the tree-like mathematical texts was caused to a large extent by the absence of modern mathematical notations, such as that of *general coefficients* a, b, c (instead of specific numerical coefficients) in quadratic or linear equations, like $ax+b = c$ or $ax^2+bx+c = 0$. With such modern notations many of the tree-like texts would loose much of their meaning, as well as, of course, the catalog texts with variations of parameters discussed in Sec. 10.3.

It remains to say a few words about the numbering of the "series texts". Such numbering may have made sense for original collections of catalog texts or even tree-like texts. However, after a while, numbering could simply have become a fashion, a distinguishing trait of texts with a certain shared provenance. That could explain why even two of the recombination texts in the list above were numbered.

Speaking about provenance, note that all the texts in the expanded list above are unprovenanced and undated. However, Neugebauer mentions, in *MKT I*, 387, that VAT 7528 and VAT 7537 were bought from Gejou in Paris and arrived in Berlin between 1911 and 1919, and that both tablets were from Uruk according to the dealer. Incidentally, in his linguistic classification of the Old Babylonian mathematical texts in *MCT*, Goetze writes that his Group 2 (BM 13901 , *MCT* J, *MCT* H, *MCT* L) is a "southern group" (*MCT*, 148-9). More precisely, it was claimed in Friberg, *RA* 94 (2000), 165 that terminological similarities indicate that "the series texts are closely related to the southern groups 1, 2, and 3", where according to Goetze Group 1 tablets are from Larsa and Group 2 tablets from Uruk.

Moreover, in *MKT I*, 384, Neugebauer makes the remark that judging from the script, the series texts (by which he meant, of course, those appearing in *MKT I-III*) appear to be post-Old Babylonian, from the Kassite period. Compare with the remark in Friberg, *MSCT 1*, 343 that the extraordinarily interesting single problem text MS 3876 may be from the Kassite period because it is of the same very unusual square format as AO 17264, another extraordinarily interesting single problem text which, according to Neugebauer (*MKT I*, 126), is Kassite, and possibly from Uruk. Also the two series texts A 24194 and A 24195 seem to have the same unusual square format, which means that they, too, may be from the Kassite period, partly in support of Neugebauer's opinion mentioned above, that all the series texts (in *MKT*) may be Kassite.

12. An Early Dynastic/Early Sargonic Metro-Mathematical Recombination Text from Umma with Commercial Exercises

12.1 CUNES 52-18-035. An Early Dynastic/Early Sargonic Recombination Text

CUNES 52-18-035 is a fairly well preserved Early Dynastic III/Early Sargonic clay tablet (2350-2300 BC), published by Vitali Bartash in *CUSAS 23* (2013), no. 77, as a text from the Umma region. It is a metro-mathematical recombination text with (at least) seven simple but closely related exercises, complete with questions and answers, but with no solution procedures. The theme of the text is commercial mathematics. The text is of particular interest here, because *it makes use of non-positional number notations similar to those in the atypical Sumerian table of reciprocals SM 2685 in Ch. 13 below*.

The hand copy below was kindly supplied by Bartash. I want to thank him warmly for letting me see and work with this amazing text.

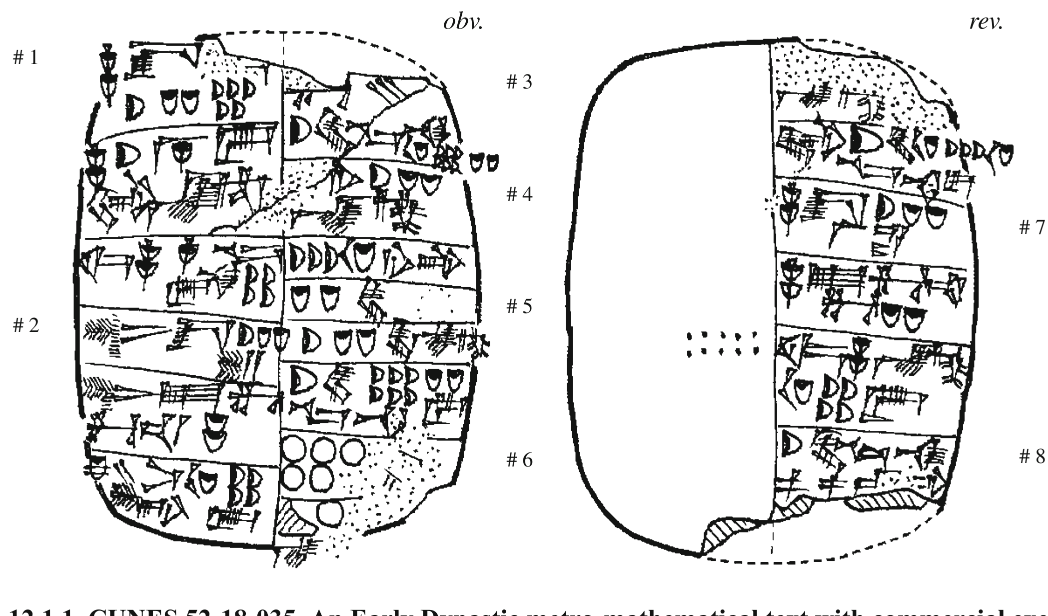


Fig. 12.1.1. CUNES 52-18-035. An Early Dynastic metro-mathematical text with commercial exercises.

The upper and lower right corners of the clay tablet are lost. This is unfortunate, because it makes it difficult to see how many exercises there were in the text, originally. In addition, the exercises in this theme text are much too concisely formulated, and the terminology is partly unfamiliar. Therefore, the following attempted interpretation of the exercises can only be tentative.

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7_12

# 1	nunuz.da 1 2⁄3 5 [kù.2]? / nunuz 1 lá 1⁄2 gín kù.bi en.nam / sag nunuz 1⁄2 ma.na 4 gín	Beads at? 1 2/3 5 [silver.2]?. / The beads 1 minus 1/2 shekel. Their silver, / what? The original beads 1/2 mina 4 shekels.
# 2	naga.da 1 2⁄3 še.2 / naga e_2.mu mu.de_6 2(barig) / 2(bán) 2 sìla 1⁄3 4 gín še.bi	Potash at? 1 2/3 grain.2. The potash was brought to my house, 2 barig. 2 bán 2 1/3 sìla 4 shekels its grain.
# 3	[………] / mu 3 / 1 kù ma.na 1⁄3 6 2⁄3	[……] / 3 years. / 1 silver mina 1/3 6 2/3.
# 4	[2] ninda 1 2⁄3 en.nam / 3 1⁄3 ninda.bi	[2] ninda, 1 2/3, what? / 3 1/3 its ninda.
# 5	2⁄3 kù <ma.na> / 1 2⁄3 kù en.nam / 1 kù ma.na 6 2⁄3 gín	2⁄3 <mina> of silver. / 1 2/3 of the silver, what? 1 silver mina 6 2/3 shekels.
# 6	50 […] x […] / ⸢10⸣ [……] en.[nam] / [………] en.nam / kù.bi 1 kù ma.na 1⁄3 3 1⁄3 gín	50 […] x […] / ⸢10⸣ [……] is w[hat]? / [………] is what? / Its silver is 1 mina of silver 1/3 3 1/3 shekels.
# 7	nunuz.da 1 2⁄3 lá.e / nunuz e_2.mu mu.de_6 2⁄3 / sag nunuz en.nam 1⁄3 4 gín	Beads weighed out at? 1 2/3. / The beads brought to my house, 2/3. / The original beads, what? 1/3 4 shekels.
# 8	1 kù ma.na PA.PA.⸢x⸣.bi [………]	1 mina of silver ??? [………]

The term en.nam occurs at least five times in this text, which is written entirely in Sumerian. It is well known that the term was used frequently in the questions in Old Babylonian mathematical problem texts, with the meaning ‘what?’. Therefore, its presence here at so many various places suggests immediately that also this text is mathematical and that it can be divided into several brief exercises, (which was done above).

Also the presence of the term sag ‘head, front’ in two places in the text is revealing. You can find this term with the meaning ‘original quantity’, in several Old Babylonian mathematical texts. In particular, the term is used in exercises involving division problems or single linear equations, such as, for instance, the single exercise MS 3976 (Friberg and George, *PF 40* (2010), Sec. 2). The question in that exercise is stated (in somewhat unusual words) as follows:

MS 3976

[še-*i*] ⸢*i-na*⸣ *ba-ab* ganba *sà-ri-iq-ma* 10-*ti-šu il-qé* / *ù* 3-*ta-šu el-qé-ma*
íb.tag_4 še-*ia ú-ša-*⸢*an*⸣*-ni-*⸢*ma*⸣ / 1-*ma*
sag níg.ga-*ia mi-nu*

[My grain] was cast down in the market gate, then he took a 10th of it, / and I took a 3rd of it, then
the remainder of my grain I measured again, then / precisely 1(bariga).
My original amount was what?

In quasi-modern symbolic notations, the stated problem in MS 3976 can be rephrased as follows: Let *C* be the original amount of grain. Then

$(C - C \cdot 1/10) - (C - C \cdot 1/10) \cdot 1/3$ = 1 bariga = 1 00 sìla.

The explicitly given solution in this exercise begins by assuming that C = '1', meaning 1 (00). Then

1 (00) – 1 (00) · rec.10 = 1 (00) – 6 = 54 and 54 – 54 · rec. 3 = 54 – 18 = 36.

Since the remainder should be 1 (00) sìla, not 36, the true value of C can then be calculated as

1 (00) sìla · rec. 36 = 1 40 sìla = 1 bariga 4 bán.

It is also instructive to look at the example provided by YBC 4652 (= *MCT* R, Neugebauer and Sachs (1945)), exercise # 22, because there are certain similarities between the terminology used in that exercise and the terminology used in CUNES 52-18-035:

YBC 4652 # 22

na_4 ì.pà ki.lá nu.na.tag
2/3 igi.6.[gál ba.zi] / igi.3.gál igi.8.gál bi.daḫ-*ma* ì.l[á 1 ma.na] /
sag na_4 en.nam sag na_4 1 ma.[na 4 2/3 gín 24 še]

I found a stone, the weight not marked.
2/3 (of) rec. 6 [I tore off] / rec. 3 (of) rec. 8 I joined, then I wei[ghed it, 1 mina]. /
The original stone was what? The original stone was 1 mi[na 4 2/3 shekels 24 grains].

The question in this exercise can be reformulated, in quasi-modern terms, as the following equation:

$(n - 2/3 \cdot 1/6 \cdot n) + 1/3 \cdot 1/8 \cdot (n - 2/3 \cdot 1/6 \cdot n)$ = 1 mina.

No solution algorithm is provided by the text itself. Anyway, a solution algorithm of the same kind as in MS 3976 would proceed, essentially, as follows:

If n = 1 (00), then $n - 2/3 \cdot 1/6 \cdot n$ = 1 (00) – 6;40 = 53;20, and
$(n - 2/3 \cdot 1/6 \cdot n) + 1/3 \cdot 1/8 \cdot (n - 2/3 \cdot 1/6 \cdot n)$ = 53;20 + 2;13 20 = 55;33 20, which is not the wanted result.
If instead n = rec. 55;33 20 = 1 04;48, then $(n - 2/3 \cdot 1/6 \cdot n) + 1/3 \cdot 1/8 \cdot (n - 2/3 \cdot 1/6 \cdot n)$ = 1(00).
Therefore, n = 1;04 48 minas = 1 mina 4 2/3 shekels 24 grains.

After this digression, it is time to look again at the exercises in CUNES 52-18-035:

Exercise # 1. The meaning of the term nunuz here is not clear. The term can stand for 'egg', and also for 'precious bead' (for instance of lapis lazuli). (See Steinkeller, *BiOr* 52 (1995), 706.) Since eggs were not weighed and were not valuable, a reasonable assumption is that the term in this text stands for 'bead'.

The translation of the brief text of this exercise is problematic. Nevertheless, the presence of the term sag nunuz 'original bead', suggests that exercise # 1 is *a division problem*. A reasonable conjecture seems to be that if it is a division problem it can be reformulated, in quasi-modern terms, as a metro-mathematical equation:

1 2/3 5 · b = 1 mina – 1/2 shekel.

It also seems reasonable to assume that '1 2/3 5' stands for '1 2/3 + 5 sixtieths'. Note that

4 · 1 2/3 + 5 · 1/60 = 6 2/3 + 20 · 1/60 = 7.

This means that the number 1 2/3 5 is *sexagesimally non-regular*, unlike the numbers in the mentioned Sumerian table of reciprocals (SM 2685 in Ch. 13 below), which are all regular. (You cannot multiply 1 2/3 5 by an integer and get some power of 60.) As a matter of fact, there are other known examples of even earlier metro-mathematical exercises involving division by non-regular numbers, namely division by the *sexagesimally non-regular* capacity number 7 sìla in the Early Dynastic Shuruppak texts *TSS* 50 and *TSS* 671 (Friberg, *MSCT 1* (2007), 414), and division by the *decimally non-regular* capacity number 24 níg.sagšu in the Ebla text TM.75.G.2346 (*op. cit.*, 413), which counts with non-positional decimal numbers. In metro-mathematical exercises from the later Old Akkadian/Sargonic period, on the other hand, the rule seems instead to have been to consider only division by *sexagesimally regular* numbers! Examples are division by the regular length numbers 2(60) 40 rods in *DPA* 38, 1(60) 7 1/2 rods in HS 815, and 4(60) 3 rods in *DPA* 39 (*op. cit*., 408-409).

Since 1 2/3 5 is non-regular, the equation in # 1,

1 2/3 5 shekels · n = 1 mina – 1/2 shekel

cannot be solved through multiplication by the reciprocal of 1 2/3 5, which does not exist. Instead, the solution to the equation can be found in the following way: Let, for instance, n^* = 4 be a tentative, "false solution". Then

1 2/3 5 · n^* = 1 2/3 · 4 + 5 · 4 · 1/60 = 6 1/3 + 1/3 = 7.

However, the desired result of the multiplication was instead

1 mina – 1/2 shekel = 59 1/2 shekel = 7 · 8 1/2 shekels.

Therefore, obviously, the true solution can be obtained through multiplication by the "correction factor" 8 1/2 shekels, in the following way:

$n = n^*$ · 8 1/2 shekels = 4 · 8 1/2 shekels = 34 shekels = 1/2 mina 4 shekels.

This is the weight number mentioned in the text of the exercise as the 'original' (weight) of the beads.

What it all means is not clear. *Maybe the 'beads', originally weighing 1/2 mina 4 shekels, were invested in some economic enterprise 'in the market gate', with an agreed payback of 1 2/3 5 times the investment*.

Exercise # 2. This, too, is a division exercise, although there is a strangely written word in the first line of the exercise, which is transcribed here, tentatively as 'grain.2'. Maybe it indicates the grain which is received as payback for the investment of an original amount of grain. (The conjectural restoration [kù.2][?] in the first line of exercise # 1 would then mean the silver received as payback for the investment of an original amount of silver.)

The stated answer to exercise # 2 seems to contain a miscalculation. What is even more confusing is that the text seems to count with units of capacity measure such that 1 bán = 6 sìla. Most of the contemporary texts in the Cornell collection are from Umma/Zabala or Adab, where the bán was equal to 10 sìla (Bartash, personal communication). Only texts from Girsu count with a bán of 6 sìla. Maybe this means that CUNES 52-18-035 is from Girsu, not Umma.

Anyway, assume that exercise # 2 actually is a division problem, and that 1 bán = 6 sìla (and as usual 1 bariga = 6 bán). Then the question in the exercise can be reformulated, in quasi-modern terms, as the following metro-mathematical problem: Potash is invested on the market with an agreed payback of 1 2/3 times the invested amount. If the capacity measure of what is brought back to the house after the transaction is 2 bariga, what was then the original amount p of potash? (The statement of the problem is vaguely formulated. It is possible that še here means 'capacity measure' rather than the usual 'grain'.)

It is interesting to observe that the payback rate 1 2/3 in this exercise is a *regular* sexagesimal number, unlike the non-regular payback rate 1 2/3 5 in exercise # 1. Actually, *the regular payback rate 1 2/3 is used also in all the remaining exercises* on CUNES 52-18-035

In quasi-modern symbolic notations, the stated problem in exercise # 2 can be formulated as follows:

1 2/3 · p = 2 bariga.

This seems to be an exceedingly simple problem which can be solved through multiplication of 2 bariga with the reciprocal of 1 2/3. However this simplicity is an illusion. How could the reciprocal of 1 2/3 be computed without the use of our modern common fractions? Or of sexagesimal numbers in place value notation. Or, at least, of a table of reciprocals of the early type discussed in Ch. 13 below? And even if the reciprocal of 1 2/3 could be computed, how could it be multiplied with the capacity number 2 bariga? *It was clearly because division by 1 2/3 was a non-trivial task at the time that almost all the exercises on CUNES 52-18-035 were devoted to examples of such divisions*.

Maybe it is a good idea to proceed precisely as in the suggested solution to the stated problem in exercise # 1. So, let n^* = 3 be a tentative, false solution. Then

1 2/3 · n^* = 1 2/3 · 3 = 5.

Since the desired result of the multiplication by 1 2/3 was 2 bariga, the needed correction factor is

1/5 of 2 bariga = 1/5 of 12 bán = 2 bán + 1/5 of 12 sìla = 2 bán 2 sìla + 1/5 of 2 · 60 shekels = 2 bán 2 sìla 1/3 (sìla) 4 shekels.

This is precisely the answer given in the text of exercise # 2. However, this is not the correct result, since the author of the text *forgot to multiply the correction factor with the false solution* n^* = 3! The correct answer should have been, instead,

p = 3 · 2 bán 2 sìla 1/3 (sìla) 4 shekels = 1 bariga 7 sìla 12 shekels.

It is easy to show that this answer is correct. (That may be a suitable task for the interested reader!)

Exercise # 3. The text of this exercise is broken, so that only the answer remains, and a small part of the question, '3 years'. The stated answer is

1 mina of silver 1/3 6 2/3 shekels.

It is tempting to investigate if what is going on here is that a certain amount of silver was invested with an agreed (yearly) payback of 1 2/3 not once but three times. If that is so, then is not difficult to find out, by counting backwards from the answer, that the original capital was [18 shekels]. Indeed, then the calculation of the answer would have proceeded as follows:

1 2/3 · 18 shekels = 30 shekels = 1/2 mina,
1 2/3 · 30 shekels = 50 shekels = 2/3 mina 10 shekels,
1 2/3 · 50 shekels = 1 2/3 · (48 + 2) shekels = 1(60) 20 shekels + 1 2/3 · 2 shekels = 1 mina 1/3 + 3 1/3 shekels.

Apparently the one who wrote the text made a mistake in the very last step of this calculation, multiplying 1 2/3 by 2 not once, but twice. In that way was obtained the result recorded in the text, namely

1 mina 1/3 + 2 · 3 1/3 shekels = 1 mina 1/3 6 2/3 shekels.

Exercise # 4. This is a quite uninteresting multiplication exercise. A broken number, here reconstructed as [2], of ninda (bread?) is given, and shall be multiplied by 1 2/3. The answer is, of course,

1 2/3 · 2 ninda = 3 1/3 ninda.

Exercise # 5. This is another multiplication exercise, which is interesting only because of the needed counting with fractions, which is tentatively reconstructed below. Given is 2/3 mina of silver, and this shall again be multiplied by 1 2/3. The result can be computed as follows:

1 2/3 · 2/3 mina = 2/3 mina 4 · 1/9 mina = 2/3 mina 4 · 6 2/3 shekels = 2/3 mina 26 2/3 shekels = 1 mina 6 2/3 shekels.

Exercise # 6. Much of the text of this exercise is lost. The correctness of the reading of the damaged second line of the exercise is doubtful. Anyway, this appears to be yet another simple multiplication exercise of the same kind as in the preceding two exercises, ## 4 and 5. The calculation performed in this case seems to be as follows:

1 2/3 shekels per x · 50 x = (50 + 33 1/3) shekels = 1 1/3 minas 3 1/3 shekels.

Exercise # 7. This exercise is of the same kind as exercise # 1 above, except that the non-regular multiplication factor 1 2/3 5 has been replaced by the regular multiplication factor 1 2/3. In this new case, the division problem can be reformulated, in quasi-modern terms, as the following metro-mathematical equation:

1 2/3 · *n* minas = 2/3 mina.

Precisely as in the case of exercise # 1, n^* = 3 can be chosen as a tentative, false solution. Then

1 2/3 · n^* = 1 2/3 · 3 = 5   instead of the required 2/3 mina.

The needed correction factor is, of course,

$1/5 \cdot 2/3$ mina $= 1/5 \cdot 40$ shekels $= 8$ shekels.

Consequently, the true answer is that

$n = 3 \cdot 8$ shekels $= 24$ shekels $= 1/3$ (mina) 4 shekels.

This is also the answer given in the text.

Exercise # 8. The text of this exercise is too damaged to allow a meaningful discussion of what is going on.

13. An Ur III Table of Reciprocals without Place Value Numbers

13.1 SM 2685. An Atypical Ur III Table of Reciprocals

13.1.1 The Table of Reciprocals

In Fig. 13.1.1 below is shown a hand copy and conform transliteration of SM 2685, a clay tablet from the Suleimaniyah Museum in the Kurdistan region in northeastern Iraq. The clay tablets in the Suleimaniyah Museum are acquired in the antiquities market and are therefore unprovenanced, but in most cases probably from Old Babylonian Larsa. However, the writing on SM 2685 is such that the text can be either from the Neo-Sumerian Ur III period or Early Old Babylonian, and, as will be shown below, the atypical table of reciprocals inscribed on the tablet is clearly older than all earlier known Ur III tables of reciprocals.

The top half of the table text on the obverse of SM 2685 is unreadable. The first well preserved lines of the table are

igi	5	gál.bi	12	meaning something like 'the reciprocal of 5 exists and is 12'
igi	5 1/3	gál.bi	11 15	meaning something like 'the reciprocal of 5 1/3 exists and is 11 15'

Earlier in this book, a sexagesimal number *n* was called a "regular sexagesimal number", whenever there existed a "reciprocal" sexagesimal number rec. *n,* such that, in Babylonian relative (floating) place value notation, $n \cdot$ rec. n = '1'. However, in the present situation, a "number" must be understood as either *an integer or an integer plus a fraction*, like the numbers appearing in the table of reciprocals on SM T. 2685, and *the definition of a* (*sexagesimally*) *regular number must be modified* so that a number *n* is called "regular" if there exists a "reciprocal" number rec. *n* such that

$n \cdot$ rec. n = 1(60).

Here, 1(60) is a convenient transliteration of the Sumerian cuneiform sign for géš = '60', a somewhat enlarged variant of the upright wedge diš = 1.

It is easy to check that 12 and 11 15 really are reciprocals of 5 and 5 1/3 in this sense, provided that 11 15 is interpreted as meaning 11 plus 15 shekels (sixtieths). Indeed, let sh. be a suitable abbreviation for shekel. Then,

$5 \cdot 12$ = 1(60) and 5 1/3 $\cdot$ (11 + 15 sh.) = 58 2/3 + (1(60) sh. + 20 sh.) = 58 2/3 + 1 1/3 = 1(60).

This means that, apparently, *the two mentioned lines are lines of a table of reciprocals, although not the Old Babylonian standard table of reciprocals* (see Neugebauer and Sachs, *MCT* (1945), 11, and Friberg, *MSCT 1* (2007), Sec. 2.5), which does not contain pairs of reciprocals like 5 20, 11 15.

Instead, *the table of reciprocals on T. 2856 is constructed after a principle similar to the principle after which Old Babylonian metrological lists are organized*. Each preserved line of the table is of the format

igi *n* gál.bi rec. *n*

where, with two or three exceptions, to be discussed later in this sub-section, the numbers *n* form an *ascending* sequence of *regular* numbers, which are all *either integers or integers plus fractions*. With the same exceptions, the reciprocal numbers rec. *n* form a *descending* sequence of *regular* numbers, which in the same

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way (with only one exception, also to be discussed later in this sub-section) are either integers or integers plus fractions.

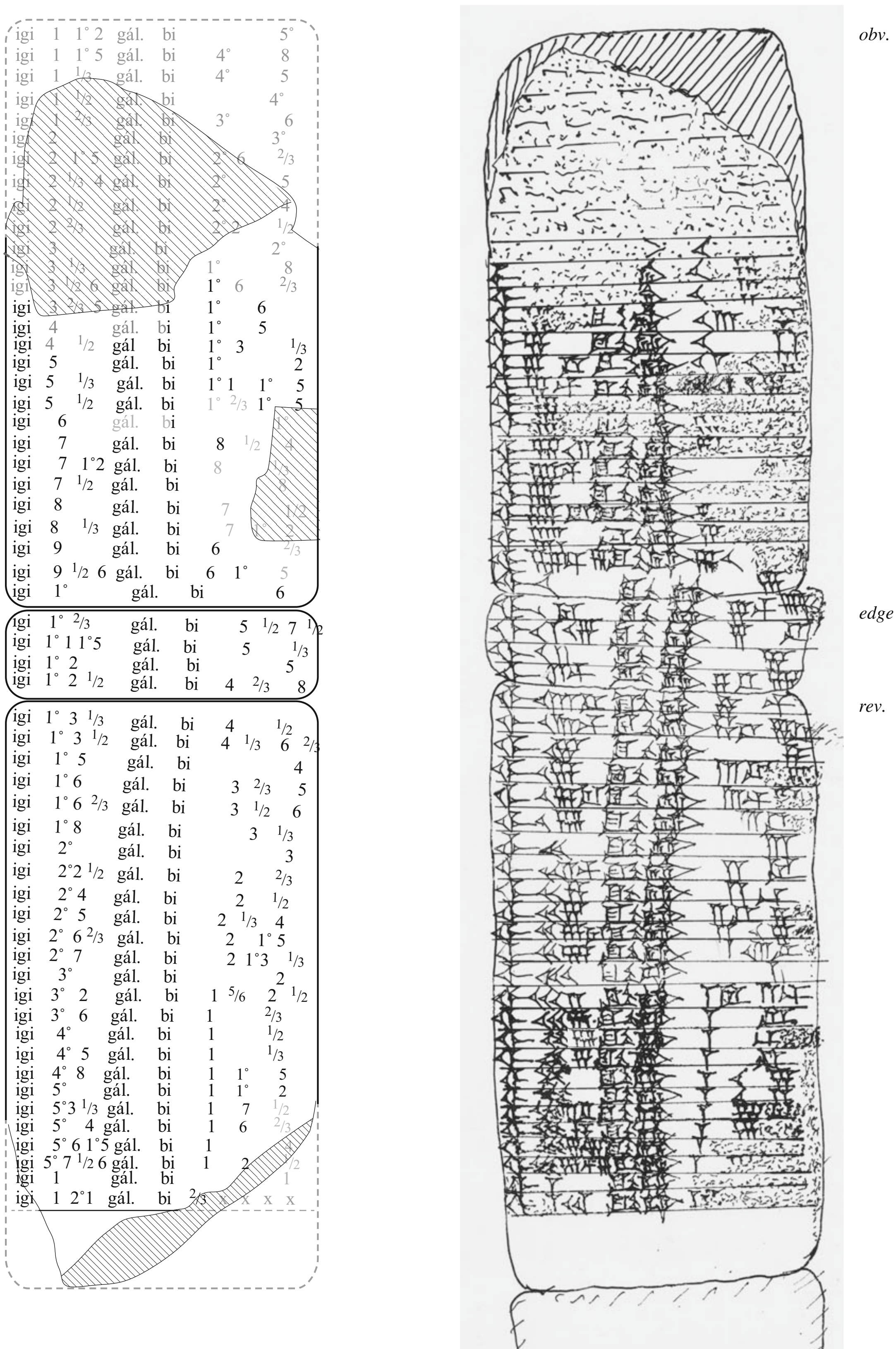


Fig. 13.1.1. SM 2685. Hand copy and conform transliteration.

obv.	[igi	1 12	gál.bi	50]	rec.	1;12	50	
	[igi	1 15	gál.bi	48]	rec.	1;15	48	
	[igi	1 1/3	gál.bi	45]	rec.	1;20	45	
	[igi	1 1/2	gál.bi	40]	rec.	1;30	40	
	[igi	1 2/3	gál.bi	36]	rec.	1;40	36	
	[igi	2	gál.bi	30]	rec.	2	30	
	[igi	2 15	gál.bi	26 2/3]	rec.	2;15	26;40	
	[igi	2 1/3 4	gál.bi	25]	rec.	2;24	25	
	[igi	2 1/2	gál.bi	24]	rec.	2;30	24	
	[igi	2 2/3	gál.bi	22 1/2]	rec.	2;40	22;30	
	[igi	3	gál.bi	20]	rec.	3	20	
	[igi	3 1/3	gál.bi	18]	rec.	3;20	18	
	[igi	3 1/2 6	gál.bi	16 2/3]	rec.	3;36	16;40	
	igi	[3 2/3 5	gál.bi]	16	rec.	3;45	16	
	igi	[4	gál.b]i	15	rec.	4	15	
	igi	[4 1/2]	gál.b]i	13 [1/3]	rec.	4;30	13;20	
	igi	5	gál.bi	12	rec.	5	12	
	igi	5 1/3	gál.bi	11 15	rec.	5;20	11;15	
	igi	5 1/2	gál.bi	[10 2/3] 15	rec.	5;30	10;55	(appr.)
	igi	6	⸢gál.bi⸣	[10]	rec.	6	10	
	igi	7	⸢gál.bi⸣	[8 1/2 4]	rec.	7	8;34	(appr.)
	igi	7 12	gál.bi	[8 1/3]	rec.	7;12	8;20	
	igi	7 1/2	gál.bi	[8]	rec.	7;30	8	
	igi	8	gál.bi	[7 1/2]	rec.	8	7;30	
	igi	8 1/3	gál.bi	[7 12]	rec.	8;20	7;12	
	igi	9	gál.bi	6 [2/3]	rec.	9	6;40	
	igi	9 1/2 6	gál.bi	6 1[5]	rec.	9;36	6;15	
	igi	10	gál.bi	6	rec.	10	6	
edge	igi	10 2/3	gál.bi	5 1/2 7 1/2	rec.	10;40	5;37 30	
	igi	11 15	gál.bi	5 1/3	rec.	11;15	5;20	
	igi	12	gál.bi	5	rec.	12	5	
	igi	12 1/2	gál.bi	4 2/3 8	rec.	12;30	4;48	
rev.	igi	13 1/3	gál.bi	4 1/2	rec.	13;20	4;30	
	igi	13 1/2	gál.bi	4 1/3 6 2/3	rec.	13;30	4;26 40	
	igi	15	gál.bi	4	rec.	15	4	
	igi	16	gál.bi	3 2/3 ⸢5⸣	rec.	16	3;45	
	igi	16 2/3	gál.bi	3 1/2 ⸢6⸣	rec.	16;40	3;36	
	igi	18	gál.bi	⸢3⸣ 1/3	rec.	18	3;20	
	igi	20	gál.bi	3	rec.	20	3	
	igi	22 1/2	gál.bi	2 2/3	rec.	22;30	2;40	
	igi	24	gál.bi	2 1/2	rec.	24	2;30	
	igi	25	gál.bi	2 1/3 4	rec.	25	2;24	
	igi	26 2/3	gál.bi	2 15	rec.	26;40	2;15	
	igi	27	gál.bi	2 13 [1/3]	rec.	27	2;13 20	
	igi	30	gál.bi	2	rec.	30	2	
	igi	32	gál.bi	1 ⸢5/6⸣ 2 1/2	rec.	32	1;52 30	
	igi	36	gál.bi	1 2/3	rec.	36	1;40	
	igi	40	gál.bi	1 1/2	rec.	40	1;30	
	igi	45	gál.bi	1 1/3	rec.	45	1;20	
	igi	4[8]	gál.bi	1 15	rec.	48	1;15	
	igi	50	gál.bi	1 12	rec.	50	1;12	
	igi	53 1/3	gál.bi	1 7 ⸢1/2⸣	rec.	53;20	1;07 30	
	igi	54	gál.bi	1 6 [2/3]	rec.	54	1;06 40	
	igi	56 15	gál.bi	1 [4]	rec.	56;15	1;04	
	igi	57 1/2 6	gál.bi	1 [2 1/2]	rec.	57;36	1;02 30	
	igi	1(60)	gál.bi	[1]	rec.	1 00	1	
	igi	1(60) 21	gál.bi	2/3 [x x x x]	rec.	1 21	[;44 26 40]	

SM 2685. Transliteration and an explanation in terms of sexagesimal numbers in place value notation.

All fractions in the table are, apparently, silently understood to be *combinations of the basic fractions, for which there existed special cuneiform signs, of multiples of the shekel (a sixtieth), and of basic fractions of the shekel. The basic fractions in question were 1/3, 1/2, 2/3, 5/6 , written as*

The most complex examples of such fractions in the table occur in the lines

igi $10\ {}^{2}/_{3}$	gál.bi	$5\ {}^{1}/_{2}\ 7\ {}^{1}/_{2}$
igi $13\ {}^{1}/_{2}$	gál.bi	$4\ {}^{1}/_{3}\ 6\ {}^{2}/_{3}$
igi $57\ {}^{1}/_{2}\ 6$	gál.bi	$1\ 2\ {}^{1}/_{2}$

In terms of sexagesimal numbers in *absolute* place value notation (that is, sexagesimal numbers in quasi-modern place value notation where a "sexagesimal semi-colon" separates integers from fractions), and without the superfluous gál.bi, the three cited lines of the table of reciprocals on SM 2685 can be read as

igi 10;40	5;37 30
igi 13;30	4;26 40
igi 57;36	1;02 30

Such reformulations of the lines of the table of reciprocals on SM 2685 in terms of sexagesimal numbers in absolute place value notation are collected together in the alternative table of reciprocals which is displayed to the right of the transcription above of the table on SM 2685. They are given here for the readers' convenience, and in order to emphasize the oddity of the table on SM 2685.

The mentioned two or three exceptions to the otherwise prevailing principle of the table of reciprocals on SM 2685, that all numbers appearing in it should be regular, are the lines

igi $5\ {}^{1}/_{2}$	gál.bi	$10\ {}^{2}/_{3}\ 15$	'the reciprocal of 5 1/2 is 10 2/3 + 15 shekels'
igi 7	gál.bi	$8\ {}^{1}/_{2}\ 4$	'the reciprocal of 7 is 8 1/2 + 4 shekels'

Here 5 1/2 = 11/2, and 7 are *non-regular* sexagesimal numbers. Consequently, 10 2/3 15 and 8 1/2 4 are only "approximate reciprocals". Indeed, it is easy to check that

5 1/2 · 10 2/3 15 sh. = 53 1/3 15 sh. + 5 1/3 7 1/2 sh. = 59 2/3 + 22 1/2 sh. = 1(60) + 2 1/2 sh. = appr. 1(60)

7 · 8 1/2 4 sh. = 59 1/2 28 sh. = 1(60) – 2 sh. = appr. 1(60).

A previously known attestation of approximate reciprocals to non-regular sexagesimal numbers in a cuneiform mathematical text appears in the small Old Babylonian tablet M 10, published by Sachs in *JCS* 6 (1952), which lists approximate reciprocals of the non-regular numbers 7, 11, 13, 14, 17. (In the last two cases, the approximate reciprocals are, for some unknown reason, multiplied by 10.) A second attestation is provided by the Old Babylonian fragment YBC 10529 (Neugebauer and Sachs, *MCT* (1945), 16)), with 22 preserved lines of the type igi *n* gál.bi rec. *n*, where *n* proceeds through all the integers from 58 to 1 20, and where each rec. *n* is an exact or approximate reciprocal of *n*. A third and final example is the small fragment Ist. Ni 10240 (Proust, *TMN* (2007), 128), which asks for the reciprocals of the six non-regular numbers 17, 22, 23, 26, 28, 34, without providing any answers.

The mentioned single exception to the rule that all reciprocal numbers mentioned on T. 2856 are either integers or integers plus fractions is the last line of the table, with rec. *n* = 5/6. More about that below.

A plausible reconstruction of the missing lines in the upper half of the table on the obverse of SM 2685, and of various badly readable numbers, is shown in Fig. 13.1.1 above. The reconstruction is based on the double assumption that the missing lines were constructed in agreement with the method apparently used for the construction of the other preserved lines in the table (see below), and that the reconstructed lines would precisely fill up the available space on the upper half of the obverse.

13.1.2 A Proposed Explanation of the Construction of the Atypical Table of Reciprocals

As mentioned, the numbers *n* appearing in the table of reciprocals on SM 2685 are (nearly) all either integers or integers plus fractions. The fractions are of a restricted number of types, some of which can best be understood (anachronistically) in terms of modern common fractions. Indeed, the only fractions appearing in the numbers *n* are:

12 (shekels) = 1/5, 15 (shekels) = 1/4, 1/3, 1/3 4 (shekels) = 24 (shekels) = 2/5,
1/2, 1/2 6 (shekels) = 36 (shekels) = 3/5, 2/3, [2/3 5 (shekels) = 45 (shekels) = 3/4].

This observation can prove to be an important clue in the search for the numerical algorithm by use of which the table of reciprocals was constructed. Indeed, a reasonable conjecture is that the first step of the algorithm was simply the construction of *a table of reciprocals for all regular integers between 1 and 1(60)*, and that afterwards, in the next steps of the algorithm, additional lines of the table were constructed through *multiplication of n by 2/3, or 1/2, or 1/3, or 15 sh. (= 1/4), or 12 sh. (= 1/5).* It will be shown below precisely how far an algorithm like that will lead.

To construct the initial table of reciprocals of sexagesimally regular integers between 1 and 1(60) is, in most cases, quite straightforward. It can be done, for instance, as follows:

rec. 2	=	30	since	2 · 30 = 60
rec. 3	=	20	since	3 · 20 = 60
rec. 4	=	15	since	4 · 15 = 60
rec. 5	=	12	since	5 · 12 = 60
rec. 6	=	10	since	6 · 10 = 60
rec. 8	=	7 1/2	since	8 · 7 1/2 = 56 + 4 = 60
rec. 10	=	6	since	10 · 6 = 60
rec. 12	=	5	since	12 · 5 = 60
rec. 15	=	4	since	15 · 4 = 60
rec. 16	=	3 2/3 5	since	16 · 3 2/3 5 = 58 2/3 1(60) 20 sh. = 60p
rec. 18	=	3 1/3	since	18 · 3 1/3 = 54 + 6 = 60
rec. 20	=	3	since	20 · 3 = 60
rec. 24	=	2 1/2	since	24 · 2 1/2 = 48 + 12 = 60
rec. 25	=	2 1/3 4	since	25 · 2 1/3 4 sh. = 58 1/3 + 1(60) 40 sh. = 58 1/3 + 1 2/3 = 60
rec. 27	=	2 13 1/3	since	27 · 2 13 1/3 sh. = 54 + 5(60) 51 sh. + 9 sh. = 54 + 6(60) sh. = 60
rec. 30	=	2	since	30 · 2 = 60
rec. 32	=	1 5/6 2 1/2	since	32 · 1 5/6 2 1/2 sh. = 32 + 26 2/3 + 1(60) 20 sh. = 58 2/3 + 1 1/3 = 60
rec. 36	=	1 2/3	since	36 · 1 2/3 = 36 + 24 = 60
rec. 40	=	1 1/2	since	40 · 1 1/2 = 40 + 20 = 60
rec. 45	=	1 1/3	since	45 · 1 1/3 = 45 + 15 = 60
rec. 48	=	1 15	since	48 · 1 15 sh. = 48 + 12 = 60
rec. 50	=	1 12	since	50 · 1 12 sh. = 50 + 10 = 60
rec. 54	=	1 6 2/3	since	54 · 1 6 2/3 sh. = 54 + 5(60) 24 sh. + 36 sh. = 54 + 6(60) sh. = 54 + 6 = 60
rec. 1(60)	=	1	since	60 · 1 = 60

Now it is time to see what can be achieved if the numbers *n* in the table of reciprocals of regular integers resulting from the first step of the algorithm are multiplied by suitable *regular fractions*, such as 2/3, 1/2, 1/3, 1/4 = 15 sh., and 1/5 = 12 sh. Note that if *n* is multiplied by 2/3, then rec. *n* has to be multiplied by 3/2 = 1 1/2, and so on, so that the product of *n* and rec. *n* will remain equal to 1(60). The procedure may be called "reciprocal compensation", and is related to the Old Babylonian doubling and halving algorithm, which is best known from the text CBS 1215 (Friberg, *MSCT 1* (2007), 369). (See also Sec. 1.2.4 above.) Of course, only new pairs with *n* strictly between 1 and 1(60) are accepted, and no duplicates.

Consider, for instance, the first pair in the table of reciprocals of integers between 1 and 1(60), that is, the pair *n*, rec. *n* = 2, 30. If 2 is multiplied by 2/3, and if 30 is multiplied by rec. 2/3 = 1 1/2 (= 3/2), the result is the new pair 2/3 · 2 = 1 1/3, 1 1/2 · 30 = 45. However, if 2 is multiplied by 1/3, the result is a number smaller than 1, which must be rejected.

Similarly, the second pair in the initial table of reciprocals of integers is *n*, rec. *n* = 3, 20. If 3 is multiplied by 2/3, and 20 by 1 1/2, the result is the pair 2, 30, which must be rejected, since it is a duplicate of a pair that is already known. However, if 3 is multiplied by 1/2 and 20 by 2, the result is the new pair 1/2 · 3 = 1 1/2, 2 · 20 = 40. And so on.

The result of an extension in this way of the original table of reciprocals of integers between 1 and 1(60) is shown in the table below. *New pairs constructed by this algorithm but which do not correspond to pairs tabulated in the (reconstructed) table of reciprocals on SM 2685 are put within brackets.* They will be called "missing pairs".

n	rec. *n*	· 2/3	· 1 1/2	· 1/2	· 2	· 1/3	· 3	· 15 (sh.)	· 4
2	30	1 1/3	45						
3	20			1 1/2	40				
4	15	2 2/3	22 1/2						
5	12	3 1/3	18	2 1/2	24	1 2/3	36	1 15	48
6	10								
8	7 1/2	5 1/3	11 15						
9	6 2/3			4 1/2	13 1/3			2 15	26 2/3
10	6	**(6 2/3**	**9)**						
12	5								
15	4			7 1/2	8			3 2/3 5	16
16	3 2/3 5	10 2/3	5 1/2 7 1/2						
18	3 1/3								
20	3	13 1/3	4 1/2						
24	2 1/2								
25	2 1/3 4	16 2/3	3 1/2 6	12 1/2	4 2/3 8	8 1/3	7 12	**(6 15**	**9 1/2 6)**
27	2 13 1/3			13 1/2	4 1/2 6 2/3			**(6 2/3 5**	**8 5/6 3 1/3)**
30	2								
32	1 5/6 2 1/2	((21 1/3	2 2/3 8 1/2 1/4))						
36	1 2/3								
40	1 1/2	26 2/3	2 15						
45	1 1/3			22 1/2	2 2/3			11 15	5 1/3
48	1 15								
50	1 12	(33 1/3	1 2/3 8)						
54	1 6 2/3								
1(60)	1								
25 original pairs		8 new pairs (2 missing)		7 new pairs		2 new pairs		4 new pairs (2 missing)	

(Note that the computed pair 21 1/3, 2 2/3 8 1/2 1/4 must be rejected, because 1/2 1/4 (sh.) = 3/4 (sh.) is not a basic fraction of a shekel.)

The construction of almost all the lines in the table of reciprocals on SM 2685 has now been explained. For a complete explanation, a few more steps of calculation are needed. First, consider the new pairs of reciprocals which can be constructed with departure from pairs *n*, rec. *n* with *n* an integer between 6 and 1(60) through multiplication of *n* by rec. 5 = 12 sh. and of rec. *n* by 5

n	rec. *n*	· 12 (sh.)	· 5
6	10	1 12	50
8	7 1/2	(1 1/2 6	37 1/2)
9	6 2/3	(1 2/3 8	33 1/3)
10	6		
12	5	2 1/3 4	25
15	4		
16	3 2/3 5	(3 12	18 2/3 5)
18	3 1/3	3 1/2 6	16 2/3
20	3		
24	2 1/2	(4 2/3 8	12 1/2)
25	2 1/3 4		
27	2 13 1/3	(5 1/3 4	11 6 2/3)
30	2		
32	1 5/6 2 1/2	**(6 1/3 4**	**9 1/3 2 1/2)**
36	1 2/3	7 12	8 1/3
40	1 1/2		
45	1 1/3		
48	1 15	9 1/2 6	6 15
50	1 12		
54	1 6 2/3	(10 2/3 8	5 1/2 3 1/3)
1(60)	1		
		5 new pairs (7 missing)	

Conspicuously, of the missing pairs put within brackets in the tables above, there are *four missing pairs with n between 6 and 7*, corresponding to the following four *missing line*s in SM 2685:

igi 6 15 gál.bi 9 1/2 6
igi 6 1/3 4 gál.bi 9 1/3 2 1/2
igi 6 2/3 gál.bi 9
igi 6 2/3 5 gál.bi 8 5/6 3 1/3

These four missing lines would have been consecutive lines in the table of reciprocals on SM 2685. This fact suggests that their absence is due to a single copying error, made by some careless student making a copy of an atypical table of reciprocals on an older tablet. A fourth omission, also probably inadvertent, is the missing line

igi 33 1/3 gál.bi 1 2/3 8

The many missing pairs in the last table above, the one for multiplication by 12 sh. and 5, can possibly be explained by assuming that the one who constructed the table on SM 2685 simply got tired when he had come this far in his long series of calculations. Or else, he may have observed that the extended table was becoming too unbalanced, with many new pairs with *n* below 30 but no new pairs with *n* above 30!

13.1.3 Four Extra Pairs with n above 50. The Notion of Regular Twins

Now the construction of 51 of the 57 lines on the (partly reconstructed) table of reciprocals on T. 2685 has been explained. In addition, there are two lines in the table where *n* is non-regular, possibly added for pedagogical reasons. What remains to explain are the following four "extra" lines with *n* above 50:

igi 53 1/3 gál.bi 1 7 1/2
igi 56 15 gál.bi 1 4
igi 57 1/2 6 gál.bi 1 2 1/2
igi 1(60) 21 gál.bi 2/3 [x x x x]

They may have been added because *the construction by reciprocal compensation demonstrated above failed to yield any new pairs n, rec. n with n greater than 33 1/3.* Through the addition of the mentioned four extra lines the table became less top-heavy.

The first three of these extra lines can be explained in the following interesting way. Look at the eight last pairs of reciprocals *n*, rec. *n* in the table SM 2685 with *n* less than 1(60). They can be rewritten, anachronistically, as in the columns below, to the right of the frame:

40	1 1/2	(1 – rec. 3)	· 60	1 + rec. 2	or, in terms of common fractions,	2/3	3/2
45	1 1/3	(1 – rec. 4)	· 60	1 + rec. 3		3/4	4/3
48	1 15	(1 – rec. 5)	· 60	1 + rec. 4		4/5	5/4
50	1 12	(1 – rec. 6)	· 60	1 + rec. 5		5/6	6/5
53 1/3	1 7 1/2	(1 – rec. 9)	· 60	1 + rec. 8		8/9	9/8
54	1 6 2/3	(1 – rec. 10)	· 60	1 + rec. 9		9/10	10/9
56 15	1 4	(1 – rec. 16)	· 60	1 + rec. 15		15/16	16/15
57 1/2 6	1 2 1/2	(1 – rec. 25)	· 60	1 + rec. 24		24/25	25/24

This stunning parade of fractions of the two types 1 – rec. *r* and 1 + rec. (*r* – 1), where both *r* and *r* – 1 are (sexagesimally) regular integers, makes it clear that the author of the table on SM 2685 was familiar with what may be called the concept of "regular twins", pairs of the form (*r* – 1, *r*), where both *r* – 1 and *r* are (sexagesimally) regular numbers. It is easy to start enumerating the sequence of all pairs of *integers* that are regular twins. If one disregards the uninteresting special case (1, 2) the sequence begins with

(2, 3), (3, 4), (4, 5), (5, 6), (8, 9), (9, 10), (15, 16), (24, 25).

The brief table above demonstrates that when (r – 1, r) is any one of these pairs, then

((1 – rec. r) · 60) · (1 + rec. (r – 1)) = 1(60)

or, simply,

(1 – rec. r) · (1 + rec. (r – 1)) = 1

(As noted, in terms of modern common fractions, 1 – rec. r = (r – 1)/r and 1 + rec. (r – 1) = r/(r – 1).)

The mentioned arithmetic rule was, apparently, well known in Old Babylonian mathematics. That fact is shown by, for instance, a curious calculation in the Old Babylonian "broken reed exercise" VAT 7535 # 1 (Neugebauer, *MKT 1* (1935), 303; Friberg, *UL* (2005), 119). Towards the end of that exercise, it has been established that the length of a broken reed, shortened by 1/5, is 20. Therefore, the text says

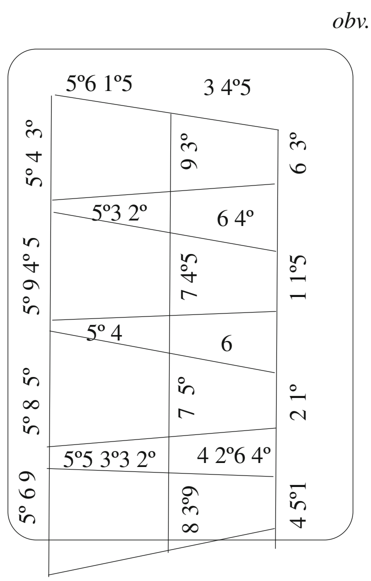
Since at first a 5th-part was broken off,
5 write down, then 1 remove, 4 is the remainder.
Compute the reciprocal of 4, then 15 it gives.
To 20 carry, 5 it gives. 5 to 20 join, then 25, the original reed.

In quasi-modern terms, this means that the equation (1 – rec. 5) · r = 20 has the solution

r = rec. (5 – 1) · 20 + 20 = (1 + rec. (5 – 1)) · 20 = 20 + 5 = 25.

(In terms of common fractions, the equation 4/5 r = 20 has the solution r = 5/4 · 20 = 25.)

Also the Old Babylonian hand tablet Ist. Si 269, from Sippar (Friberg, *AT* (2007), 279), is of interest in the present connection.



a) P = (54;30 + 9;30)/2 · 56;15 = 32 · 56;15 = 30 · 60
Q = (9;30 + 6;30)/2 · 3;45 = 8 · 3;45 = 30

b) P = (59;45 + 7;45)/2 · 53;20 = 33;45 · 53:20 = 30 · 60
Q = (7;45 + 1;15)/2 · 6;40 = 4;30 · 6;40 = 30

c) P = (58;50 + 7;50)/2 · 54 = 33;20 · 54 = 30 · 60
Q = (7;50 + 2;10)/2 · 6 = 5 · 6 = 30

d) P = (56;09 + 8;39)/2 · 55;33 20 = 32;24 · 55;33 20 = 30 · 60
Q = (8;39 + 4;51)/2 · 4;26 40 = 6;45 · 4;26 40 = 30

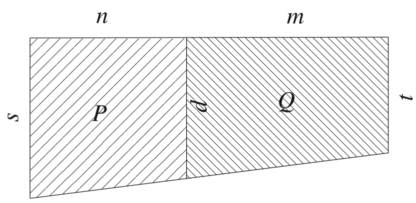
Fig. 13.1.2. Ist. Si 269. Four rational trapezoids with their areas divided in the ratio 60 : 1.

On the obverse of that clay tablet, there are four diagrams showing four trapezoidal fields, divided by transversals into pairs of sub-trapezoids, the ones to the left all with the area 30 · 60, the ones to the right all with the area 30. Since lengths of fields in Old Babylonian mathematical exercises typically are counted in tens of the basic length unit (the rod), the "partial lengths" of the the four divided trapezoids are

a) (56;15, 3;45) = ((1 – rec. 16) · 60, rec. 16 · 60) sum: 1 · 60
b) (53;20, 6;40) = ((1 – rec. 9) · 60, rec. 9 · 60) sum: 1 · 60
c) (54, 6) = ((1 – rec. 10) · 60, rec. 10 · 60) sum: 1 · 60
d) (55;33 20, 4;26 40) = ((1 – rec. 13;30) · 60, rec. 13;30 · 60) sum: 1 · 60

Apparently, the one who constructed the data for the divided trapezoids on Ist. Si 269 was familiar with the concept of regular twins. Since, 16 – 1 = 15, 9 – 1 = 8, 10 – 1 = 9, and 13;30 – 1 = 12;30, the regular twins involved in this construction are (15, 16), (8, 9), (9, 10), and (12;30, 13;30).

Here is why regular twins play a role in the construction of divided trapezoids like the ones depicted on Ist. Si 269. Consider a divided trapezoid with given partial lengths n and m and given partial areas P and Q, as in Fig. 13.1.3 below.



n, m, P, and Q are given
$s + t = 2\,(P + Q)/(n + m)$
$s + d = 2\,P/n$
$d + t = 2\,Q/m$

$s + d + t = (P + Q)/(n + m) + P/n + Q/m$
$s = (P + Q)/(n + m) + P/n - Q/m$
$d = P/n + Q/m - (P + Q)/(n + m)$
$t = (P + Q)/(n + m) - P/n + Q/m$

Fig. 13.1.3. The data for a trapezoid division problem.

The area of the whole trapezoid is $(n + m) \cdot (s + t)/2 = P + Q$. Since both $n + m$ and $P + Q$ are known, this equation for the total area can be reformulated as the following linear equation for $s + t$:

$s + t = 2\,(P + Q)/(n + m)$.

Similarly, the equations for the two partial areas can be reformulated as linear equations for $s + d$ and for $d + t$. In this way, three area equations lead to a system of three linear equations for the three unknowns s, d, t. Computing the solution to this system of linear equations, one finds the solution indicated in Fig. 13.1.3 above, with, for instance,

$s = (P + Q)/(n + m) + P/n - Q/m$.

For an Old Babylonian student of mathematics, a solution of this kind would make sense only if the reciprocals of $(n + m)$, n, and m exist, in other words, only if $(n + m)$, n, and m all are regular sexagesimal numbers. Therefore, if $n + m = r \cdot m$, then $r \cdot m$, $(r - 1) \cdot m$, and m must all be regular numbers, and r and $r - 1$ must be regular twins. In the first of the four trapezoids above, for instance, m = 3;45 and $m + n$ = 3;45 · 16 = 1(60).

The outcome of this discussion makes it tempting to conjecture that the one who constructed the table of reciprocals on SM 2685, with the mentioned curious extra lines near the end of the table, was familiar with trapezoid divisions like the ones depicted on Ist. Si 269!

Interestingly, regular twins may still have been a familiar concept also in Seleucid mathematics, near the end of the first millennium BC. This is demonstrated by a series of four "igi-igi.bi problems" in §§ 7 a–d of the Seleucid mathematical recombination text AO 6484 (Neugebauer, *MKT 1* (1935), 98: rev. 10 ff.; Friberg, *MSCT 1* (2007), 445). An igi-igi.bi problem is, essentially, a quadratic problem of the following very special kind:

igi *u* igi.bi (*a*) (In modern terms, n + rec. $n = a$, where it is silently understood that n · rec. n = 1 (precisely).)

It is easy to demonstrate that when igi and igi.bi satisfy the mentioned equations, then

sq. ((igi – igi.bi)/2) = sq. ((igi + igi.bi)/2) – 1 = sq. a/2 – 1.

Therefore, it is a simple task to compute first (igi – igi.bi)/2, and then also igi and igi.bi.

In AO 6484 § 7 a, for instance, the given parameter is a = 2;00 00 33 20. (Here the fraction separation ; is inserted at the appropriate place for the modern readers' convenience. In the original cuneiform text, the two double zeroes in this number were indicated by small double wedges.) The computations proceed as follows:

(igi + igi.bi)/2 = a/2 = 1;00 00 16 40,
sq. (igi + igi.bi)/2 – 1 = 1; 00 00 33 20 04 37 46 40 – 1 = ;00 00 33 20 04 37 46 40 = sq. ; 00 44 33 20,
(igi – igi.bi)/2 = ;00 44 33 20,
igi = 1;00 00 16 40 + ;00 44 33 20 = 1;00 45, igi.bi = 1;00 00 16 40 – ;00 44 33 20 = ;59 15 33 20.

Presumably, the author of this mathematical problem was familiar with the concept of regular twins and wanted to use it *in order to find two regular numbers, igi and igi.bi very close to 1*, possibly with the further purpose of calculating the (rational) sides and (rational) diagonal of an extremely thin rectangle. The trick used was first to find a pair of regular twins (r – 1, r) with r quite large, such as (1 20, 1 21), and then to set

igi = 1 + rec. (r – 1), and, consequently, as observed above, igi.bi = 1 – rec. r.

Then,

(igi + igi.bi)/2 = 1 + (rec. (r – 1) – rec. r)/2 = 1 + 1/2 rec. ($r \cdot (r – 1)$).

True enough, when r = 1 21, then

igi = 1 + rec. 1 20 = 1;00 45, igi.bi = 1 – rec. 1 21 = 1 – ;00 44 26 40 = ;59 15 33 20, and

(igi + igi.bi)/2 = 1 + 1/2 rec. (1 21 · 1 20) = 1;00 00 16 40.

Now, return to the question about the construction of the pairs of reciprocals in the atypical table of reciprocals SM 2685. So far, it has been possible to explain the construction of the majority of those pairs as what one gets by starting with a table of reciprocal pairs (n, rec. n) for integers n between 1 and 1(60), and then multiplying n and dividing rec. n by 2/3, 1/2, 1/3, rec. 4 = 15 sh., and rec. 6 = 12 sh. The curious extra lines near the end of the table were explained by assuming that the one who constructed the table had some knowledge of and interest in regular twins.

It remains to explain only the reason for the inclusion in the table of the last line of the table,

igi 1(60) 21 gál.bi $^2/_3$ [x x x x].

It is, of course, well known that in sexagesimal place value notation

rec. 1 21 = ;44 26 40.

Therefore, a possible reconstruction of the mentioned last line of the table on SM 2685 could be

igi 1(60) 21 gál.bi $^2/_3$ [4 $^1/_3$ 6 $^2/_3$].

This would mean something like

rec. 1(60) 21 = $^2/_3$ + [4 $^1/_3$ shekels + 6 $^2/_3$ shekels of a shekel].

(Note that the correctness of this equation can be demonstrated as follows:

1(60) 21 = 3 · 3 · 3 · 3,

so that

1(60) 21 · $^2/_3$ 4 $^1/_3$ sh. 6 $^2/_3$ sh. sh. = 3 · 3 · 3 · 2 13 $^1/_3$ sh. = 3 · 3 · 6 $^2/_3$ = 3 · 20 = 1(60).)

This conjectured reconstruction ought to be compared with the observation above that, apparently, all fractions in the table SM 2685 are silently understood to be *combinations of basic fractions. of multiples of the shekel (a sixtieth), and of basic fractions of the shekel.* It is not unlikely that the purpose of this exceptional last line in the table was precisely to show what can happen in more complicated situations than those studied in the main part of the table.

In connection with the proposed reconstruction of the last line of the table on SM 2685 it is interesting to note that shekels of a shekel (and even shekels of a shekel of a shekel) appear in CUNES 50-08-001, a table of areas of squares from the Early Dynastic period around the middle of the millennium, thus much older than SM 2685. In particular, in sub-table E of CUNES 50-08-001 (Friberg, *MSCT* 1 (2007), 424), the area of a square of side 1 šu.bad (1/24 of a ninda) is given as

6 gín.bi 15 gín.ba.gín = 6 'its shekel' + 15 shekels of 'its shekel ' (= ;00 06 15 sq. nindan).

In the same sub-table, the area of a square of side 9 šu.bad, is given as

8 $^1/_3$ gín gín.bi.ta 6 15 gín.ba.gín.ta = 8 1/3 shekels + 6 of 'its shekel' + 15 shekels of 'its shekel' (= ;08 26 15 sq. nindan).

In addition to the mentioned possibility that the destroyed reciprocal in the last line of SM 2685 can have been expressed in terms of shekels and shekels of shekels, there is also the equally plausible possibility that it can have been expressed in terms of fractions borrowed from the Sumerian system of weight numbers, namely shekels and grains (še), with 3(60) grains = 1 shekel. In that case the reconstructed reciprocal would take the following form:

rec. 1(60) 21 = $^2/_3$ [4 $^1/_3$ 20 še] = $^2/_3$ + [4 $^1/_3$ shekels + 20 grains].

13.2 SU 52/5. A Late Assyrian Table of Reciprocals with Sumerian Number Words

obv.

1.da šá. bi. u ni. im. ni	[of sixty]$^{??}$ its two-thirds	fourty		(2/3 of [60] = 40)
igi šu. ri. a. bi sa. a	its half-part	half		(half of it = 30)
igi me. nu šu. u	rec. two	thirty		(rec. 2 = 30)
igi e. šá gál. bi šu.šá. an	rec. three	one-third		(rec. 3 = 20)
igi 4 gál. bi 1˚5	rec. 4	15		(rec. 4 = 15)
igi ia gál. bi 1˚5 2	rec. five	15$^{!?}$-2		(rec. 5 = 12)
igi e. šá gál. bi 1˚	rec. six	10		(rec. 6 = 10)
igi ú. sa gál. bi e ba.a	rec. eight	7$^{??}$-half		(rec. 8 = 7 30)
igi. lam. mu u. šá. na. bi	rec. nine$^{!}$	six-two-thirds		(rec. 9 = 6 40)
igi 1˚ gál. bi a. ši	rec. 10	six		(rec. 10 = 6)
igi 1˚5 2 gál. bi ia	rec. 15$^{!?}$-2	five		(rec. 12 = 5)
igi 1˚5 gál. bi lam.mu.u ba.a.a	rec. 15	four-half	(error)	(rec. 15 = 4)
igi 1˚5 a.šú gál. bi e. ši. na.bi en. ga	rec. 15$^{!?}$-six	three-two-thirds-five$^{??}$		(rec. 16 = 3 45)
igi 1˚5 u. sa gál.bi e. šá. an	rec. 15$^{!?}$-eight	three-one-third		(rec. 18 = 3 20)
igi ni. ši gál. bi e. šá	rec. twenty	three		(rec. 20 = 3)
igi ni. ši 4 diri.da.ga gál.bi 2 3˚.a	rec. twenty 4-more	2 30		(rec. 24 = 2 30)
ḫe-pé diri da gál.bi e.šá.an ši lam	rec. [twenty five]$^{?}$-more	three-two-thirds-four$^{?}$	(error)	(rec. 25 = 2 24)

rev.

igi *ḫe-pé* gál.bi 2 1˚5 e. šá.an	rec. [twenty seven-more]$^{?}$	2 15$^{!?}$-three-one-third		(rec. 27 = 2 13 20)
igi *ḫe-pé* *pé*	rec. [thirty]$^{?}$	[two]$^{?}$		(rec. 30 = 2)
igi *pé* gál.bi 2 3˚. a	rec. [thirty-two-more]$^{?}$	2 30	(error)	(rec. 32 = 1 52 30)
igi *ḫe-pé* gál.bi diš.na. bi	rec. [thirty six-more]$^{?}$	one-two-thirds		(rec. 36 = 1 40)
igi *ḫe-pé* gál.bi 1 3˚. a	rec. [fourty]$^{?}$	1 30		(rec. 40 = 1 30)
igi ni da diri.ga gál.bi diš.šá.a.ni	rec. fourty$^{??}$-five$^{??}$-more	one-one-third		(rec. 45 = 1 20)
igi ni il da diri.ga gál.bi gi.lid 1˚5	rec. fourty$^{??}$-eight$^{??}$-more	sixty$^{??}$ 15		(rec. 48 = 1 15)
igi ni.na.u gál.bi gi. lid 1˚5 2	rec. fifty$^{??}$	sixty$^{??}$ 15$^{!?}$-2		(rec. 50 = 1 12)
igi gu.si. il.la lam.mu gál.bi gi.lid a. šá.na. bi	rec. five-sixths four	sixty$^{??}$ six-two-thirds		(rec. 54 = 1 06 40)
igi gi.^{lu}lud. u gál.bi diš.šu	rec. sixty$^{??}$	one		(rec. 1 = 1)
igi gi.lid lam.mu gál.bi gu.si.la a.šú 1˚5	rec. sixty$^{??}$ four	five-sixths six 15		(rec. 1 04 = 56 15)
igi di.šá.a.ni di. še. da diri gál.bi šá.na.a.be lam.mu šu.šá.na a.sa šá.na.bi	rec. one-one-third one-more two-thirds four one-third six two-thirds			(rec. 1 21 = 44 26 40)

Fig. 13.2.1. SU 52/5. A Late Assyrian table of reciprocals with Sumerian spellings of number words.

The Late Assyrian text SU 52/5 (Hulin, *JCS* 17(1963)) was found in Huzurīna (Sultantepe). It contains a table of reciprocals written on a well preserved baked tablet. The table of reciprocals is basically of the standard Old Babylonian type, but it is not in the usual way written in terms of sexagesimal numbers in place value notation. Instead it is is written, predominantly, with a strangely corrupt form of Sumerian spellings of number words. Compare with the following list of known forms of Sumerian number words:

šušana	1/3	limmu	4	udiliya	15
sa, ba	1/2	ya	5	niš	20
šanabi	2/3	aš	6	ušu	30
kingusila$^{?}$, gigusila$^{?}$	5/6	imin	7	nimin	40
diš	1	ussu	8	ninnu	50
min	2	ilimmu	9	geš	60
eš	3	u	10		

What is most interesting in the present connection is that the numbers appearing in SU 52/5 are expressed as a curious mix of number signs, Sumerian words for integers, and *Sumerian words for basic fractions*. In Fig. 13.2.1 above is shown a conform transliteration of the cuneiform text within an outline of the clay tablet and an attempt to explain what the text tries to say. The transliteration is Hulin's, and the attempted explanations to the right of the outline are based on Hulin's detailed linguistic analysis of the text. Note that Su 52/5 clearly was a copy of an older tablet missing the upper left corner of the reverse. The one who made the copy inserted the Akkadian word *ḫe-pé* 'broken', or simply *pé*, when he could not restore lost Sumerian number words. The use of (words for) basic fractions is not as well organized in SU 52/5 as in SM 2685, and in spite of some similarities between the two texts, there is no reason to believe that there was any kind of distant historical connection between the Neo-Sumerian table of reciprocals T. 2685 and the Neo-Assyrian table of reciprocals SU 52/5. After all, SM 2685 is totally atypical, while SU 52/5 is, at least essentially, a standard table of reciprocals.

Note the appearance in five places of what may be an otherwise unattested writing gi.lid/gi.lud for 'sixty', or possibly for 'sixty-plus'. Note also that in three places in column 1 and in three places in column 2 in this text, what should have been '10-and' is written as '15'! This six times repeated curious error may, conceivably, have been caused by a consistent mis-hearing during dictation. This explanation for the erroneous appearances of '15' is the present author's, not Hulin's, although Hulin understandably suspected that the "scribe was working from dictation, perhaps without understanding what he was writing".

The poor organization of the text is apparent from the fact that among the numbers *n*, which ought to be a sequence of integers *increasing* from 2 to 1 21 (= 81), there are two deviations (54 is written as 'five-sixths four', and 1 21 is written as 'one-one-third one-more'). Similarly, among the reciprocal numbers *n*´, which should have *decreased* from 20 to ;44 26 40, there are several deviations ('one-third' instead of 20, 'sixty$^{??}$ 15' instead of 'one 15', and so on).

Another observation of interest in the present context was made by Steinkeller, in the second part of his paper *ZA* 69 (1979). There Steinkeller was able to reconstruct the first line of Ist. S 485, a badly damaged fragment of a table of reciprocals from Sippar with Sumerian spellings (Neugebauer, *MKT 1* (1935), 27). The fragment had been published originally by Scheil in *ZA* 9 (1894), as an "Assyrian syllabary". Steinkeller interpreted the extensively damaged first line of the table as meaning

/gešt.ak šanabi.bi 40 of sixty, its two-thirds is forty.

Neugebauer's transliteration of Ist. S 485 in *MKT 1*, 27 is reproduced below, complemented with Steinkeller's improved reading of the first line of the table.

Ist. S 485

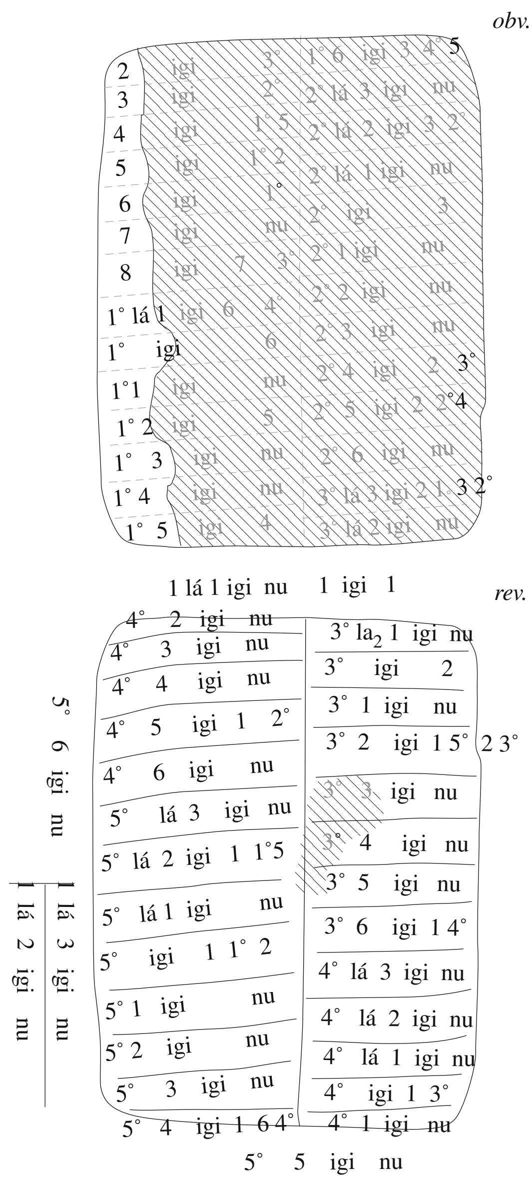
gi.[eš.da] [šá].[na.b]i.[b]i	[x x x]	40 [àm]
šu.[ri.a].bi	[x.b]i	[30.àm]$^{?}$
i.gi mi.in.n[u]	igi 2 [gál.bi]	[30]
" eš g[a]l.bi	igi 3 [gál.bi]	[20]
" lim.mu "	igi 4 gál.bi	15
" ia "	igi 5 gál,bi	12
" á[š].ša "	igi 6 gál,bi	10
" [us].su "	igi 8 gál,bi	7 30
" e.lim.mu	igi 9 <gál,bi>	6 40
" ú	gal.bi	igi 10 gál,bi [6]
" ú *ù* mi.in.n[u] "	igi 12 gál,bi	[5]
" ú *ù* ia	"	igi 15 gál,bi [4]
" ú *ù* áš.šá "	igi 16 gál,b[i	3 45]
" ú *ù* us.su "	igi 18 gá[l.bi	3 20]
[" ni.i]š	"	igi 20 gál,bi 3]
………………………	……………	……

The second and third columns in this fragmentary text constitute (the beginning of) a table of reciprocals of Old Babylonian standard type, with the Sumerian pronunciations of the entries in the second column given in the first column. The interested reader is asked to compare this (partly) syllabically written table of reciprocals with the equally (partly) syllabically written table of reciprocals SU 52/5 in Fig. 13.2.1 above.

13.3 Ist. L 7375. An Ur III Table of Reciprocals

One of several known Sumerian tables of reciprocals *with sexagesimal place value numbers* is Ist. L 7375 (Delaporte, *RA* 8 (1911); Friberg, *MSCT 1* (2007), 356; below Fig. 13.3.1). It is potentially significant that this Sumerian table of reciprocals, unlike all the others, but just like the table of reciprocals SM 2685 (minus the catch line), ends with n = 1(60).

On the other hand, the reciprocal table Ist. L 7375 has a number of characteristic traits that distinguishes it clearly from from the reciprocal table SM 2685. See the tabular survey in Section 13.5 below of the many common or diverse characteristic traits in all known Ur III or "intermediate" tables of reciprocals.

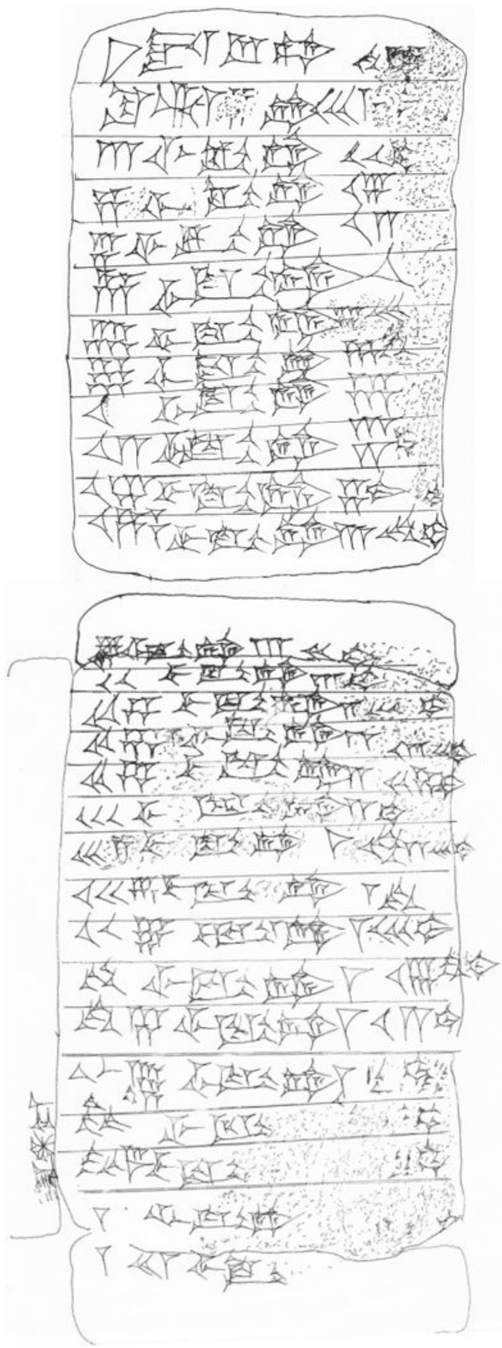


2	30	16	3 45
3	20	20-3	no
4	15	20-2	3 20
5	12	20-1	no
6	10	20	3
7	no	21	no
8	7 30	22	no
10-1	6 40	23	no
10	6	24	2 30
11	no	25	2 24
12	5	26	no
13	no	30-3	2 13 20
14	no	30-2	no
15	4		

42	no	30-1	no
43	no	30	2
44	no	31	no
45	1 20	32	1 52 30
46	no	33	no
50-3	no	34	no
50-2	1 15	35	no
50-1	no	36	1 40
50	1 12	40-3	no
51	no	40-2	no
52	no	40-1	no
53	no	40	1 30
54	1 06 40	41	no
55	no		
56	no		
1-3	no	1-2	no
1-1	no	1	1

Fig. 13.3.1. Ist. L 7375. An Ur III table of reciprocals, with *n* proceeding from 2 to 1(60).

13.4 SM 2574. A Table of Reciprocals of Intermediate Type



	1(60).da $\frac{2}{3}$.bi	40.	⌜àm⌝
	šu.ri.⌜a⌝.bi	30.	⌜àm⌝
3	igi.gál.bi	20.	[kam]
4	igi.gál.bi	15.	[kam]
5	igi.gál.bi	12.	[kam]
6	igi.gál.bi	10.	[kam]
8	igi.gál.bi	7 3[0].	[kam]
9	igi.gál.bi	6 ⌜40⌝.	[kam]
10	igi.gál.bi	6.	[kam]
12	igi.gál.bi	5.	[kam]
15	igi.gál.bi	4.	⌜kam⌝
16	igi.gál.bi	3 45.	[kam]
18	igi.gál.bi	3 20.	⌜kam⌝
20	igi.gál.bi	⌜3⌝.	⌜kam⌝
24	igi.gál.bi	⌜2 30⌝.	[kam]
25	igi.gál.bi	2 24.	kam
27	igi.gál.bi	13 20.	kam
30	igi.gál.bi	2.	kam
32	igi.gál.bi	1 52 30.	kam
36	igi.gál.bi	1 40.	<kam>
40	igi.gál.bi	1 30.	kam
45	igi.gál.bi	120.	kam
48	igi.gál.bi	1 15.	kam
50	igi.gál.[bi	1 12].	kam
54	igi.gál.[bi	1 06 40].	kam
1	igi.gál.bi	[1].	kam
1 21	igi.gál.[bi	44 26 40.	kam]

Fig. 13.4.1. SM 2574. A table of reciprocals of intermediate type.

The table of reciprocals SM 2574 (Fig. 13.4.1 above) is very different from the atypical table of reciprocals T. 2685 (Fig. 12.1), and also from the Ur III table of reciprocals Ist. L 7375 (Fig. 12.5). Instead it is, essentially, of the Old Babylonian standard type. (See, for instance, Neugebauer and Sachs, *MCT* (1934), 9 ## 32-35; or Friberg, *MSCT 1* (2007), Sec. 2.5.) However, it differs from the Old Babylonian standard type in certain ways.

Like SM 2685, SM 2574 makes use of archaic-looking “variant cuneiform number signs” for 4, 7, 8, and 9 (but not for 40!) and an early form of the cuneiform sign bi. This shows that probably, at the very least, SM 2547 is older than most Old Babylonian tables of reciprocals.

Moreover, in two ways SM 2574 differs from all other known cuneiform tables of reciprocals. Namely, in the main part of the table, each line is of the following form

n igi.gál.bi *n´*.kam (kam is normally a determinative after ordinal numbers)

The affixed .kam occurs almost nowhere else in a cuneiform table of reciprocals (the only known exception is discussed immediately below, in the discussion of *RA* 12, 197). Moreover, in all other known cases (see, for instance, Neugebauer, *MKT 1* (1935), 10, ## 4-9, and Neugebauer and Sachs, *MCT* (1945), 12. ## 32-37), the igi and the gál.bi are on different sides of *n*. The presence of these two unique features supports a cautious assumption that SM 2574 is Neo-Sumerian rather than Early Old Babylonian.

The first two lines of the table of reciprocals SM 2574 have the following form

1(60).da $^{2}/_{3}$.bi 40.⌜àm⌝ ‘of 60, its 2/3 is 40’
šu.ri.a.bi 30.⌜àm⌝ ‘its half is 30’

It is probably significant that in SM 2574 the first sign in line 1 is an oversize diš sign, a clear indication that it stands for géš(t) = 1(60), and not for diš = ‘1’. Therefore, the mentioned first lines of the table can be interpreted as saying that *2/3 of 60 is 40, one half of 60 is 30.* This is potentially important, in view of the observation above that $n \cdot n' = 60$ in each line of the atypical table of reciprocals SM 2685.

Only one other occurrence is known of the unusual phrase *n* igi.gál.bi *n´* in SM 2574; i_t appears also in the well known algorithm text CBS 10201 (Hilprecht, *BE 20/1* (1906), text. 25; Friberg, *MSCT 1* (2007), 367; Fig. 13.4.2 below).

2 05 12
igi.gál.bi 28 48
4 10 6
igi.gál.bi 14 24
8 20 3
igi.gál.bi 7 12
16 40 1 30

igi.gál.bi 3 36
33 20 18
igi.gál.bi 1 48
1 06 40 9
igi.gál.bi 54
2 13 20 18
igi.gál.bi 27
4 26 40 9
igi.gál.bi 13 30

Fig. 13.4.2. CBS 10201. An Ur III/Early Old Babylonian algorithm text. Hilprecht’s hand copy.

In this text, the cuneiform signs for gál and bi are archaic-looking, but the number signs are standard Old Babylonian. The text may be either Ur III or Early Old Babylonian. In other words, it may be called “intermediate,” just like SM 2574.

13.5 Some Other Ur III or Early Old Babylonian Tables of Reciprocals

See the interesting discussion of cuneiform tables of reciprocals in Proust, *TMN* (2007), Sec. 5.2.1, in particular p. 123.

In this connection, it is interesting to recall that in *RA* 12 (1915), Scheil was the first one to officially refute Hilprecht's absurd explanation of the first published cuneiform tables of reciprocals (published by himself in *BE 20/1* (1906), texts 20-25), as lists of "divisors of 12,960,000". (In Hilprecht's opinion, 12,960,000 = 60^4 was the "Number of Plato".)

Fig. 13.5.1. ***RA*** **12, 197. Scheil's hand copy of a privately owned table of reciprocals.**

Of particular interest in the present connection is that in *RA* 12, Scheil cleverly observed that in a certain, privately owned (and now, a hundred years later, still unavailable) table of reciprocals (Neugebauer, *MKT 1* (1935), 10, n. 4; Fig. 13.5.1 above), *the form of the sign for 40 is different in integers and in sexagesimal fractions*. More precisely, the archaic-looking, variant form of 40 is used in the following lines of the table, which are *reproduced here side by side with Scheil's own interpretations:*

1(60).da $^2/_3$	<u>40</u>	of 60 the 2/3 is	40
igi <u>40</u> gá[l.bi]	1 30	—— a 40th is	1.30 = 1 1/2
igi <u>4</u>5 gál.bi	1 20	—— a 45th is	1.20 = 1 1/3
igi <u>4</u>8 gál.bi	1 15	—— a 48th is	1.15 = 1 1/4

On the other hand, the "normal" (Old Babylonian) form of 40 is used in the lines

igi 9 gál.bi	6 40	—— a 9th is	6.40 = 6 2/3
igi 16 gál.bi	3 45	—— a 16th is	3.45 = 3 3/4
igi 54 gál.bi	1 06 40	—— a 54th is	1.6.40 = 1.111
igi 1 20 gál.bi	45	—— an 80th is	—45 = 3/4
igi 1 21 gál.bi	44 26 40	—— an 81th is	—44.26.40 = 0.74

Scheil drew the conclusion that the table in question was a table of fractions of the number 1(60), with 40 written in one way when it appeared in one of the integers *n*, and in another way when it occurred in the fractional parts of some of the reciprocals. He also thought that this conclusion was confirmed by the following unique final line of the table:

igi.gál 1(60).da.kam	fractions of 60

Scheil would have been happy if he had lived to see his suspicions confirmed by the atypical table of reciprocals SM 2685 with its explicitly written cuneiform signs for basic fractions! Note also that the suffix .kam in the last line of Scheil's text echoes the use of the same suffix in the table of reciprocals SM 2574.

In *Changing Views* (2001), Oelsner discussed HS 201 (Neugebauer, *MKT 1* (1935), 10 n. 2), a table of reciprocals from Ur III Nippur. In that connection he described how difficult it is to try to date cuneiform texts with departure from their use of normal or variant cuneiform number signs. (Oelsner's discussion of this issue is continued in Proust, *TMH 8* (2008), 20.) A photo of HS 201 was published for the first time in Oelsner's mentioned paper.

Not much later, in *AfO* 50 (2003/2004), Robson published two new tables of reciprocals from the collections of the British Museum, which she claimed to be Ur III, arguing as follows:

> "While it is theoretically possible that the tablets are early OB it is on the balance of probabilities unlikely.
> On prosopographical grounds, almost all the entire 3000-strong batch of tablets accessioned as lot number 1913–4–16 are from Ur III Umma ……"

One of Robson's new Ur III tables of reciprocals is BM 106425. The first line of that text is

1(60).da igi.2.bi 30 of sixty, its rec. 2 is 30

This line has kindly been collated for me by I. Finkel, with the purpose of checking if the number sign before the .da in this line is a normal upright wedge for diš ='1' or a bigger upright wedge for géš = '1(60)'. Finkel's answer to this question was that "géš is written differently from a single upright, such as occurs twice in MIN in the same line. The wedge is not as tall as a 'normal' upright and has a wide head that sweeps up to the right, and is thus altogether distinct from DISH = l". (Compare with the form of the first wedge in line 1 of the hand copy of SM 2574 in Fig. 13.4.1 above.)

Parenthetically, it may be remarked here that, while it appears that a line for the reciprocal of 48 is completely missing in BM 106425, the truth is that the two lines for the reciprocals of 48 and 50 have been conflated in that table of reciprocals. Of the first line, only the second half remains (1 15), and of the second line only the first half remains (igi 50 gál).

Interestingly, the Ur III table of reciprocals BM 106425 (but not Robson's other Ur III table of reciprocals, BM 106444!) distinguishes between normal and variant cuneiform number signs for 40 in the same way as Scheil's table of reciprocals in *RA* 12, 197. Thus, in the two lines igi.<u>40</u>.bi 1 30 and igi.<u>4</u>5.bi 1 20, the 40 is written in variant form, but in igi.9.bi 6 40, igi.16.bi 3 45, igi.36.bi 1 40, igi.54.bi 1 06 40, and igi.1 21.bi 44 26 40, the 40 signs are written in normal form. However, a difference between the two tables of reciprocals is that the line 1(60).da 2/3 40 in *RA* 12, 197 has no counterpart in BM 106425, which instead begins with the line [1(60)].da igi.2.gál 30. Therefore, the number 40 does not at all appear in the first line of *RA* 12, 197.

Yet another interesting peculiarity of the table of reciprocals BM 106425 is that it includes two extra lines between the lines for $n = 6$ and $n = 8$, namely

igi.7 12.bi 8 20
igi.7 30.bi 8

This is an echo of the atypical table of reciprocals SM 2685, which also includes lines for the reciprocals of the two non-integers 7 12 (meaning 7 12 sh.) and 7 1/2.

A similar peculiarity of Robson's Ur III table of reciprocals BM 106444 (Fig. 13.5.2 below) is that it includes the following two lines:

igi 7 1/2.bi 8
igi 8.bi 7 1/2

Not only is there an extra line with n equal to a non-integer, but this non-integer is in two places written as an integer plus a basic fraction. This is yet another echo of the atypical table on SM 2685!

Note finally the following interesting line on BM 106444, without any known counterpart elsewhere:

1(60).da igi.1(60).gál.bi 1.àm of sixty, a sixtieth is 1

Unfortunately, BM 106444 is presently in a showcase and therefore not available for collation, so it remains only a plausible assumption that the sign géš = 1(60), and not diš = 1, occurs twice in this line.

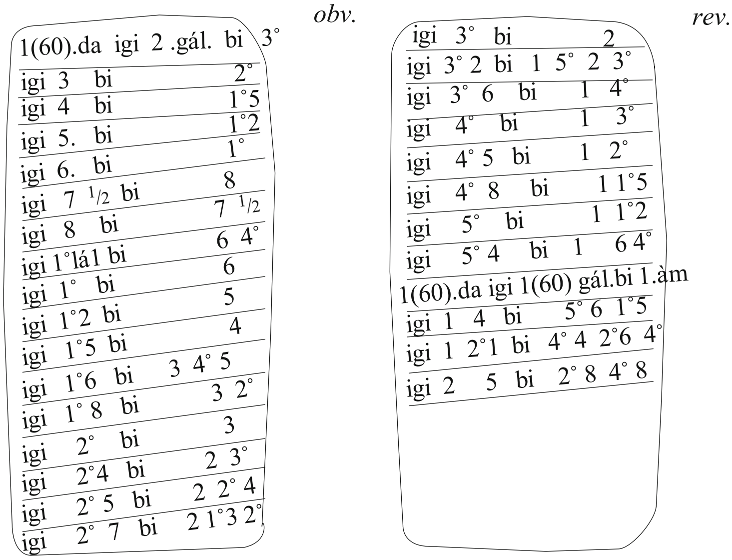


Fig. 13.5.2. BM 106444. An intermediate table of reciprocals from Ur III Umma. Conform transliteration.

In addition to BM 106425, there are two other Ur III tables of reciprocals, which distinguish between normal and variant cuneiform number signs for 40 in the same way as Scheil's table of reciprocals in *RA* 12. One is Erm 14645 (Friberg, *CDLJ* 2009:3, Fig. 8; on-line photo: cdli.ucla.edu no. P211991). Thus, in the two lines igi.40.bi 1 30 and igi.45.bi 1 20, the 40 is written in variant form, but in igi.9.bi 6 40, igi.16.bi 3 45, igi.36.bi 1 40, igi.54.bi 1 06 40, and igi.1 21.bi 44 26 40, the 40 signs are written in normal form. The third Ur III table of reciprocals writing 40 in these two ways, depending on the context, is Ist. L 9007+9005 (Lafont, *to be published*).

Is this an indication that also Scheil's text in *RA* 12, 197 is an Ur III table of reciprocals? A further link between BM 106425 and Erm 14645 is, by the way, that there is an extra line for the reciprocal of 7 30 also in Erm 14645.

The same question can be asked about the fragment Ist. Ni 3854 (Proust, *TMN* (2007), pl. 48), which also writes 40 in two ways, depending on the context. The first line of Ist. Ni 3854 is [1(60) 2/3].bi 40. à[m], with 40 in the variant form, while in the line [igi.36.gál.bi] 1 40, 40 is written in normal form. Also 4, 8, and 9 are written in variant form.

Another member of the same group of early tables of reciprocals as *RA* 12, 197 and Ist. Ni 3854 may be MS 3874 (Friberg, *MSCT 1* (2007), 69), although it is not quite clear how 40 is written in some places in that table of reciprocals.

As mentioned above, in three of the known Ur III tables of reciprocals, namely BM 106425, Erm 14645, and Ist. L 9007+9005, 40 is written in variant form when it is part of an integer, but in normal form when it is part of a fraction. What about the remaining known Ur III tables of reciprocals? This question has a somewhat surprising answer. Indeed, in Ist. Ni 374 (Proust, *TMN* (2007), pl. 1), the number 40 is written as one row of four oblique wedges when it is part of an integer, but it is written in variant form, as two rows of two oblique wedges, when it is part of a fraction, namely in the lines [igi 9 6] 40, [igi 1]6 3 45, igi 36 1 40, and [igi 54] 1 06 40! Similarly, in Ist. L 9006 (Lafont, to be published), where 50 as part of an integer is written as one row of five oblique wedges (all instances of 40 as a part of an integer are lost), and 40 as part of a fraction is written in variant form, namely in the lines 10.lá.1 igi 6 40 and 16 igi 3 45.

In the case of the Ur III table of reciprocals Ist. L 7375 (Delaporte, *RA* 8 (1911); Friberg, *MSCC 1* (2007), 356), 40 is written everywhere as a row of 4 oblique wedges, and in HS 201 (Proust, *TMH 8* (2008), 123), 40 as a part of a fraction is written in variant form, as in Ist. Ni 374, but the end of the table is reached before there can be any instances of 40 as part of an integer.

In *TMH 8* (2008), 19-21, Proust gave a survey of the six Ur III tables of reciprocals then known, two from Nippur and four from Girsu. (Note that "batch L" (L for Lagash) is the name traditionally given to the collection of tablets from Girsu (Telloh) at Istanbul, although it has been known now for a long time that those tablets are not from Lagash (Al-Hiba).) Since then four more Ur III tables of reciprocals have become known, the two from Umma and two from unknown sites (Erm 14645 and SM 2685). In addition, there are four tables of reciprocals which are either Early Old Babylonian or Ur III (*RA* 12, 197, Ist. Ni 3854, MS 3874, and SM 2574). These four texts may, deliberately vaguely, be called "intermediate". Altogether, *there are now ten known Ur III tables of reciprocals, and four intermediate tables of reciprocals*. A new survey below is intended to list a full set of characteristics for all the mentioned early tables of reciprocals, except the atypical SM 2685.

The survey shows, for instance, that all the Ur III tables of reciprocals below which have a (partly) preserved *incipit* (the more or less distinct first part of the table) begin with the line

géš.da igi.2.(gál).bi 30,

although in the texts from Girsu in the abbreviated version

2 igi 30.

The intermediate tables of reciprocals, on the other hand, all have some version of the following incipit, which is also common in standard Old Babylonian tables of reciprocals:

géš.da 2/3.bi 40.àm
šu.ri.(a).bi 30.àm

With one exception (Ist. L 9007+), the Ur III tables of reciprocals from Nippur and Girsu, but not the tables from Umma, and not Erm 14645, include lines of the type igi.n nu 'rec. *n* does not exist'.

Variant number signs are used, more or less consistently, for 4, 7, 8, 9, 40, 50, in all the listed tables of reciprocals, both Ur III and intermediate.

Subtractive notations for all numbers *n* ending with 7, 8, 9, in some cases only for *n* = 9, are used in all the Ur III tables, except BM 106425, but not in the intermediate tables.

Different number signs are used for 40 as part of an integer and for 40 as part of a fraction, in all the tables of reciprocals, both Ur III and intermediate, except Ist. L 7375, BM 106444, and SM 2574.

Two of the Ur III tables (Erm 14645 and BM 106425), for which the last lines are preserved, end with *n*, rec. *n* = 1 21, 44 26 40, just like all standard Old Babylonian tables of reciprocals. However, Ist. L 7375 ends with *n*, rec. *n* = 1, 1, and BM 106444 ends with *n*, rec. *n* = 2 05, 28 48. (The well known explanation for last lines of those types is that 1 21 = 3 · 3 · 3 · 3 and 2 05 = 5 · 5 · 5.)

In Proust, *TMN* (2007), 126, it is claimed that the last two lines of Ist. Ni 374 are

[igi 1 36 3]7 30
[igi 1 40] 36.

However, the hand copy of Ist. Ni 374 (*op. cit*., pl. 1) appears to show instead

[igi 1 40 3]⌜6⌝
[igi 2 05 28]⌜48⌝.

If this changed reading is correct, then the presence of a line for *n*, rec. *n* = 1 40, 36 is not surprising, in view of the fact that 1 40 = 100.

Below is given in tabular form a survey of the various interesting properties of all the 10 known "Ur III" tables of reciprocals, except the atypical SM 2685, and all the 4 "intermediate" tables of reciprocals.

Ist. Ni 374 **Nippur** **Ur III**	**HS 201** **Nippur** **Ur III**	**Ist. L 7375** **Girsu** **Ur III**	**Ist. L 9006** **Girsu** **Ur III**	**Ist. L 9008** **Girsu** **Ur III**	**Ist. L 9007+** **Girsu** **Ur III**	**Erm 14645** **?** **Ur III**
2 columns	2 columns	2 columns damaged	2 columns damaged	2 columns? fragmentary	2 columns fragmentary	1 column
[gèš.da igi 2] / [gál.b]i 30	[gèš].da igi 2 / [g]ál.bi 30	2 [igi 30]	2 [igi 30]	?	?	[gèš].da igi 2 gál.bi 30
igi *n n´* igi *n* nu	*n* igi *n´* *n* igi nu	*n* igi *n´* *n* igi nu	*n* igi *n´* *n* igi nu	*n* igi *n´* *n* igi nu	*n* igi *n´*	igi *n n´*
variant: 4, 7, 8, 9, 40, 50	variant: 4 ——	variant: 4, 7, 8, 40, 50	variant: 4, 7, 8, [40], 50	variant: [4], 7, 8, ?	variant: 4, 7, 8, ?	variant: 4?, 7, 8, 9 40, 50
subtractive: [9]?	subtractive: [9] and 10 to 20 + 7, 8, 9	subtractive: 9 and [10] to 50 + 7, 8, 9	subtractive: 9 and 10 to [50] + 7, 8, 9	subtractive: 9 and ?	subtractive: 9	subtractive: 18, 27, 48
ends with: [igi 1 40 3]6 [igi 2 05 28]48	ends with: 31 igi nu 32 igi 1 52 30	ends with: 1 lá 1 igi nu 1 igi 1	ends with: ?	ends with: ?	ends with: ?	ends with: 1 21 igi.gál.bi 44 26 40
special feature: 40 in two ways, both archaic	———		special feature: 40 in two ways, both archaic	?	special features: 40 in two ways, also an *inverted* table of reciprocals	special features: 40 in two ways, non-integral *n* = 7 30

BM 106444 **Umma** **Ur III**	**BM 106425** **Umma** **Ur III**	***RA* 12, 197** **?** **intermediate**	**Ist. Ni 3854** **Nippur** **intermediate**	**MS 3874** **?** **intermediate**	**SM 2574** **?** **intermediate**
1 column	1 column	1 column	1 column fragmentary	1 column	1 column
gèš.da igi 2 gál.bi 30	gèš.da igi 2 bi 30	géš.da 2/3 40 šu.ri.bi 30	[géš 2/3].bi 40.àm šu.ri.a.bi 30.à[m]	géš.da.ám 2/3.bi 40.[àm] šu.ri.a.bi 30.àm	géš.da 2/3.bi 40.àm šu.ri.a.bi 30.àm
igi *n*.bi *n´*	igi *n*.bi *n´*	igi.*n*.gál.bi *n´*	igi.*n*.gál.bi *n´*	igi.*n*.gál.bi *n´*	*n* igi.gál.bi *n´*.kam
variant: 4, 7, 8, 40, 50	variant: 4, 7, 8, 9, 40, 50	variant: 4, 7, 8, 9 40, 50	variant: 4, [7], 8, 9 40, [50]	variant: 4, [7], 8, 9 40, [50]	variant: 4, 7, 8, 9, but not 40, 50
subtractive: 9					
ends with: igi 2 05.bi 28 48	ends with: igi 1 21.bi 44 26 40	ends with: 1 21 igi.gál.bi 44 26 40 igi.gál gèš.da.kam	ends with: ?	ends with: 1 21 igi.gál.bi 44 26 40	ends with: 1 21 igi.gál.bi [44 26 40]
special feature: géš.da igi 1 gál.bi 1.àm, non-integral: *n* = 7 1/2 and *n´* = 7 1/2	special feature: 40 in two ways, non-integral: *n* = 7 12 and *n* = 7 30	special feature: 40 in two ways	special feature: 40 in two ways	special feature: 40 in two ways(?)	

13.6 The Historical Importance of SM 2685 as a Missing Link

Only a small number of suspected mathematical school texts from the 3rd millennium BC in Mesopotamia are known. Practically all of them should rather be called "metro-mathematical", because it is evident that their purpose was just as much training in "metrology", that is in the use of the various systems of measures and measure notations, as training in "mathematics", that is in counting with sexagesimal numbers and performing various kinds of computations.

As a matter of fact, the handful of known tables of reciprocals from the Neo-Sumerian Ur III period at the very end of the 3rd millennium are the *earliest known examples of purely mathematical texts*. That is what makes the table of reciprocals SM 2685 so interesting; it is a kind of "missing link" between the metro-mathematical texts from the preceding part of the millennium and all other known Neo-Sumerian tables of reciprocals or Old Babylonian tables of reciprocals, multiplication tables, tables of squares, and tables of square roots or cube roots, which are all based on the use of sexagesimal numbers in place value notation.

Take, for instance, the four known "tables of squares or rectangles" from the 3rd millennium, all of them from the Early Dynastic period around the middle of the millennium: VAT 12593, Nissen, Damerow, Englund, *ABK* (1993), Fig. 119; A 681, Friberg, *MSCT 1* (2007), 358; CUNES 50-08-001, Friberg, *op. cit.* (2007), 420, and, most recently, the privately owned tablet published by Feliu in *AoF* 39 (2012). All four should, more precisely, be called "tables of areas of squares or rectangles", since they list the *area measures* of squares or rectangles with given sides *in current units of area measure,* when the side lengths of the squares are expressed *in current units of length measure*.

The biggest of the three mentioned Early Dynastic tables of areas of squares is CUNES 50-08-001, which contains five sub-tables, A-E. In sub-table A, side lengths are measured in the basic unit nindan 'rod', in sub-table B in *nikkas* = 1/4 nindan, in sub-table C in kùš.numun 'seed.cubit' = 1/6 nindan, in sub-table D in geš.bad = 1/12 nindan, and in sub-table E in šu.bad = 1/24 nindan. In the present connection, sub-table C is of particular interest, because it expresses areas of squares in terms of the basic area unit sar = 1 sq. nindan and sixtieths of a sar, called gín 'shekel'. See below. *It looks very much like the table of reciprocals SM 2685*, with all coefficients before (or after!) the sar and the gín in the form of *integers and basic fractions*!

CUNES 50–08–001, C (Friberg *MSCT 1*, 423)

1 kùš.numun square	1 2/3 gín	1 sq. k.n.	= 1/36 sar	= 1 2/3 shekel
2 kùš.numun square	6 2/3 gín	4 sq. k.n.	= 1/9 sar	= 6 2/3 shekel
3 kùš.numun square	sar 15 gín	9 sq. k.n.	= 1/4 sar	= 15 shekel
4 kùš.numun square	sar 1/3 6 2/3 gín	16 sq. k.n.	= 16 · 1 2/3 shekel	= 1/3 sar 6 2/3 shekel
5 kùš.numun square	sar 2/3 1 2/3 gín	25 sq. k.n.	= 25 · 1 2/3 shekel	= 2/3 sar 1 2/3 shekel
6 kùš.numun square	1 sar	36 sq. k.n.		= 1 sar
7 kùš.numun square	1 sar 1/3 1 2/3 gín	49 sq. k.n.	= 49 · 1 2/3 shekel	= 1 1/3 sar 1 2/3 shekel
8 kùš.numun square	1 sar 2/3 6 2/3 gín	64 sq. k.n.	= 64 · 1 2/3 shekel	= 1 2/3 sar 6 2/3 shekel
9 kùš.numun square	2 sar 15 gín	81 sq. k.n.	= 81 · 1 2/3 shekel	= 2 sar 15 shekel
10 kùš.numun square	2 sar 2/3 6 2/3 gín	100 sq. k.n.	= 100 · 1 2/3 shekel.	= 2 2/3 sar 6 2/3 shekel

Sub-table D of CUNES 50-08-001, the table of squares with side lengths measured in geš.bad = 1/12 nindan, is just as interesting. It is shown below side by side with the parallel text A 681, which is from Adab in the Early Dynastic IIIb period, 2500-2334 BC, and therefore roughly contemporaneous with CUNES 50-08-001. A 681 is a table of squares with side lengths measured in kùš 'cubits', another name for 1/12 nindan.

In sub-table D, sq. (1 geš.bad) = sq. (1/12 nindan), for instance, is written as

gín 1/3 5 gín.bi meaning 1/3 shekel 5 its shekel (= 1/3 shekel and 5 sixtieths of a shekel).

In A 681, on the other hand, the area of the same square is written as

1 sa$_{10}$ ma.na 15 gín = 1 exchange-mina and 15 shekel (of the exchange-mina).

For an explanation of the name sa$_{10}$ ma.na 'exchange-mina' for 1/3 shekel, see Friberg, *CDLJ* 2005:2, § 2.28. Briefly, in the ED IIIa period silver was 180 times more expensive than copper, and therefore, originally, copper was measured in terms of minas, while silver was measured in terms of the 180 times lighter exchange-minas. Consequently, the exchange mina was equal to 1/3 shekel (of the ordinary mina), and 15 shekels of the exchange-mina were equal to 15 shekels of 1/3 ordinary shekel, or 5 gín.bi (shekels of the ordinary shekel). In other words, as indicated by the comparison above of corresponding lines in the two tables,

1 sa$_{10}$ ma.na 15 gín = gín 1/3 5 gín.bi (1/3 shekel 5 its shekel).

CUNES 50–08–001, D (Friberg *MSCT 1*, 425)

1 geš.bad square	gín 1/3 5 gín.bi
2 geš.bad square	1 2/3 gín
3 geš.bad square	3 gín 2/3 5 gín.bi
4 geš.bad square	6 2/3 gín
5 geš.bad square	10 gín 1/3 5 gín.bi
6 geš.bad square	sar 15 gín
7 geš.bad square	sar 1/3 gín 1/3 5 gín.bi
8 geš.bad square	sar 1/3 6 2/3 gín
9 geš.bad square	sar 1/2 3 2/3 gín 5 gín.bi
10 geš.bad square	sar 2/3 1 2/3 gín

A 681 (Friberg *MSCT 1*, 358)

1 cubit	1 sa$_{10}$ ma.na 15 gín
2 cubits	2 gín – 1 <sa$_{10}$> ma.na
3 cubits	4 gín – igi 4
4 cubits	6 gín 2/3 ma.na (sic!)
5 cubits	10 gín 1/3 5
6 cubits	15 gín
7 cubits	1/3 sar 1/3 5
8 cubits	1/2 sar – (3 gín 1 sa$_{10}$ ma.na)
10–1 cubits	1/2 sar 4 gín – igi 4
10 cubits	2/3 sar 2 gín – 1 sa$_{10}$ ma.na

(Note the curious inconsistencies in A 681, where the scribe twice forgot that he was counting with exchange-minas and (exchange-)shekels and wrote 1/3 gín 5 in line 5 and just 1/3 5 in line 7, as if he was counting with gín and gín.bi, instead of 1 sa$_{10}$ ma.na 15 gín as in line 1 of the table.)

The only known "metrological list" from the 3rd millennium is CUNES 47-12-176 (Friberg, *MSCT 1* (2007), 427), dating from the Old Akkadian period, 2334-2154 BC. It is a descending list of weight measures, beginning with minas, basic fractions of minas, and shekels. The remainder of the list, which is of a curiously complicated structure, counts with exchange-minas, basic fractions of the exchange-mina, and multiples of the (exchange-)shekel, once called gín.tur 'small shekel'.

There are, of course, no known "metrological tables" from the 3rd millennium, since in a metrological table a list of measures of a certain kind is shown side by side with a corresponding list of sexagesimal multiples of some basic (but never mentioned) measure, with the sexagesimal numbers in place value notation. In this respect, metrological tables are like tables of reciprocals of the standard Old Babylonian kind, where a list of regular sexagesimal numbers is shown side by side with the corresponding list of reciprocal sexagesimal numbers, in place value notation.

The concept of regular sexagesimal numbers is essential for the construction of tables of reciprocals. The earliest clear occurrences of regular numbers (in some sense) in (metro-)mathematical texts can be found in three "metric division problems" from the Old Akkadian period, c. 2500-2334 BC, namely 2(60) 40 (rods) in *DPA* 38, 1(60) 7 1/2 (rods) in HS 815 (*TMH 5*, 65), and 4(60) 3 (rods) in *DPA* 39. See Friberg, *MSCT 1* (2007), 408-9, or Friberg, *CDLJ* 2005:2, § 3. Note, by the way, that reasonably *the reciprocal of a length number n should be another length number n′ such that n · n′ = 1 area unit!*

It is easy to find the reciprocals of the mentioned counting numbers 2(60) 40 and 1(60) 7 1/2. They are 1/3 2 1/2, and 5/6 3 1/3, respectively. It is not so easy to find a nice expression for the reciprocal of 4(60) 3 (= the 5th power of 3) without the use of sexagesimal numbers in place value notation. (As a sexagesimal number in relative place value notation, that reciprocal is

1/(3 · 3 · 3 · 3 · 3) = 20/(3 · 3 · 3 · 3) = 6 40/(3 · 3 · 3) = 2 13 20/(3 · 3) = 44 26 40/3 = 14 48 53 20.)

It is instructive to see the following example of how an Old Akkadian metric division problem could (hypothetically) be solved without the use of sexagesimal numbers in place value notation.

HS 815 (Westenholz, *ECTJ*, 65; Friberg, *MSCT 1*, 408; Proust, *TMH 8*, 1)

1(géš) 7 1/2 nindan^{du} uš / sag 1(iku) ašag / 1 nindan^{du} 5 kùš 2 šu.dù.a 3 šu.si 1/3 šu.si

1(60) 7 1/2 rods. the length. / Front, 1 iku field measure? / 1 rod 5 cubits 2/3 cubits 3 fingers 1/3 finger

This brief text contains only a question and its (correct) answer. (Note that šu.dù.a was an Old Akkadian name for 1/3 cubit.) The obscurely formulated question can be interpreted as meaning

A rectangle has the length u = 1(60) 7 1/2 (rods) and the area A = 1(iku) = sq. (10 rods).

What is the front s (the short side) of the rectangle?

A reasonable conjecture is that an Old Akkadian student, without recourse to sexagesimal numbers in place value notation, would have attacked this problem in something like the following way: First, it is easy to find a useful *factorization* of the given length number, namely

u = 1(60) 7 1/2 (rods) = 9 · 7 1/2 (rods) = 3 · 3 · 3 · 2 1/2 (rods).

Therefore, the problem may be solved algorithmically, in a number of easy steps, by first solving the corresponding problem when u = 2 1/2 n. (rods), then when u = 3 · 2 1/2 n., and so on, as follows:

1(iku) = 1(60) 40 sq. n.

If u = 2 1/2 n.,	then s = 40 n.
If u = 3 · 2 1/2 n. = 7 1/2 n.,	then s = 1/3 · 40 n. = 13 1/3 n. = 13 n. 4 c.
If u = 3 · 7 1/2 n. = 22 1/2 n.,	then s = 1/3 · 13 n. 4 c. = 4 1/3 n. 1 1/3 c. = 4 n. 5 c. 10 f.
If u = 3 · 22 1/2 n. = 1(60) 7 1/2 n.,	then s = 1/3 · 4 n. 5 c. 10 f. = 1 n. 1/3 n. 1 2/3 c. 3 1/3 f. = 1 n. 5 2/3 c. 3 1/3 f.

This algorithmic procedure resembles the proposed explanation in Sec. 13.1.2 above of the construction of the entries in the table of reciprocals SM 2685 by use of *reciprocal compensation*.

(Actually, the oldest known example of a metric division problem with *an explicitly given* algorithmic solution procedure occurs in TM.75.G.1392 (Friberg, *VO* (1986); *MSCT 1* (2007), 411), a cuneiform metro-mathematical text from the ancient city Ebla in Syria, c. 2250 BC)

Now return to the metric division problem in HS 815. Would it be easier to solve it by use of the (non-place-value) reciprocal of 1(60) 7 1/2? In quasi-modern terms the problem in HS 815 can be reformulated as an equation of the following kind:

1(60) 7 1/2 rods · s = 1(60) 40 sar.

Here 1 sar = 1 sq. rod. Therefore, an equivalent, but simpler equation is

1(60) 7 1/2 · s = 1(60) 40 rods .

Now, the reciprocal of 1(60) 7 1/2 is 5/6 3 1/3 (shekel). (Compare with the reciprocal pair 53 1/3, 1 7 1/2 in SM 2685.) Therefore, another equivalent equation is

1(60) · s = 5/6 3 1/3 sh. · 1(60) 40 rods = 53 1/3 rods · 1 2/3 = (53 + 35 1/3) rods + 1/3 rod · 1 2/3
= 1(60) 28 2/3 rods + 1/3 rod · 2/3 = 1(60) 28 1/2 rods 4 2/3 cubits.

After another tedious series of computations, one finally gets the correct result (the same as above) that

s = 1 rod 5 2/3 cubits 3 1/3 fingers.

This little example demonstrates that counting with reciprocal numbers without recourse to sexagesimal numbers in place value notation was *not* an attractive option for the solution of *metric* division problems!

Incidentally, the task would have been much easier for an Old Babylonian student of mathematics, who would have answered the question in HS 815 without any great effort, as follows:

If u = 1 07 30 and A = 1 40, then s = rec. $u \cdot A$ = 53 20 · 1 40 = 1 28 53 20.

Therefore s = 1 rod 5 2/3 cubits (1 28 20) + 3 1/3 fingers (33 20).

Now, consider again the Neo-Sumerian table of reciprocals SM 2685. As already mentioned, it, or some older table of reciprocals of which it may have been a copy or further development, was possibly *the first mathematical cuneiform text which was not metro-mathematical.* This was a great breakthrough. Working for the first time with *abstract numbers,* without any metrological constraints, must have made a great difference. It was now possible, for instance, to understand much better the crucial concept of the reciprocal of a regular sexagesimal number.

An Old Babylonian metric division problem can be found in the exercise **VAT 8522 # 4** (Fig. 11.3.5 above) where the stated question is the following:

5/6 gín a.šag$_4$ / *si-bi-at si-bi-at* šu.si sag / uš en.nam

5/6 shekel the field. / A seventh of a seventh of a finger the front. / The length what?

As in the preceding three exercises in VAT 8522, the solution procedure is chaotic also in this fourth exercise. Therefore, it may be a good idea to start by proposing a working solution procedure, and only afterwards compare with the scribbled solution procedure in the text.

First, a few words about the meaning of the expression 5/6 gín (incorrectly transliterated by Neugebauer as 5/6 gán, which later led to unnecessary difficulties in the course of his commentary). As was pointed out already by Thureau-Dangin, it is not unusual in Old Babylonian mathematical texts that an expression like 5/6 gín should be understood as meaning '5/6 (of 1 sar), in the range of shekels' = ;50 (sar). Therefore, the answer to the question in exercise # 4 can have been obtained in (essentially) the following way:

5/6 gín = ;50 sar = ;50 nindan · 1 nindan = 49 · ;50 nindan · 1/7 of 1/7 of 1 nindan = 40;50 nindan · 1/7 of 1/7 of 1 nindan.

Now, 1 nindan = 12 · 30 fingers = 6 (00) fingers, so that 1 finger = (;00) 10 nindan. Therefore,

40;50 nindan · 1/7 of 1/7 of 1 nindan = 6 (00) · 40;50 nindan · 1/7 of 1/7 of 1 finger = 4 05 (00) nindan · 1/7 of 1/7 of 1 finger.

(In modern metric units of measure, 5/6 sar = 6 meters · 5 meters, which is the area of an ordinary room, while 1/7 of 1/7 of 1 finger = 1/49 · 10 centimeters = approximately half a millimeter, which is a ridiculously short length for one side of a rectangle, when the length of the other side is 245 · 60 · 6 meters = 88.2 kilometers.)

The solution procedure suggested above agrees fairly well with the first half of the scribbled solution procedure of VAT 8522 # 4. After that comes the inverted question

šum-ma 4 05 uš 50 a.šag$_4$ sag$^{!}$.ki en.<nam>

If 4 05 (00) is the length, ;50 the field, the front is what?

However, what comes after that question is a sketch of a solution procedure for a third related question, namely

šum-ma 4 05 uš *si-bi-at si-bi-at* šu.si sag a.šag$_4$ en.<nam>

If 4 05 (00) is the length, a seventh of a seventh of a finger the front, the field is what?

A severe limitation for a table of reciprocals like SM 2685 is that it makes use of only integers or shekels (sixtieths) and their basic fractions. Recall that it was observed above that the algorithmic method, based on "reciprocal compensation", which clearly was used to construct the table of reciprocals on SM 2685, should have lead to, among others, the reciprocal pair (21 1/3, 2 2/3 8 1/2 1/4). However, this pair does not occur in the table, possibly because 1/2 1/4 (shekel) is not a basic fraction of a shekel. If the author of SM 2685 had really wished to include the mentioned reciprocal pair in his table, he would, conceivably, have had to express 2 2/3 8 1/2 1/4 in one of the four following ways, all of them rather inelegant:

2 2/3 8 1/2 igi 4	cf. lines 3 and 9 of A 681 (see above)
2 2/3 8 2/3 5 (gín.bi)	cf. line 9 of CUNES 50–08–001, D (see above)
2 2/3 8 2 sa$_{10}$ ma.na 15 gín	cf. line 1 of A 681 (see above)
2 2/3 8 2/3 15 še	with 1 še 'grain' = 1/180 gín 'shekel'

Anyway, it is clear that all kinds of experimenting with expressions of this kind for reciprocals in more and more difficult cases, would soon have had to be abandoned. Out of sheer frustration with the situation, the concept of sexagesimal numbers in place value notation may have been invented.

Maybe it was just in order to demonstrate the superiority of the newly invented sexagesimal numbers in place value notation that an Ur III table of reciprocals such as Erm 14645 (Friberg, *CDLJ* 2009:3, Fig. 8) contained the two extra pairs of reciprocals (1 04, 56 15) and (1 21, 44 26 40). Indeed, without the use of place value notation, those two pairs would have had to be expressed inelegantly, for instance as

(1(60) 4, 5/6 6 45 še) and (1 1/3 1, 2/3 4 2/3 20 še).

Actually, it is not at all unlikely that the person who first got the idea of working with abstract numbers and to construct a table of reciprocals like SM 2685 was also the one who ultimately invented sexagesimal numbers in place value notation, and who made the first combined multiplication table.

This bold conjecture is, to some extent, supported by the following series of observations: First, in her survey in *TMH 8* (2008), 19-21, of the six then known Ur III tables of reciprocals, Proust writes:

> "Another characteristic of the Neo-Sumerian tablets concerns their material aspect. They are very small tablets, extremely carefully fabricated, and with a particularly minute writing. They are written in a crowded fashion in two columns. One notes, moreover, a quite large variability in these old tablets, which is in contrast to the very strong standardization of the very numerous Old Babylonian tables of reciprocals .... The characteristic traits revealed above, in particular their finesse and their variability, suggest that the tables of reciprocals from the Neo-Sumerian period belong to the sphere of advanced scholarship, maybe they are witnesses of a phase of elaboration of a new scholarly system. It was not until at a later stage, in the Old Babylonian period, that these scholarly elaborations would pass over into the domain of elementary education." (My translation.)

Ideas of a similar kind were aired by Proust already in *TMN* (2007), 134, in a paragraph with the title "More ancient tables", where she considers the two tablets Ist. Ni 2208 (a multiplication table for the non-regular head number 7) and Ist. Ni 5173 (a "numerical table", in this case a table of reciprocals followed by a combined multiplication table). In both texts, variant number signs are used for 4, 7, 8, and 9 (but not for 40 and 50). About these texts Proust wrote:

> "By the way, these two tablets are of a superb quality of manufacture, and as in the case of the table of reciprocals Ni 374 described above, it is difficult to believe that they were the work of children. The numerical tables seem, in the Ur III period, to belong to the field of scholarship and not to that of school exercises." (My translation.)

Thus, in Proust's opinion, Ist. Ni 2208 may be the oldest known example of a cuneiform multiplication table, possibly from the Ur III period, which is a quite reasonable hypothesis. More surprising is the claim that also Ist. Ni 5173 (Fig. 13.6.1 below) may be from the Ur III period, but it too may very well be true.

Unfortunately, only traces remain of the table of reciprocals Ist. Ni 5173. With the incipit and the end of the table missing, there is no difference between the layout of Ist. Ni 5173 and the layout of a standard Old Babylonian table of the same kind. Ist. Ni 5173 is even like all Old Babylonian combined multiplication tables from Nippur in that it lacks a multiplication table for the head number 48! (See the survey in Proust, *TMN* (2007), 133, of the few known multiplication tables for 48 from other Mesopotamian sites.)

As mentioned, according to Proust, Ist. Ni 2208 may be an Ur III single multiplication table, in view of its superb quality of manufacture, and because it makes use of variant number signs. Yet, its layout is like that of an Old Babylonian multiplication table of type A (Neugebauer and Sachs, *MCT* (1945), 20), with the first line in the form 7 a.rá [1 7], and with all the other lines of the form a.rá *n* 7 *n*. Also the layout of the combined multiplication table on Ist. Ni 5173 (Fig. 13.6.1 below) is indistinguishable from the layout of an Old Babylonian combined multiplication table.

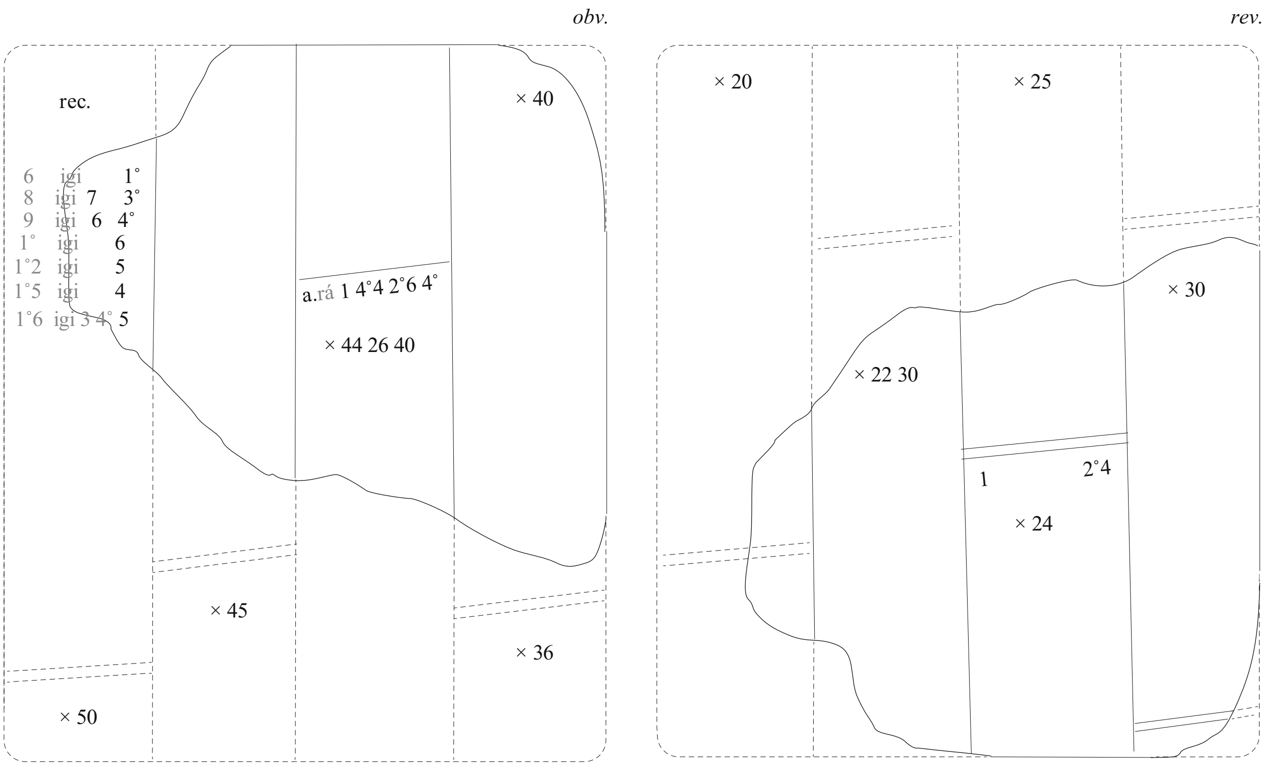


Fig. 13.6.1. Ist. Ni 5173. A table of reciprocals and a combined multiplication table, from Ur III(?) Nippur.

Incidentally, there are two other known multiplication tables with variant number signs, which do have layouts deviating from the Old Babylonian norm. One of them is MS 3849 (Friberg, *MSCT 1* (2007), 92; Fig. 13.6.2 below). The obverse of that tablet is divided into two columns. In the first column is a multiplication table for the head number 50, the second column is empty. The reverse is blank.

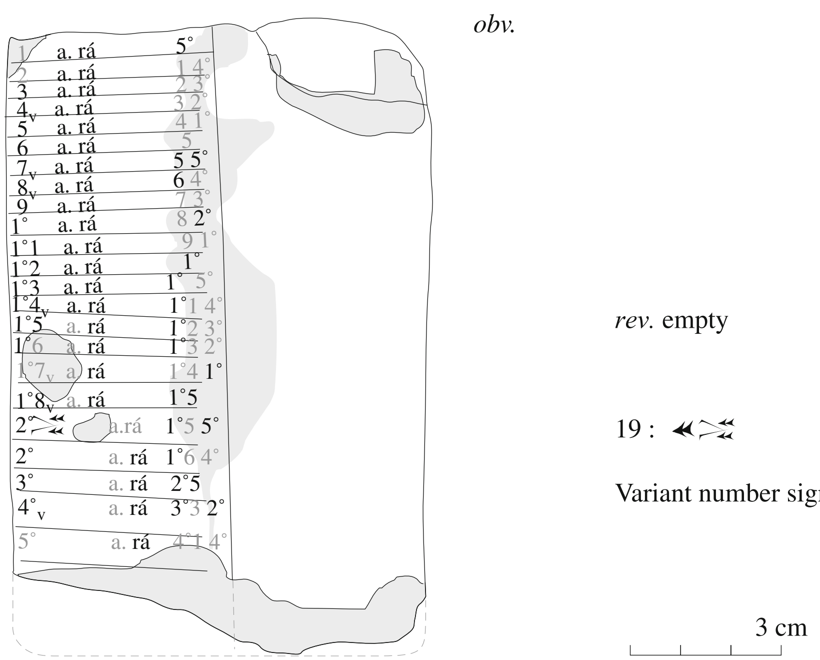


Fig. 13.6.2. MS 3849. An Ur III(?) multiplication table for the head number 50.

MS 3849 is not a tablet of type II, inscribed in the first column of the obverse with the teacher's model text (see the discussion of such tablets in Proust, *TMN* (2007), 21). Indeed, the table in the first column was probably produced by an inexperienced student. Instead, it is likely that is *an unfinished version of what was intended to be a multiple multiplication table, written in two columns like six of the eight known Ur III tables of reciprocals.* (Compare with the three Old Babylonian multiple multiplication tables in Friberg, *MSCT 1*, Figs. 2.6.8-9, which are written on single column tablets.) In addition, the multiplication table in the first column of MS 3849 is of *a new type, not known from any other cuneiform multiplication tables*, with each line of the form *n* a.rá 50 *n*. (See the survey of all previously known types in *MCT,* 20.)

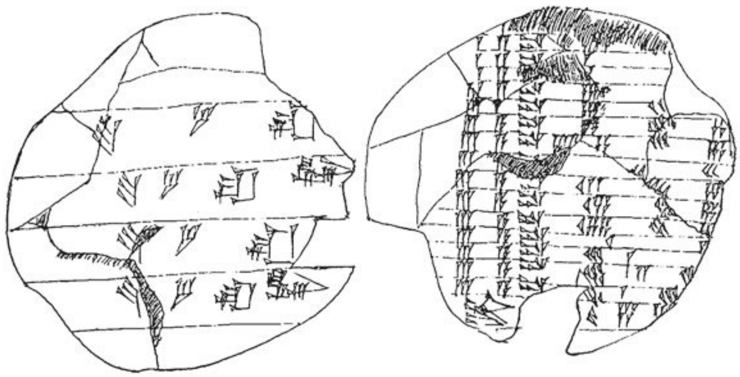
The other known example of a multiplication table with variant number signs and a layout deviating from the Old Babylonian norm is van der Meer's *MMAP 27* (1935), no. 61. This is a round hand tablet from Susa with a simple writing exercise on the obverse, and a multiplication table for the head number 3 on the reverse. The multiplication table is followed by the extra line

íb.[sá] 25 [5]	the square-side of 25 is [5]

This text was discussed already by Neugebauer, in *MKT 3* (1937), 50. The hand copy made by van der Meer (Fig. 13.6.3 below) has several (probable) copying errors. Those errors have been corrected in the transliteration to the right of the hand copy.

obv. *rev.*

šà gar.ra
šà gar.ra lugal
šà gar.ra
šà gar.ra lugal



⸢3⸣	a.rá	[3	9]
⸢3⸣	a.rá	4	12
⸢3⸣	a.rá	[5	15]
3	a.rá	6	[18]
3	a.rá	7	2[1]
3	a.rá	8	24
3	a.rá	9	27
3	[a.rá]	10	30
3	a.rá	12	36
3	a.rá	15	45
3	a.rá	18	54
3	a.rá	20	1
3	a.rá	30	1 [30]
3	a.rá	40	2
3	a.rá	50	2 30
íb[sá]		25	⸢5⸣

Fig. 13.6.3. ***MMAP 27*, 61. An Ur III(?) atypical multiplication table from Susa, head number 3.**

In this multiplication table, variant number signs are used for 7, 8, 9, but not for 40 and 50. What is particularly interesting is that this is the only known cuneiform multiplication table where the head number is repeated in all lines of the table. Moreover, in this multiplication table only the regular numbers 12, 15, and 18, of all integers between 10 and 20, are used as multipliers of 3. (The regular number 16 is missing, which may be an unintended oversight.) With the exception of MS 3849 above, this is the *only known atypical cuneiform multiplication table*, which makes it natural to suspect that the text is from the Ur III period, possibly the earliest known multiplication table. Indeed, Neugebauer mentioned (*op. cit.*) that "according to the writing style the tablet is from about the Ur III time".

Neugebauer also made the remark that *even the extra line in this text is atypical*, in that it mentions a square root different from the head number, and makes use of the previously undocumented layout íb.sá sq. *n* *n*, instead of the normal layout sq. *n*.e *n* íb.sá (see Neugebauer, *MKT 1* (1935), 75).

13.7 Counting with Loan and Interest in a Proto-Cuneiform Text

In YBC 4698 §§ 1a-b (see Sec. 11.1.2 above) two common Old Babylonian interest rates are mentioned, namely 1/3 and 1/5 of the principal. This corresponds to a rate of *loan plus interes*t of 1 1/3 in the first case and of 1 1/5 in the second case. Interestingly, 1 1/3 and 1 1/5 are both *sexagesimally regular numbers of the form integer plus fraction*, just like the numbers in the first column of the atypical table of reciprocals SM 2685.

Numbers of the form integer plus fraction occur also in the Early Dynastic/Early Sargonic metro-mathematical recombination text CUNES 52-18-035 discussed in Sec. 12.1 above. The number occurring as a multiplier in the first exercise of that text is the sexagesimally non-regular number 1 2/3 5, but in all the subsequent mathematical exercises of the text the multiplier is instead the sexagesimally regular number 1 2/3. It is not clear if the mentioned numbers in this case, too, can be explained as loan plus interest.

Recently, a number of proto-cuneiform texts from the Uruk III period (ca. 3200-3000 BC), allegedly concerned with loan and interest as indicated by the signs UR_{5a} and TAR_a, have been identified and discussed in Monaco *ZA* 102 (2012). One particularly interesting such text, published by Monaco in *CUSAS 1* (2007) (see the photos at cdli.ucla.edu/P283918), is shown below.

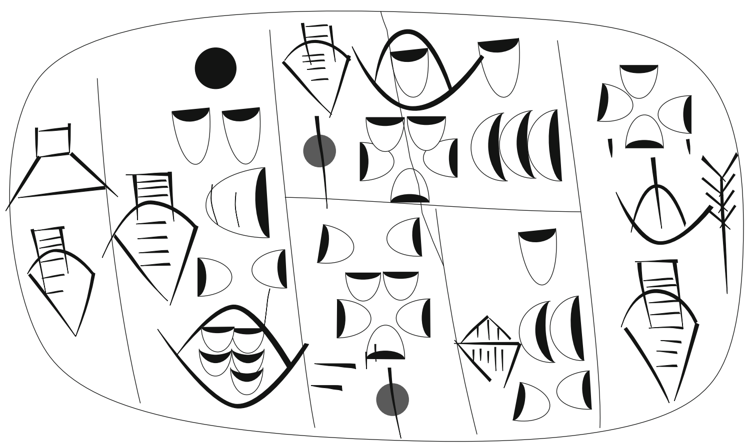


Fig. 13.7.1. *CUSAS 1*, 143. A proto-cuneiform text with a complicated calculation of a loan plus interest.

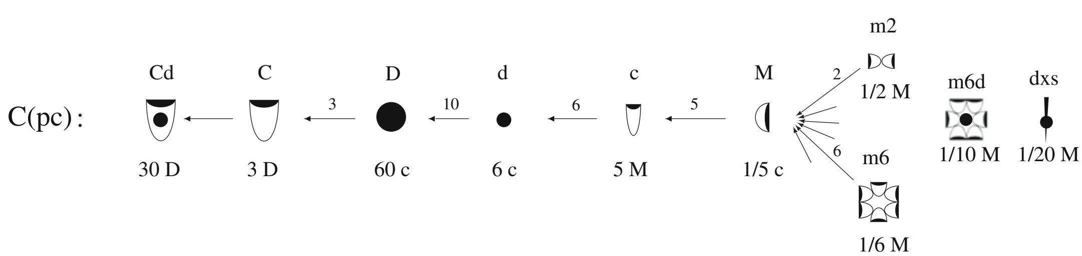


Fig. 13.7.2. Factor diagram for the proto-cuneiform system C of capacity numbers.

In this text, two exclusively proto-cuneiform systems of measure notations are used. One is the proto-cuneiform system T of notations for *time numbers*, composed of the sign U_4 'sun, day' and various ordinary number notations. Thus, in column *i* of *CUSAS 1*, 143, the sign u_4™stroke in column *i* stands for 1year, while the signs u_4™1c in column *ii* and u_4™5c in column *iii* stand for 1month and 5months, respectively. The other system of measure notations figuring in *CUSAS 1*, 143 is the proto-cuneiform system C of number notations for *capacity measures*. The structure of that system of number notations is succinctly described by the factor diagram in Fig. 13.7.2 above. Note that the meaning of the sign d™s(stroke) was only recently recognized, by Monaco in *CDLB* 2006:1.

The meaning of text box *i* of *CUSAS 1*, 143 appears to be that for the whole year the daily ration of some non-liquid barley product, kept in jars without a spout, was

1m4 = 1/4 M = 1/20 c.

(See Monaco, *ZA* 102, footnote 50 for a discussion of the strange way in which the sign for '1year' may be used to refer to daily rations.)

In text box *ii*:a, the following amount of the same barley product is recorded:

1c 3M 1m5 1d×s.

As indicated explicitly in the two sub-boxes *ii*:b1-2, this number can be explained as

1c 2M 1m2 + 1m2 1m5 1d×s = (1 1/10) · 1c 2M 1m2.

Indeed,

1c 2M 1m2 = 7 M 1m2 = 15 m2, and 1/10 · 15 · m2 = 15 · 1/20 M = (10 + 4 + 1) d×s = 1m2 1m5 1 d×s.

In text box *ii*:b1, the amount 1c 2M 1m is qualified as UR_{5a} 'loan', while in text box *ii*:b2 the tenth of that amount is qualified as TAR_a 'interest'.

Now, note the mention of the time number 1month in the text box *ii*:a. This explains the number 1c 2M 1m2 in the sub-box *ii*:b1, since, as is well known, the structure of the system T of proto-cuneiform time numbers is such that 1month = 30 · 1day. Therefore,

1month · (1m4 per day) = 30 days · (1 m4 per day) = 30 · 1m4 = 7 1/2 · 1M = 1c 2M 1m2.

Clearly, then. the meaning of the text in column *ii* on *CUSAS 1*, 143 is that 1 month's worth of the daily rations of 1m4 of a certain barley product mentioned in column *i*, plus an interest of a tenth, is the number recorded in text box *ii*:a.

However, the purpose of the whole text seems to have been to calculate *5 month's worth of the mentioned daily rations, plus an interest of a tenth*. The result of this calculation is recorded in text box *iii*, which mentions the time number 5months, the capacity number 1d 2c 1M 1m2, and the barley product represented by a spoutless jar. Presumably, the calculation would have been performed as follows:

5 · 1c 3M 1m5 1d×s = 5c + 3c + 1M + 1m4 = 1d 2c 1M 1m4.

Note that the capacity number recorded in text box *iv* is 1d 2c 1M 1m2. The discrepancy is probably due to a scribal mistake done when copying the number after a computation.

13.8 A New Explanation of the Head Numbers in Combined Multiplication Tables

It is important that texts like Ist. Ni 2208, Ist. Ni 5173, MS 3849, and *MMAP* 27, 61 seem to indicate that *not only tables of reciprocals but also multiplication tables, combined multiplication tables, and even tables of square-sides, may have been around already in the Ur III period*. That means that *all these various kinds of arithmetical tables may have been invented simultaneously.*

Furthermore, an intriguing possibility is that *there may be some kind of connection between the atypical table of reciprocals SM 2685 discussed in Sec. 1 above and the list of head numbers in a combined multiplication table* like the one in Fig. 13.6.1 above. It is, of course, already well known that there is some kind of connection between that list of head numbers and the reciprocals rec. *n* in a cuneiform table of reciprocals.

Here is the list of the 39 documented head numbers in Old Babylonian multiplication tables:

50, 48 (rare), 45, 44 26 40, 40, 36, (not 32!), 30, (not 27!), 25, 24, 22 30, 20, 18, 16 40, 16, 15, 12 30, 12, 10, 9, 8 20, 8, 7 30, 7 12, 7, 6 40, 6, 5, 4 30, 4, 3 45, 3 20, 3, 2 30, 2 24 (rare), 2, 1 40, 1 30, 1 15 (rare), 1 12 (rare).

In Friberg, *MSCT 1* (2007), Sec. 2.6 f, an attempt was made to explain this list of documented Old Babylonian head numbers with departure from the entries in the Old Babylonian standard table of reciprocals. The argumentation proceeded as follows:

1. 50 was the first head number because a multiplication table was needed for the recently introduced basic fraction 5/6 = '50'.
2. Most of the head numbers were taken, in a quite complicated way, from numbers *n* or rec. *n* in the standard table of reciprocals. Special reasons were given for why 1 21 and both *n* and rec. *n* in the pairs (27, 2 13 20), (32, 1 52 30), (54, 1 06 40) were not chosen as head numbers.
3. Various contrived reasons were given for the inclusion of the following seven "extra head numbers": 22 30, 16 40, 12 30, 8 20, 7 12, 7, 4 30.

Now, however, a much simpler explanation of the head numbers in in the Old Babylonian combined multiplication table can be given. Indeed, *suppose that the first combined multiplication tables, just like the first tables of reciprocals, appeared in the Neo-Sumerian Ur III period soon after, or perhaps somewhat before the invention of sexagesimal numbers in place value notation. Then the list of head numbers may have been based, not on the numbers in a standard table of reciprocals of the Old Babylonian type, but on an atypical table of reciprocals of the same type as SM 2685* in Sec. 13.1 above.

More precisely, as will be shown below, the list of head numbers seems to have been created with departure from the reciprocal numbers in the right hand part of a *precursor* of SM 2685. *In that precursor table, the four extra lines close to the end of SM 2685 had not yet been included* (see Sec. 13.1.3 above), so that the table of reciprocals ended with the pairs (54, 1 6 2/3), (1, 1) and (1(60) 21, 2/3 4 1/3 6 2/3). *Neither were the non-regular numbers 5 1/2 and 7 and their approximate reciprocals included*. On the other hand, in the presumed precursor table, *two reciprocal pairs which apparently had been lost in SM 2685 were still present*, namely (4 2/3 8, 12 1/2), and (6 2/3, 9). See the reproduction below of this assumed precursor table.

In this scenario, the construction of the list of head numbers can be explained as follows: There were 54 reciprocal numbers rec. *n* in the right column of this atypical table of reciprocals. Of these 54 reciprocal numbers, *6 were rejected because they were either too complex (involving fractions of a shekel) or too simple (the uninteresting number 1)*. There remained 48 reciprocal numbers in the right column of the presumed atypical table of reciprocals. Then *one extra head numbers* was added, namely 7, possibly because a multiplication table for this special number was already in circulation. Remember that Ist. Ni 2208 is an Ur III multiplication table for the non-regular number 7, published and discussed by Proust in *TMN* (2007), 134 and pl. III.

The proposed connection between the hypothetical precursor of the atypical table of reciprocals SM 2685 and the list of head numbers is clarified by the comparison below.

the *original table of reciprocals			head numbers	
1 12	50		50	
1 15	48		lost	
1 1/3	45		45	
1 1/2	40		40	
1 2/3	36		36	
2	30		30	
2 15	26 2/3		lost	
2 1/3 4	25		25	
2 1/2	24		24	
2 2/3	22 1/2		22 1/2	
3	20		20	
3 1/3	18		18	
3 1/2 6	16 2/3		16 2/3	
3 2/3 5	16		16	
4	15		15	
4 1/2	13 1/3		lost	
4 2/3 8	12 1/2	lost in SM 2685	12 1/2	
5	12		12	
5 1/2	11 15		lost	
6	10		10	
6 2/3	9	lost in SM 2685	9	
7 12	8 1/3		8 1/3	
7 1/2	8		8	
8	7 1/2		7 1/2	
8 1/3	7 12		7 12	
——	——		**7**	extra
9	6 2/3		6 2/3	
9 1/2 6	6 15		lost	
10	6		6	
10 2/3	5 1/2 7 1/2	too complex	rejected	
11 15	5 1/3		lost	
12	5		5	
12 1/2	4 2/3 8		lost	
13 1/3	4 1/2		4 1/2	
13 1/2	4 1/3 6 2/3	too complex	rejected	
15	4		4	
16	3 2/3 5		3 2/3 5	
16 2/3	3 1/2 6		lost	
18	3 1/3		3 1/3	
20	3		3	
22 1/2	2 2/3		lost	
24	2 1/2		2 1/2	
25	2 1/3 4		2 1/3 4	
26 2/3	2 15		lost	
27	2 13 1/3	too complex	rejected	
30	2		2	
32	1 5/6 2 1/2	too complex	rejected	
36	1 2/3		1 2/3	
40	1 1/2		1 1/2	
45	1 1/3		lost	
48	1 15		1 15	
50	1 12		1 12	
54	1 6 2/3	too complex	rejected	
1(60)	1	too simple	rejected	
1(60) 21	2/3 4 1/3 6 2/3		2/3 4 1/3 6 2/3	

54 reciprocals
5 of them too complex
1 of them too simple

38 head numbers
11 lost head numbers
1 extra head number

The result of the mentioned construction was a list of *48 head numbers in this (hypothetical) original Ur III (or earlier) combined multiplication table.* Presumably, this combined multiplication table was itself combined with an initial atypical table of reciprocals. However, rather soon after the construction of the first combined multiplication table with these 48 head numbers, *the head number 48 and ten more seem to have been lost in the process of copying and re-copying the combined multiplication table*. *Ultimately, the original table of reciprocals was replaced by its successor, the (Ur III)/Old Babylonian standard table of reciprocals, and sexagesimal place value numbers were introduced in the combined multiplication table.* The result was a combined "igi-a.rá table" like Ist. Ni 5173 in Fig. 13.6.1 above. (The few known instances, where 48 is actually used as a head number can be explained as later additions to the standard list of head numbers.)

One advantage of this new explanation of the construction of the 38 head numbers of the (Ur III)/Old Babylonian combined multiplication table is that *no further explanation is needed for why the head numbers form a decreasing list of numbers beginning with 50 and ending with 1 12. There is no need to explain why the numbers 32 and 27 are conspicuously absent from the list of known head numbers, and there is also no need to explain why the head numbers 22 30, 16 40, 12 30, 8 20, 7 12, and 4 30 are included in the list, although those numbers are not present in the Old Babylonian standard table of reciprocals.*

These new findings strongly suggest that the invention in the Ur III period (or maybe even earlier) of the atypical table of reciprocals making use of basic fractions ultimately lead to the further invention of combined multiplication tables, sexagesimal place value numbers, and probably also all the other types of arithmetical and metrological tables, known so far only from the Old Babylonian period..

In this connection it is worth mentioning that it is now known that difficult problems involving quadratic equations were investigated and solved as early as in the Ur III period, possibly already in the Old Akkadian period (and maybe even earlier). See the discussion of the division of an irregularly formed orchard into five lots, all of the same area measure, in the legal document YBC 3879, and the related discussion of the division of a trapezoid into two equal parts in the Old Akkadian hand tablet IM 58045 (Friberg, *CDLJ* 2009:3).

It is also known (see Friberg, *MSCT 1* (2007), 192-195 and 401-403) that for more than a thousand years before the Ur III period, a method which may be called the "proto-literate field expansion procedure" was used on clay tablets from various periods in Mesopotamia in order to solve quadratic "form and magnitude problems" of the kind appearing in the exercises IM 121613 # 1 (Sec. 5.1.1 above) and IM 52685 § 1e (Sec. 10.2.3).

Various other "metric algebra problems" are known from small Old Akkadian metro-mathematical clay tablets (see Friberg, *op. cit.*, A.6.1-A.6.4). The oldest known examples of regular sexagesimal (length) numbers appear in some of those texts. Also from the Old Akkadian period is CUNES 47-12-176, the oldest known metrological list, a decreasing list of weight measures finishing with an amazing sequence of complicated expressions for fractions of a shekel (Friberg, *op. cit.*, A.7.4).

Quite a few metro-mathematical "tables of areas of squares or rectangles" from the Old Dynastic III and Old Akkadian periods are now known. In particular one of them, CUNES 50-08-001, displays an even more impressive variety of expressions for fractions of basic length and area numbers (Friberg, *op. cit.*, App. 7).

On top of all this, the Early Dynastic/Early Old Akkadian metro-mathematical text CUNES 52-18-035 with its commercial division and multiplication problems (Ch. 12 above) is now the earliest known combined mathematical problem text. It is of particular interest not least because it counts with numbers of (essentially) the same type as the numbers in the atypical table of reciprocals SM 2685, and because it makes use of terms like sag 'initial amount' and en.nam 'what?' which appear frequently in Old Babylonian mathematical texts.

To sum it all up, all these recent developments in the study of the history of Mesopotamian mathematics makes it increasingly evident that *a well organized education in mathematics and metrology was a significant part of the Mesopotamian culture not only during the Old Babylonian period, but also continually during the whole preceding millennium, or more.*

14. Fragments of Three Tablets from Ur III Nippur with Drawings of Labyrinths

By Jöran Friberg and Anthony Phillips

The two almost perfectly preserved clay tablets MS 3194 and MS 4515, with drawings of a rectangular labyrinth in the former case and of a square labyrinth in the latter, were published and thoroughly analyzed in Friberg, *MSCT 1* (2007), Sec. 8.3. Since then, the two clay tablets have been baked and cleaned, and new, much improved photos of them have been published in George, *CUSAS 18* (2013), nos. 39-40, and online at cdli.ucla.edu, nos. P253616 and P274587. In addition, it has been realized that the hand copies of the two tablets presented in *MSCT 1* contain small but significant errors: it is now clear that for both kinds of labyrinth it is not correct that there is one “good” path from an entrance on one side ending close to the center of the labyrinth and a second “bad” path from an entrance on the opposite side, ending not so close to the center. Instead, in both cases, there is only one convoluted path, from one entrance to the other.

This new result has already been announced in Friberg, *NAMS* (2008), in Brunke, *MPIHS 430* (2012), 77, and in George, *CUSAS 18* (2013), 277. According to Brunke, there is only one path through the “array” on both MS 4515 and MS 3194 , “entering the array on one side, spiraling towards the center, turning around, spiraling out again and leaving the array on the other side”. As Brunke remarks, that makes the path in both cases, using a modern mathematical term, into a “surface-filling band” (in fact, this property holds for almost every labyrinth ever drawn). According to George, on the other hand,

> “The central part of both designs is a square coil that loops back on itself at the centre. This is in essence a rectangular version of a feature found in many drawings of lambs’ colons. For that reason, it might be considered that these designs are schematic depictions of imaginary colons for use in instruction in extispicy. However, there are two serious objections: (a) in the spiral drawings that certainly represent lambs’ colons, entry and exit fall together, in accordance with a sheep’s anatomy, whereas here they are diametrically opposed; (b) the colon count of both, measured across the diagram from entry to exit, would be unusually high—nineteen on MS 4515, twenty-one on MS 3194.”

As mentioned in *MSCT 1*, there is no way of directly dating the two tablets MS 3194 and MS 4515, but since the overwhelmingly great majority of mathematical clay tablets in the Schøyen Collection are unmistakably Old Babylonian, it is quite likely that these two labyrinth texts are also Old Babylonian.

In view of what is said above, it may be a good idea to republish MS 3194 and MS 4515. This is done below, in Secs. 14.1 and 14.2. A further reason for republishing those two Old Babylonian(?) labyrinth texts is that they can now be compared with what appear to be fragments of three Neo-Sumerian labyrinth drawings, but of a wholly different kind, from Ur III Nippur. These are discussed below in Secs. 14.3 and 14.4.

A hand copy of the largest of these fragments, 6N-T 428, has already been published, by Heinrich and Seidl in *MDOG 98* (1967), Fig. 11. They interpreted the drawing on the fragment as part of a house plan with measurements, observing that the longish rooms shown in the house plan could be taken for storage rooms, were not their mutual arrangement unsuitable for the purpose. The fragment 6N-T 428, as well as a few similar but smaller fragments, together called 6N-T 478, belong to the Iraq Museum, Baghdad, and are not readily accessible for study. Fortunately however, thanks to McGuire Gibson, we have a good photograph of 6N-T 428 from the Oriental Institute in Chicago. It is reproduced in Fig. 14.3.1 below.

Three additional similar, small fragments from Nippur make up item 6N-T 669 in the Yale Babylonian Collection. Their photographs (Fig. 14.4.1 below), as well as scanned copies of gypsum casts of the fragments 6N-T 478 (Fig. 14.4.3), were kindly made available to us by Ulla Kasten.

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7_14

14.1 MS 3194. A New Interpretation of the Old Babylonian(?) Rectangular Labyrinth

In Fig. 14.1.1 below, the new photo of MS 3194 after cleaning is shown to the right; to the left is shown a hand copy of the tablet with an obvious reconstruction of some missing parts near the upper edge.

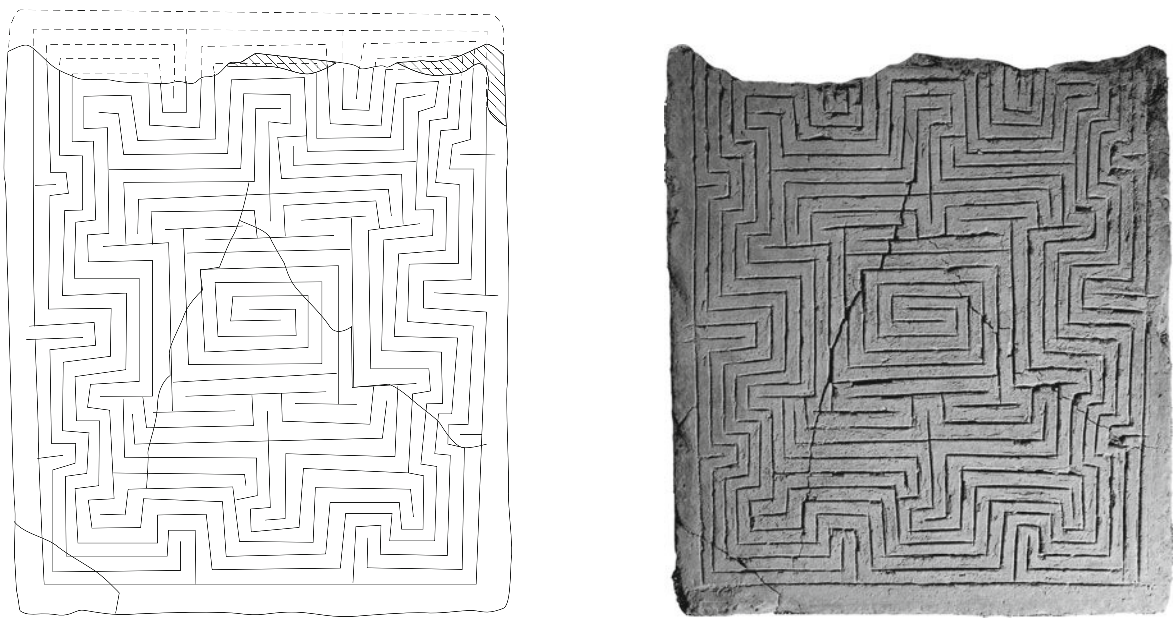


Fig. 14.1.1. MS 3149. Photograph of the tablet after cleaning.

The walls of the labyrinth have two connected components (to use the language of topology; this is a consequence of there being a single path traversing the entire labyrinth: one wall must be on the left, and the other on the right, all the way through). This is shown in Figure 14.1.2 below.

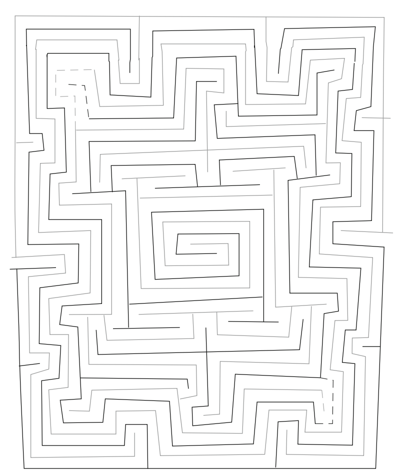


Fig. 14.1.2. MS 3194. The labyrinth walls have two connected components. Above, one is drawn in black, the other in gray.

It can be noted that the walls of this labyrinth were very skillfully *drawn by hand*, *without the support of an underlying grid of lines.* In fact, an attempt to give the labyrinth a constant path width by following an underlying grid is doomed to fail. This is so because the labyrinth has a different number of horizontal levels in the center and towards the sides: an overall rectangular shape is incompatible with constant path width, as illustrated in Fig. 14.1.3 below. An incidental consequence is that Brunke's interpretation of the path through the labyrinth as a surface-filling curve of constant width cannot be completely correct.

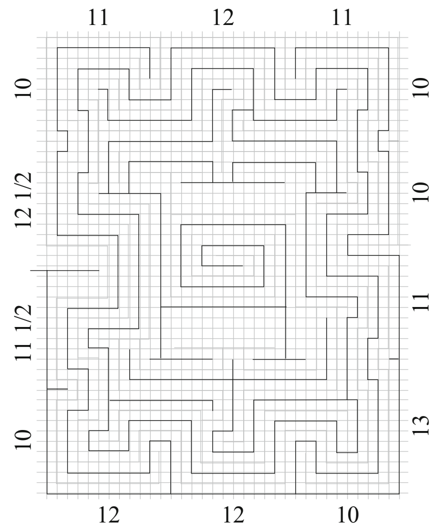


Fig. 14.1.3. MS 3194. MS 3149. The number of horizontal levels varies across the labyrinth. This figure, comparing the labyrinth with a uniform rectangular grid, shows that it could not have been drawn with uniform path width.

The rectangular labyrinth has *nearly perfect 180º rotational symmetry*. In fact, a rotation of the labyrinth by 180˚ has essentially no other effect than letting the two walls replace each other. This remarkable feature of the labyrinth is illustrated in Figure 14.1.4 below.

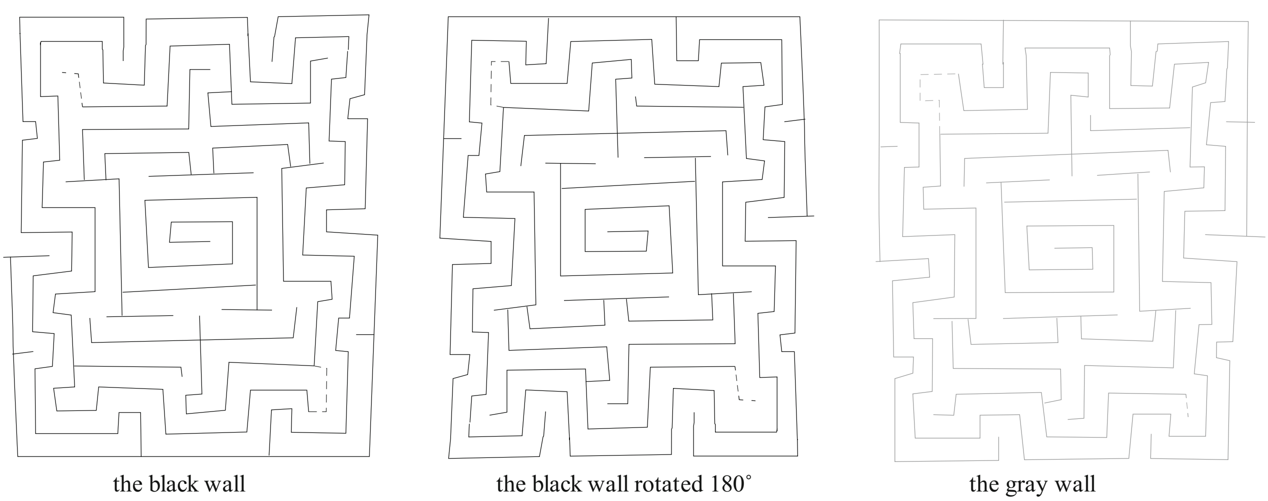


Fig. 14.1.4. MS 3194. This illustration shows the black wall, the black wall rotated 180˚, and the gray wall. The second and third diagrams are essentially identical. The only (insignificant) exceptions are the small pieces shown with dotted lines.

In addition to the rotational symmetry, the labyrinth on MS 3149 has an additional, purely aesthetic feature. The top and bottom walls are punctuated by the emergence of branches, which cause crenelation in the nearby paths, similar to the perturbation caused by the entrances. To carry this wavy effect uniformly around the border, the designers have added two spikes to each of the side walls, halfway between corners and entrances. They do not contribute to the complication of the maze, but add to the elegance of the overall pattern.

Note that the way in which the drawing of the rectangular labyrinth on MS 3194 may have been constructed, by use of an intricate algorithmic procedure in 36 steps, is shown in *MSCT 1*, Figs. 8.3.10-11. Only the final step of the algorithmic procedure now has to be changed for the convoluted path through the labyrinth to go all the way from one entrance to the other.

14.2 MS 4515. A New Interpretation of the Old Babylonian(?) Square Labyrinth

In Fig. 14.2.1 below, the new photo of MS 4515 after cleaning is shown to the right, and to the left is shown a hand copy of the tablet with the drawing.

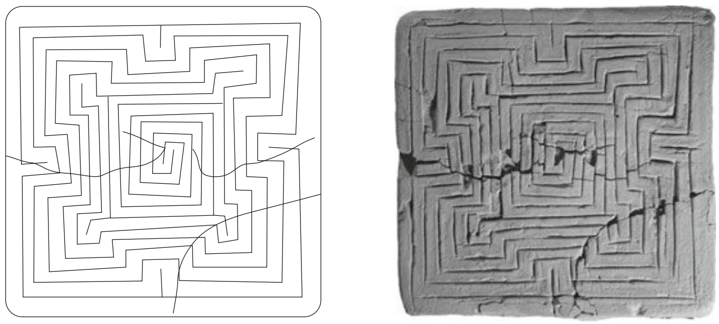


Fig. 14.2.1. MS 4515. An Old Babylonian(?) square labyrinth. Photograph of the tablet after cleaning.

This labyrinth has many common features with the one on MS 3194, but there are also some significant differences. The overall right-angled, crenelated organization is the same; in both cases there are two walls (drawn as black and gray in Figs. 14.1.2, 14.1.3, 14.1.4, 14.2.2, 14.2.4), which enter from opposite sides and incorporate a counter-clockwise spiral into the center. The resulting path enters on one side, follows a path into and then away from the center, and ends up exiting from the opposite side. In both cases the pattern was clearly drawn free-hand, without following an underlying rectangular grid. (This is in contrast with the neo-Sumerian labyrinthine tablet fragments considered in Sections 14.3 and 14.4). Again, the topology of the pattern precludes a constant path width (Fig. 14.2.2, below).

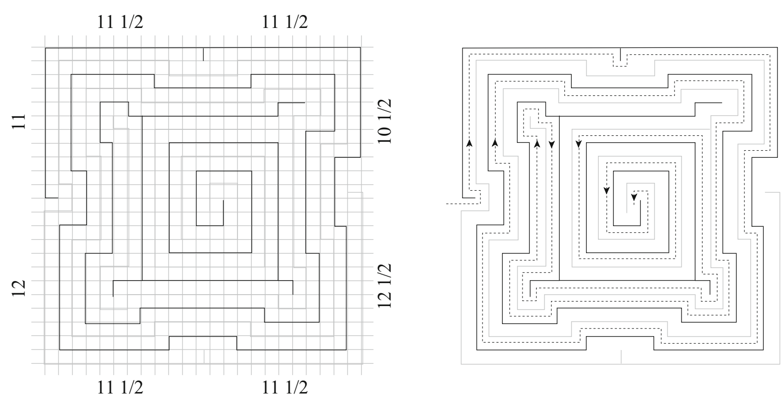
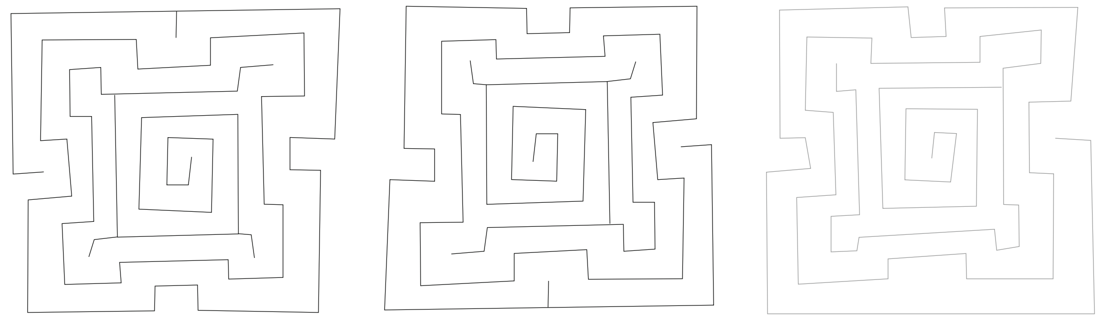


Fig. 14.2.2. MS 4515. Left: superposition of the labyrinth pattern on a uniform rectangular grid shows that constant path width is impossible. Right: the first half of the path through the labyrinth, entering from the left.

The labyrinth on MS 4515 is square, the one on MS 3194 is not. The labyrinth on MS 3194 is considerably more complicated: the top and bottom walls are punctuated by the emergence of branches, which penetrate deep into the pattern. On MS 4515 these branches are reduced to vestigial twigs, stretching just far enough to mimic the indentations caused by the entrances. Besides, it is clear that *the labyrinth on MS 3194 was drawn with a sure hand by someone with a considerable experience of drawing labyrinths, while the labyrinth on MS 4515 was drawn clumsily by a beginner.* In fact, it is extremely likely that MS 4515 was meant to be drawn to have 180° rotational symmetry, following an algorithm like that used for MS 3194, but that an error occurred in the execution, resulting in the unsymmetrical dead-end for the black wall shown in the upper left-hand corner in Fig. 14.2.2, right. When the two walls are copied separately, as in Fig. 14.2.3 below, the lack of symmetry becomes obvious.



the black wall | the black wall rotated 180° | the gray wall

Fig. 14.2.3. MS 4515. The black wall, the black wall rotated 180°, and the gray wall. This labyrinth does not have the same rotational symmetry as the labyrinth on MS 3194.

The exact place at which this error occurred can be pinpointed by examining the algorithm that was almost certainly used to generate the pattern. A reconstruction of the way the actual tablet presumably was drawn is presented in Friberg, *NAMS* (2008) (available online at http://www.ams.org/notices/200809/200809-full-issue.pdf) Fig. 13. Fig. 14.2.4 below contrasts the end of that story with the steps that would have generated a symmetrical pattern.

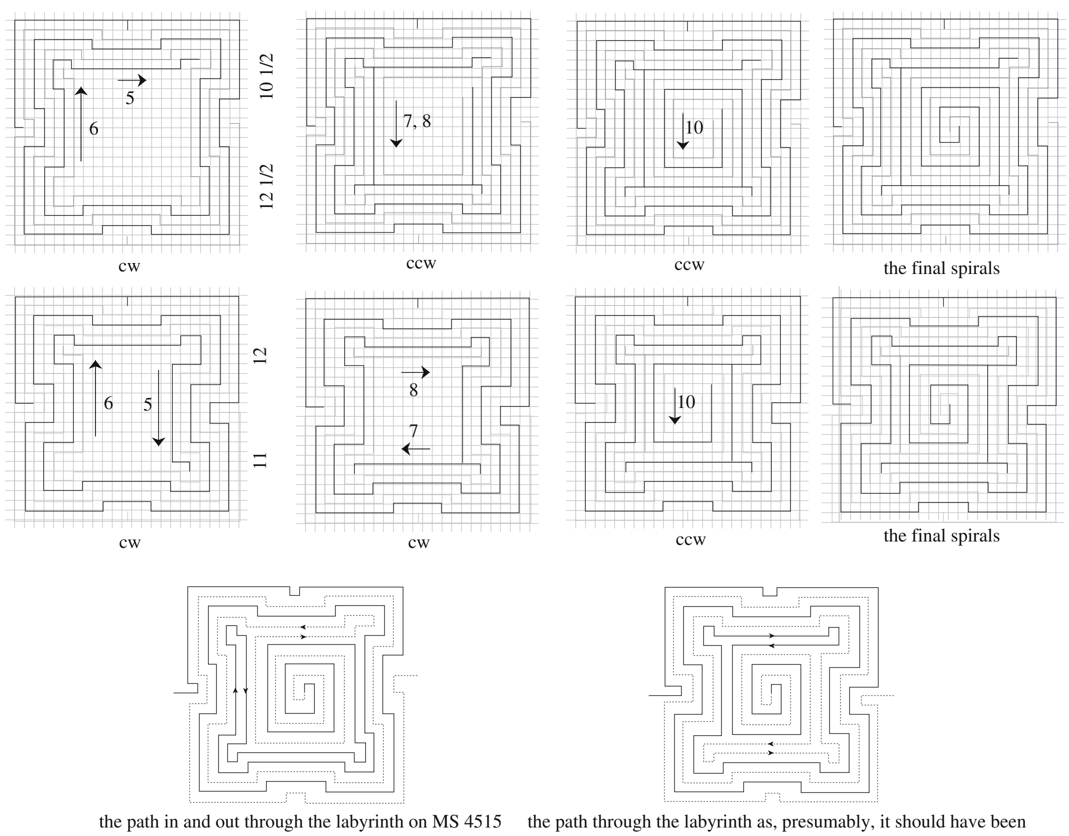


Fig. 14.2.4. MS 4515. Top row: The final steps of the algorithm presumably followed by the executor of MS 4515.
Second row: In parallel, the steps that would have led to a symmetrical labyrinth.
Bottom row: The path through the labyrinth before and after the correction.

Another interesting observation is that if one tries to draw a labyrinth without the indentations around the four gates, the construction will not be easier, and the final result will not be as visually appealing as one would have wished.

It should be noted that the two complete labyrinths on MS 3194 and MS 4515 testify to the existence in deep antiquity of a tradition of labyrinth design that is radically different from anything else that has come down to us. The many labyrinths occurring on loom weights from Francavilla, Cretan coins, and Roman mosaics (see below Sec. 14.6), medieval manuscripts, anything up to the Renaissance, when the first non-labyrinth mazes were invented, are almost without exception derivable from meander designs. The "Cretan maze," a design that probably goes back to prehistory but that has a nicely datable instantiation on the reverse of a Linear B accounting tablet, baked when King Nestor's palace in Pylos (western Peloponnese) burned around 1200 BC (Kern, *LED* (1983), item 104). It is often considered the archetypal labyrinth; it, too, can be read as a double meander in rolled up form (see Figs. 14.2.5 and 14.4.6). Sometimes these labyrinths have no entrance or exit. This happens when the meander path is closed, as on the Francavilla loom weights (Sec. 14.6), or on some of the coins from Knossos (see Fig. 14.6.5). Otherwise, and more commonly, the labyrinth path leads from the outside of the design to a point in the center. The organization we encounter in MS 3194 and MS 4414, where the path enters in the center of one side and exits through the center of the opposite side, in such a way as to give the entire pattern an 180-degree rotational symmetry, is completely alien to this tradition.

names of 10 persons (written in Linear B), each one receiving or delivering one goat, or two

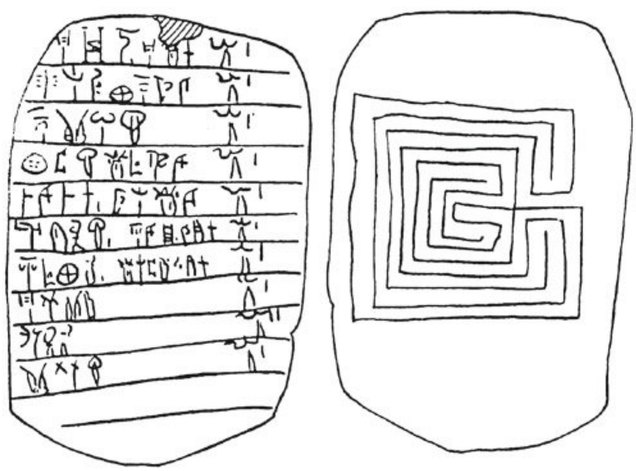
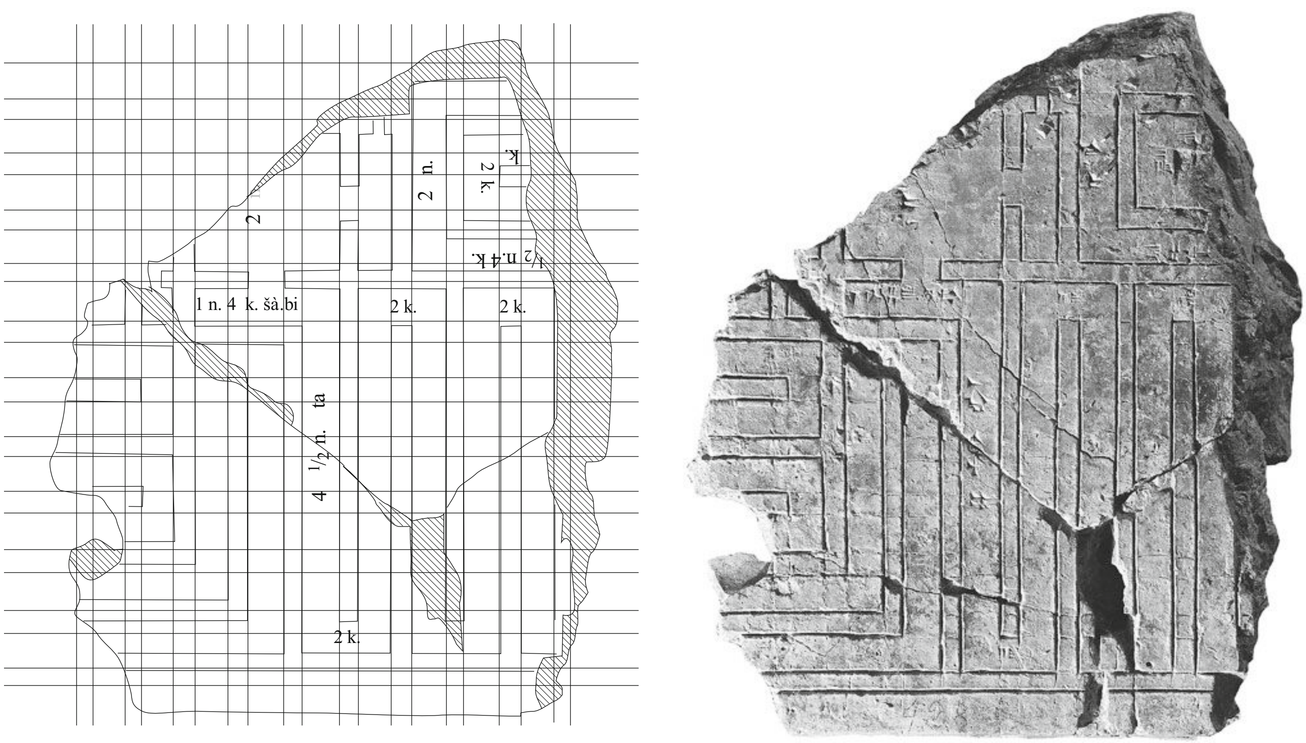


Fig. 14.2.5. Cn 1287. A Cretan maze as a doodle on the reverse of a Linear B accounting tablet, from around 1200 BC, preserved in the National Museum, Athens. Blegen 1958, pl. 46 (drawing by M. Lang).
Courtesy American Journal of Archaeology and Archaeological Institute of America.

14.3 6N-T 428. A Fragment of an Ur III(?) Tablet from Nippur with a Drawing of a Labyrinth

The fragment 6N-T 428 in the Iraq Museum, Baghdad (Heinrich and Seidl, *MDOG 98* (1967), Fig. 11; Fig. 14.3.1 below), is a substantial part of a clay tablet inscribed with what is obviously some kind of labyrinth, disguised as a house plan of the usual kind, as the one in Fig. 14.5.1 below. The walls of the labyrinth are not just drawn as thin lines, as in the case of the Old Babylonian(?) labyrinths on MS3994 and MS 4515. Instead they are very carefully drawn as walls of a constant thickness, given in several places as 2 kùš (2 cubits, equal to about 1 meter). The convoluted path through the labyrinth is shown as a corridor of constant width, clearly 4 kùš (about 2 meters), although this width is not explicitly indicated anywhere. Many traces of an underlying rectangular grid are clearly visible, and the outlines of the walls follow the lines of the grid meticulously. The lengths of the various corridors into which the path is divided are indicated in several places.

In the middle of the plan of the labyrinth there is a square open court, and next to the central court are shown two short pieces of corridor, not part of the path through the labyrinth, leading to two small inner chambers.



6N-T 428 (15.5 x 13.3)

Fig. 14.3.1. 6N-T 428. Fragment of an Ur III(?) Tablet from Nippur with a drawing of a labyrinth.

Concerning the somewhat problematic dating of the large fragment 6N-T 428, McGuire Gibson was kind enough to write the following (personal communication in an email, Oct. 17, 2013):

> "(The fragment 6N-T 428) was found in Parthian fill under the latest version of the Inanna temple (at Nippur). The Parthians built an Inanna Temple above the stack of temples that goes down to the Early Dynastic. This Parthian version was the last traditional Babylonian temple at the site, as far as we know. The Parthians, when they built large buildings, would excavate huge and deep holes down into earlier levels of the site. They then built very deep foundations (4-5 meters) of mud brick and would gradually fill in the space with the debris they had removed from the hole they had made originally or would bring in additional material from other parts of the site. Therefore, you can expect to find in Parthian fill

hundreds of artifacts from whatever periods they took the dirt from. At Nippur, the Parthians dug deep holes in the area north of the ziggurat to make their mud bricks. They went down to as early as the Ubaid period, and they included in their bricks not just the dirt they had excavated but any potsherds and other artifacts they encountered, which gave strength to the bricks. This has resulted in the fact that in any place on the mounds where Parthians built, there are sherds from as early as the Ubaid period."

That leaves us with only the form of the cuneiform signs in the brief lines of text on 6N-T 428 as an indication of the probable date of the fragment, notably the two lines

1 nindan 4 kùš šag$_4$.bi '1 rod 4 cubits its interior (measure)'
4 1/2 nindan.ta 'of 4 1/2 rods '

The language used here is Sumerian. More importantly, the elaborate sign forms used for nindan , kùš, šag$_4$, bi, and ta are all either Ur III (Neo-Sumerian) or Early Old Babylonian. The same can possibly be said of the sign form used for the number 4, which is written as 2+2 wedges, rather than 3+1. Therefore, it feels safe to draw the conclusion that the fragment is from the Ur III period or, at least, Early Old Babylonian.

It is interesting that an *attempted complete reconstruction* of the whole drawing of a labyrinth on the clay tablet, of which 6N-T 428 is a fragment, can be based on *the assumption that the whole labyrinth would look the same after a rotation by 180˚*. In Fig 14.3.2 below is shown (with lines drawn in black) the result when the existing fragment is joined with a copy of the fragment rotated by 180 degrees. Adding *the further assumption that the original labyrinth was a square labyrinth* leads almost inevitably to the full reconstruction shown in both black and dashed gray lines in the same figure.

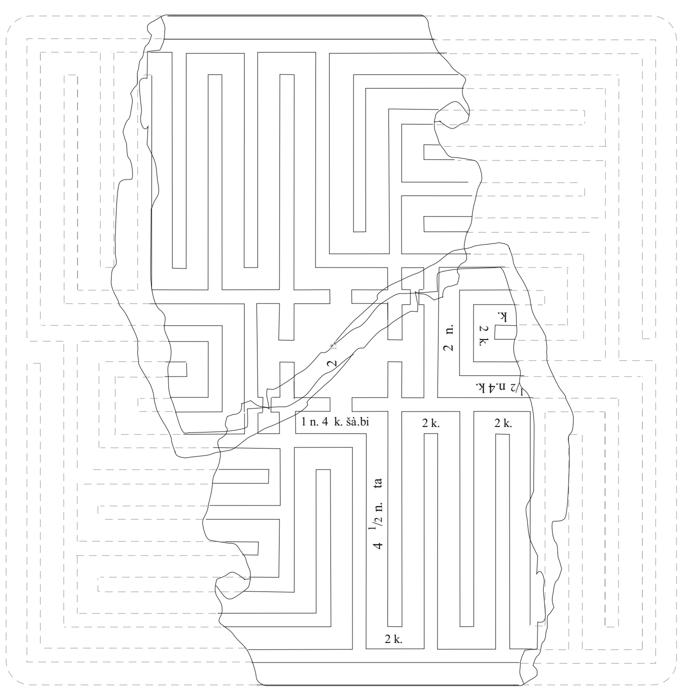


Fig. 14.3.2. 6N-T 428. The proposed reconstruction of an Ur III square labyrinth with 180˚ rotational symmetry.

The form of the whole (tentatively) reconstructed Ur III(?) labyrinth is shown in Fig. 14.3.3 below. According to the suggested reconstruction, the labyrinth has two entrances (open gates), but no spikes (closed gates), causing crenelation in the nearby paths, as in the case of the two kinds of Old Babylonian(?) labyrinths.

One set of walls emanates from the left entrance (black), and another set of walls (gray) emanates from the right entrance. The gray wall is a rotated exact copy of the black wall. As mentioned before, in the middle of the labyrinth there is a square open court with the side 1 rod 4 cubits = 16 cubits (8 meters), and next to it are two short pieces of corridor leading to two small secluded chambers. The whole has the appearance of a plan for a very special kind of temple building, with the small chambers playing the role of the most holy places. Note, by the way that the small chambers are squares with the side 4 cubits (2 meters), while the entrances to the chambers measure 2 cubits (1 meter). There is a path through the whole labyrinth leading from one entrance to the other.

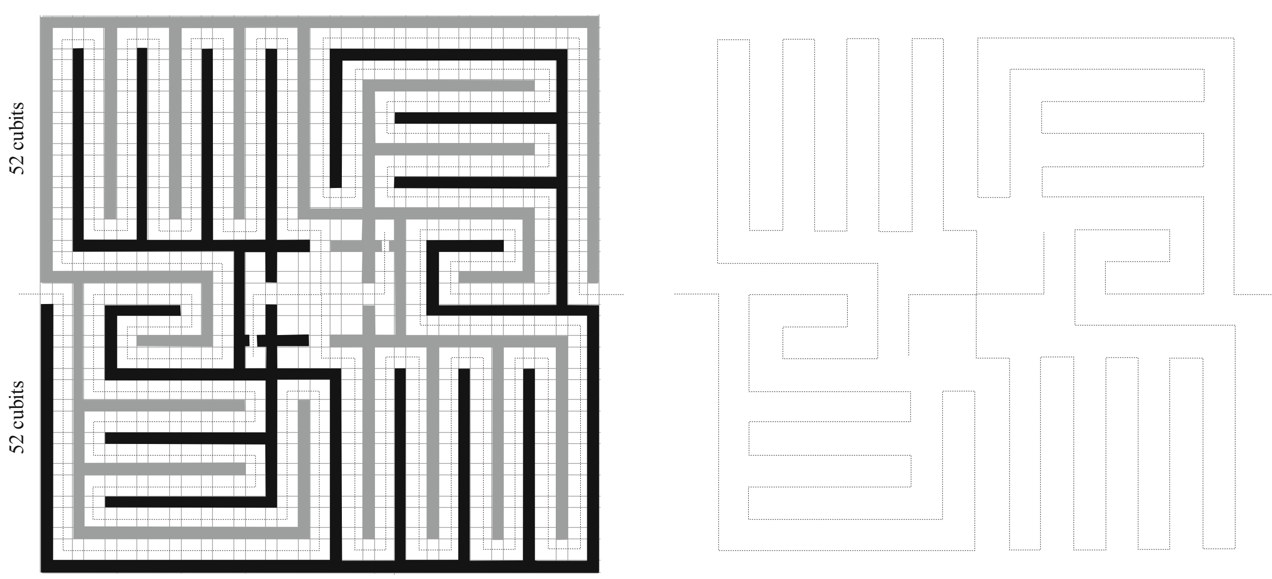
Curiously, in spite of the meticulousness of the outlines of the walls, the indicated lengths of the corridors (see Fig. 14.3.1) are not correctly computed. (The various lengths are written with shallowly impressed cuneiform signs, as if in almost dry clay.) Call for a moment the width of the corridors *a* and the thickness of the walls *b*. (As mentioned already, it is explicitly stated several times in the house plan that $b = 2$ cubits, and it definitely looks as if $a = 2b = 4$ cubits.) Then the five inscribed lengths that can be checked in 6N-T 428 are

$7a\ 6b$	= 4 1/2 rods	= 54 cubits = $27b$	
[$6a\ 5b$]	= [2]1/2 rods 4 cubits	= 34 cubits = $17b$	probably correct
$4a\ 3b$	= 2 rods	= 24 cubits = $12b$	
[$3a\ 2b$]	= 2 [rods]	= 24 cubits = $12b$	
$3a\ 2b$	= 1 rod 4 cubits	= 16 cubits = $8b$	correct

Note that the length of any piece of corridor must always be of the form

$$n\,a + (n-1)\,b = (n-1)\cdot 6 \text{ cubits} + 4 \text{ cubits} = (n-1)\cdot 1/2 \text{ rod} + 4 \text{ cubits}.$$

In the first of the enumerated cases, the equation $7a\ 6b = 27\,b$ implies that $7a = 21b$, and so $a = 3b$. Thus, in this case the one who constructed the labyrinth incorrectly counted with $a = 3b$ instead of $a = 2b$. In the second case the length measure is only partly preserved but appears to be correct. In the third case, the author of the text seems to have counted incorrectly with $4a\ 4b$ instead of $4a\ 3b$. In the fourth case, the text is broken but the inscribed length measure cannot ha<zave been correct. In the fifth and final case, the inscribed length measure is correct. So, only two out of five length measures were correctly computed.



8 1/2 nindan 2 cubits = 104 cubits (52 meters) — the path through the labyrinth (note the rotational symmetry)

Fig. 14.3.3. 6N-T 428. The proposed reconstruction of an Ur III labyrinth. Two sets of connected walls (black and gray).

In Fig. 14.3.4 below is shown an attempt to explain how a design like the (reconstructed) design on 6N-T 428 can have been developed through a series of more and more complicated designs of square labyrinths, all with a single path through the labyrinth from one entrance to the other, and all with a central square court.

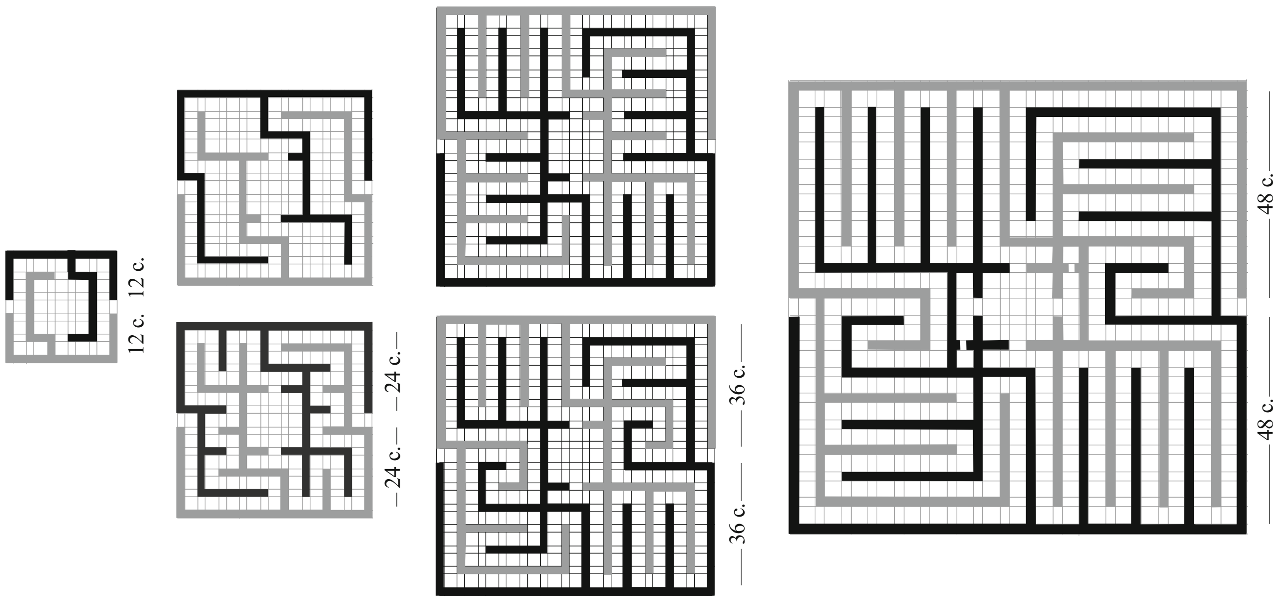


Fig. 14.3.4. 6N-T 428. A possible development of increasingly more complicated designs of a common type.

Note. Whether or not the suggested reconstruction of 6N-T 428 is exact, the unmistakably labyrinthine nature of the drawing on the fragment, together with its dating, again extend the history of labyrinths in Middle Eastern cultures by a significant amount. The two clay tablets MS 3194 and MS 4515 and the fragment 6N-T 428 together suggest that, quite independent of the Cretan maze tradition (where mazes are drawn outward from a nucleus, and the maze path typically runs from the outside to the center), another and perhaps more ancient culture of labyrinths flourished in Mesopotamia. The designs in these three instances are well developed conceptions, and testify to a history that must go much farther back in time. Actually, it is commonplace in the study of art in the ancient world, that the earliest objects we have come across already show a complete mastery of form and technique.

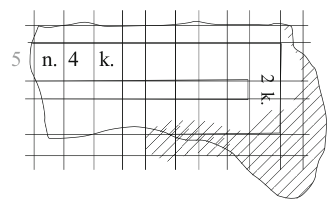
14.4 Fragments of Two Other Ur III(?) Drawings of Labyrinths from Nippur

Two fragments from the Babylonian Collection at Yale, shown in Fig. 14.4.1 below, show great similarities with the fragment 6N-T 428 from the Iraq Museum discussed in the previous section and shown in Fig. 14.3.1 above. The constant thickness of the walls is still 2 cubits, it is likely that the width of the corridors was intended to be 4 cubits, and the cuneiform sign forms are Ur III or, at least, Early Old Babylonian. Moreover, in both cases, the inscribed length measures are not always correctly computed. Indeed, the preserved length measures in the present case are

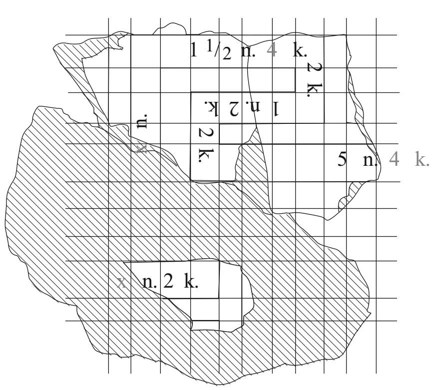
[11*a* 10*b*]	= 5 rods 4 cubits	= 64 cubits	probably correct
4*a* 3*b*	= 1 1/2 rods 4 cubits	= 22 cubits	probably correct
3*a* 2*b*	= 1 rod 2 cubits	= 14 cubits	not correct, should be 1 nindan 4 cubits
[?*a* ?*b*]	= [? rods] 2 cubits		not correct, should be a length measure ending with 4 cubits

Unfortunately, the fragments shown in Fig. 14.4.1 are not large enough to allow even a tentative reconstruction of the original drawing of a labyrinth. The most can be done is shown in Fig. 14.4.2 below. Clearly, the fragments in Fig. 14.4.1 come from a clay tablet with a drawing of a labyrinth of another type than the one shown in Figs. 14. 3.1-2.

Note also that in the case of the fragments in Fig. 14.4.1, the surface is smoother, the drawn lines and the cuneiform signs in the inscribed length measures are incised deeper, and the grid lines are less precisely spaced than in the case of the fragments in Fig. 14.3.1. On the other hand, the constant thickness of the walls is still 2 cubits, it is likely that the width of the corridors was intended to be 4 cubits, and the cuneiform sign forms are Ur III or, at least, Early Old Babylonian.



6N-T 739 (50×30 mm)



6N-T 669 (66×62 mm)

Fig. 14.4.1. 6N-T 669, etc. Fragments of another Ur III(?) Tablet from Nippur with a drawing of a labyrinth.

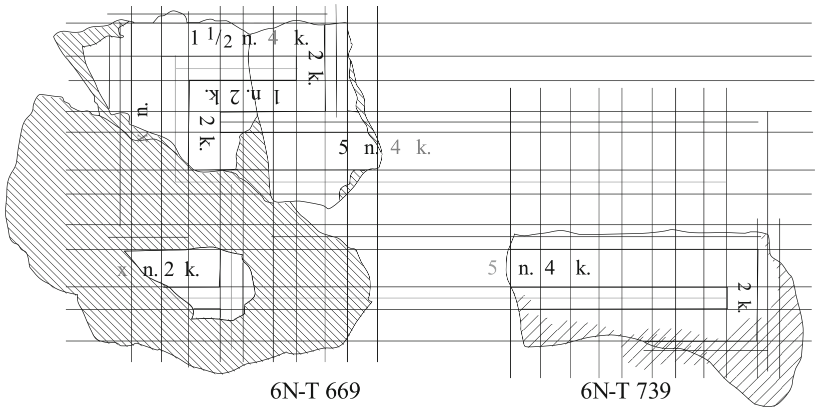


Fig. 14.4.2. 6N-T 669 + 6N-T 739. An attempted reconstruction of a part of the original drawing of a labyrinth.

The small fragments 6N-T 478 a-c, shown in Fig. 14.4.3 below, are parts of another labyrinthine "house plan", also in this case with corridors twice as wide as the walls, and showing traces of an underlying rectangular grid. Notably, however, in the case of 6N-T 478 b-c, the outlines of the walls do not follow the grid lines precisely. Obviously, this implies that the small fragments 6N-T 478 a-c are quite certainly neither from the same clay tablet as 6N-T 428 nor from the same clay tablet as 6N-T 669, etc.

Fig. 14.4.3. 6N-T 478 a-c. Three small fragments of an Ur III labyrinth.

Nevertheless, an attempt has been made to see if thee small fragments 6N-T 478 a-c, fit into the same reconstructed labyrinth as the larger fragment 6N-T 428 in Fig. 14.3.2. Actually, there is at least one place where the fragment 6N-T 478 b fits in, and the fragment 6N-T 478 a fits in in various places. However, the remaining small fragment, 6N-T 478 c does not fit in anywhere.

The conclusion of the discussion in the present section is that the fragments in Figs. 14.3.1, 14.4.1, and 14.4.3 appear to derive from three different original Ur III(?) clay tablets from Nippur with drawings of at least two different types of labyrinths in the form of house plans.

14.5 *JSS* 7, 184. An Ur III(?) House Plan with Inscribed Length Measures

In Fig 14.5.1 below is shown an unprovenanced Ur III(?) house plan from John Rylands Library, University of Manchester, published in the form of a hand copy in Trevor, *JSS* 7 (1962), 184. An excellent photo of the clay tablet with the house plan is published online at cdli.ucla.edu/P 112404. The house plan will be presented again here because of its obvious similarity with the Ur III(?) labyrinths depicted on the fragments discussed in Sections 14.3-4 above. It shows a rectangular house or temple with the given length 3 1/2 nindan 1 cubit = 43 cubits (21,5 meters) and width 2 1/2 nindan [3 cubits] = 33 cubits (16,5 meters). Inside are shown three rows of rooms, most of them with given lengths and widths. Also the thickness of the walls is given, namely by the following Sumerian line of text, inscribed inside the leftmost wall of the house:

[3] kùš iz.zi.ta   '[3] cubits of wall'

As in the case of the mentioned labyrinth fragments, the dating of the tablet to Ur III(?) is suggested by the form of the cuneiform signs for kùš, nindan, zi, ta, and for the numbers 7, 8, 9. As already observed by R. K. Englund on the mentioned cdli page, it is likely that the thickness of all the outer and inner walls is, indeed, 3 cubits, as indicated by the cited line of text, except that the width of the inner walls of the third row of rooms is only 2 cubits! This can be shown by means of summations of widths of rooms and walls, compared with the given length and width of the entire house or temple. For the three rows of rooms, the total lengths can be calculated as

(3 + 17 + 3 + 9 + 3 + 5 + 3) cubits = 43 cubits = 3 1/2 nindan 1 cubit
(3 + 5 + 3 + 12 + 3 + 6 + 3 + 5 + 3) cubits = 43 cubits
(3 + 8 + 2 + 11 + 2 + 7 + 2 + 5 + 3) cubits = 43 cubits

The common width of the rooms in the third row is missing, but it is likely that it should be [5] cubits so that the following computation of the total width is correct: (3 + 6 + 3 + 10 + 3 + [5] + 3) cubits = 33 cubits = 2 1/2 nindan [3 cubits]. Note the staircase to the roof in the upper left room, and note also the central nearly square court in the middle of the building, which is reminiscent of the central square court in the reconstructed plan of the square labyrinth in Fig. 14.3.4 above.

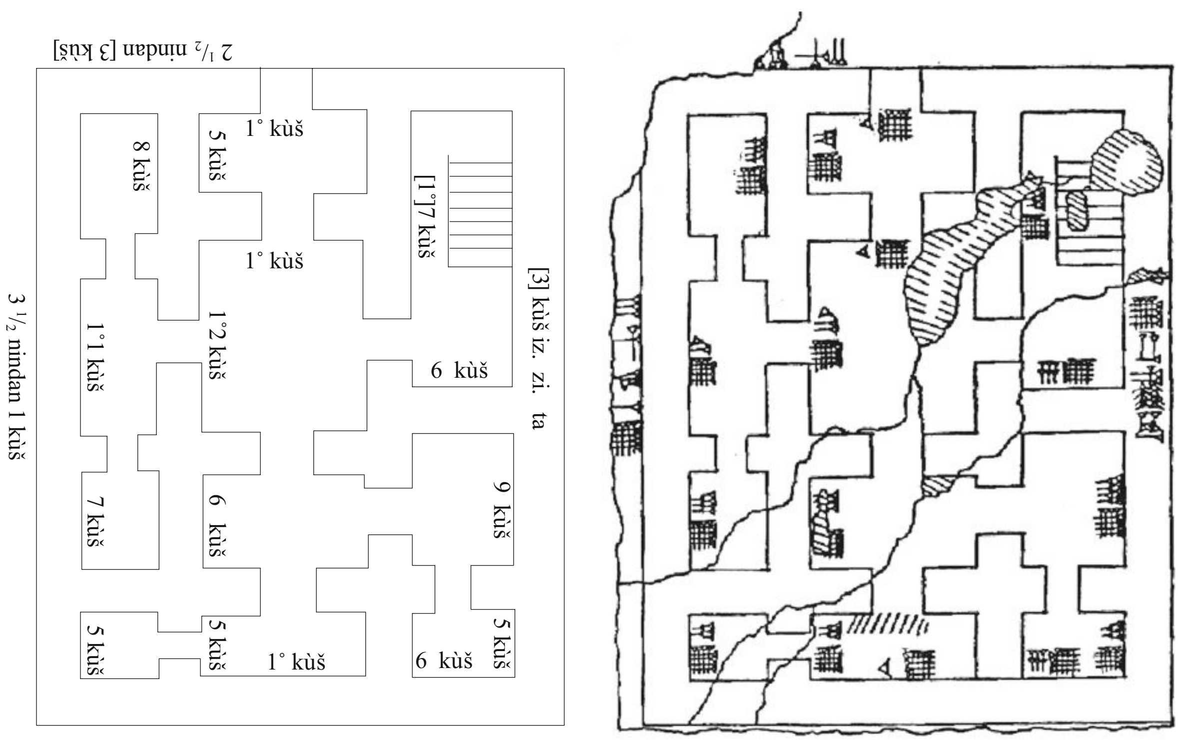
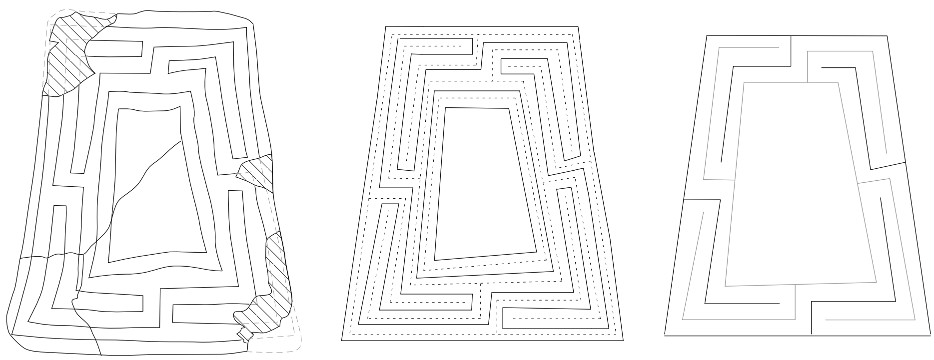


Fig. 14.5.1. *JSS* 7, 184. An Ur III house plan with a central court and inscribed length numbers.

14.6 Other Known Labyrinths or Mazes with a Square or Rectangular Central Court

Loom Weights from Francavilla with Engraved Fourfold Symmetric Paths (Ariadne Threads)

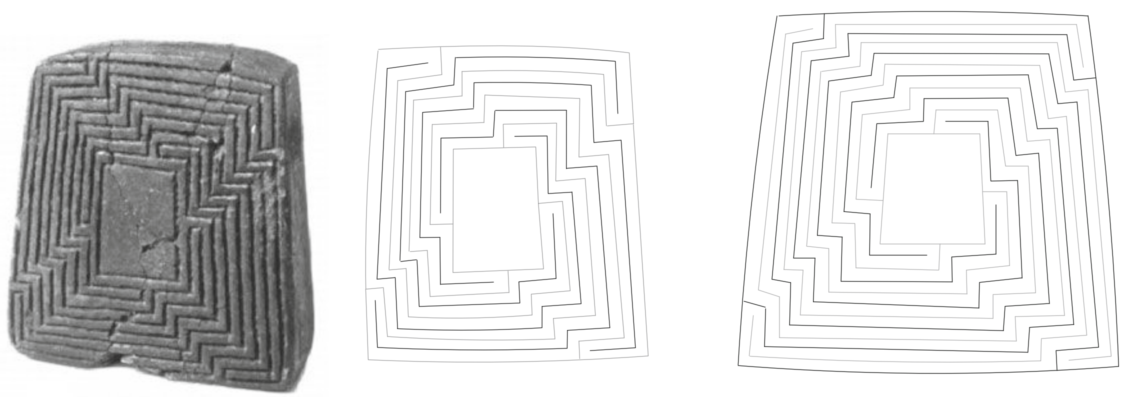
At the Timpone della Motta, a site near Francavilla Marittima in Calabria, Italy, already inhabited before the founding of the Greek colony Sybaris (c. 720 BC - 510 BC), excavations have found what remains of a "Weaving House" from about 800 BC. In the ruins of the weaving house were about 40 loom weights made of clay, many of them decorated with more or less intricate engraved patterns. One of them is shown in Fig. 14.6.1 below, left (copied from a photograph in Stoop, *ASMG* 11-12 (1976), pl. XXVI, D).



a b c

Fig. 14.6.1. a) A decorated loom weight from Francavilla, c. 800 BC. b) An interpretation of the engraved lines on the loom weight as a path (black) through an invisible maze (dotted). c) The two connected walls of the maze, one black, the other gray.

If the path through the imagined maze is thought of as an *Ariadne's thread*, it makes sense that it appears as a talisman on a loom weight in a weaving house. Note, however, that the implied maze is not a proper labyrinth, in the sense that it has no entrance and no terminus, since the path on the loom weight has neither a beginning nor an end. On the other hand, it has (topologically) perfect 4-fold symmetry, which would be impossible in a proper labyrinth.



a b c

Fig. 14.6.2. Two implied mazes of two further decorated loom weights from Francavilla, c. 800 BC. a) and b): weight from the Francavilla website, with interpretation. c) weight illustrated in Stoop, pl. XXVI, A.

Fig. 14.6.2 above shows two additional complicated designs of mazes with a central court on loom weights from Francavilla.

The photograph in a) to the left is also displayed online at museumfrancavilla.com/index.php?module=9&action=showProduct&groupID=20&productID=18&lang=en. It shows four separate paths through an invisible maze. The maze in question is shown in b). Note that there are four separate gray walls but only one connected black wall. Fig. 14.6.2 c shows a maze with strong analogies to the maze of Fig. 14.6.1. The corresponding path appears on the loom weight Stoop XXVI, A. Here, as in the example of Fig.

14.6.1, there are two connected walls, one (black) consisting of an outer near-square with four branches extending inwards, and another connected wall (gray) consisting of an inner square with four branches extending outwards. In fact these three mazes can be thought of as part of a sequence, in which the maze of Fig. 14.6.1 (Stoop XXVI, D) is expanded externally by more and more spiral levels. See Fig. 14.6.3 below. The maze c) implied in Stoop XXVI, A has 8 new levels, and the maze b) has 5. The maze corresponding to Stoop XXVI, B is from the same sequence, with 2 new levels; Stoop XXVI, C, although quite damaged, certainly is of the same type, with 2 new levels (and some extra rectangular paths around the central court).

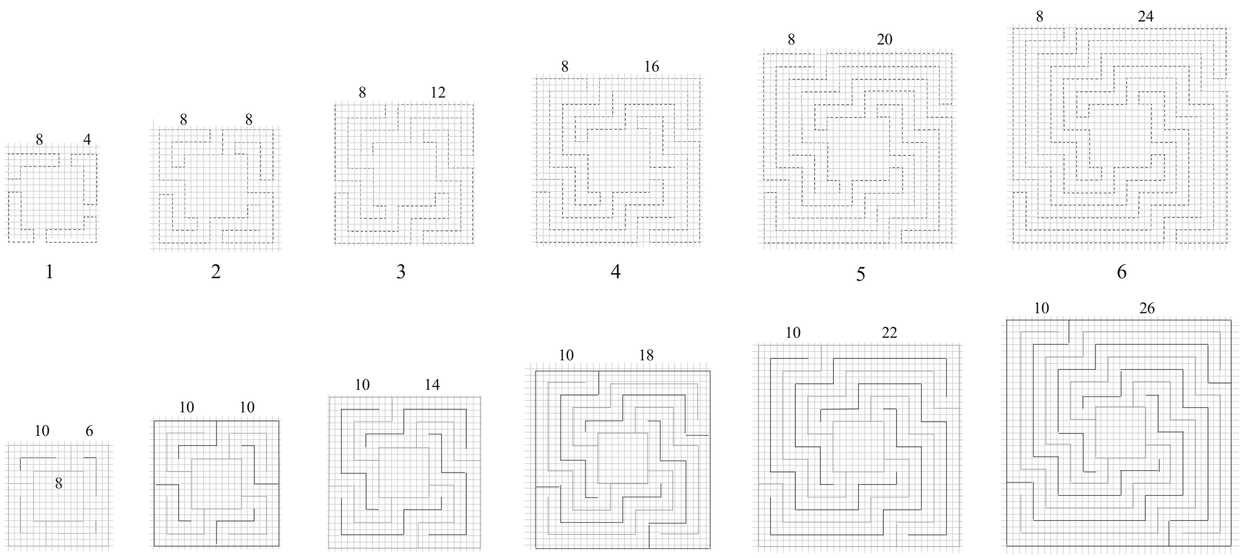


Fig. 14.6.3. The beginning of a sequence of paths (and implied mazes) to which the patterns with four-fold rotation symmetry on the three Francavilla loom weights illustrated above belong. The even-numbered mazes can be traversed by a single path. In the odd-numbered mazes, the path breaks up into four separate loops. The distance is the same everywhere between the black and the gray walls. The paths can be constructed easily in a regular grid by starting with a square of side 8+2+4 = 14 and then adding one outer level at a time. Similarly for the mazes, starting with a square of side 10+6 = 16.

The paths on the Francavilla loom weights are evidence of a labyrinth-drawing tradition related initially to the meander-patterns that occur on (later) Greek coins, and thus to the Cretan maze, but structurally different from the elaborations of that design found in Roman mosaic mazes (see below). There is some visual similarity to the Mesopotamian mazes considered earlier in this chapter, and clearly a shared interest in labyrinthine designs, but the patterns are organized in fundamentally different ways. There is certainly evidence of a considerable mathematical sophistication in the generation of both the Sumerian and the Francavilla-type patterns. Note, by the way, that if the algorithmic construction of the (corrected) Old Babylonian square labyrinth with rotational symmetry in Fig. 14.2.4 had been stopped after step 8, the result would have been a labyrinth with a central square court, instead of a central spiral.

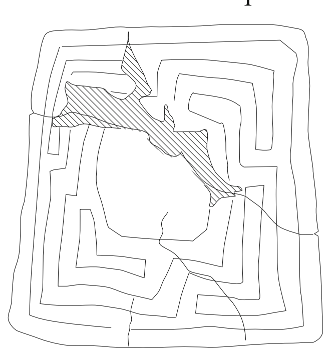


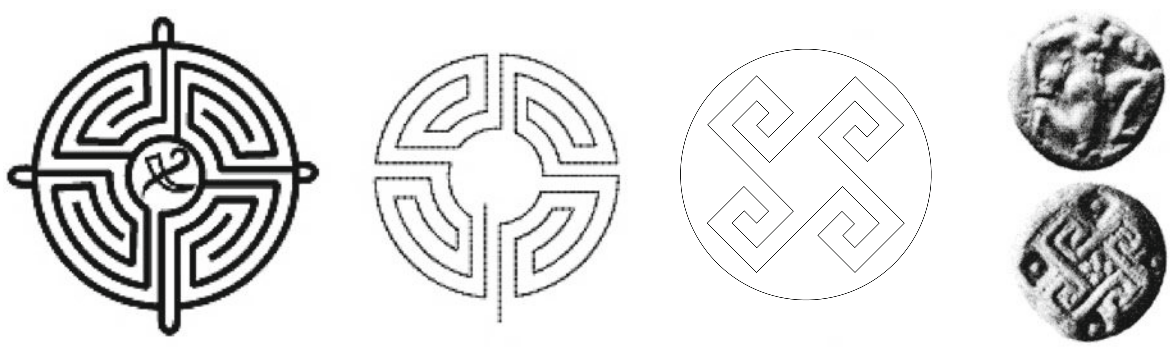
Fig. 14.6.4. A loom weight from Montuoro with a crudely drawn path of Francavilla-type # 3.

The pattern on a loom weight published by Montuoro (*RAA* 50 (1975); excavated at Macchiabate, near the Timpone dell Motta, with a secure dating to no later than 800 BC) is crudely drawn and partly unintelligible. Nevertheless, it can be identified as belonging to the Francavilla-type, # 3 in Fig. 14.6.3 above.

Roman Mosaic Mazes or Labyrinths

Mosaic floors are characteristic of Roman interiors. The medium is fragile but imperishable; many examples, more or less intact, still exist throughout the lands once ruled by the Romans. In some 57 of the examples, the design features or incorporates a labyrinth. These labyrinths were catalogued and classified by Daszewski in *LMT* (1977). They were itemized again by Kern in *TL* (2000), as part of a general survey of the history of the labyrinth motif.

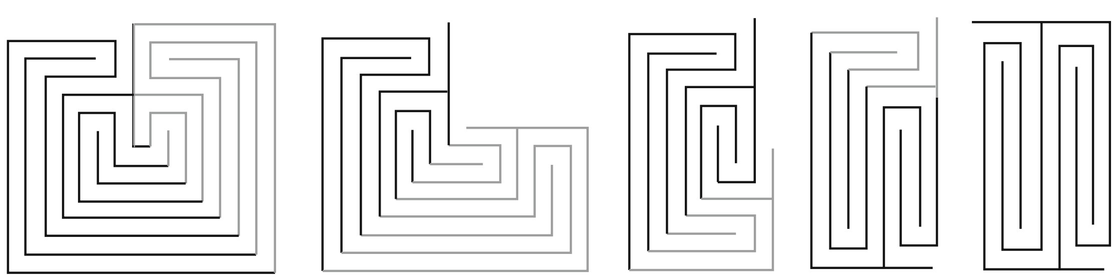
The simplest example, and one that illustrates the continuity between the Greek/Cretan meander-maze/labyrinth tradition and Roman mosaic labyrinth designs is a circular mosaic labyrinth in Avenches, Switzerland (about 250 AD; photograph: Kern, *TL* (2000) # 120). See Fig. 14.6.3 a below. The center of the mosaic shows a club crossed with a horn, symbolic of the combat between Theseus and the Minotaur; in fact references to the *Minotauromachy* appear in many, but not all, of the Roman mosaic mazes we know. This design is closely related to the meander-pattern on the reverse of coins from Knossos in the collection of The British Museum and others like it (for instance Kern, *TL* ## 39-40). See Fig. 14.6.4 below, right. Thematically, they both refer to the Minotaur (interestingly, *the Minotaur may appear on the obverse of the coin*); topologically, *the meander-pattern on the coin is almost exactly the path through the Avenches labyrinth* (*an unusually simple Ariadne's thread*). Almost, because the meander-pattern has no start or end (a possible link with the labyrinth tradition manifest at Francavilla), whereas the path traversing the Avenches labyrinth leads from the outside of the maze to the central court. The central court is round in this example, but otherwise almost always square.



a b c

Fig. 14.6.5. a) The pattern of a Roman circular labyrinth in Avenches (Kern *TL* (2000) # 120).
b) The path through the labyrinth.
c) A path (Ariadne's thread) on the reverse of a silver coin from Knossos (c. 425-360 BC).
The obverse is adorned with an image of the Minotaur.

In most cases the Roman mosaic labyrinth is square and organized as follows: The path enters at the center of one of the sides and proceeds through four identical copies of a partially "unrolled" Cretan-type labyrinth, one in each corner. Each copy is traversed from the inside to the outside; after the fourth copy an extra segment leads back to the central court. (See Phillips, *Leonardo* 25 (1992), for a discussion of exceptions to this rule).



a b c d e

Figure 14.6.6. The Cretan labyrinth in its standard form (a) and its unrolled double meander form (e).

In all forms of the Cretan maze, levels (counted from the outside) are encountered in the following order by the path through the maze: 0 3 2 1 4 7 6 5 8. The standard form is found on coins of Knossos (14 of the items on Wroth's Plates V and VI bear this design), and on the tablet Cn 1287 from Pylos shown in Fig 14.2.5 above.

Fig. 14.6.5 shows how the Cretan maze a) can be continuously deformed to a double meander e), once the central axis has been doubled. Four steps of the process are illustrated: to get from a) to b), from b) to c), etc., the essential move is to rotate the part marked in gray by 90˚, clockwise for the first three steps, counterclockwise for the last. The unrolled form can be rolled up again by holding the top fixed, stretching the bottom edge around the maze on one side or the other and gluing the two edges together.

A standard Roman mosaic labyrinth incorporates four copies of a meander-maze M, arranged in the four corners of a square, and linked as shown in Fig. 14. 6.6, left, where the fourth copy is slightly squeezed to make room for the final approach to the center. (See, Phillips, *Leonardo* 25 (1993).) On the right is an example from Pont Chevron in France. (A photograph appears in Kern, *TL* (2000) # 161 (84 cm on the side; the center shows a house with its door facing the path.). Here M is an unrolled Cretan maze, distorted to fit around the corner.

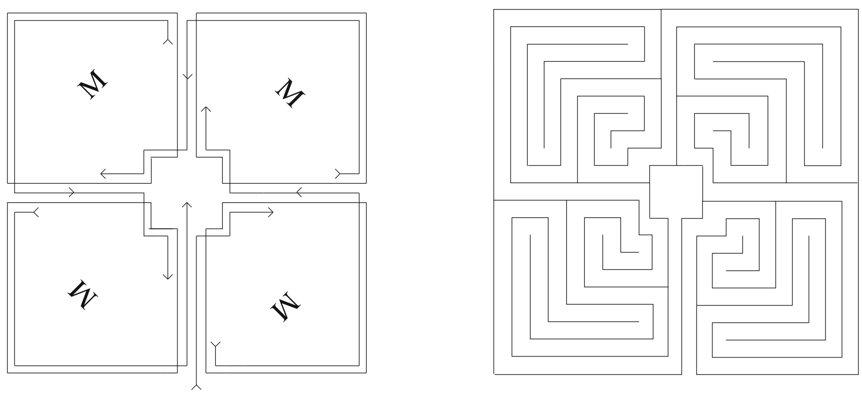


Fig. 14.6.7. Left: A typical Roman mosaic labyrinth incorporates four copies of a meander-maze M, arranged in the four corners of a square, and linked as shown here. Right: An example, copied from Kern, *TL* # 161.

Kern's encyclopedic survey of labyrinths from antiquity through the middle ages is subtitled "Designs and Meanings over 5,000 Years", the (exaggerated) 5000-year presence of an archetype. This archetype has its graphic roots in the Greek key meander pattern, and was, or became, intimately associated to the legend of the Minotaur. In these labyrinths the path either leads from the outside to the center (often accompanied by references to Ariadne's thread), or forms a circular, meandering path (perhaps evocative of the dance of the liberated Athenian hostages, see Kern, pp. 48-55). The Mesopotamian labyrinths presented in this chapter show the presence in antiquity of a completely different labyrinth archetype, in which the path leads from an entrance on one side of the design to an exit on the opposite side; this also allows 180-degree rotational symmetry (perfect in one case, presumably attempted or suggested in the others), unique to these specimens.

Added in the proofs:

A paper by R. M. Shelton with the title "The Babylonian Labyrinths" was published recently in *Caerdroia, The Journal of Mazes & Labyrinths* 42 (2013), 7-29. It contains, among other things, a renewed interesting discussion of both the Old Babylonian rectangular labyrinth MS 3194 and the Old Babylonian square labyrinth MS 4515 (both in Secs. 14.1-2 above). It contains also an equally interesting renewed discussion of the geometric theme text MS 4516 with its eight assorted mazes (Friberg, *MSCT 1* (2007), Fig. 8.3.13; CDLI no. P253617).

Fig. 14.6.8. MS 4516. A geometric theme text with eight assorted mazes.

Index of Texts

6N-T 428	a large fragment of an Ur III(?) labyrinth	Sec. 14.3
6N-T 478	small fragments of an Ur III(?) labyrinth	Sec. 14.3
6N-T 669	fragments of an Ur III(?) labyrinth	Sec. 14.4
A 681	a table of areas of squares, from ED IIIb Adab	Sec. 13.6
AO 6456	a Seleucid many-place table of reciprocals, with *n* from 1 to 3	Sec. 1.5
AO 6555	the “Esagila tablet”, a Neo-Babylonian metro-mathematical text	Sec. 3.1
AO 10822	a fragment of a large catalog text with problems for 4 brick types	Sec. 8.5.3
Ass. 13956dr	a Neo-Assyrian structure table for traditional length measure	Fig. 3.1.5
BM 34517	a fragment of a Late Babylonian descending table of powers of 9	Sec. 2.3.1
BM 34907	a fragment of a Late Babylonian direct and inverse last place factorization algorithm	Sec. 2.3.3
BM 34958	a fragment of a Late Babylonian last place factorization algorithm	Sec. 2.3.2
BM 35568	a Late Babylonian list of squares of many-place regular sexagesimal numbers	Sec. 1.2.1
BM 36776	a fragment of a table of constants concerned with wet, dry, and baked bricks, from LB Babylon	Fig. 8.1.2
BM 46550	a Neo-Babylonian tablet with direct and indirect factorization algorithms	Sec. 2.1
BM 54320	a small OB fragment with homogeneous quadratic problem for three squares	Sec. 8.3
BM 54779	a small OB fragment with homogeneous quadratic problems for squares	Sec. 8.2
BM 78822	a Neo-Babylonian metro-mathematical exercise, concerned with common and major seed measure	Fig. 4.1.6
BM 80078	an OB fragment of a recombination text with problems for bricks, from OB Sippar	Sec. 8.1
BM 80088	a catalog text with rectangular-linear systems of equations of types B1a-b, from OB Sippar	Sec. 9.3
BM 80209	a catalog text with metric algebra problems for squares and circles, from OB Sippar	Sec. 9.5
BM 80209	a catalog text with metric algebra problems for squares and circles, from OB Sippar	Sec. 9.5
BM 85194	a mathematical recombination text with exercises of mixed content, from OB Sippar	—
—— # 4	an exercise about circular defense works with trapezoidal cross sections	Sec. 8.5.12
—— ## 10-13	exercises expressed in terms of the term im.lá	Sec. 8.1.4
—— ## 22-24	counting with cylinder sìla in exercises about cylindrical containers	Sec. 11.3.1
—— # 28	an isolated exercise dealing with a truncated pyramid	Sec. 11.3.1
—— ## 34-35	counting with cylinder sìla in exercises about cylindrical containers	Sec. 11.3.1
BM 85196	a mathematical recombination text with exercises of mixed content, from OB Sippar	—
—— # 2	an approximate computation rule for the volume of a reed bundle (a truncated cone)	Sec 8.511
—— # 5	a problem for a banked canal (a trapezoid within a trapezoid)	Sec. 10.1.3
—— # 11	an approximate computation rule for the volume of a truncated square pyramid	Sec. 8.5.10
—— # 18	a system of linear equations for the weights of two silver rings	Sec.11.1.3
BM 96954+	a mathematical recombination text with the theme pyramids and cones, from OB Sippar	Sec. 8.5
BM 96957+	a mathematical recombination text about brick walls, a brick pyramid, and rectangles with diagonals	Sec. 8.6
BM 106425	a table of reciprocals from Ur III Umma	Sec. 13.5
BM 106444	a table of reciprocals from Ur III Umma	Fig. 13.5.2
Böhl 1821	an metrical algebra problem for two concentric circular towns, from OB Sippar	Sec. 9.1
CBS 165	a fragment of a catalog of rectangular-linear systems of equations of types B1a-b, from OB Sippar	Sec. 9.4
CBS 1215	an OB combined factorization and doubling and halving algorithm	Sec. 2.2
CBS 8539	a mixed metrological table text from Achaemenid Nippur	Sec. 4.1
CBS 10201	an Ur III/early OB text with an example of the doubling and halving algorithm	Fig. 13.4.2

J. Friberg, F.N.H. Al-Rawi, *New Mathematical Cuneiform Texts*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-3-319-44597-7

CBS 11019	a small Neo-Babylonian table for shekel fractions	Fig. 4.2.2
CBS 11032	a small Neo-Babylonian table for shekel fractions	Fig. 4.2.1
CUNES 50-08-001	a large Early Dynastic table of areas of squares with many sub-sections	Sec. 13.6
CUNES 52-06-31	a proto-cuneiform explicit computation of the areas of 5 quadrilaterals	Sec. 6.2
CUNES 52-18-035	a recombination text from Early Dynastic/Early Sargonic Umma with commercial exercises	Sec. 12.1
Erm 14645	an Ur III table of reciprocals	Sec. 3.5
Haddad 104	a work norm for spreading plaster in a recombination text of varied content, from OB Mê-Turran	Sec. 6.1.4
Haddad 3657	a brief OB multiplication table for 27 times 9-11	Fig. 6.4.1
Haddad 3661	a brief OB multiplication table for 30 times 1-3	Fig. 6.4.1
Haddad 3662	a brief OB multiplication table for 18 times 14-16	Fig. 6.4.1
Haddad 3669	a brief OB multiplication table for 22 30 times 1-5	Fig. 6.4.1
Haddad 3694	a brief OB multiplication table for 25 times 15-17	Fig. 6.4.1
Haddad 3717	a brief OB multiplication table for 10 times 1-11	Fig. 6.4.1
Haddad 3718	a brief OB multiplication table for 50 times 1-6	Fig. 6.4.1
Haddad 3739	a brief OB multiplication table for 6 times 1-10	Fig. 6.4.1
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HS 201	an Ur III table of reciprocals from Nippur	Sec. 13.5
IM 18232	an OB multiplication table with head number 8 20	Fig. 6.4.1
IM 18236	an OB table of cube sides	Fig. 6.4.1
IM 31210	a large fragment of a recombination text from OB Shaduppûm with economic transactions	Sec. 7.1
IM 31247	a recombination text from OB Shaduppûm with metrical algebra problems for rectangles	Sec. 5.3
IM 43993	a small text from OB Shaduppûm(?) with an interesting metric algebra problem for a rectangle	Sec. 5.2
IM 52301	a small text from OB Shaduppûm with two metric algebra problems and a computation rule	Sec. 6.2.7
IM 52685+	Goetze's compendium II from OB Shaduppûm: a recombination text of mixed content	Sec. 10.2
IM 52916	Goetze's compendium I from OB Shaduppûm: a recombination text of mixed content	Sec. 10.1
IM 53963	a single problem text from OB Shaduppûm with a metric algebra problem for a right triangle	Sec. 5.5
IM 54559	a single problem text from OB Shaduppûm with a metric algebra problem for a rectangle	Sec. 5.4
IM 95771	a fragment of a recombination text from OB Mê-Turran with geometric exercises	Sec. 6.1
IM 121512	a recombination text from OB Mê-Turran with metric algebra problems for semicircles	Sec. 6.3
IM 121565	a recombination text from OB Mê-Turran with metric algebra problems for quadrilateral fields	Sec. 6.2
IM 121613	a recombination text from OB Mê-Turran with metric algebra problems for rectangles	Sec. 5.1
IM 630174	an ascending table of powers from OB Bikasi	Sec. 2.4
Ist. L 7375	an Ur III table of reciprocals	Sec. 13.3
Ist. L 9006	an unpublished table of reciprocals from Ur III Girsu	Sec. 13.5
Ist. L 9007+	an unpublished table of reciprocals from Ur III Girsu	Sec. 13.5
Ist. L 9008	an unpublished table of reciprocals from Ur III Girsu	Sec. 13.5
Ist. Ni 374	an Ur III table of reciprocals from Nippur	Sec. 13.5
Ist. Ni 3854	an Ur III/early OB table of reciprocals from Nippur	Sec. 13.5
Ist. Ni 5173	an Ur III/early OB combined multiplication table from Nippur	Fig. 13.6.1
Ist. S 485	a fragment of a table of reciprocals from Sippar with Sumerian spellings of number words	Sec. 13.2
Ist. Si 269	bisected trapezoids, variations of parameters, OB(?)	Fig. 13.1.2
MLC 1345	a badly conceived metric algebra problem for a semicircle	Fig. 6.3.2
MLC 1842	an OB metric algebra problem concerning two market rates	Sec. 11.1
***MMAP 27*, 61**	an Ur III atypical multiplication table, head number 3	Fig. 13.6.3
MS 3050	a square inscribed in a circle, OB(?)	Fig. 10.1.5
MS 3051	an equilateral triangle inscribed in a circle, OB(?)	Fig. 10.1.3
MS 3194	an OB(?) rectangular labyrinth	Sec. 14.2
MS 3849	an Ur III(?) multiplication table, head number 50	Fig. 13.6.2

MS 3874	an Ur III/early OB intermediate table of reciprocals	Sec. 13.5
MS 4515	an OB(?) square labyrinth	Sec. 14.1
***RA* 12, 197**	an early OB table of reciprocals	Fig. 13.5.1
Sb 13293	a ridge pyramid	Sec. 8.5.8
Sippar 2175/12	a Neo-Babylonian many-place table of reciprocals, with *n* from 1 to 3	Sec. 1.4
SM T. 2574	an Ur III/early OB intermediate table of reciprocals	Sec. 13.4
SM T. 2685	an Ur III(?) table of reciprocals without place value numbers	Sec. 13.1
Str 364	a catalog of metric algebra problems for striped triangles, OB(?)	Fig. 10.1.7
SU 52/5	a Late Assyrian table of reciprocals with syllabic Sumerian number words	Sec. 13.2
***TMS* III**	constants related to bricks, from OB Susa	Sec. 11.3
***TMS* V**	a large catalog text with metric algebra problems for squares, from OB Susa	Sec. 10.3
***TMS* VI**	another catalog text with metric algebra problems for squares, from OB Susa	Sec. 10.4
***TMS* VIII**	a fragment with metric algebra problems, from OB Susa	Sec. 9.4
***TMS* XXII**	a hand tablet with an interest and annuity problem, from OB Susa	Fig. 11.1.2
VAT 6505	an OB combined factorization and doubling and halving algorithm	Fig. 8.4.1
VAT 7530	an OB catalog text with market rate problems	Fig. 11.1.6
VAT 8522	an OB recombination text of mixed content with chaotically organized notes of solution procedures	Fig. 11.3.5
——— # 1	an approximate computation rule for the volume of a cedar tree (a truncated cone)	Sec. 11.3.1
——— # 2	division of shares between five brothers	Sec. 7.1.9
——— # 3	a system of linear problems for ewes and lambs in two sheep-folds	Sec. 11.3.8
——— # 4	a metric division problem for a rectangle	Sec. 13.6
W 22260	a fragment of a mixed metrological text from Achaemenid Uruk	Sec. 3.4
W 23273	a metrological recombination text from Achaemenid Uruk	Sec. 3.2
W 22309	a small fragment of a metrological recombination text from Achaemenid Uruk	Sec. 3.3
W 23016	a factorization algorithm on a round hand tablet from Late Babylonian Uruk	Fig. 2.1.8
W 23021	direct and inverse factorization algorithms on a round tablet, from Achaemenid Uruk	Fig. 2.1.6
W 23281, *obv.*	a metrological recombination text	Sec. 3.1
———, *rev.*	a many-place table of reciprocals, from Achaemenid Uruk	Sec. 1.6
W 23283+	a many-place table of reciprocals, with *n* from 1 to 4, from Achaemenid Uruk	Sec. 1.3
YBC 4607	a brief catalog text with problems for 5 brick types	Sec, 8.5.3
YBC 4669	an OB recombination text with mixed problems	Sec. 11.3
YBC 4673	an OB recombination text with problems about bricks, mud, and reeds	Sec. 11.2
YBC 4698	an OB recombination text with commercial problems	Sec. 11.1
YBC 6492	a small OB catalog text with simple form and magnitude problems, variations of parameters	Fig. 9.2.1
YBC 7326	an OB square hand tablet with a system of linear problems for ewes and lambs in two sheep-folds	Sec. 11.3

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