PM 811 H^p Spaces **Assignment 2** **Due Thursday October 28.**

1. Let $Z(B)$ denote the zeros of a Blaschke product B including multiplicity, and let S_i be singular inner functions corresponding to singular measures $\mu_i \in M(\mathbb{T})_+$. Show that B_1S_1 divides B_2S_2 if and only if $Z(B_1) \subset Z(B_2)$ and $\mu_1 \le \mu_2$.

2. (a) Let $\omega(z) = \exp\left(\frac{z+1}{z-1}\right)$. Describe the sets $\{z : |\omega(z)| = r\}$. Hence show that for $a \in \mathbb{D}$, the zeros of $\omega_a(z) = \dfrac{\omega(z) - a}{1 - \bar{a}\omega(z)}$ approach 1 tangentially.
   (b) Let T be the operator on $\mathcal{K} = H^2 \ominus \omega H^2$ given by $Tf = P_{\mathcal{K}} zf$ for $f \in \mathcal{K}$. Find all invariant subspaces of T.
   (c) **Bonus.** Is ω_a a Blaschke product for all $a \in \mathbb{D} \setminus \{0\}$? I suspect this is true.

3. (a) If $f(z)$ and $1/f(z)$ are both in $H^1(\mathbb{D})$, prove that f is outer.
   (b) If $f \in H^1(\mathbb{D})$ and $\operatorname{Re} f(z) > 0$ on $\mathbb{D}$, prove that f is outer.
   Hint: $f+\varepsilon$ is outer. Split $\int \log|f+\varepsilon|\,dt$ over $E = \{t : |f(e^{it})| \ge 1/2\}$ and $\mathbb{T} \setminus E$.
   (c) If $\omega(z)$ is a non-constant inner function, for which $a \in \mathbb{C}$ is $\omega(z) - a$ outer?

4. If $f, g \in H^1$, f is outer and $h = g/f \in L^1$, show that $h \in H^1$.
   Hint: reduce to the case g outer and use the integral formula.

5. Suppose that $f(z) = \sum_{n\ge 0} a_n z^n$ belongs to $H^1(\mathbb{D})$. Prove that $\sum_{n=1}^{\infty} \frac{1}{n}|a_n| \le \pi\|f\|_1$.
   Hint: Factor f into two H^2 functions, replace their Taylor coefficients by their absolute values, so that the product F has positive coefficients and $\|f\|_1 = \|F\|_1$. Compute $\int_0^{2\pi} (\pi - t)\operatorname{Im} F(re^{it})\,dt$.

6. Let $\tau(z) = \lambda \dfrac{z-a}{1-\bar{a}z}$ for $a \in \mathbb{D}$ and $|\lambda| = 1$.
   (a) Show that if B is a Blaschke product, so is $B\circ\tau$; and if S is a singular inner function, so is $S \circ \tau$.
   (b) Hence deduce that if F is an outer function in H^p, then $F(\tau(z))$ is outer.
   Hint: By Assignment 1, 5(a), if $f \in H^p$, so is $f(\tau(z))$.
   (c) Show that $\alpha_\tau(f) = f \circ \tau$ defines a continuous automorphism of $\mathrm{A}(\mathbb{D})$.
   (d) Let α be an automorphism of $\mathrm{A}(\mathbb{D})$.
   (i) Show that $\operatorname{Ran}\alpha(f) = \operatorname{Ran} f$ for $f \in \mathrm{A}(\mathbb{D})$.
   (ii) Show that $\tau = \alpha(z)$ must be a conformal map of $\mathbb{D}$ onto itself.
   (iii) Show that $\alpha = \alpha_\tau$.