Notes on Tensor Products and the Exterior Algebra

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1 Tensor Products

1.1 Axiomatic definition of the tensor product

In linear algebra we have many examples of products. For example,

- The scalar product: $V \times \mathbb{F} \to V$
- The dot product: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
- The cross product: $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}$
- Matrix products: $\mathsf{M}_{m\times k} \times \mathsf{M}_{k\times n} \to \mathsf{M}_{m\times n}$

Note the three vector spaces involved aren't necessarily the same. What these examples have in common, is that in each case, the product is a bilinear map. The tensor product is just another example of a product like this. If V_1 and V_2 are any two vector spaces over a field $\mathbb{F}$, the tensor product is a bilinear map:

$$V_1 \times V_2 \to V_1 \otimes V_2\,,$$

where $V_1 \otimes V_2$ is a vector space over $\mathbb{F}$. The tricky part is that in order to define this map, we first need to construct this vector space $V_1 \otimes V_2$.

We give two definitions. The first is an axiomatic definition, in which we specify the properties that $V_1 \otimes V_2$ and the bilinear map must have. In some sense, this is all we need to work with tensor products in a practical way. Later we'll show that such a space actually exists, by constructing it.

Definition 1.1. Let V_1, V_2 be vector spaces over a field $\mathbb{F}$. A pair (Y, μ), where Y is a vector space over $\mathbb{F}$ and $\mu : V_1 \times V_2 \to Y$ is a bilinear map, is called the **tensor product** of V_1 and V_2 if the following condition holds:

(*) Whenever β_1 is a basis for V_1 and β_2 is a basis for V_2, then

$$\mu(\beta_1 \times \beta_2) := \{\mu(x_1, x_2) \mid x_1 \in \beta_1,\ x_2 \in \beta_2\}$$

is a basis for Y.

Notation: We write $V_1 \otimes V_2$ for the vector space Y, and $x_1 \otimes x_2$ for $\mu(x_1, x_2)$.

The condition (*) does not actually need to be checked for every possible pair of bases β_1, β_2: it is enough to check it for any single pair of bases.

Theorem 1.2. *Let Y be a vector space, and $\mu : V_1 \times V_2 \to Y$ a bilinear map. Suppose there exists a bases γ_1 for V_1, γ_2 for V_2 such that $\mu(\gamma_1 \times \gamma_2)$ is a basis for Y. Then condition (*) holds (for any choice of basis).*

Proof. Let β_1, β_2 be bases for V_1 and V_2 respectively. We first show that $\mu(\beta_1 \times \beta_2)$ spans Y. Let $y \in Y$. Since $\mu(\gamma_1 \times \gamma_2)$ spans Y, we can write

$$y = \sum_{j,k} a_{jk}\mu(z_{1j}, z_{2k})$$

where $z_{1j} \in \gamma_1$, $z_{2k} \in \gamma_2$. But since β_1 is a basis for V_1 $z_{1j} = \sum_l b_{jl}x_{1l}$, where $x_{1l} \in \beta_1$, and similarly $z_{2k} = \sum_m c_{km}x_{2m}$, where $x_{2m} \in \beta_2$. Thus

$$\begin{aligned} y &= \sum_{j,k} a_{jk}\mu(\sum_l b_{jl}x_{1l}\,,\, \sum_m c_{km}x_{2m}) \\ &= \sum_{j,k,l,m} a_{jk}b_{jl}c_{km}\mu(x_{1l}, x_{2j}) \end{aligned}$$

so $y \in \text{span}(\mu(\beta_1 \times \beta_2))$.

Now we need to show that $\mu(\beta_1 \times \beta_2)$ is linearly independent. If V_1 and V_2 are both finite dimesional, this follows from the fact that $|\mu(\beta_1 \times \beta_2)| = |\mu(\gamma_1 \times \gamma_2)|$ and both span Y. For infinite dimensions, a more sophisticated change of basis type of argument is needed. Suppose

$$\sum_{l,m} d_{lm}\mu(x_{1l}, x_{2m}) = 0\,,$$

where $x_{1l} \in \beta_1$, $x_{2m} \in \beta_2$. Then by change of basis, $x_{1l} = \sum_j e_{lj}z_{1j}$, where $z_{1j} \in \gamma_1$, and $x_{2m} = \sum_k f_{mk}z_{2k}$, where $z_{2k} \in \gamma_2$. Note that the e_{lj} form an inverse "matrix" to the b_{jl} above, in that $\sum_j e_{lj}b_{jl'} = \delta_{ll'}$, and similarly $\sum_k f_{mk}c_{km'} = \delta_{mm'}$. Thus we have

$$\begin{aligned} 0 &= \sum_{l,m} d_{lm}\mu(x_{1l}, x_{2m}) = 0 \\ &= \sum_{l,m} d_{lm}\mu(\sum_j e_{lj}z_{1j}, \sum_k f_{mk}z_{2k}) \\ &= \sum_{j,k}\sum_{l,m} d_{lm}e_{lj}f_{mk}\mu(z_{1j}, z_{2k}) \end{aligned}$$

Since $\mu(\gamma_1 \times \gamma_2)$ is linearly independent, $\sum_{l,m} d_{lm} e_{lj} f_{mk} = 0$ for all j, k. But now

$$\begin{aligned} d_{l'm'} &= \sum_{l,m} d_{lm} \delta_{ll'} \delta_{mm'} \\ &= \sum_{l,m} d_{lm} \left(\sum_j b_{jl'} e_{lj} \right) \left(\sum_k c_{km'} f_{mk} \right) \\ &= \sum_{j,k} b_{jl'} c_{km'} \left(\sum_{l,m} d_{lm} e_{lj} f_{mk} \right) \\ &= 0 \end{aligned}$$

for all l', m'. □

The tensor product can also be defined for more than two vector spaces.

Definition 1.3. Let $V_1, V_2, \ldots V_k$ be vector spaces over a field $\mathbb{F}$. A pair (Y, μ), where Y is a vector space over $\mathbb{F}$ and $\mu : V_1 \times V_2 \times \cdots \times V_k \to Y$ is a k-linear map, is called the **tensor product** of $V_1, V_2, \ldots, V_k$ if the following condition holds:

(*) Whenever β_i is a basis for V_i, $i = 1, \ldots, k$,

$$\{\mu(x_1, x_2, \ldots, x_k) \mid x_i \in \beta_i\}$$

is a basis for Y.

We write $V_1 \otimes V_2 \otimes \cdots \otimes V_k$ for the vector space Y, and $x_1 \otimes x_2 \otimes \cdots \otimes x_k$ for $\mu(x_1, x_2, \ldots, x_k)$.

1.2 Examples: working with the tensor product

Let V, W be vector spaces over a field $\mathbb{F}$. There are two ways to work with the tensor product. One way is to think of the space $V \otimes W$ abstractly, and to use the axioms to manipulate the objects. In this context, the elements of $V \otimes W$ just look like expressions of the form

$$\sum_i a_i v_i \otimes w_i \, ,$$

where $a_i \in \mathbb{F}$, $v_i \in V$, $w_i \in W$.

Example 1.4. Show that

$$(1, 1) \otimes (1, 4) + (1, -2) \otimes (-1, 2) = 6(1, 0) \otimes (0, 1) + 3(0, 1) \otimes (1, 0)$$

in $\mathbb{R}^2 \otimes \mathbb{R}^2$.

Solution: Let $x = (1, 0)$, $y = (0, 1)$. We rewrite the left hand side in terms of x and y, and use bilinearity to expand.

$$\begin{aligned} (1, 1) \otimes (1, 4) + (1, -2) \otimes (-1, 2) &= (x + y) \otimes (x + 4y) + (x - 2y) \otimes (-x + 2y) \\ &= (x \otimes x + 4x \otimes y + y \otimes x + 4y \otimes y) \\ &\quad + (-x \otimes x + 2x \otimes y - 2y \otimes x - 4y \otimes y) \\ &= 6x \otimes y + 3y \otimes x \\ &= 6(1, 0) \otimes (0, 1) + 3(0, 1) \otimes (1, 0) \end{aligned}$$

The other way is to actually identify the space $V_1 \otimes V_2$ and the map $V_1 \times V_2 \to V_1 \otimes V_2$ with some familiar object. There are many examples in which it is possible to make such an identification naturally. Note, when doing this, it is crucial that we not only specify the vector space we are identifying as $V_1 \otimes V_2$, but also the product (bilinear map) that we are using to make the identification.

Example 1.5. Let $V = \mathbb{F}^n_{\text{row}}$, and $W = \mathbb{F}^m_{\text{col}}$. Then $V \otimes W = \mathsf{M}_{m\times n}(\mathbb{F})$, with the product defined to be $v \otimes w = wv$, the matrix product of a column and a row vector.

To see this, we need to check condition (*). Let $\{e_1, \dots, e_m\}$ be the standard basis for W, and $\{f_1, \dots, f_n\}$ be the standard basis for V. Then $\{f_j \otimes e_i\}$ is the standard basis for $\mathsf{M}_{m\times n}(\mathbb{F})$. So condition (*) checks out.

Note we can also say that $W \otimes V = \mathsf{M}_{m\times n}(\mathbb{F})$, with the product defined to be $w \otimes v = wv$. From this, it looks like $W \otimes V$ and $V \otimes W$ are the same space. Actually there is a subtle but important difference: to define the product, we had to switch the two factors. The relationship between $V \otimes W$ and $W \otimes V$ is very closely analogous to that of the Cartesian products $A \times B$ and $B \times A$, where A and B are sets. There's an obvious bijection between them and from a practical point of view, they carry the same information. But if we started conflating the two, and writing (b, a) and (a, b) interchangeably it would cause a lot of problems. Similarly $V \otimes W$ and $W \otimes V$ are naturally isomorphic to each other, so in that sense they are the same, but we would never write $v \otimes w$ when we mean $w \otimes v$.

Example 1.6. Let $V = \mathbb{F}[x]$, vector space of polynomials in one variable. Then $V \otimes V = \mathbb{F}[x_1, x_2]$ is the space of polynomials in two variables, the product is defined to be $f(x) \otimes g(x) = f(x_1)g(x_2)$.

To check condition (*), note that $\beta = \{1, x, x_2, x_3, \dots\}$ is a basis for V, and that

$$\begin{aligned}\beta \otimes \beta &= \{x^i \otimes x^j \mid i, j = 0, 1, 2, 3, \dots\} \\ &= \{x_1^i x_2^j \mid i, j = 0, 1, 2, 3, \dots\}\end{aligned}$$

is a basis for $\mathbb{F}[x_1, x_2]$.

Note this is not a commutative product, because in general

$$f(x) \otimes g(x) = f(x_1)g(x_2) \neq g(x_1)f(x_2) = g(x) \otimes f(x)\,.$$

Example 1.7. Let V, W be finite dimensional vector spaces over $\mathbb{F}$. Let $V^* = \mathsf{L}(V, \mathbb{F})$ be the dual space of V. Then $V^* \otimes W = \mathsf{L}(V, W)$, with multiplication defined as $f \otimes w \in \mathsf{L}(V, W)$ is the linear transformation $(f \otimes w)(v) = f(v) \cdot w$, for $f \in V^*$, $w \in W$.

This is just the abstract version of Example 1.5. If $V = \mathbb{F}^n_{\text{col}}$ and $W = \mathbb{F}^m_{\text{col}}$, then $V^* = \mathbb{F}^n_{\text{row}}$. Using Example 1.5, $V \otimes W$ is identified with $\mathsf{M}_{m\times n}(\mathbb{F})$, which is in turn identified with $\mathsf{L}(V, W)$.

Exercise: Prove that condition (*) holds.

Note: If V and W are both infinite dimensional then $V^* \otimes W$ is a subspace of $\mathsf{L}(V, W)$ but not equal to it. Specifically, $V^* \otimes W = \{T \in \mathsf{L}(V, W) \mid \dim \mathsf{R}(T) < \infty\}$ is the set of finite rank linear transformations in $\mathsf{L}(V, W)$. This follows from the fact that $f \otimes w$ has rank 1, for any $f \in V^*$, $w \in W$, and a linear combination of such transformations must have finite rank.

Example 1.8. Let V be the space of velocity vectors in Newtonian 3-space. Let T be the vector space time measurements. Then $V \otimes T$ is the space of displacement vectors in Newtonian 3-space. This is because velocity times time equals displacement.

The point of this example is that physical quantities have units associated with them. Velocity is not a vector in $\mathbb{R}^3$. It's an element of a totally different 3 dimensional vector space over $\mathbb{R}$. To perform calculations, we identify it with $\mathbb{R}^3$ by choosing coordinate directions such as up forward and right and units such as m/s, but these are artificial constructions. Displacement again lives in a different vector space, and the tensor product allows us to relate elements in these different physical spaces.

Example 1.9. Consider $\mathbb{Q}^n$ and $\mathbb{R}$ as vector spaces over $\mathbb{Q}$. Then $\mathbb{Q}^n \otimes \mathbb{R} = \mathbb{R}^n$ as vector spaces over $\mathbb{Q}$, where the multiplication is just scalar multiplication on $\mathbb{R}^n$.

Proof the condition (*) holds. Let $\beta = e_1, \ldots, e_n$ be the standard basis for $\mathbb{Q}^n$, and let γ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. First we show that $\beta \otimes \gamma$ is spans $\mathbb{R}^n$. Let $(a_1, \ldots a_n) \in \mathbb{R}^n$. We can write $a_i = \sum b_{ij} x_j$, where $x_j \in \gamma$. Thus $(a_1, \ldots, a_n) = \sum_{i,j} b_{ij} e_i \otimes x_j$. Next we show that $\beta \otimes \gamma$ is linearly independent. Suppose $\sum_{i,j} c_{ij} e_i \otimes x_j = 0$. Since

$$\sum_{i,j} c_{ij} e_i \otimes x_j = (\sum_j c_{1j} x_j, \ldots, \sum_j c_{nj} x_k)\,,$$

we have $\sum_j c_{ij} x_j = 0$ for all i. Since $\{x_j\}$ are linearly independent, $c_{ij} = 0$, as required.

1.3 Constructive definition of the tensor product

To give a construction of the tensor product, we need the notion of a free vector space.

Definition 1.10. Let A be a set, and $\mathbb{F}$ a field. The **free vector space** over $\mathbb{F}$ generated by A is the vector space $\mathrm{Free}(A)$ consisting of all formal finite linear combinations of elements of A.

Thus, A is always a basis of $\mathrm{Free}(A)$.

Example 1.11. Let $A = \{\spadesuit, \heartsuit, \clubsuit, \diamondsuit\}$, and $\mathbb{F} = \mathbb{R}$. Then the $\mathrm{Free}(A)$ is the four-dimensional vector space consisting of all elements $a\spadesuit + b\heartsuit + c\clubsuit + d\diamondsuit$, where $a, b, c, d \in \mathbb{R}$. Addition and scalar multiplication are defined in the obvious way:

$$(a\spadesuit + b\heartsuit + c\clubsuit + d\diamondsuit) + (a'\spadesuit + b'\heartsuit + c'\clubsuit + d'\diamondsuit) = (a + a')\spadesuit + (b + b')\heartsuit + (c + c')\clubsuit + (d + d')\diamondsuit$$

$$k(a\spadesuit + b\heartsuit + c\clubsuit + d\diamondsuit) = (ka)\spadesuit + (kb)\heartsuit + (kc)\clubsuit + (kd)\diamondsuit$$

When the elements of the set A are numbers or vectors, the notation gets tricky, because there is a danger of confusing the operations of addition and scalar multiplication and the zero-element in the vector space $\mathrm{Free}(A)$, and the operations of addition and multiplication and the zero element in A (which are irrelevant in the definition of $\mathrm{Free}(A)$). To help keep these straight in situations where there is a danger of confusion, we'll write $\boxplus$, and $\boxdot$ when we mean the operations of addition and scalar multiplication in $\mathrm{Free}(A)$. We'll denote the zero vector of $\mathrm{Free}(A)$ by $0_{\mathrm{Free}(A)}$.

Example 1.12. Let $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$, and $\mathbb{F} = \mathbb{R}$. Then Free($\mathbb{N}$) is the infinite dimensional vector space whose elements are of the form

$$(a_0 \boxdot 0) \boxplus (a_1 \boxdot 1) \boxplus \cdots \boxplus (a_m \boxdot m)\,,$$

for some $m \in \mathbb{N}$.

Note that the element 0 here is *not* the zero vector in Free($\mathbb{N}$). It's called 0 because it happens to be the zero element in $\mathbb{N}$, but this is completely irrelevant in the construction of the free vector space.

If we wanted we could write this a little differently by putting x^i in place of $i \in \mathbb{N}$. In this new notation, the elements of Free($\mathbb{N}$) would look like

$$a_0x^0 + a_1x^1 + \dots a_mx^m\,,$$

for some $m \in \mathbb{N}$, in other words elements of the vector space of polynomials in a single variable.

Definition 1.13. Let V and W be vector spaces over a field $\mathbb{F}$. Let

$$\mathcal{P} := \mathrm{Free}(V \times W)\,,$$

the free vector space over $\mathbb{F}$ generated by the set $V \times W$. Let $\mathcal{R} \subset \mathcal{P}$ be the subspace spanned by all vectors of the form

$$(u + kv, w + lx) \boxplus ((-1) \boxdot (u, w)) \boxplus ((-k) \boxdot (v, w)) \boxplus ((-l) \boxdot (u, x)) \boxplus ((-kl) \boxdot (v, x))$$

with $k, l \in \mathbb{F}$, $u, v \in V$, and $w, x \in W$.

Let $\pi_{\mathcal{P}/\mathcal{R}} : \mathcal{P} \to \mathcal{P}/\mathcal{R}$ be the quotient map. Let $\mu : V \times W \to \mathcal{P}/\mathcal{R}$ be the map $\mu(v, w) = \pi_{\mathcal{P}/\mathcal{R}}\big((v, w)\big)$.

The pair $(\mathcal{P}/\mathcal{R}\,,\, \mu)$ is the **tensor product** of V and W. We write $V \otimes W$ for $\mathcal{P}/\mathcal{R}$, and $v \otimes w$ for $\mu(v, w)$.

We need to show that our two definitions agree, i.e.. that tensor product as defined in Definition 1.13 satisfies the conditions of Definition 1.1. In particular, we need to show that μ is bilinear, and that the pair $(\mathcal{P}/\mathcal{R}\,,\, \mu)$ satisfies condition (*).

We can show the bilinearity immediately. Essentially bilinearity is built into the definition. If the $\mathcal{P}$ is the space of all linear combinations of symbols (v, w), then $\mathcal{R}$ is the space of all those symbols that can be simplified to the zero vector using bilinearity. Thus $\mathcal{P}/\mathcal{R}$ is the set of all expressions, where two expressions are equal one can be simplified to the other using bilinearity.

Proposition 1.14. *The map μ in Definition 1.13 is bilinear.*

Proof. Let $k, l \in \mathbb{F}$, $u, v \in V$, and $w, x \in W$. We must show that

$$\mu(u + kv, w + lx) = \mu(u, w) + k\mu(v, w) + l\mu(u, x) + kl\mu(v, x)\,.$$

We know that $\pi_{\mathcal{P}/\mathcal{R}}(z) = 0$ for all $z \in \mathcal{R}$. In particular, we have

$$\begin{aligned}
0 &= \pi_{\mathcal{P}/\mathcal{R}}\big((u+kv, w+lx) \\
&\qquad\qquad \boxplus ((-1) \boxdot (u,w)) \boxplus ((-k) \boxdot (v,w)) \boxplus ((-l) \boxdot (u,x)) \boxplus ((-kl) \boxdot (v,x))\big) \\
&= \pi_{\mathcal{P}/\mathcal{R}}\big((u+kv, w+lx)\big) \\
&\qquad\qquad - \pi_{\mathcal{P}/\mathcal{R}}\big((u,w)\big) - k\pi_{\mathcal{P}/\mathcal{R}}\big((v,w)\big) - l\pi_{\mathcal{P}/\mathcal{R}}\big((u,x)\big) - kl\pi_{\mathcal{P}/\mathcal{R}}\big((v,x)\big) \\
&= \mu(u+kv, w+lx) - \mu(u,w) - k\mu(v,w) - l\mu(u,x) - kl\mu(v,x)\,,
\end{aligned}$$

and the result follows. □

To prove that condition (*) holds we need some technology.

Lemma 1.15. *Suppose V and W are vector spaces over a field $\mathbb{F}$ and $T : V \to W$ is a linear transformation. Let S be a subspace of V. Then there exists a linear transformation $\overline{T} : V/S \to W$ such that $\overline{T}(x+S) = T(x)$ for all $x \in V$ if and only if $T(s) = 0$ for all $s \in S$.*

Moreover, if $\overline{T}$ exists it is unique, and very linear transformation $V/S \to W$ arises in this way from a unique T.

Proof. Suppose that $\overline{T}$ exists. Then for all $s \in S$ we have $T(s) = \overline{T}(s+S) = \overline{T}(0_{V/S}) = 0$.

Now suppose that $T(s) = 0$ for all $s \in S$. We must show that $\overline{T}(x+S) = T(x)$ makes $\overline{T}$ well defined. In other words, we must show that if $v, v' \in V$ are such that $v + S = v' + S$, then $\overline{T}(v+S) = T(v) = T(v') = \overline{T}(v'+S)$. Now, if $v + S = v' + S$, then $v - v' \in S$. Thus $T(v) - T(v') = T(v - v') = 0$ and so $T(v) = T(v')$ as required.

The final remarks are clear, since the statement $\overline{T}(x+S) = T(x)$ uniquely determines either of T, $\overline{T}$ from the other. □

Theorem 1.16 (Universal property of tensor products)**.** *Let V, W, M be vector spaces over a field $\mathbb{F}$. Let $V \otimes W = \mathcal{P}/\mathcal{R}$ be the tensor product, as defined in Definition 1.13. For any bilinear map $\phi : V \times W \to M$, there is a unique linear transformation $\overline{\phi} : V \otimes W \to M$, such that $\overline{\phi}(v \otimes w) = \phi(v,w)$ for all $v, w \in V$. Moreover, every linear transformation in $\mathsf{L}(V \otimes W, M)$ arises in this way.*

Proof. Since $V \times W$ is a basis for $\mathcal{P}$, we can extend any map $\phi : V \times W \to M$ to a linear map $\phi : \mathcal{P} \to M$. By a similar argument to Proposition 1.14, one can see that ϕ is bilinear if and only if $\phi(s) = 0$ for all $s \in \mathcal{R}$. Thus by Lemma 1.15, there exists a linear map $\overline{\phi} : \mathcal{P}/\mathcal{R} \to M$ if and only if ϕ is bilinear, and every such linear transformation arises from a bilinear map in this way. □

More generally, every linear map $V_1 \otimes V_2 \otimes \cdots \otimes V_k \to M$ corresponds to a k-linear map $V_1 \times V_2 \times \cdots \times V_k \to M$.

Theorem 1.17. *Condition (*) holds for the tensor product as defined in Definition 1.13.*

Proof. Let $\beta =$ be a basis for V, and γ a basis for W. We must show that $\mu(\beta \times \gamma)$ is a basis for $\mathcal{P}/\mathcal{R}$. First we show it is spanning. Let $z \in \mathcal{P}/\mathcal{R}$. Then we can write

$$\begin{aligned}
z &= a_1 \pi_{\mathcal{P}/\mathcal{R}}\big((u_1, x_1)\big) + \cdots + a_m \pi_{\mathcal{P}/\mathcal{R}}\big((u_m, x_m)\big) \\
&= a_1 \mu(u_1, x_1) + \cdots + a_m \mu(u_m, x_m)
\end{aligned}$$

with $a_1, \dots a_m \in \mathbb{F}$, $u_1, \dots u_m \in V$, and $x_1, \dots x_m \in W$. But now $u_i = \sum_j b_{ij} v_j$ with $v_j \in \beta$ and $x_i = \sum_k c_{ij} w_k$ with $w_k \in \gamma$. Hence

$$\begin{aligned} z &= a_1 \mu(u_1, x_1) + \cdots + a_m \mu(u_m, x_m) \\ &= \sum_{i=1}^{m} a_i \mu(\sum_j b_{ij} v_j \,, \sum_k b_{ik} w_k) \\ &= \sum_{i=1}^{m} \sum_{j,k} a_i b_{ij} c_{ik} \mu(v_j, w_k) \,. \end{aligned}$$

Next we show linear independence. Suppose that

$$\sum_{i,j} d_{ij} \mu(v_i, w_j) = 0$$

with $v_i \in \beta$, $w_j \in \gamma$. Let $f_k \in V^*$ be the linear functional for which $f_k(v_k) = 1$, and $f_k(v) = 0$, for $v \in \beta \setminus \{v_k\}$. Define $F_k : V \times W \to W$ to be $F_k(v, w) = f_k(v) w$. Then F_k is bilinear, so by Theorem 1.16 there is a induced linear transformation $\overline{F_k} : V \otimes W \to W$ such that $\overline{F_k}(\mu(u, k)) = f_k(u) x$. Applying $\overline{F_k}$ to the equation $\sum_{i,j} d_{ij} \mu(v_i, w_j) = 0$, we have

$$\begin{aligned} 0 &= \overline{F_k} \left(\sum_{i,j} d_{ij} \mu(v_i, w_j) \right) \\ &= \sum_{i,j} d_{ij} f_k(v_i) w_j \\ &= \sum_j d_{kj} w_j \,, \end{aligned}$$

for all k. But $\{w_j\}$ are linearly independent, so $d_{kj} = 0$. □

Exercise: Let $V \otimes W$ be the tensor product as defined in Definition 1.13. Let (Y, μ) be a pair satisfying the definition of the tensor product of V and W as in Definition 1.1. Prove that there is a natural isomorphism $\varphi : V \otimes W \to Y$ such that $\varphi(v \otimes w) = \mu(v, w)$ for all $v \in V$, $w \in W$. (In other words, show that the tensor product is essentially unique.)

1.4 The trace and composition of linear transformations

Let V be a finite dimensional vector space over a field $\mathbb{F}$. Recall from Example 1.7 that $V^* \otimes V$ is identified with the space $\mathsf{L}(V)$ of linear operators on V.

Proposition 1.18. *There is a natural linear transformation* $\mathrm{tr} : V^* \otimes V \to \mathbb{F}$*, such that* $\mathrm{tr}(f \otimes v) = f(v)$ *for all* $f \in V^*$, $v \in V$.

Proof. The map $(f, v) \mapsto f(v)$ is bilinear, so the result follows by Theorem 1.16. □

The map tr is called the **trace** on V. We can also define the trace from $V \otimes V^* \to \mathbb{F}$, using either the fact that $V = V^{**}$, or by switching the factors in the definition (both give the same answer).

Example 1.19. Let $V = \mathbb{F}^n_{\text{col}}$. Let $\{e_1, \ldots e_n\}$ be the standard basis for V, and let $\{f_1, \ldots, f_n\}$ be the dual basis (standard basis for row vectors).

From Example 1.5, we can identify $V^* \otimes V$ with $\mathsf{M}_{n\times n}$. A matrix $A \in \mathsf{M}_{n\times n}$ can be written as

$$\begin{aligned} A &= \sum_{i,j} A_{ij} e_j f_i \\ &= \sum_{i,j} A_{ij} f_i \otimes e_j \,. \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{tr}(A) &= \sum_{i,j} A_{ij} \operatorname{tr}(f_i \otimes e_j) \\ &= \sum_{i,j} A_{ij} f_i(e_j) \\ &= \sum_{i,j} A_{ij} \delta_{ij} \\ &= A_{11} + A_{22} + \cdots + A_{nn} \,. \end{aligned}$$

Now, let $V_1, V_2, \ldots, V_k$ be finite dimensional vector spaces over $\mathbb{F}$, and consider the tensor product space $V_1 \otimes V_2 \otimes \cdots \otimes V_k$. Suppose that $V_i = V_j^*$ for some i, j. Using the trace, we can define a linear transformation

$$\operatorname{tr}_{ij} : V_1 \otimes \cdots \otimes V_k \to V_1 \otimes \cdots \otimes V_{i-1} \otimes V_{i+1} \otimes \cdots \otimes V_{j-1} \otimes V_{j+1} \cdots \otimes V_k$$

satisfying

$$\operatorname{tr}_{ij}(v_1 \otimes \cdots \otimes v_k) = \operatorname{tr}(v_i \otimes v_j)(v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes \cdots \otimes v_{j-1} \otimes v_{j+1} \cdots \otimes v_k)\,.$$

This transformation is sometimes called **contraction**. The most basic example of this is something familiar in disguise.

Example 1.20. Let V_1, V_2, V_3 be finite dimensional vector spaces. From Example 1.7 we have $V_1^* \otimes V_2 = \mathsf{L}(V_1, V_2)$, $V_2^* \otimes V_3 = \mathsf{L}(V_2, V_3)$ and $V_1^* \otimes V_3 = \mathsf{L}(V_1, V_3)$. The map

$$\operatorname{tr}_{23} : V_1^* \otimes V_2 \otimes V_2^* \otimes V_3 \to V_1^* \otimes V_3$$

can therefore be identified with a map from

$$\operatorname{tr}_{23} : \mathsf{L}(V_1, V_2) \otimes \mathsf{L}(V_2, V_3) \to \mathsf{L}(V_1, V_3)\,.$$

By theorem 1.16, this arises from a bilinear map

$$\mathsf{L}(V_1, V_2) \times \mathsf{L}(V_2, V_3) \to \mathsf{L}(V_1, V_3)\,.$$

This map is nothing other than the composition of linear transformations: $(T, U) \mapsto U \circ T$.

Proof. Since composition is bilinear, and the space of all linear transformations is spanned by rank 1 linear transformations, it is enough to prove this in the case where $T \in \mathsf{L}(V_1, V_2)$, and $U \in \mathsf{L}(V_2, V_3)$ have rank 1. If $T \in \mathsf{L}(V_1, V_2)$ has rank 1, then $T(x) = f(x)v$, where $v \in V_2$, $f \in V_1^*$. As in Example 1.7 we identify this with $f \otimes v \in V_1^* \otimes V_2$. Similarly $U(y) = g(y)w$ where $w \in V_3$, $g \in V_2^*$, and U is identified with $g \otimes w \in V_2^* \otimes V_3$. Then

$$U \circ T(x) = f(x)g(v)w = \mathrm{tr}(v \otimes g)f(x)w$$

so $U \circ T$ is identified with

$$\mathrm{tr}(v \otimes g)(f \otimes w) = \mathrm{tr}_{23}(f \otimes v \otimes g \otimes w)\,,$$

as required. □

2 The polynomial algebra

The universal property for tensor products says that every bilinear map $V \times W \to M$ is essentially the same as a linear map $V \otimes W \to M$. If that linear map is something reasonably natural and surjective, we may be able to view M as a quotient space of $V \otimes W$. Most interesting products in algebra can be somehow regarded as quotient spaces tensor products. One of the most familiar is the polynomial algebra.

Let V be a finite dimensional vector space over a field $\mathbb{F}$.) Let

$$\mathsf{T}^0(V) = \mathbb{F}\,, \quad \mathsf{T}^1(V) = V\,, \quad \text{and} \quad \mathsf{T}^k(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k} = V^{\otimes k}\,.$$

Let $\mathcal{C}^k$ be the subspace of $\mathsf{T}^k(V)$ spanned by all vectors of the form

$$(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_k) - (x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_k)\,,$$

with $x_1, \dots x_k \in V$, and $i, j \in \{1, \dots, k\}$.

The tensor product is not commutative, but when we quotient by the space $\mathcal{C}^k$, we impose the commutativity relation and obtain a commutative product.

Definition 2.1. The k^{th} **symmetric power** of V is the quotient space $\mathsf{Sym}^k(V) = \mathsf{T}^k(V)/\mathcal{C}^k$. We write $x_1 \cdot x_2 \cdots x_k$ (or $x_1 x_2 \cdots x_k$) for $\pi_{\mathsf{T}^k(V)/\mathcal{C}^k}(x_1 \otimes x_2 \otimes \cdots \otimes x_k)$.

Note the following facts.

1. The map $V^k \to \mathsf{Sym}^k(V)$, defined by $(x_1, x_2, \dots, x_k) \mapsto x_1 \cdot x_2 \cdots x_k$ is k-linear.

2. $x_1 \cdots x_i \cdots x_j \cdots x_k = x_1 \cdots x_j \cdots x_i \cdots x_k$ for all $i, j \in \{1, \dots, k\}$ and all $x_1, \dots x_k \in V$. (In other words "$\cdot$" is commutative.)

3. If $\{v_1, v_2, \dots, v_n\}$ is a basis for V, than any element of $\mathsf{Sym}^k(V)$ can be written as a polynomial of degree k in $v_1, v_2, \dots, v_n$.

4. There is a bilinear map $\mathsf{Sym}^k(V) \times \mathsf{Sym}^l(V) \to \mathsf{Sym}^{k+l}(V)$ such that

$$(x_1 \cdot x_2 \cdots x_k, y_1 \cdot y_2 \cdots y_l) \mapsto x_1 \cdot x_2 \cdots x_k \cdot y_1 \cdot y_2 \cdots y_l\,.$$

   When expressed in terms of a basis, this map is just polynomial multiplication.

In other words, $\mathsf{Sym}^k(V)$ can be thought of as the space of polynomials of degree k in the elements of V, and the operation "$\cdot$" is just ordinary commutative polynomial multiplication. When the elements are expressed in terms of a basis, these look exactly like polynomials.

Example 2.2. Consider the product $(1,1,1)\cdot(-1,1,1)\cdot(0,1,-1) \in \mathsf{Sym}^3(\mathbb{R}^3)$ Let $x = (1,0,0)$, $y = (0,1,0)$, $z = (0,0,1)$. When we expand this product in terms of x, y, z we obtain:

$$\begin{aligned}(1,1,1)\cdot(-1,1,1)\cdot(0,1,-1) &= (x+y+z)(-x+y+z)(y-z)\\ &= x^2y - x^2z + y^3 + y^2z - yz^2 - z^3\,.\end{aligned}$$

If we want to consider polynomials of mixed degrees, we can look at the direct sum of all the symmetric products of V:

$$\mathsf{Sym}^\bullet(V) := \mathsf{Sym}^0(V) \oplus \mathsf{Sym}^1(V) \oplus \mathsf{Sym}^2(V) \oplus \mathsf{Sym}^3(V) \oplus \cdots$$

The multiplication $\mathsf{Sym}^k(V) \times \mathsf{Sym}^l(V) \to \mathsf{Sym}^{k+l}(V)$ makes $\mathsf{Sym}^\bullet(V)$ a commutative ring.

3 The exterior algebra

3.1 Definition and examples

The exterior algebra is constructed similarly to the polynomial algebra. Its properties are analogous to those of the polynomial algebra, but the multiplication that arises is not commutative. As before V will be a vector space over a field $\mathbb{F}$.

Let $\mathcal{A}^k$ be the subspace of $\mathsf{T}^k(V)$ spanned by all vectors of the form

$$x_1 \otimes x_2 \otimes \cdots \otimes x_k$$

with $x_1, \dots x_k \in V$ where $x_i = x_j$ for some $i \neq j$.

Definition 3.1. The k^{th} **exterior power** of V is the quotient space $\bigwedge^k(V) = \mathsf{T}^k(V)/\mathcal{A}^k$. We write $x_1 \wedge x_2 \wedge \cdots \wedge x_k\ \pi_{\mathsf{T}^k(V)/\mathcal{A}^k}(x_1 \otimes x_2 \otimes \cdots \otimes x_k)$.

Example 3.2. If V is any vector space, then $\mathcal{A}^0 = \{0\}$ and $\mathcal{A}^1 = \{0\}$. Therefore $\bigwedge^0(V) = \mathsf{T}^0(V) = \mathbb{F}$, and $\bigwedge^1(V) = \mathsf{T}^1(V) = V$.

We have the following properties:

1. The map $V^k \to \bigwedge^k(V)$, defined by $(x_1, x_2, \dots, x_k) \mapsto x_1 \wedge x_2 \wedge \cdots \wedge x_k$ is k-linear.

2. For all $x_1, \dots x_k \in V$, $x_1 \wedge x_2 \wedge \dots x_k = 0$ if $x_i = x_j$ for some $i \neq j$.

3. For all $x_1, \dots x_k \in V$ and all $i \neq j$, we have

$$x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_j \wedge \cdots \wedge x_k = -x_1 \wedge \cdots \wedge x_j \wedge \cdots \wedge x_i \wedge \cdots \wedge x_k \,.$$

Proof.

$$\begin{aligned} 0 &= x_1 \wedge \cdots \wedge (x_i + x_j) \wedge \cdots \wedge (x_i + x_j) \wedge \cdots \wedge x_k \\ &= x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_i \wedge \cdots \wedge x_k \;+\; x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_j \wedge \cdots \wedge x_k \\ &\quad + \; x_1 \wedge \cdots \wedge x_j \wedge \cdots \wedge x_i \wedge \cdots \wedge x_k \;+\; x_1 \wedge \cdots \wedge x_j \wedge \cdots \wedge x_j \wedge \cdots \wedge x_k \\ &= x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_j \wedge \cdots \wedge x_k \;+\; x_1 \wedge \cdots \wedge x_j \wedge \cdots \wedge x_j \wedge \cdots \wedge x_k \end{aligned}$$ □

We can use these properties to simplify expressions.

Example 3.3. Let $V = \mathbb{R}^2$, $x = (1,0)$, $y = (0,1)$. Then

$$\begin{aligned} (a,b) \wedge (c,d) &= (ax + by) \wedge (cx + dy) \\ &= ac(x \wedge x) + ad(x \wedge y) + bc(y \wedge x) + bd(y \wedge y) \\ &= 0 + ad(x \wedge y) + bc(y \wedge x) + 0 \\ &= ad(x \wedge y) - bc(y \wedge x) \\ &= (ad - bc)(x \wedge y) \end{aligned}$$

Note that the coefficient of $x \wedge y$ is $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

On the other hand, if we do a similar calculation with three terms,

$$\begin{aligned} (a,b) \wedge (c,d) \wedge (e,f) &= (ax + by) \wedge (cx + dy) \wedge (ex + fy) \\ &= ace(x \wedge x \wedge x) + ade(x \wedge y \wedge x) + bce(y \wedge x \wedge x) + bde(y \wedge y \wedge x) \\ &\quad + acf(x \wedge x \wedge y) + adf(x \wedge y \wedge y) + bcf(y \wedge x \wedge y) + bdf(y \wedge y \wedge y) \\ &= 0 \end{aligned}$$

we get exactly zero. It looks the the dimensions of $\dim \bigwedge^k(\mathbb{R}^2)$ are given by the table below.

k	0	1	2	3	4	5	$\cdots$
$\dim \bigwedge^k(\mathbb{R}^2)$	1	2	1	0	0	0	$\cdots$
basis of $\bigwedge^k(\mathbb{R}^2)$	$\{1\}$	$\{x, y\}$	$\{x \wedge y\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\cdots$

However, at this point we haven't completely proved this. We still need to show that $x \wedge y$ is not the zero vector in $\bigwedge^2(\mathbb{R}^2)$.

As with the symmetric products, there is a multiplication map between the exterior products. This allows us to think of "$\wedge$" as a multiplication operator. This product is not commutative.

Proposition 3.4. *There is a natural bilinear map* $\bigwedge^k(V) \times \bigwedge^l(V) \to \bigwedge^{k+l}(V)$ *such that*

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_k, w_1 \wedge w_2 \wedge \cdots \wedge w_l) \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_k \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_l$$

Proof. Bilinearity means linearity in each component, with the other held fixed. Therefore it is enough to check that there is a linear map satisfying

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_k \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_l$$

for fixed $w_1, \dots, w_l \in V$. First note that the map

$$\phi(v_1, v_2, \dots, v_k) = v_1 \wedge v_2 \wedge \cdots \wedge v_k \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_l$$

is k-linear, so by Theorem 1.16, there is an induced linear map $\overline{\phi} : \mathsf{T}^k(V) \to \bigwedge^{k+l}(V)$, such that

$$\overline{\phi}(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_1 \wedge v_2 \wedge \cdots \wedge v_k \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_l$$

Note that for all $s \in \mathcal{A}$ we have $\overline{\phi}(s) = 0$. Since $\bigwedge^k(V) = \mathsf{T}^k(V)/\mathcal{A}$, by Lemma 1.15, there is an induced linear map from $\bigwedge^k(V)$ to $\bigwedge^{k+l}(V)$, as required. □

Definition 3.5. The collection of the spaces $\bigwedge^k(V)$, for $k = 0, 1, 2, \dots$, together with the operation $\wedge$ is called the **exterior algebra** on V.

More specifically, the space

$$\textstyle\bigwedge^{\bullet}(V) := \bigwedge^0(V) \oplus \bigwedge^1(V) \oplus \bigwedge^2(V) \oplus \bigwedge^3(V) \oplus \cdots$$

is a ring, with multiplication defined by the wedge product. A vector space over $\mathbb{F}$ which also has the structure of a ring is called an $\mathbb{F}$-algebra.

Example 3.6. Let $V = \mathbb{R}^3$, $x = (1, 0, 0)$, $y = (0, 1, 0)$, $z = (0, 0, 1)$. We have

$$\begin{aligned}(a, b, c) \wedge (d, e, f) &= (ax + by + cz) \wedge (dx + ey + fz) \\ &= ae(x \wedge y) + bd(y \wedge x)af(x \wedge z) + cd(z \wedge x)bf(y \wedge z) + ce(z \wedge y) \\ &= (ae - bd)(x \wedge y) + (af - cd)(x \wedge z) + (bf - ce)(y \wedge z)\,.\end{aligned}$$

To compute $(a, b, c) \wedge (d, e, f) \wedge (g, h, i)$ we just multiply this resulting expression by $gx + hy + iz$.

$$\begin{aligned}&(a, b, c) \wedge (d, e, f) \wedge (g, h, i) \\ &\quad= ((ae - bd)(x \wedge y) + (af - cd)(x \wedge z) + (bf - ce)(y \wedge z)) \wedge (gx + hy + iz) \\ &\quad= (ae - bd)i(x \wedge y \wedge z) + (af - cd)h(x \wedge z \wedge y) + (bf - ce)z(y \wedge z \wedge x) \\ &\quad= ((ae - bd)i - (af - cd)h + (bf - ce)z)(x \wedge y \wedge z)\end{aligned}$$

Here we've used $y \wedge z \wedge x = -x \wedge z \wedge y = x \wedge y \wedge z$. Note that the coefficient is equal to

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and moreover, the expression we obtained is exactly the cofactor expansion along the bottom row.

Example 3.7. Let $V = \mathbb{R}^4$ with basis $\{x, y, z, w\}$. We have

$$\begin{aligned}(x \wedge y + z \wedge w) \wedge (x \wedge y + z \wedge w) &= x \wedge y \wedge x \wedge y \ + \ x \wedge y \wedge z \wedge w \\ &\quad + \ z \wedge w \wedge x \wedge y \ + \ z \wedge w \wedge z \wedge w \\ &= x \wedge y \wedge z \wedge w \ - \ x \wedge w \wedge z \wedge y \\ &= x \wedge y \wedge z \wedge w \ + \ x \wedge y \wedge z \wedge w \\ &= 2x \wedge y \wedge z \wedge w\end{aligned}$$

It may at first be surprising that the answer we obtained is not zero, but as we will soon see, this product really is non-zero. What this proves is that $x \wedge y + z \wedge w$ is not a pure wedge product, i.e. $x \wedge y + z \wedge w \neq u \wedge v$ for any $u, v \in \mathbb{R}^4$. Indeed, $(u \wedge v) \wedge (u \wedge v) = 0$ for all u, v. However, expressions which are not pure wedge products can be multiplied with themselves to give a non-zero vector.

Proposition 3.8. *If* $\alpha \in \bigwedge^k(V)$, $\omega \in \bigwedge^l(V)$, *then* $\alpha \wedge \omega = (-1)^{kl} \omega \wedge \alpha$.

Thus although the wedge product is not commutative, it is reasonably close commutative. A product with this property said to **supercommutative**.

Proof. If $\alpha = v_1 \wedge v_2 \wedge \cdots \wedge v_k$, $\omega = w_1 \wedge w_2 \wedge \cdots \wedge w_l$, then we can get from $v_1 \wedge v_2 \wedge \cdots \wedge v_k \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_l$, to $\pm w_1 \wedge w_2 \wedge \cdots \wedge w_l \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_k$ by swapping adacent v's and w's. Each swap contributes a factor of -1, and it takes exactly kl swaps. □

3.2 Linear independence

As we have already seen, the exterior algebra has many determinant-like properties. One of the key properties of the determinant, is that the determinant of a square matrix is non-zero if and only if its rows (or columns) are linearly independent. Our next goal is to prove the following theorem, which says that the wedge product generalizes this idea.

Theorem 3.9. *Let V be a vector space over a field $\mathbb{F}$. A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if $v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0 \in \bigwedge^k(V)$.*

The "if" direction is easy. Unfortunately, at the moment we don't have any tools for proving that two vectors in $\bigwedge^k(V)$ are not equal. One of the best ways to prove that a vector is non-zero is to apply a linear transformation to it. If the result is non-zero, the vector must be non-zero. This will be our strategy.

The following theorem is analogous to the universal property for tensor products.

Theorem 3.10. *Let V, M be vectors spaces over $\mathbb{F}$. For any k-linear, alternating function, $\phi : V^k \to M$, there exists a unique map $\overline{\phi} : \bigwedge^k(V) \to M$ such that $\overline{\phi}(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \phi(v_1, v_2, \ldots, v_k)$ for all $v_1, v_2, \ldots, v_k \in V$. Moreover every linear map $\bigwedge^k(V) \to M$ arises in this way.*

Proof. Since ϕ is k-linear, by Theorem 1.16, there is an equivalent linear map $\widetilde{\phi} : \mathsf{T}^k(V) \to M$. We must show that $\widetilde{\phi}$ descends to a map from $\bigwedge^k(V) = \mathsf{T}^k(V)/\mathcal{A}^k$ to M. By Lemma 1.15, such a map descends if and only if $\widetilde{\phi}(s) = 0$ for all $s \in \mathcal{A}^k$, and moreover every such linear map arises in this way. But by the definition of $\mathcal{A}^k$ this is equivalent to the fact that ϕ is alternating, as required. □

We can use Theorem 3.10 to obtain linear functionals on $\bigwedge^k(V)$.

Theorem 3.11. *Let $f_1, f_2, \ldots, f_k \in V^*$, where V^* is the dual space to V. There is a well defined linear functional $F_{f_1,f_2,\ldots,f_k} : \bigwedge^k(V) \to \mathbb{F}$, such that*

$$F_{f_1,f_2,\ldots,f_k}(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \det \begin{pmatrix} f_1(v_1) & f_1(v_2) & \cdots & f_1(v_k) \\ f_2(v_1) & f_2(v_2) & \cdots & f_2(v_k) \\ \vdots & \vdots & & \vdots \\ f_k(v_1) & f_k(v_2) & \cdots & f_k(v_k) \end{pmatrix}.$$

Moreover, if $f'_1, f'_2, \ldots, f'_k \in V^*$, and $f_1 \wedge f_2 \wedge \cdots \wedge f_k = f'_1 \wedge f'_2 \wedge \cdots \wedge f'_k \in \bigwedge^k(V^*)$, then $F_{f_1,f_2,\ldots,f_k} = F_{f'_1,f'_2,\ldots,f'_k}$.

In light of this second part, we write $f_1 \wedge f_2 \wedge \cdots \wedge f_k(v_1 \wedge v_2 \wedge \cdots \wedge v_k)$ for $F_{f_1,f_2,\ldots,f_k}(v_1 \wedge v_2 \wedge \cdots \wedge v_k)$.

Proof. The first statement follows from Theorem 3.10 and the fact that determinant is k-linear alternating in the columns. The second statement is equivalent to saying that for any given $v_1, v_2, \ldots v_k$, there is a well defined linear map from $G : \bigwedge^k(V^*) \to \mathbb{F}$ such that $G(f_1 \wedge f_2 \wedge \cdots \wedge f_k) = F_{f_1,f_2,\ldots,f_k}(v_1 \wedge v_2 \wedge \cdots \wedge v_k)$. This follows from Theorem 3.10 and the fact that the determinant is k-linear and alternating in the rows. □

Proof of Theorem 3.9. First, suppose that $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent. Then we can write one of these vectors as a linear combination of the others; without loss of generality, say $v_k = a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1}$. Then

$$\begin{aligned} v_1 \wedge v_2 \wedge \cdots \wedge v_k &= v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge (a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1}) \\ &= a_1v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v_1 \;+\; a_2v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v_2 \\ &\qquad + \;\cdots\; + \; a_{k-1}v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1} \wedge v_{k-1} \\ &= 0 \end{aligned}$$

Now, suppose that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. Since this set can be extended to a basis, we can define linear functionals $f_1, \ldots, f_k \in V^*$ such that $f_i(v_j) = \delta_{ij}$. But then by Theorem 3.11 we have

$$f_1 \wedge f_2 \wedge \cdots \wedge f_k(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = 1\,,$$

so $v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0$. □

Corollary 3.12. *If* $\dim V < k$ *then* $\dim \bigwedge^k(V) = 0$.

Proof. If $\dim V < k$ then any k vectors will be linearly dependent, so any k-fold wedge product is zero. □

Using similar ideas, we can compute the dimensions of $\bigwedge^k(V)$ when V is finite dimensional, for all $k < \dim V$.

Theorem 3.13. *Let V be an n-dimensional vector space over a field $\mathbb{F}$, and let $\beta_1 = \{e_1, \ldots e_n\}$ be a basis for V. For each $k \leq n$, let*

$$\beta_k = \{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

Then β_k is a basis for $\bigwedge^k(V)$. In pariticular, $\dim \bigwedge^k(V) = \binom{n}{k}$.

Exercise: Prove Theorem 3.13.

3.3 Functoriality and the determinant

Let $T : V \to W$ be a linear transformation. For any non-negative integer k, we can define a linear transformation $T_* : \bigwedge^k(V) \to \bigwedge^k(W)$ satisfying

$$T_*(v_1 \wedge v_2 \wedge \cdots \wedge v_k) := T(v_1) \wedge T(v_2) \wedge \cdots \wedge T(v_k),$$

for all $v_1, v_2, \ldots, v_k \in V$. To see that T_* exists, by Theorem 3.10 it is enough to check that the map

$$(v_1, v_2, \ldots, v_k) \mapsto T(v_1) \wedge T(v_2) \wedge \cdots \wedge T(v_k)$$

is bilinear and alternating, which is straightforward.

Proposition 3.14. *Let V and W be vector spaces over a field $\mathbb{F}$, and let $T \in \mathsf{L}(V, W)$. For any non-negative integer k, the following are true.*

(i) If T is surjective then $T_* \in \mathsf{L}(\bigwedge^k(V), \bigwedge^k(W))$ is surjective.

(ii) If T is injective then $T_* \in \mathsf{L}(\bigwedge^k(V), \bigwedge^k(W))$ is injective.

(iii) If T is an isomorphism then $T_* \in \mathsf{L}(\bigwedge^k(V), \bigwedge^k(W))$ is an isomorphism.

Proof. For (i) suppose T is surjective. It suffices to show that every vector $w_1 \wedge w_2 \wedge \ldots w_k \in \bigwedge^k(W)$ is in the range of T_*, which is clear.

For (ii) suppose T is injective. Let γ be a basis for V, and extend $\{T(w) \mid w \in \gamma\}$ to a basis β for W. If

$$\begin{aligned} 0 &= T_*\left(\sum_{1\leq i_1<i_2<\cdots<i_k\leq n} a_{i_1 i_2 \ldots i_k} w_{i_1} \wedge w_{i_2} \wedge \cdots \wedge w_{i_k}\right) \\ &= \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} a_{i_1 i_2 \ldots i_k} T(w_{i_1}) \wedge T(w_{i_2}) \wedge \cdots \wedge T(w_{i_k}) \end{aligned}$$

where $\{w_1, w_2, \ldots, w_n\}$ is an ordered finite subset of γ then all $a_{i_1 i_2 \ldots i_k}$ must be equal to zero. (*Exercise:* Prove this.) Thus T_* is injective.

Finally (iii) follows from (i) and (ii). □

One consequence of this is that if W is a subspace of V, then $\bigwedge^k(W)$ can be regarded as a subspace of $\bigwedge^k(V)$. More formally, let $i : W \to V$ be the inclusion map. This is injective, so $i_* : \bigwedge^k(W) \to \bigwedge^k(V)$ is injective. Thus we can identify $\bigwedge^k(W)$ with $\mathsf{R}(i_*) \subset \bigwedge^k(V)$.

Proposition 3.15. *Let V_1, V_2, V_3 be vector spaces over $\mathbb{F}$, and let $T \in \mathsf{L}(V_1, V_2)$, $U \in \mathsf{L}(V_2, V_3)$. For any non-negative integer k, we have:*

(i) $(UT)_* = U_* T_* \in \mathsf{L}(\bigwedge^k(V_1), \bigwedge^k(V_3))$.

(ii) If T is an isomorphism, then $(T_*)^{-1} = (T^{-1})_*$.

Exercise: Prove Proposition 3.15.

Theorem 3.16. *Let $A \in \mathsf{M}_{n\times n}(\mathbb{F})$, and let $V = \mathbb{F}^n_{col}$. Then for all $\omega \in \bigwedge^n(V)$, we have*

$$(L_A)_*(\omega) = \det(A)\omega\,,$$

where $L_A \in \mathsf{L}(V)$ is the linear transformation $L_A(x) = Ax$.

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for V; let $\{f_1, f_2, \ldots, f_n\}$ be the dual basis. Since $\dim \bigwedge^n(V) = 1$, it suffices to prove the result for $\omega = e_1 \wedge e_2 \wedge \cdots \wedge e_n$. We have

$$\begin{aligned} f_1 \wedge f_2 \wedge \cdots \wedge f_n((L_A)_*(e_1 \wedge e_2 \wedge \cdots \wedge e_n)) &= f_1 \wedge f_2 \wedge \cdots \wedge f_n(Ae_1 \wedge Ae_2 \wedge \cdots \wedge Ae_n) \\ &= \det \begin{pmatrix} f_1(Ae_1) & f_1(Ae_2) & \cdots & f_1(Ae_n) \\ f_2(Ae_1) & f_2(Ae_2) & \cdots & f_2(Ae_n) \\ \vdots & \vdots & & \vdots \\ f_n(Ae_1) & f_n(Ae_2) & \cdots & f_n(Ae_n) \end{pmatrix} = \det A \end{aligned}$$

Since $f_1 \wedge f_2 \wedge \cdots \wedge f_n(e_1 \wedge e_2 \wedge \cdots \wedge e_n) = 1$, the result follows. □

Corollary 3.17. *Let V be an n-dimensional vector space over $\mathbb{F}$, and $T \in \mathsf{L}(V)$. Then for all $\omega \in \bigwedge^n(V)$, we have $T_*(\omega) = \det(T)\omega$.*

Proof. Let $\Phi : V \to \mathbb{F}^n_{\text{col}}$ be any isomorphism. Then by definition, $\det T = \det(\Phi T \Phi^{-1})$.

$$\Phi_*(T_*(\omega))) = (\Phi T \Phi^{-1})_*(\Phi_*(\omega)) = (\det T)(\Phi_*(\omega)) = \Phi_*((\det T)\omega)\,.$$

Since Φ_* is an isomorphism, the result follows. □

3.4 Plücker relations

The exterior algebra is useful in studying finite dimensional subspaces of a vector space.

Theorem 3.18. *Let V be an vector space over a field $\mathbb{F}$. Let W_1, W_2 be k-dimensional subspaces of V, with bases $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_k\}$ respectively. Then*

$$x_1 \wedge x_2 \wedge \cdots \wedge x_k = c\, y_1 \wedge y_2 \wedge \cdots \wedge y_k$$

for some non-zero scalar $c \in \mathbb{F}$ if and only if $W_1 = W_2$.

Exercise: Prove Theorem 3.18.

In other words, vectors in $\bigwedge^k(V)$ which are pure wedges represent k-dimensional subspaces of V, up to a non-zero constant. It is therefore an important questiont to ask which vectors in $\bigwedge^k(V)$ are pure wedges.

Our first criterion for determining when a vector $\omega \in \bigwedge^k(V)$ is a pure wedge product is the following.

Theorem 3.19. *Let V vector space over $\mathbb{F}$, and let $\omega \in \bigwedge^k(V)$ be a non-zero vector. Let $W_\omega = \{x \in V \mid x \wedge \omega = 0\}$. Then $\dim W_\omega \leq k$, and $\dim W_\omega = k$ if and only if ω is a pure wedge.*

Before we can prove this, we need to develop some further results.

Lemma 3.20. *Let V be a vector space over $\mathbb{F}$. Let $f \in V^*$. For each k, there is a map $\chi_f : \bigwedge^k(V) \to \bigwedge^{k-1}(V)$ such that for all $v_1, v_2, \ldots, v_k \in V$,*

$$\chi_f(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = f(v_1)v_2 \wedge v_3 \wedge \cdots \wedge v_k \;-\; f(v_2)v_1 \wedge v_3 \wedge \cdots \wedge v_k \\ + \;\cdots\; + \;(-1)^{k-1} f(v_k)v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1}\,.$$

Proof. The right hand side is k-linear and alternating in $(v_1, v_2, \ldots v_k)$, so this follows by Theorem 3.10. □

More generally, we have the following important theorem.

Theorem 3.21. *Let V be a vector space over $\mathbb{F}$, and let $k \geq l$ be a positive integers. There exists a bilinear map:*

$$\chi : \textstyle\bigwedge^l(V^*) \times \bigwedge^k(V) \to \bigwedge^{k-l}(V)$$

such that

$$\chi(f_1 \wedge f_2 \wedge \cdots \wedge f_l\,,\, v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \chi_{f_l} \circ \chi_{f_{l-1}} \circ \cdots \circ \chi_{f_1}(v_1 \wedge v_2 \wedge \cdots \wedge v_k)$$

for all $f_1, f_2, \ldots, f_l \in V^*$, $v_1, v_2, \ldots, v_k \in V$.

Proof. We need to show that for fixed $v_1, v_2, \ldots, v_k \in V$, the map

$$(f_1, \ldots, f_l) \mapsto \chi_{f_l} \circ \chi_{f_{l-1}} \circ \cdots \circ \chi_{f_1}(v_1 \wedge v_2 \wedge \cdots \wedge v_k)$$

is l-linear and alternating. The l-linearity is clear. To see that this map is alternating, we need to check that the right hand side is 0 if $f_i = f_j$ for some $i \neq j$. To prove this, first one checks that $\chi_f \circ \chi_f = 0$ for all $f \in V^*$. This implies all that $\chi_f \circ \chi_g = -\chi_g \circ \chi_f$, so we can swap adjacent pairs (picking up a factor of -1 with each swap) until f_i and f_j are next to each other. But then we have

$$\chi_{f_l} \circ \chi_{f_{l-1}} \circ \cdots \circ \chi_{f_1} = \pm \chi_{f_l} \circ \cdots \circ \chi_{f_i} \circ \chi_{f_j} \circ \cdots \circ \chi_{f_1} = 0\,,$$

as required. □

Exercise: If $k = l$, prove that

$$\chi(f_1 \wedge f_2 \wedge \cdots \wedge f_k\,,\, v_1 \wedge v_2 \wedge \cdots \wedge v_k) = f_1 \wedge f_2 \wedge \cdots \wedge f_k(v_1 \wedge v_2 \wedge \cdots \wedge v_k)\,.$$

Lemma 3.22. *Let V' be a subspace of V, and let $x \in V \setminus V'$. There is a linear functional $f \in V^*$ such that $\chi_f(x \wedge \theta) = \theta$ for all $\theta \in \bigwedge^j(V') \subset \bigwedge^j(V)$.*

Proof. Let $f \in V^*$ be a linear functional such that $f(x) = 1$, $f(y) = 0$ for all $y \in V'$. Then we have $0 = \chi_f(x \wedge \theta) = f(x)\theta + 0 = \theta$. □

Lemma 3.23. *Let V, ω and W_ω be as in Theorem 3.19, and let $x \in W_\omega$ be a non-zero vector. Then $\omega = x \wedge \alpha$ for some $\alpha \in \bigwedge^{k-1}(V')$, where V' is a codimension one subspace of V that does not contain x.*

Proof. Let V' be any codimension one subspace of V which does not contain x. Let $f \in V^*$ be the linear functional of Lemma 3.22. We can write $\omega = x \wedge \alpha + \beta$, uniquely, where $\alpha \in \bigwedge^{k-1}(V')$, and $\beta \in \bigwedge^k(V')$. But $0 = \chi_f(x \wedge \omega) = \chi_f(x \wedge \beta) = \beta$, so $\omega = x \wedge \alpha$, as required. □

Proof of Theorem 3.19. Suppose $\omega = v_1 \wedge v_2 \wedge \cdots \wedge v_k$. By Theorem 3.9, since $\omega \neq 0$, $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. Again by Theorem 3.9, $W_\omega = \operatorname{span}\{v_1, v_2, \ldots, v_k\}$, hence $\dim W_\omega = k$.

Now suppose that $\dim W_\omega = k$. We proceed by induction. The result is trivial for $k = 1$. Assume the result is true for all smaller values of k. Let $x \in W_\omega$ be a non-zero vector. By Lemma 3.23 we can write $\omega = x_1 \wedge \alpha$, where $\alpha \in \bigwedge^{k-1}(V')$, and V' is a codimension one subspace of V that does not contain x. Let $W' = W_\omega \cap V'$. We claim that $x' \wedge \alpha = 0$ for all $x' \in W'$. To see this let $f \in V^*$ be the linear functional of Lemma 3.22. Then we have

$$0 = \chi_f(x' \wedge \omega) = \chi_f(-x \wedge x' \wedge \alpha) = -x' \wedge \alpha\,.$$

Since V' has codimension one, $\dim W' = \dim W - 1 = k - 1$. Thus the inductive hypthesis, $\alpha = v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1}$, so $\omega = x \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_{k-1}$.

Finally, if $\dim W_\omega > k$ we could apply this second argument with two different k-dimensional subspaces $W_1, W_2 \subset W_\omega$. But then we would have $\omega \in \bigwedge^k(W_1) \cap \bigwedge^k(W_2)$, which contradicts Theorem 3.18. Therefore we must have $\dim W_\omega \leq k$ for all $\omega \in \bigwedge^k(V)$. □

For some purposes, Theorem 3.19 is not the best answer to the problem of determining which elements of $\bigwedge^k(V)$ are pure wedges. The following criterion has the advantage that it can be used to give explicit necessary and sufficient equations that the coefficients of a pure wedge product must satisfy (when everything is expanded in terms of a basis). These equations are called the **Plücker relations**.

Theorem 3.24 (Plücker relations)**.** *Let V be a vector space over $\mathbb{F}$, and let k be a positive integer. A vector $\omega \in \bigwedge^k(V)$ is a pure wedge product if and only if, for all $\xi \in \bigwedge^{k-1}(V^*)$ we have*

$$\chi(\xi, \omega) \wedge \omega = 0\,.$$

We need two lemmas.

Lemma 3.25. *Let V be a vector space over $\mathbb{F}$, and let $k \geq l$ be a positive integers. Let $W = \operatorname{span}\{v_1, \ldots, v_k\}$, where $v_1, v_2, \ldots v_k \in V$, and let $f_1, f_2, \ldots f_l \in V^*$. Then*

$$\chi(f_1 \wedge f_2 \wedge \cdots \wedge f_l\, , \, v_1 \wedge v_2 \wedge \cdots \wedge v_k) \in \bigwedge^{k-l}(W)\,.$$

Proof. For $l = 1$ the statement is immediate from the definition of χ_f. Using

$$\chi(f_1 \wedge f_2 \wedge \cdots \wedge f_l\, , \, v_1 \wedge v_2 \wedge \cdots \wedge v_k) = \chi_{f_l} \circ \chi_{f_{l-1}} \circ \cdots \circ \chi_{f_1}(v_1 \wedge v_2 \wedge \cdots \wedge v_k)\,,$$

the general case follows by induction. □

Lemma 3.26. *Let V be a vector space over $\mathbb{F}$, and let $\omega \in \bigwedge^k(V)$ be a non-zero vector. Let $\widetilde{W}_\omega = \{\chi(\xi, \omega) \mid \xi \in \bigwedge^{k-1}(V^*)\}$. Then $\dim \widetilde{W}_\omega \geq k$.*

Proof. Let $f_1, f_2, \ldots f_k \in V^*$ be linear functionals such that $f_1 \wedge f_2 \wedge \cdots \wedge f_k(\omega) = 1$. (*Exercise:* Prove that these exist.) Let $\xi_i = (-1)^{k-i} f_1 \wedge \cdots \wedge f_{i-1} \wedge f_{i+1} \wedge \cdots \wedge f_k$. Then,

$$\begin{aligned} f_j(\chi(\xi_i, \omega)) &= (-1)^{k-i} f_j \circ \chi_{f_k} \circ \cdots \circ \chi_{f_{i+1}} \circ \chi_{f_{k-i}} \circ \cdots \circ \chi_{f_1}(\omega) \\ &= (-1)^{k-i} f_1 \wedge \cdots \wedge f_{i-1} \wedge f_{i+1} \wedge \cdots \wedge f_k \wedge f_j(\omega) \\ &= \delta_{ij}\,. \end{aligned}$$

In particular, the set $\{\chi(\xi_1, \omega), \chi(\xi_2, \omega), \ldots, \chi(\xi_k, \omega)\}$ is linearly independent. □

Proof of Theorem 3.24. Suppose that $\omega = v_1 \wedge v_2 \wedge \cdots \wedge v_k$ is a pure wedge. Let $W = \operatorname{span}\{v_1, \ldots, v_k\}$. Then $\omega \in \bigwedge^k(W)$. By Lemma 3.25, $\chi(\xi, \omega) \in \bigwedge^1(W)$, so $\chi(\xi, \omega) \wedge \omega \in \bigwedge^{k+1}(W)$. Since $\dim(W) \leq k$, by Corollary 3.12 this implies $\chi(\xi, \omega) \wedge \omega = 0$.

Suppose that for all $\xi \in \bigwedge^{k-1}(V^*)$ we have $\chi(\xi, \omega) \wedge \omega = 0$. Let $\widetilde{W}_\omega$ be as in Lemma 3.26, and let W_ω be as in Theorem 3.19. Then $x \wedge \omega = 0$ for all $x \in \widetilde{W}_\omega$. So $\widetilde{W}_\omega \subset W_\omega$. By Lemma 3.26 $\dim \widetilde{W}_\omega \geq k$, whereas $\dim W_\omega \leq k$. Thus $\dim W_\omega = k$, which by Theorem 3.19 implies that ω is a pure wedge product. □

Example 3.27. Let $V = \mathbb{R}^5$, with basis $\{e_1, e_2, \ldots, e_5\}$. Let $\{f_1, f_2, \ldots, f_5\}$ be the dual basis. Consider a general vector $\omega \in \bigwedge^2(V)$.

$$\omega = \sum_{1 \leq i < j \leq 5} a_{ij} e_i \wedge e_j\,.$$

In this case the Plücker relations are:

$$\chi(f_k, \omega) \wedge \omega = 0\,, \qquad \text{for } k = 1, 2, 3, 4, 5\,.$$

Expanding this out, we obtain

$$\begin{aligned} 0 &= \chi(f_k, \omega) \wedge \omega \\ &= \chi(f_k, \sum_{1 \leq i < j \leq 5} a_{ij} e_i \wedge e_j) \wedge \Big(\sum_{1 \leq l < m \leq 5} a_{lm} e_l \wedge e_m \Big) \\ &= \Big(\sum_{j=k+1}^{5} a_{kj} e_j - \sum_{i=1}^{k} a_{ik} e_i \Big) \wedge \Big(\sum_{1 \leq l < m \leq 5} a_{lm} e_l \wedge e_m \Big) \\ &= \sum_{j=k+1}^{5} \sum_{\substack{l<m \\ \{l,m\} \cap \{k,j\} = \emptyset}} a_{kj} a_{lm}\, e_j \wedge e_l \wedge e_m \; - \; \sum_{i=1}^{k} \sum_{\substack{l<m \\ \{l,m\} \cap \{i,k\} = \emptyset}} a_{ik} a_{lm}\, e_i \wedge e_l \wedge e_m \end{aligned}$$

It is possible to simplify this further, but at the expense of messier notation. However, from here we can see quite explicitly what happens for any specific value of k. For example, if $k = 1$ we get.

$$
\begin{aligned}
0 &= \sum_{j=2}^{5} \sum_{\substack{l<m \\ \{l,m\}\cap\{1,j\}=\emptyset}} a_{1j}a_{lm}\, e_j \wedge e_l \wedge e_m \\
&= \sum_{\substack{l<m \\ \{l,m\}\cap\{1,2\}=\emptyset}} a_{12}a_{lm}\, e_2 \wedge e_l \wedge e_m \;+\; \sum_{\substack{l<m \\ \{l,m\}\cap\{1,3\}=\emptyset}} a_{13}a_{lm}\, e_3 \wedge e_l \wedge e_m \\
&\quad + \sum_{\substack{l<m \\ \{l,m\}\cap\{1,4\}=\emptyset}} a_{14}a_{lm}\, e_4 \wedge e_l \wedge e_m \;+\; \sum_{\substack{l<m \\ \{l,m\}\cap\{1,5\}=\emptyset}} a_{15}a_{lm}\, e_5 \wedge e_l \wedge e_m \\
&= (a_{12}a_{34}\, e_2 \wedge e_3 \wedge e_4 \;+\; a_{12}a_{35}\, e_2 \wedge e_3 \wedge e_5 \;+\; a_{12}a_{45}\, e_2 \wedge e_4 \wedge e_5) \\
&\quad + \; (a_{13}a_{24}\, e_3 \wedge e_2 \wedge e_4 \;+\; a_{13}a_{25}\, e_3 \wedge e_2 \wedge e_5 \;+\; a_{13}a_{45}\, e_3 \wedge e_4 \wedge e_5) \\
&\quad + \; (a_{14}a_{23}\, e_4 \wedge e_2 \wedge e_3 \;+\; a_{14}a_{25}\, e_4 \wedge e_2 \wedge e_5 \;+\; a_{14}a_{35}\, e_4 \wedge e_3 \wedge e_5) \\
&\quad + \; (a_{15}a_{23}\, e_5 \wedge e_2 \wedge e_3 \;+\; a_{15}a_{24}\, e_5 \wedge e_2 \wedge e_4 \;+\; a_{15}a_{34}\, e_5 \wedge e_3 \wedge e_4) \\
&= (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}) e_2 \wedge e_3 \wedge e_4 \\
&\quad + \; (a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23}) e_2 \wedge e_3 \wedge e_5 \\
&\quad + \; (a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24}) e_2 \wedge e_4 \wedge e_5 \\
&\quad + \; (a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34}) e_3 \wedge e_4 \wedge e_5
\end{aligned}
$$

Thus, we must have

$$
\begin{aligned}
a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} &= 0 \\
a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23} &= 0 \\
a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24} &= 0 \\
a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34} &= 0\,.
\end{aligned}
$$

Each other value of k produces four similar looking equations, but in fact there are only 5 equations in total, because each equation comes up four times in this process. The last equation is (unsurprisingly):

$$
a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34} = 0\,.
$$

By Theorem 3.24, ω is a pure wedge if an only if its coefficients satisfy these five equations.

3.5 Inner products and volumes

Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$, and Let V be an n-dimensional vector space over $\mathbb{F}$, and let $\langle \cdot, \cdot \rangle$ be an inner product on V.

Recall that an inner product on a V satisfies three conditions:

1. For all $x, y, z \in V$, and $c \in \mathbb{F}$, we have $\langle x + cy, z \rangle = \langle x, z \rangle + c\langle y, z \rangle$.

2. For $x, y \in V$, $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

3. For all non-zero vectors $x \in V$, $\langle x, x\rangle > 0$.

If $\mathbb{F}$ is the field of real numbers, then the complex conjugation is trivial.

Theorem 3.28. *There is an inner product $\langle \cdot, \cdot \rangle_k$ on $\bigwedge^k(V)$ such that*

$$\langle x_1 \wedge x_2 \wedge \cdots \wedge x_k \, , \, y_1 \wedge y_2, \wedge \ldots y_k\rangle_k = \det \begin{pmatrix} \langle x_1, y_1\rangle & \langle x_1, y_2\rangle & \cdots & \langle x_1, y_k\rangle \\ \langle x_2, y_1\rangle & \langle x_2, y_2\rangle & \cdots & \langle x_2, y_k\rangle \\ \vdots & \vdots & & \vdots \\ \langle x_k, y_1\rangle & \langle x_k, y_2\rangle & \cdots & \langle x_k, y_k\rangle \end{pmatrix} .$$

As usual, we define the norm of a vector $\|\omega\|_k = \sqrt{\langle \omega, \omega\rangle_k}$.

Proof. The existence of such a form satisfying condition 1 follows from Theorem 3.10. Condition 2 follows from this holds for the inner product on V, and the fact that $\overline{\det A} = \det \overline{A}$. To prove condition 3, it is enough to show that there is a basis for $\bigwedge^k(V)$ which is orthonormal. *Exercise:* If $\{e_1, \ldots, e_n\}$ is an orthonormal basis for V, then the basis for $\bigwedge^k(V)$ constructed in Theorem 3.13 is orthonormal. □

Recall that if $v_1, v_2, \ldots v_n \in \mathbb{R}^n$ are linearly independent vectors, there is a parallelopiped determined by these vectors whose volume is $|\det A|$, where $v_1, \ldots, v_n$ are the columns of A. This works more generally on any vector space V with an orthonormal basis $\{e_1, \ldots, e_n\}$. If $v_i = T(e_i)$ for some linear transforomation T, then the volume of the parallelopiped determined by $v_1, v_2, \ldots, v_n$ is $|\det T|$.

This inner product on $\bigwedge^k(\mathbb{R}^n)$ can be used to compute the volume of a parallelopiped in $\mathbb{R}^n$, when $k < n$.

Theorem 3.29. *Let $v_1, \ldots v_k \in \mathbb{R}^n$ be linearly independent vectors. Then the k-dimensional volume of the parallelopiped determined by $v_1, v_2, \ldots, v_k$ is $\|v_1 \wedge v_2 \wedge \cdots \wedge v_k\|_k$.*

Proof. Let $W = \operatorname{span}\{v_1, \ldots, v_k\}$. Let $\{w_1, w_2, \ldots, w_k\}$ be an orthonormal basis for W. Note that the linear transformation from $W \to \mathbb{R}^k$ which takes w_i to the standard basis vectors is volume preserving. Then $v_i = Tw_i$ for some linear transformation $T \in \mathsf{L}(W)$. Then

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = T_*(w_1 \wedge w_2 \wedge \cdots \wedge w_k) = \det(T) w_1 \wedge w_2 \wedge \cdots \wedge w_k \, .$$

Therefore, $\|v_1 \wedge v_2 \wedge \cdots \wedge v_k\|_k = |\det T|$, which equals the volume of the parallelopiped. □

We can also compute volumes of simplices.

Definition 3.30. A **k-simplex** in $\mathbb{R}^n$ is a k-dimensional polyhedron, with $k+1$ vertices $v_0, v_1, \ldots, v_k \in \mathbb{R}^n$, where $\{v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\}$ is a linearly independent set.

For example, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron.

Theorem 3.31. *Let S be the k-simplex with vertices $0, v_1, v_2, \ldots, v_k$. Let P be the parallelopiped determined by $v_1, v_2, \ldots, v_k$. Then $\operatorname{vol}(S) = \frac{1}{k!}\operatorname{vol}(P)$.*

Proof. Let W be the subspace spanned by $\{v_1, \ldots, v_k\}$. Since $\dim W = k$, any linear transformation from W to $\mathbb{R}^k$ preserves the ratios of k-dimensional volumes. Thus it enough to prove this for the case where the vertices are $n = k$, and $v_i = e_1 + e_2 + \cdots + e_i$, where $\{e_1, \ldots, e_k\}$ is the basis for $\mathbb{R}^k$. In this case, the volume of the parallelopiped is 1, and the volume of the simplex is

$$\int_{x_1=0}^{1} \int_{x_2=0}^{x_1} \int_{x_3=0}^{x_2} \cdots \int_{x_k=0}^{x_{k-1}} 1 \, dx_k dx_{k-1} \ldots dx_2 dx_1 \,,$$

which one can easily show is equal to $\frac{1}{k!}$. $\square$