Home assignment 1 - solutions

Question 1

(a) From the figure,

$$\begin{aligned} y[n] &= (x[n] + x[n] * h_1[n]) * h_2[n] \\ &= (x[n] * (\delta[n] + h_1[n])) * h_2[n]. \end{aligned}$$

Let $h[n]$ be the impulse response of the overall system,

$$y[n] = x[n] * h[n].$$

Comparing with the above expression,

$$\begin{aligned} h[n] &= (\delta[n] + h_1[n]) * h_2[n] \\ &= h_2[n] + h_1[n] * h_2[n] \\ &= \alpha^n u[n] + \underset{\beta\alpha^{n-1}}{\cancel{\beta^{(n-1)}}} u[n-1]. \end{aligned}$$

(b) Taking the Fourier transform of $h[n]$ from part (a),

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\omega n} + \beta \sum_{n=-\infty}^{\infty} \alpha^{(n-1)} u[n-1] e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} + \beta \sum_{\ell=0}^{\infty} \alpha^{(\ell-1)} e^{-j\omega \ell}, \end{aligned}$$

where we have used $\ell = (n-1)$ in the second sum.

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{1-\alpha e^{-j\omega}} + \frac{\beta e^{-j\omega}}{1-\alpha e^{-j\omega}} \\ &= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}, \text{ for } |\alpha| < 1. \end{aligned}$$

Note that the Fourier transform of $\alpha^n u[n]$ is well known, and the second term of $h[n]$ (see part (a)) is just a scaled and shifted version of $\alpha^n u[n]$. So, we could have used the properties of the Fourier transform to reduce the algebra.

(c) We have

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}, \end{aligned}$$

cross multiplying,

$$Y(e^{j\omega})[1 - \alpha e^{-j\omega}] = X(e^{j\omega})[1 + \beta e^{-j\omega}]$$

taking the inverse Fourier transform, we have

$$y[n] - \alpha y[n-1] = x[n] + \beta x[n-1].$$

(d) From part (a):

$$h[n] = 0, \text{ for } n < 0.$$

This implies that the system is CAUSAL.

If the system is stable, its Fourier transform exists. Therefore, the condition for stability is the same as the condition imposed on the frequency response of part (b). That is, STABLE, if $|\alpha| < 1$.

Question 2

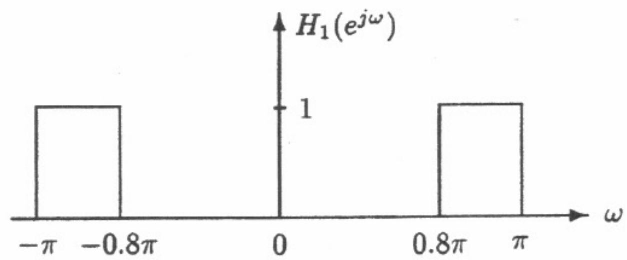
(a) $H_1(e^{j\omega})$ corresponds to a frequency shifted version of $H(e^{j\omega})$, specifically:

$$H_1(e^{j\omega}) = H(e^{j(\omega-\pi)}).$$

We thus have:

$$H_1(e^{j\omega}) = \begin{cases} 0 & , \quad |\omega| < 0.8\pi \\ 1 & , \quad 0.8\pi \leq |\omega| \leq \pi. \end{cases}$$

This is a highpass filter.



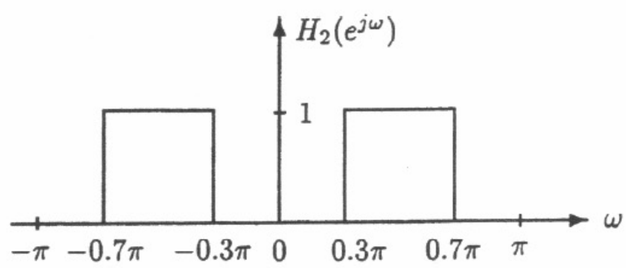
(b) $H_2(e^{j\omega})$ corresponds to a frequency modulated version of $H(e^{j\omega})$, specifically:

$$H_2(e^{j\omega}) = H(e^{j\omega}) * (\delta(\omega - 0.5\pi) + \delta(\omega + 0.5\pi)) \qquad \text{where } |\omega| \leq \pi.$$

We thus have:

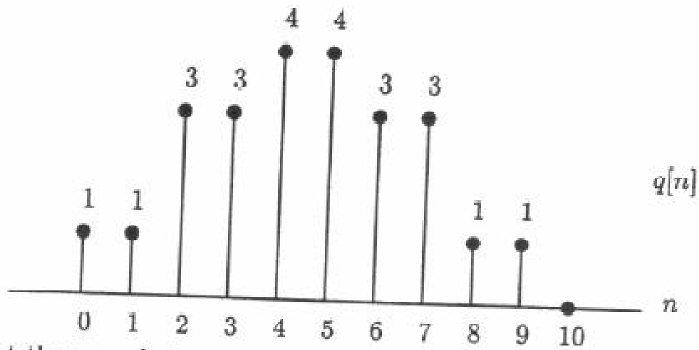
$$H_2(e^{j\omega}) = \begin{cases} 0 & , \quad |\omega| < 0.3\pi \\ 1 & , \quad 0.3\pi \leq |\omega| \leq 0.7\pi \\ 0 & , \quad 0.7\pi < |\omega| \leq \pi. \end{cases}$$

This is a bandpass filter.

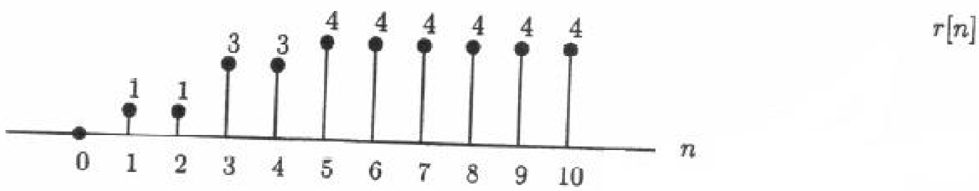


Question 3

(a) Carrying out the convolution sum, we get the following sequence $q[n]$:



b) Again carrying out the convolution sum, we get the following sequence $r[n]$:



(c) Let $a[n] = v[-n]$ and $b[n] = w[-n]$, then:

$$
\begin{aligned}
a[n] * b[n] &= \sum_{k=-\infty}^{+\infty} a[k]b[n-k] \\
&= \sum_{k=-\infty}^{+\infty} v[-k]w[k-n] \\
&= \sum_{r=-\infty}^{+\infty} v[r]w[-n-r] \text{ where } r=-k \\
&= q[-n].
\end{aligned}
$$

We thus conclude that $q[-n] = v[-n] * w[-n]$.

<u>Question 4</u>

The causal system has system function

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}$$

and the input is $x[n] = \left(\frac{1}{3}\right)^n u[n] + u[-n-1]$. Therefore the z-transform of the input is

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{1 - z^{-1}} = \frac{-\frac{2}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - z^{-1})} \qquad \frac{1}{3} < |z| < 1$$

(a) $h[n]$ causal $\Rightarrow$

$$h[n] = \left(-\frac{3}{4}\right)^n u[n] - \left(-\frac{3}{4}\right)^{n-1} u[n-1]$$

(b)

$$\begin{aligned} Y(z) &= X(z)H(z) = \frac{-\frac{2}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 + \frac{3}{4}z^{-1})} \qquad \frac{3}{4} < |z| \\ &= \frac{-\frac{8}{13}}{1 - \frac{1}{3}z^{-1}} + \frac{\frac{8}{13}}{1 + \frac{3}{4}z^{-1}} \end{aligned}$$

Therefore the output is

$$y[n] = -\frac{8}{13}\left(\frac{1}{3}\right)^n u[n] + \frac{8}{13}\left(-\frac{3}{4}\right)^n u[n]$$

(c) For $h[n]$ to be causal the ROC of $H(z)$ must be $\frac{3}{4} < |z|$ which includes the unit circle. Therefore, $h[n]$ absolutely summable.

Question 5

(a)

$$
\begin{aligned}
H(z) &= \frac{1 - \frac{1}{2}z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} \\
&= -4 + \frac{5 + \frac{7}{2}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \\
&= -4 - \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{7}{1 - \frac{1}{4}z^{-1}} \\
h[n] &= -4\delta[n] - 2\left(\frac{1}{2}\right)^n u[n] + 7\left(\frac{1}{4}\right)^n u[n]
\end{aligned}
$$

(b)

$$
y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n] - \frac{1}{2}x[n-2]
$$