THE DISCRETE FOURIER TRANSFORM

REPRESENTATION OF PERIODIC SEQUENCES: THE DISCRETE FOURIER SERIES

Consider a sequence $\tilde{x}[n]$ that is periodic[1] with period N, so that $\tilde{x}[n] = \tilde{x}[n + rN]$ for any integer values of n and r.

$$\tilde{x}[n] = \frac{1}{N}\sum_{k} \tilde{X}[k]e^{j(2\pi/N)kn}.$$

$e_k[n]$ in Eq. (8.1) are identical for values of k separated by N; i.e., $e_0[n] = e_N[n]$, $e_1[n] = e_{N+1}[n]$, and, in general,

$$e_{k+\ell N}[n] = e^{j(2\pi/N)(k+\ell N)n} = e^{j(2\pi/N)kn}e^{j2\pi \ell n} = e^{j(2\pi/N)kn} = e_k[n], \qquad (8.3)$$

where ℓ is an integer. Consequently, the set of N periodic complex exponentials $e_0[n]$, $e_1[n], \ldots, e_{N-1}[n]$ defines all the distinct periodic complex exponentials with frequencies that are integer multiples of $(2\pi/N)$.

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j(2\pi/N)kn}.$$

Thus, the Fourier series coefficients $\tilde{X}[k]$ in Eq. (8.4) are obtained from $\tilde{x}[n]$ by the relation

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)kn}. \qquad (8.9)$$

Note that the sequence $\tilde{X}[k]$ is periodic with period N; i.e., $\tilde{X}[0] = \tilde{X}[N]$, $\tilde{X}[1] = \tilde{X}[N+1]$.

For convenience in notation, these equations are often written in terms of the complex quantity

$$W_N = e^{-j(2\pi/N)}. \qquad (8.10)$$

With this notation, the DFS analysis–synthesis pair is expressed as follows:

$$\text{Analysis equation:} \quad \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]W_N^{kn}. \qquad (8.11)$$

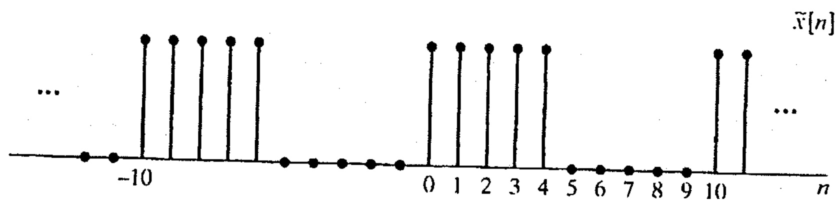
$$\text{Synthesis equation:} \quad \tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]W_N^{-kn}. \qquad (8.12)$$

In both of these equations, $\tilde{X}[k]$ and $\tilde{x}[n]$ are periodic sequences. We will sometimes find it convenient to use the notation

$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k] \qquad (8.13)$$

The Discrete Fourier Series of a Periodic Rectangular Pulse Train

For this example, $\tilde{x}[n]$ is the sequence shown in Figure 8.1, whose period is $N = 10$.

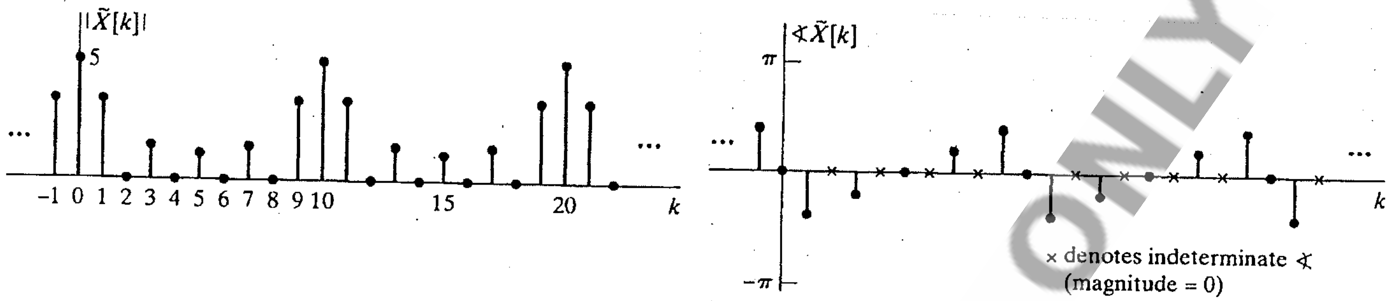


$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \sum_{n=0}^{4} e^{-j(2\pi/10)kn}. \tag{8.17}$$

This finite sum has the closed form

$$\tilde{X}[k] = \frac{1 - W_{10}^{5k}}{1 - W_{10}^{k}} = e^{-j(4\pi k/10)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}. \tag{8.18}$$

The magnitude and phase of the periodic sequence $\tilde{X}[k]$ are



PROPERTIES OF THE DISCRETE FOURIER SERIES

Linearity

Consider two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, both with period N, such that

$$\tilde{x}_1[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k] \tag{8.19a}$$

and

$$\tilde{x}_2[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_2[k] \tag{8.19b}$$

Then

$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} a\tilde{X}_1[k] + b\tilde{X}_2[k]. \tag{8.20}$$

Shift of a Sequence

If a periodic sequence $\tilde{x}[n]$ has Fourier coefficients $\tilde{X}[k]$, then $\tilde{x}[n-m]$ is a shifted version of $\tilde{x}[n]$, and

$$\tilde{x}[n-m] \stackrel{\mathcal{DFS}}{\longleftrightarrow} W_N^{km}\tilde{X}[k]. \tag{8.21}$$

$$W_N^{-n\ell}\tilde{x}[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k-\ell].$$

Duality

Because of the strong similarity between the Fourier analysis and synthesis equations in continuous time, there is a duality between the time domain and frequency domain.

If

$$\tilde{x}[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k], \tag{8.25a}$$

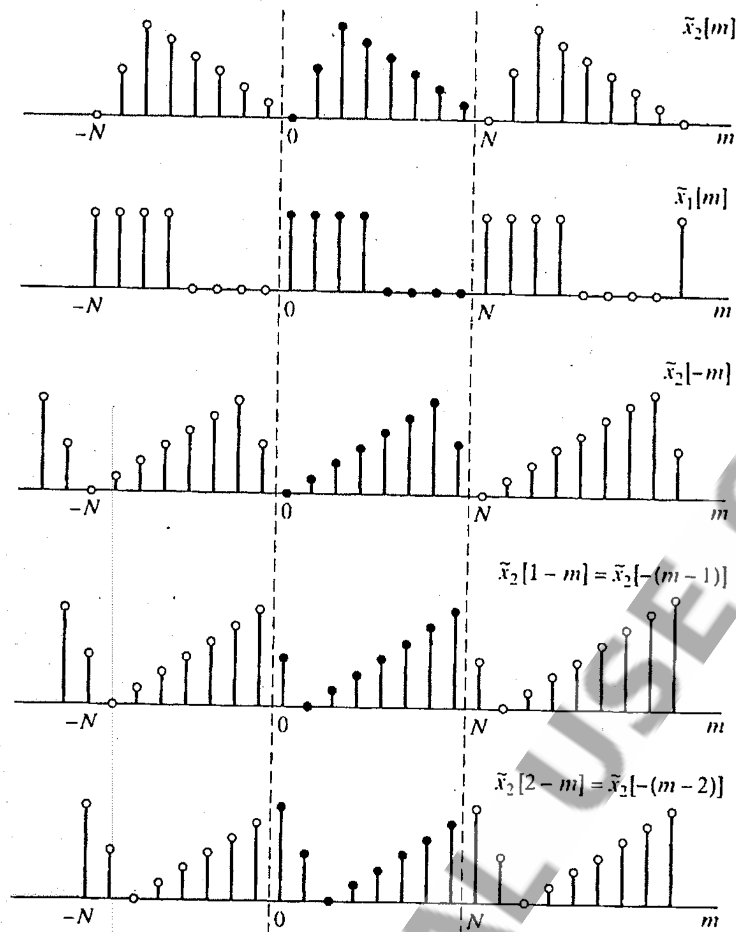
then

$$\tilde{X}[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k]. \tag{8.25b}$$

Periodic Convolution

In summary,

$$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \stackrel{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k]\tilde{X}_2[k].$$



The duality theorem (Section 8.2.3) suggests that if

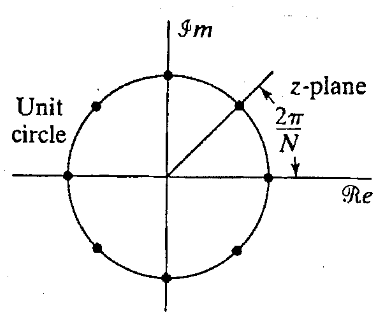
$$\tilde{x}_3[n] = \tilde{x}_1[n]\tilde{x}_2[n], \tag{8.33}$$

where $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ are periodic sequences, each with period N, has the discrete Fourier series coefficients given by

$$\tilde{X}_3[k] = \frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell], \tag{8.34}$$

SAMPLING THE FOURIER TRANSFORM

In this section, we discuss with more generality the relationship between an aperiodic sequence with Fourier transform $X(e^{j\omega})$ and the periodic sequence for which the DFS coefficients correspond to samples of $X(e^{j\omega})$ equally spaced in frequency.



Consider an aperiodic sequence $x[n]$ with Fourier transform $X(e^{j\omega})$, and assume that a sequence $\tilde{X}[k]$ is obtained by sampling $X(e^{j\omega})$ at frequencies $\omega_k = 2\pi k/N$; i.e.,

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=(2\pi/N)k} = X(e^{j(2\pi/N)k}). \tag{8.49}$$

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Since the Fourier transform is periodic in ω with period 2π, the resulting sequence is periodic in k with period N.

Note that the sequence of samples $\tilde{X}[k]$, being periodic with period N, *could* be the sequence of discrete Fourier series coefficients of a sequence $\tilde{x}[n]$. To obtain that sequence, we can simply substitute $\tilde{X}[k]$ obtained by sampling into Eq. (8.12):

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}. \tag{8.51}$$

Substituting Eq. (8.52) into Eq. (8.49) and then substituting the resulting expression for $\tilde{X}[k]$ into Eq. (8.51) gives

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j(2\pi/N)km}\right] W_N^{-kn}, \tag{8.53}$$

which, after we interchange the order of summation, becomes

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right] = \sum_{m=-\infty}^{\infty} x[m]\tilde{p}[n-m]. \tag{8.54}$$

$$\tilde{p}[n-m] = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)} = \sum_{r=-\infty}^{\infty} \delta[n-m-rN] \tag{8.55}$$

and therefore,

$$\tilde{x}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta[n-rN] = \sum_{r=-\infty}^{\infty} x[n-rN], \tag{8.56}$$

If $x[n]$ has finite length and we take a sufficient number of equally spaced samples of its Fourier transform (specifically, a number greater than or equal to the length of $x[n]$), then the Fourier transform is recoverable from these samples, and, equivalently, $x[n]$ is recoverable from the corresponding periodic sequence $\tilde{x}[n]$ through the relation

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.57}$$

FOURIER REPRESENTATION OF FINITE-DURATION SEQUENCES: THE DISCRETE FOURIER TRANSFORM

We begin by considering a finite-length sequence $x[n]$ of length N samples such that $x[n] = 0$ outside the range $0 \le n \le N-1$. In many instances, we will want to assume that a sequence has length N even if its length is $M \le N$. In such cases, we simply recognize that the last $(N-M)$ samples are zero. To each finite-length sequence of length N, we can always associate a periodic sequence

$$\tilde{x}[n] = \sum^{\infty} x[n-rN]. \tag{8.58a}$$

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \quad (8.58b)$$

Recall from Section 8.4 that the DFS coefficients of $\tilde{x}[n]$ are samples (spaced in frequency by $2\pi/N$) of the Fourier transform of $x[n]$. Since $x[n]$ is assumed to have finite length N, there is no overlap between the terms $x[n - rN]$ for different values of r. Thus,

$$\tilde{x}[n] = x[((n))_N].$$

Thus, the DFT, $X[k]$, is related to the DFS coefficients, $\tilde{X}[k]$, by

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases} \quad (8.61)$$

and

$$\tilde{X}[k] = X[(k \text{ modulo } N)] = X[((k))_N]. \quad (8.62)$$

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases} \quad (8.65)$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \quad (8.66)$$

$$\text{Analysis equation:} \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}. \quad (8.67)$$

$$\text{Synthesis equation:} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}. \quad (8.68)$$

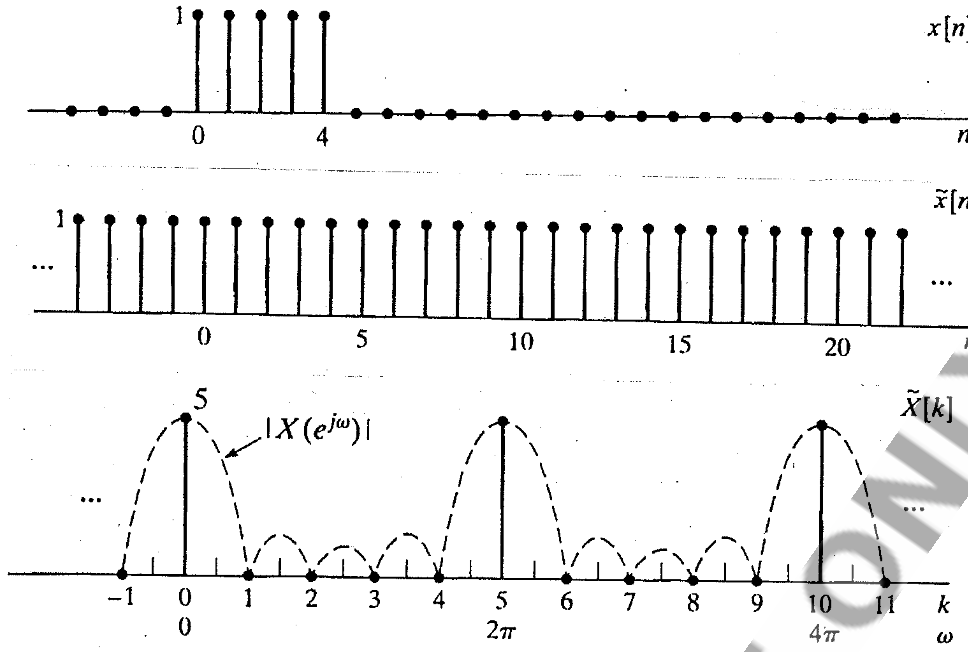
That is, the fact that $X[k] = 0$ for k outside the interval $0 \le k \le N-1$ and that $x[n] = 0$ for n outside the interval $0 \le n \le N-1$ is implied, but not always stated explicitly.

In defining the DFT representation, we are simply recognizing that we are *interested* in values of $x[n]$ only in the interval $0 \le n \le N-1$ because $x[n]$ is really zero outside that interval, and we are *interested* in values of $X[k]$ only in the interval $0 \le k \le N-1$ because these are the only values needed in Eq. (8.68).

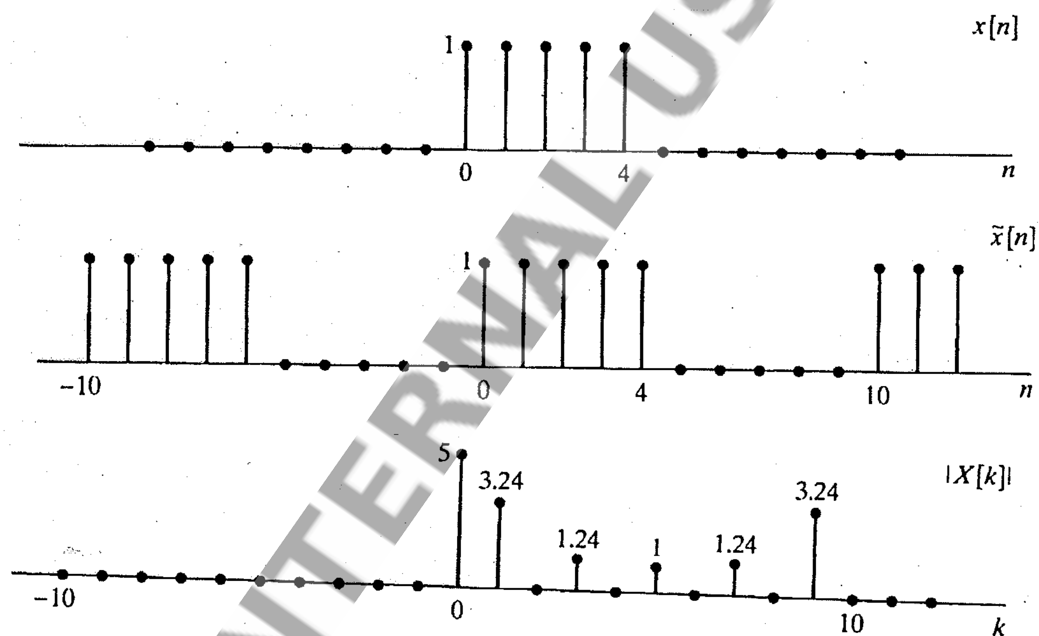
The DFT of a Rectangular Pulse

$$\tilde{X}[k] = \sum_{n=0}^{4} e^{-j(2\pi k/5)n} = \frac{1 - e^{-j2\pi k}}{1 - e^{-j(2\pi k/5)}}$$

$$= \begin{cases} 5, & k = 0, \pm 5, \pm 10, \ldots, \\ 0, & \text{otherwise}; \end{cases}$$



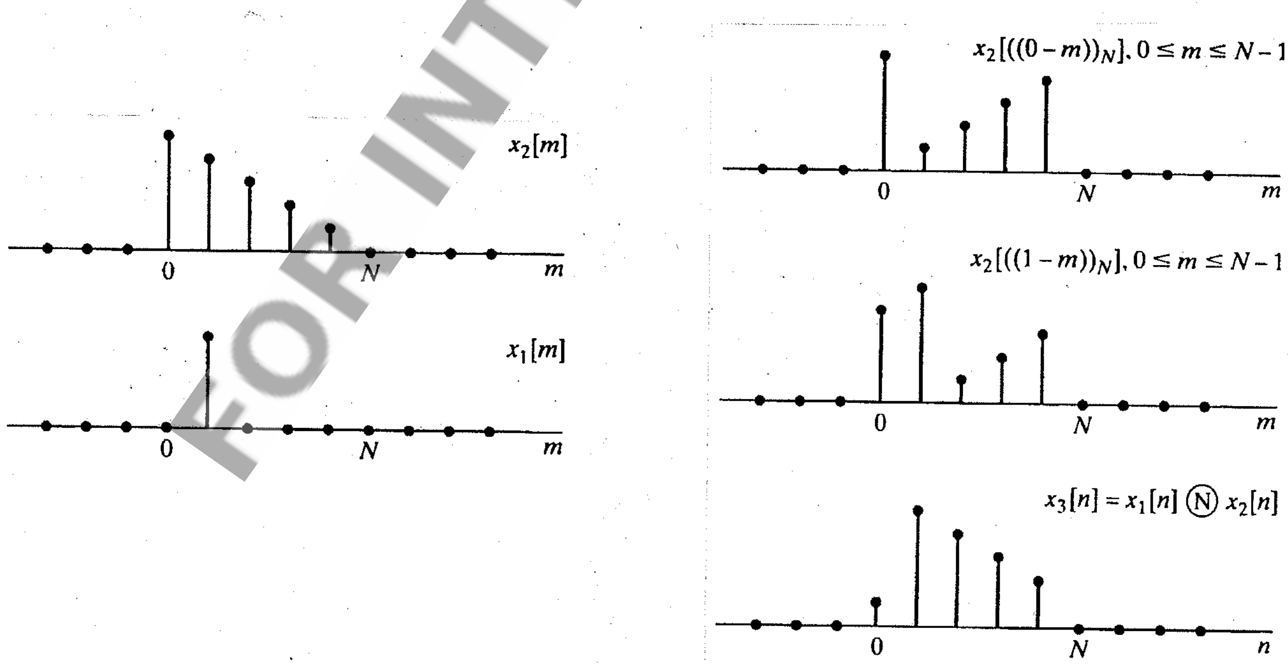
If, instead, we consider $x[n]$ to be of length $N = 10$, then the underlying periodic sequence is that shown in



PROPERTIES OF THE DISCRETE FOURIER TRANSFORM

Finite-Length Sequence (Length N)	N-point DFT (Length N)
1. $x[n]$	$X[k]$
2. $x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3. $ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4. $X[n]$	$Nx[((-k))_N]$
5. $x[((n-m))_N]$	$W_N^{km}X[k]$
6. $W_N^{-\ell n}x[n]$	$X[((k-\ell))_N]$
7. $\sum_{m=0}^{N-1} x_1(m)x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8. $x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell)X_2[((k-\ell))_N]$
9. $x^*[n]$	$X^*[((-k))_N]$
10. $x^*[((-n))_N]$	$X^*[k]$
11. $\mathcal{R}e\{x[n]\}$	$X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12. $j\mathcal{J}m\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{J}m\{X[k]\} = -\mathcal{J}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$

Circular Convolution with a Delayed Impulse



Circular Convolution of Two Rectangular Pulses

As another example of circular convolution, let

$$x_1[n] = x_2[n] = \begin{cases} 1, & 0 \le n \le L-1, \\ 0, & \text{otherwise}, \end{cases} \tag{8.122}$$

If we let N denote the DFT length, then, for $N = L$, the N-point DFTs are

$$X_1[k] = X_2[k] = \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N, & k=0, \\ 0, & \text{otherwise}. \end{cases} \tag{8.123}$$

If we explicitly multiply $X_1[k]$ and $X_2[k]$, we obtain

$$X_3[k] = X_1[k]X_2[k] = \begin{cases} N^2, & k=0, \\ 0, & \text{otherwise}, \end{cases} \tag{8.124}$$

from which it follows that

$$x_3[n] = N, \qquad 0 \le n \le N-1. \tag{8.125}$$

It is, of course, possible to consider $x_1[n]$ and $x_2[n]$ as $2L$-point sequences by augmenting them with L zeros.

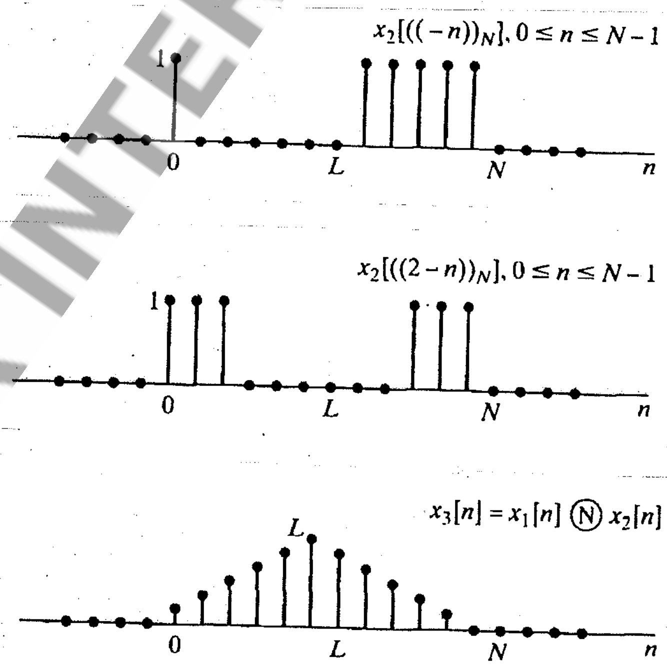
Note that for $N = 2L$,

$$X_1[k] = X_2[k] = \frac{1 - W_N^{Lk}}{1 - W_N^k},$$

so the DFT of the triangular-shaped sequence $x_3[n]$ in Figure 8.16(e) is

$$X_3[k] = \left(\frac{1 - W_N^{Lk}}{1 - W_N^k}\right)^2,$$

with $N = 2L$.



LINEAR CONVOLUTION USING THE DISCRETE FOURIER TRANSFORM

(a) Compute the N-point discrete Fourier transforms $X_1[k]$ and $X_2[k]$ of the two sequences $x_1[n]$ and $x_2[n]$, respectively.

(b) Compute the product $X_3[k] = X_1[k]X_2[k]$ for $0 \le k \le N-1$.

(c) Compute the sequence $x_3[n] = x_1[n] \circledN x_2[n]$ as the inverse DFT of $X_3[k]$.

In most applications, we are interested in implementing a linear convolution of two sequences; i.e., we wish to implement a linear time-invariant system.

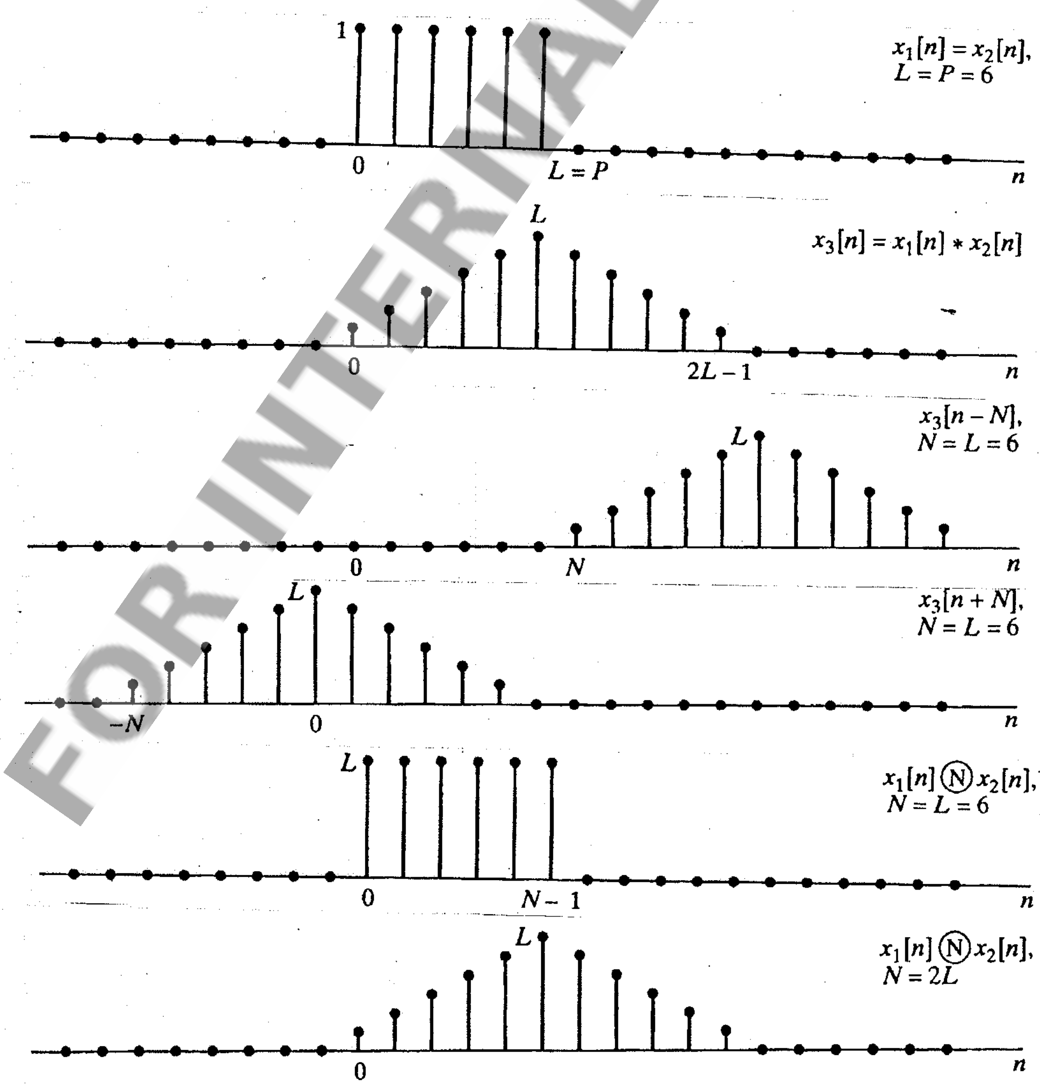
To obtain a linear convolution, we must ensure that circular convolution has the effect of linear convolution.

Consider a sequence $x_1[n]$ whose length is L points and a sequence $x_2[n]$ whose length is P points, and suppose that we wish to combine these two sequences by linear convolution to obtain a third sequence

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]. \tag{8.129}$$

Therefore, $(L+P-1)$ is the maximum length of the sequence $x_3[n]$ resulting from the linear convolution of a sequence of length L with a sequence of length P.

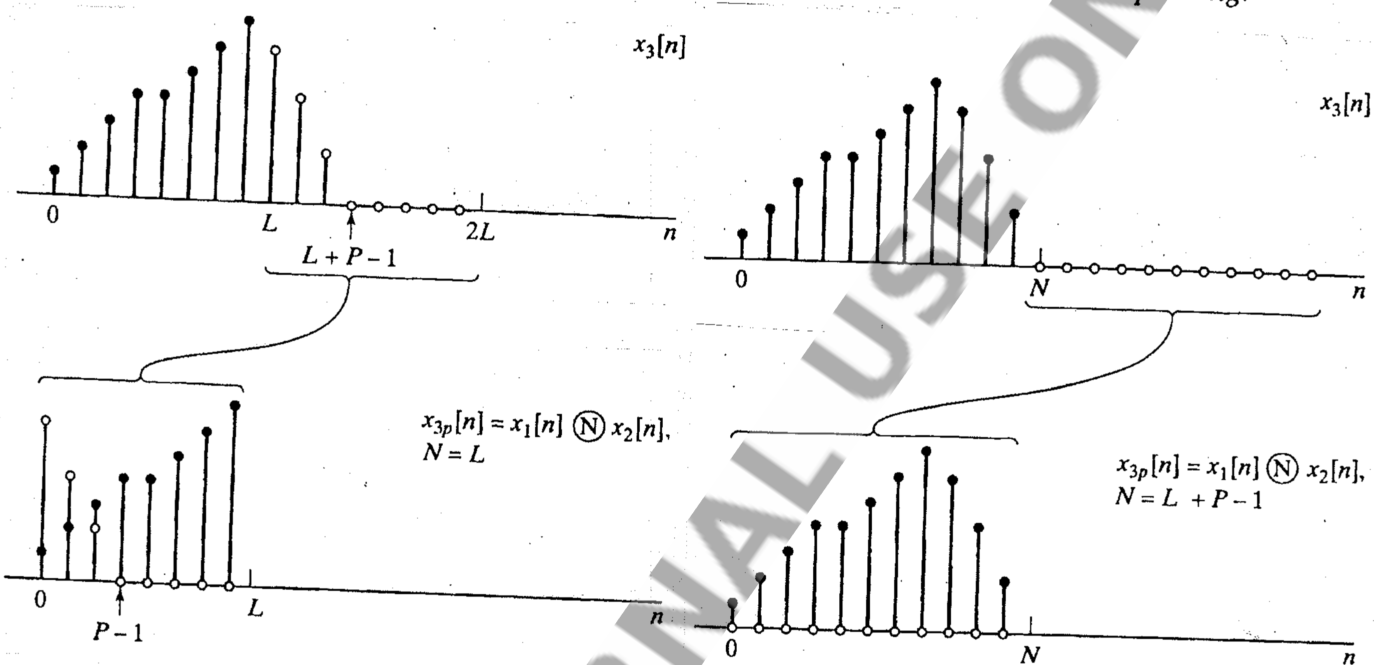
As we showed , if $x_1[n]$ has length L and $x_2[n]$ has length P, then $x_3[n]$ has maximum length $(L+P-1)$. Therefore, the circular convolution corresponding to $X_1[k]X_2[k]$ is identical to the linear convolution corresponding to $X_1(e^{j\omega})X_2(e^{j\omega})$ if N, the length of the DFTs, satisfies $N \ge L+P-1$.



As Example points out, time aliasing in the circular convolution of two finite-length sequences can be avoided if $N \geq L + P - 1$. Also, it is clear that if $N = L = P$, all of the sequence values of the circular convolution may be different from those of the linear convolution. However, if $P < L$, some of the sequence values in an L-point circular convolution will be equal to the corresponding sequence values of the linear convolution. The time-aliasing interpretation is useful for showing this.

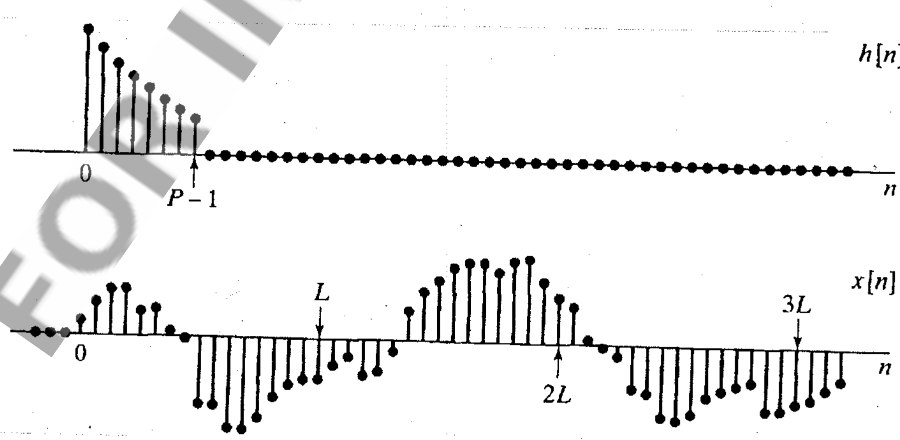
Implementing Linear Time-Invariant Systems Using the DFT

multiplying the DFTs of $x[n]$ and $h[n]$. Since we want the product to represent the DFT of the linear convolution of $x[n]$ and $h[n]$, which has length $(L + P - 1)$, the DFTs that we compute must also be of at least that length, i.e., both $x[n]$ and $h[n]$ must be augmented with sequence values of zero amplitude. This process is often referred to as *zero-padding*.



This procedure permits the computation of the linear convolution of two finite-length sequences using the discrete Fourier transform; i.e., the output of an FIR system whose input also has finite length can be computed with the DFT.

block convolution



Henceforth, we will assume that $x[n] = 0$ for $n < 0$ and that the length of $x[n]$ is much greater than P. The sequence $x[n]$ can be represented as a sum of shifted finite-length segments of length L; i.e.,

$$x[n] = \sum_{r=0}^{\infty} x_r[n - rL], \tag{8.140}$$

where

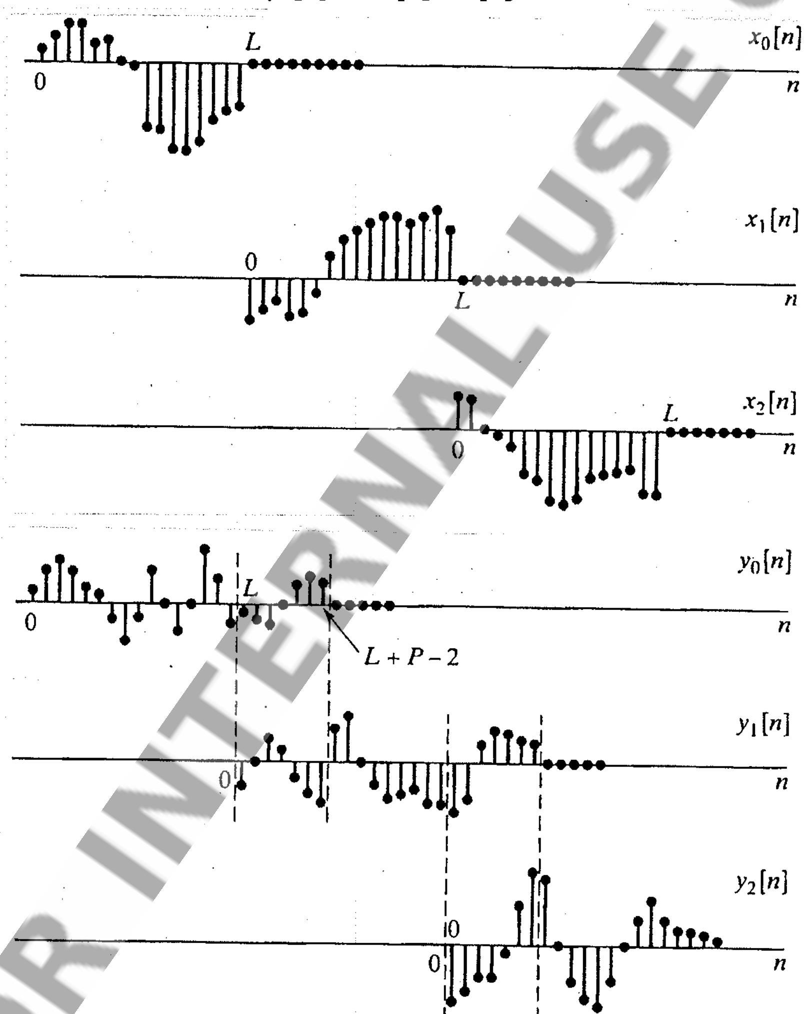
$$x_r[n] = \begin{cases} x[n + rL], & 0 \le n \le L - 1, \\ 0, & \text{otherwise.} \end{cases} \tag{8.141}$$

Because convolution is a linear time-invariant operation, it follows from Eq. (8.140) that

$$y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} y_r[n - rL], \tag{8.142}$$

where

$$y_r[n] = x_r[n] * h[n]. \tag{8.143}$$



Since the sequences $x_r[n]$ have only L nonzero points and $h[n]$ is of length P, each of the terms $y_r[n] = x_r[n] * h[n]$ has length $(L + P - 1)$. Thus, the linear convolution $x_r[n] * h[n]$ can be obtained by the procedure described earlier using N-point DFTs, where $N \ge L + P - 1$. Since the beginning of each input section is separated from its neighbors by L points and each filtered section has length $(L + P - 1)$, the nonzero points in the filtered sections will overlap by $(P - 1)$ points, and these overlap samples must be added in carrying out the sum required