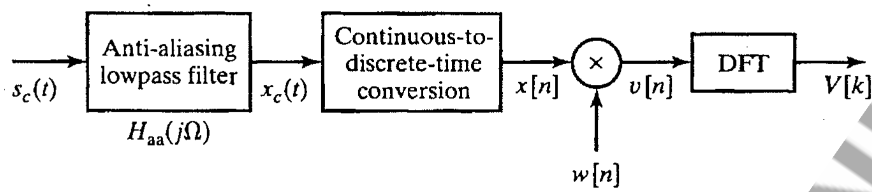
FOURIER ANALYSIS OF SIGNALS USING THE DISCRETE FOURIER TRANSFORM



The conversion of $x_c(t)$ to the sequence of samples $x[n]$ is represented in the frequency domain by periodic replication and frequency normalization, i.e.,

$$X(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} + j\frac{2\pi r}{T}\right). \tag{10.1}$$

As indicated, the sequence $x[n]$ is typically multiplied by a finite-duration window $w[n]$, since the input to the DFT must be of finite duration. This produces the finite-length sequence $v[n] = w[n]x[n]$. The effect in the frequency domain is a periodic convolution, i.e.,

$$V(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta. \tag{10.2}$$

If $w[n]$ is constant over the range of n for which it is nonzero, it is referred to as a *rectangular window*. However, as we will see, there are good reasons to taper the window at its edges.

At this point, it is sufficient to observe that convolution of $W(e^{j\omega})$ with $X(e^{j\omega})$ will tend to smooth sharp peaks and discontinuities in $X(e^{j\omega})$.

The DFT of the windowed sequence $v[n] = w[n]x[n]$ is

$$V[k] = \sum_{n=0}^{N-1} v[n]e^{-j(2\pi/N)kn}, \qquad k = 0, 1, \ldots, N-1, \tag{10.3}$$

where we assume that the window length L is less than or equal to the DFT length N. $V[k]$, the DFT of the finite-length sequence $v[n]$, corresponds to equally spaced samples of the Fourier transform of $v[n]$; i.e.,

$$V[k] = V(e^{j\omega})\big|_{\omega=2\pi k/N}. \tag{10.4}$$

Since the spacing between DFT frequencies is $2\pi/N$, and the relationship between the normalized discrete-time frequency variable and the continuous-time frequency variable is $\omega = \Omega T$, the DFT frequencies correspond to the continuous-time frequencies

$$\Omega_k = \frac{2\pi k}{NT}. \tag{10.5}$$

The discrete-time Fourier transform of a sinusoidal signal $A\cos(\omega_0 n+\phi)$ is a pair of impulses at $+\omega_0$ and $-\omega_0$ (repeating periodically with period 2π). In analyzing sinusoidal signals using the DFT, windowing and spectral sampling have an important effect.

Let us consider a continuous-time signal consisting of the sum of two sinusoidal components; i.e.,

$$s_c(t) = A_0\cos(\Omega_0 t + \theta_0) + A_1\cos(\Omega_1 t + \theta_1), \qquad -\infty < t < \infty. \tag{10.6}$$

Assuming ideal sampling with no aliasing and no quantization error, we obtain the discrete-time signal

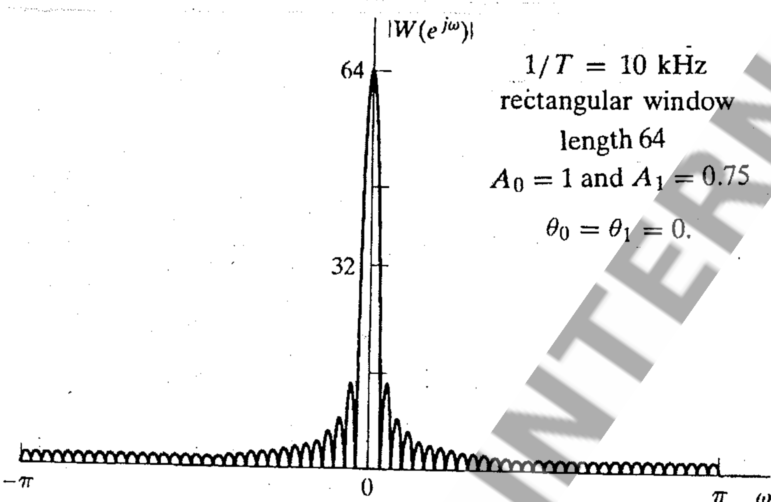
$$x[n] = A_0\cos(\omega_0 n + \theta_0) + A_1\cos(\omega_1 n + \theta_1), \qquad -\infty < n < \infty, \tag{10.7}$$

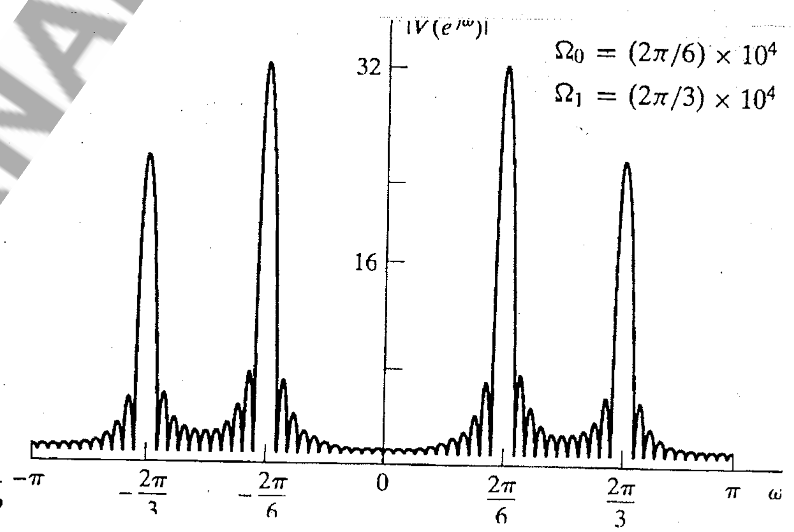
where $\omega_0 = \Omega_0 T$ and $\omega_1 = \Omega_1 T$. The windowed sequence $v[n]$

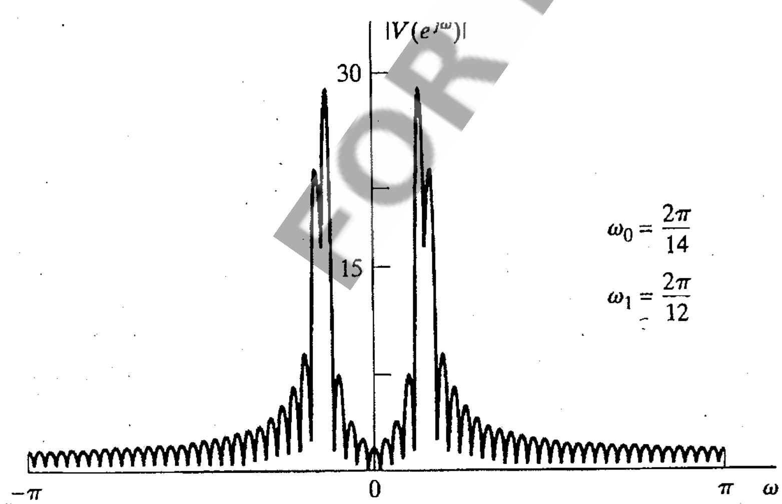
$$v[n] = A_0 w[n]\cos(\omega_0 n + \theta_0) + A_1 w[n]\cos(\omega_1 n + \theta_1). \tag{10.8}$$

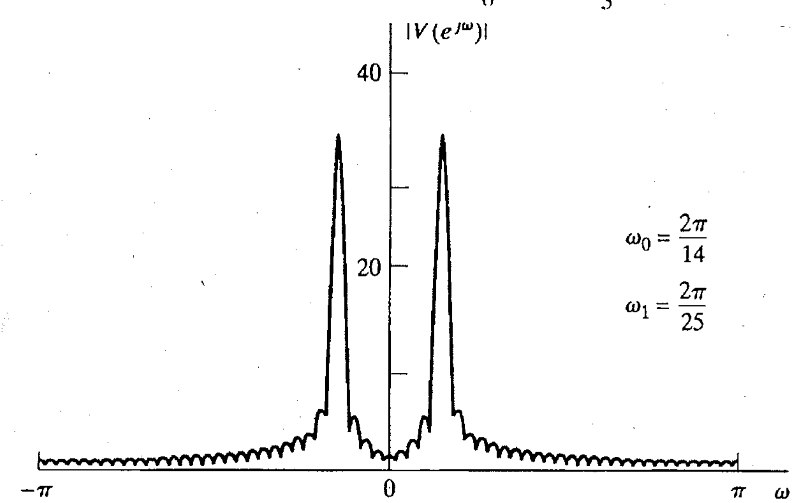
$$\begin{aligned} v[n] = {} & \frac{A_0}{2}w[n]e^{j\theta_0}e^{j\omega_0 n} + \frac{A_0}{2}w[n]e^{-j\theta_0}e^{-j\omega_0 n} \\ & + \frac{A_1}{2}w[n]e^{j\theta_1}e^{j\omega_1 n} + \frac{A_1}{2}w[n]e^{-j\theta_1}e^{-j\omega_1 n}, \end{aligned} \tag{10.9}$$

$$\begin{aligned} V(e^{j\omega}) = {} & \frac{A_0}{2}e^{j\theta_0}W(e^{j(\omega-\omega_0)}) + \frac{A_0}{2}e^{-j\theta_0}W(e^{j(\omega+\omega_0)}) \\ & + \frac{A_1}{2}e^{j\theta_1}W(e^{j(\omega-\omega_1)}) + \frac{A_1}{2}e^{-j\theta_1}W(e^{j(\omega+\omega_1)}). \end{aligned} \tag{10.10}$$









Reduced resolution and leakage are the two primary effects on the spectrum as a result of applying a window to the signal. The resolution is influenced primarily by the width of the main lobe of $W(e^{j\omega})$, while the degree of leakage depends on the relative amplitude of the main lobe and the side lobes of $W(e^{j\omega})$.

The rectangular window, which has Fourier transform

$$W_r(e^{j\omega}) = \sum_{n=0}^{L-1} e^{-j\omega n} = e^{-j\omega(L-1)/2}\frac{\sin(\omega L/2)}{\sin(\omega/2)}, \tag{10.11}$$

has the narrowest main lobe for a given length, but it has the largest side lobes of all the commonly used windows. As defined in Chapter 7, the Kaiser window is

$$w_K[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le L-1, \\ 0, & \text{otherwise,} \end{cases} \tag{10.12}$$

where $\alpha = (L-1)/2$ and $I_0(\cdot)$ is the zeroth-order modified Bessel function of the first kind. We have already seen in the context of the filter design problem that this window has two parameters, β and L, which can be used to trade between main-lobe width and relative side-lobe amplitude. (Recall that the Kaiser window reduces to the rectangular window when $\beta = 0$.) The main-lobe width Δ_{ml} is defined as the symmetric distance between the central zero-crossings. The relative side-lobe level A_{sl} is defined as the ratio in dB of the amplitude of the main lobe to the amplitude of the largest side lobe.

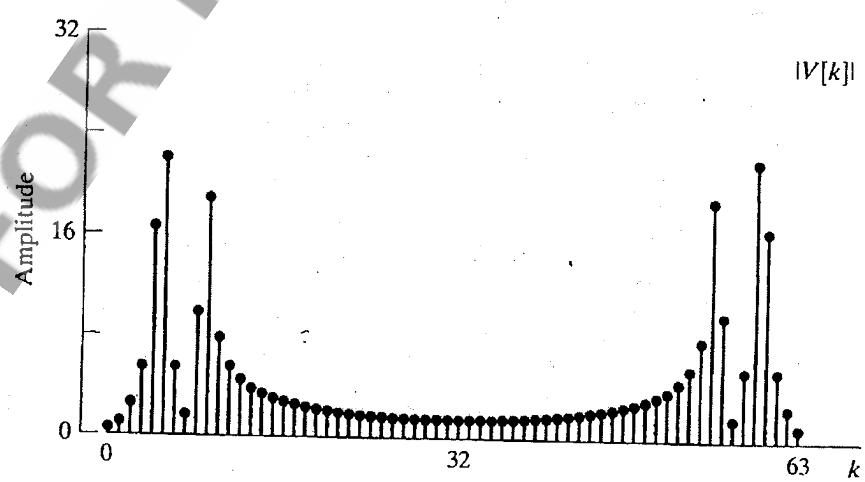
The trade-off between main-lobe width, relative side-lobe amplitude, and window length is displayed by the approximate relationship

$$L \simeq \frac{24\pi(A_{sl}+12)}{155\Delta_{ml}} + 1, \tag{10.14}$$

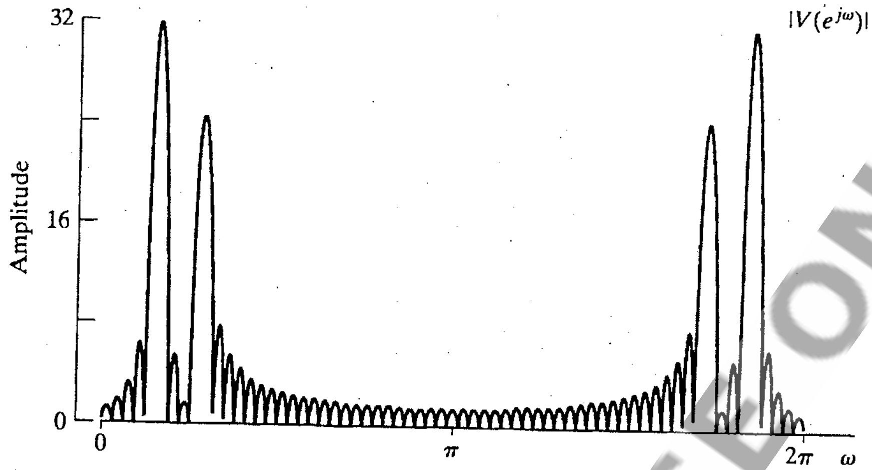
which was also given by Kaiser and Schafer (1980).

Let us consider the same parameters , i.e., $A_0 = 1$, $A_1 = 0.75$, $\omega_0 = 2\pi/14$, $\omega_1 = 4\pi/15$, and $\theta_1 = \theta_2 = 0$ in Eq. (10.8). $w[n]$ is a rectangular window of length 64. Then

$$v[n] = \begin{cases} \cos\left(\dfrac{2\pi}{14}n\right) + 0.75\cos\left(\dfrac{4\pi}{15}n\right), & 0 \le n \le 63, \\ 0, & \text{otherwise.} \end{cases} \tag{10.15}$$

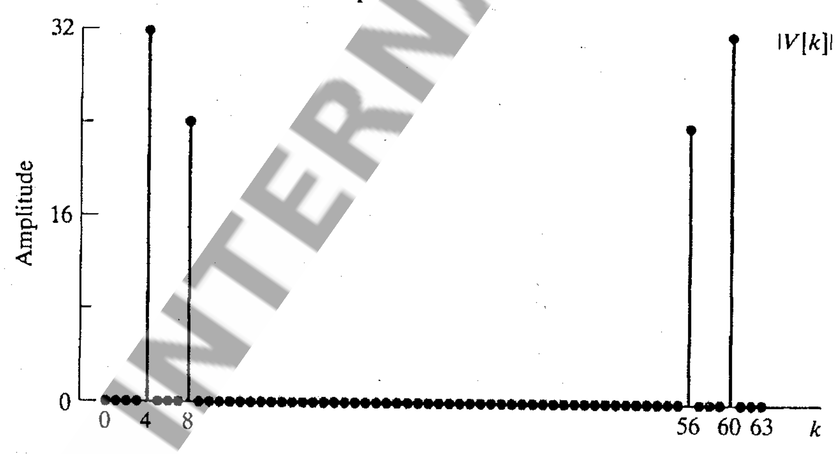


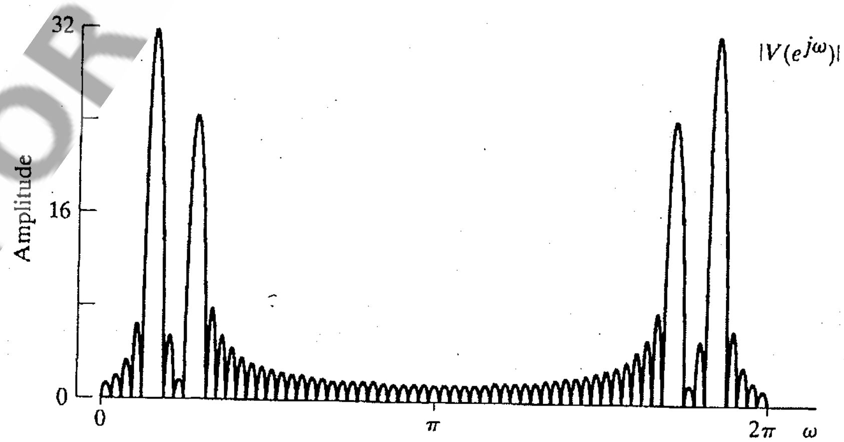
As is the usual convention in displaying the DFT of a time sequence, we display the DFT values in the range from $k = 0$ to $k = N - 1$, corresponding to displaying samples of the discrete-time Fourier transform in the frequency range 0 to 2π. Because of the inherent periodicity of the discrete-time Fourier transform, the first half of this range corresponds to the positive continuous-time frequencies, i.e., Ω between zero and π/T, and the second half of the range to the negative frequencies, i.e., Ω between $-\pi/T$ and zero.



$$v[n] = \begin{cases} \cos\left(\dfrac{2\pi}{16}n\right) + 0.75\cos\left(\dfrac{2\pi}{8}n\right), & 0 \le n \le 63, \\ 0, & \text{otherwise,} \end{cases} \tag{10.16}$$

Again, a rectangular window is used with $N = L = 64$. This is very similar to the previous example, except that in this case, the frequencies of the cosines coincide exactly with two of the DFT frequencies. Specifically, the frequency $\omega_1 = 2\pi/8 = 2\pi 8/64$ corresponds exactly to the DFT sample $k = 8$ and the frequency $\omega_0 = 2\pi/16 = 2\pi 4/64$ to the DFT sample $k = 4$.



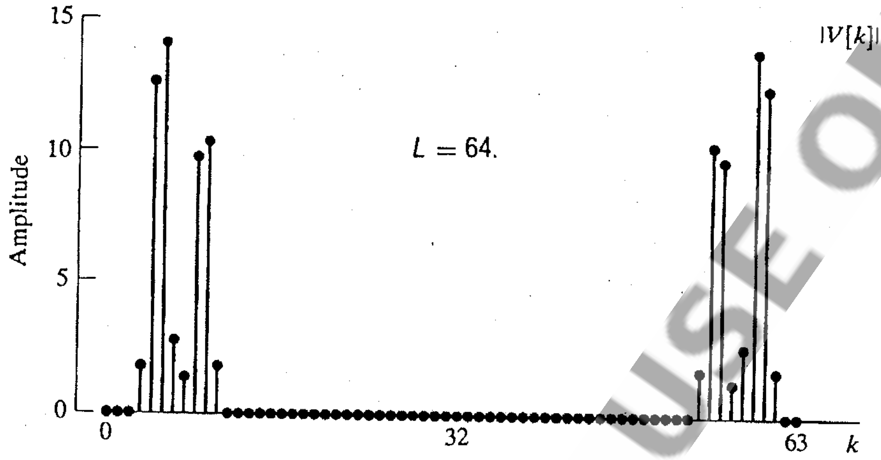


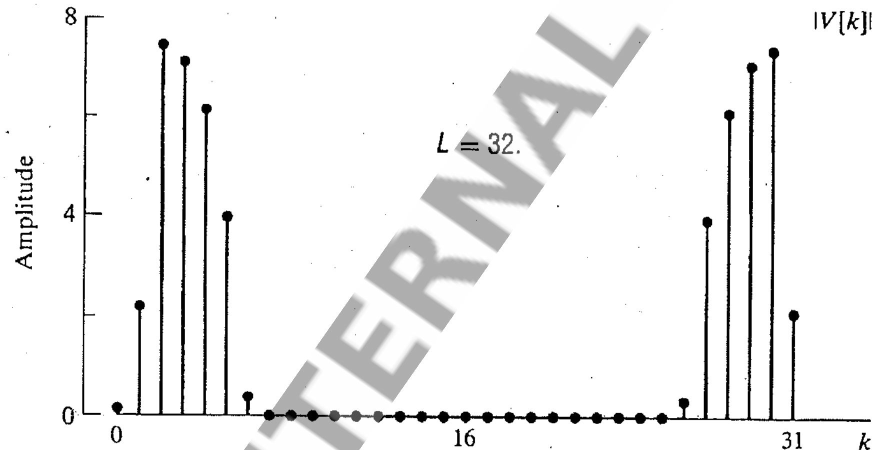
DFT Analysis of Sinusoidal Signals Using a Kaiser Window

Let us return to the frequency, amplitude, and phase parameters of Example 10.3, but now with a Kaiser window applied, so that

$$v[n] = w_K[n]\cos\left(\frac{2\pi}{14}n\right) + 0.75w_K[n]\cos\left(\frac{4\pi}{15}n\right), \tag{10.17}$$

where $w_K[n]$ is the Kaiser window as given by Eq. (10.12). We will select the Kaiser window parameter β to be equal to 5.48.





For a complete representation of a sequence of length L, the L-point DFT is sufficient, since the original sequence can be recovered exactly from it. However, as we saw in the preceding examples, simple examination of the L-point DFT can result in misleading interpretations. For this reason, it is common to apply zero-padding so that the spectrum is sufficiently oversampled and important features are therefore readily apparent. With a high degree of time-domain zero-padding or frequency-domain oversampling, simple interpolation (e.g., linear interpolation) between the DFT values provides a reasonably accurate picture of the Fourier spectrum, which can then be used, for example, to estimate the locations and amplitudes of spectral peaks.

THE TIME-DEPENDENT FOURIER TRANSFORM

The time-dependent Fourier transform of a signal $x[n]$ is defined as

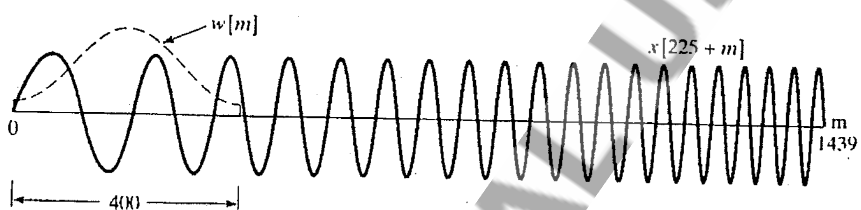
$$X[n, \lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}, \tag{10.18}$$

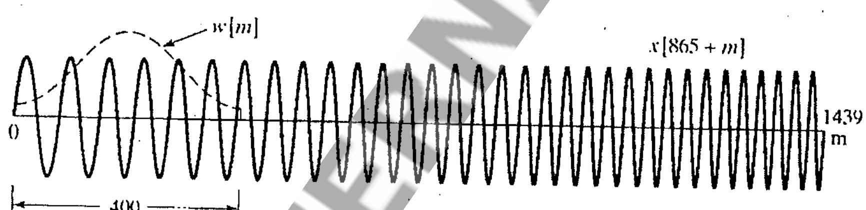
where $w[n]$ is a window sequence. In the time-dependent Fourier representation, the one-dimensional sequence $x[n]$, a function of a single discrete variable, is converted into a two-dimensional function of the time variable n, which is discrete, and the frequency variable λ, which is continuous.[2] Note that the time-dependent Fourier transform is periodic in λ with period 2π, and therefore, we need consider only values of λ for $0 \le \lambda < 2\pi$ or any other interval of length 2π.

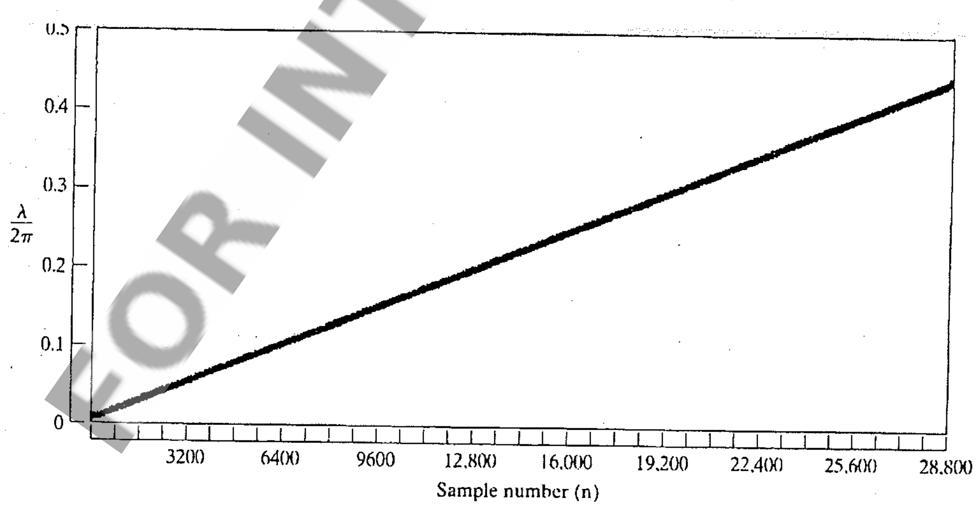
$$x[n] = \cos(\omega_0 n^2), \qquad \omega_0 = 2\pi \times 7.5 \times 10^{-6}, \tag{10.19}$$

corresponding to a linear frequency modulation (i.e., the "instantaneous frequency" is $2\omega_0 n$).

Typically, $w[m]$ in Eq. (10.18) has finite length around $m = 0$, so that $X[n, \lambda)$ displays the frequency characteristics of the signal around time n.







The magnitude of the time-dependent Fourier transform of $x[n] = \cos(\omega_0 n^2)$ using a Hamming window of length 400.

Since $X[n, \lambda)$ is the discrete-time Fourier transform of $x[n+m]w[m]$, the time-dependent Fourier transform is invertible if the window has at least one nonzero sample.

$$x[n+m]w[m] = \frac{1}{2\pi}\int_0^{2\pi} X[n, \lambda)e^{j\lambda m}d\lambda, \qquad -\infty < m < \infty, \tag{10.20}$$

from which it follows that

$$x[n] = \frac{1}{2\pi w[0]}\int_0^{2\pi} X[n, \lambda)d\lambda \tag{10.21}$$

if $w[0] \neq 0$. $X[n, \lambda)$ can be written as

$$X[n, \lambda) = \sum_{m'=-\infty}^{\infty} x[m']w[-(n-m')]e^{j\lambda(n-m')}. \tag{10.22}$$

Equation (10.22) can be interpreted as the convolution

$$X[n, \lambda) = x[n] * h_\lambda[n], \tag{10.23a}$$

where

$$h_\lambda[n] = w[-n]e^{j\lambda n}. \tag{10.23b}$$

$$H_\lambda(e^{j\omega}) = W(e^{j(\lambda-\omega)}). \tag{10.24}$$

In general, a window that is nonzero for positive time will be called a *noncausal window,* since the computation of $X[n, \lambda)$ using Eq. (10.18) requires samples that *follow* sample n in the sequence. Equivalently, in the linear-filtering interpretation, the impulse response $h_\lambda[n] = w[-n]e^{j\lambda n}$ is noncausal.

Another possibility is to shift the window as n changes, keeping the time origin for Fourier analysis fixed

This leads to a definition for the time-dependent Fourier transform of the form

$$\check{X}[n, \lambda) = \sum_{m=-\infty}^{\infty} x[m]w[m-n]e^{-j\lambda m} = e^{-j\lambda n}X[n, \lambda) \tag{10.25}$$

The Effect of the Window

The primary purpose of the window in the time-dependent Fourier transform is to limit the extent of the sequence to be transformed so that the spectral characteristics are reasonably stationary over the duration of the window. The more rapidly the signal characteristics change, the shorter the window should be.

If we consider the time-dependent Fourier transform for fixed n, then it follows from the properties of Fourier transforms that

$$X[n, \lambda) = \frac{1}{2\pi}\int_0^{2\pi} e^{j\theta n}X(e^{j\theta})W(e^{j(\lambda-\theta)})d\theta; \tag{10.28}$$

i.e., the Fourier transform of the shifted signal is convolved with the Fourier transform of the window. In Section 10.2 we saw that the ability to resolve two narrowband signal components depends on the width of the main lobe of the Fourier transform of the window, while the degree of leakage of one component into the vicinity of the other depends on the relative side-lobe amplitude. The case of no window at all corresponds to $w[n] = 1$ for all n. In this case $W(e^{j\omega}) = 2\pi\delta(\omega)$ for $-\pi \leq \omega \leq \pi$, which gives precise frequency resolution, but no time resolution.

The preceding discussion suggests that if we are using the time-dependent Fourier transform to obtain a time-dependent estimate of the frequency spectrum of a signal, it is desirable to taper the window to lower the side lobes and to use as long a window as feasible to improve the frequency resolution.

Sampling in Time and Frequency

$$w[m] = 0 \quad \text{outside the interval } 0 \le m \le L-1. \tag{10.29}$$

If we sample $X[n, \lambda)$ at N equally spaced frequencies $\lambda_k = 2\pi k/N$, with $N \ge L$, then we can still recover the original sequence from the sampled time-dependent Fourier transform. Specifically, if we define $X[n, k]$ to be

$$X[n,k] = X[n, 2\pi k/N) = \sum_{m=0}^{L-1} x[n+m]w[m]e^{-j(2\pi/N)km}, \qquad 0 \le k \le N-1, \tag{10.30}$$

then $X[n, k]$ is the DFT of the windowed sequence $x[n+m]w[m]$. Using the inverse DFT, we obtain

$$x[n+m]w[m] = \frac{1}{N}\sum_{k=0}^{N-1} X[n,k]e^{j(2\pi/N)km}, \qquad 0 \le m \le L-1. \tag{10.31}$$

Since we assume that the window $w[m] \ne 0$ for $0 \le m \le L-1$, the sequence values can be recovered in the interval from n through $(n+L-1)$ using the equation

$$x[n+m] = \frac{1}{Nw[m]}\sum_{k=0}^{N-1} X[n,k]e^{j(2\pi/N)km}, \qquad 0 \le m \le L-1, \tag{10.32}$$

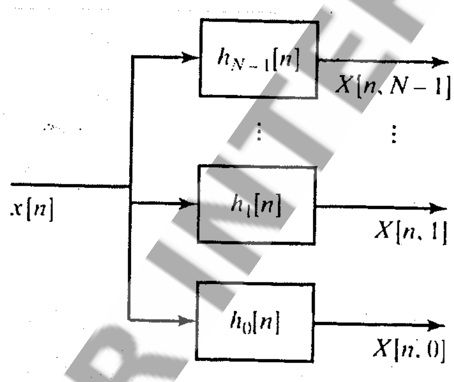
where it is assumed that $w[m] \ne 0$ for $0 \le m \le L-1$.

Eq. (10.30) can be rewritten as

$$X[n,k] = x[n] * h_k[n], \qquad 0 \le k \le N-1, \tag{10.33a}$$

where

$$h_k[n] = w[-n]e^{j(2\pi/N)kn}. \tag{10.33b}$$



Filter bank representation of the time-dependent Fourier transform.

$$H_k(e^{j\omega}) = W(e^{j[(2\pi k/N)-\omega]}). \tag{10.34}$$

Our discussion suggests that $x[n]$ for $-\infty < n < \infty$ can be reconstructed if $X[n, \lambda)$ or $X[n, k]$ is sampled in the time dimension as well. Specifically, using Eq. (10.32), we can reconstruct the signal in the interval $n_0 \le n \le n_0 + L - 1$ from $X[n_0, k]$, and we can reconstruct the signal in the interval $n_0 + L \le n \le n_0 + 2L - 1$ from $X[n_0 + L, k]$, etc. Thus, $x[n]$ can be reconstructed exactly from the time-dependent Fourier transform sampled in both the frequency and the time dimension.

we define this sampled time-dependent Fourier transform as

$$X[rR,k] = X[rR,2\pi k/N) = \sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j(2\pi/N)km}, \tag{10.35}$$

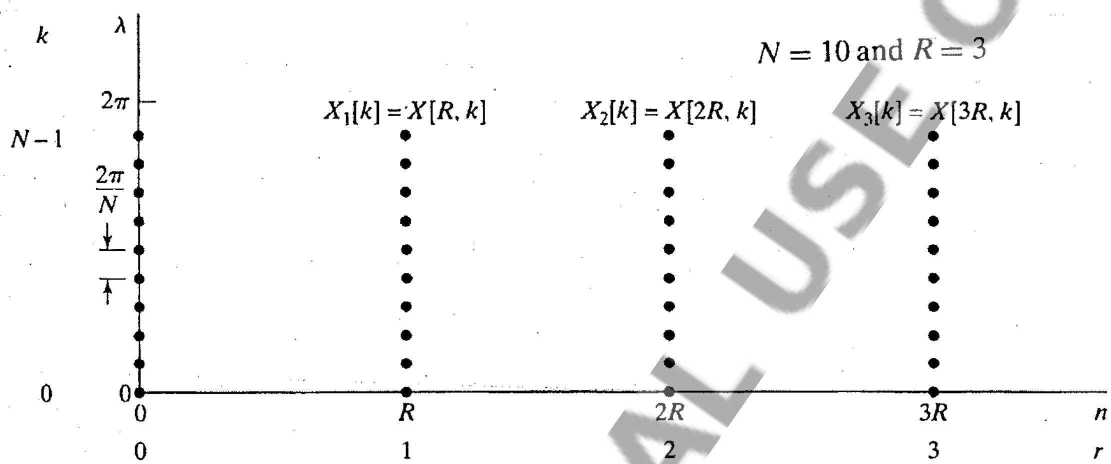
where r and k are integers such that $-\infty < r < \infty$ and $0 \le k \le N-1$. To further simplify our notation, we define

$$X_r[k] = X[rR,k] = X[rR,\lambda_k), \qquad -\infty < r < \infty, \quad 0 \le k \le N-1, \tag{10.36}$$

where $\lambda_k = 2\pi k/N$. This notation denotes explicitly that the sampled time-dependent Fourier transform is simply a sequence of N-point DFTs of the windowed signal segments

$$x_r[m] = x[rR+m]w[m], \qquad -\infty < r < \infty, \quad 0 \le m \le L-1, \tag{10.37}$$

with the window position moving in jumps of R samples in time.



Equation (10.35) involves the following integer parameters: the window length L; the number of samples in the frequency dimension, or the DFT length N; and the sampling interval in the time dimension, R. However, not all choices of these parameters will permit exact reconstruction of the signal. The choice $L \le N$ guarantees that we *can* reconstruct the windowed segments $x_r[m]$ from the block transforms $X_r[k]$. If $R < L$, the segments overlap, but if $R > L$, some of the samples of the signal are not used and therefore cannot be reconstructed from $X_r[k]$. Thus, in general, the three sampling parameters should satisfy the relation $N \ge L \ge R$.

BLOCK CONVOLUTION USING THE TIME-DEPENDENT FOURIER TRANSFORM

Assume that $x[n] = 0$ for $n < 0$, and suppose that we compute the time-dependent Fourier transform for $R = L$ and a rectangular window. In other words, the sampled time-dependent Fourier transform $X_r[k]$ consists of a set of N-point DFTs of segments of the input sequence

$$x_r[m] = x[rL+m], \qquad 0 \le m \le L-1. \tag{10.38}$$

Since each sample of the signal $x[n]$ is included and the blocks do not overlap, it follows that

$$x[n] = \sum_{r=0}^{\infty} x_r[n-rL]. \tag{10.39}$$

Now suppose that we define a new time-dependent Fourier transform

$$Y_r[k] = H[k]X_r[k], \qquad 0 \le k \le N-1, \tag{10.40}$$

where $H[k]$ is the N-point DFT of a finite-length unit sample sequence $h[n]$ such that $h[n] = 0$ for $n < 0$ and for $n > P-1$. If we compute the inverse DFT of $Y_r[k]$, we obtain

$$y_r[m] = \frac{1}{N}\sum_{k=0}^{N-1} Y_r[k]e^{j(2\pi/N)km} = \sum_{\ell=0}^{N-1} x_r[\ell]h[((m-\ell))_N]. \tag{10.41}$$

That is, $y_r[m]$ is the N-point circular convolution of $h[m]$ and $x_r[m]$. Since $h[m]$ has length P samples and $x_r[m]$ has length L samples, it follows from the discussion of Section 8.7 that if $N \ge L+P-1$, then $y_r[m]$ will be identical to the linear convolution of $h[m]$ with $x_r[m]$ in the interval $0 \le m \le L+P-2$, and it will be zero otherwise. Thus, it follows that if we construct an output signal

$$y[n] = \sum_{r=0}^{\infty} y_r[n-rL], \tag{10.42}$$

then $y[n]$ is the output of a linear time-invariant system with impulse response $h[n]$. The procedure just described corresponds exactly to the *overlap–add* method of block convolution.

FOURIER ANALYSIS OF STATIONARY RANDOM SIGNALS: THE PERIODOGRAM

Let us consider the problem of estimating the power density spectrum $P_{ss}(\Omega)$ of a continuous-time signal $s_c(t)$.

The antialiasing lowpass filter creates a new stationary random signal whose power spectrum is bandlimited, so that the signal can be sampled without aliasing. Then $x[n]$ is a stationary discrete-time random signal whose power density spectrum $P_{xx}(\omega)$ is proportional to $P_{ss}(\Omega)$ over the bandwidth of the antialiasing filter; i.e.,

$$P_{xx}(\omega) = \frac{1}{T}P_{ss}\left(\frac{\omega}{T}\right), \qquad |\omega| < \pi, \tag{10.50}$$

where we have assumed that the cutoff frequency of the antialiasing filter is π/T and that T is the sampling period.

Consequently, a reasonable estimate of $P_{xx}(\omega)$ will provide a reasonable estimate of $P_{ss}(\Omega)$. The window $w[n]$ in Figure 10.1 selects a finite-length segment (L samples) of $x[n]$, which we denote $v[n]$, the Fourier transform of which is

$$V(e^{j\omega}) = \sum_{n=0}^{L-1} w[n]x[n]e^{-j\omega n}. \tag{10.51}$$

Consider as an estimate of the power spectrum the quantity

$$I(\omega) = \frac{1}{LU}|V(e^{j\omega})|^2, \tag{10.52}$$

where the constant U anticipates a need for normalization to remove bias in the spectral estimate. When the window $w[n]$ is the rectangular window sequence, this estimator for the power spectrum is called the *periodogram*. If the window is not rectangular, $I(\omega)$ is called the *modified periodogram*. Clearly, the periodogram has some of the basic properties of the power spectrum. It is nonnegative, and for real signals, it is a real and even function of frequency.

Specifically, samples of the periodogram are given by

$$I(\omega_k) = \frac{1}{LU}|V[k]|^2, \qquad U = \frac{1}{L}\sum_{n=0}^{L-1}(w[n])^2 \tag{10.55}$$

where $V[k]$ is the N-point DFT of $w[n]x[n]$. If we want to choose N to be greater than the window length L, appropriate zero-padding would be applied to the sequence $w[n]x[n]$.

If a random signal has a nonzero mean, its power spectrum has an impulse at zero frequency. If the mean is relatively large, this component will dominate the spectrum estimate, causing low-amplitude, low-frequency components to be obscured by leakage. Therefore, in practice the mean is often estimated using Eq. (10.48), and the resulting estimate is subtracted from the random signal before computing the power spectrum estimate.

However, it has been shown (see Jenkins and Watts, 1968) that over a wide range of conditions, as the window length increases,

$$\mathrm{var}[I(\omega)] \simeq P_{xx}^2(\omega). \tag{10.65}$$

That is, the variance of the periodogram estimate is approximately the same size as the square of the power spectrum that we are estimating. Therefore, since the variance does not asymptotically approach zero with increasing window length, the periodogram is not a consistent estimate.

Periodogram Averaging

The averaging of periodograms in spectrum estimation was first studied extensively by Bartlett (1953); later, after fast algorithms for computing the DFT were developed, Welch (1970) combined these computational algorithms with the use of a data window $w[n]$ to develop the method of averaging modified periodograms. In periodogram averaging, a data sequence $x[n], 0 \le n \le Q-1$, is divided into segments of length-L samples, with a window of length L applied to each; i.e., we form the segments

$$x_r[n] = x[rR+n]w[n], \qquad 0 \le n \le L-1. \tag{10.67}$$

The periodogram of the rth segment is

$$I_r(\omega) = \frac{1}{LU}|X_r(e^{j\omega})|^2, \tag{10.68}$$

where $X_r(e^{j\omega})$ is the discrete-time Fourier transform of $x_r[n]$. Each $I_r(\omega)$ has the properties of a periodogram, as described previously. Periodogram averaging consists of averaging together the K periodogram estimates $I_r(\omega)$; i.e., we form the time-averaged periodogram defined as

$$\bar{I}(\omega) = \frac{1}{K}\sum_{r=0}^{K-1} I_r(\omega). \tag{10.69}$$

To examine the variance, we use the fact that, in general, the variance of the average of K independent identically distributed random variables is $1/K$ times the variance of each individual random variable. (See Papoulis, 1991.) Therefore,

$$\mathrm{var}[\bar{I}(\omega)] \simeq \frac{1}{K}P_{xx}^2(\omega). \tag{10.76}$$

Consequently, the variance of $\bar{I}(\omega)$ is inversely proportional to the number of periodograms averaged, and as K increases, the variance approaches zero.