University of Waterloo
Department of Electrical and Computer Engineering

ECE 413 – Digital Signal Processing
Midterm Exam, Spring 2008

June 18, 2008, 5:30-6:45 PM

Instructor: Dr. Oleg Michailovich

Student's name: ____

Student's ID #: ____

Instructions:

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Problem №1 (35%)

The continuous-time signal $x(t) = v_1(t) \cdot v_2(t)$ is sampled with an impulse train:

$$p_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - Tn).$$

a) Assuming $v_1(t)$ and $v_2(t)$ are band-limited to 200 Hz and 450 Hz, respectively, compute the minimum value of the sampling rate f_s that does not introduce any aliasing.

b) Repeat part (a) for $v_1(t) = \mathrm{sinc}(600\,t)$ and $v_2(t) = \mathrm{sinc}(1000\,t)$. Assuming that a sampling interval of $T = 2\,\mathrm{ms}$ is used to sample $x(t) = v_1(t) \cdot v_2(t)$, sketch the discrete-time Fourier transform $X^{\mathrm{f}}(\theta)$ of the sampled signal. Can $x(t)$ be accurately recovered from its samples?

c) Repeat part (b) for a sampling interval of $T = 0.1\,\mathrm{ms}$.

Problem №2 (35%)

An alternative to impulse train sampling is *sawtooth wave* sampling. In this case, the continuous time signal $x(t)$, whose Fourier transform $X(\omega)$ is shown in Fig.1(a) below, is multiplied with a periodic sawtooth wave $s(t)$ (shown in Fig.1(b)). Denote the resulting signal by $z(t) = x(t) \cdot s(t)$.

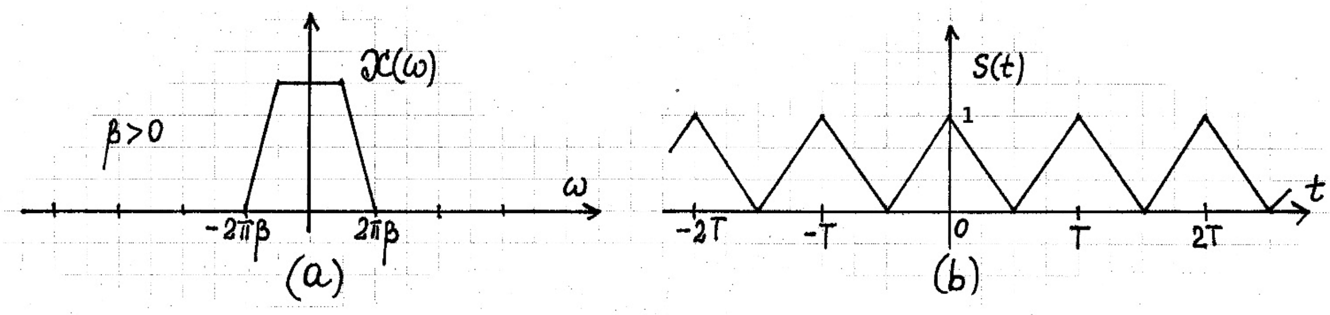


Figure 1: Pertaining to Problem 2.

a) Derive an expression for the continuous-time Fourier transform $Z(\omega)$ of $z(t)$ in terms of $X(\omega)$. Sketch $Z(\omega)$.

b) Based on your answer to part (a), can $x(t)$ be reconstructed from $z(t)$? If yes, state the conditions under which the reconstruction of $x(t)$ is possible, and provide reconstruction formulas.

c) State how $z(t)$ is related to the sampled signal $x_T(t) = x(t) \cdot p_T(t)$ obtained by ideal impulse train sampling.

Problem №3 (30%)

Recall that a linear time-invariant (LTI) system with impulse response $h[n]$ is called invertible, if there exists $h_i[n]$ such that $(h * h_i)[n] = \delta[n]$. Determine which of the following pair of impulse responses correspond to inverse systems (with respect to each other):

a)
$$h_1[n] = u[-n-1], \qquad h_2[n] = \delta[n-1] - \delta[n];$$

b)
$$h_1[n] = 0.5^n u[n], \qquad h_2[n] = \delta[n] - 0.5\,\delta[n-1];$$

c)
$$h_1[n] = n\,u[n], \qquad h_2[n] = \delta[n+1] - 2\,\delta[n] + \delta[n-1].$$

d)
$$h_1[n] = n\,0.8^n\,u[n], \qquad h_2[n] = 0.8\,\delta[n-1] - 2\,\delta[n] + 1.25\,\delta[n+1].$$

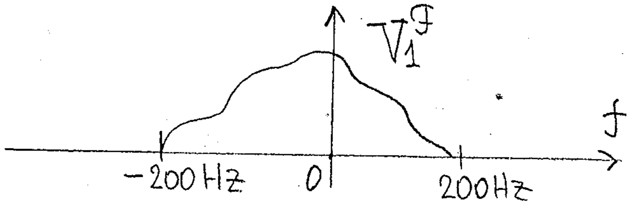
where $\delta[n]$ is the unit impulse sequence, and $u[n]$ is the unit step sequence.

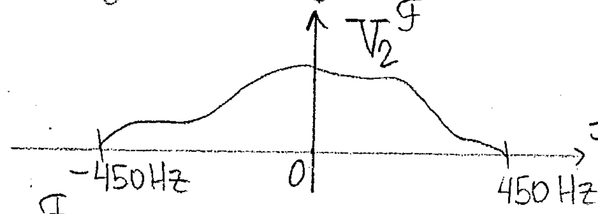
Hint: You may want (or not) to use the following facts:

- If the discrete-time Fourier transform (DTFT) of $x[n]$ is $X^{\mathrm{f}}(\theta)$, then the DTFT of $n\,x[n]$ can be derived via taking the first derivative of $X^{\mathrm{f}}(\theta)$ with respect to θ.
- The DTFT of $\alpha^n u[n]$ (for $|\alpha| < 1$) can be easily computed using the geometric series $\sum_{n=0}^{\infty} \alpha^n = (1-\alpha)^{-1}$.

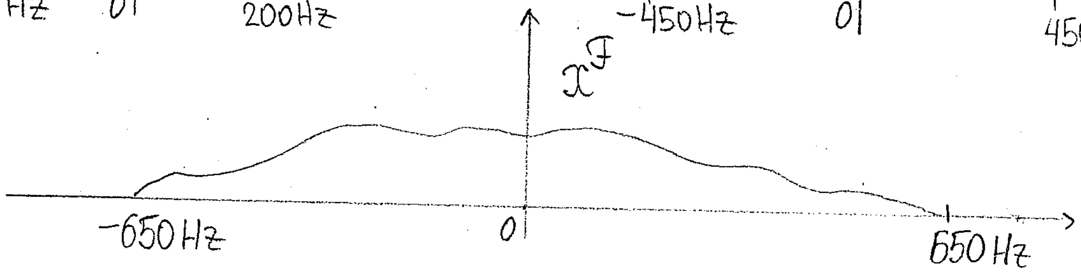
Problem #1

a) Let $x(t) = v_1(t) \cdot v_2(t)$, then $x^{\mathcal{F}}(\omega) = \frac{1}{2\pi}\{V_1^{\mathcal{F}} * V_2^{\mathcal{F}}\}(\omega)$



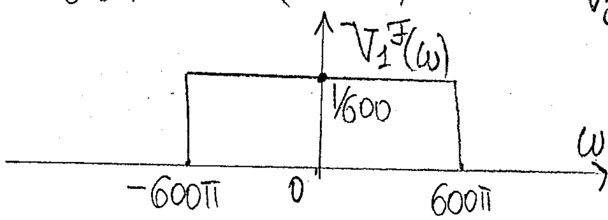


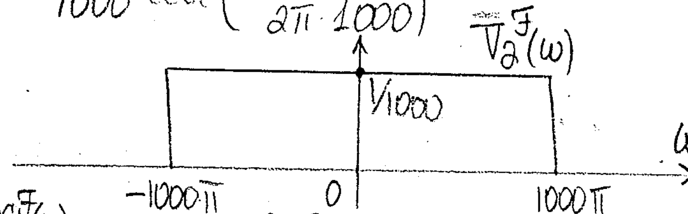
Then:



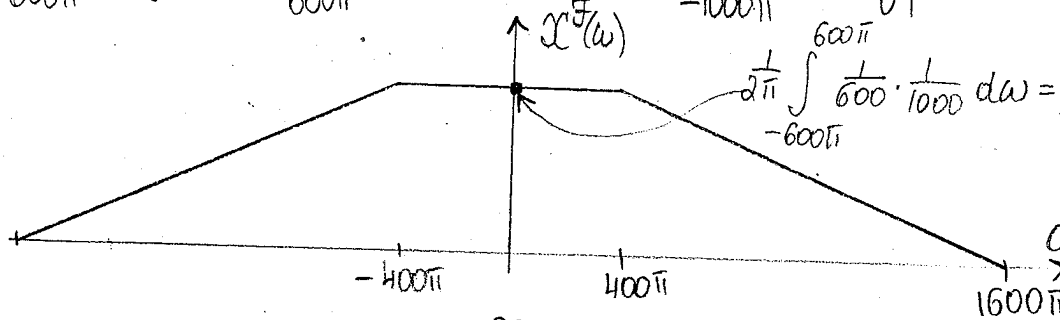
b) $v_1(t) = \text{sinc}(600t) \Rightarrow V_1^{\mathcal{F}}(\omega) = \frac{1}{600}\text{rect}\left(\frac{\omega}{2\pi \cdot 600}\right)$

$v_2(t) = \text{sinc}(1000t) \Rightarrow V_2^{\mathcal{F}}(\omega) = \frac{1}{1000}\text{rect}\left(\frac{\omega}{2\pi \cdot 1000}\right)$





Then:



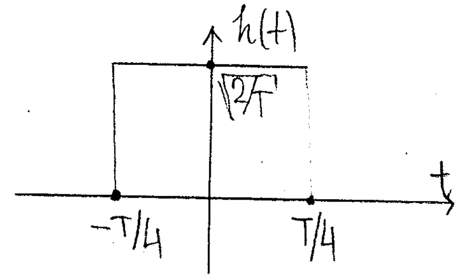
$\frac{1}{2\pi}\int_{-600\pi}^{600\pi} \frac{1}{600}\cdot\frac{1}{1000}\,d\omega = 10^{-3}$

If $T = 2ms \Rightarrow x_p^{\mathcal{F}}(\omega) = 500 \cdot \sum_{n=-\infty}^{\infty} x^{\mathcal{F}}(\omega - 1000\pi k) \Rightarrow$ aliasing

If $T = 0.1ms \Rightarrow x_p^{\mathcal{F}}(\omega) = 10^4 \sum_{n=-\infty}^{\infty} x^{\mathcal{F}}(\omega - 20000\pi) \Rightarrow$ no aliasing

Problem # 2

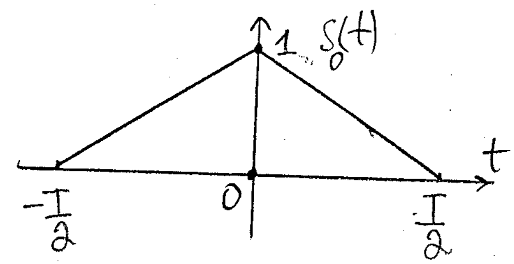
Let $h(t)$ be given by: $h(t) = \begin{cases} \sqrt{\frac{2}{T}}, & |t| \le \frac{T}{4} \\ 0, & |t| > \frac{T}{4} \end{cases}$



Alternatively, $h(t) = \sqrt{\frac{2}{T}}\, \text{rect}\left(\frac{t}{T/2}\right) = \sqrt{\frac{2}{T}}\, \text{rect}\left(\frac{2}{T}\cdot t\right)$

Therefore, $H^{\mathcal{F}}(\omega) = \sqrt{\frac{2}{T}} \cdot \frac{T}{2} \cdot \text{sinc}\left(\frac{\omega}{2\pi}\cdot\frac{T}{2}\right) = \sqrt{\frac{T}{2}}\, \text{sinc}\left(\frac{\omega T}{4\pi}\right)$

Let $S_0(t)$ be given as: $s_0(t) = h(t) * h(t)$



Moreover: $S_0^{\mathcal{F}}(\omega) = H^{\mathcal{F}}(\omega) \cdot H^{\mathcal{F}}(\omega) =$

$$= \frac{T}{2}\, \text{sinc}^2\left(\frac{\omega T}{4\pi}\right)$$

One can then see that: $s(t) = s_0(t) * p_T(t)$, and hence

$$S^{\mathcal{F}}(\omega) = S_0^{\mathcal{F}}(\omega) \cdot P_T^{\mathcal{F}}(\omega) = S_0^{\mathcal{F}}(\omega) \cdot \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T}\cdot k\right)$$

Note that $\left. S_0^{\mathcal{F}}(\omega) \right|_{\omega = \frac{2\pi}{T}k} = \frac{T}{2}\, \text{sinc}^2\left(\frac{2\pi}{T}\cdot\frac{T}{4\pi}\cdot k\right) = \frac{T}{2}\, \text{sinc}^2\left(\frac{k}{2}\right)$

Thus:

$$S^{\mathcal{F}}(\omega) = \pi \cdot \sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{k}{2}\right) \delta\left(\omega - \frac{2\pi}{T}\cdot k\right)$$

Consequently:

$$z(t) = x(t) \cdot s(t)$$

$$\Rightarrow Z^{\mathcal{F}}(\omega) = \frac{1}{2\pi}\left\{X^{\mathcal{F}} * S^{\mathcal{F}}\right\}(\omega) =$$

$$= 0.5 \sum_{k=-\infty}^{\infty} \text{sinc}^2\left(\frac{k}{2}\right) X^{\mathcal{F}}\left(\omega - \frac{2\pi}{T}\cdot k\right)$$

Note that: $\text{sinc}^2\left(\frac{k}{2}\right) = \left[\frac{\sin \frac{\pi}{2}\cdot k}{\frac{\pi}{2}\cdot k}\right]^2 = \begin{cases} 1, & k = 0 \\ \frac{4}{\pi^2 k^2}, & k\text{-odd} \\ 0, & k\text{-even} \end{cases}$

So, if $2\pi\beta < \frac{\pi}{T}$ $(T < 2\beta)$, then $Z^{\mathcal{F}}(\omega) = \frac{1}{2} X^{\mathcal{F}}(\omega)$, where $\omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$

a) $$h_1 * h_2 = \underbrace{h_1 * \delta[n-1]}_{u[-n]} - \underbrace{h_1 * \delta[n]}_{u[-n-1]} = u[-n] - u[-n-1] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases} = \delta[n]$$

b) $$H_1^f(\theta) = \frac{1}{1-0.5\, e^{-j\theta}}, \quad H_2^f(\theta) = 1 - 0.5 e^{-j\theta}$$

$$\Rightarrow \quad H_1^f(\theta) \cdot H_2^f(\theta) = H_2^f(\theta) \cdot H_1^f(\theta) = 1, \quad \forall \theta$$

$$\Rightarrow \quad h_1 * h_2 = \delta[n]$$

c) $$x^f(\theta) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\theta n} \Rightarrow \frac{dx^f(\theta)}{d\theta} = \sum_{n=-\infty}^{\infty} -j\, x[n] \cdot n\, e^{-j\theta n}$$

$$\Rightarrow \quad j\frac{dx^f(\theta)}{d\theta} = \sum_{n=-\infty}^{\infty} (n\, x[n])\, e^{-j\theta n}$$

Hence,

$$H_1^f(\theta) = j\, \frac{d}{d\theta}\left(\frac{1}{1-e^{-j\theta}}\right) = \frac{e^{-j\theta}}{(1-e^{-j\theta})^2} = \frac{e^{-j\theta}}{1-2e^{-j\theta}+e^{-j2\theta}}.$$

$$H_2^f(\theta) = e^{+j\theta} - 2 + e^{-j\theta}$$

$$H_1^f(\theta) \cdot H_2^f(\theta) = \frac{(e^{+j\theta}-2+e^{-j\theta})\, e^{-j\theta}}{1-2e^{-j\theta}+e^{-j2\theta}} = \frac{1-2e^{-j\theta}+e^{-j2\theta}}{1-2e^{-j\theta}+e^{-j2\theta}} = 1, \quad \forall\theta$$

$$\Rightarrow \quad h_1 * h_2 = \delta[n].$$

d) $$H_1^f(\theta) = j\, \frac{d}{d\theta}\left(\frac{1}{1-0.8\, e^{-j\theta}}\right) = \frac{0.8 e^{-j\theta}}{(1-0.8e^{-j\theta})^2}$$

$$H_2^f(\theta) = 0.8\, e^{-j\theta} - 2 + 1.25\, e^{j\theta}$$

$$H_1^f(\theta)\, H_2^f(\theta) = 1, \quad \forall\theta \quad \Rightarrow \quad h_1 * h_2 = \delta[n]$$

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ECE 413 – Digital Signal Processing
Final Exam, Spring 2008

August 16, 2008, 7:30-10:00 PM

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Student's ID #: ____

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Problem №1 (20%)

The input to a *causal* linear time-invariant (LTI) system is

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n],$$

where $u[n]$ is a unit step function. The z-transform of the output of this system is

$$Y(z) = \frac{-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)}.$$

a) Determine the z-transform $H(z)$ of the system impulse response. Be sure to specify the region of convergence.

b) What is the region of convergence (ROC) for $Y(z)$?

c) Determine $y[n]$.

Hint: Recall that the z-transform of $a^n u[n]$ is equal to $\frac{1}{1-az^{-1}}$ and its ROC is defined as $|z| > |a|$. On the other hand, the z-transform of $-a^n u[-n-1]$ is also equal to $\frac{1}{1-az^{-1}}$, while its ROC is defined as $|z| < |a|$.

Problem №2 (20%)

An LTI system is characterized by the transfer function

$$H(z) = \frac{\left(1-\frac{1}{2}z^{-2}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}, \quad |z| > \frac{1}{2}.$$

a) Determine the impulse response $h[n]$ of the system.

b) Determine the difference equation relating the system input $x[n]$ and the system output $y[n]$.

Hint: As the first step, you may want to represent $H(z)$ in the following form

$$H(z) = c_0 + \frac{1+d_0 z^{-1}}{1+a_1 z^{-1} + a_2 z^{-2}},$$

and then apply the method of partial fractions to the second term.

Problem №3 (20%)

Recall that an LTI system is said to have a generalized linear phase if its transfer function $H(\theta)$ can be represented as $H(\theta) = A(\theta)e^{\jmath(\varphi_0 - \theta\tau_g)}$, where $A(\theta)$ is a real-valued amplitude, φ_0 is an initial phase, and τ_g is a group delay. Let $H_1(\theta)$ and $H_2(\theta)$ be generalized linear-phase systems. Which, if any, of the following systems also must be generalized linear-phase systems?

a)
$$G_1(\theta) = H_1(\theta) + H_2(\theta),$$

b)
$$G_2(\theta) = H_1(\theta) \cdot H_2(\theta),$$

c)
$$G_3(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_1(\eta)\, H_2(\theta - \eta)\, d\eta.$$

Explain your answer and, if needed, provide a couter-example if your answer is negative.

Problem №4 (20%)

Consider a continuous-time system with transfer function

$$H_c(s) = \frac{1}{s}.$$

Suppose a discrete time system is obtained by applying the bilinear transformation to $H_c(s)$, *viz.*

$$H(z) = H_c(s)\big|_{s=(2/T)[(1-z^{-1})/(1+z^{-1})]}.$$

a) What is the transfer function $H(z)$ of the resulting discrete-time system? What is the corresponding impulse response $h[n]$?

b) If $x[n]$ is the input and $y[n]$ is the output of the resulting discrete-time system, write the difference equation that is satisfied by the input and output. What problems do you anticipate in implementing the discrete-time system using this difference equation?

Problem №5 (20%)

A discrete-time filter with transfer function $H(z)$ is designed by transforming a continuous-time filter with transfer function $H_c(s)$. It is desired that

$$H(\theta)|_{\theta=0} = H(\omega)|_{\omega=0}. \tag{1}$$

a) Could this condition hold for a filter designed by the impulse invariance method? If so, what condition(s), if any, would $H_c(\omega)$ have to satisfy?

b) Could this condition hold for a filter designed using the bilinear transformation? If so, what condition(s), if any, would $H_c(\omega)$ have to satisfy?

Hint: Recall that, in the impulse invariance method, the impulse response of the discrete-time filter is chosen proportionally to equally spaced samples of the impulse response of the continuous-time filter, i.e., $h[n] = T\,h_c(n\,T)$.

Solutions

Problem # 1:

(a)
$$x[n] = u[-n-1] + \left(\tfrac{1}{2}\right)^n u[n]$$

$$\Rightarrow \quad X(z) = \frac{-1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \tfrac{1}{2} < |z| < 1$$

Now to find $H(z)$ we simply use $H(z) = Y(z)/X(z)$; i.e.,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})} \cdot \frac{(1-z^{-1})(1-\frac{1}{2}z^{-1})}{-\frac{1}{2}z^{-1}} = \frac{1-z^{-1}}{1+z^{-1}}$$

$H(z)$ - causal $\Rightarrow$ ROC: $|z| > 1$.

(b) Since one of the poles of $X(z)$, which limited the ROC of $X(z)$ to be less than 1, is cancelled by the zero of $H(z)$, the ROC of $Y(z)$ is the region in the z-plane that satisfies the remaining two constraints $|z| > \frac{1}{2}$ and $|z| > 1$. Hence $Y(z)$ converges on $|z| > 1$.

(c)
$$Y(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1+z^{-1}} \qquad |z| > 1$$

Therefore,
$$y[n] = -\tfrac{1}{3}\left(\tfrac{1}{2}\right)^n u[n] + \tfrac{1}{3}(-1)^n u[n].$$

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Problem #2:

(a)

$$H(z) = \frac{1 - \frac{1}{2}z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} =$$

$$= -4 + \frac{5 + 7/2 z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} =$$

$$= -4 - \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{7}{1 - \frac{1}{4}z^{-1}}$$

$$\Rightarrow \quad h[n] = -4\,\delta[n] - 2\left(\tfrac{1}{2}\right)^n u[n] + 7\left(\tfrac{1}{4}\right)^n u[n].$$

(b)

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] =$$
$$= x[n] - \frac{1}{2}x[n-2]$$

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Problem #3:

(a) This system does not necessarily have generalized linear phase. The phase response,

$$G_1(\theta) = \tan^{-1}\left(\frac{Im\{H_1(\theta) + H_2(\theta)\}}{Re\{H_1(\theta) + H_2(\theta)\}}\right)$$

is not necessarily linear. As a counter-example, consider the systems

$$h_1[n] = \delta[n] + \delta[n-1]$$
$$h_2[n] = 2\delta[n] - 2\delta[n-1]$$

$$g_1[n] = h_1[n] + h_2[n] = 3\delta[n] - \delta[n-1]$$

Then, $G_1(\theta) = 3 - e^{-j\theta} = 3 - \cos\theta + j\sin\theta$

$$\Rightarrow \quad \measuredangle G_1(\theta) = \tan^{-1}\left(\frac{\sin\theta}{3-\cos\theta}\right).$$

Clearly, $G_1(\theta)$ does not have linear phase.

(b) This system must have generalized linear phase.

$$G_2(\theta) = H_1(\theta) \cdot H_2(\theta)$$

$$\Rightarrow \quad \measuredangle G_2(\theta) = \measuredangle H_1(\theta) + \measuredangle H_2(\theta).$$

The sum of two linear phase responses is also a linear phase response.

(c) This system does not necessarily have linear phase. Consider the systems

$$h_1[n] = \delta[n] + \delta[n-1]$$

$$h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

$$g_3[n] = h_1[n] \cdot h_2[n] = \delta[n] + 2\delta[n-1]$$

$$\Rightarrow \quad G_3(\theta) = 1 + 2e^{-j\theta} = 1 + 2\cos\theta - j\,2\sin\theta$$

$$\Rightarrow \quad \measuredangle G_3(\theta) = \tan^{-1}\left(\frac{2\sin\theta}{1+2\cos\theta}\right).$$

Clearly, $G_3(\theta)$ does not have linear phase.

Problem # 4:

Applying the bilinear transform yields

(a) $$H(z) = H_c(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right), \quad |z|>1$$

which has the impulse response

$$h[n] = \frac{T}{2}\left(u[n]+u[n-1]\right)$$

(b) The difference equation is

$$y[n] = \frac{T}{2}\left(x[n]+x[n-1]\right) + y[n-1]$$

This system is not implementable since it has a pole on the unit circle and is therefore not stable.

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Problem #5:

(a) Since $H(\theta) = \sum_{k=-\infty}^{\infty} H_c\left(\frac{\theta+2\pi k}{T}\right)$

and we desire

$$H(\theta)\Big|_{\theta=0} = H_c(\omega)\Big|_{\omega=0}$$

We see that

$$H(\theta)\Big|_{\theta=0} = \sum_{k=-\infty}^{\infty} H_c\left(\frac{\theta+2\pi k}{T}\right)\Big|_{\theta=0} = H_c(\omega)\Big|_{\omega=0}$$

requires

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} H_c\left(\frac{2\pi k}{T}\right) = 0.$$

(b) Since the bilinear transform maps $\omega=0$ to $\theta=0$, the condition will hold for any choice of $H_c(\omega)$.