Project №2 - Solution

Exercise 1

The Fourier transform of the signal at the output of the analog filter is $Y^F(\Omega) = H^F(\Omega)X^F(\Omega)$.
Therefore, the frequency response of the sampled signal $y[n]$ is given by the sampling theorem:

$$Y^F(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H^F(\frac{\omega-2\pi k}{T})X^F(\frac{\omega-2\pi k}{T}).$$

For the system with the digital filter we have

$$Z^F(\omega) = \frac{1}{T}\sum_{l=-\infty}^{\infty} H^F(\frac{\omega-2\pi l}{T})\sum_{k=-\infty}^{\infty} X^F(\frac{\omega-2\pi k}{T}).$$

The two outputs will be identical if and only if

$$H^F(\frac{\omega-2\pi l}{T})X^F(\frac{\omega-2\pi k}{T}) = 0 \text{ for } k \neq l.$$

We have

$$X^F(\frac{\omega}{T}) = 0 \text{ for } |w| > \Omega_1 T.$$

$$H^F(\frac{\omega-2\pi k}{T}) = 0 \text{ for } |w-2\pi k| > \Omega_2 T.$$

Therefore, the condition will be met if and only if

$$\Omega_1 T < 2\pi - \Omega_2 T$$

That is,

$$T < \frac{2\pi}{\Omega_1+\Omega_2}$$

Exercise 2

The Fourier transform of the signal at the output of the analog filter is $Y_c(\Omega) = H_c(\Omega)X_c(\Omega)$. For the system with the digital filter we have

$$Y(\omega) = \frac{1}{T}\sum_{l=-\infty}^{\infty} H_c(\frac{\omega-2\pi l}{T})\sum_{k=-\infty}^{\infty} X_c(\frac{\omega-2\pi k}{T}).$$

In order to have no aliasing, demand:

$$H^F(\frac{\omega-2\pi l}{T})X^F(\frac{\omega-2\pi k}{T}) = 0 \text{ for } k \neq l.$$

Which from exercise 1, results in:

$$T < \frac{2\pi}{\Omega_1+\Omega_2}$$

Exercise 3

The low pass filter will allow aliasing of the input signal in the range $\frac{\pi}{L} < |w| < \pi$. $\frac{\pi}{L}$ corresponds to an angular frequency of $\frac{\Omega_s}{2L}$ where Ω_s is the sampling angular frequency. To have no aliasing in the range $|w| < \frac{\pi}{L}$ we have to make sure that:

$$\frac{\Omega_s}{2L} < \Omega_s - \Omega_1$$

Which leads to

$$\Omega_s > \frac{2L}{2L-1}\Omega_1$$

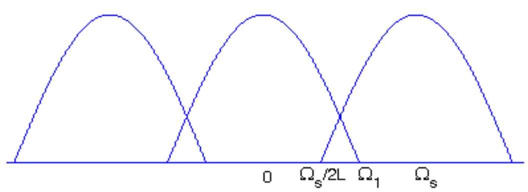

Figure 1: Frequency response after the sampler