Home Assignment №1

Due on May 20, 2008

Exercise 1

In this problem we derive the Fourier transforms of the sign function and the unit-step function.

Part A

Find the Fourier transform of the signal (for $\alpha > 0$)

$$x(t) = \begin{cases} e^{-\alpha t}, & t > 0, \\ 0, & t = 0, \\ -e^{\alpha t}, & t < 0. \end{cases}$$

Part B

The *sign function* $\operatorname{sign}(t)$ is defined by

$$\operatorname{sign}(t) = \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

Find the Fourier transform of the sign function.

Part C

The *unit-step function* $u(t)$ is defined by

$$u(t) = \begin{cases} 1, & t > 0, \\ 0.5, & t = 0, \\ 0, & t < 0. \end{cases}$$

Find the Fourier transform of the unit-step function. Hint: Express $u(t)$ in terms of $\mathrm{sign}(t)$ and another signal.

Exercise 2

Part A

For a real signal $x(t)$ we define

$$
\begin{aligned}
x_e(t) &= \frac{x(t)+x(-t)}{2}, \\
x_o(t) &= \frac{x(t)-x(-t)}{2}.
\end{aligned}
$$

The signals $x_e(t)$ and $x_o(t)$ are called the *even* and the *odd* parts of $x(t)$, respectively. Prove that

$$X_e^F(\omega) = \Re\{X^F(\omega)\}, \quad X_o^F(\omega) = \jmath\Im\{X^F(\omega)\},$$

with $\Re$ and $\Im$ denoting the real and the imaginary parts of the Fourier transform, respectively.

Part B

Let

$$x(t) = \cos(\omega_m t) \tag{1}$$

and

$$y(t) = [1 + a x(t)] \sin(\omega_c t) \tag{2}$$

where $0 < a < 1$ and $\omega_m \ll \omega_c$[1]. In this problem we assume, for simplicity, that $\omega_c = M\omega_m$, where M is an integer much larger than 1.

- Show that $y(t)$ is periodic with period $2\pi/\omega_m$.
- Define[2]

$$z(t) = \begin{cases} y(t), & y(t) \geq 0, \\ 0, & y(t) < 0. \end{cases} \tag{3}$$

[1] The operation of constructing $y(t)$ from $x(t)$ is called *amplitude modulation* (AM).
[2] The signal $z(t)$ is called half-wave rectification of $y(t)$.

Since $a < 1$, we also have

$$z(t) = \begin{cases} y(t), & \sin(\omega_c t) \geq 0, \\ 0, & \sin(\omega_c t) < 0. \end{cases} \tag{4}$$

Show that $z(t)$ is periodic with period $2\pi/\omega_m$.

- Suggest a way to extract the signal $x(t)$ from $z(t)$.

Solutions to Assignment #1

Exercise 1:

(a)

$$X^{\mathrm{F}}(\omega) = -\int_{-\infty}^{0} e^{(\alpha - j\omega)t}\,dt + \int_{0}^{\infty} e^{-(\alpha + j\omega)t}\,dt = -\frac{1}{\alpha - j\omega} + \frac{1}{\alpha + j\omega} = \boxed{-\frac{j2\omega}{\alpha^2 + \omega^2}}.$$

(b) We have

$$\mathrm{sign}(t) = \lim_{\alpha \to 0} x(t),$$

where $x(t)$ is the signal in part a. Therefore, assuming that the order of the limit and integral operations can be interchanged, we get

$$\{\mathcal{F}\,\mathrm{sign}\}(\omega) = -\lim_{\alpha \to 0} \frac{j2\omega}{\alpha^2 + \omega^2} = \boxed{\frac{2}{j\omega}}.$$

(c) We have

$$v(t) = 0.5\mathrm{sign}(t) + 0.5\,I(t).$$

Therefore,

$$\mathcal{F}\{v\}(\omega) = \boxed{\frac{1}{j\omega} + \pi\delta(\omega).}$$

Exercise 2:
Part A

If $x(t)$ is real then it is equal to its conjugate, so

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^F(w)\,e^{-jwt}\,dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X^F(w)}\,e^{jwt}\,dw$$

So the Fourier transform of $x(-t)$ is $\bar{X}^F(w)$. Therefore

$$X_e^F(w) = 0.5X^F(w) + 0.5\bar{X}^F(w) = \Re\{X^F(w)\}$$

and,

$$X_o^F(w) = 0.5X^F(w) - 0.5\bar{X}^F(w) = j\,\Im\{X^F(w)\}$$

part B :

(1) $y(t) = [1+a\,x(t)]\sin w_c t = [1+a\cos(w_m t)]\sin(M w_m t)$

$= \sin(M w_m t) + a\cos w_m t \sin(M w_m t)$

$= \sin(M w_m t) + 0.5a\sin[(M+1)w_m t] + 0.5a\sin[(M-1)w_m t]$

Thus $y(t)$ is periodic with period $\frac{2\pi}{w_m}$.

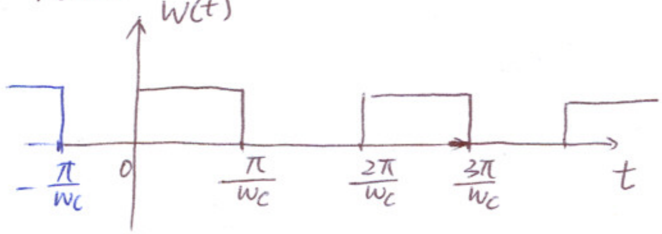
(2) $z(t) = \begin{cases} y(t) & \sin(w_c t) \geq 0 \\ 0 & \sin(w_c t) < 0 \end{cases}$

$\Leftrightarrow z(t) = \begin{cases} x(t) & \frac{2k\pi}{w_c} \leq t \leq \frac{(2k+1)\pi}{w_c} \\ 0 & \text{otherwise} \end{cases}$ $k \in \mathbb{Z}$

$\Leftrightarrow z(t) = y(t)\sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{\pi}{w_c}(t-\frac{1}{2}) - \frac{2\pi}{w_c}\cdot k\right)$

Let $w(t) = \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{\pi}{w_c}(t-\frac{1}{2}) - \frac{2\pi}{w_c}k\right)$. shows in Fig. 1

below



We can see that $w(t)$ is periodic with period $\frac{2\pi}{w_c}$

of course, it is periodic with period $M\cdot\frac{2\pi}{w_c} = \frac{2\pi}{w_m}$

Since $y(t)$ is periodic with period $\frac{2\pi}{w_m}$, $z(t)$ is periodic with period $\frac{2\pi}{w_m}$

(3) Extracting $x(t)$ from $z(t)$.

Since $z(t) = y(t)\, w(t) = y(t) \sum_{k=-\infty}^{\infty} \delta(t - \frac{2\pi}{w_c}k) * rect\left(\frac{\pi}{w_c}(t-\frac{1}{2})\right)$

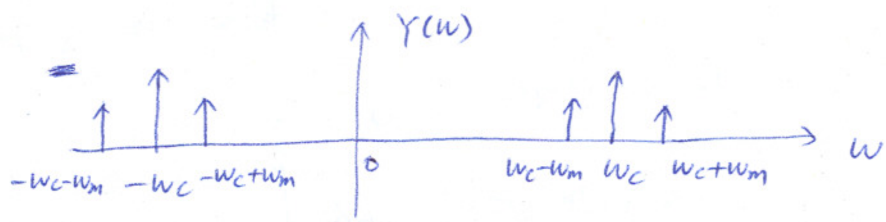
we have.

$$z^F(w) = \frac{w_c^2}{2\pi^2}\, Y^F(w) * \sum_{k=-\infty}^{\infty} sinc\left(\frac{w_c^2 k}{2\pi^2}\right) e^{-j\frac{w_c k}{2}}\, \delta(w - w_c k)$$

$$= \frac{w_c^2}{2\pi^2} \sum_{k=-\infty}^{\infty} sinc\left(\frac{w_c^2 k}{2\pi^2}\right) e^{-j\frac{w_c k}{2}}\, Y(w - w_c k)$$

And $y(t) = [1 + a\,x(t)] \sin w_c t$, so $Y^F(w) = \frac{1}{2\pi}[1 + a\,X(w)]$ $* \left[\frac{\pi}{j}\left(\delta(w-w_c) - \delta(w+w_c)\right)\right]$

$\Rightarrow$ $Y^F(w) = \frac{1}{2\pi}\left[1 + a\pi\left(\delta(w-w_m) + \delta(w+w_m)\right)\right] * \frac{\pi}{j}$ $\left(\delta(w-w_c) - \delta(w+w_c)\right)$ see as below



Notice that in frequency domain $Z(w)$ is just copying $Y(w)$, scaling it with a constant and shifting it with period w_c. So to obtain $x(t)$ from $z(t)$, what we need is just a $\boxed{\text{low-pass filter}}$ with cut-off frequency around w_m followed by some simple scaling pro

Home Assignment №2

Due on May 29, 2008

Exercise 1

Given a continuous-time signal $x(t)$ with $X^F(\omega) = 0$ for $|\omega| > \omega_m$ determine the minimum sampling rate f_s for a signal $y(t)$ defined by:

$$\textbf{a)}\ \ x^2(t), \quad \textbf{b)}\ \ x(2t), \quad \textbf{c)}\ \ x(t)\cos(6\pi\omega_m t).$$

Exercise 2 (*Natural Sampling*)

Suppose the signal $x(t)$ is band-limited with $X^F(\omega) = 0$ for $|\omega| \geq B$. Instead of sampling with a train of δ's we sample $x(t)$ with a train of very narrow-supported pulses. The pulse is given by a function $p(t)$, we sample at a rate T, and the sampled signal then has the form

$$g(t) = x(t) \sum_{k=-\infty}^{\infty} T\, p(t - kT)$$

- Is it possible to recover the original signal $x(t)$ from the signal $g(t)$?
- If not, why not. If it is possible, what conditions on the parameters T and B, and on the pulse $p(t)$ make it possible.

Exercise 3

Reconstruction of continuous-time signals from their samples using the classical formula is impractical, since the *sinc* function cannot be implemented as a causal filer. Hence, developing different reconstruction methods has been a topic of intensive research. One of the possible methods is so-called *zero-order hold* that interpolates the signal over $[nT, nT+T]$ by a constant value equal to $x(nT)$. Formally it can be represented by the following formula:

$$x_{ZOH}(t) = \sum_{-\infty}^{\infty} x(nT)h_T(t - nT),$$

where

$$h_T(t) = \begin{cases} 1, & \text{if } t \in [0, T), \\ 0, & \text{otherwise.} \end{cases}$$

The figure below shows an output of a reconstructor (i.e., D/A converter) which linearly interpolates between successive samples:

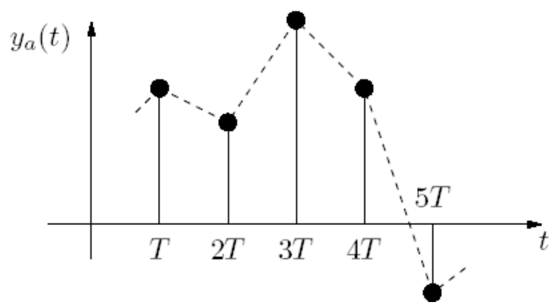


Figure 1: Pertaining to Exercise 3.

Suppose the impulse response of this D/A converter is $h_a(t)$, i.e.

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(nT)h_a(t - nT)$$

- Plot the impulse response $h_a(t)$ and determine whether this reconstruction can be performed as a causal operation.
- Is there a filter, by using which the result of the above reconstruction can be transformed to the original signal $x(t)$ (assuming the latter is band-limited and sampled well above its Nyquist rate)? If there is, find the frequency response of this analog filter.

Solutions to Assignment #2

<u>Exercise 1:</u>

a) $y(t) = x^2(t) \Rightarrow Y^F(w) = \frac{1}{2\pi} X^F(w) * X^F(w)$

we have $x(t)$ is band-limited with $Y^F(w) = 0$ for $|w| > 2W_m$

so the minimum sampling rate f_s (Nyquist rate) will be

$2\pi f_s = 4W_m$ i.e. $f_s = \frac{2}{\pi} W_m$. $\boxtimes$

b) $y(t) = x(2t) \Rightarrow Y^F(w) = \frac{1}{2} X^F(\frac{w}{2})$

so $y(t)$ is band-limited with $Y^F(w) = 0$ for $|w| > 2W_m$

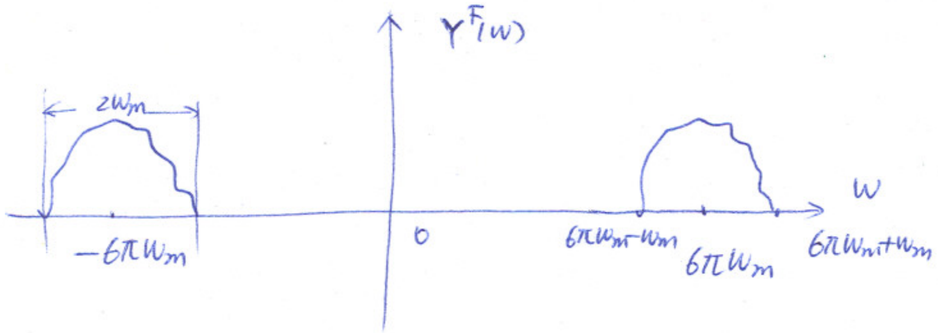
Thus Nyquist rate f_s is $f_s = \frac{2}{\pi} W_m$. $\boxtimes$

c) $y(t) = x(t) \cos(6\pi W_m t) \Rightarrow Y^F(w) = \pi [X^F(w - 6\pi W_m) +$

$X^F(w + 6\pi W_m)]$

$y(t)$ is band-limited with $Y^F(w) = 0$ for $|w| > (6\pi+1)W_m$

and $|w| < (6\pi-1)W_m$



Thus ~~the~~ the minimum sampling rate

$$f_s = \frac{(6\pi+1)W_m}{\pi \left\lfloor \frac{(6\pi+1)W_m}{2W_m} \right\rfloor} = \frac{6\pi+1}{9\pi} W_m$$

(See Problem #3 in Tutorial 3)

Exercise 2 :

Solution : The trick to this problem is noting that our natural sampling function is periodization of the pulse and can be expressed as a convolution of $g(t)$ with implinse train.

$$g(t) = x(t) \sum_{k=-\infty}^{\infty} T\, p(t-kT) = x(t)\, T \sum_{k=-\infty}^{\infty} \delta(t-kT) * p(t)$$

$$= x(t)\, T \left(p(t) * \sum_{k=-\infty}^{\infty} \delta(t-kT) \right)$$

Now do Fourier transform on both sides

$$G^F(w) = \frac{1}{2\pi} x^F(w) * T \left(p^F(w) \cancel{*} \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(w - \frac{2\pi n}{T}) \right)$$

$$= \frac{1}{2\pi} \cdot 2\pi \; x^F(w) * \left(\sum_{n=-\infty}^{\infty} p^F(\frac{2n\pi}{T})\, \delta(w - \frac{2\pi n}{T}) \right)$$

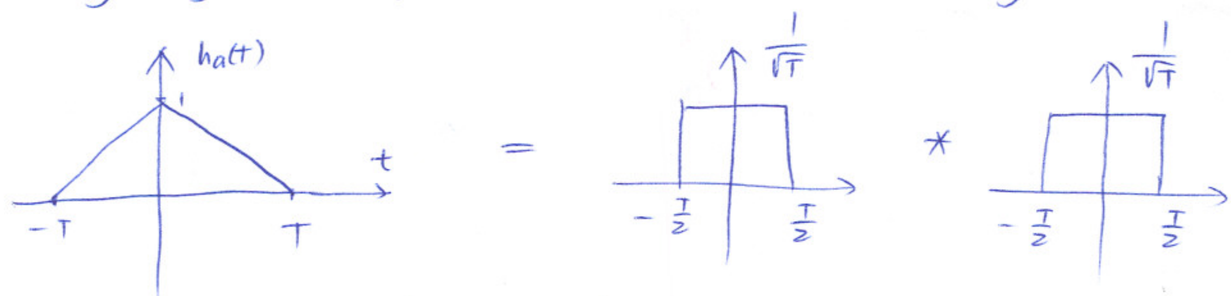
$$= \cancel{2\pi} \sum_{n=-\infty}^{\infty} p^F(\frac{2n\pi}{T})\, x^F(w - \frac{2\pi n}{T})$$

a) So we see that, just like with ideal sampling, the Fourier transform of the sampled signal is an infinite sum of shifted copies of original signal's Fourier transform. The only difference in the case when realistic pulse are used is that each of these copies is scaled by the value of pulse's Fourier transform at the center frequency of the copy. Scaling factors are easy to be taken out by adjusting the gain of the low-pass filter we use to reconstructe ~~f~~ $x(t)$. So $\boxed{\text{yes,}}$ under certain conditions it is possible to recover $x(t)$ from $g(t)$

b) As with ideal sampling, we need $\boxed{T < \frac{1}{2B}}$. Surprisingly there are no conditions on the pulse $p(t)$ as long as it has a Fourier transform so that all the steps we took before make sense.

Exercise 3 :

a) Consider the unit pulse response of the D/A. The linear interpolation implies that the response ($h_a(t)$) will be a triangle function from $-T$ to T with height 1, it is



no causal since $h_a(t) > 0$, for $t < 0$.

b) We need a filter, passing through which, the undersirable effect of ~~th~~ the above reconstructor can be overcome.

since the frequency response of $h_a(t)$ is

$$H_a^F(w) = \frac{1}{T}\left(\frac{T\sin\frac{wT}{2}}{\frac{wT}{2}}\right)^2 = T\,\mathrm{sinc}^2\left(\frac{wT}{2\pi}\right)$$

Using a filter $G^F(w)$ at the output of the D/A and made frequency response of the cascade of the two filters be an ideal D/A

$$G^F(w)\cdot H_a^F(w) = G_{ideal}(w) = \begin{cases} T & |w| \le \frac{\pi}{T} \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow \quad G^F(w) = \begin{cases} \dfrac{T}{Ha^F(w)} = \left(\dfrac{\frac{wT}{2}}{\sin\frac{wT}{2}}\right)^2 & |w| \le \frac{\pi}{T} \\ 0 & \text{else.} \end{cases}$$

So $\boxed{\text{yes,}}$ there is an analog filter satisfying above conditions and its frequency response is $G^F(w)$

$$G^F(w) = \begin{cases} \left(\dfrac{\frac{wT}{2}}{\sin\frac{wT}{2}}\right)^2 & |w| \le \frac{\pi}{T} \\ 0 & \text{else.} \end{cases} \quad \boxtimes$$

Home Assignment №3

Due on June 9, 2008

Exercise 1

Consider the finite-length sequence

$$x[n] = 2\delta[n] + \delta[n-1] + \delta[n-3],$$

We perform the following operation on this sequence:

- We compute the five-point DFT $X^d[k]$.
- We compute a five-point inverse DFT of $Y^d[k] = (X^d[k])^2$ to obtain a sequence $y[n]$.

a) Determine the sequence $y[n]$ for $n = 0, 1, 2, 3, 4$.

b) If N-point DFTs are used in the two-step procedure, how should we choose N so that $y[n] = \{x * x\}[n]$ for $0 \leq n \leq N-1$?

Exercise 2 (Inverse DFT)

A sequence $x[n]$ with $n = 0, 1, 2, \ldots, N-1$ is zero-padded to $2N$ points (i.e., N zeros are appended to $x[n]$). Let $x_a[n]$ be this $2N$-point sequence and let $X_a^d[k]$ be the $2N$-point DFT of $x_a[n]$.

a) Determine the inverse N-point DFT of a N-point sequence consisting of the even-index components of $X_a^d[k]$.

b) Determine the inverse N-point DFT of a N-point sequence consisting of the odd-index components of $X_a^d[k]$.

Exercise 3 (Circular Convolution)

Let X_1 and X_2 be two $N \times N$ *circulant* matrices corresponding to finite-dimensional sequences $x_1[n]$ and $x_2[n]$ of length N, respectively. Find a general expression for the (k, n)-th element of matrix Y that is equal to the product of X_1 and X_2, viz. $Y = X_1 X_2$.

Solution to Assignment #3

Exercise 1

We have the finite-length sequence:

$$x[n] = 2\delta[n] + \delta[n-1] + \delta[n-3]$$

(i) Suppose we perform the 5-point DFT:

$$X[k] = 2 + W_5^{-k} + W_5^{-3k}$$

where $W_5^k = e^{j\frac{2\pi k}{5}}$

(ii) Now we square the DFT of $x[n]$

$$Y[k] = X^2[k] = \left(2 + W_5^{-k} + W_5^{-3k}\right)^2$$

$$= 4 + W_5^{-2k} + W_5^{-6k} + 4W_5^{-k} + 4W_5^{-3k} + 2W_5^{-4k}$$

Using the fact $W_5^{-6k} = W_5^{-k}$,

$$Y[k] = 4 + 5W_5^{-k} + W_5^{-2k} + 4W_5^{-3k} + 2W_5^{-4k}$$

(a) By inspection

$$y[n] = 4\delta[n] + 5\delta[n-1] + \delta[n-2] + 4\delta[n-\cancel{4}3] + 2\delta[n-4]$$

(b) Using the properties of the DFT, an alternative method may be achieved with circular convolution:

$$y[n] = IDFT\{X^2[k]\} = x[n] \circledast x[n] =$$

If we want

$$y[n] = x[n] * x[n] = x[n] \circledast x[n]$$

then the length of $y[n]$ $N \geq 2M-1$, where M is the length of $x[n]$. Since $M = 4$, $N \geq 2\times 4 - 1 = 7$. $\boxtimes$

Exercise 2.

Since

$$x_a[n] = \begin{cases} x[n], & 0 \leq n \leq N-1 \\ 0 & N \leq n \leq 2N-1 \end{cases}$$

$$X_a^d[k] = DFT_{2N}\{x_a[n]\} = \sum_{n=0}^{2N-1} x_a[n]\, W_{2N}^{-kn} = \sum_{n=0}^{N-1} x[n]\, W_{2N}^{-kn}$$

Let H_K be the N-point sequence consisting of the even index components of $X_a^d[k]$. Then $H_K = X_a^d[2k]$ for $k = 0, 1, 2 \ldots N-1$

From the inverse DFT formula:

$$h[n] = IDFT_N\{H_k\} = \frac{1}{N}\sum_{k=0}^{N-1} H_k W_N^{kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_a^d[2k]\, W_N^{kn} = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{\ell=0}^{N-1} x[\ell]\, W_{2N}^{-2k\ell}\right) W_N^{kn}$$

$$\left(W_{2N}^{-2k\ell} = W_N^{-k\ell}\right) \quad = \frac{1}{N}\sum_{\ell=0}^{N-1} x[\ell]\left(\sum_{k=0}^{N-1} W_N^{k(n-\ell)}\right)$$

$$= \frac{1}{N}\sum_{\ell=0}^{N-1} x[\ell]\, N \cdot \delta[(n-\ell) \bmod N]$$

$$= x[n]$$

b) This time, let H_k be the N-point sequence consisting of odd index components of $X_{da}^{d}[k]$. Then $H_k = X_a^d[2k+1]$, for $k = 0, 1, \dots, N-1$. So

$$h[n] = IDFT_N\{H_k\} = \frac{1}{N}\sum_{k=0}^{N-1} H_k W_N^{kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X_a^d[2k+1] W_N^{kn} = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{\ell=0}^{N-1} x[\ell] W_{2N}^{-(2k+1)\ell}\right) \cdot W_N^{kn}$$

$$= \frac{1}{N}\sum_{\ell=0}^{N-1} x[\ell]\left(\sum_{k=0}^{N-1} W_N^{k(n-\ell)}\right) \cdot W_{2N}^{-\ell}$$

$$= \frac{1}{N}\sum_{\ell=0}^{N-1} x[\ell] \cdot N \cdot \delta[n-\ell] \cdot W_{2N}^{-\ell}$$

$$= x[n] W_{2N}^{-n} = x[n]\exp\left(-\frac{2\pi j n}{2N}\right) = x[n] e^{-\frac{\pi n j}{N}}$$

$\boxtimes$

Exercise 3

We know $y[n] = \{x_1 \circledast x_2\}[n]$. Also, $Y = X_1 X_2$, where X_1 and X_2 are circulant matrices corresponding to $x_1[n]$ and $x_2[n]$. We prove that Y must be the circulant matrix corresponding to $y[n]$.

Suppose we have a column vector z corresponding to a sequence $z[n]$. Let $w[n] = \{y \circledast z\}[n] = \{x_1 \circledast x_2 \circledast z\}[n]$. Therefore $w = X_1 X_2 z = Y' z$, where w is column vector corresponding to $w[n]$. So $Y' = X_1 X_2 = Y$. Y must be the circlant matrix corresponding to $y[n]$. So $Y_{k,n} = y[(k-n) \bmod N]$. $\boxtimes$

Home Assignment №4

Due on June 16, 2008

Exercise 1

Let $\omega_0 = \frac{\pi}{16}$. Consider three values $N = 16$, $N = 64$, and $N = 256$. Let $x_N[n] = \cos(\omega_0 n)$ for $n = 0, \ldots, N-1$, and $X_N[k]$ be its N-point DFT coefficients.

a) Plot $|X_N[k]|$ for the above three values of N. Use `subplot()` to compare the results for different values of N.

b) Explain the behavior of $|X_N[k]|$ as a function of N.

Exercise 2

In some applications in coding theory, it is necessary to compute a 63-point circular convolution of two 63-points sequences $x[n]$ and $h[n]$. Suppose that the only computational devices available for us are multipliers, adders and processors that compute N-point DFTs, and N restricted to be a power of 2.

a) Create two 63-length random sequences $x[n]$ and $h[n]$ using the MATLAB function `randn()` and write a matlab program to compute an M-point circular convolution of two sequences by computing their linear convolution in time domain. Verify your program by computing the 63-point circular convolution of $x[n]$ and $h[n]$.

b) Write a MATLAB program that computes the 63-point circular convolution of the above random sequences $x[n]$ and $h[n]$ using two 128-point DFTs and one 128-point inverse DFT. Compare the results and computation complexity (in terms of multiplications) of this method with that in (a). (Assume that one complex multiplication requires four real multiplications and both $x[n]$ and $h[n]$ are real.)

Exercise 3 (Zero Padding in frequency domain)

Assume we are given the DFT of a length-N sequence (where N is odd), and define the *zero-padded* in frequency domain as

$$X_i^d[k] = \begin{cases} LX^d[k], & 0 \leq k \leq \frac{N-1}{2}, \\ LX^d[k-M+N], & M - \frac{N-1}{2} \leq k \leq M-1, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

for $M = LN$ and $L > 0$. The new DFT has zero values at high frequencies and conjugate symmetry is preserved if processed by $X^d[k]$.

a) Modify the definition of zero padding in the frequency domain (1) to the case of even N.

b) Write a MATLAB program that implements interpolation of a finite sequence $x[n]$ by zero padding in the frequency domain. The program should treat both even and odd lengths of $x[n]$ and not be limited to M which is an integer multiple of N.

Note:

- Remember to attach the MATLAB code and the generated plots to your homework submission.
- To get info about MATLAB functions, type `help function_name` at the MATLAB command prompt.

Solutions to assignment 4

Note: The following answers are not unique, we hereby provide referenced code.

Exercise 1

```
close all;
clear all;
omega = pi/16;
j = [16, 64, 256];
for i = 1:1:length(j)
N = j(i);
x(i, 1:N) = ones(1, N) .* cos(omega .*(0:1:N-1));
y(i, 1:N) = fft(x(i, 1:N));
end
figure;
subplot(3, 1, 1),
plot(x(1, 1:j(1)), 'o');
xlabel('k+1');
ylabel('x_1_6[n]');
subplot(3, 1, 2)
plot(x(2, 1:j(2)), 'o');
xlabel('k+1');
ylabel('x_6_4[n]');
subplot(3, 1, 3)
plot(x(3, 1:j(3)), 'o');
xlabel('k+1');
ylabel('x_2_5_6[n]');
figure;
subplot(3, 1, 1);
plot(abs(y(1, 1:j(1))), 'o');
xlabel('k+1');
ylabel('|X_1_6[k]|');
subplot(3, 1, 2)
plot(abs(y(2, 1:j(2))), 'o');
xlabel('k+1');
ylabel('|X_6_4[k]|');
subplot(3, 1, 3)
plot(abs(y(3, 1:j(3))), 'o');
xlabel('k+1');
ylabel('|X_2_5_6[k]|');
```

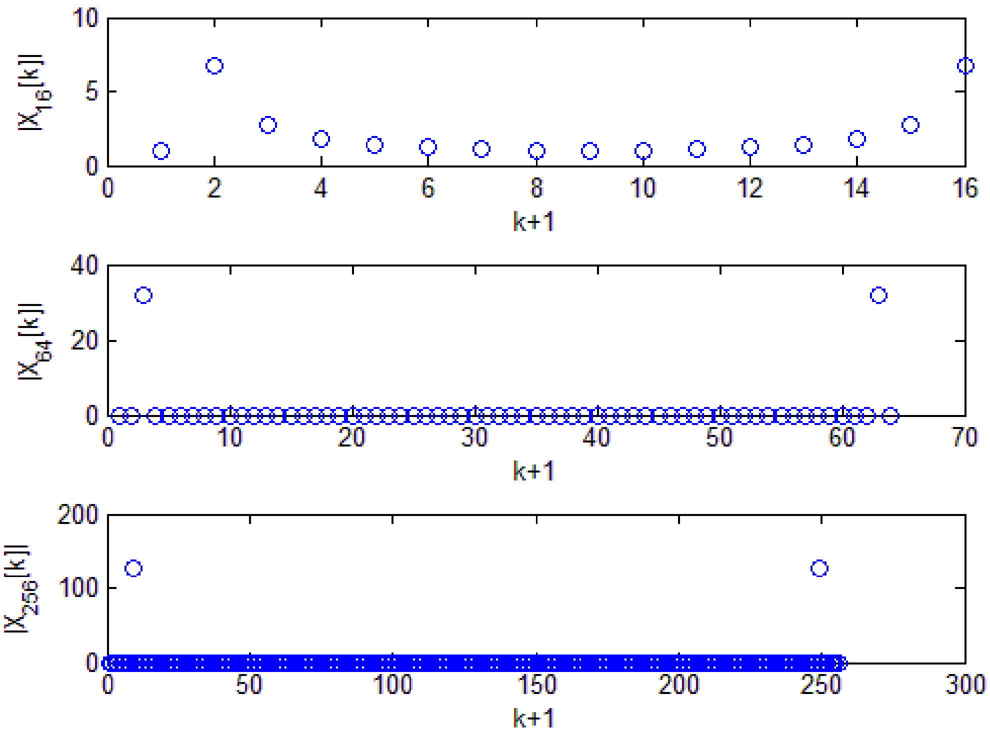


Figure 1: Figure for Problem 1

Exercise 2

(a)
```
function z=circonv(x,y)
%synopsis: z=circonv(x,y)
%performs circular convolution by means of linear convolution
% input: x,y are two vectors; output: z the result of the circular
% convolution
N=length(x);
if (length(y)~=N)
    error('Vectors of unequal lengths in circonv');
else
    z=conv(reshape(x,1,N),reshape(y,1,N));
    z=z(1:N)+[z(N+1:2*N-1),0];
end
```

(b)
```
function z=circonv63(x,y)
% Implement 63-point cicular convolution using 128-point DFT
N=length(x); M=length(y);
if (length(y)~=N)
    error('Vectors of unequal lengths in circonv63');
elseif (M~=63)&&(N~=63)
    error('input vector length does not equal to 63');
else
    % zero padded to 128 points sequences
```

```
    Xa=[reshape(x,1,N),zeros(1,128-N)];
    Ya=[reshape(y,1,M),zeros(1,128-M)];
    Za=ifft(fft(Xa,128).*fft(Ya,128),128);
    z=Za(1:63)+[Za(64:125),0];
end
```

Complexity: in terms of multiplications

Part (a) implement it using linear convolution: # mult= $2\times\sum_{n=1}^{63} n^2 - 63 = 3969$

Part (b) implement it using 128-point DFTs: # mult= $4\times 3\times(128\log_2 128) = 10752$

Exercise 3

(a) when N is even, $X_i^d[k] = \begin{cases} LX^d[k] & 0 \le k \le \frac{N}{2}-1 \\ LX^d[k+N-M] & M+1-\frac{N}{2} \le k \le M-1 \end{cases}$

(b)

```
function z=zero_pad_freq(x,M)
% Implement zero padding in frequency domain
% x is input vector with length N
% M length of zero padded sequence in frequency domain
N=length(x);
L=M/N;
Xk=fft(x,N);
if (N-2*floor(N/2)==0)
    % when N is even
    Zk=[L*Xk(1:N/2),zeros(1,M-N),L*Xk(N/2+1:N)];
else
    % when N is odd
    Zk=[L*Xk(1:(N+1)/2),zeros(1,M-N),L*Xk((N+1)/2+1:N)];
end
z=ifft(Zk,M);
```

Home Assignment №5

Due on June 30, 2008

Exercise 1

Consider the window

$$w[n] = \sin^4\left(\frac{\pi n}{N-1}\right).$$

Explore the properties of this window: main-lobe width and side-lobe level. How is this window related to the Hann window?

Exercise 2

We are given 128 samples of the signal

$$x[n] = \sin\left(2\pi\frac{6.3}{128}n\right) + 0.001\sin\left(2\pi\frac{56}{128}n\right).$$

a) Explain why a rectangular window is not adequate for detecting the second component.

b) Of the Hann and Hamming windows, which one is better in this case for detecting the second component? Explain you answer and illustrate it on a computer.

Exercise 3

Consider the system of Figure 1 with input $x(t) = e^{j\phi_0 t}$ that is sampled with period T[1].

[1] T is chosen such that no aliasing occurs during the sampling process.

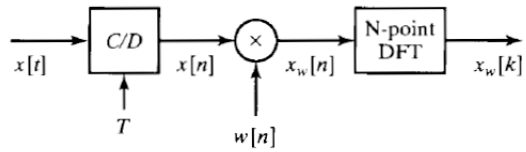


Figure 1: System figure for Exercise 3.

Suppose that now $N = 32$, Figure 2 and Figure 3 show the magnitude of the sequence $X_w[k]$ for $k = 0, 1, \ldots, 31$ for the following two different choices of $w[n]$:

$$w_1[n] = \begin{cases} 1, & 0 \le n \le 31, \\ 0, & \text{otherwise.} \end{cases}$$

$$w_2[n] = \begin{cases} 1, & 0 \le n \le 7, \\ 0, & \text{otherwise.} \end{cases}$$

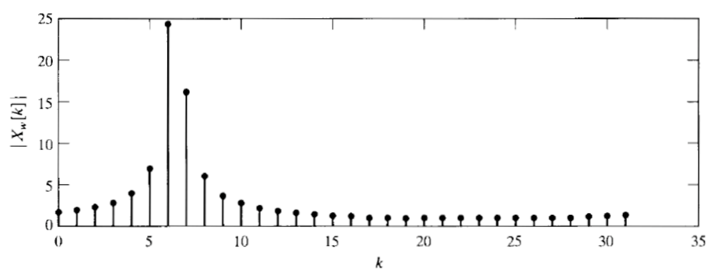


Figure 2: Pertaining to Exercise 3.

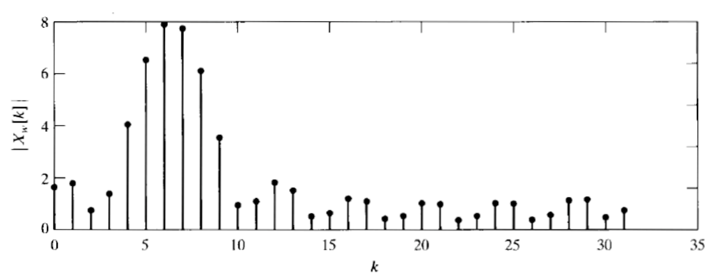


Figure 3: Pertaining to Exercise 3.

a) Indicate which figure corresponds to which choice of $w[n]$. State your reasoning clearly.

b) For the input signal and system parameters in part (A), we would like to estimate the value of ϕ_0 from Figure 2 when the sampling period is $T = 10^{-4}$. Assuming that

$$w[n] = \begin{cases} 1, & 0 \leq n \leq 31, \\ 0, & \text{otherwise.} \end{cases}$$

and that the sampling period is sufficient to ensure that no aliasing occurs during sampling, estimate the value of ϕ_0. Is your estimate exact? If it is not, what is the maximum possible error of your frequency estimate?

c) Suppose you were provided with exact values of the 32-point DFT $X_w[k]$ for the windows choices $w_1[n]$ and $w_2[n]$. Briefly describe a procedure to obtain a precise estimate of ϕ_0.

Solutions to assignment ~~4~~ 5

Exercise 1

By the identity $\sin^2\alpha = 0.5(1-\cos(2\alpha))$ we can write

$$w[n] = 0.25\left[1-\cos\left(\frac{2\pi n}{N-1}\right)\right]^2$$

Therefore by (6.14) we have

$$w[n] = (w_{hn}[n])^2$$

This implies that

$$W^f(\theta) = \frac{1}{2\pi}\left\{W^f_{hn} \circledast W^f_{hn}\right\}(\theta)$$

Using MATLAB, we can see that the main-lobe width of this window is $\frac{12\pi}{N}$. This is 1.5 times wider then the main-lobe width of the Hann window. The side-lobe level is approximately -47 dB.

Exercise 2

A) We recall from solution 6.1 [redacted] that the side-lobe of the rectangular window nearest to $\theta = \pi$ has magnitude N times smaller than the main lobe. In this case the ratio is approximately 0.01. Since the second sinusoidal component has amplitude 0.001, it is completely masked by the side lobe of the rectangular window.

B) The side lobes of the Hann window decay much faster than those of the Hamming window, therefore the Hann window is much better for detecting the second component in this case. This is confirmed by testing the windowed DFTs on MATLAB

Exercise 3

a)

The rectangular windows, $w_1[n]$ and $w_2[n]$, differ only in their lengths, which are 32 and 8 respectively. Recall that the Fourier transform of a shorter window has a wider mainlobe and higher sidelobes compared to that of a longer window. Since the DFT is a sampled version of the DTFT, we try to use these features to distinguish the two plots. We notice that the second plot, Figure [redacted]-3, appears to have a wider mainlobe and higher sidelobes. As a result, we conclude that Figure [redacted]-2 corresponds to $w_1[n]$, and Figure [redacted]-3 corresponds to $w_2[n]$.

b)

A simple technique to estimate the value of ω_0 is to find the value of k where $|X_w[k]|$ is largest. Call this index $\hat{k}_0$. The estimate is then:

$$\widehat{\omega}_0 = \frac{2\pi\hat{k}_0}{N}$$

The corresponding value of $\widehat{\Omega}_0$ is

$$\widehat{\Omega}_0 = \frac{2\pi\hat{k}_0}{NT}$$

This estimate is not exact, since the peak of the Fourier transform magnitude $|X_w(e^{j\omega})|$ could occur between two DFT samples. The maximum possible error $\Delta\Omega_{\max}$ in the estimate is one half of the frequency resolution of the DFT.

$$\Delta\Omega_{\max} = \frac{1}{2}\frac{2\pi}{NT} = \frac{\pi}{NT}$$

From Figure P[redacted]-2, $k = 6$, and with the system parameters $N = 32$ and $T = 10^{-4}$,

$$\widehat{\Omega}_0 \pm \Delta\Omega_{\max} = 11781 \pm 982 \text{ rad/s} = 1875 \pm 156 \text{ Hz}$$

c)

The following procedure provides a precise estimate of Ω_0, starting from the coarse estimate in part (c). Other procedures are also possible.

We seek an algebraic expression for the N-point DFT $X_w[k]$. We first find the Fourier transform of $x_w[n] = x[n]w[n]$, where $w[n]$ is an M-point rectangular window and M is not necessarily equal to N. Since $x[n]$ is a pure complex exponential with frequency ω_0, $X_w(e^{j\omega})$ is equal to the Fourier transform of an M-point rectangular window shifted in frequency by ω_0:

$$X_w(e^{j\omega}) = \frac{\sin\left(\frac{(\omega-\omega_0)M}{2}\right)}{\sin\left(\frac{(\omega-\omega_0)}{2}\right)} e^{-j\frac{(\omega-\omega_0)(M-1)}{2}}$$

Note that $X_w(e^{j\omega})$ has generalized linear phase. We find $X_w[k]$ by evaluating the above expression at frequencies $\omega = \frac{2\pi k}{N}$ for $k = 0, 1, \ldots, N-1$:

$$X_w[k] = \frac{\sin\left(\frac{\left(\frac{2\pi k}{N}-\omega_0\right)M}{2}\right)}{\sin\left(\frac{\left(\frac{2\pi k}{N}-\omega_0\right)}{2}\right)} e^{-j\frac{\left(\frac{2\pi k}{N}-\omega_0\right)(M-1)}{2}}$$

We know the wrapped phase of $X_w[k]$, given by:

$$\measuredangle X_w[k] = \left(\omega_0 - \frac{2\pi k}{N}\right)\left(\frac{M-1}{2}\right) + m\pi$$

where the $m\pi$ term accounts for possible sign changes in the amplitude of $X_w[k]$ as well as phase wrapping, so that $\measuredangle X_w[k]$ stays in the range $[-\pi, \pi]$.

From part (c) we know roughly where ω_0 should lie. Substituting $k = \hat{k}_0$ into the phase expression,

$$\begin{aligned}\measuredangle X_w[\hat{k}_0] &= \left(\omega_0 - \frac{2\pi \hat{k}_0}{N}\right)\left(\frac{M-1}{2}\right) + m\pi \\ &= (\omega_0 - \widehat{\omega}_0)\left(\frac{M-1}{2}\right) + m\pi\end{aligned}$$

The magnitude of the error $|\omega_0 - \widehat{\omega}_0|$ is bounded by π/N, so the first term lies within the range $[-\pi, \pi]$ even for the case $M = N$. In addition, $\widehat{\omega}_0$ lies within the main lobe of $X_w(e^{j\omega})$ bounded by $\omega_0 - \frac{2\pi}{M}$ and $\omega_0 + \frac{2\pi}{M}$, so the amplitude at $\omega = \widehat{\omega}_0$ is positive. We can therefore set $m = 0$ in the phase equation.

Solving the phase equation for ω_0 with $m = 0$,

$$\omega_0 = \widehat{\omega}_0 + \frac{2\measuredangle X_w[\hat{k}_0]}{M-1}$$

and $\Omega_0 = \frac{\omega_0}{T}$. We can obtain two estimates of Ω_0 for the two window choices $w_1[n]$ ($M = 32$) and $w_2[n]$ ($M = 8$), using the values of $\widehat{\omega}_0$ and $\hat{k}_0$ from part (c) in both cases, and check that they are consistent.

Home Assignment №6

Due on July 9, 2008

Exercise 1

A causal LTI system has impulse response $h[n]$, for which the z-transform is

$$H(z) = \frac{1 + z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{4}z^{-1})}$$

a) What is the region of convergence of $H(z)$?

b) Is the system stable? Explain.

c) Find the z-transform $X(z)$ of an input $x[n]$ that will produce the output

$$y[n] = -\frac{1}{3}\left(\frac{-1}{4}\right)^n u[n] - \frac{4}{3}\,2^n\,u[-n-1].$$

Exercise 2

If the input to an LTI system is $x[n] = u[n]$, the output is

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1].$$

a) Find $H(z)$, the z-transform of the system input response.

b) Find the input response $h[n]$.

c) Is the system causal? Is it stable?

Exercise 3

In Figure 1, $H(z)$ is the system function of a causal LTI system.

a) Using z-transform of the signals show in the figure, obtain an expression for $W(z)$ in the form

$$W(z) = H_1(z)X(z) + H_2(z)E(z),$$

where both $H_1(z)$ and $H_2(z)$ are expressed in terms of $H(z)$.

b) For special case $H(z) = \frac{z^{-1}}{1-z^{-1}}$, determine $H_1(z)$ and $H_2(z)$.

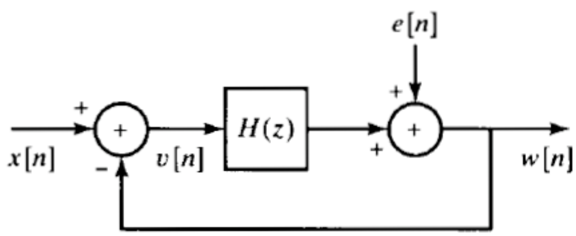


Figure 1: Pertaining to Exercise 3.

Solutions to assignment 6

Exercise 1

A causal LTI system has impulse response $h[n]$, for which the z-transform is

$$H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$$

a) What is the region of convergence of $H(z)$?

b) Is the system stable? Explain.

c) Find the z-transform $X(z)$ of an input $x[n]$ that will produce the output

$$y[n] = -\frac{1}{3}\left(\frac{-1}{4}\right)^n u[n] - \frac{4}{3}2^n u[-n-1].$$

Solution:

$$H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{2}{(1-\frac{1}{2}z^{-1})} - \frac{1}{(1+\frac{1}{4}z^{-1})}$$

(a) $h[n]$ causal $\Rightarrow$ ROC outside $|z| = \frac{1}{2} \Rightarrow$ $|z| > \frac{1}{2}$.

(b) ROC includes $|z| = 1$ $\Rightarrow$ stable.

(c)

$$\begin{aligned}
y[n] &= -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1] \\
Y(z) &= \frac{-\frac{1}{3}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} \\
&= \frac{1+z^{-1}}{(1+\frac{1}{4}z^{-1})(1-2z^{-1})} \qquad \frac{1}{4} < |z| < 2 \\
X(z) &= \frac{Y(z)}{H(z)} = \frac{(1-\frac{1}{2}z^{-1})}{(1-2z^{-1})} \qquad |z| < 2 \\
x[n] &= -(2)^n u[-n-1] + \frac{1}{2}(2)^{n-1}u[-n]
\end{aligned}$$

Exercise 2

If the input to an LTI system is $x[n] = u[n]$, the output is

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1].$$

a) Find $H(z)$, the z-transform of the system input response.

b) Find the input response $h[n]$.

c) Is the system causal? Is it stable?

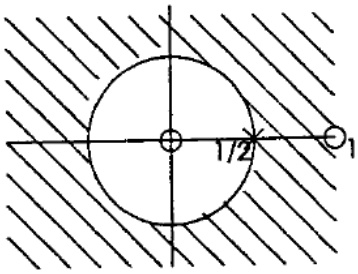
Solution:

$$x[n] = u[n] \Leftrightarrow X(z) = \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1] = 4\left(\frac{1}{2}\right)^{n+1} u[n+1] \Leftrightarrow Y(z) = \frac{4z}{1 - \frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

(a)

$$H(z) = \frac{Y(z)}{X(Z)} = \frac{4z(1 - z^{-1})}{1 - \frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$



(b)

$$\begin{aligned} H(z) &= \frac{4z}{1 - \frac{1}{2}z^{-1}} - \frac{4}{1 - \frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2} \\ h[n] &= 4\left(\frac{1}{2}\right)^{n+1} u[n+1] - 4\left(\frac{1}{2}\right)^{n} u[n] \\ &= 4\delta[n+1] - 2\left(\frac{1}{2}\right)^{n} u[n] \end{aligned}$$

(c) The ROC of $H(z)$ includes $|z| = 1 \Rightarrow$ stable.

(d) From part (b) we see that $h[n]$ starts at $n = -1 \Rightarrow$ not causal

Exercise 3

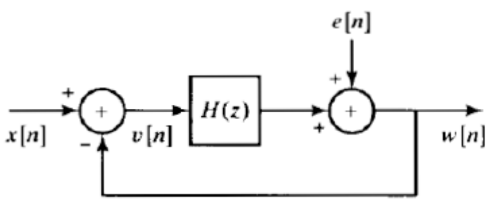
In Figure 1, $H(z)$ is the system function of a causal LTI system.

a) Using z-transform of the signals show in the figure, obtain an expression for $W(z)$ in the form

$$W(z) = H_1(z)X(z) + H_2(z)E(z),$$

where both $H_1(z)$ and $H_2(z)$ are expressed in terms of $H(z)$.

b) For special case $H(z) = \frac{z^{-1}}{1-z^{-1}}$, determine $H_1(z)$ and $H_2(z)$.



Solution:

(a) After writing the following equalities:

$$\begin{aligned} V(z) &= X(z) - W(z) \\ W(z) &= V(z)H(z) + E(z) \end{aligned}$$

we solve for W(z):

$$W(z) = \frac{H(z)}{1+H(z)}X(z) + \frac{1}{1+H(z)}E(z)$$

(b)

$$\begin{aligned} H_1(z) &= \frac{H(z)}{1+H(z)} = \frac{\frac{z^{-1}}{1-z^{-1}}}{1+\frac{z^{-1}}{1-z^{-1}}} = z^{-1} \\ H_2(z) &= \frac{1}{1+\frac{z^{-1}}{1-z^{-1}}} = 1 - z^{-1} \end{aligned}$$

(c) $H(z)$ is not stable due to its pole at $z = 1$, but $H_1(z)$ and $H_2(z)$ are.

Home Assignment №8

Due on July 17, 2008

Exercise 1

Design a symmetric FIR filter with group delay $N/2$ according to the ideal magnitude response

$$|H_d^f(\theta)| = \begin{cases} 1, & 0 \leq |\theta| \leq \frac{\pi}{3}, \\ 0, & \frac{\pi}{3} < |\theta| < \frac{2\pi}{3}, \\ 0.5, & \frac{2\pi}{3} \leq |\theta| \leq \pi. \end{cases}$$

A. Compute $h_d[n]$.

B. If the filter is designed with the Hamming window and its order is $N = 40$, what are the values of

$$\theta_{p,1},\ \theta_{s,1},\ \theta_{s,2},\ \theta_{p,2},\ \delta_{p,1},\ \delta_s,\ \delta_{p,2}?$$

C. Suppose we want to have $\delta_{p,1} = \delta_{p,2} = 0.01$, and $\delta = 0.005$. Is it possible to achieve this with a Hamming window filter of order $N = 39$?

Exercise 2

Fig.1 shows the ideal magnitude and phase responses of a filter that is a differentiator at low frequencies and high pass at high frequencies.

A. Compute the desired impulse response $h_d[n]$. Does the truncated impulse response have linear phase? Explain why or why not.

B. Design an FIR filter of order $N = 128$ and having $\theta_0 = 0.5\pi$, using: (1) a rectangular window; (2) a Hamming window; (3) a Kaiser window with $\alpha = 6$; (4) a Kaiser window with $\alpha = 12$. Compute and plot the magnitude and phase responses of the four filters.

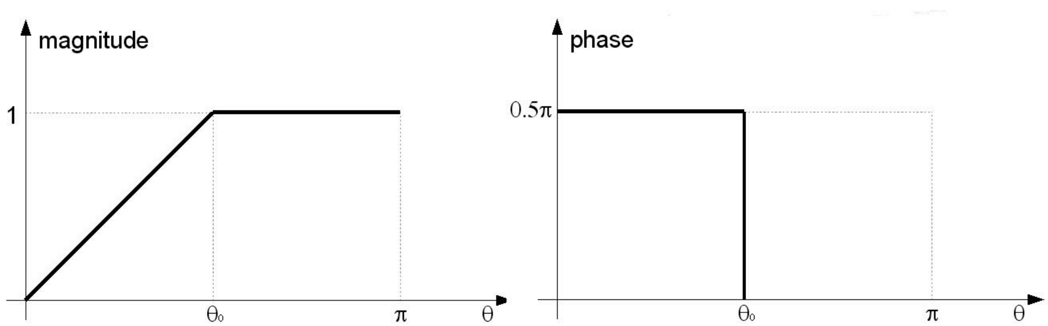


Figure 1: Pertaining to Exercise 2.

Exercise 1

(A) From the general multiband filter formula we get

$$h_d[n] = \frac{\sin[(n-0.5N0(\pi/3)]}{\pi(n-0.5N)} - 0.5\frac{\sin[\{n-0.5N)(2\pi/3)]}{\pi(n-0.5N)}, \quad n \neq 0.5N$$

and

$$h_d[0.5N] = 0.5$$

(B) Using the Hamming window parameters, we get

$$\theta_{p,1} = \frac{\pi}{3} - \frac{4\pi}{41}, \quad \theta_{s,1} = \frac{\pi}{3} + \frac{4\pi}{41},$$
$$\theta_{s,2} = \frac{2\pi}{3} - \frac{4\pi}{41}, \quad \theta_{p,2} = \frac{2\pi}{3} + \frac{4\pi}{41},$$
$$\delta_{p,1} = \delta_s = 0.0022, \quad \delta_{p,2} = 0.0011.$$

(C) No, since a type II filter cannot meet the pass-band specifications near $\theta = \pi$.

Exercise 2

(A) The desired frequency response is

$$H_d^f(\theta) = \begin{cases} j\dfrac{\theta}{\theta_0}, & |\theta| \leq \theta_0 \\ 1, & |\theta| > \theta_0 \end{cases}$$

Therefore, the desired impulse response is

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{-\theta_0} e^{j\theta n}\,d\theta + \frac{j}{2\pi\theta_0}\int_{-\theta_0}^{\theta_0} \theta e^{j\theta n}\,d\theta + \frac{1}{2\pi}\int_{\theta_0}^{\pi} e^{j\theta n}\,d\theta \\ &= \frac{1}{\pi(n-0.5N)}\left[Sinc\left(\frac{\theta_0(n-0.5N)}{\pi}\right) - \cos(\theta_0(n-0.5N))\right] \\ &\quad + \delta[n-0.5N] - \frac{\theta_1}{\pi} Sinc\left[\frac{\theta_0(n-0.5N)}{\pi}\right] \end{aligned}$$

The impulse response is neither symmetric nor antisymmetric, therefore the IRT filter does not have linear phase. This is also evident from $H_d^f(\theta)$, since this frequency response is neither purely real nor purely imaginary.

(B) The following MATLAB code performs the required computations and plots the results. The line computing the window w should be edited as needed. Note that the phase is advanced by $\theta(n\text{-}0.5N)$ before plotting, to compensate for the filter's group delay.

```
w = window(129, 'rect'); % change as needed to 'hamm'; 'kais',6;
'kais',12.
theta0 = 0.5*pi
n = -64:64;
h = (sinc((theta0/pi)*n)-cos(theta0*n)) ./ (pi*n) - ...
    theta0/pi)*sinc((theta0/pi)*n);
h(65) = 1 - (theta0/pi);
h = h.*w';
H = frqresp(h,1,501);(in the textbook P235)
theta = (1/500)*(0:500);
plot(theta,abs(H)),grid,figure(1),pause
plot(theta,(180/pi)*angle(H.*exp(j*64*pi*theta))),grid,figure(1)
```

Home Assignment №8

Due on July 30, 2008

Exercise 1

A Chebyshev-I filter of order $N = 3$ and $\omega_0 = 1$ is known to have a pole at $s = -1$ rad/s.

a) Find the other two poles of the filter and its parameter ϵ.

b) The filter is transformed to the z domain using a bilinear transform with $T = 2$. Compute the transfer function of the digital filter $H^Z(z)$.

Exercise 2

A first-order analog filter $H^L(s)$ has a zero at $s = -2$, a pole at $s = -2/3$, and its DC gain is $H^L(0) = 1$. Bilinear transformation of $H^L(s)$ yields the digital filter $H^Z(z) = K/(1 - \alpha z^{-1})$. Find K, α, and the sampling interval T.

Solutions to Assignment #8

Exercise 1

10.20 A Chebyshev-I filter of order $N = 3$ and $\omega_0 = 1$ is known to have a pole at $s = -1$ rad/s.

(a) Find the other two poles of the filter and its parameter ε.

(b) The filter is transformed to the z domain using a bilinear transform with $T = 2$. Compute the transfer function of the digital filter $H^z(z)$.

Solution :

(a) We have from the given information

$$\sinh\left(\frac{1}{3}\operatorname{arcsinh}\frac{1}{\varepsilon}\right) = 1,$$

which gives $\varepsilon = 1/7$. Also,

$$\cosh\left(\frac{1}{3}\operatorname{arcsinh}\frac{1}{\varepsilon}\right) = \sqrt{2}.$$

Therefore, the other two poles are

$$s_{0,2} = -\sin 30^\circ \pm j\sqrt{2}\cos 30^\circ = 0.5(-1 \pm j\sqrt{6}).$$

(b)

$$H^L(s) = \frac{1.75}{(s+1)(s^2+s+1.75)}.$$

$$H^z(z) = \frac{0.2333(1+z^{-1})^3}{1+0.4z^{-1}+0.4666z^{-2}}.$$

Exercise 2

10.21 A first-order analog filter $H^L(s)$ has a zero at $s = -2$, a pole at $s = -2/3$, and its DC gain is $H^L(0) = 1$. Bilinear transformation of $H^L(s)$ yields the digital filter $H^z(z) = K/(1 - \alpha z^{-1})$. Find K, α, and the sampling interval T.

Solution :

The analog filter is

$$H^L(s) = \frac{s+2}{3s+2}.$$

Therefore, the digital filter is

$$H^z(z) = \frac{(T+1)z + (T-1)}{(T+3)z + (T-3)}.$$

This gives,

$$T = 1, \quad K = 0.5, \quad \alpha = 0.5.$$