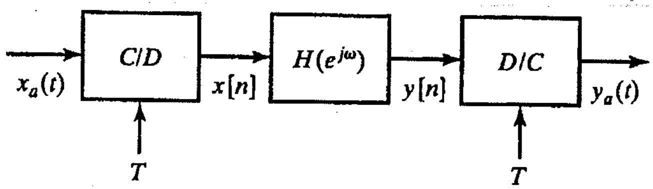
FILTER DESIGN TECHNIQUES

Filters are a particularly important class of linear time-invariant systems. Strictly speaking, the term *frequency-selective filter* suggests a system that passes certain frequency components and totally rejects all others, but in a broader context any system that modifies certain frequencies relative to others is also called a filter.



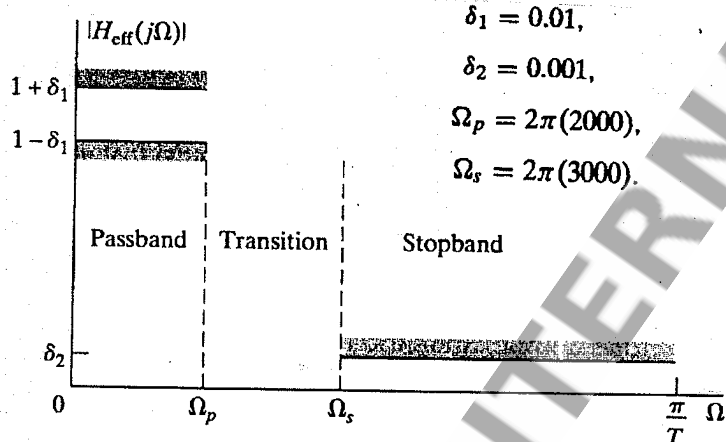
if a linear time-invariant discrete-time system is used as in Figure 7.1, and if the input is bandlimited and the sampling frequency is high enough to avoid aliasing, then the overall system behaves as a linear time-invariant continuous-time system with frequency response

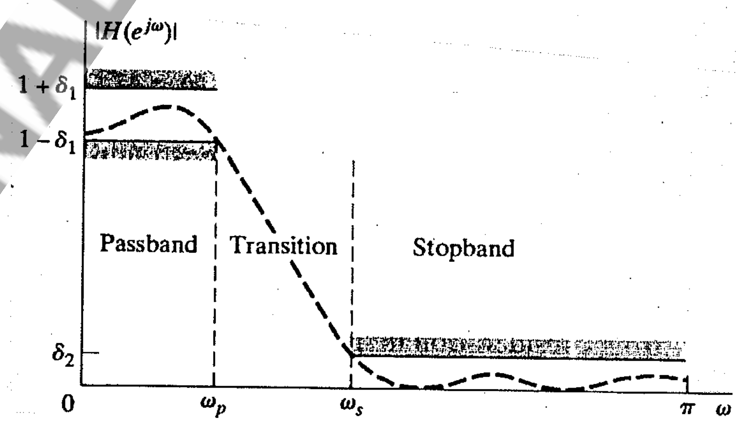
$$H_{\mathrm{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T, \\ 0, & |\Omega| > \pi/T. \end{cases} \tag{7.1a}$$

In such cases, it is straightforward to convert from specifications on the effective continuous-time filter to specifications on the discrete-time filter through the relation $\omega = \Omega T$. That is, $H(e^{j\omega})$ is specified over one period by the equation

$$H(e^{j\omega}) = H_{\mathrm{eff}}\left(j\frac{\omega}{T}\right), \qquad |\omega| < \pi. \tag{7.1b}$$

$$\delta_1 = 0.01,$$
$$\delta_2 = 0.001,$$
$$\Omega_p = 2\pi(2000),$$
$$\Omega_s = 2\pi(3000).$$





ideal passband gain in decibels $= 20\log_{10}(1) = 0$ dB
maximum passband gain in decibels $= 20\log_{10}(1.01) = 0.086$ dB
maximum stopband gain in decibels $= 20\log_{10}(0.001) = -60$ dB

Filter Design by Impulse Invariance

In the impulse invariance design procedure for transforming continuous-time filters into discrete-time filters, the impulse response of the discrete-time filter is chosen proportional to equally spaced samples of the impulse response of the continuous-time filter; i.e.,

$$h[n] = T_d h_c(nT_d), \tag{7.4}$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right). \tag{7.5}$$

If the continuous-time filter is bandlimited, so that

$$H_c(j\Omega) = 0, \qquad |\Omega| \geq \pi/T_d, \tag{7.6}$$

then

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right), \qquad |\omega| \leq \pi; \tag{7.7}$$

However, if the continuous-time filter approaches zero at high frequencies, the aliasing may be negligibly small, and a useful discrete-time filter can result from the sampling of the impulse response of a continuous-time filter.

In the impulse invariance design procedure, the discrete-time filter specifications are first transformed to continuous-time filter specifications through the use of Eq. (7.7). Assuming that the aliasing involved in the transformation from $H_c(j\Omega)$ to $H(e^{j\omega})$ will be negligible, we obtain the specifications on $H_c(j\Omega)$ by applying the relation

$$\Omega = \omega/T_d \tag{7.8}$$

While the impulse invariance transformation from continuous time to discrete time is defined in terms of time-domain sampling, it is easy to carry out as a transformation on the system function.

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}. \tag{7.9}$$

The corresponding impulse response is

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{7.10}$$

The impulse response of the discrete-time filter obtained by sampling $T_d h_c(t)$ is

$$\begin{aligned} h[n] = T_d h_c(nT_d) &= \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n] \\ &= \sum_{k=1}^{N} T_d A_k (e^{s_k T_d})^n u[n]. \end{aligned} \tag{7.11}$$

The system function of the discrete-time filter is therefore given by

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}. \tag{7.12}$$

If the continuous-time filter is stable, corresponding to the real part of s_k being less than zero, then the magnitude of $e^{s_k T_d}$ will be less than unity, so that the corresponding pole in the discrete-time filter is inside the unit circle.

Impulse Invariance with a Butterworth Filter

Let us consider the design of a lowpass discrete-time filter by applying impulse invariance to an appropriate Butterworth continuous-time filter.[2] The specifications for the discrete-time filter are

$$0.89125 \leq |H(e^{j\omega})| \leq 1, \qquad 0 \leq |\omega| \leq 0.2\pi, \tag{7.13a}$$

$$|H(e^{j\omega})| \leq 0.17783, \qquad 0.3\pi \leq |\omega| \leq \pi. \tag{7.13b}$$

Since the parameter T_d cancels in the impulse invariance procedure, we can choose

Specifically, the magnitude-squared function of a Butterworth filter is of the form

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}, \tag{7.16}$$

so that the filter design process consists of determining the parameters N and Ω_c to meet the desired specifications.

Because of the preceding considerations, we want to design a continuous-time Butterworth filter with magnitude function $|H_c(j\Omega)|$ for which

$$0.89125 \le |H_c(j\Omega)| \le 1, \qquad 0 \le |\Omega| \le 0.2\pi, \tag{7.14a}$$

$$|H_c(j\Omega)| \le 0.17783, \qquad 0.3\pi \le |\Omega| \le \pi. \tag{7.14b}$$

Since the magnitude response of an analog Butterworth filter is a monotonic function of frequency, Eqs. (7.14a) and (7.14b) will be satisfied if

$$|H_c(j0.2\pi)| \ge 0.89125 \tag{7.15a}$$

and

$$|H_c(j0.3\pi)| \le 0.17783. \tag{7.15b}$$

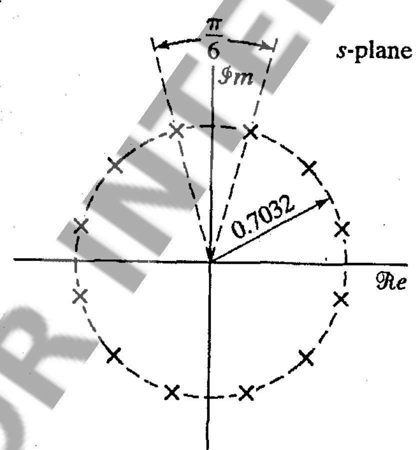
$$1+\left(\frac{0.2\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \tag{7.17a}$$

and

$$1+\left(\frac{0.3\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2. \tag{7.17b}$$

The solution of these two equations is $N = 5.8858$ and $\Omega_c = 0.70474$. The parameter N, however, must be an integer. Therefore, so that the specifications are met or exceeded, we must round N up to the nearest integer, $N = 6$.

With $\Omega_c = 0.7032$ and with $N = 6$, the 12 poles of the magnitude-squared function $H_c(s)H_c(-s) = 1/[1 + (s/j\Omega_c)^{2N}]$ are uniformly distributed in angle on a circle of radius $\Omega_c = 0.7032$.



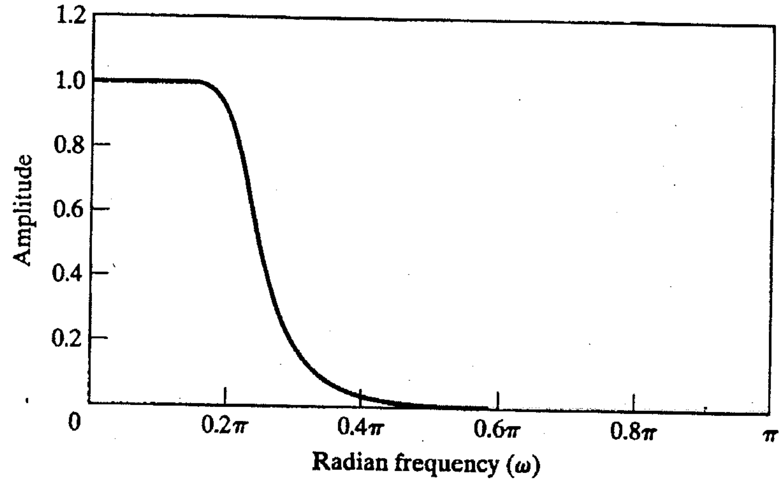
Pole pair 1: $-0.182 \pm j(0.679)$,

Pole pair 2: $-0.497 \pm j(0.497)$,

Pole pair 3: $-0.679 \pm j(0.182)$.

Therefore,

$$H_c(s) = \frac{0.12093}{(s^2+0.3640s+0.4945)(s^2+0.9945s+0.4945)(s^2+1.3585s+0.4945)}$$



$$H(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.1455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8557 - 0.6303z^{-1}}{1 - 0.9972z^{-1} + 0.2570z^{-2}}.$$

Bilinear Transformation

The technique discussed in this subsection avoids the problem of aliasing by using the bilinear transformation, an algebraic transformation between the variables s and z that maps the entire $j\Omega$-axis in the s-plane to one revolution of the unit circle in the z-plane.

With $H_c(s)$ denoting the continuous-time system function and $H(z)$ the discrete-time system function, the bilinear transformation corresponds to replacing s by

$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right); \tag{7.20}$$

that is,

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right]. \tag{7.21}$$

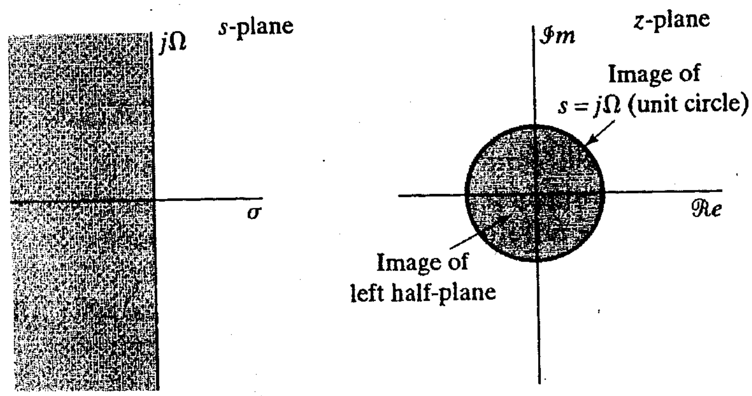
To develop the properties of the algebraic transformation specified in Eq. (7.20), we solve for z to obtain

$$z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s}, \tag{7.22}$$

and, substituting $s = \sigma + j\Omega$ into Eq. (7.22), we obtain

$$z = \frac{1 + \sigma T_d/2 + j\Omega T_d/2}{1 - \sigma T_d/2 - j\Omega T_d/2}. \tag{7.23}$$

If $\sigma < 0$, then, from Eq. (7.23), it follows that $|z| < 1$ for any value of Ω. Similarly, if $\sigma > 0$, then $|z| > 1$ for all Ω. That is, if a pole of $H_c(s)$ is in the left-half s-plane, its image in the z-plane will be inside the unit circle. Therefore, causal stable continuous-time filters map into causal stable discrete-time filters.



Next, to show that the $j\Omega$-axis of the s-plane maps onto the unit circle, we substitute $s = j\Omega$ into Eq. (7.22), obtaining

$$z = \frac{1 + j\Omega T_d/2}{1 - j\Omega T_d/2}. \tag{7.24}$$

From Eq. (7.24), it is clear that $|z| = 1$ for all values of s on the $j\Omega$-axis. That is, the $j\Omega$-axis maps onto the unit circle, so Eq. (7.24) takes the form

$$e^{j\omega} = \frac{1 + j\Omega T_d/2}{1 - j\Omega T_d/2}. \tag{7.25}$$

$$s = \frac{2}{T_d}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right), \tag{7.26}$$

or, equivalently,

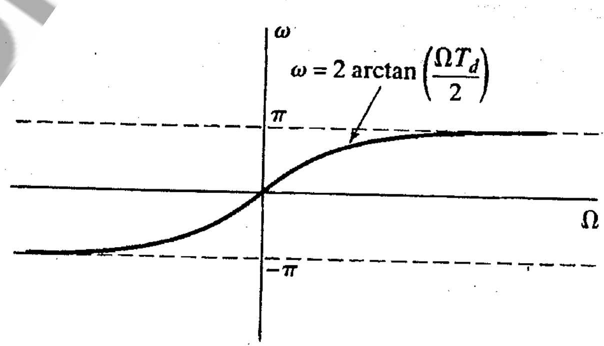
$$s = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}(j \sin\omega/2)}{2e^{-j\omega/2}(\cos\omega/2)}\right] = \frac{2j}{T_d}\tan(\omega/2). \tag{7.27}$$

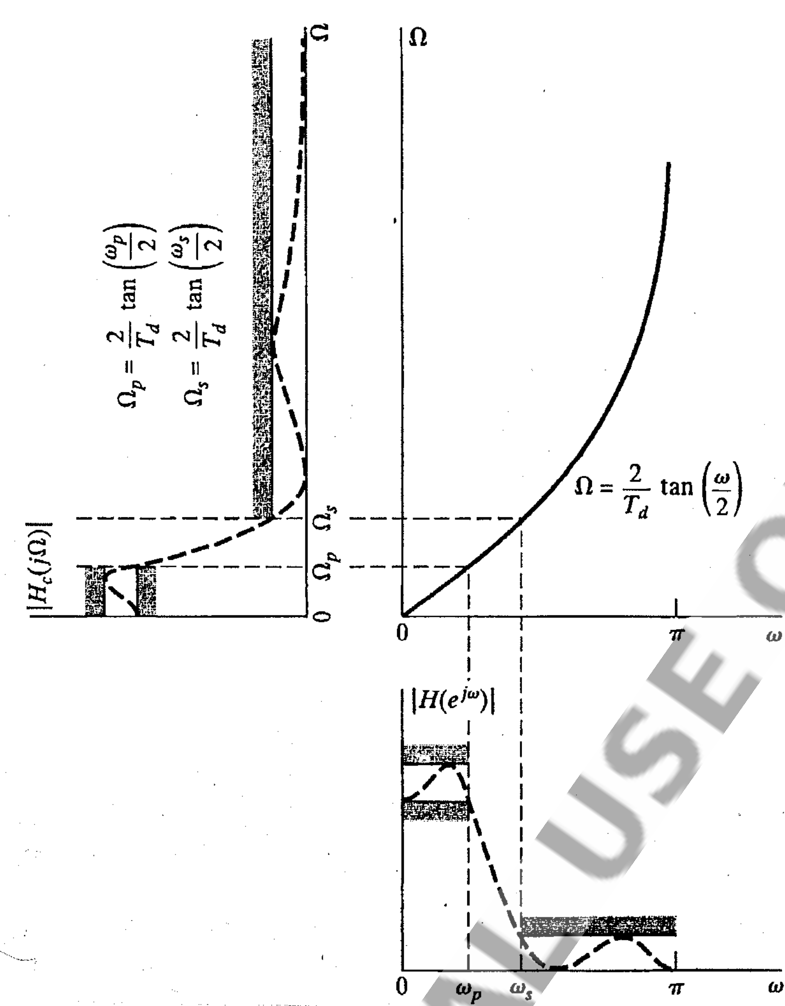
Equating real and imaginary parts on both sides of Eq. (7.27) leads to the relations $\sigma = 0$ and

$$\Omega = \frac{2}{T_d}\tan(\omega/2), \tag{7.28}$$

or

$$\omega = 2\arctan(\Omega T_d/2). \tag{7.29}$$





Bilinear Transformation of a Butterworth Filter

Consider the discrete-time filter specifications of Example 7.2, in which we illustrated the impulse invariance technique for the design of a discrete-time filter. The specifications on the discrete-time filter are

$$0.89125 \le |H(e^{j\omega})| \le 1, \qquad 0 \le \omega \le 0.2\pi, \tag{7.30a}$$

$$|H(e^{j\omega})| \le 0.17783, \qquad 0.3\pi \le \omega \le \pi. \tag{7.30b}$$

$$0.89125 \le |H_c(j\Omega)| \le 1, \qquad 0 \le \Omega \le \frac{2}{T_d}\tan\left(\frac{0.2\pi}{2}\right), \tag{7.31a}$$

$$|H_c(j\Omega)| \le 0.17783, \qquad \frac{2}{T_d}\tan\left(\frac{0.3\pi}{2}\right) \le \Omega \le \infty. \tag{7.31b}$$

For convenience, we choose $T_d = 1$.

and
$$|H_c(j2\tan(0.1\pi))| \ge 0.89125$$

$$|H_c(j2\tan(0.15\pi))| \le 0.17783. \tag{7.32b}$$

The form of the magnitude-squared function for the Butterworth filter is

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}. \tag{7.33}$$

Solving for N and Ω_c with the equality sign in Eqs. (7.32a) and (7.32b), we obtain

$$1+\left(\frac{2\tan(0.1\pi)}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.89}\right)^2 \tag{7.34a}$$

and

$$1+\left(\frac{2\tan(0.15\pi)}{\Omega_c}\right)^{2N}=\left(\frac{1}{0.178}\right)^2, \tag{7.34b}$$

and solving for N in Eqs. (7.34a) and (7.34b) gives

$$\begin{aligned} N &= \frac{\log\left[\left(\left(\frac{1}{0.178}\right)^2-1\right)\Big/\left(\left(\frac{1}{0.89}\right)^2-1\right)\right]}{2\log[\tan(0.15\pi)/\tan(0.1\pi)]} \\ &= 5.305. \end{aligned} \tag{7.35}$$

Since N must be an integer, we choose $N = 6$. Substituting $N = 6$ into Eq. (7.34b), we obtain $\Omega_c = 0.766$. For this value of Ω_c, the passband specifications are exceeded and the stopband specifications are met exactly. This is reasonable for the bilinear transformation, since we do not have to be concerned with aliasing. That is, with proper prewarping, we can be certain that the resulting discrete-time filter will meet the specifications exactly at the desired stopband edge.

$$H_c(s)=\frac{0.20238}{(s^2+0.3996s+0.5871)(s^2+1.0836s+0.5871)(s^2+1.4802s+0.5871)}. \tag{7.36}$$

The system function for the discrete-time filter is then obtained by applying the bilinear transformation to $H_c(s)$ with $T_d = 1$. The result is

$$\begin{aligned} H(z) &= \frac{0.0007378(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})} \\ &\quad\times\frac{1}{(1-0.9044z^{-1}+0.2155z^{-2})}. \end{aligned} \tag{7.37}$$