DISCRETE-TIME SIGNALS AND SYSTEMS

2.0 INTRODUCTION

The term *signal* is generally applied to something that conveys information. Signals generally convey information about the state or behavior of a physical system, and often, signals are synthesized for the purpose of communicating information between humans or between humans and machines.
either continuous or discrete. *Continuous-time signals* are defined along a continuum of times and thus are represented by a continuous independent variable. Continuous-time signals are often referred to as *analog signals*. *Discrete-time signals* are defined at discrete times, and thus, the independent variable has discrete values; i.e., discrete-time signals are represented as sequences of numbers. Signals such as speech or images may have either a continuous- or a discrete-variable representation, and if certain conditions hold, these representations are entirely equivalent. Besides the independent variables being either continuous or discrete, the signal amplitude may be either continuous or discrete. *Digital signals* are those for which both time and amplitude are discrete.

Discrete-time signals may arise by sampling a continuous-time signal, or they may be generated directly by some discrete-time process. Whatever the origin of the discrete-time signals, discrete-time signal-processing systems have many attractive features. They can be realized with great flexibility with a variety of technologies, such as charge transport devices, surface acoustic wave devices, general-purpose digital computers, or high-speed microprocessors. Complete signal-processing systems can be implemented using VLSI techniques. Discrete-time systems can be used to simulate analog systems or, more importantly, to realize signal transformations that cannot be implemented with continuous-time hardware. Thus, discrete-time representations of signals are often desirable when sophisticated and flexible signal processing is required.

2.1 DISCRETE-TIME SIGNALS: SEQUENCES

Discrete-time signals are represented mathematically as sequences of numbers. A sequence of numbers x, in which the nth number in the sequence is denoted $x[n]$,[1] is formally written as

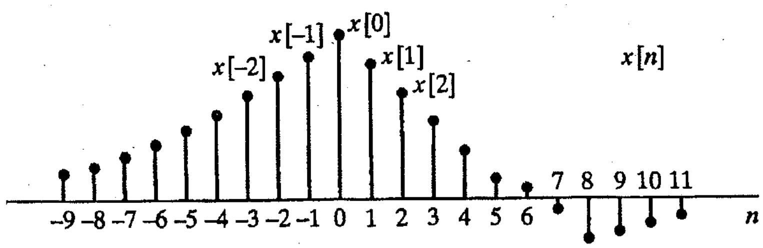
$$x = \{x[n]\}, \qquad -\infty < n < \infty, \tag{2.1}$$

where n is an integer. In a practical setting, such sequences can often arise from periodic

sampling of an analog signal. In this case, the numeric value of the nth number in the sequence is equal to the value of the analog signal, $x_a(t)$, at time nT; i.e.,

$$x[n] = x_a(nT), \qquad -\infty < n < \infty. \tag{2.2}$$

The quantity T is called the *sampling period*, and its reciprocal is the *sampling frequency*.



2.1.1 Basic Sequences and Sequence Operations

In the analysis of discrete-time signal-processing systems, sequences are manipulated in several basic ways. The product and sum of two sequences $x[n]$ and $y[n]$ are defined as the sample-by-sample product and sum, respectively. Multiplication of a sequence $x[n]$ by a number α is defined as multiplication of each sample value by α. A sequence $y[n]$ is said to be a delayed or shifted version of a sequence $x[n]$ if

$$y[n] = x[n - n_0], \tag{2.3}$$

where n_0 is an integer.

The *unit sample sequence* (Figure 2.3a) is defined as the sequence

$$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases} \tag{2.4}$$

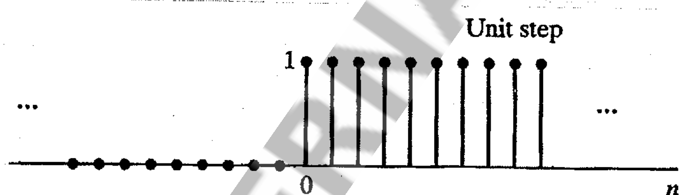
As we will see, the unit sample sequence plays the same role for discrete-time signals and systems that the unit impulse function (Dirac delta function) does for continuous-time signals and systems.

More generally, any sequence can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{2.6}$$

The *unit step sequence* (Figure 2.3b) is given by

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{2.7}$$



The unit step is related to the impulse by

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]; \tag{2.8}$$

or

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{2.9b}$$

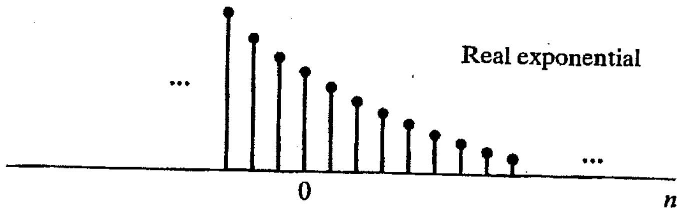
Conversely, the impulse sequence can be expressed as the first backward difference of the unit step sequence, i.e.,

$$\delta[n] = u[n] - u[n-1]. \tag{2.10}$$

Exponential sequences are extremely important in representing and analyzing linear time-invariant discrete-time systems. The general form of an exponential sequence is

$$x[n] = A\alpha^n. \tag{2.11}$$

If A and α are real numbers, then the sequence is real. If $0 < \alpha < 1$ and A is positive, then the sequence values are positive and decrease with increasing n, as in Figure 2.3(c).

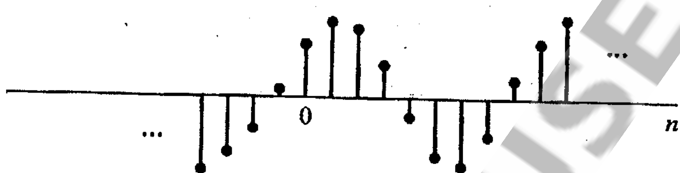


For $-1 < \alpha < 0$, the sequence values alternate in sign, but again decrease in magnitude with increasing n. If $|\alpha| > 1$, then the sequence grows in magnitude as n increases.

Sinusoidal sequences are also very important. A sinusoidal sequence has the general form

$$x[n] = A\cos(\omega_0 n + \phi), \qquad \text{for all } n, \tag{2.13}$$

with A and ϕ real constants, and is illustrated in Figure 2.3(d).



Specifically, if $\alpha = |\alpha|e^{j\omega_0}$ and $A = |A|e^{j\phi}$, the sequence $A\alpha^n$ can be expressed in any of the following ways:

$$\begin{aligned} x[n] = A\alpha^n &= |A|e^{j\phi}|\alpha|^n e^{j\omega_0 n} \\ &= |A|\,|\alpha|^n e^{j(\omega_0 n+\phi)} \\ &= |A|\,|\alpha|^n \cos(\omega_0 n + \phi) + j|A|\,|\alpha|^n \sin(\omega_0 n + \phi). \end{aligned} \tag{2.14}$$

The sequence oscillates with an exponentially growing envelope if $|\alpha| > 1$ or with an exponentially decaying envelope if $|\alpha| < 1$. (As a simple example, consider the case $\omega_0 = \pi$.)

When $|\alpha| = 1$, the sequence is referred to as a *complex exponential sequence* and has the form

$$x[n] = |A|e^{j(\omega_0 n+\phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi); \tag{2.15}$$

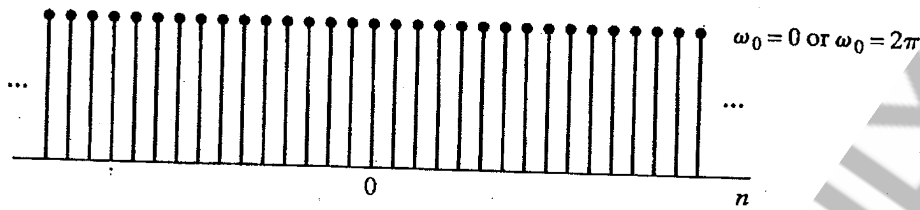
that is, the real and imaginary parts of $e^{j\omega_0 n}$ vary sinusoidally with n.

An important difference between continuous-time and discrete-time complex sinusoids is seen when we consider a frequency $(\omega_0 + 2\pi)$. In this case,

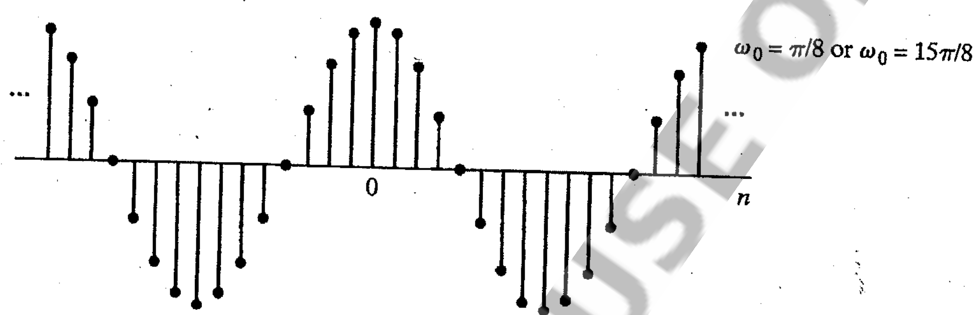
$$\begin{aligned} x[n] &= Ae^{j(\omega_0+2\pi)n} \\ &= Ae^{j\omega_0 n}e^{j2\pi n} = Ae^{j\omega_0 n}. \end{aligned} \tag{2.16}$$

More generally, we can easily see that complex exponential sequences with frequencies $(\omega_0 + 2\pi r)$, where r is an integer, are indistinguishable from one another.

For now, we simply conclude that, when discussing complex exponential signals of the form $x[n] = Ae^{j\omega_0 n}$ or real sinusoidal signals of the form $x[n] = A\cos(\omega_0 n + \phi)$, we need only consider frequencies in an interval of length 2π, such as $-\pi < \omega_0 \le \pi$ or $0 \le \omega_0 < 2\pi$.


$\omega_0 = 0$ or $\omega_0 = 2\pi$
...
...
0
n

(a)


$\omega_0 = \pi/8$ or $\omega_0 = 15\pi/8$
...
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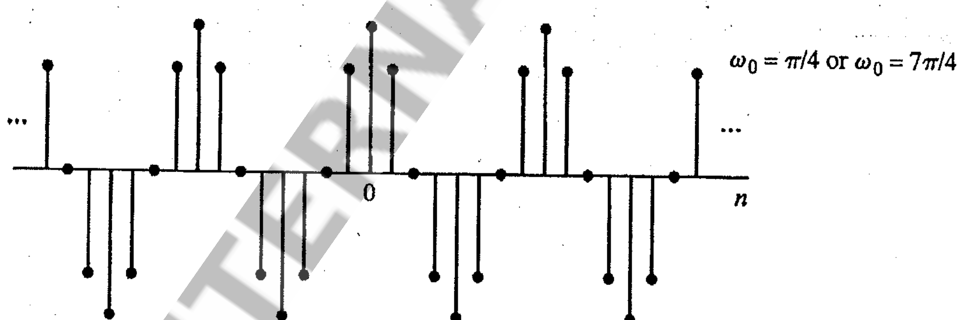
0
n

(b)


$\omega_0 = \pi/4$ or $\omega_0 = 7\pi/4$
...
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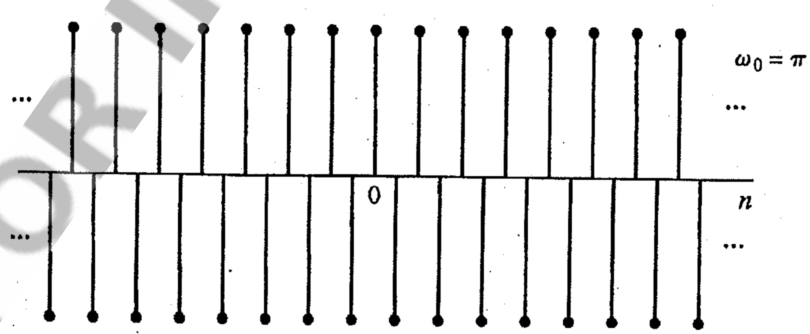
0
n

(c)


$\omega_0 = \pi$
...
...
...
...
0
n

(d)

In the discrete-time case, a periodic sequence is a sequence for which

$$x[n] = x[n+N], \qquad \text{for all } n, \tag{2.18}$$

where the period N is necessarily an integer. If this condition for periodicity is tested for the discrete-time sinusoid, then

$$A\cos(\omega_0 n + \phi) = A\cos(\omega_0 n + \omega_0 N + \phi), \tag{2.19}$$

which requires that

$$\omega_0 N = 2\pi k, \tag{2.20}$$

where k is an integer.

The integer restriction on n causes some sinusoidal signals not to be periodic at all. For example, there is no integer N such that the signal $x_3[n] = \cos(n)$ satisfies the condition $x_3[n+N] = x_3[n]$ for all n. These and other properties of discrete-time sinusoids that run counter to their continuous-time counterparts are caused by the limitation of the time index n to integers for discrete-time signals and systems.

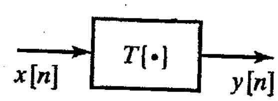
When we combine the condition of Eq. (2.20) with our previous observation that ω_0 and $(\omega_0 + 2\pi r)$ are indistinguishable frequencies, it becomes clear that there are N distinguishable frequencies for which the corresponding sequences are periodic with period N. One set of frequencies is $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. These properties of complex exponential and sinusoidal sequences are basic to both the theory and the design of computational algorithms for discrete-time Fourier analysis.

For the discrete-time sinusoidal signal $x[n] = A\cos(\omega_0 n + \phi)$, as ω_0 increases from $\omega_0 = 0$ toward $\omega_0 = \pi$, $x[n]$ oscillates more and more rapidly. However, as ω_0 increases from $\omega_0 = \pi$ to $\omega_0 = 2\pi$, the oscillations become slower. As a consequence, for sinusoidal and complex exponential signals, values of ω_0 in the vicinity of $\omega_0 = 2\pi k$ for any integer value of k are typically referred to as low frequencies (relatively slow oscillations), while values of ω_0 in the vicinity of $\omega_0 = (\pi + 2\pi k)$ for any integer value of k are typically referred to as high frequencies (relatively rapid oscillations).

2.2 DISCRETE-TIME SYSTEMS

A discrete-time system is defined mathematically as a transformation or operator that maps an input sequence with values $x[n]$ into an output sequence with values $y[n]$. This can be denoted as

$$y[n] = T\{x[n]\} \tag{2.22}$$



The ideal delay system is defined by the equation

$$y[n] = x[n - n_d], \qquad -\infty < n < \infty, \tag{2.23}$$

where n_d is a fixed positive integer called the delay of the system.

The general moving-average system is defined by the equation

$$\begin{aligned} y[n] &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k] \\ &= \frac{1}{M_1 + M_2 + 1} \{x[n+M_1] + x[n+M_1-1] + \cdots + x[n] \\ &\quad + x[n-1] + \cdots + x[n-M_2]\} \end{aligned} \tag{2.24}$$

2.2.1 Memoryless Systems

A system is referred to as memoryless if the output $y[n]$ at every value of n depends only on the input $x[n]$ at the same value of n.

$$y[n] = (x[n])^2, \qquad \text{for each value of } n.$$

2.2.2 Linear Systems

The class of *linear systems* is defined by the principle of superposition. If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs, then the system is linear if and only if

$$T\{x_1[n] + x_2[n]\} = T\{x_2[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n] \tag{2.26a}$$

and

$$T\{ax[n]\} = aT\{x[n]\} = ay[n], \tag{2.26b}$$

where a is an arbitrary constant.

This equation can be generalized to the superposition of many inputs. Specifically, if

$$x[n] = \sum_k a_k x_k[n], \tag{2.28a}$$

then the output of a linear system will be

$$y[n] = \sum_k a_k y_k[n], \tag{2.28b}$$

where $y_k[n]$ is the system response to the input $x_k[n]$.

The system defined by the input–output equation

$$y[n] = \sum_{k=-\infty}^{n} x[k] \tag{2.29}$$

is called the *accumulator* system, since the output at time n is just the sum of the present and all previous input samples. The accumulator system is a linear system.

Consider the system defined by

$$w[n] = \log_{10}(|x[n]|). \tag{2.36}$$

This system is not linear.

2.2.3 Time-Invariant Systems

Then the system is said to be time invariant if, for all n_0, the input sequence with values $x_1[n] = x[n - n_0]$ produces the output sequence with values $y_1[n] = y[n - n_0]$.

Consider the accumulator from Example 2.6. We define $x_1[n] = x[n - n_0]$.

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k] \tag{2.38}$$

$$= \sum_{k=-\infty}^{n} x[k - n_0]. \tag{2.39}$$

Substituting the change of variables $k_1 = k - n_0$ into the summation gives

$$y_1[n] = \sum_{k_1=-\infty}^{n-n_0} x[k_1] = y[n - n_0]. \tag{2.40}$$

Thus, the accumulator is a time-invariant system.

The system defined by the relation

$$y[n] = x[Mn], \qquad -\infty < n < \infty, \tag{2.41}$$

with M a positive integer, is called a *compressor.*

Comparing these two outputs, we see that $y[n-n_0]$ is not equal to $y_1[n]$ for all M and n_0, and therefore, the system is not time invariant.

2.2.4 Causality

A system is causal if, for every choice of n_0, the output sequence value at the index $n = n_0$ depends only on the input sequence values for $n \le n_0$. This implies that if $x_1[n] = x_2[n]$ for $n \le n_0$, then $y_1[n] = y_2[n]$ for $n \le n_0$. That is, the system is *nonanticipative.*

Consider the *forward difference system* defined by the relationship

$$y[n] = x[n+1] - x[n]. \tag{2.44}$$

This system is not causal, since the current value of the output depends on a future value of the input.

The *backward difference system,* defined as

$$y[n] = x[n] - x[n-1], \tag{2.45}$$

has an output that depends only on the present and past values of the input. Because there is no way for the output at a specific time $y[n_0]$ to incorporate values of the input for $n > n_0$, the system is causal.

2.2.5 Stability

A system is stable in the bounded-input, bounded-output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence. The input $x[n]$ is bounded if there exists a fixed positive finite value B_x such that

$$|x[n]| \le B_x < \infty, \qquad \text{for all } n. \tag{2.46}$$

The accumulator, as defined in Example 2.6 by Eq. (2.29), is also not stable. For example, consider the case when $x[n] = u[n]$, which is clearly bounded by $B_x = 1$. For this input, the output of the accumulator is

$$y[n] = \sum_{k=-\infty}^{n} u[k] \tag{2.48}$$

$$= \begin{cases} 0, & n < 0, \\ (n+1), & n \ge 0. \end{cases} \tag{2.49}$$

There is no finite choice for B_y such that $(n+1) \le B_y < \infty$ for all n; thus, the system is unstable.

2.3 LINEAR TIME-INVARIANT SYSTEMS

$$y[n] = T\left\{ \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right\}. \tag{2.50}$$

From the principle of superposition in Eq. (2.27), we can write

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h_k[n]. \tag{2.51}$$

According to Eq. (2.51), the system response to any input can be expressed in terms of the responses of the system to the sequences $\delta[n-k]$.

The property of time invariance implies that if $h[n]$ is the response to $\delta[n]$, then the response to $\delta[n-k]$ is $h[n-k]$. With this additional constraint, Eq. (2.51) becomes

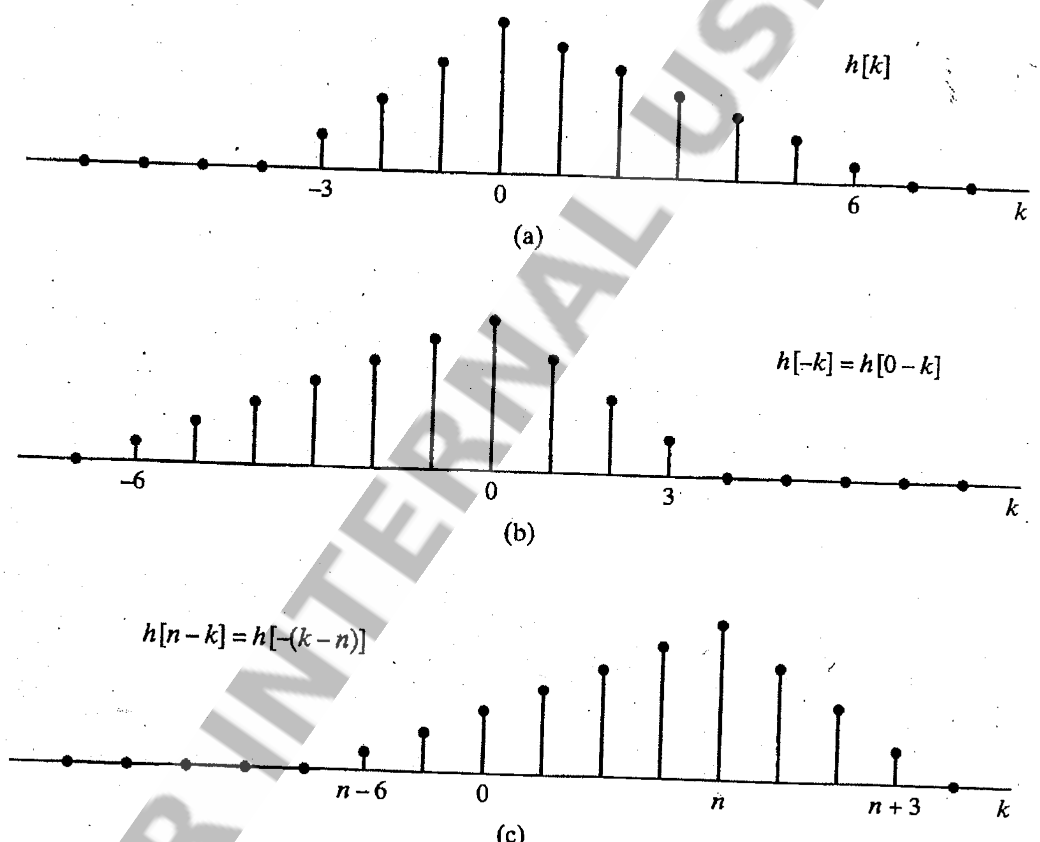
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{2.52}$$

As a consequence of Eq. (2.52), a linear time-invariant system (which we will sometimes abbreviate as LTI) is completely characterized by its impulse response $h[n]$ in the sense that, given $h[n]$, it is possible to use Eq. (2.52) to compute the output $y[n]$ due to *any* input $x[n]$.

Equation (2.52) is commonly called the *convolution sum*. If $y[n]$ is a sequence whose values are related to the values of two sequences $h[n]$ and $x[n]$ as in Eq. (2.52), we say that $y[n]$ is the convolution of $x[n]$ with $h[n]$ and represent this by the notation

$$y[n] = x[n] * h[n]. \tag{2.53}$$

Example 2.12 Computation of the Convolution Sum



From Example 2.3, it should be clear that, in general, the sequence $h[n-k]$, $-\infty < k < \infty$, is obtained by

1. reflecting $h[k]$ about the origin to obtain $h[-k]$;
2. shifting the origin of the reflected sequence to $k = n$.

2.4 PROPERTIES OF LINEAR TIME-INVARIANT SYSTEMS

Some general properties of the class of linear time-invariant systems can be found by considering properties of the convolution operation. For example, the convolution operation is commutative:

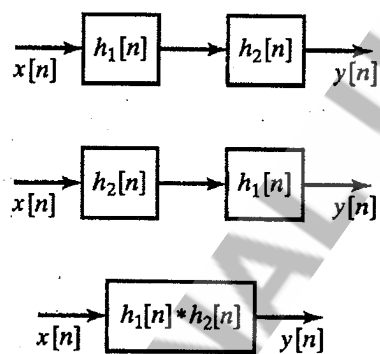
$$x[n] * h[n] = h[n] * x[n]. \qquad (2.61)$$

The convolution operation also distributes over addition; i.e.,

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

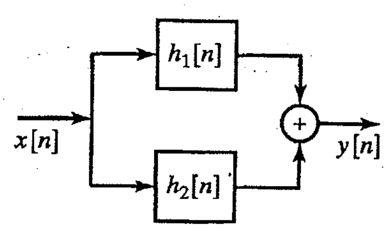
This follows in a straightforward way from Eq. (2.52) and is a direct result of the linearity and commutativity of convolution.

In a *cascade connection* of systems, the output of the first system is the input to the second, the output of the second is the input to the third, etc. The output of the last system is the overall output.

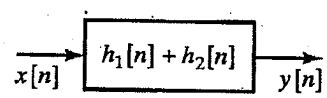


In a *parallel connection,* the systems have the same input, and their outputs are summed to produce an overall output. It follows from the distributive property of convolution that the connection of two linear time-invariant systems in parallel is equivalent to a single system whose impulse response is the sum of the individual impulse responses; i.e.,

$$h[n] = h_1[n] + h_2[n]. \qquad (2.64)$$



(a)



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Linear time-invariant systems are stable if and only if the impulse response is absolutely summable, i.e., if

$$S = \sum_{k=-\infty}^{\infty} |h[k]| < \infty. \tag{2.65}$$

This can be shown as follows. From Eq. (2.62),

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|. \tag{2.66}$$

If $x[n]$ is bounded, so that

$$|x[n]| \leq B_x,$$

then substituting B_x for $|x[n-k]|$ can only strengthen the inequality. Hence,

$$|y[n]| \leq B_x \sum_{k=-\infty}^{\infty} |h[k]|. \tag{2.67}$$

The class of causal systems was defined in Section 2.2.4 as those systems for which the output $y[n_0]$ depends only on the input samples $x[n]$, for $n \leq n_0$. It follows from Eq. (2.52) or Eq. (2.62) that this definition implies the condition

$$h[n] = 0, \qquad n < 0, \tag{2.70}$$

for causality of linear time-invariant systems. (See Problem 2.62.) For this reason, it is sometimes convenient to refer to a sequence that is zero for $n < 0$ as a *causal sequence*, meaning that it could be the impulse response of a causal system.

Ideal Delay (Example 2.3)

$$h[n] = \delta[n - n_d], \qquad n_d \text{ a positive fixed integer.} \tag{2.71}$$

Moving Average (Example 2.4)

$$\begin{aligned} h[n] &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k] \\ &= \begin{cases} \dfrac{1}{M_1 + M_2 + 1}, & -M_1 \leq n \leq M_2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.72}$$

Accumulator (Example 2.6)

$$\begin{aligned} h[n] &= \sum_{k=-\infty}^{n} \delta[k] \\ &= \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases} \\ &= u[n]. \end{aligned} \tag{2.73}$$

Forward Difference (Example 2.10)

$$h[n] = \delta[n+1] - \delta[n]. \tag{2.74}$$

Backward Difference (Example 2.10)

$$h[n] = \delta[n] - \delta[n-1]. \tag{2.75}$$

The impulse response of the accumulator is infinite in duration. This is an example of the class of systems referred to as *infinite-duration impulse response* (IIR) systems. An example of an IIR system that is stable is a system whose impulse response is $h[n] = a^n u[n]$ with $|a| < 1$. In this case,

$$S = \sum_{n=0}^{\infty} |a|^n. \tag{2.76}$$

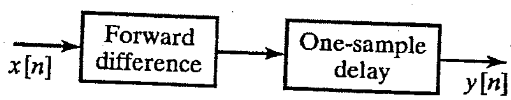
If $|a| < 1$, the formula for the sum of the terms of an infinite geometric series gives

$$S = \frac{1}{1 - |a|} < \infty. \tag{2.77}$$

If, on the other hand, $|a| \geq 1$, the sum is infinite and the system is unstable.

Since the output of the delay system is $y[n] = x[n-n_d]$, and since the delay system has impulse response $h[n] = \delta[n - n_d]$, it follows that

$$x[n] * \delta[n - n_d] = \delta[n - n_d] * x[n] = x[n - n_d]. \tag{2.78}$$



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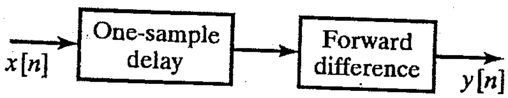
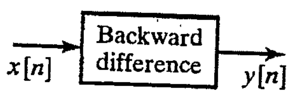


(b)

$$\begin{aligned} h[n] &= (\delta[n+1] - \delta[n]) * \delta[n-1] \\ &= \delta[n-1] * (\delta[n+1] - \delta[n]) \\ &= \delta[n] - \delta[n-1]. \end{aligned}$$



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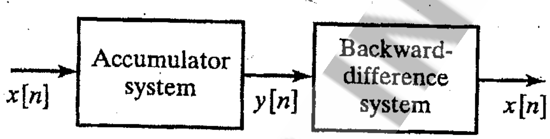


$$\begin{aligned} h[n] &= u[n] * (\delta[n] - \delta[n-1]) \\ &= u[n] - u[n-1] \\ &= \delta[n]. \end{aligned}$$

In general, if a linear time-invariant system has impulse response $h[n]$, then its inverse system, if it exists, has impulse response $h_i[n]$ defined by the relation

$$h[n] * h_i[n] = h_i[n] * h[n] = \delta[n]. \tag{2.81}$$

Inverse systems are useful in many situations in which it is necessary to compensate for the effects of a linear system.

2.5 LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

An important subclass of linear time-invariant systems consists of those systems for which the input $x[n]$ and the output $y[n]$ satisfy an Nth-order linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]. \tag{2.82}$$

An example of the class of linear constant-coefficient difference equations is the accumulator system defined by

$$y[n] = \sum_{k=-\infty}^{n} x[k]. \tag{2.83}$$

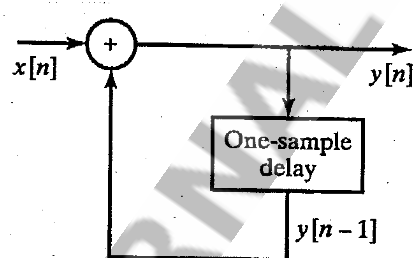
By separating the term $x[n]$ from the sum, we can rewrite Eq. (2.83) as

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k]. \tag{2.85}$$

$$y[n] = x[n] + y[n-1], \tag{2.86}$$

from which the desired form of the difference equation can be obtained by grouping all the input and output terms on separate sides of the equation:

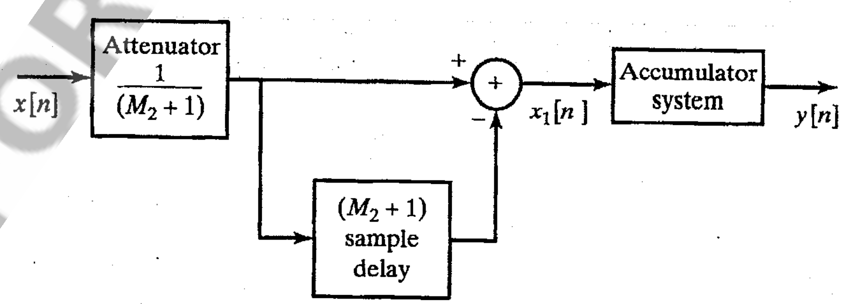
$$y[n] - y[n-1] = x[n]. \tag{2.87}$$



Consider the moving-average system of Example 2.4, with $M_1 = 0$ so that the system is causal. In this case,

$$y[n] = \frac{1}{(M_2+1)} \sum_{k=0}^{M_2} x[n-k], \tag{2.89}$$

which is a special case of Eq. (2.82), with $N = 0$, $a_0 = 1$, $M = M_2$, and $b_k = 1/(M_2+1)$ for $0 \le k \le M_2$.



Alternatively, if the auxiliary conditions are a set of auxiliary values of $y[n]$, the other values of $y[n]$ can be generated by rewriting Eq. (2.82) as a recurrence formula, i.e., in the form

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]. \tag{2.97}$$

If the input $x[n]$, together with a set of auxiliary values, say, $y[-1], y[-2], \ldots, y[-N]$, is specified, then $y[0]$ can be determined from Eq. (2.97). With $y[0], y[-1], \ldots, y[-N+1]$ available, $y[1]$ can then be calculated, and so on. When this procedure is used, $y[n]$ is said to be computed *recursively;* i.e., the output computation involves not only the input sequence, but also previous values of the output sequence.

To summarize, for a system for which the input and output satisfy a linear constant-coefficient difference equation:

- The output for a given input is not uniquely specified. Auxiliary information or conditions are required.
- If the auxiliary information is in the form of N sequential values of the output, later values can be obtained by rearranging the difference equation as a recursive relation running forward in n, and prior values can be obtained by rearranging the difference equation as a recursive relation running backward in n.
- Linearity, time invariance, and causality of the system will depend on the auxiliary conditions. If an additional condition is that the system is initially at rest, then the system will be linear, time invariant, and causal.

$$h[n] = \begin{cases} \left(\dfrac{b_n}{a_0}\right), & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases} \tag{2.107}$$

The impulse response is obviously finite in duration.

2.6 FREQUENCY-DOMAIN REPRESENTATION OF DISCRETE-TIME SIGNALS AND SYSTEMS

To demonstrate the eigenfunction property of complex exponentials for discrete-time systems, consider an input sequence $x[n] = e^{j\omega n}$ for $-\infty < n < \infty$, i.e., a complex exponential of radian frequency ω. From Eq. (2.62), the corresponding output of a linear time-invariant system with impulse response $h[n]$ is

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)} \\ &= e^{j\omega n} \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right). \end{aligned} \tag{2.108}$$

If we define

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}, \tag{2.109}$$

Eq. (2.108) becomes

$$y[n] = H(e^{j\omega}) e^{j\omega n}. \tag{2.110}$$

The eigenvalue $H(e^{j\omega})$ is called the *frequency response* of the system. In general, $H(e^{j\omega})$ is complex and can be expressed in terms of its real and imaginary parts as

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}) \tag{2.111}$$

or in terms of magnitude and phase as

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\sphericalangle H(e^{j\omega})}. \tag{2.112}$$

As a simple example of how we can find the frequency response of a linear time-invariant system, consider the ideal delay system defined by

$$y[n] = x[n - n_d], \tag{2.113}$$

$$y[n] = e^{j\omega(n-n_d)} = e^{-j\omega n_d}e^{j\omega n}.$$

$$H(e^{j\omega}) = e^{-j\omega n_d}. \tag{2.114}$$

The magnitude and phase are

$$|H(e^{j\omega})| = 1, \tag{2.116a}$$

$$\sphericalangle H(e^{j\omega}) = -\omega n_d. \tag{2.116b}$$

Since it is simple to express a sinusoid as a linear combination of complex exponentials, let us consider a sinusoidal input

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}. \tag{2.119}$$

Thus, the total response is

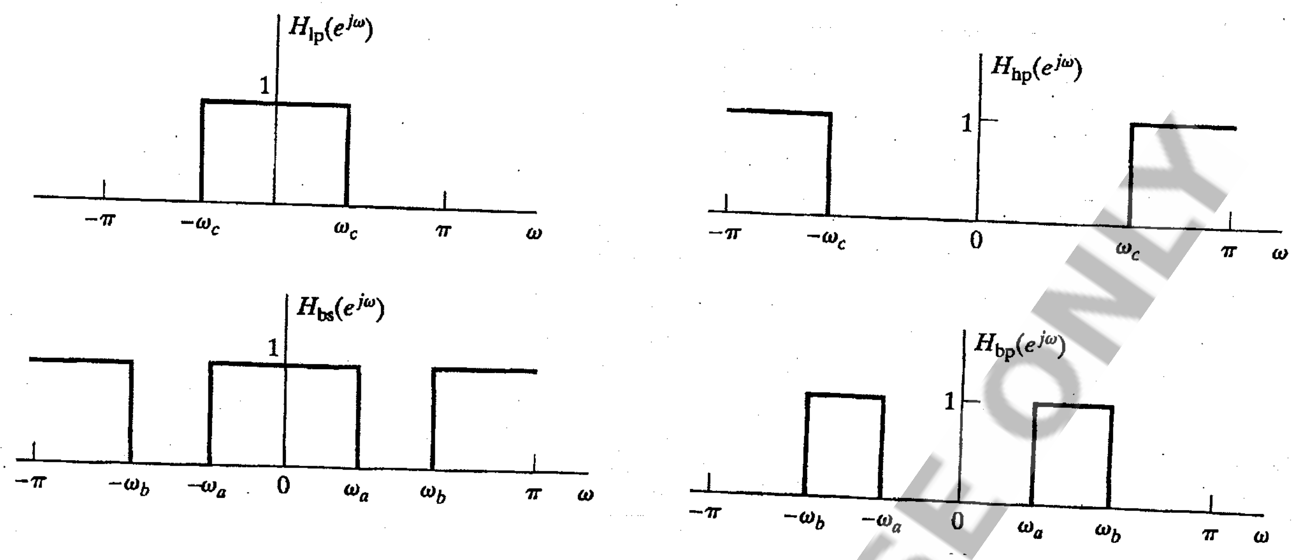
$$y[n] = \frac{A}{2}[H(e^{j\omega_0})e^{j\phi}e^{j\omega_0 n} + H(e^{-j\omega_0})e^{-j\phi}e^{-j\omega_0 n}]. \tag{2.121}$$

If $h[n]$ is real, it can be shown (see Problem 2.71) that $H(e^{-j\omega_0}) = H^*(e^{j\omega_0})$. Consequently,

$$y[n] = A|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta), \tag{2.122}$$

where $\theta = \sphericalangle H(e^{j\omega_0})$ is the phase of the system function at frequency ω_0.

Since $H(e^{j\omega})$ is periodic with period 2π, and since the frequencies ω and $\omega + 2\pi$ are indistinguishable, it follows that we need only specify $H(e^{j\omega})$ over an interval of length 2π, e.g., $0 \le \omega \le 2\pi$ or $-\pi < \omega \le \pi$. The inherent periodicity defines the frequency response everywhere outside the chosen interval. For simplicity and for consistency with the continuous-time case, it is generally convenient to specify $H(e^{j\omega})$ over the interval $-\pi < \omega \le \pi$. With respect to this interval, the "low frequencies" are frequencies close to zero, while the "high frequencies" are frequencies close to $\pm\pi$. Recalling that frequencies differing by an integer multiple of 2π are indistinguishable, we might generalize the preceding statement as follows: The "low frequencies" are those that are close to an even multiple of π, while the "high frequencies" are those that are close to an odd multiple of π.



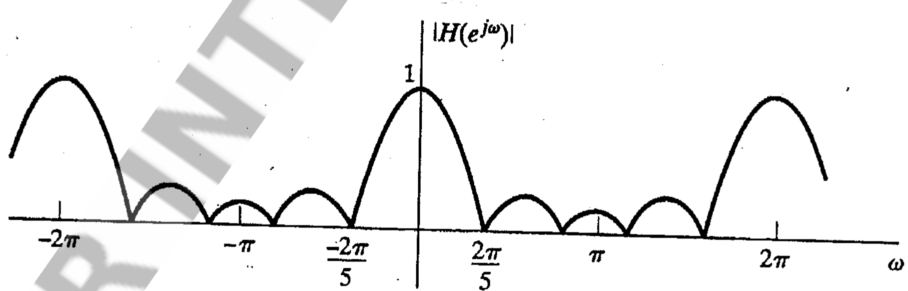
The impulse response of the moving-average system of Example 2.4 is

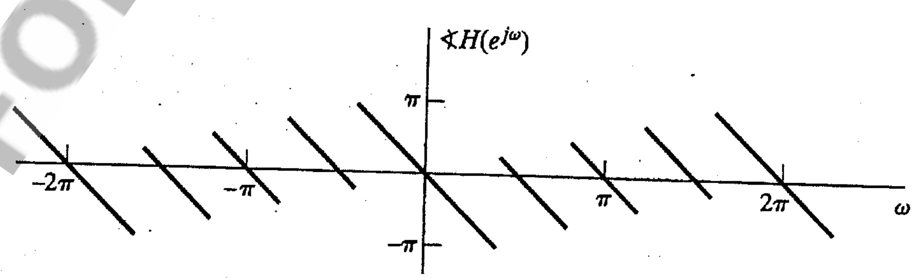
$$h[n] = \begin{cases} \dfrac{1}{M_1 + M_2 + 1}, & -M_1 \leq n \leq M_2, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the frequency response is

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega n}. \tag{2.127}$$

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} \frac{e^{j\omega M_1} - e^{-j\omega(M_2+1)}}{1 - e^{-j\omega}} = \frac{1}{M_1 + M_2 + 1} \frac{\sin[\omega(M_1 + M_2 + 1)/2]}{\sin(\omega/2)} e^{-j\omega(M_2 - M_1)/2}.$$





The magnitude and phase of $H(e^{j\omega})$ are plotted in Figure 2.19 for $M_1 = 0$ and $M_2 = 4$. Note that $H(e^{j\omega})$ is periodic, as is required of the frequency response of a discrete-time system. Note also that $|H(e^{j\omega})|$ falls off at "high frequencies" and $\sphericalangle H(e^{j\omega})$, i.e., the phase of $H(e^{j\omega})$, varies linearly with ω.

Many sequences can be represented by a Fourier integral of the form

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \tag{2.133}$$

where

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{2.134}$$

Equations (2.133) and (2.134) together form a *Fourier representation* for the sequence. Equation (2.133), the *inverse Fourier transform,* is a *synthesis* formula.

Equation (2.134), the *Fourier transform,*[3] is an expression for computing $X(e^{j\omega})$ from the sequence $x[n]$, i.e., for *analyzing* the sequence $x[n]$ to determine how much of each frequency component is required to synthesize $x[n]$ using Eq. (2.133).

[3]Sometimes we will refer to Eq. (2.134) more explicitly as the discrete-time Fourier transform, or DTFT, particularly when it is important to distinguish it from the continuous-time Fourier transform.

In general, the Fourier transform is a complex-valued function of ω. As with the frequency response, we may either express $X(e^{j\omega})$ in rectangular form as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) \tag{2.135a}$$

or in polar form as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}. \tag{2.135b}$$

The phase $\sphericalangle X(e^{j\omega})$ is not uniquely specified by Eq. (2.135b), since any integer multiple of 2π may be added to $\sphericalangle X(e^{j\omega})$ at any value of ω without affecting the result of the complex exponentiation. When we specifically want to refer to the principal value, i.e., $\sphericalangle X(e^{j\omega})$ restricted to the range of values between $-\pi$ and $+\pi$, we will denote this as ARG[$X(e^{j\omega})$]. If we want to refer to a phase function that is a continuous function of ω for $0 < \omega < \pi$, we will use the notation arg[$X(e^{j\omega})$].

By comparing Eqs. (2.109) and (2.134), we can see that the frequency response of a linear time-invariant system is simply the Fourier transform of the impulse response and that, therefore, the impulse response can be obtained from the frequency response by applying the inverse Fourier transform integral; i.e.,

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega. \tag{2.136}$$

Thus, if $x[n]$ is *absolutely summable,* then $X(e^{j\omega})$ exists. Furthermore, in this case, the series can be shown to converge uniformly to a continuous function of ω.

Let $x[n] = a^n u[n]$. The Fourier transform of this sequence is

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{1}{1 - ae^{-j\omega}} \qquad \text{if } |ae^{-j\omega}| < 1 \quad \text{or} \quad |a| < 1.$$

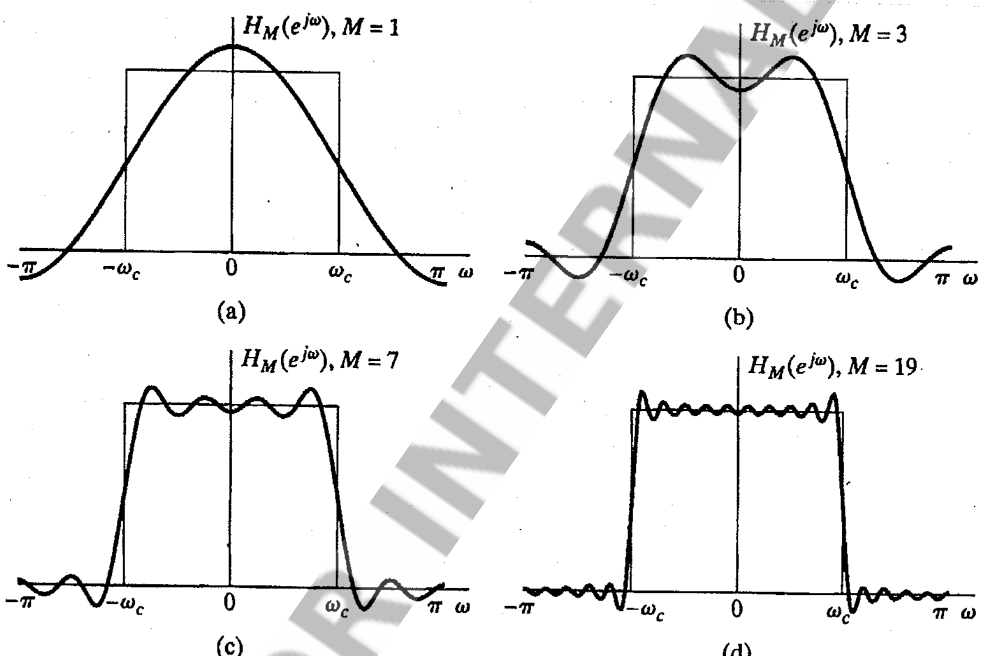
Clearly, the condition $|a| < 1$ is the condition for the absolute summability of $x[n]$; i.e.,

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty \qquad \text{if } |a| < 1. \tag{2.140}$$

Let us determine the impulse response of the ideal lowpass filter discussed in Example 2.19. The frequency response is

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases} \tag{2.144}$$

$$\begin{aligned} h_{\text{lp}}[n] &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi jn} \left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn}(e^{j\omega_c n} - e^{-j\omega_c n}) \\ &= \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty. \end{aligned} \tag{2.145}$$



$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}.$$

However, $h_{\text{lp}}[n]$, as given in Eq. (2.145), is square summable, and correspondingly, $H_M(e^{j\omega})$ converges in the mean-square sense to $H_{\text{lp}}(e^{j\omega})$; i.e.,

$$\lim_{M \to \infty} \int_{-\pi}^{\pi} |H_{\text{lp}}(e^{j\omega}) - H_M(e^{j\omega})|^2 d\omega = 0.$$

Fourier Transform of a Constant

Consider the sequence $x[n] = 1$ for all n. This sequence is neither absolutely summable nor square summable, and Eq. (2.134) does not converge in either the uniform or mean-square sense for this case. However, it is possible and useful to define the Fourier transform of the sequence $x[n]$ to be the periodic impulse train[4]

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r). \qquad (2.147)$$

Another sequence that is neither absolutely summable nor square summable is the unit step sequence $u[n]$. Although it is not completely straightforward to show, this sequence can be represented by the following Fourier transform:

$$U(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{r=-\infty}^{\infty} \pi\delta(\omega + 2\pi r). \qquad (2.153)$$

SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

A *conjugate-symmetric sequence* $x_e[n]$ is defined as a sequence for which $x_e[n] = x_e^*[-n]$, and a *conjugate-antisymmetric sequence* $x_o[n]$ is defined as a sequence for which $x_o[n] = -x_o^*[-n]$, where * denotes complex conjugation. Any sequence $x[n]$ can be expressed as a sum of a conjugate-symmetric and conjugate-antisymmetric sequence. Specifically,

$$x[n] = x_e[n] + x_o[n], \qquad (2.154a)$$

where

$$x_e[n] = \tfrac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n] \qquad (2.154b)$$

and

$$x_o[n] = \tfrac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n]. \qquad (2.154c)$$

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega}),$$

$$X_e(e^{j\omega}) = \tfrac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})]$$

$$X_o(e^{j\omega}) = \tfrac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})].$$

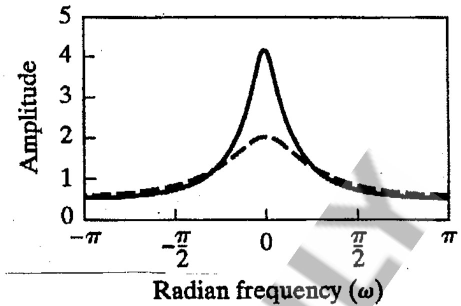
$$X_e(e^{j\omega}) = X_e^*(e^{-j\omega})$$

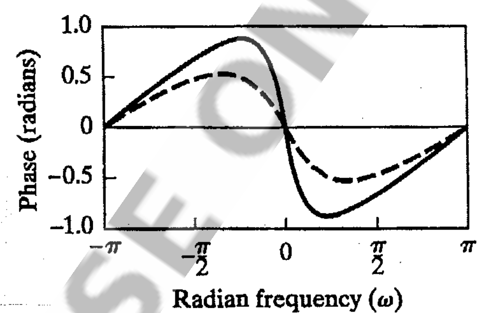
$$X_o(e^{j\omega}) = -X_o^*(e^{-j\omega}).$$

$$x[n] = a^n u[n]$$
$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} \qquad \text{if } |a| < 1.$$

Then, from the properties of complex numbers, it follows that

$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} = X^*(e^{-j\omega})$$
$$X_R(e^{j\omega}) = \frac{1-a\cos\omega}{1+a^2-2a\cos\omega} = X_R(e^{-j\omega})$$
$$X_I(e^{j\omega}) = \frac{-a\sin\omega}{1+a^2-2a\cos\omega} = -X_I(e^{-j\omega})$$
$$|X(e^{j\omega})| = \frac{1}{(1+a^2-2a\cos\omega)^{1/2}} = |X(e^{-j\omega})|$$
$$\sphericalangle X(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1-a\cos\omega}\right) = -\sphericalangle X(e^{-j\omega})$$





2.9 FOURIER TRANSFORM THEOREMS

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}).$$

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$	
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$	**Linearity**
2. $x[n-n_d]$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$	**Time Shifting**
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$	**Frequency Shifting**
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.	**Time Reversal**
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$	**Differentiation in Frequency**
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$	**The Convolution Theorem**
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$	
Parseval's theorem:		
8. $\sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$		**Parseval's Theorem**
9. $\sum_{n=-\infty}^{\infty}x[n]y^*[n] = \dfrac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})Y^*(e^{j\omega})d\omega$		

The function $|X(e^{j\omega})|^2$ is called the *energy density spectrum.*

Thus, convolution of sequences implies multiplication of the corresponding Fourier transforms. Note that the time-shifting property is a special case of the convolution property, since

$$\delta[n-n_d] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d} \tag{2.170}$$

and if $h[n]=\delta[n-n_d]$, then $y[n]=x[n]*\delta[n-n_d]=x[n-n_d]$. Therefore,

$$H(e^{j\omega})=e^{-j\omega n_d} \qquad \text{and} \qquad Y(e^{j\omega})=e^{-j\omega n_d}X(e^{j\omega}).$$

The duality inherent in most Fourier transform theorems is evident when we compare the convolution and modulation theorems.

Specifically, discrete-time convolution of sequences (the convolution sum) is equivalent to multiplication of corresponding periodic Fourier transforms, and multiplication of sequences is equivalent to *periodic* convolution of corresponding Fourier transforms.

FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty<n<\infty)$	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert<1)$	$\dfrac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\dfrac{1}{1-e^{-j\omega}}+\displaystyle\sum_{k=-\infty}^{\infty}\pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert<1)$	$\dfrac{1}{(1-ae^{-j\omega})^2}$
7. $\dfrac{r^n\sin\omega_p(n+1)}{\sin\omega_p}u[n] \quad (\lvert r\rvert<1)$	$\dfrac{1}{1-2r\cos\omega_p e^{-j\omega}+r^2e^{-j2\omega}}$
8. $\dfrac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega})=\begin{cases}1, & \lvert\omega\rvert<\omega_c,\\ 0, & \omega_c<\lvert\omega\rvert\le\pi\end{cases}$
9. $x[n]=\begin{cases}1, & 0\le n\le M\\ 0, & \text{otherwise}\end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\displaystyle\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n+\phi)$	$\displaystyle\sum_{k=-\infty}^{\infty}[\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k)+\pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$