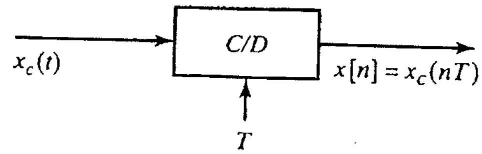
SAMPLING OF CONTINUOUS-TIME SIGNALS



$$x[n] = x_c(nT), \qquad -\infty < n < \infty. \tag{4.1}$$

In Eq. (4.1), T is the *sampling period*, and its reciprocal, $f_s = 1/T$, is the *sampling frequency*, in samples per second. We also express the sampling frequency as $\Omega_s = 2\pi/T$ when we want to use frequencies in radians per second.

In a practical setting, the operation of sampling is implemented by an analog-to-digital (A/D) converter. Such systems can be viewed as approximations to the ideal C/D converter.

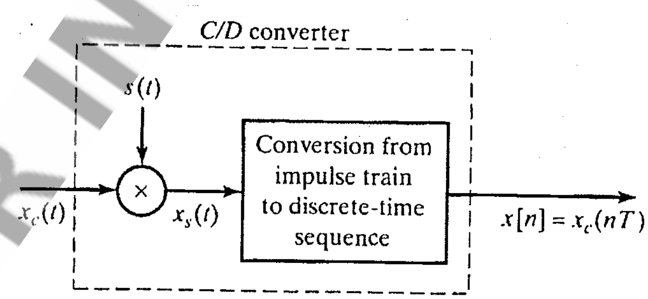
The sampling operation is generally not invertible; i.e., given the output $x[n]$, it is not possible in general to reconstruct $x_c(t)$, the input to the sampler, since many continuous-time signals can produce the same output sequence of samples. The inherent ambiguity in sampling is a fundamental issue in signal processing. Fortunately, it is possible to remove the ambiguity by restricting the input signals that go into the sampler.

To derive the frequency-domain relation between the input and output of an ideal C/D converter, let us first consider the conversion of $x_c(t)$ to $x_s(t)$ through modulation of the periodic impulse train

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \tag{4.2}$$

where $\delta(t)$ is the unit impulse function, or Dirac delta function. We modulate $s(t)$ with $x_c(t)$, obtaining

$$\begin{aligned} x_s(t) &= x_c(t)s(t) \\ &= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT). \end{aligned} \tag{4.3}$$



$x_s(t)$ can be expressed as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT). \tag{4.4}$$

The Fourier transform of a periodic impulse train is a periodic impulse train (Oppenheim and Willsky, 1997). Specifically,

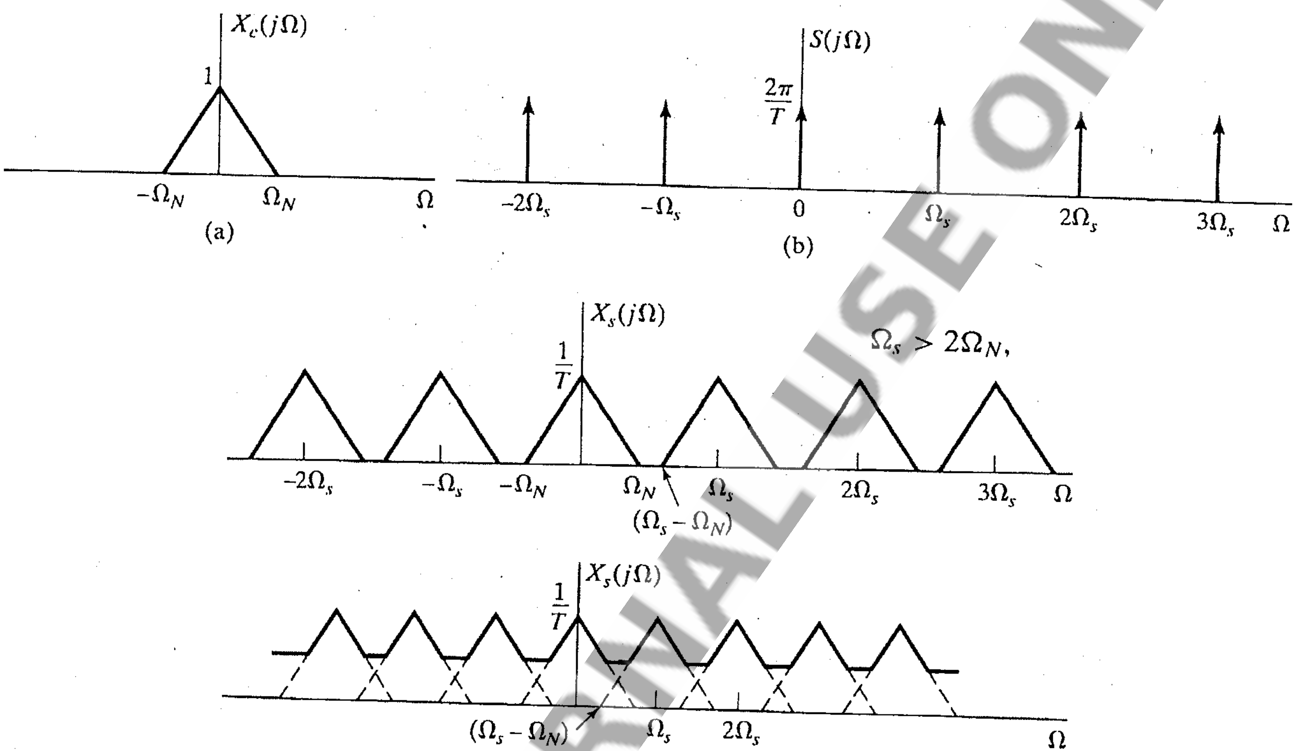
$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s), \tag{4.5}$$

where $\Omega_s = 2\pi/T$ is the sampling frequency in radians/s. Since

$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega),$$

where $*$ denotes the operation of continuous-variable convolution, it follows that

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)). \tag{4.6}$$



Consequently, $x_c(t)$ can be recovered from $x_s(t)$ with an ideal lowpass filter. This is depicted in Figure 4.4(a), which shows the impulse train modulator followed by a linear time-invariant system with frequency response $H_r(j\Omega)$.

Since

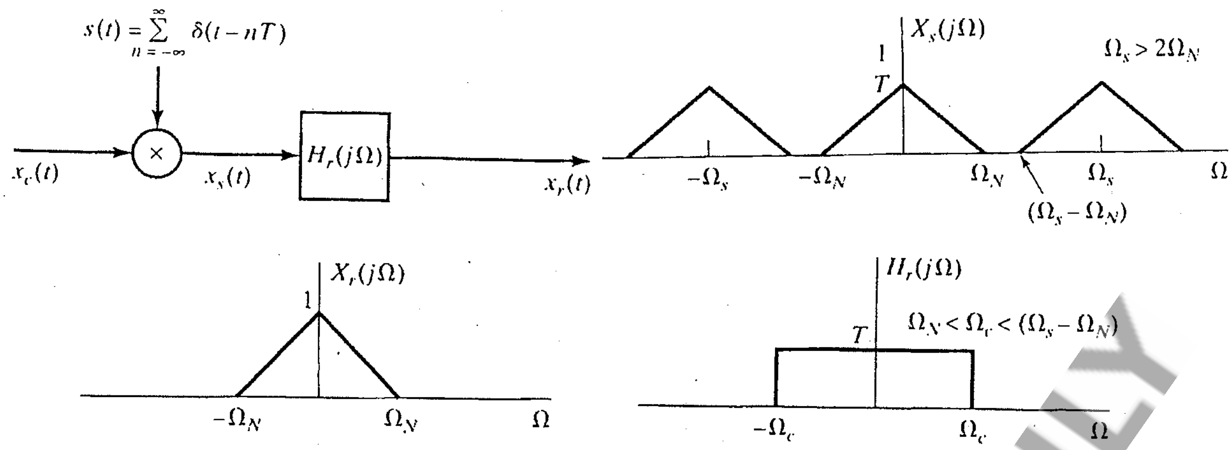
$$X_r(j\Omega) = H_r(j\Omega)X_s(j\Omega), \tag{4.8}$$

it follows that if $H_r(j\Omega)$ is an ideal lowpass filter with gain T and cutoff frequency Ω_c such that

$$\Omega_N < \Omega_c < (\Omega_s - \Omega_N), \tag{4.9}$$

then

$$X_r(j\Omega) = X_c(j\Omega), \tag{4.10}$$



Nyquist Sampling Theorem: Let $x_c(t)$ be a bandlimited signal with

$$X_c(j\Omega) = 0 \qquad \text{for } |\Omega| \geq \Omega_N. \tag{4.14a}$$

Then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT)$, $n = 0, \pm1, \pm2, \ldots$, if

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N. \tag{4.14b}$$

The frequency Ω_N is commonly referred to as the *Nyquist frequency*, and the frequency $2\Omega_N$ that must be exceeded by the sampling frequency is called the *Nyquist rate*.

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn}. \tag{4.15}$$

Since

$$x[n] = x_c(nT) \tag{4.16}$$

and

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \tag{4.17}$$

it follows that

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T}). \tag{4.18}$$

Consequently,

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)), \tag{4.19}$$

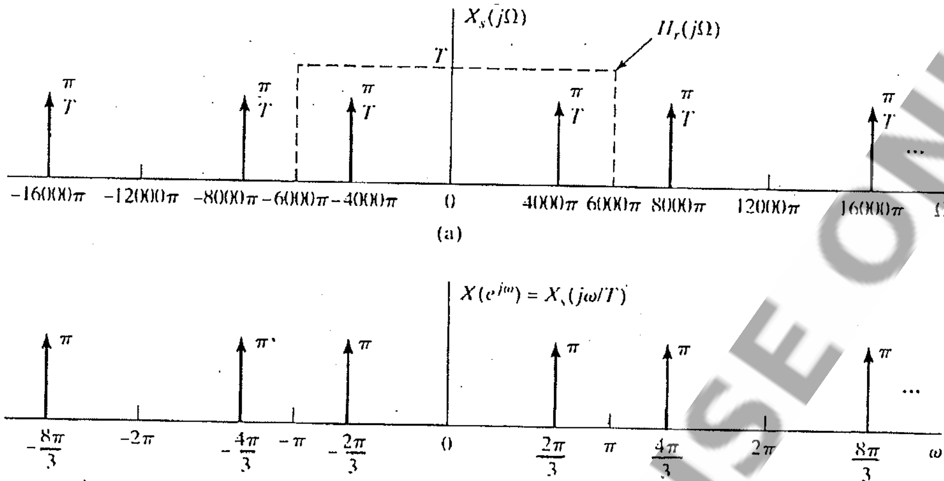
or equivalently,

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right). \tag{4.20}$$

Sampling and Reconstruction of a Sinusoidal Signal

If we sample the continuous-time signal $x_c(t) = \cos(4000\pi t)$ with sampling period $T = 1/6000$, we obtain $x[n] = x_c(nT) = \cos(4000\pi Tn) = \cos(\omega_0 n)$, where $\omega_0 = 4000\pi T = 2\pi/3$. In this case, $\Omega_s = 2\pi/T = 12000\pi$, and the highest frequency of the signal is $\Omega_0 = 4000\pi$, so the conditions of the Nyquist sampling theorem are satisfied and there is no aliasing. The Fourier transform of $x_c(t)$ is

$$X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi).$$



(a) (b)

From Eqs. (4.18)–(4.20), we see that $X(e^{j\omega})$ is simply a frequency-scaled version of $X_s(j\Omega)$ with the frequency scaling specified by $\omega = \Omega T$. This scaling can alternatively be thought of as a normalization of the frequency axis so that the frequency $\Omega = \Omega_s$ in $X_s(j\Omega)$ is normalized to $\omega = 2\pi$ for $X(e^{j\omega})$. The fact that there is a frequency scaling or normalization in the transformation from $X_s(j\Omega)$ to $X(e^{j\omega})$ is directly associated with the fact that there is a time normalization in the transformation from $x_s(t)$ to $x[n]$.

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

, where we have used the fact that scaling the independent variable of an impulse also scales its area, i.e., $\delta(\omega/T) = T\delta(\omega)$.

RECONSTRUCTION OF A BANDLIMITED SIGNAL FROM ITS SAMPLES

If we are given a sequence of samples, $x[n]$, we can form an impulse train $x_s(t)$ in which successive impulses are assigned an area equal to successive sequence values, i.e.,

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT). \tag{4.22}$$

The nth sample is associated with the impulse at $t = nT$, where T is the sampling period associated with the sequence $x[n]$. If this impulse train is the input to an ideal lowpass continuous-time filter with frequency response $H_r(j\Omega)$ and impulse response $h_r(t)$, then the output of the filter will be

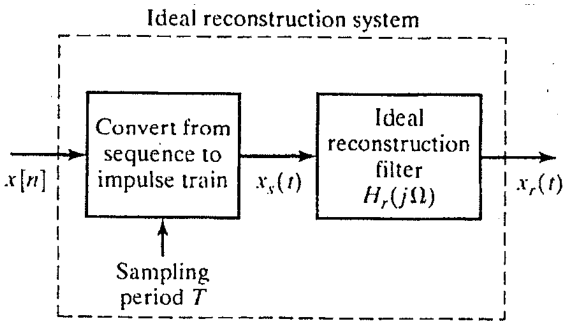
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT). \tag{4.23}$$

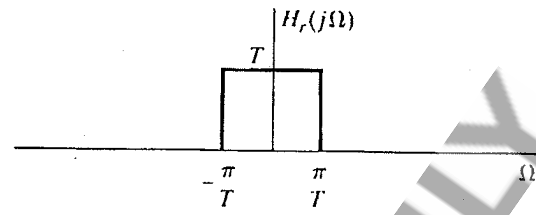
The corresponding impulse response, $h_r(t)$, is the inverse Fourier transform of $H_r(j\Omega)$, and for cutoff frequency π/T it is given by

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}. \tag{4.24}$$

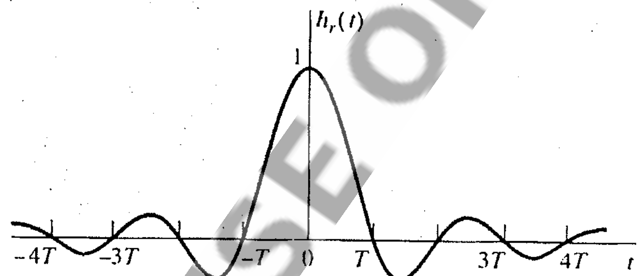
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}. \tag{4.25}$$

From the frequency-domain argument of Section 4.2, we saw that if $x[n] = x_c(nT)$, where $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, then $x_r(t)$ is equal to $x_c(t)$.

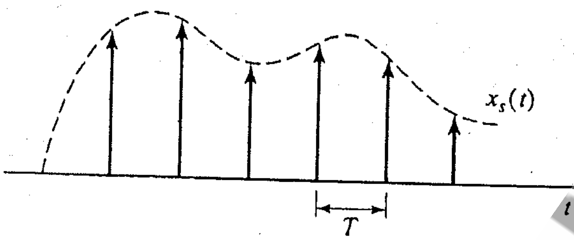


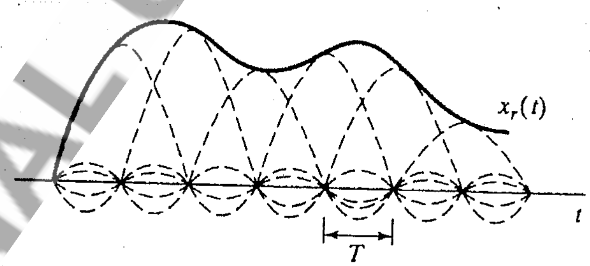


(b)

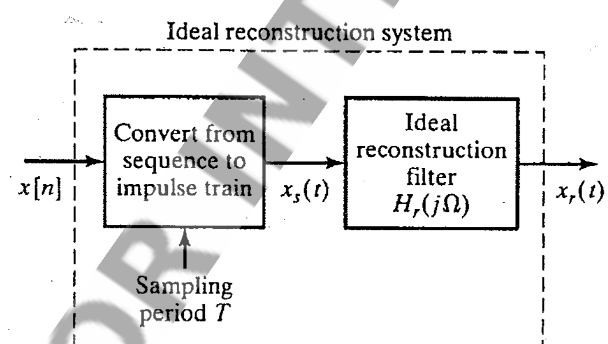


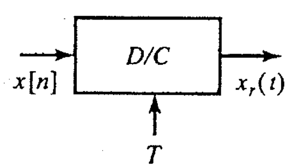
(c)





As suggested by this figure, the ideal lowpass filter *interpolates* between the impulses of $x_s(t)$ to construct a continuous-time signal $x_r(t)$.





It is useful to formalize the preceding discussion by defining an ideal system for reconstructing a bandlimited signal from a sequence of samples. We will call this system the *ideal discrete-to-continuous-time (D/C) converter*.

The properties of the ideal D/C converter are most easily seen in the frequency domain. To derive an input/output relation in this domain, consider the Fourier transform

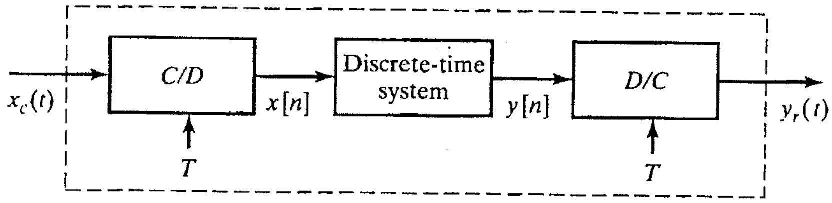
$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] H_r(j\Omega) e^{-j\Omega T n}.$$

By factoring $H_r(j\Omega)$ out of the sum, we can write

$$X_r(j\Omega) = H_r(j\Omega) X(e^{j\Omega T}). \tag{4.28}$$

Equation (4.28) provides a frequency-domain description of the ideal D/C converter. According to Eq. (4.28), $X(e^{j\omega})$ is frequency scaled (i.e., ω is replaced by ΩT). The ideal lowpass filter $H_r(j\Omega)$ selects the base period of the resulting periodic Fourier transform $X(e^{j\Omega T})$ and compensates for the $1/T$ scaling inherent in sampling. Thus, if the sequence $x[n]$ has been obtained by sampling a bandlimited signal at the Nyquist rate or higher, then the reconstructed signal $x_r(t)$ will be equal to the original bandlimited signal.

DISCRETE-TIME PROCESSING OF CONTINUOUS-TIME SIGNALS



If the discrete-time system in Figure is linear and time invariant, we then have

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \tag{4.33}$$

where $H(e^{j\omega})$ is the frequency response of the system or, equivalently, the Fourier transform of the unit sample response, and $X(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of the input and output, respectively. Combining

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})X(e^{j\Omega T}). \tag{4.34}$$

$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T})$$

$$= \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \pi/T, \\ 0, & \text{otherwise.} \end{cases}$$

$$Y_r(j\Omega) = H_r(j\Omega)H(e^{j\Omega T})\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right). \tag{4.35}$$

If $X_c(j\Omega) = 0$ for $|\Omega| \geq \pi/T$, then the ideal lowpass reconstruction filter $H_r(j\Omega)$ cancels the factor $1/T$ and selects only the term in Eq. (4.35) for $k = 0$; i.e.,

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega T})X_c(j\Omega), & |\Omega| < \pi/T, \\ 0, & |\Omega| \geq \pi/T. \end{cases} \tag{4.36}$$

Thus, if $X_c(j\Omega)$ is bandlimited and the sampling rate is above the Nyquist rate, the output is related to the input through an equation of the form

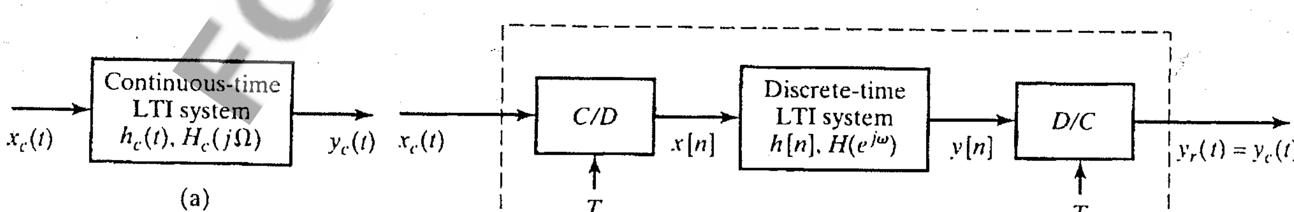
$$Y_r(j\Omega) = H_{\text{eff}}(j\Omega)X_c(j\Omega), \tag{4.37}$$

where

$$H_{\text{eff}}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi/T, \\ 0, & |\Omega| \geq \pi/T. \end{cases} \tag{4.38}$$

That is, the overall continuous-time system is equivalent to a linear time-invariant system whose *effective* frequency response is given by Eq. (4.38).

Impulse Invariance



$H_{\text{eff}}(j\Omega) = H_c(j\Omega)$

(b)

With $H_c(j\Omega)$ bandlimited, Eq. (4.38) specifies how to choose $H(e^{j\omega})$ so that $H_{\text{eff}}(j\Omega) = H_c(j\Omega)$. Specifically,

$$H(e^{j\omega}) = H_c(j\omega/T), \qquad |\omega| < \pi, \tag{4.49}$$

with the further requirement that T be chosen such that

$$H_c(j\Omega) = 0, \qquad |\Omega| \geq \pi/T. \tag{4.50}$$

Under the constraints of Eqs. (4.49) and (4.50), there is also a straightforward and useful relationship between the continuous-time impulse response $h_c(t)$ and the discrete-time impulse response $h[n]$. In particular, as we shall verify shortly,

$$h[n] = Th_c(nT); \tag{4.51}$$

i.e., the impulse response of the discrete-time system is a scaled, sampled version of $h_c(t)$. When $h[n]$ and $h_c(t)$ are related through Eq. (4.51), the discrete-time system is said to be an *impulse-invariant* version of the continuous-time system.

A Discrete-Time Lowpass Filter Obtained By Impulse Invariance

Suppose that we wish to obtain an ideal lowpass discrete-time filter with cutoff frequency $\omega_c < \pi$. We can do this by sampling a continuous-time ideal lowpass filter with cutoff frequency $\Omega_c = \omega_c/T < \pi/T$ defined by

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c, \\ 0, & |\Omega| \leq \Omega_c. \end{cases}$$

The impulse response of this continuous-time system is

$$h_c(t) = \frac{\sin(\Omega_c t)}{\pi t},$$

so we define the impulse response of the discrete-time system to be

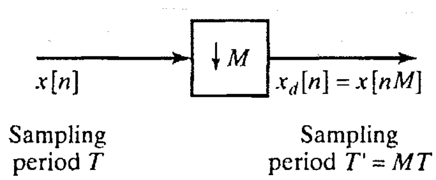
$$h[n] = Th_c(nT) = T\frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n},$$

where $\omega_c = \Omega_c T$. We have already shown that this sequence corresponds to the discrete-time Fourier transform

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi, \end{cases}$$

which is identical to $H_c(j\omega/T)$, as predicted by Eq. (4.56).

CHANGING THE SAMPLING RATE USING DISCRETE-TIME PROCESSING



Sampling Rate Reduction by an Integer Factor

The sampling rate of a sequence can be reduced by "sampling" it, i.e., by defining a new sequence

$$x_d[n] = x[nM] = x_c(nMT). \tag{4.71}$$

sampling with period $T' = MT$. Furthermore, if $X_c(j\Omega) = 0$ for $|\Omega| \geq \Omega_N$, then $x_d[n]$ is an exact representation of $x_c(t)$ if $\pi/T' = \pi/(MT) \geq \Omega_N$. That is, the sampling rate can be reduced by a factor of M without aliasing if the original sampling rate was at least M times the Nyquist rate or if the bandwidth of the sequence is first reduced by a factor of M by discrete-time filtering. In general, the operation of reducing the sampling rate (including any prefiltering) will be called *downsampling*.

First recall that the discrete-time Fourier transform of $x[n] = x_c(nT)$ is

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right). \tag{4.72}$$

Similarly, the discrete-time Fourier transform of $x_d[n] = x[nM] = x_c(nT')$ with $T' = MT$ is

$$X_d(e^{j\omega}) = \frac{1}{T'}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T'} - \frac{2\pi r}{T'}\right)\right). \tag{4.73}$$

Now, since $T' = MT$, we can write Eq. (4.73) as

$$X_d(e^{j\omega}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right). \tag{4.74}$$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT}\right)\right)\right]. \tag{4.76}$$

The term inside the square brackets in Eq. (4.76) is recognized from Eq. (4.72) as

$$X(e^{j(\omega-2\pi i)/M}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi k}{T}\right)\right). \tag{4.77}$$

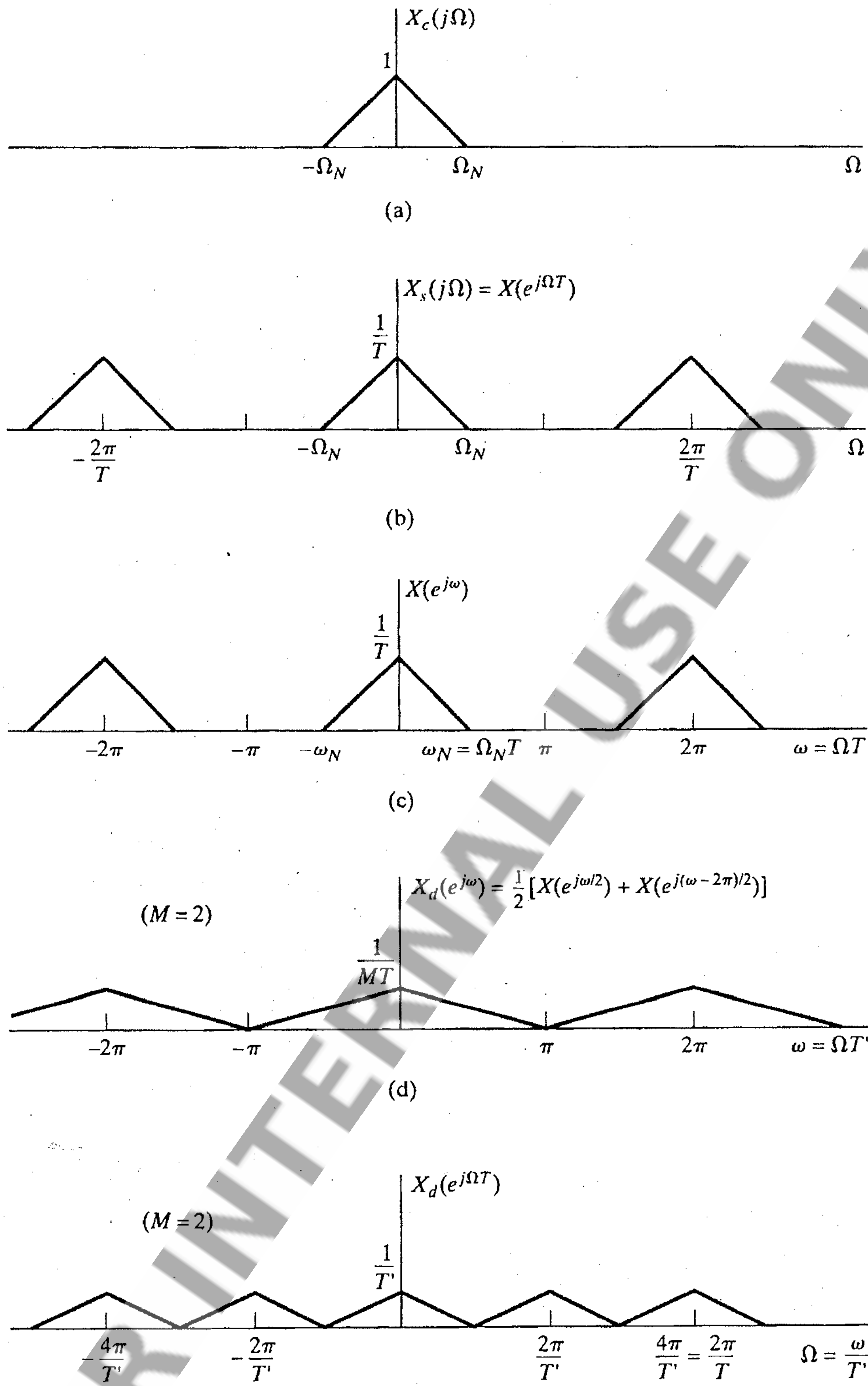
Thus, we can express Eq. (4.76) as

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega/M - 2\pi i/M)}). \tag{4.78}$$

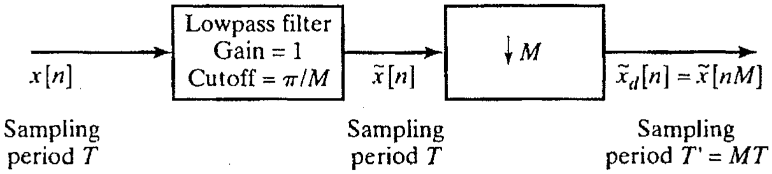
If we compare Eqs. (4.73) and (4.78), we see that $X_d(e^{j\omega})$ can be thought of as being composed of either an infinite set of copies of $X_c(j\Omega)$, frequency scaled through $\omega = \Omega T'$ and shifted by integer multiples of $2\pi/T'$ (Eq. (4.73)), or M copies of the periodic Fourier transform $X(e^{j\omega})$, frequency scaled by M and shifted by integer multiples of 2π (Eq. (4.78)). Either interpretation makes it clear that $X_d(e^{j\omega})$ is periodic with period 2π (as are all discrete-time Fourier transforms) and that aliasing can be avoided by ensuring that $X(e^{j\omega})$ is bandlimited, i.e.,

$$X(e^{j\omega}) = 0, \qquad \omega_N \leq |\omega| \leq \pi, \tag{4.79}$$

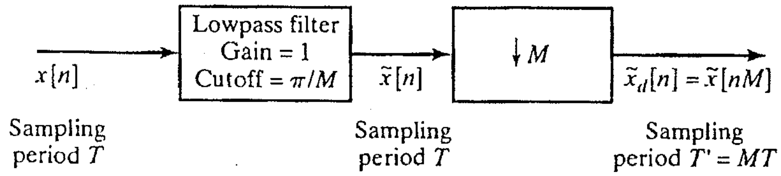
and $2\pi/M \geq 2\omega_N$.



In this example, $2\pi/T = 4\Omega_N$; i.e., the original sampling rate is exactly twice the minimum rate to avoid aliasing. Thus, when the original sampled sequence is down-sampled by a factor of $M = 2$, no aliasing results.



From the preceding discussion, we see that a general system for downsampling by a factor of M is the one shown in Figure . Such a system is called a *decimator,* and downsampling by lowpass filtering followed by compression has been termed *decimation* (Crochiere and Rabiner, 1983).



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Increasing the Sampling Rate by an Integer Factor

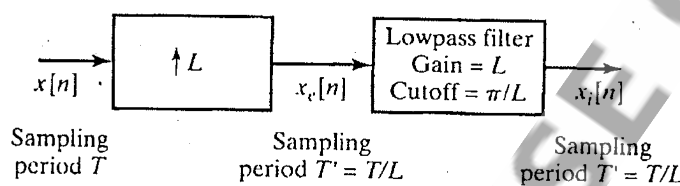


Figure shows a system for obtaining $x_i[n]$ from $x[n]$ using only discrete-time processing. The system on the left is called a *sampling rate expander* (see Crochiere and Rabiner, 1983) or simply an *expander*. Its output is

$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots, \\ 0, & \text{otherwise}, \end{cases} \tag{4.84}$$

or equivalently,

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]. \tag{4.85}$$

The Fourier transform of $x_e[n]$ can be expressed as

$$\begin{aligned} X_e(e^{j\omega}) &= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]\right)e^{-j\omega n} \\ &= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega Lk} = X(e^{j\omega L}). \end{aligned} \tag{4.86}$$

Thus, the Fourier transform of the output of the expander is a frequency-scaled version of the Fourier transform of the input; i.e., ω is replaced by ωL so that ω is now normalized by

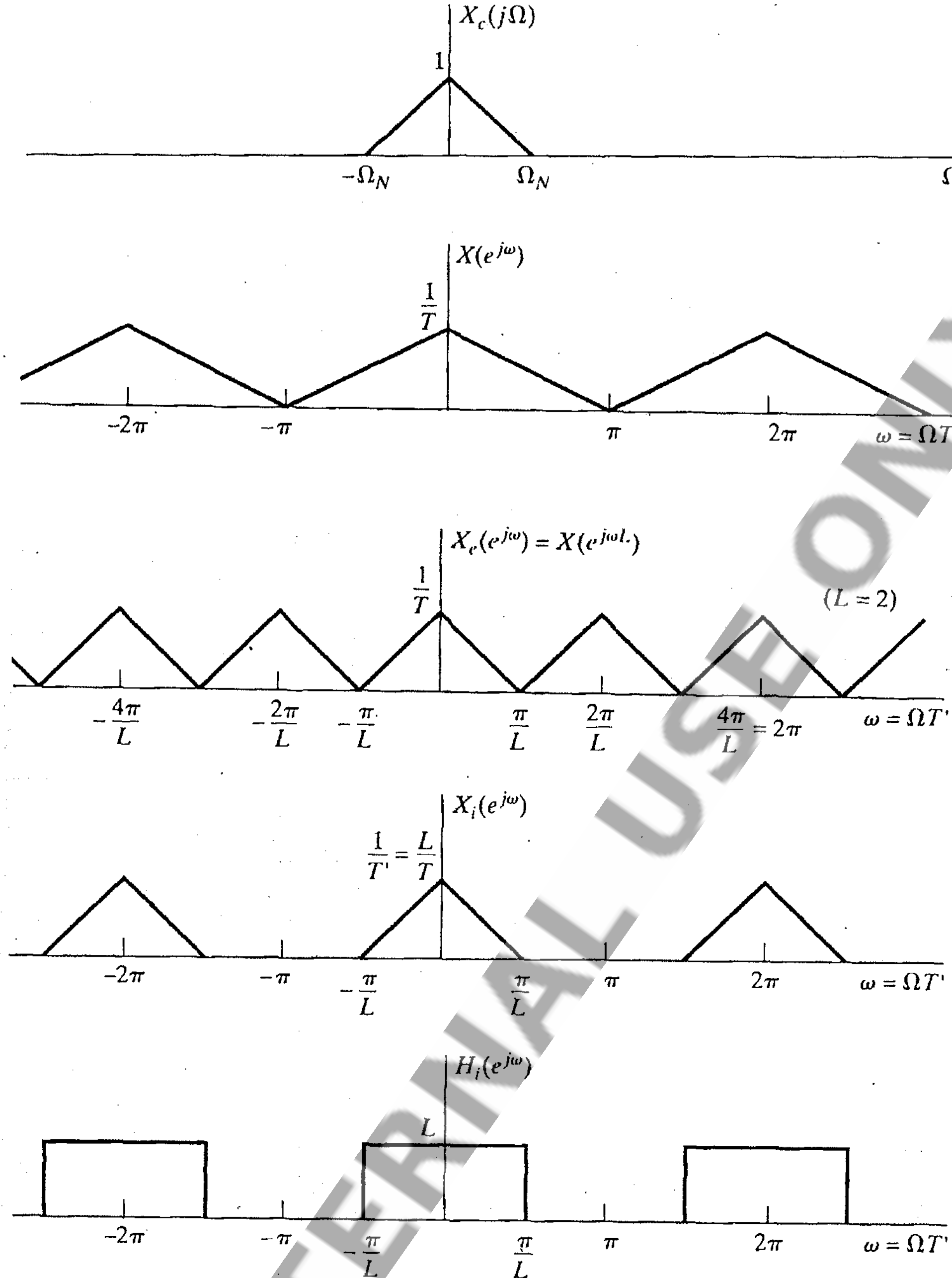
$$\omega = \Omega T'. \tag{4.87}$$

As in the case of the D/C converter, it is possible to obtain an interpolation formula for $x_i[n]$ in terms of $x[n]$. First note that the impulse response of the lowpass filter is

$$h_i[n] = \frac{\sin(\pi n/L)}{\pi n/L}. \tag{4.88}$$

Using Eq. (4.85), we obtain

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k]\frac{\sin[\pi(n-kL)/L]}{\pi(n-kL)/L}. \tag{4.89}$$

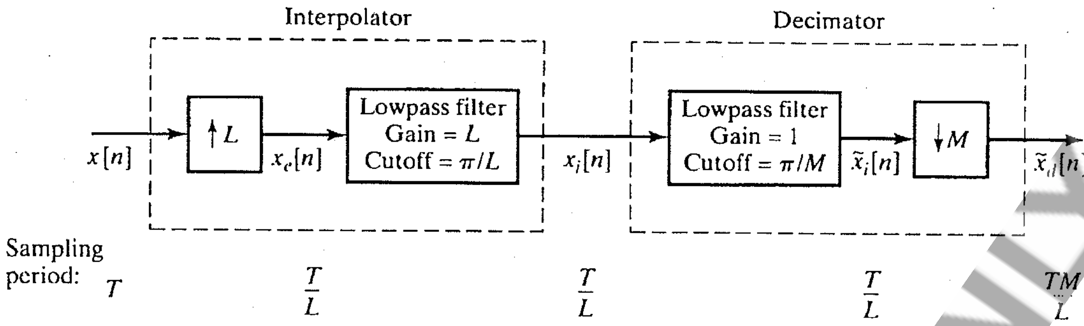


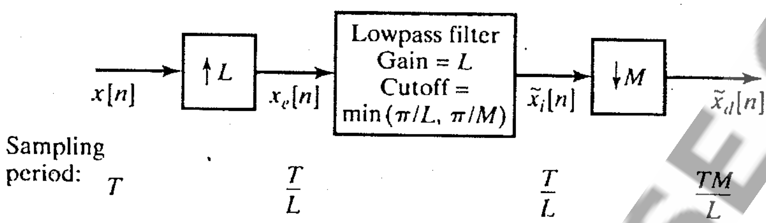
We see that $X_i(e^{j\omega})$ can be obtained from $X_e(e^{j\omega})$ by correcting the amplitude scale from $1/T$ to $1/T'$ and by removing all the frequency-scaled images of $X_c(j\Omega)$ except at integer multiples of 2π.

In general, the required gain would be L, since $L(1/T) = [1/(T/L)] = 1/T'$, and the cutoff frequency would be π/L.

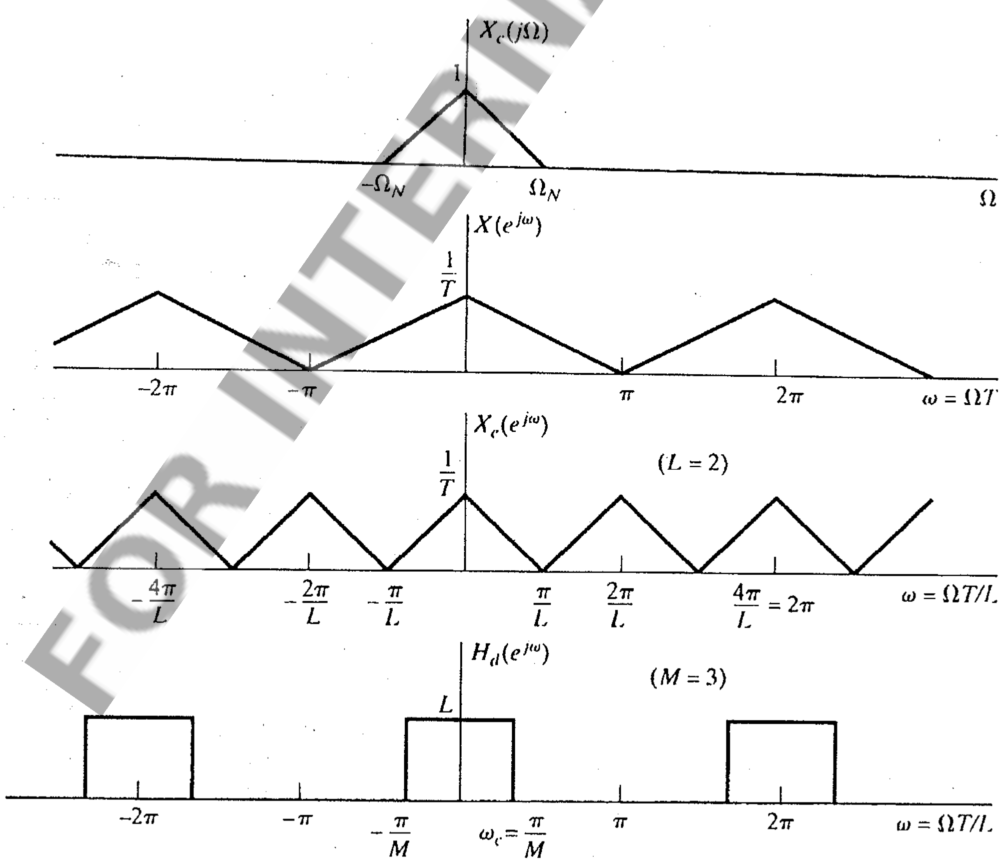
That system is therefore called an *interpolator*, since it fills in the missing samples, and the operation of upsampling is therefore considered to be synonymous with *interpolation*.

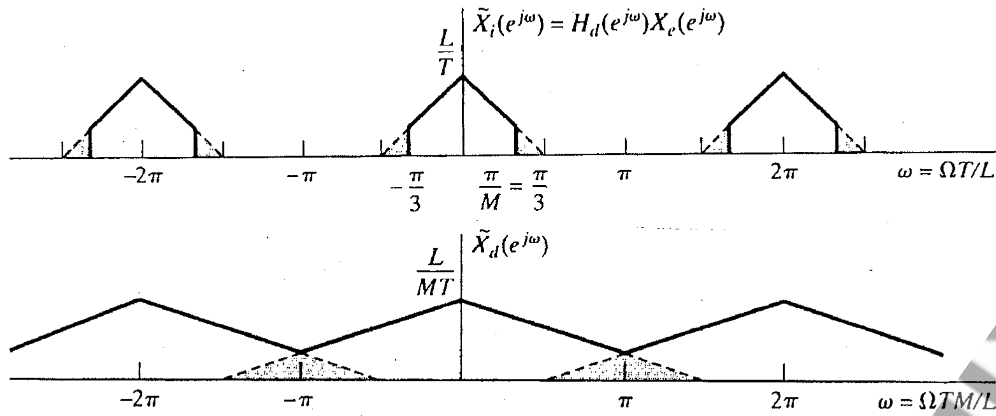
Changing the Sampling Rate by a Noninteger Factor





If we wish to change the sampling period to $T' = (3/2)T$, we must first interpolate by a factor $L = 2$ and then decimate by a factor of $M = 3$. Since this implies a net decrease in sampling rate, and the original signal was sampled at the Nyquist rate, we must incorporate additional lowpass filtering in order to avoid aliasing. If we were interested only in interpolating by a factor of 2, we could choose the lowpass filter to have a cutoff frequency of $\omega_c = \pi/2$ and a gain of $L = 2$. However, since the output of the filter will be decimated by $M = 3$, we must use a cutoff frequency of $\omega_c = \pi/3$, but the gain of the filter should still be 2 as in Figure

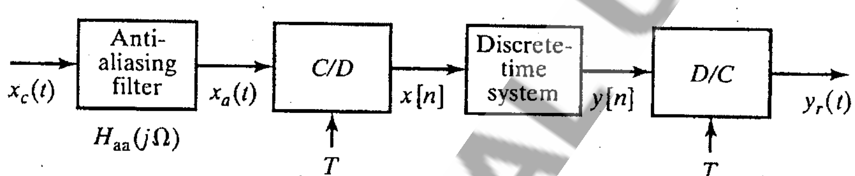




Note that the shaded regions show the aliasing that would have occurred if the cutoff frequency of the interpolation lowpass filter had been $\pi/2$ instead of $\pi/3$.

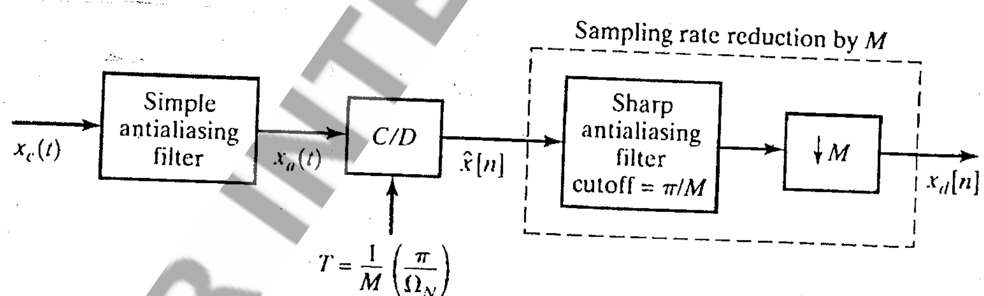
Prefiltering to Avoid Aliasing

In processing analog signals using discrete-time systems, it is generally desirable to minimize the sampling rate. This is because the amount of arithmetic processing required to implement the system is proportional to the number of samples to be processed. If the input is not bandlimited or if the Nyquist frequency of the input is too high, prefiltering may be necessary.



In this context, the lowpass filter that precedes the C/D converter is called an *antialiasing filter*. Ideally, the frequency response of the antialiasing filter would be

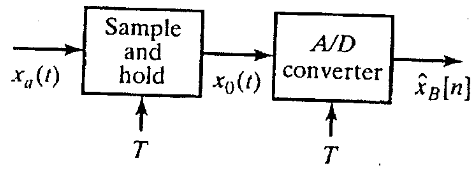
$$H_{aa}(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c < \pi/T, \\ 0, & |\Omega| > \Omega_c. \end{cases} \tag{4.108}$$



With Ω_N denoting the highest frequency component to eventually be retained after the antialiasing filtering is completed, we first apply a very simple antialiasing filter that has a gradual cutoff with significant attenuation at $M\Omega_N$. Next, implement the C/D conversion at a sampling rate much higher than $2\Omega_N$, e.g., at $2M\Omega_N$. After that, sampling rate reduction by a factor of M that includes sharp antialiasing filtering is implemented in the discrete-time domain. Subsequent discrete-time processing can then be done at the low sampling rate to minimize computation.

Analog-to-Digital (A/D) Conversion

An ideal C/D converter converts a continuous-time signal into a discrete-time signal, where each sample is known with infinite precision. As an approximation to this for digital signal processing, the system of Figure converts a continuous-time (analog) signal into a digital signal, i.e., a sequence of finite-precision or quantized samples.



However, the conversion is not instantaneous, and for this reason, a high-performance A/D system typically includes a sample-and-hold. The ideal sample-and-hold system is the system whose output is

$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t - nT), \tag{4.111}$$

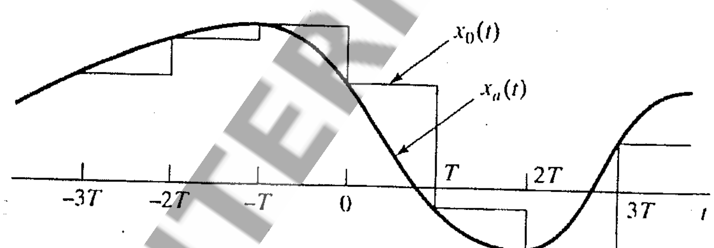
where $x[n] = x_a(nT)$ are the ideal samples of $x_a(t)$ and $h_0(t)$ is the impulse response of the zero-order-hold system, i.e.,

$$h_0(t) = \begin{cases} 1, & 0 < t < T, \\ 0, & \text{otherwise.} \end{cases} \tag{4.112}$$

If we note that Eq. (4.111) has the equivalent form

$$x_0(t) = h_0(t) * \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t - nT), \tag{4.113}$$

we see that the ideal sample-and-hold is equivalent to impulse train modulation followed by linear filtering with the zero-order-hold system.

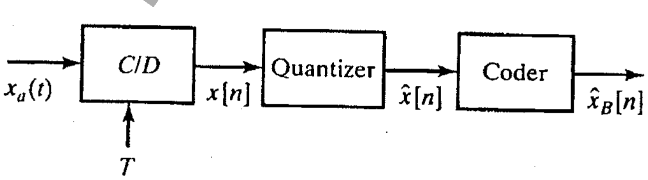


Analysis of Quantization Errors

The quantizer is a nonlinear system whose purpose is to transform the input sample $x[n]$ into one of a finite set of prescribed values. We represent this operation as

$$\hat{x}[n] = Q(x[n]) \tag{4.114}$$

and refer to $\hat{x}[n]$ as the quantized sample. Quantizers can be defined with either uniformly or nonuniformly spaced quantization levels; however, when numerical calculations are to be done on the samples, the quantization steps usually are uniform.



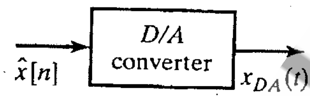
From Figure we see that the quantized sample $\hat{x}[n]$ will generally be different from the true sample value $x[n]$. The difference between them is the *quantization error*, defined as

$$e[n] = \hat{x}[n] - x[n]. \tag{4.117}$$

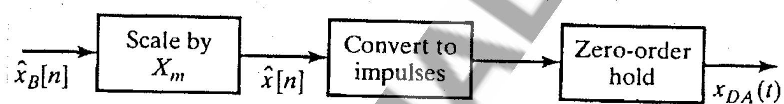
A simplified, but useful, model of the quantizer is depicted in Figure In this model, the quantization error samples are thought of as an additive noise signal. The model is exactly equivalent to the quantizer if we know $e[n]$. The statistical representation of quantization errors is based on the following assumptions:

1. The error sequence $e[n]$ is a sample sequence of a stationary random process.
2. The error sequence is uncorrelated with the sequence $x[n]$.
3. The random variables of the error process are uncorrelated; i.e., the error is a white-noise process.
4. The probability distribution of the error process is uniform over the range of quantization error.

D/A Conversion



(a)



(b)

In terms of Fourier transforms, the reconstruction is represented as

$$X_r(j\Omega) = X(e^{j\Omega T})H_r(j\Omega), \tag{4.127}$$

where $X(e^{j\omega})$ is the discrete-time Fourier transform of the sequence of samples and $X_r(j\Omega)$ is the Fourier transform of the reconstructed continuous-time signal. The ideal reconstruction filter is

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| < \pi/T, \\ 0, & |\Omega| > \pi/T. \end{cases} \tag{4.128}$$

$$\begin{aligned} x_{DA}(t) &= \sum_{n=-\infty}^{\infty} X_m \hat{x}_B[n] h_0(t - nT) \\ &= \sum_{n=-\infty}^{\infty} \hat{x}[n] h_0(t - nT), \end{aligned} \tag{4.130}$$

To simplify our discussion, we define

$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]h_0(t-nT), \tag{4.132}$$

$$e_0(t) = \sum_{n=-\infty}^{\infty} e[n]h_0(t-nT), \tag{4.133}$$

so that Eq. (4.131) can be written as

$$x_{DA}(t) = x_0(t) + e_0(t). \tag{4.134}$$

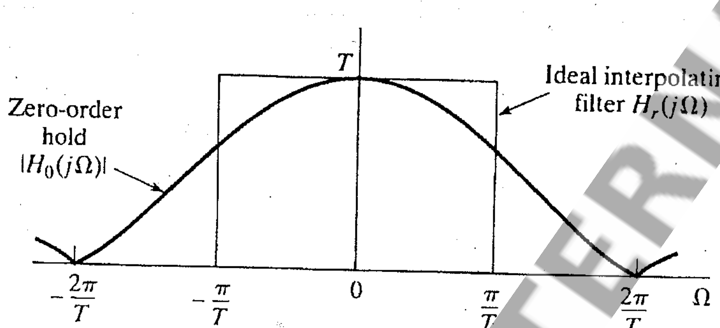
The signal component $x_0(t)$ is related to the input signal $x_a(t)$, since $x[n] = x_a(nT)$. The noise signal $e_0(t)$ depends on the quantization-noise samples $e[n]$ in the same way that $x_0(t)$ depends on the unquantized signal samples.

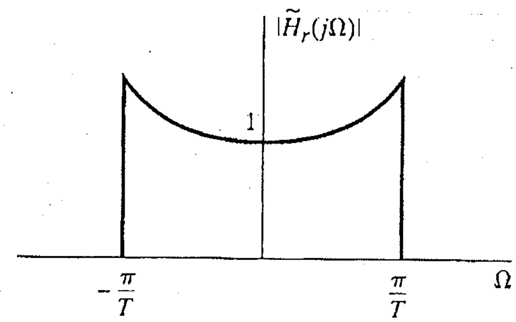
$$X_0(j\Omega) = \left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)\right] H_0(j\Omega). \tag{4.137}$$

if we define a compensated reconstruction filter as

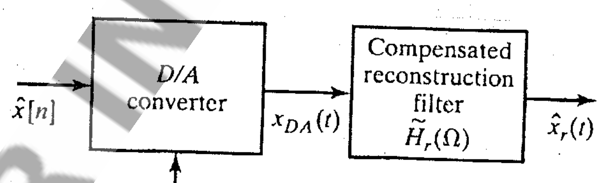
$$\tilde{H}_r(j\Omega) = \frac{H_r(j\Omega)}{H_0(j\Omega)}, \tag{4.138}$$

then the output of the filter will be $x_a(t)$ if the input is $x_0(t)$. The frequency response of the zero-order-hold filter is easily shown to be





$$\tilde{H}_r(j\Omega) = \begin{cases} \dfrac{\Omega T/2}{\sin(\Omega T/2)} e^{j\Omega T/2}, & |\Omega| < \pi/T, \\ 0, & |\Omega| > \pi/T. \end{cases}$$



In other words, the output would be

$$\hat{x}_r(t) = x_a(t) + e_a(t), \tag{4.142}$$

where $e_a(t)$ would be a bandlimited white-noise signal.

$H(e^{j\Omega T})$ is the frequency response of the discrete-time system. Similarly, assuming that the quantization noise introduced by the A/D converter is white noise with variance $\sigma_e^2 = \Delta^2/12$, it can be shown that the power spectrum of the output noise is

$$P_{e_a}(j\Omega) = |\tilde{H}_r(j\Omega)H_0(j\Omega)H(e^{j\Omega T})|^2\sigma_e^2, \tag{4.145}$$

i.e., the input quantization noise is changed by the successive stages of discrete- and continuous-time filtering.