Lecture 12

Wave equation: vibrating string (cont'd)

`Sections 4.2-4.4 of text by Haberman`

In the previous lecture, we arrived at the general one-dimensional PDE for the vibrating string:

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) + \rho_0(x)Q(x,t). \tag{1}$$

We'll now make some additional simplifications in order to come up with a simpler PDE. First, we'll assume homogeneity of the string, i.e., $\rho_0(x) = \rho_0$ constant.

Secondly, we'll assume that $T(x,t) = T_0$ constant, i.e., constant tension throughout the string. This is due to homogeneity plus the additional assumption that the string is *perfectly elastic* and tightly stretched so that variations in T are negligible. (It is the variations in the *direction* of the tension vector that are responsible for motion.)

With these assumptions, the above PDE becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho_0}\frac{\partial^2 u}{\partial x^2} + Q(x,t). \tag{2}$$

We now assume that there are no external forces except gravity acting on the string. And if the string is so tightly stretched that its equilibrium position is horizontal, then the gravity term can be ignored – there are no net forces on the string when it is in its horizontal equilibrium position. Thus we set $Q = 0$ and the above equation becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad c = \sqrt{\frac{T_0}{\rho_0}}. \tag{3}$$

This is known as the *one-dimensional wave equation.*

Let's examine the dimensionality of c: Since T_0 is tension, i.e., force (mass $\times$ acceleration), and ρ is mass per unit length, we have

$$\left[\frac{T_0}{\rho_0}\right] = \frac{ML}{T^2}\cdot\frac{L}{M} = \frac{L^2}{T^2}. \tag{4}$$

This implies that c has the dimensions of *velocity.* We'll see below that this velocity is important in the solutions of the wave equation.

We now provide some solutions to the wave equation (3). First, however, we'll need to specify a sufficient number of conditions in order to be able to extract a unique solution. As with the heat equation, the second-order derivative in x will require two conditions, normally boundary conditions at the ends of the string. There are some very interesting physical possibilities here, and we refer the reader to Section 4.3, "Boundary Conditions", of the textbook for a discussion. In what follows, we shall examine the simplest, and perhaps most common condition, that of a string of length L and *fixed ends* (discussed in some detail in Section 4.4 of text). This would be the situation of a guitar or violin string – ignoring the fret. These boundary conditions will then take the form

$$u(0,t) = 0, \qquad u(L,t) = 0. \tag{5}$$

As for the time variable t, we now have a second-order derivative in t, implying that we shall need two initial conditions on $u(x,t)$. We shall assume that the initial position and initial velocity of each segment of the string is prescribed, i.e.,

$$\begin{aligned} u(x,0) &= f(x), \\ \frac{\partial u}{\partial t}(x,0) &= g(x), \qquad 0 \le x \le L. \end{aligned} \tag{6}$$

Recall that the second-derivative in t came from the acceleration term in Newton's equation. In particle mechanics problems, a knowledge of the initial position and velocity (or momentum) of a particle is sufficient to determine a unique trajectory of the particle as dictated by Newton's second law.

The wave equation and boundary conditions are linear and homogeneous, which means that we can try to use the method of separation of variables. As for the heat equation, we'll look for solutions of the form

$$u(x,t) = \phi(x)G(t). \tag{7}$$

(For some reason, the textbook switches to using $h(t)$ for the time-dependent part.) Substitution into Eq. (3) yields

$$\phi(x)\frac{d^2G}{dt^2} = c^2G(t)\frac{d^2\phi(x)}{dx^2}, \tag{8}$$

or simply

$$\phi(x)G''(t) = c^2\phi''(x)G(t). \tag{9}$$

We "separate the variables", putting the c^2 term with the t-dependent part:

$$\frac{G''}{c^2G} = \frac{\phi''}{\phi} = \mu = -\lambda. \tag{10}$$

We've also introduced the separation constants μ and $-\lambda$ since the LHS of the equation is solely t-dependent while the other side is solely x-dependent. For convenience, we'll use $-\lambda$ for the separation constant since the spatial equation for ϕ will be identical to that of the heat equation, and we already know that those eigenvalues were positive. The resulting separated equations for ϕ and G are

$$\begin{aligned} \phi'' + \lambda\phi &= 0, \qquad \phi(0) = \phi(L) = 0. \\ G'' + \lambda c^2 G &= 0. \end{aligned} \tag{11}$$

We know that solutions to the boundary value problem for $\phi(x)$ exist only for $\lambda > 0$. In this case, we have an infinite set of discrete eigenvalues λ_n,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, \cdots, \tag{12}$$

with associated eigenfunctions,

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right). \tag{13}$$

From these values of λ_n, we find the solutions to the corrresponding equations for $G(t)$,

$$G'' + \lambda_n c^2 G = 0, \tag{14}$$

to be

$$\begin{aligned} G_n(t) &= C_1 \cos(\sqrt{\lambda_n} c)t + C_2 \sin(\sqrt{\lambda_n} c)t, \\ &= C_1 \cos\left(\frac{n\pi}{L} ct\right) + C_2 \sin\left(\frac{n\pi}{L} ct\right). \end{aligned} \tag{15}$$

As a result, the product solutions $u(x,t)$ yielded by the separation variables technique, cf. Eq. (7), are given by

$$u_n(x,t) = \phi_n(x) G_n(t) = \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right]. \tag{16}$$

Any finite linear combination of the $u_n(x,t)$ is also a solution to the wave equation with fixed-end boundary conditions. We'll examine these solutions in more detail later.

In order to accomodate the initial two conditions, we shall generally have to resort to infinite series in the u_n, i.e.,

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\ &= \sum_{n=1}^{\infty} \left[a_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right)\right]. \end{aligned} \tag{17}$$

From the first initial condition $u(x,0) = f(x)$, we have

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \tag{18}$$

which is the Fourier expansion of $f(x)$, as encountered with the heat equation. From the second initial condition $\frac{\partial u}{\partial t}(x,0) = g(x)$, we have

$$g(x) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right), \tag{19}$$

which may also be viewed as a Fourier expansion of $g(x)$ with coefficients $b_n \frac{n\pi c}{L}$.

From Eq. (18), we have, as for the heat equation,

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx. \tag{20}$$

From Eq. (19), we have

$$b_n \frac{n\pi c}{L} = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)\, dx, \tag{21}$$

or

$$b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)\, dx. \tag{22}$$

So, in principle, we have now solved the 1D wave equation with fixed ends and two initial conditions.

Let us now return to the individual product solutions

$$u_n(x,t) = \phi_n(x) G_n(t) = \sin\left(\frac{n\pi x}{L}\right) \left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right]. \tag{23}$$

Each of these solutions is called a *normal mode* of vibration. The spatial portion, $\sin(n\pi x/L)$ defines the *profile* of the normal mode – a wave with nodes at the endpoints. The time-dependent portion is oscillatory – the frequency of this oscillation (number of oscillations in 2π units of time) is

$$\omega_n = \frac{n\pi c}{L} = \frac{n\pi}{L} \sqrt{\frac{T_0}{\rho_0}}, \qquad n = 1, 2, \cdots. \tag{24}$$

This oscillatory time-dependence *modulates* the profile of the wave, as sketched below.

The frequencies ω_n are the *natural frequencies* of the vibrating string. In practial applications, frequencies are expressed in cycles per second – in these units, the natural frequencies are

$$\nu_n = \frac{\omega_n}{2\pi} = \frac{nc}{2L} = \frac{n}{2L} \sqrt{\frac{T_0}{\rho_0}}, \qquad n = 1, 2, \cdots \quad \text{cycles/second (cps) or "Hertz" (Hz).} \tag{25}$$

The lowest frequency $\omega_1 = \pi c/L$ is called the *first harmonic* or *fundamental.* All others are multiples of ω_1. Note that ω_1 may be increased/decreased by increasing/decreasing the tension T_0.

Each normal mode may also be written in the form (exercise)

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right) A_n \sin\left(\frac{n\pi ct}{L} + \phi_n\right), \tag{26}$$

where the amplitude and phase of the time-oscillation are given by, respectively,

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \tan\phi_n = \frac{a_n}{b_n}. \tag{27}$$

The time-oscillation of a normal mode may be viewed as a *standing wave* – the wave is standing, or stationary, because its nodes, including the ones at the endpoints are fixed in time. However, each standing wave may be expressed as a sum of two travelling waves. This is possible from the addition law for sin and cos:

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) = \frac{1}{2}\cos\frac{n\pi}{L}(x-ct) - \frac{1}{2}\cos\frac{n\pi}{L}(x+ct), \tag{28}$$

and

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) = \frac{1}{2}\sin\frac{n\pi}{L}(x-ct) + \frac{1}{2}\sin\frac{n\pi}{L}(x+ct). \tag{29}$$

The terms with $(x - ct)$ represent waves that are travelling to the right with velocity c – to see this, examine the position of $x(t)$ for which the argument $x - ct$ is zero:

$$x(t) = ct, \tag{30}$$

so that $x(t)$ is increasing linearly in time.

Likewise, the terms with $(x + ct)$ represent waves that are travelling to the left with velocity c.

In fact, it is rather easy to show (via Chain Rule – exercise) that

$$u(x,t) = f(x-ct) + g(x+ct) \tag{31}$$

is a solution to the 1D wave equation for any functions $f(x)$ and $g(x)$.

Lecture 13

The energy of a vibrating string

We now derive a result for the total energy of a vibrating string, in terms of the solution $u(x,t)$ of the 1D wave equation. Obviously, the string has kinetic energy – the velocity of each segment is $v(x,t) = \dfrac{\partial u}{\partial t}(x,t)$ so that the kinetic energy of each segment is $\dfrac{1}{2}\rho_0(x)\Delta x \left(\dfrac{\partial u}{\partial t}(x,t)\right)^2$. Integrating over all segments comprising the string yields the total kinetic energy

$$K = \frac{1}{2}\rho_0 \int_0^L \left(\frac{\partial u}{\partial t}(x,t)\right)^2 \, dx, \tag{32}$$

where we have once again assumed that the density ρ_0 is constant.

But that is not all – the string must have potential energy. After all, as the vibrating string moves toward its profile position, where maximum amplitude is achieved at each point x, it will slow down and "turn around", moving in the opposite direction. This slowing down and reversing is done by exchanging kinetic energy for potential energy, just as in the case of an oscillatory mass-spring system, or a pendulum. The problem is to compute the potential energy. One could try to integrate the force that would have to be exerted on a segment against its outermost tensions in order to move it from its equilibrium position $u = 0$ to a position $u > 0$. But this is rather complicated. We'll try another method that uses the result for the kinetic energy, as well as the fact that the string position $u(x,t)$ satisfies the wave equation.

First, let us differentiate the total kinetic energy w.r.t. time t:

$$\begin{aligned} \frac{dK}{dt} &= \frac{1}{2}\rho_0 \frac{d}{dt}\int_0^L v^2 \, dx \\ &= \rho_0 \int_0^L v\frac{\partial v}{\partial t} \, dx \\ &= \rho_0 \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} \, dx \\ &= \rho_0 c^2 \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} \, dx, \end{aligned} \tag{33}$$

where the final step comes from the fact that u is a solution to the wave equation. We now integrate by parts, letting

$$f = \frac{\partial u}{\partial t}, \qquad g' = \frac{\partial^2 u}{\partial x^2}. \tag{34}$$

This gives

$$\frac{dK}{dt} = T\left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_0^L - T\int_0^L \frac{\partial^2 u}{\partial x \partial t}\frac{\partial u}{\partial x} \, dx. \tag{35}$$

We have used the fact that $T = \rho_0 c^2$. The integrand on the right can be expressed as a time derivative:

$$\frac{\partial^2 u}{\partial x \partial t}\frac{\partial u}{\partial x} = \left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\right]\frac{\partial u}{\partial x} = \frac{d}{dt}\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2. \tag{36}$$

(Verify this.) Using this result, Eq. (35) can be rewritten as

$$\frac{d}{dt}\left[K + \frac{1}{2}T\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right] = T\left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_0^L. \tag{37}$$

For the clamped string problem, the term $\partial u/\partial t$ vanishes at the endpoints, so that the above time derivative is zero. Therefore, the term in brackets is constant in time. It represents the total energy of the vibrating string – the second integral is (up to a constant) the *potential energy.* As in particle mechanics, the total mechanical energy $E(t)$ is determined by the initial conditions. Here,

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^L \left[\rho_0\left(\frac{\partial u}{\partial t}\right)^2 + T\left(\frac{\partial u}{\partial x}\right)^2\right] dx \\ &= \frac{1}{2}\rho_0 \int_0^L [g(x)^2 + c^2 f'(x)^2]\, dx \\ &= E(0). \end{aligned} \tag{38}$$

If you knew that the energy $E(t)$ was given by the above expression, then you could differentiate w.r.t. time and use the wave equation to show that $E'(t) = 0$.

A note on the above derivation: The above derivation may seem somewhat mysterious. How did we know to compute the time derivative of the kinetic energy $K(t)$? The answer lies in proof that total mechanical energy is conserved when a particle is moving in $\mathbf{R}^n$ according to Newton's Law $\mathbf{F} = m\mathbf{a}$ when the force $\mathbf{F}$ is conservative. Let's review that proof briefly.

Firstly, the total mechanical energy of the particle along the trajectory is

$$\begin{aligned} E(t) &= \frac{1}{2}mv^2(t) + V(\mathbf{x}(t)) \\ &= \frac{1}{2}m\mathbf{v}(t)\cdot\mathbf{v}(t) + V(\mathbf{x}(t)). \end{aligned} \tag{39}$$

Now differentiate w.r.t t:

$$\begin{aligned} E'(t) &= \frac{1}{2}m(\mathbf{v}'\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}') + \vec{\nabla}V(\mathbf{x})\cdot\mathbf{x}' \\ &= m\mathbf{a}\cdot\mathbf{v} + \vec{\nabla}V(\mathbf{x})\cdot\mathbf{v} \\ &= \mathbf{v}\cdot(m\mathbf{a} - \mathbf{F}) \\ &= 0. \end{aligned} \tag{40}$$

In the second-to-last line, we have used the fact that $\mathbf{F}$ is conservative, i.e., there exists a potential energy function V such that $\mathbf{F} = -\vec{\nabla} V$.

Now let's try to prove this result in a slightly different way. Let's simply compute the time derivative of the kinetic energy of the particle:

$$\begin{aligned}
\frac{dK(t)}{dt} &= \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) \\
&= m\mathbf{a}\cdot\mathbf{v} \\
&= \mathbf{F}\cdot\mathbf{v} \\
&= -\vec{\nabla}V(\mathbf{x})\cdot\mathbf{x}'(t) \\
&= -\frac{d}{dt}V(\mathbf{x}(t)). \qquad (41)
\end{aligned}$$

This implies that

$$\frac{d}{dt}[K(t) + V(t)] = 0, \qquad (42)$$

from which we conclude that the total mechanical energy $E(t) = K(t) + V(t)$ is constant over the trajectory.

This derivation is quite similar in form to that for the vibrating string. In the case of the vibrating string, the wave equation represents Newton's Law and, up to a constant, the right hand side, $\dfrac{\partial^2 u}{\partial x^2}$ represents the force.

Solution of Laplace's equation using separation of variables

`Section 2.5 of text by Haberman`

Recall that Laplace's equation for a function $u : \mathbf{R}^n \to \mathbf{R}$ is given by

$$\nabla^2 u = 0. \tag{43}$$

The solution of this equation, with appropriate boundary conditions, is important in the determination of steady-state or equilibrium temperature distributions for the heat equation. In $\mathbf{R}^n$, the heat equation with sources will assume the general form

$$\frac{\partial u}{\partial t} = k\nabla^2 u + Q. \tag{44}$$

We also assume the existence of boundary conditions appropriate to the problem of concern. If there exists a steady-state temperature distribution $u(x,t) = u_{eq}(x)$, then it will satisfy the PDE (or ODE in $\mathbf{R}$)

$$\nabla^2 u = -\frac{Q}{k}. \tag{45}$$

This is known as *Poisson's equation*. In the case that there are no sources, i.e., $Q(x) = 0$, Poisson's equation becomes Laplace's equation.

The above discussion also applies to electrostatics. Recall that the electrostatic potential function $V(\mathbf{r})$ associated with a charge distribution $\rho(\mathbf{r})$ satisfies Poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}. \tag{46}$$

In the case that there is no charge, then V satisfies Laplace's equation. For this reason, all the heat problems that we examine in this section can also be viewed as electrostatic potential problems.

We have already considered Laplace's equation in one-dimension, i.e.,

$$\frac{d^2 u}{dx^2} = 0, \tag{47}$$

along with various boundary conditions (e.g., fixed endpoint temperatures, zero flux, mixed conditions), for the determination of steady-state temperature distributions on a rod. The determination of these distributions was relatively straightforward. It is more complicated in higher dimensions. In what follows, we consider some rather simple, yet illustrative cases in $\mathbf{R}^2$.

Laplace's equation over a rectangular region

Here, $u = u(x, y)$ and we consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{48}$$

over the rectangular region $0 \le x \le L$, $0 \le y \le L$. As for boundary conditions, we consider the case of fixed boundary temperature distributions, i.e.,

$$\begin{aligned} u(x,0) &= f_1(x), \\ u(L,y) &= g_2(y), \\ u(x,H) &= f_2(x), \\ u(0,y) &= g_1(y). \end{aligned} \tag{49}$$

These four conditions are *nonhomogeneous* – as a result, the technique of superposition of solutions/separation of variables will not work. However, there is a "trick" that will allow us to use S of V: We'll divide the solution $u(x, y)$ into four components, i.e.,

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y). \tag{50}$$

Each of the components u_i will satisfy one non-zero BC and three zero BCs. For example, we'll let $u_1(x, y)$ be the solution to Eq. (48) that satisfies the following BCs:

$$\begin{aligned} &\text{BC1:} \quad u_1(x,0) = f_1(x), \\ &\text{BC2:} \quad u_1(L,y) = 0, \\ &\text{BC3:} \quad u_1(x,H) = 0, \\ &\text{BC4:} \quad u_1(0,y) = 0. \end{aligned} \tag{51}$$

We now apply separation of variables to each u_i function. For $u_1(x, y)$, let

$$u_1(x,y) = h(x)\phi(y). \tag{52}$$

We're adopting the notation used in the book, in an effort to reduce confusion. (By the way, the textbook provides the solution for $u_4(x, y)$.)

The three homogeneous BCs from above will yield the following conditions on h and ϕ:

$$\begin{aligned} &\text{BC2:} \quad h(L)\phi(y) = 0 \quad \Rightarrow \quad h(L) = 0, \\ &\text{BC3:} \quad h(x)\phi(H) = 0 \quad \Rightarrow \quad \phi(H) = 0, \\ &\text{BC4:} \quad h(0)\phi(y) = 0 \quad \Rightarrow \quad h(0) = 0. \end{aligned} \tag{53}$$

We see that there are two BCs for $h(x)$ and only one for $\phi(y)$.

Substitution of (52) into (48) yields

$$h''(x)\phi(y) + h(x)\phi''(y) = 0, \tag{54}$$

which can be separated to

$$\frac{h''(x)}{h(x)} = -\frac{\phi''(y)}{\phi(y)} = \mu. \tag{55}$$

We do not yet know whether μ should be positive or negative, so we just leave it for now. The separation yields the following problems for h and ϕ:

$$h'' - \mu h = 0, \qquad h(0) = 0, \quad h(L) = 0, \tag{56}$$

and

$$\phi'' + \mu\phi = 0, \qquad \phi(H) = 0. \tag{57}$$

We've seen the BVP for $h(x)$ before: nontrivial solutions exist only for $\mu < 0$, so we let $\mu = -\lambda$, $\lambda > 0$ to give

$$h'' + \lambda h = 0, \qquad h(0) = 0, \quad h(L) = 0, \tag{58}$$

with eigenvalues

$$\lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, \cdots, \tag{59}$$

and associated eigenfunctions

$$h_n(x) = \sin\left(\frac{n\pi}{L}\right). \tag{60}$$

Lecture 14

Laplace's equation over a rectangular region (cont'd)

We now consider the ϕ equation, recalling that $\mu = -\lambda$, so that $\mu_n = -\lambda_n$. The associated ϕ_n functions will satisfy the DE

$$\phi_n''(y) - \left(\frac{n\pi}{L}\right)^2 \phi_n(y) = 0, \qquad \phi(H) = 0. \tag{61}$$

This is not a boundary value problem but an initial value problem, with only one condition. The general solution could be written as

$$\phi_n(y) = C_1 e^{n\pi y/L} + C_2 e^{-n\pi y/L} \tag{62}$$

but it will be more convenient to use hyperbolic functions, i.e.,

$$\phi_n(y) = D_1 \cosh(n\pi y/L) + D_2 \sinh(-n\pi y/L). \tag{63}$$

In order to impose the boundary condition at $y = H$, it is even more convenient to use shifted hyperbolic functions, which also satisfy the DE:

$$\phi_n(y) = A_1 \cosh \frac{n\pi}{L}(y - H) + A_2 \sinh \frac{n\pi}{L}(y - H). \tag{64}$$

The condition $\phi_n(H) = 0$ implies that $A_1 = 0$ (Exercise) so that the $\phi_n(x)$ functions associated with the $h_n(x)$ functions are

$$\phi_n(y) = A_2 \sinh \frac{n\pi}{L}(y - H). \tag{65}$$

As a result, the product solutions yielded by separation of variables are (up to a constant)

$$\begin{aligned} u_{1,n}(x, y) &= h_n(x)\phi_n(y) \\ &= \sin\left(\frac{n\pi x}{L}\right) \sinh \frac{n\pi}{L}(y - H). \end{aligned} \tag{66}$$

Note that these functions are oscillatory in the x-direction but nonoscillatory in the y-direction. Each of the $u_{1,n}(x, y)$ functions satisfies the three zero BCs (2-4) but will not, generally, satisfy the nonzero BC $u_1(x, 0) = f_1(x)$. As we have done before, we look for an appropriate superposition of these solutions, i.e.,

$$\begin{aligned} u_1(x, y) &= \sum_{n=1}^{\infty} a_n u_{1,n}(x, y) \\ &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \sinh \frac{n\pi}{L}(y - H). \end{aligned} \tag{67}$$

It follows that

$$u_1(x,0) = \sum_{n=1}^{\infty} a_n \sinh \frac{n\pi}{L}(-H) \sin\left(\frac{n\pi x}{L}\right) = f(x). \tag{68}$$

This is just a Fourier expansion of $f(x)$ in the complete basis $\phi_n(x)$, where the coefficients are $a_n \sinh(-n\pi H/L)$. If we multiply the middle and right sides of the equation by $\sin(k\pi x/L)$ and integrate over $[0, L]$, we obtain, by virtue of the orthogonality of the $\phi_n(x)$,

$$a_k \left(\frac{L}{2}\right) \sinh \frac{k\pi}{L}(-H) = \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right)\, dx, \quad k = 1, 2, \cdots. \tag{69}$$

We then rearrange to solve for the a_k:

$$a_k = \frac{2}{L \sinh \frac{k\pi}{L}(-H)} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right)\, dx, \quad k = 1, 2, \cdots. \tag{70}$$

Now we repeat the procedure for $u_2(x,y)$, $u_3(x,y)$ and $u_4(x,y)$! We can then construct the solution $u(x,y)$ from Eq. (50).

Example: Here we determine the steady-state temperature distribution on $[0, L] \times [0, H]$ which satisfies the boundary conditions,

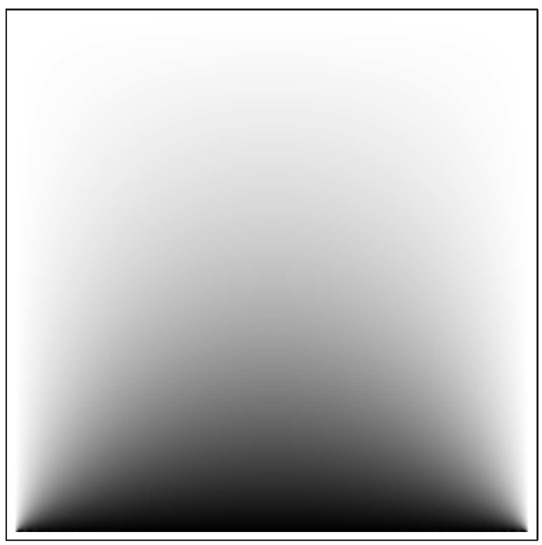
$$u(x, 0) = f_1(x) = 100, \quad u(L, y) = 0, \quad u(x, H) = 0, \quad u(0, y) = 0. \tag{71}$$

(This boundary value problem corresponds to the function $u_1(x, y)$ discussed in class.) The Fourier expansion coefficients for the constant function $f_1(x) = 100$ were computed earlier in the course. But now they have to be modified by the sinh term in Eq. (70):

$$a_k = \begin{cases} \frac{400}{k\pi \sinh(-k\pi L/H)}, & \text{k even,} \\ 0, & \text{k odd.} \end{cases} \tag{72}$$

In the figure below, $L = H = 1$ and $M = 100$ terms were used in the expansion of $u(x, y)$, i.e.,

$$u(x, y) \approx \sum_{n=1}^{100} a_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi}{L}(y - H)\right). \tag{73}$$

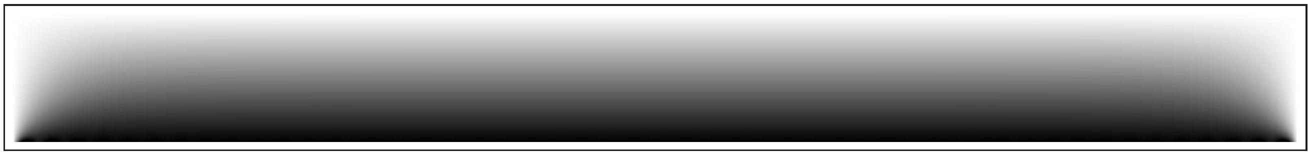


Steady-state heat distribution $u(x, y)$: $L = H = 1$. The shading at a point (x, y) is proportional to its temperature $0 \leq u(x, y) \leq 100$. The darker the shade, the higher the temperature.

As $L \to \infty$, the effects of the vertical boundaries at $x = 0$ and $x = L$ will become negligible, and the distribution should approach the one-dimensional case

$$u_{eq}(y) = \frac{100}{H} y. \tag{74}$$

Below is shown the distribution for $L = 10$, $H = 1$.



Steady-state heat distribution $u(x, y)$: $L = 10$, $H = 1$.

Laplace's equation over a circular region

For physical problems with circular symmetry, e.g. a circular disk, it is convenient to work in planar polar coordinates (r, θ) so that $u = u(r, \theta)$. In polar coordinates, Laplace's equation becomes

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{75}$$

There are several conventions for the domain of definition of the angular coordinate θ. Here, we shall use $-\pi \leq \theta \leq \pi$.

Circular disk, radius a

We shall solve Laplace's equation $\nabla^2 u = 0$ over the region $0 \leq r \leq a$, $-\pi \leq \theta \leq \pi$. There is only one boundary for this region, the outer perimeter $r = a$. Over this boundary, we shall impose the boundary condition

$$\text{BC1:} \qquad u(a, \theta) = f(\theta). \tag{76}$$

Of particular interest will be the case $f(\theta) = T$, constant.

The polar coordinates of any point $(x, y) \neq (0, 0)$ are unique. This is not the case at $(0, 0)$, for which $r = 0$ but θ is not unique. This is simply an illustration of the fact that polar coordinates are singular at $r = 0$. This is also reflected in the Laplacian in Eq. (75) – the point $r = 0$ is a singular point of the differential equation in r that will result from separation of variables.

Because of this singularity at $r = 0$, we'll also need a condition on solutions there: With an eye to physical applications, we impose the condition of boundedness,

$$\text{C2:} \qquad |u(0, \theta)| < \infty. \tag{77}$$

We shall also need periodicity conditions on u that imply continuity and continuous differentiability across the ray $\theta = \pi = -\pi$:

$$\text{C3:} \qquad u(r, -\pi) \;\; = \;\; u(r, \pi) \tag{78}$$

$$\text{C4:} \quad \frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \tag{79}$$

$$\tag{80}$$

As a result, there are *four* conditions on the solution $u(r,\theta)$. Only the outer boundary condition in Eq. (76) is nonhomogeneous. (Exercise.) As a result, we shall try to use the separation of variables technique to construct solutions to (75) with the above boundary conditions.

Following the notation in the book, we write

$$u(r,\theta) = G(r)\phi(\theta). \tag{81}$$

The conditions $C2 - C4$ translate to the following:

$$\text{C2':} \qquad |G(0)\phi(\theta)| < \infty \quad \Rightarrow \quad |G(0)| < \infty, \tag{82}$$

$$\text{C3':} \qquad G(r)\phi(-\pi) = G(r)\phi(\pi) \quad \Rightarrow \quad \phi(-\pi) = \phi(\pi), \tag{83}$$

$$\text{C4':} \qquad G(r)\phi'(-\pi) = G(r)\phi'(\pi) \quad \Rightarrow \quad \phi'(-\pi) = \phi'(\pi). \tag{84}$$

Substitution of (81) into (75) yields

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG(r)}{dr}\right)\phi(\theta) + \frac{1}{r^2}G(r)\frac{d^2\phi}{d\theta^2} = 0. \tag{85}$$

We can separate variables by moving the second term on the LHS to the RHS, multiplying by r^2 and dividing by $G(r)$:

$$\frac{1}{G}r\frac{d}{dr}\left(r\frac{dG(r)}{dr}\right)\phi(\theta) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \mu, \tag{86}$$

where, once again, μ is the separation constant. We'll determine the restrictions on μ shortly.

First, we examine the resulting ϕ-equation,

$$\phi'' + \mu\phi = 0, \qquad \phi(\pi) = \phi(-\pi), \qquad \phi'(\pi) = \phi'(-\pi). \tag{87}$$

The general solution to this DE is

$$\phi(\theta) = C_1\cos(\sqrt{\mu}\theta) + C_2\sin(\sqrt{\mu}\theta). \tag{88}$$

The first condition (C3') implies that

$$C_1\cos(\sqrt{\mu}\pi) + C_2\sin(\sqrt{\mu}\pi) = C_1\cos(\sqrt{\mu}(-\pi)) + C_2\sin(\sqrt{\mu}(-\pi)). \tag{89}$$

Since cos is an even function, we have that

$$C_2\sin(\sqrt{\mu}\pi) = 0. \tag{90}$$

The second condition (C4') implies that

$$-C_1 \sin(\sqrt{\mu}\pi) + C_2 \cos(\sqrt{\mu}\pi) = -C_1 \sin(\sqrt{\mu}(-\pi)) + C_2 \cos(\sqrt{\mu}(-\pi)). \tag{91}$$

Once again, since cos is an even function, we have that

$$C_1 \sin(\sqrt{\mu}\pi) = 0. \tag{92}$$

Therefore, conditions (90) and (92) must be satisfied simultaneously. Since C_1 and C_2 cannot be both zero (otherwise the solution is the trivial zero solution), it follows that

$$\sqrt{\mu}\pi = n\pi, \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{93}$$

This, in turn, implies that the separation constant may assume the discrete values

$$\mu = \mu_n = n^2, \qquad n = 0, 1, 2, \cdots. \tag{94}$$

(The negative values of n yield the same values of μ.)

We may separate the sin and cos solutions into the sets:

$$\sin(n\theta), \qquad n = 1, 2, \cdots, \tag{95}$$

and

$$\cos(n\theta), \qquad n = 0, 1, 2, \cdots. \tag{96}$$

(Note that $n = 0$ is excluded from the sin case since it yields the trivial zero solution). Another way to express this set is as follows:

$$1, \quad \cos(n\theta), \quad \sin(n\theta), \qquad n = 1, 2, \cdots. \tag{97}$$

Note that these functions form an *orthogonal set* on $[-\pi, \pi]$.