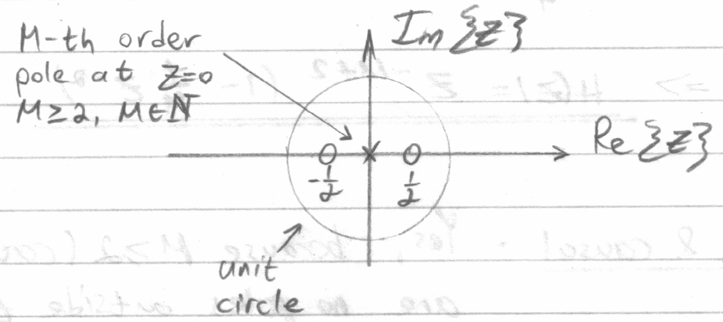
# ECE413 - Final Exam.

Topics: Transform analysis, DFT, spectral analysis, STFT, filter design.

## Q1 - Transform analysis

Consider the following zero-pole plot of $H(z)$:




It is given that $H(1) = 3/4$ and the ROC is $|z| > 0$.

(a) Determine $H(z)$.

(b) Determine whether $H(z)$ is:

stable & causal, IIR, FIR, minimum-phase, all-pass and GLP (if so determine its type). Explain your answer.

(c) Consider the following systems:

$$x[n] \longrightarrow \boxed{H(z)} \xrightarrow{W[n]} \boxed{\downarrow 2} \longrightarrow y[n]$$

$$x[n] \longrightarrow \boxed{\downarrow 2} \xrightarrow{V[n]} \boxed{G(z)} \longrightarrow r[n]$$

Find $G(z)$ such that $y[n] = r[n]$.

## Answer

(a) $H(z) = A \cdot z^{-M}\left(z - \frac{1}{2}\right)\left(z + \frac{1}{2}\right) =$

$$A \cdot z^{-M}\left(z^2 - \frac{1}{4}\right) =$$

$$A \cdot z^{-M+2}\left(1 - \frac{1}{4} z^{-2}\right)$$

$$H(1) = A \cdot \left(1 - \frac{1}{4}\right) = 3/4 \quad \Rightarrow A = 1.$$

$$\Rightarrow \underline{H(z) = z^{-M+2}\left(1 - \frac{1}{4} z^{-2}\right)}$$

(b) Stable & causal - Yes, because $M \geq 2$ (causality) and there are no poles outside the unit circle.

IIR - No, no poles involved.

FIR - Yes, two tap response.

Minimum-phase - Yes, all zeros and poles are inside the unit circle.

All-pass - No, zeros and poles should be conjugate reciprocal pairs.

GLP - No, If $z_0 \in \mathbb{R}$ & $z_0 \neq 1$ then $z_0$ and $(z_0)^{-1}$ should both be zeros for GLP.

(c) Solve by using $X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j\frac{\omega-2\pi i}{M}})$

$$Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\frac{\omega}{2}})\cdot H(e^{j\frac{\omega}{2}}) + X(e^{j\frac{\omega-2\pi}{2}})\, H(e^{j\frac{\omega-2\pi}{2}})\right]$$

$$r(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\frac{\omega}{2}}) + X(e^{j\frac{\omega-2\pi}{2}})\right] G(e^{j\omega})$$

$$H(e^{j\omega}) = e^{j\omega(2-M)}\cdot\left(1-\frac{1}{4}e^{-j2\omega}\right)$$

$$H(e^{j\frac{\omega}{2}}) = e^{j\omega(1-\frac{M}{2})}\left(1-\frac{1}{4}e^{-j\omega}\right)$$

$$H(e^{j\frac{\omega-2\pi}{2}}) = e^{j\omega(1-\frac{M}{2})}\cdot e^{-j\pi(2-M)}\left(1-\frac{1}{4}e^{-j\omega}\cdot e^{j2\pi}\right)$$

$$= e^{j\omega(1-\frac{M}{2})}\, e^{jM\pi}\left(1-\frac{1}{4}e^{-j\omega}\right)$$

<u>If M is even</u> $H(e^{j\frac{\omega-2\pi}{2}}) = H(e^{j\frac{\omega}{2}})$

and $\underline{\underline{G(e^{j\omega}) = H(e^{j\frac{\omega}{2}})}}$

## $Q_2$ - DFT

Consider the following two sequences:

$$x[n] = \{-1, -3, -1, 5\}_{n=0}^{3} \qquad w[n] = \{0, 7, 0, 5\}_{n=0}^{3}$$

(a) Determine if a sequence $y[n]$ that satisfies

$$x \circledast y = w \quad \text{can be fund.}$$

If so, find $y[n]$. If not prove it does not exist.

(b) Given that $W[k] = DFT_4\{w[n]\}_0^3 = Im\{G[k]\}_0^3$

Find $g[3] - g[1]$.

(c) Define $Q[k] = W[(k-2) \bmod 4] \cdot W_4^{2k}$

Find $\{q[n]\}_{n=0}^{3}$.

Answer

(a) Solve by using DFT: $X[k] \cdot Y[k] = W[k]$

$$X[k=0] = \sum_{n=0}^{3} x[n] = 0$$

$$W[k=0] = 12$$

$y[0]$ that satisfies $0 \cdot y[0] = 12$ does not exist.

$\Rightarrow$ No such sequence exists!

(b) $Im\{G[k]\} = \frac{1}{2}(G[k] - G^*[k])$

$$w[n] = IDFT_4\{\tfrac{1}{2}(G[k] - G^*[k])\}$$

$$= \tfrac{1}{2}(g[n] - g^*[-n \bmod 4])$$

Proof: $IDFT_N\{G^*[k]\} = \frac{1}{N}\sum_{k=0}^{N-1} G^*[k]e^{j\frac{2\pi}{N}nk}$

$$= \left(\frac{1}{N}\sum_{k=0}^{N-1} G[k]e^{j\frac{2\pi}{N}(-n)k}\right)^* = g^*[-n \bmod N].$$

$$w[3] = \tfrac{1}{2}(g[3] - g[1]) = 5$$

$$\Rightarrow \underline{\underline{g[3] - g[1] = 10}}$$

(c) $W[(k-2) \bmod 4] \xrightarrow{IDFT_4} w[n]\cdot e^{j\frac{2\pi}{4}\cdot 2\cdot n} = w[n]e^{jn\pi}$

$$W[k]\cdot W_4^{2k} = W[k]e^{-j\frac{2\pi}{4}\cdot 2k} \xrightarrow{IDFT_4} w[(n-2) \bmod 4]$$

$$\Rightarrow W[(k-2)\bmod 4]\cdot W_4^{2k} \xrightarrow{IDFT_4} w[(n-2)\bmod 4]\cdot e^{j[(n-2)\bmod 4]\pi}$$

$q[0] = 0$ $\qquad q[1] = w[3]\cdot e^{j3\pi} = -5$

$q[2] = 0$ $\qquad q[3] = w[1]\cdot e^{j\pi} = -7$

$$\underline{\underline{\{q[n]\}_{n=0}^{3} = \{0, -5, 0, -7\}_{n=0}^{3}}}$$

## Q3 - spectral analysis

A DSP engineer would like to use windowing and zero-padding to analize the DFT of a sampled signal.

(a) Given that the signal was sampled using a sampling frequency of 128 Hz, and the engineer intends to apply the DFT of length 64 to the signal, what is the minimal distance (in Hz) between two <u>distinguishable</u> frequency components?

(b) Which of the following two is the proper procedure:
(1) Zero-pad, multiply by a window, apply FFT.
(2) multiply by a window, zero-pad, apply DFT.

<u>Answer</u>

(a) The engineer is using the rectangular window.
The main lobe width is given by

$$\frac{4\pi}{N+1} \quad \text{where } N+1 \text{ is the sequence length.}$$

$$\frac{1}{128} \cdot 2\pi \cdot \Delta f = \frac{4\pi}{64}$$

$$\Delta f = \frac{4 \cdot 128k}{2 \cdot 64} = 4 \text{ Hz}$$

(b) The proper procedure would be to multiply the sequence by a window, zero-pad and apply the DFT transform.

Zero padding first will multiply a part of the window by zero and will create a new window with unknown and undesired properties.

Q4 - STFT

In STFT analysis the DTFT is calculated for windowed portions of a sequence $x[n]$, namely

$$X[n,\lambda) = \text{DTFT}\{x[n+m]\cdot w[m]\} = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$

Instead, one can apply the DTFT to the autocorrelation sequence of the windowed sequence namely,

$$a[n,m] = \sum_{r=-\infty}^{\infty} x[n+r]\,w[r]\cdot x[n+m+r]\,w[m+r]$$

Find $A[n,\lambda)$, the DTFT of $a[n,m]$, in terms of $X[n,\lambda)$.

Assume real sequence and window.

Answer

$$\sum_{m=-\infty}^{\infty} a[n,m] e^{-j\lambda m} =$$

$$= \sum_{m=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} X[n+r] W[r] X[n+m+r] W[m+r] e^{-j\lambda m}$$

$$= \sum_{r=-\infty}^{\infty} X[n+r] W[r] \sum_{m=-\infty}^{\infty} X[n+m+r] W[m+r] e^{-j\lambda m}$$

$m+r=l$

$$= \sum_{r=-\infty}^{\infty} X[n+r] W[r] \underbrace{\sum_{l=-\infty}^{\infty} X[n+l] W[l] e^{-j\lambda l}}_{X[n,\lambda)} \cdot e^{+j\lambda r}$$

$$= \sum_{r=-\infty}^{\infty} X[n+r] W[r] \cdot X[n,\lambda) \cdot e^{j\lambda r} =$$

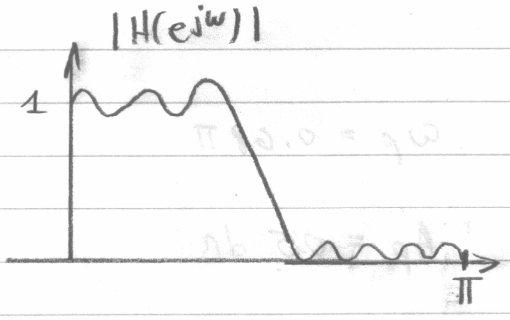
$$= X[n,\lambda) \cdot \sum_{r=-\infty}^{\infty} X[n+r] W[r] e^{j\lambda r} =$$

$$= X[n,\lambda) \Big( \underbrace{\sum_{r=-\infty}^{\infty} X[n+r] W[r] e^{-j\lambda r}}_{X[n,\lambda)} \Big)^* =$$

$$= X[n,\lambda) \cdot (X[n,\lambda))^* = \underline{\underline{|X[n,\lambda)|^2}}$$

Q5- Filter design

Consider the following response of an FIR GLP filter:




Which of the following is correct:

(a) The filter is type I.

(b) The filter is type II.

(c) The filter is type III.

(d) The type can not be determined from the graph.

Answer

(d)

Type III can not be LPF because $H(0) = H(\pi) = 0$.
Type I and II can both be LPF, Type II must have $H(\pi)=0$ which can be seen in the sketch.
⇒ can not determine between Type I and Type II.