# TRANSFORM ANALYSIS OF LINEAR TIME-INVARIANT SYSTEMS

LTI system can be completely characterized in the time domain by its impulse response $h[n]$, with the output $y[n]$ due to a given input $x[n]$ specified through the convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \tag{5.1}$$

transform, and we showed that $Y(z)$, the $z$-transform of the output of an LTI system, is related to $X(z)$, the $z$-transform of the input, and $H(z)$, the $z$-transform of the system impulse response, by

$$Y(z) = H(z)X(z), \tag{5.2}$$

$H(z)$ is referred to as the *system function.* Since the $z$-transform and a sequence form a unique pair, it follows that any LTI system is completely characterized by its system function, again assuming convergence.

## THE FREQUENCY RESPONSE OF LTI SYSTEMS

The frequency response $H(e^{j\omega})$ of an LTI system was defined as the complex gain (eigenvalue) that the system applies to the complex exponential input (eigenfunction) $e^{j\omega n}$. Furthermore,

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \tag{5.3}$$

where $X(e^{j\omega})$ and $Y(e^{j\omega})$ are the Fourier transforms of the system input and output, respectively. With the frequency response expressed in polar form, the magnitude and phase of the Fourier transforms of the system input and output are related by

$$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|, \tag{5.4a}$$

$$\sphericalangle Y(e^{j\omega}) = \sphericalangle H(e^{j\omega}) + \sphericalangle X(e^{j\omega}). \tag{5.4b}$$

$|H(e^{j\omega})|$ is referred to as the *magnitude* response or the *gain* of the system, and $\sphericalangle H(e^{j\omega})$ is referred to as the *phase response* or *phase shift* of the system. For example, the ideal lowpass filter was defined as the discrete-time linear time-invariant system whose frequency response is

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases} \tag{5.5}$$

The corresponding impulse response was shown in to be

$$h_{\text{lp}}[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty. \tag{5.6}$$

Analogously, the *ideal highpass filter* is defined as

$$H_{\text{hp}}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c, \\ 1, & \omega_c < |\omega| \le \pi, \end{cases} \tag{5.7}$$

and since $H_{\text{hp}}(e^{j\omega}) = 1 - H_{\text{lp}}(e^{j\omega})$, its impulse response is

$$h_{\text{hp}}[n] = \delta[n] - h_{\text{lp}}[n] = \delta[n] - \frac{\sin \omega_c n}{\pi n}. \tag{5.8}$$

The ideal lowpass filters are noncausal, and their impulse responses extend from $-\infty$ to $+\infty$. Therefore, it is not possible to compute the output of either the ideal lowpass or the ideal highpass filter either recursively or nonrecursively; i.e., the systems are not *computationally realizable.*

### Phase Distortion and Delay

To understand the effect of the phase of a linear system, let us first consider the ideal delay system.

$$|H_{\text{id}}(e^{j\omega})| = 1, \tag{5.11a}$$

$$\sphericalangle H_{\text{id}}(e^{j\omega}) = -\omega n_d, \qquad |\omega| < \pi, \tag{5.11b}$$

with periodicity $2\pi$ in $\omega$ assumed. For now, we will assume that $n_d$ is an integer.

In many applications, delay distortion would be considered a rather mild form of phase distortion, since its effect is only to shift the sequence in time.

For example, an ideal lowpass filter with linear phase would be defined as

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi. \end{cases} \tag{5.12}$$

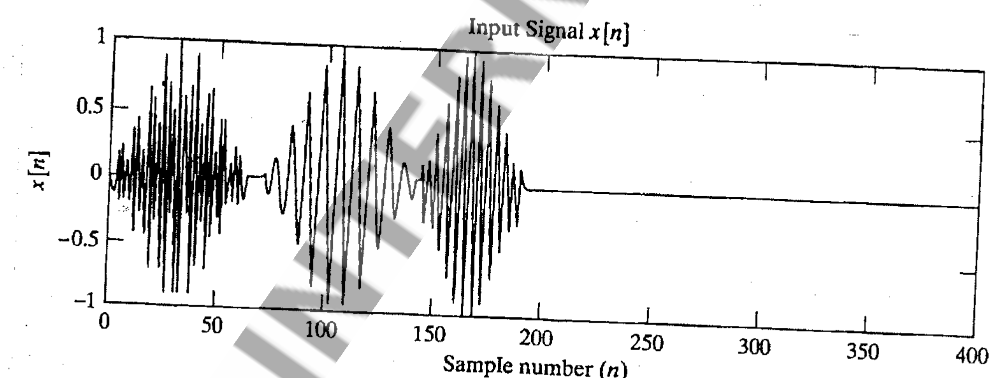
Its impulse response is

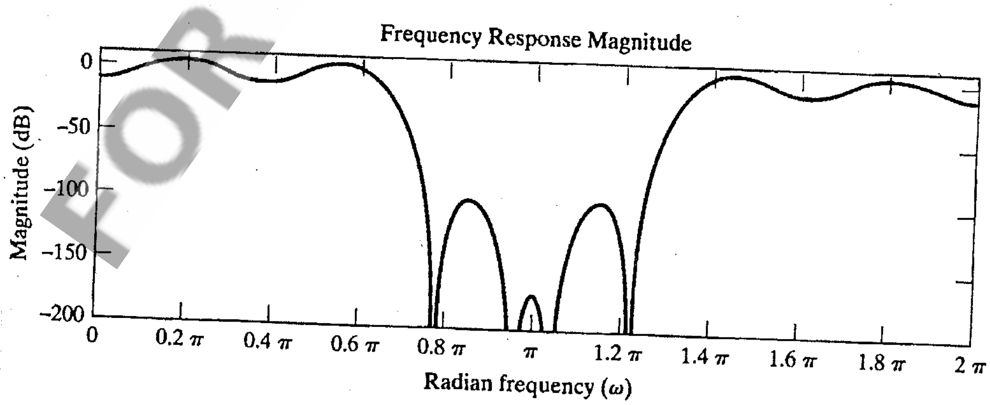
$$h_{\text{lp}}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \qquad -\infty < n < \infty. \tag{5.13}$$

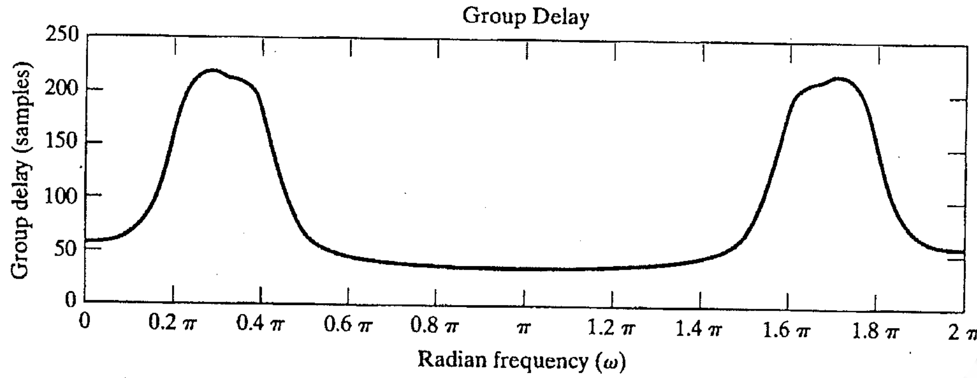
With phase specified as a continuous function of $\omega$, the group delay of a system is defined as

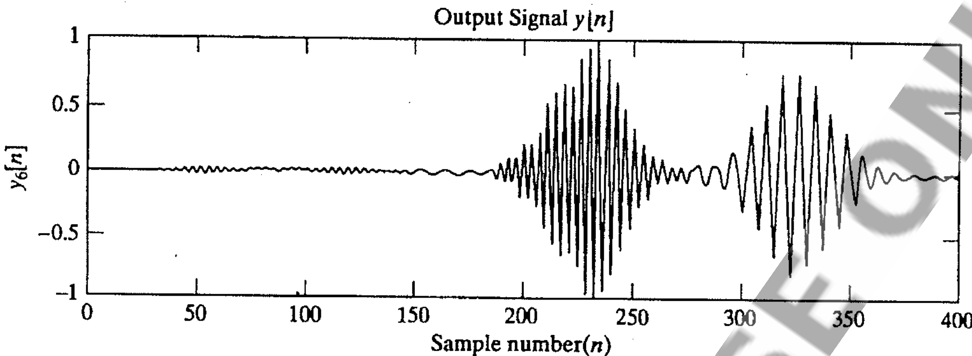
$$\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}. \tag{5.15}$$

The deviation of the group delay from a constant indicates the degree of nonlinearity of the phase.








## SYSTEM FUNCTIONS FOR SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

While ideal frequency-selective filters are useful conceptually, they cannot be implemented with finite computation. Therefore, it is of interest to consider a class of systems that can be implemented as approximations to ideal frequency-selective filters.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z),$$

$$\left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z).$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}. \qquad (5.19)$$

Each of the factors $(1 - c_k z^{-1})$ in the numerator contributes a zero at $z = c_k$ and a pole at $z = 0$. Similarly, each of the factors $(1 - d_k z^{-1})$ in the denominator contributes a zero at $z = 0$ and a pole at $z = d_k$.

Suppose that the system function of a linear time-invariant system is

$$H(z) = \frac{(1+z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)}. \tag{5.20}$$

$$H(z) = \frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}} = \frac{Y(z)}{X(z)}.$$

Thus,

$$\left(1+\tfrac{1}{4}z^{-1}-\tfrac{3}{8}z^{-2}\right)Y(z) = (1+2z^{-1}+z^{-2})X(z),$$

and the difference equation is

$$y[n] + \tfrac{1}{4}y[n-1] - \tfrac{3}{8}y[n-2] = x[n] + 2x[n-1] + x[n-2].$$

## Stability and Causality

For a given ratio of polynomials, each possible choice for the region of convergence will lead to a different impulse response, but they will all correspond to the same difference equation. However, if we assume that the system is causal, it follows that $h[n]$ must be a right-sided sequence, and therefore, the region of convergence of $H(z)$ must be outside the outermost pole. Alternatively, if we assume that the system is stable, then

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{5.23}$$

Since Eq. (5.23) is identical to the condition that

$$\sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty \tag{5.24}$$

for $|z| = 1$, the condition for stability is equivalent to the condition that the ROC of $H(z)$ include the unit circle.

In order for a linear time-invariant system whose input and output satisfy a difference equation of the form of Eq. (5.16) to be both causal and stable, the ROC of the corresponding system function must be outside the outermost pole *and* include the unit circle. Clearly, this requires that all the poles of the system function be inside the unit circle.

## Inverse Systems

For a given linear time-invariant system with system function $H(z)$, the corresponding inverse system is defined to be the system with system function $H_i(z)$ such that if it is cascaded with $H(z)$, the overall effective system function is unity; i.e.,

$$G(z) = H(z)H_i(z) = 1. \tag{5.27}$$

Many systems do have inverses, and the class of systems with rational system functions provides a very useful and interesting example. Specifically, consider

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_kz^{-1})}{\prod_{k=1}^{N}(1-d_kz^{-1})}, \tag{5.31}$$

with zeros at $z = c_k$ and poles at $z = d_k$, in addition to possible zeros and/or poles at $z = 0$ and $z = \infty$.

Then

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}; \tag{5.32}$$

i.e., the poles of $H_i(z)$ are the zeros of $H(z)$ and vice versa.

Let $H(z)$ be

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}$$

with ROC $|z| > 0.9$. Then $H_i(z)$ is

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}.$$

Since $H_i(z)$ has only one pole, there are only two possibilities for its ROC, and the only choice for the ROC of $H_i(z)$ that overlaps with $|z| > 0.9$ is $|z| > 0.5$. Therefore, the impulse response of the inverse system is

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1].$$

In this case, the inverse system is both causal and stable.

Then

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}; \qquad (5.32)$$

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In this case, the inverse system is both causal and stable.

A generalization is that if $H(z)$ is a causal system with zeros at $c_k$, $k = 1, \ldots, M$, then its inverse system will be causal if and only if we associate the region of convergence,

$$|z| > \max_k |c_k|,$$

with $H_i(z)$. If we also require that the inverse system be stable, then the region of convergence of $H_i(z)$ must include the unit circle. Therefore, it must be true that

$$\max_k |c_k| < 1;$$

i.e., all the zeros of $H(z)$ must be inside the unit circle. Thus, a linear time-invariant system is stable and causal and also has a stable and causal inverse if and only if both the poles and the zeros of $H(z)$ are inside the unit circle. Such systems are referred to as *minimum-phase* systems.

Recall that any rational function of $z^{-1}$ with only first-order poles can be expressed in the form

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}, \tag{5.34}$$

where the terms in the first summation would be obtained by long division of the denominator into the numerator and would be present only if $M \geq N$.

If the system is assumed to be causal, then the ROC is outside all of the poles in Eq. (5.34), and it follows that

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n], \tag{5.35}$$

where the first summation is included only if $M \geq N$.

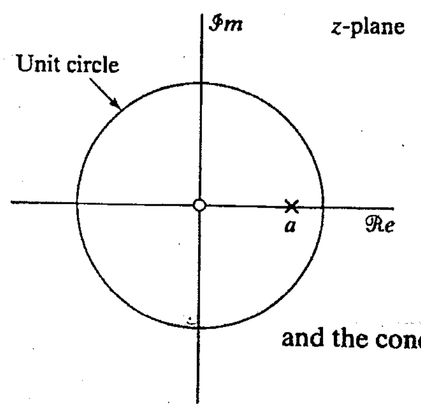
### A First-Order IIR System

Consider a causal system whose input and output satisfy the difference equation

$$y[n] - ay[n-1] = x[n]. \tag{5.36}$$

The system function is (by inspection)

$$H(z) = \frac{1}{1 - az^{-1}}. \tag{5.37}$$




The region of convergence is $|z| > |a|$, and the condition for stability is $|a| < 1$. The inverse $z$-transform of $H(z)$ is

$$h[n] = a^n u[n]. \tag{5.38}$$

For the second class of systems, $H(z)$ has no poles except at $z = 0$; i.e., $N = 0$,

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}. \tag{5.39}$$

(We assume, without loss of generality, that $a_0 = 1$.) In this case, $H(z)$ is determined to within a constant multiplier by its zeros. From Eq. (5.39), $h[n]$ is seen by inspection to be

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \leq n \leq M, \\ 0, & \text{otherwise.} \end{cases} \tag{5.40}$$

In this case, the impulse response is finite in length; i.e., it is zero outside a finite interval. Consequently, these systems are called *finite impulse response* (FIR) systems. Note that for FIR systems, the difference equation of Eq. (5.16) is identical to the convolution sum, i.e.,

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]. \tag{5.41}$$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}.\qquad H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1-c_k e^{-j\omega})}{\prod_{k=1}^{N}(1-d_k e^{-j\omega})}.$$

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}|1-c_k e^{-j\omega}|}{\prod_{k=1}^{N}|1-d_k e^{-j\omega}|}.$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}),$$

$$|H(e^{j\omega})|^2 = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1-c_k e^{-j\omega})(1-c_k^* e^{j\omega})}{\prod_{k=1}^{N}(1-d_k e^{-j\omega})(1-d_k^* e^{j\omega})}.$$

The function $20\log_{10}|H(e^{j\omega})|$ is referred to as the *log magnitude* of $H(e^{j\omega})$ and is expressed in *decibels* (dB). Sometimes this quantity is called the *gain in dB;* i.e.,

$$\text{Gain in dB} = 20\log_{10}|H(e^{j\omega})|. \qquad (5.51)$$

When $|H(e^{j\omega})| < 1$, the quantity $20\log_{10}|H(e^{j\omega})|$ is negative. This would be the case, for example, in the stopband of a frequency-selective filter. It is common practice to define

$$\begin{aligned}\text{Attenuation in dB} &= -20\log_{10}|H(e^{j\omega})|\\ &= -\text{Gain in dB}.\end{aligned} \qquad (5.52)$$

Another advantage to expressing the magnitude in decibels stems from Eq. (5.4a), which, after taking logarithms of both sides, becomes

$$20\log_{10}|Y(e^{j\omega})| = 20\log_{10}|H(e^{j\omega})| + 20\log_{10}|X(e^{j\omega})|. \qquad (5.53)$$

The phase response for a rational system function has the form

$$\sphericalangle H(e^{j\omega}) = \sphericalangle\left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M}\sphericalangle[1-c_k e^{-j\omega}] - \sum_{k=1}^{N}\sphericalangle[1-d_k e^{-j\omega}]. \qquad (5.54)$$

The corresponding group delay for a rational system function is

$$\text{grd}[H(e^{j\omega})] = \sum_{k=1}^{N}\frac{d}{d\omega}(\arg[1-d_k e^{-j\omega}]) - \sum_{k=1}^{M}\frac{d}{d\omega}(\arg[1-c_k e^{-j\omega}]), \qquad (5.55)$$

where arg[ ] represents the continuous phase.

When the angle of a complex number is computed, with the use of an arctangent subroutine on a calculator or with a computer system subroutine, the principal value is obtained. The principal value of the phase of $H(e^{j\omega})$ is denoted as $\text{ARG}[H(e^{j\omega})]$, where

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi. \tag{5.57}$$

Any other angle that gives the correct complex value of the function $H(e^{j\omega})$ can be represented in terms of the principal value as

$$\sphericalangle H(e^{j\omega}) = \text{ARG}[H(e^{j\omega})] + 2\pi r(\omega), \tag{5.58}$$

where $r(\omega)$ is a positive or negative integer that can be different at each value of $\omega$.

If the principal value is used to compute the phase response as a function of $\omega$, then $\text{ARG}[H(e^{j\omega})]$ may be a discontinuous function. The discontinuities introduced by taking the principal value will be jumps of $2\pi$ radians.

Except at the discontinuities of $\text{ARG}[H(e^{j\omega})]$ corresponding to jumps between $+\pi$ and $-\pi$,

$$\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\} = \frac{d}{d\omega}\{\text{ARG}[H(e^{j\omega})]\}. \tag{5.62}$$

Consequently, the group delay can be obtained from the principal value by differentiating, except at the discontinuities. Similarly, we can express the group delay in terms of the ambiguous phase $\sphericalangle H(e^{j\omega})$ as

$$\text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}[\sphericalangle H(e^{j\omega})], \tag{5.63}$$

with the interpretation that impulses caused by discontinuities of size $2\pi$ in $\sphericalangle H(e^{j\omega})$ are ignored.

The square of the magnitude of such a factor is

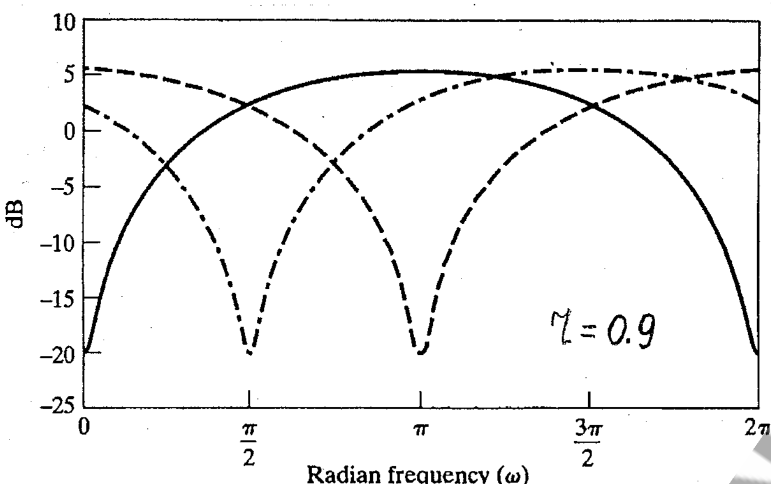
$$|1 - re^{j\theta}e^{-j\omega}|^2 = (1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{j\omega}) = 1 + r^2 - 2r\cos(\omega - \theta). \quad (5.64)$$

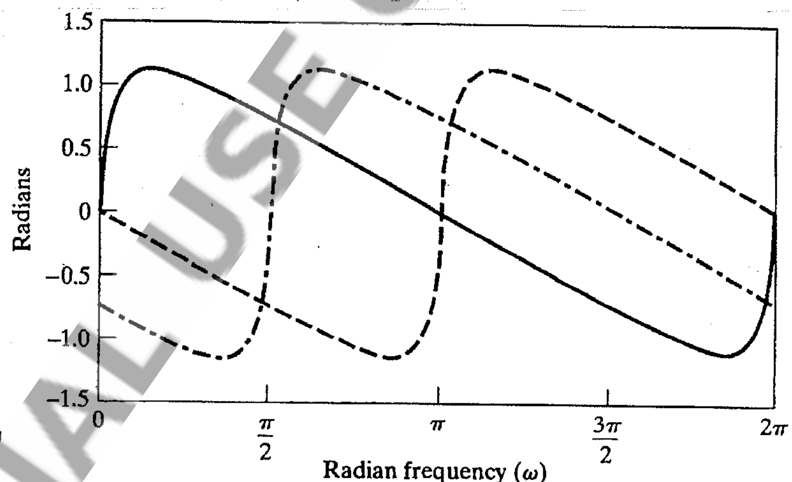
The principal value phase for such a factor is

$$\text{ARG}[1 - re^{j\theta}e^{-j\omega}] = \arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]. \quad (5.66)$$

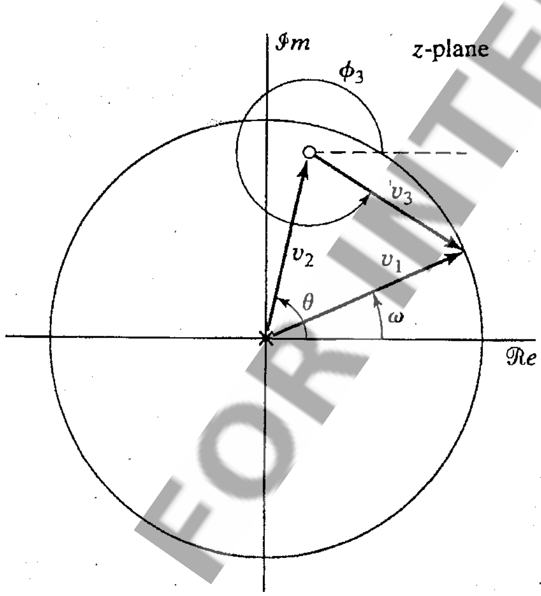
Differentiating the right-hand side of Eq. (5.66) (except at discontinuities) gives the group delay of the factor as

$$\text{grd}[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega - \theta)}{1 + r^2 - 2r\cos(\omega - \theta)} = \frac{r^2 - r\cos(\omega - \theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}. \quad (5.67)$$




A simple geometric construction is often very useful in approximate sketching of frequency-response functions directly from the pole–zero plot.



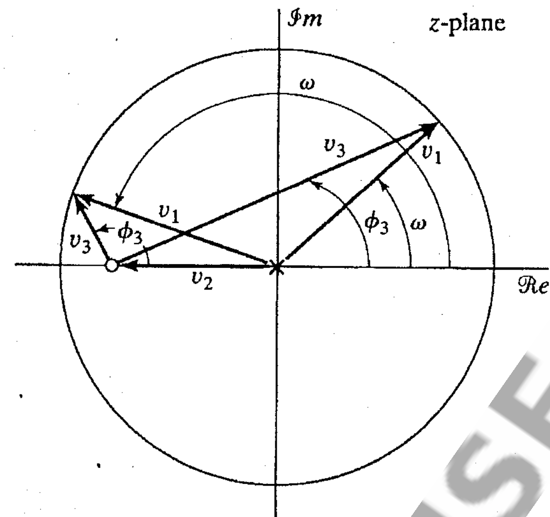

$$|1 - re^{j\theta}e^{-j\omega}| = \left|\frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}\right| = \frac{|v_3|}{|v_1|}.$$

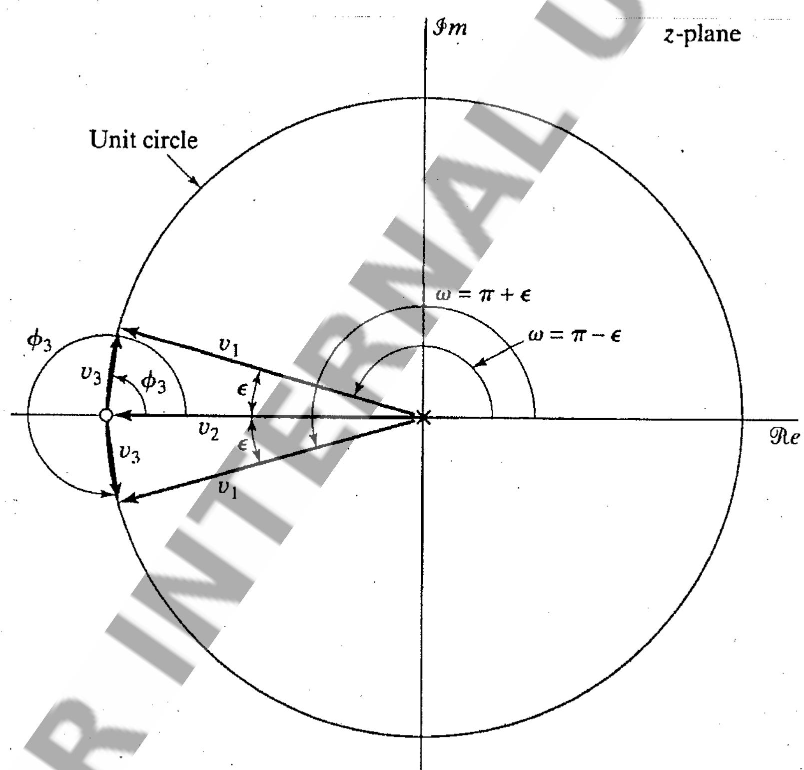
$$H(z) = (1 - re^{j\theta}z^{-1}) = \frac{(z - re^{j\theta})}{z}, \qquad r < 1.$$

The corresponding phase is

$$\begin{aligned}\sphericalangle(1 - re^{j\theta}e^{-j\omega}) &= \sphericalangle(e^{j\omega} - re^{j\theta}) - \sphericalangle(e^{j\omega}) = \sphericalangle(v_3) - \sphericalangle(v_1) \\ &= \phi_3 - \phi_1 = \phi_3 - \omega.\end{aligned} \quad (5.70)$$

Typically, a vector such as $v_3$ from a zero to the unit circle is referred to as a zero vector, and a vector from a pole to the unit circle is called a pole vector. Thus, the contribution of a single zero factor $(1-re^{j\theta}z^{-1})$ to the magnitude function at frequency $\omega$ is the length of the zero vector $v_3$ from the zero to the point $z = e^{j\omega}$ on the unit circle. The vector has minimum length when $\omega = \theta$. This accounts for the sharp dip in the magnitude function at $\omega = \theta$. Note that the pole vector $v_1$ from the pole at $z = 0$ to $z = e^{j\omega}$ always has unit length. Thus, it does not have any effect on the magnitude response.

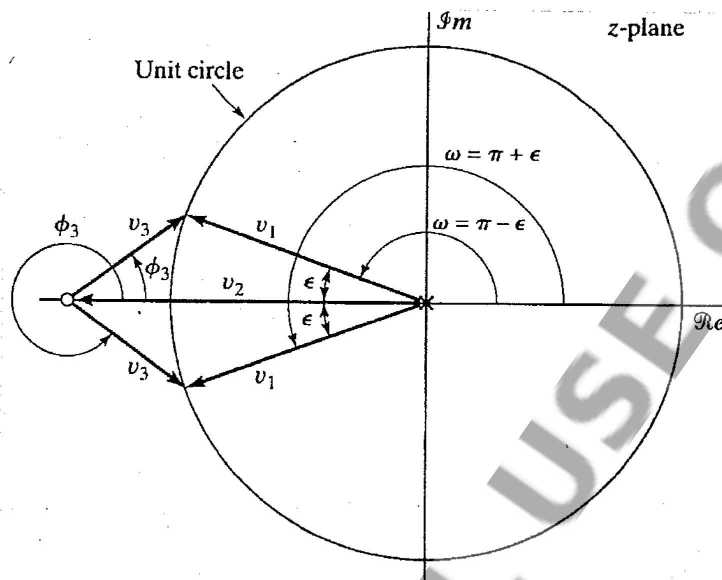


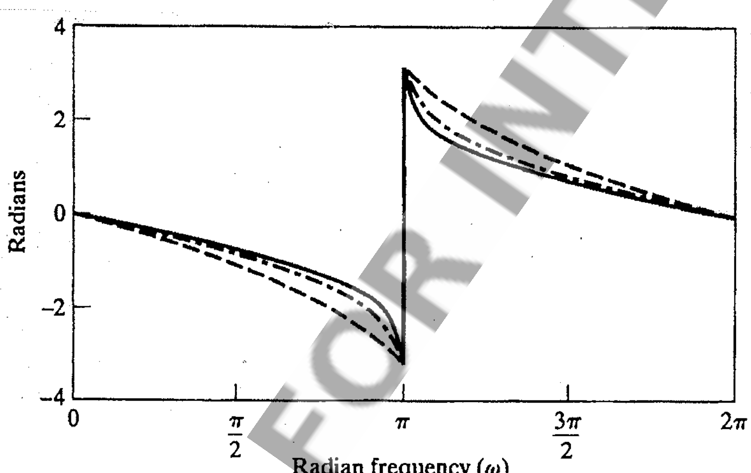

The geometric construction for a zero on the unit circle at $z = -1$ is shown. Indicated are vectors for two different frequencies, $\omega = (\pi - \varepsilon)$ and $\omega = (\pi + \varepsilon)$, where $\varepsilon$ is small. Two observations can be made. First, the length of the vector $v_3$ approaches zero as $\omega$ approaches the angle of the zero vector ($\varepsilon \to 0$). Therefore, the multiplicative contribution to the frequency response is zero ($-\infty$ dB). Second, the vector $v_3$ changes its angle discontinuously by $\pi$ radians as $\omega$ goes from $(\pi - \varepsilon)$ to $(\pi + \varepsilon)$.

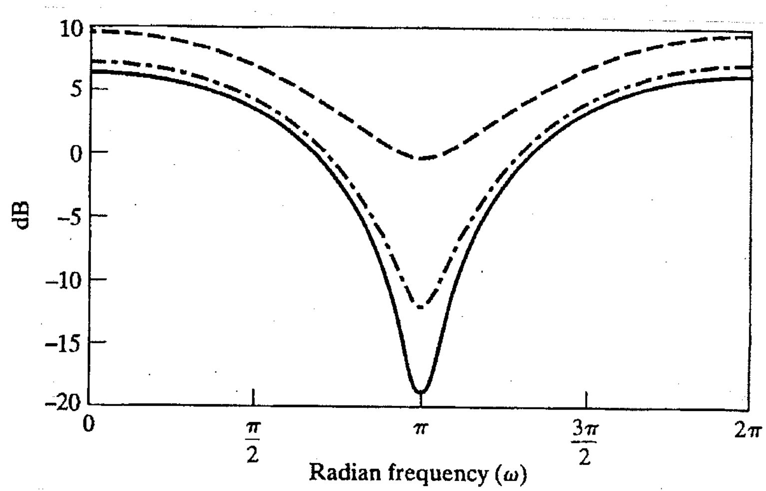
If $r > 1$, the log magnitude function behaves similarly to the case $r < 1$; i.e., it dips more sharply as $r \to 1$.

The phase function in Figure shows a discontinuity of $2\pi$ radians at $\omega = \theta$ for all values of $r > 1$. The source of this discontinuity can be seen from Figure 5.14, which shows vectors for $\omega = (\pi - \varepsilon)$ and $\omega = (\pi + \varepsilon)$. Note that the pole vector $v_1$ has an angle of $\omega$, which varies continuously from $\omega = 0$ to $\omega = 2\pi$. The angle of the zero vector $v_3$ is labeled $\phi_3$ in the figure. If this angle is measured positively in the counterclockwise direction, the figure shows that $\phi_3$ jumps from zero to $2\pi$ radians as $\omega$ goes from $(\pi - \varepsilon)$ to $(\pi + \varepsilon)$.




If the factor represents a pole of $H(z)$, then all the contributions will enter with opposite sign. Thus, the contribution of a pole $z = re^{j\theta}$ would be the negative of the curves in Figures 5.8 and 5.11. Instead of dipping toward zero ($-\infty$ dB), the magnitude function would peak around $\omega = \theta$. The dependence on $r$ would be the same as for a zero; i.e., the closer $r$ is to 1, the more peaked will be the contribution to the magnitude function. For stable and causal systems, there will, of course, be no poles outside the unit circle; i.e., $r$ will always be less than 1.



# LINEAR SYSTEMS WITH GENERALIZED LINEAR PHASE

In designing filters and other signal-processing systems that pass some portion of the frequency band undistorted, it is desirable to have approximately constant frequency-response magnitude and zero phase in that band. For causal systems, zero phase is not attainable, and consequently, some phase distortion must be allowed.

For example, consider a more general frequency response with linear phase, i.e.,

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha}, \qquad |\omega| < \pi. \tag{5.126}$$

The signal $x[n]$ is filtered by the zero-phase frequency response $|H(e^{j\omega})|$, and the filtered output is then "time shifted" by the (integer or noninteger) amount $\alpha$. Suppose, for example, that $H(e^{j\omega})$ is the linear-phase ideal lowpass filter

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi. \end{cases} \tag{5.127}$$

The corresponding impulse response is

$$h_{\text{lp}}[n] = \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}. \tag{5.128}$$

Specifically, a system is referred to as a *generalized linear-phase system* if its frequency response can be expressed in the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega+j\beta}, \tag{5.135}$$

where $\alpha$ and $\beta$ are constants and $A(e^{j\omega})$ is a real (possibly bipolar) function of $\omega$.

However, if we ignore any discontinuities that result from the addition of constant phase over all or part of the band $|\omega| < \pi$, then such a system can be characterized by constant group delay. That is, the class of systems such that

$$\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\} = \alpha \tag{5.136}$$

have linear phase of the more general form

$$\arg[H(e^{j\omega})] = \beta - \omega\alpha, \qquad 0 < \omega < \pi, \tag{5.137}$$

where $\beta$ and $\alpha$ are both real constants.

### *Type I FIR Linear-Phase Systems*

A type I system is defined as a system that has a symmetric impulse response

$$h[n] = h[M-n], \qquad 0 \le n \le M, \tag{5.148}$$

with $M$ an even integer. The delay $M/2$ is an integer. The frequency response is

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}. \tag{5.149}$$

By applying the symmetry condition, Eq. (5.148), the sum in Eq. (5.149) can be rewritten in the form

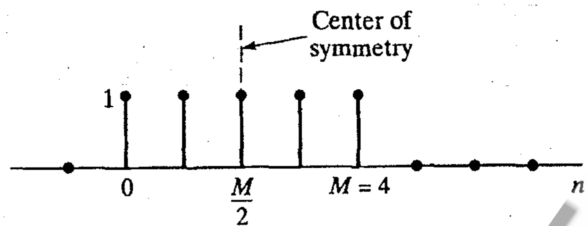
$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right), \tag{5.150a}$$

where

$\beta = 0,\ \tau = \frac{M}{2}$

$$a[0] = h[M/2], \tag{5.150b}$$

$$a[k] = 2h[(M/2) - k], \qquad k = 1, 2, \ldots, M/2. \tag{5.150c}$$




If the impulse response is

$$h[n] = \begin{cases} 1, & 0 \le n \le 4, \\ 0, & \text{otherwise}, \end{cases}$$

The frequency response is

$$H(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = \frac{1 - e^{-j\omega 5}}{1 - e^{-j\omega}}$$

$$= e^{-j\omega 2}\frac{\sin(5\omega/2)}{\sin(\omega/2)}. \tag{5.156}$$

***Type II FIR Linear-Phase Systems***

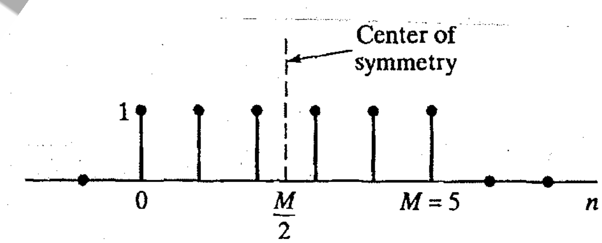
A type II system has a symmetric impulse response , with $M$ an odd integer. $H(e^{j\omega})$ for this case can be expressed as

$$H(e^{j\omega}) = e^{-j\omega M/2}\left\{\sum_{k=1}^{(M+1)/2} b[k]\cos\left[\omega\left(k - \tfrac{1}{2}\right)\right]\right\}, \tag{5.151a}$$

where

$\beta = 0,\ \tau = \frac{M}{2} = \text{"}N + \frac{1}{2}\text{"}$

$$b[k] = 2h[(M+1)/2 - k], \qquad k = 1, 2, \ldots, (M+1)/2. \tag{5.151b}$$




If the length of the impulse response of the previous example is extended by one sample, we obtain the impulse response

$$H(e^{j\omega}) = e^{-j\omega 5/2}\frac{\sin(3\omega)}{\sin(\omega/2)}. \tag{5.157}$$

***Type III FIR Linear-Phase Systems***

If the system has an antisymmetric impulse response

$$h[n] = -h[M-n], \qquad 0 \le n \le M, \tag{5.152}$$

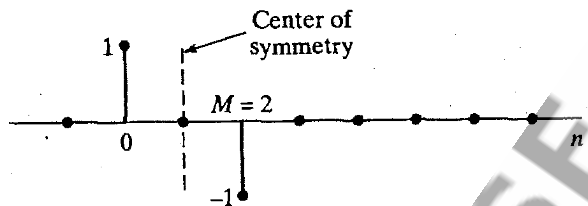
with $M$ an even integer, then $H(e^{j\omega})$ has the form

$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{M/2} c[k]\sin\omega k\right], \tag{5.153a}$$

where

$\beta = \frac{\pi}{2}, \tau = \frac{M}{2}$

$$c[k] = 2h[(M/2)-k], \qquad k = 1, 2, \ldots, M/2. \tag{5.153b}$$




If the impulse response is

$$h[n] = \delta[n] - \delta[n-2], \tag{5.158}$$

as in Figure 5.36(c), then

$$\begin{aligned} H(e^{j\omega}) &= 1 - e^{-j2\omega} \\ &= j[2\sin(\omega)]e^{-j\omega}. \end{aligned} \tag{5.159}$$

***Type IV FIR Linear-Phase Systems***

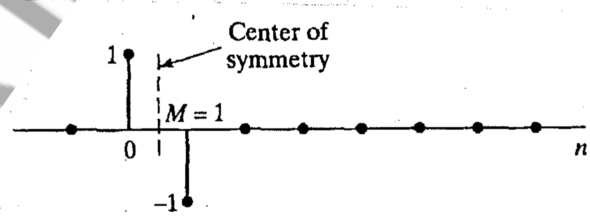
If the impulse response is antisymmetric as in Eq. (5.152) and $M$ is odd, then

$$H(e^{j\omega}) = je^{-j\omega M/2}\left[\sum_{k=1}^{(M+1)/2} d[k]\sin\left[\omega\left(k-\tfrac{1}{2}\right)\right]\right], \tag{5.154a}$$

where

$\beta = \frac{\pi}{2}, \tau = \frac{M}{2} = "N + \frac{1}{2}"$

$$d[k] = 2h[(M+1)/2-k], \qquad k = 1, 2, \ldots, (M+1)/2. \tag{5.154b}$$




In this case (Figure 5.36(d)), the impulse response is

$$h[n] = \delta[n] - \delta[n-1], \tag{5.160}$$

for which the frequency response is

$$\begin{aligned} H(e^{j\omega}) &= 1 - e^{-j\omega} \\ &= j[2\sin(\omega/2)]e^{-j\omega/2}. \end{aligned} \tag{5.161}$$