STRUCTURES FOR DISCRETE-TIME SYSTEMS

As an illustration of the computation associated with a difference equation, consider the system described by the system function

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - az^{-1}}, \qquad |z| > |a|. \tag{6.1}$$

The impulse response of this system is

$$h[n] = b_0 a^n u[n] + b_1 a^{n-1} u[n-1], \tag{6.2}$$

$$y[n] - ay[n-1] = b_0 x[n] + b_1 x[n-1]. \tag{6.3}$$

Since the system has an infinite-duration impulse response, it is not possible to implement the system by discrete convolution. However, rewriting Eq. (6.3) in the form

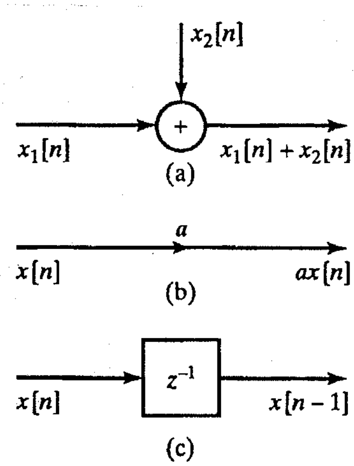
$$y[n] = ay[n-1] + b_0 x[n] + b_1 x[n-1] \tag{6.4}$$

provides the basis for an algorithm for recursive computation of the output at any time n in terms of the previous output $y[n-1]$, the current input sample $x[n]$, and the previous input sample $x[n-1]$.

However, the algorithm suggested by Eq. (6.4) and its generalization for higher order difference equations is not the only computational algorithm for implementing a particular system, and often it is not the most preferable. As we will see, an unlimited variety of computational structures result in the same relation between the input sequence $x[n]$ and the output sequence $y[n]$.

BLOCK DIAGRAM REPRESENTATION OF LINEAR CONSTANT-COEFFICIENT DIFFERENCE EQUATIONS

Therefore, the basic elements required for the implementation of a linear time-invariant discrete-time system are

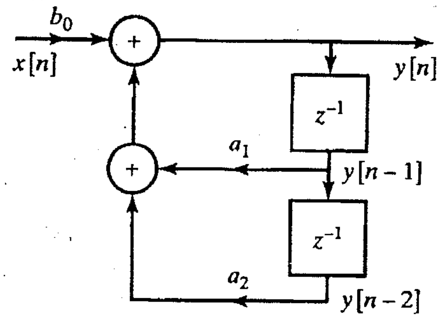


As an example of the representation of a difference equation in terms of the elements in Figure consider the second-order difference equation

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n]. \tag{6.5}$$

The corresponding system function is

$$H(z) = \frac{b_0}{1 - a_1 z^{-1} - a_2 z^{-2}}. \tag{6.6}$$



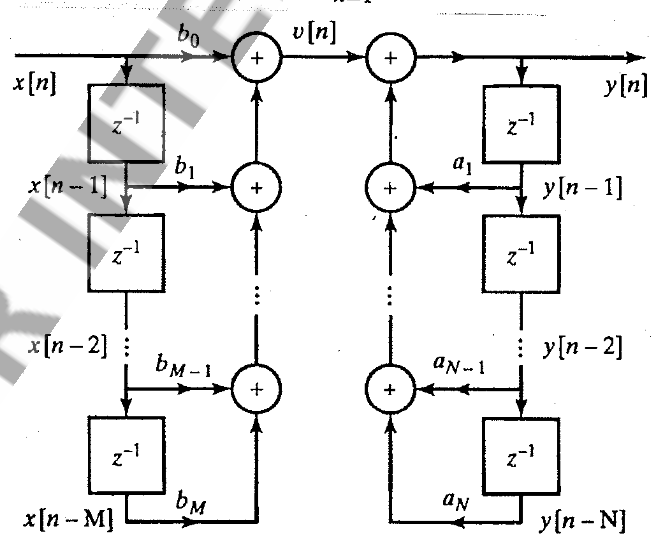
Thus, Figure conveniently depicts the complexity of the associated computational algorithm, the steps of the algorithm, and the amount of hardware required to realize the system.

Example can be generalized to higher order difference equations of the form

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \tag{6.7}$$

with the corresponding system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}. \tag{6.8}$$



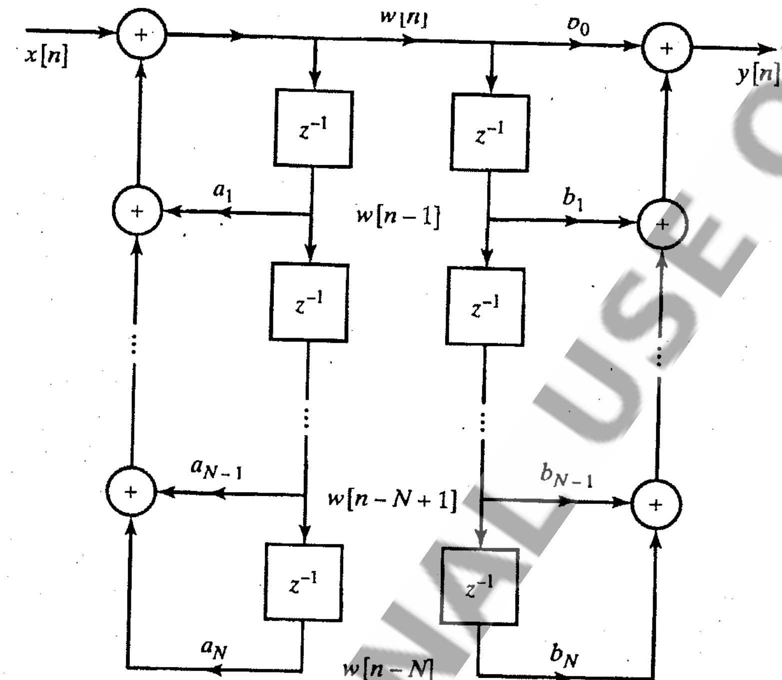
$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]. \tag{6.9}$$

More precisely, it represents the pair of difference equations

$$v[n] = \sum_{k=0}^{M} b_k x[n-k], \tag{6.10a}$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + v[n]. \tag{6.10b}$$

A block diagram can be rearranged or modified in a variety of ways without changing the overall system function. Each appropriate rearrangement represents a *different* computational algorithm for implementing the *same* system.



In terms of the system function $H(z)$, Figure can be viewed as an implementation of $H(z)$ through the decomposition

$$H(z) = H_2(z)H_1(z) = \left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right) \left(\sum_{k=0}^{M} b_k z^{-k} \right) \tag{6.11}$$

or, equivalently, through the pair of equations

$$V(z) = H_1(z)X(z) = \left(\sum_{k=0}^{M} b_k z^{-k} \right) X(z), \tag{6.12a}$$

$$Y(z) = H_2(z)V(z) = \left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right) V(z). \tag{6.12b}$$

or, equivalently, through the equations

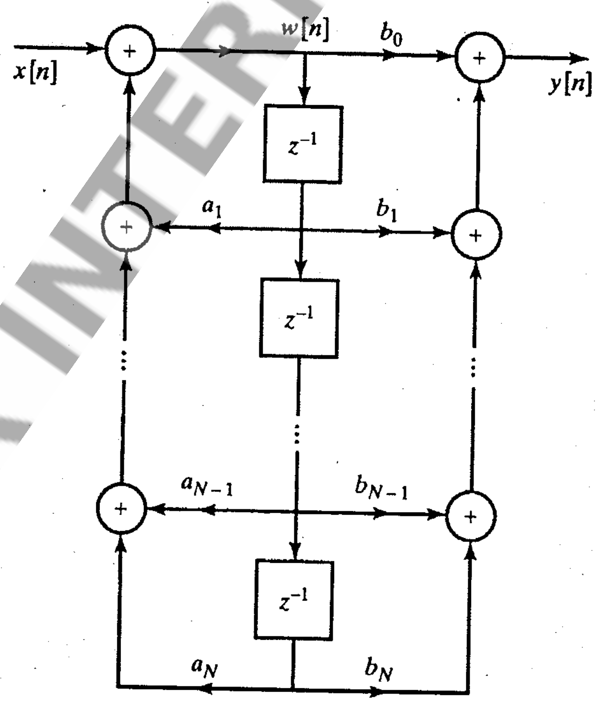
$$W(z) = H_2(z)X(z) = \left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right) X(z), \tag{6.14a}$$

$$Y(z) = H_1(z)W(z) = \left(\sum_{k=0}^{M} b_k z^{-k} \right) W(z). \tag{6.14b}$$

$$w[n] = \sum_{k=1}^{N} a_k w[n-k] + x[n], \tag{6.15a}$$

$$y[n] = \sum_{k=0}^{M} b_k w[n-k]. \tag{6.15b}$$

Theoretically, the order of implementation does not affect the overall system function. However, as we will see, when a difference equation is implemented with finite-precision arithmetic, there can be a significant difference between two systems that are theoretically equivalent. Another important point concerns the number of delay elements in the two systems. Specifically, the minimum number of delays required is, in general, max(N, M). An implementation with the minimum number of delay elements is commonly referred to as a *canonic form* implementation. The non-canonic block diagram in Figure 6.3 is referred to as the *direct form I* implementation of the general Nth-order system because it is a direct realization of the difference equation satisfied by the input $x[n]$ and the output $y[n]$, which in turn can be written directly from the system function by inspection. Figure 6.5 is often referred to as the *direct form II* or *canonic direct form* implementation.



Consider the LTI system with system function

$$H(z) = \frac{1 + 2z^{-1}}{1 - 1.5z^{-1} + 0.9z^{-2}}. \tag{6.16}$$

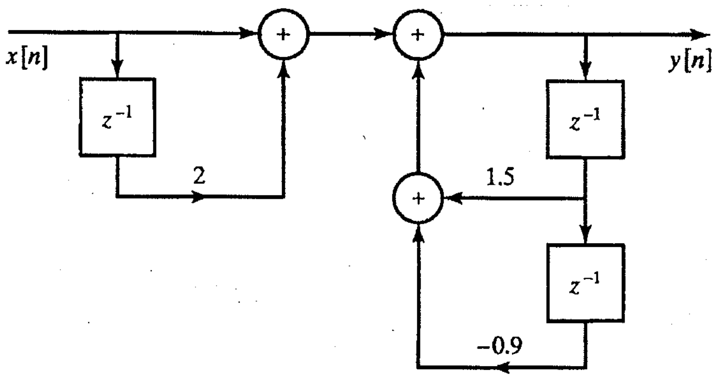


Figure Direct form I implementation

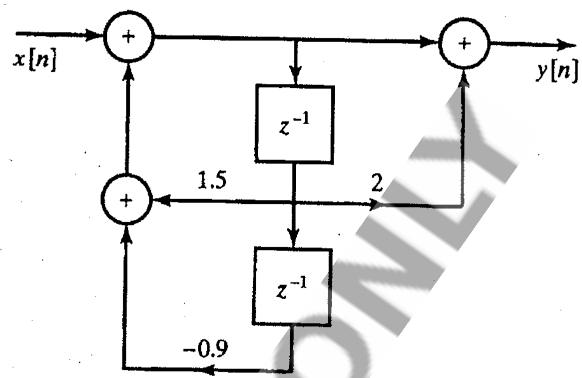
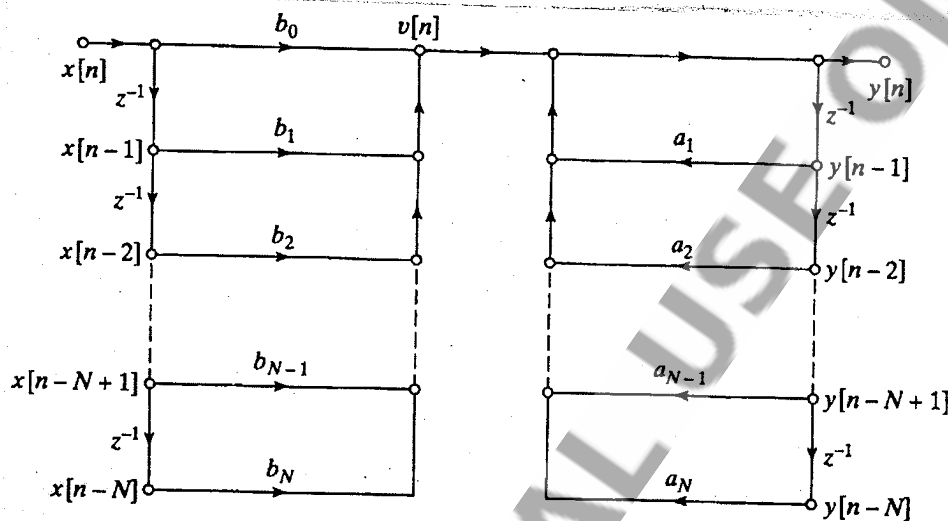


Figure Direct form II implementation

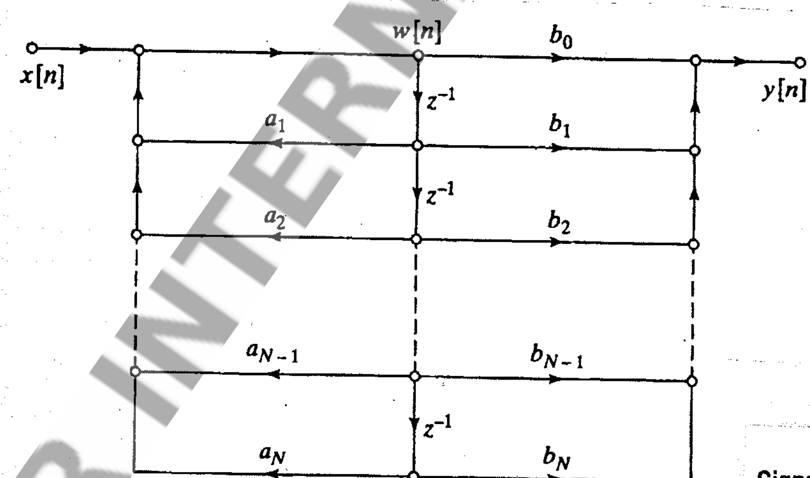
$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \tag{6.26}$$

with the corresponding rational system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}. \tag{6.27}$$



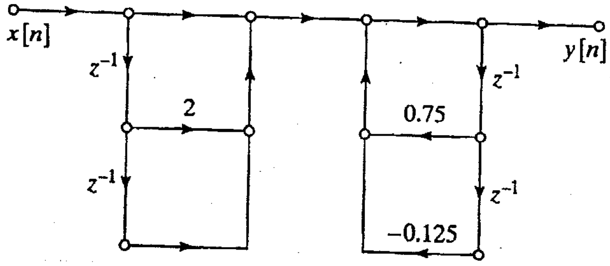
Signal flow graph of direct form I structure for an Nth-order system.

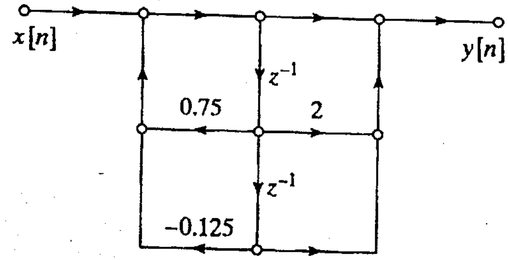


Signal flow graph of direct form II structure for an Nth-order system.

Consider the system function

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}. \tag{6.28}$$





Cascade Form

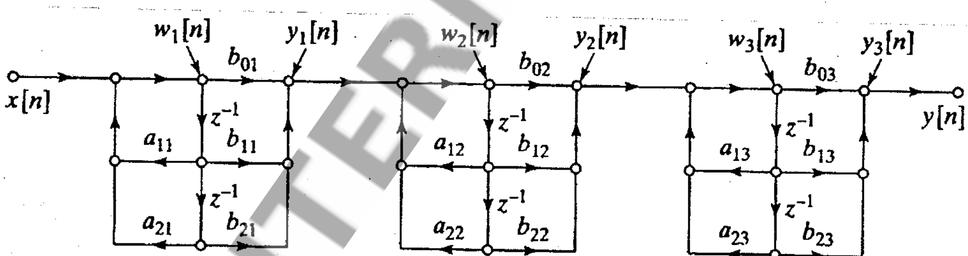
$$H(z) = A\frac{\prod_{k=1}^{M_1}(1 - f_k z^{-1})\prod_{k=1}^{M_2}(1 - g_k z^{-1})(1 - g_k^* z^{-1})}{\prod_{k=1}^{N_1}(1 - c_k z^{-1})\prod_{k=1}^{N_2}(1 - d_k z^{-1})(1 - d_k^* z^{-1})}, \tag{6.29}$$

where $M = M_1 + 2M_2$ and $N = N_1 + 2N_2$. In this expression, the first-order factors represent real zeros at f_k and real poles at c_k, and the second-order factors represent complex conjugate pairs of zeros at g_k and g_k^* and complex conjugate pairs of poles at d_k and d_k^*.

A modular structure that is advantageous for many types of implementations is obtained by combining pairs of real factors and complex conjugate pairs into second-order factors so that Eq. (6.29) can be expressed as

$$H(z) = \prod_{k=1}^{N_s}\frac{b_{0k} + b_{1k}z^{-1} + b_{2k}z^{-2}}{1 - a_{1k}z^{-1} - a_{2k}z^{-2}}, \tag{6.30}$$

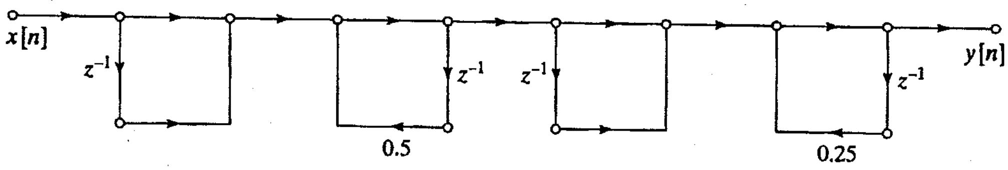
where $N_s = \lfloor (N+1)/2 \rfloor$ is the largest integer contained in $(N+1)/2$. In writing $H(z)$ in this form, we have assumed that $M \leq N$ and that the real poles and zeros have been combined in pairs. If there are an odd number of real zeros, one of the coefficients b_{2k} will be zero. Likewise, if there are an odd number of real poles, one of the coefficients a_{2k} will be zero.



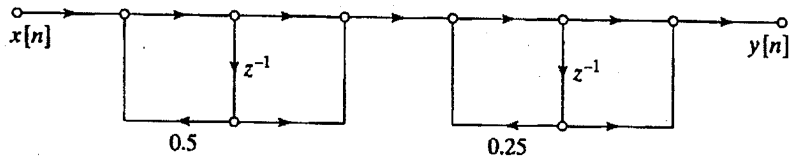
Alternatively, to illustrate the cascade structure, we can use first-order systems by expressing $H(z)$ as a product of first-order factors, as in

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})}. \tag{6.32}$$

Since all of the poles and zeros are real, a cascade structure with first-order sections has real coefficients. If the poles and/or zeros were complex, only a second-order section would have real coefficients.



(a)



(b)

Parallel Form

As an alternative to factoring the numerator and denominator polynomials of $H(z)$, we can express a rational system function as given by

$$H(z) = \sum_{k=0}^{N_p} C_k z^{-1} + \sum_{k=1}^{N_1} \frac{A_k}{1 - c_k z^{-1}} + \sum_{k=1}^{N_2} \frac{B_k(1 - e_k z^{-1})}{(1 - d_k z^{-1})(1 - d_k^* z^{-1})}, \tag{6.34}$$

where $N = N_1 + 2N_2$. If $M \geq N$, then $N_p = M - N$; otherwise, the first summation in Eq. (6.34) is not included. If the coefficients a_k and b_k are real in Eq. (6.27), then the quantities A_k, B_k, C_k, c_k, and e_k are all real. Alternatively, we may group the real poles in pairs, so that $H(z)$ can be expressed as

$$H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_s} \frac{e_{0k} + e_{1k} z^{-1}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}, \tag{6.35}$$

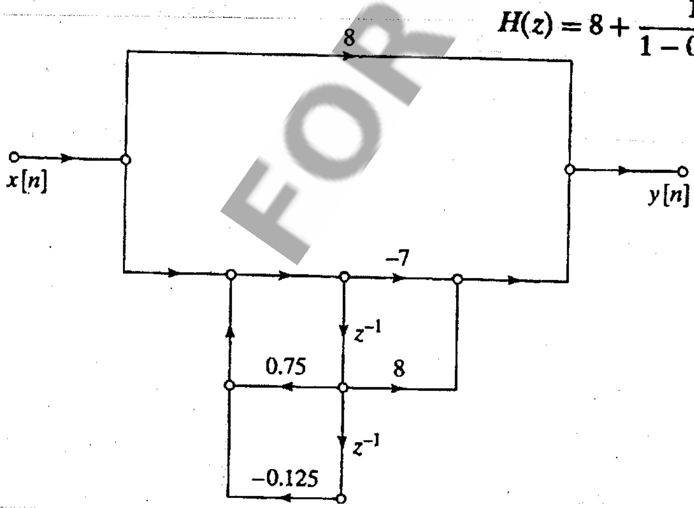
where, as in the cascade form, $N_s = \lfloor (N+1)/2 \rfloor$ is the largest integer contained in $(N+1)/2$, and if $N_p = M - N$ is negative, the first sum is not present.

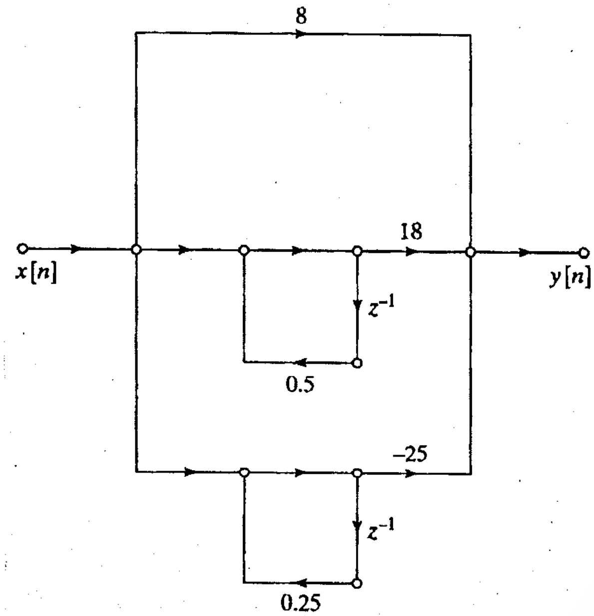
If we use second-order sections,

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = 8 + \frac{-7 + 8z^{-1}}{1 - 0.75z^{-1} + 0.125z^{-2}}. \tag{6.37}$$

Since all the poles are real, we can obtain an alternative parallel form realization by expanding $H(z)$ as

$$H(z) = 8 + \frac{18}{1 - 0.5z^{-1}} - \frac{25}{1 - 0.25z^{-1}}. \tag{6.38}$$





BASIC NETWORK STRUCTURES FOR FIR SYSTEMS

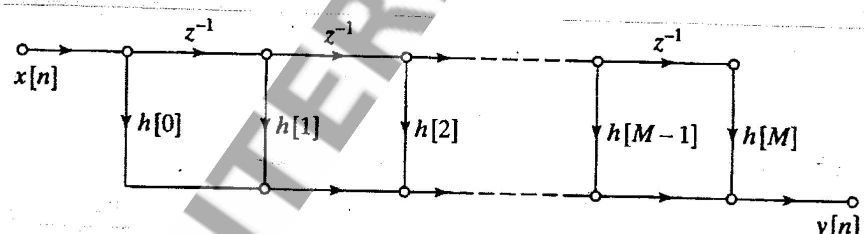
Direct Form

For causal FIR systems, the system function has only zeros (except for poles at $z = 0$), and since the coefficients a_k are all zero, the difference equation of Eq. (6.9) reduces to

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]. \tag{6.46}$$

This can be recognized as the discrete convolution of $x[n]$ with the impulse response

$$h[n] = \begin{cases} b_n & n = 0, 1, \ldots, M, \\ 0 & \text{otherwise.} \end{cases} \tag{6.47}$$



Because of the chain of delay elements across the top of the diagram, this structure is also referred to as a *tapped delay line* structure or a *transversal filter* structure.

THE EFFECTS OF COEFFICIENT QUANTIZATION

When the parameters of a rational system function or corresponding difference equation are quantized, the poles and zeros of the system function move to new positions in the z-plane. Equivalently, the frequency response is perturbed from its original value. If the system implementation structure is highly sensitive to perturbations of the coefficients, the resulting system may no longer meet the original design specifications, or an IIR system might even become unstable.

For example, the system function representation corresponding to both direct forms is the ratio of polynomials

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}. \tag{6.62}$$

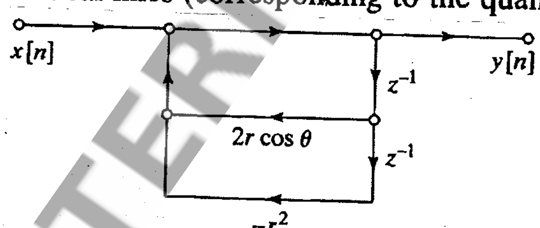
The sets of coefficients $\{a_k\}$ and $\{b_k\}$ are the ideal infinite-precision coefficients in both direct-form implementation structures. If we quantize these coefficients, we obtain the system function

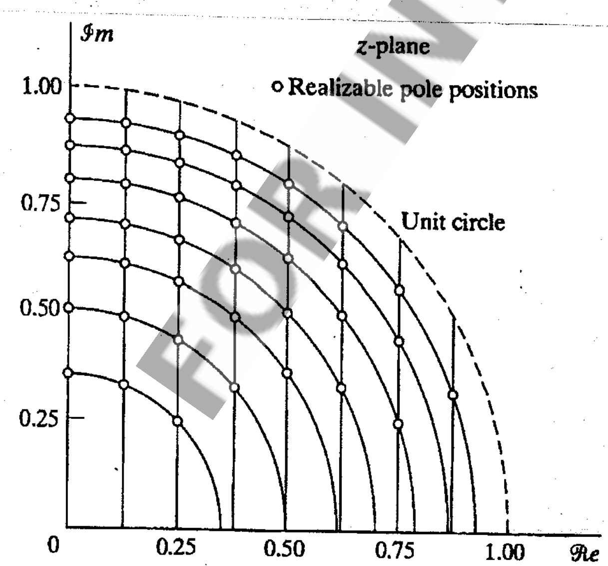
$$\hat{H}(z) = \frac{\sum_{k=0}^{M} \hat{b}_k z^{-k}}{1 - \sum_{k=1}^{N} \hat{a}_k z^{-k}}, \tag{6.63}$$

where $\hat{a}_k = a_k + \Delta a_k$ and $\hat{b}_k = b_k + \Delta b_k$ are the quantized coefficients that differ from the original coefficients by the quantization errors Δa_k and Δb_k.

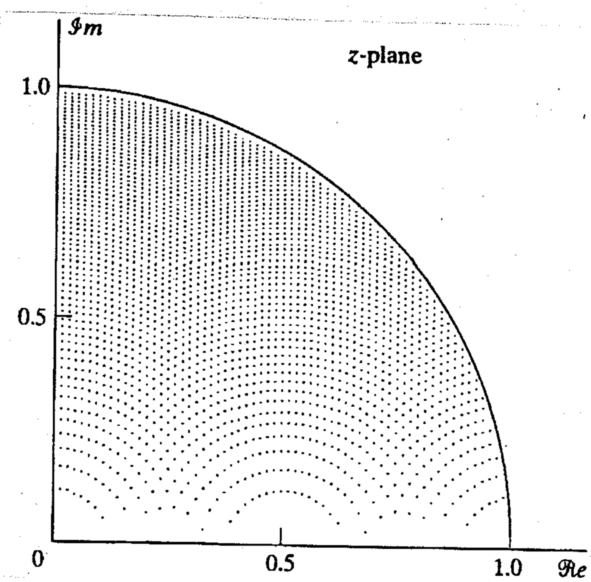
Poles of Quantized Second-Order Sections

Even for the second-order systems that are used to implement the cascade and parallel forms, there remains some flexibility to improve the robustness to coefficient quantization. Consider a complex-conjugate pole pair implemented using the direct form, as in Figure . With infinite-precision coefficients, this network has poles at $z = re^{j\theta}$ and $z = re^{-j\theta}$. However, if the coefficients $2r\cos\theta$ and $-r^2$ are quantized, only a finite number of different pole locations is possible. The poles must lie on a grid in the z-plane defined by the intersection of concentric circles (corresponding to the quantization of r^2) and vertical lines (corresponding to the quantization of $2r\cos\theta$).





(a)



Notice that for the direct form, the grid is rather sparse around the real axis. Thus, poles located around $\theta = 0$ or $\theta = \pi$ may be shifted more than those around $\theta = \pi/2$. Of course, it is always possible that the infinite-precision pole location is very close to one of the allowed quantized poles. In this case, quantization causes no problem whatsoever, but in general, quantization can be expected to degrade performance.

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Effects of Coefficient Quantization in FIR Systems

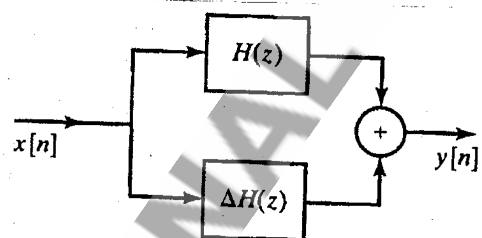
For FIR systems, we must be concerned only with the locations of the zeros of the system function, since, for causal FIR systems, all the poles are at $z = 0$. Although we have just seen that the direct-form structure should be avoided for high-order IIR systems, it turns out that the direct-form structure is commonly used for FIR systems. To understand why this is so, we express the system function for a direct-form FIR system in the form

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n}. \tag{6.64}$$

$$\hat{H}(z) = \sum_{n=0}^{M} \hat{h}[n]z^{-n} = H(z) + \Delta H(z),$$

where

$$\Delta H(z) = \sum_{n=0}^{M} \Delta h[n]z^{-n}.$$



The reason that the direct form FIR system is widely used is that, for most linear phase FIR filters, the zeros are more or less uniformly spread in the z-plane.

Maintaining Linear Phase

So far, we have not made any assumptions about the phase response of the FIR system. However, the possibility of generalized linear phase is one of the major advantages of an FIR system. Recall that a linear-phase FIR system has either a symmetric ($h[M-n] = h[n]$) or an antisymmetric ($h[M-n] = -h[n]$) impulse response. These linear-phase conditions are easily preserved for the direct-form quantized system.