THE Z-TRANSFORM

The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z). \tag{3.3}$$

With this interpretation, the z-transform operator is seen to transform the sequence $x[n]$ into the function $X(z)$, where z is a continuous complex variable.

This is one motivation for the notation $X(e^{j\omega})$ for the Fourier transform; when it exists, the Fourier transform is simply $X(z)$ with $z = e^{j\omega}$. This corresponds to restricting z to have unity magnitude; i.e., for $|z| = 1$, the z-transform corresponds to the Fourier transform. More generally, we can express the complex variable z in polar form as

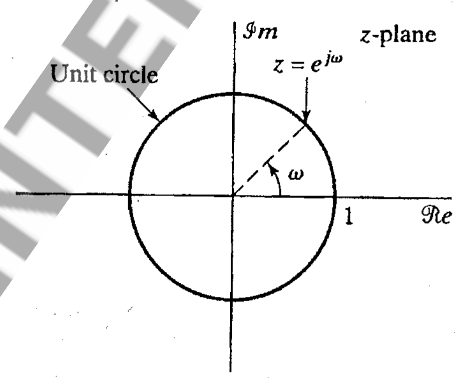
$$z = re^{j\omega}.$$

With z expressed in this form, Eq. (3.2) becomes

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}. \tag{3.6}$$

Equation (3.6) can be interpreted as the Fourier transform of the product of the original sequence $x[n]$ and the exponential sequence r^{-n}. Obviously, for $r = 1$, Eq. (3.6) reduces to the Fourier transform of $x[n]$.

The z-transform evaluated on the unit circle corresponds to the Fourier transform. Note that ω is the angle between the vector to a point z on the unit circle and the real axis of the complex z-plane.



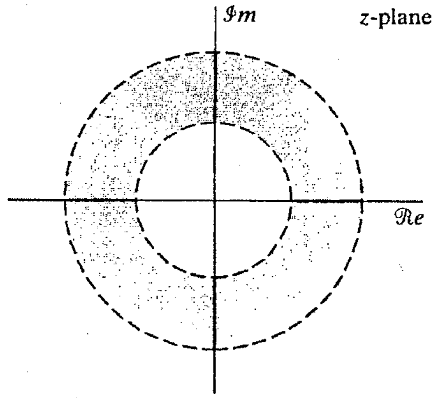
With this interpretation, the inherent periodicity in frequency of the Fourier transform is captured naturally, since a change of angle of 2π radians in the z-plane corresponds to traversing the unit circle once and returning to exactly the same point.

For any given sequence, the set of values of z for which the z-transform converges is called the *region of convergence,* which we abbreviate ROC.

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty \tag{3.7}$$

for convergence of the z-transform.

The region of convergence will consist of a ring in the z-plane.



For example, the sequence $x[n] = u[n]$ is not absolutely summable, and therefore, the Fourier transform does not converge absolutely. However, $r^{-n}u[n]$ is absolutely summable if $r > 1$. This means that the z-transform for the unit step exists with a region of convergence $|z| > 1$.

If the ROC includes the unit circle, this of course implies convergence of the z-transform for $|z| = 1$, or equivalently, the Fourier transform of the sequence converges. Conversely, if the ROC does not include the unit circle, the Fourier transform does not converge absolutely.

Among the most important and useful z-transforms are those for which $X(z)$ is a rational function inside the region of convergence, i.e.,

$$X(z) = \frac{P(z)}{Q(z)}, \tag{3.9}$$

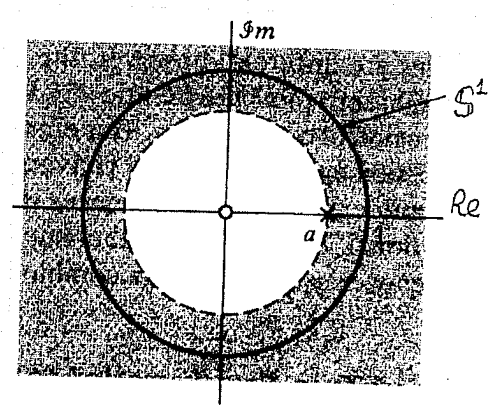
where $P(z)$ and $Q(z)$ are polynomials in z. The values of z for which $X(z) = 0$ are called the *zeros* of $X(z)$, and the values of z for which $X(z)$ is infinite are referred to as the *poles* of $X(z)$. The poles of $X(z)$ for finite values of z are the roots of the denominator polynomial. In addition, poles may occur at $z = 0$ or $z = \infty$.

Right-Sided Exponential Sequence

Consider the signal $x[n] = a^n u[n]$. Because it is nonzero only for $n \geq 0$, this is an example of a *right-sided* sequence. From Eq. (3.2),

$$X(z) = \sum_{n=-\infty}^{\infty} = a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|. \tag{3.10}$$



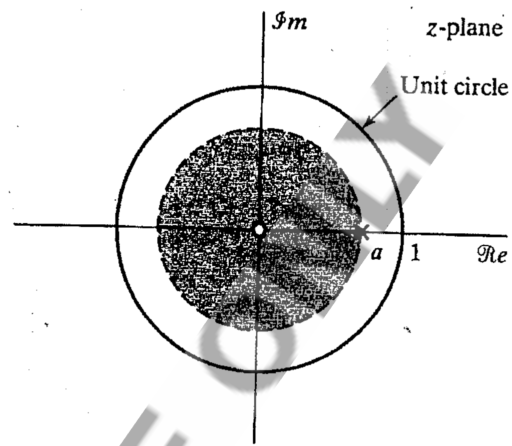
Here we have used the familiar formula for the sum of terms of a geometric series. The z-transform has a region of convergence for any finite value of $|a|$. The Fourier transform of $x[n]$, on the other hand, converges only if $|a| < 1$. For $a = 1$, $x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1. \tag{3.11}$$

Left-Sided Exponential Sequence

Now let $x[n] = -a^n u[-n-1]$. Since the sequence is nonzero only for $n \le -1$, this is a *left-sided* sequence. Then

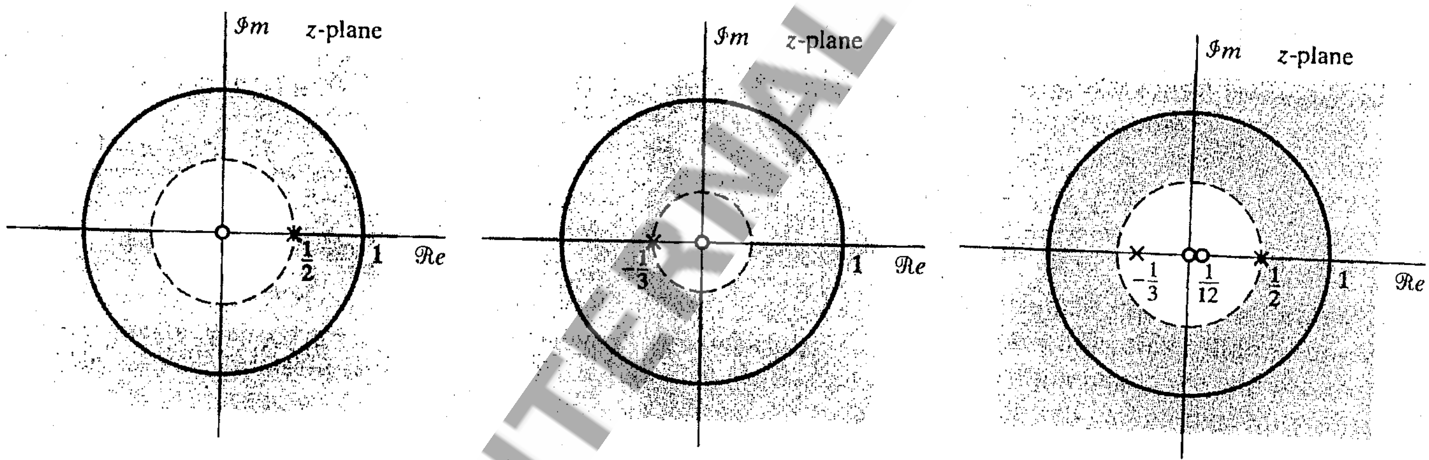
$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \qquad |z| < |a|.$$



Sum of Two Exponential Sequences

Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n]. \qquad (3.14)$$



(a) (b) (c)

$$\left(\tfrac{1}{2}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad |z| > \tfrac{1}{2}, \qquad (3.17)$$

$$\left(-\tfrac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 + \frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{3}, \qquad (3.18)$$

and, consequently,

$$\left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{2}, \qquad (3.19)$$

$$= \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}.$$

Two-Sided Exponential Sequence

Consider the sequence

$$x[n] = \left(-\tfrac{1}{3}\right)^n u[n] - \left(\tfrac{1}{2}\right)^n u[-n-1]. \tag{3.20}$$

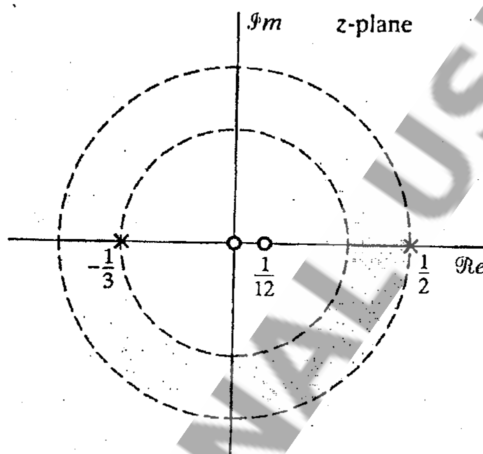
Note that this sequence grows exponentially as $n \to -\infty$.

$$\left(-\tfrac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{3},$$

$$-\left(\tfrac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \tfrac{1}{2}.$$

Thus, by the linearity of the z-transform,

$$\begin{aligned} X(z) &= \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \tfrac{1}{3} < |z|, \quad |z| < \tfrac{1}{2}, \\ &= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}. \end{aligned} \tag{3.21}$$



Finite-Length Sequence

Consider the signal

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n \\ &= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}, \end{aligned} \tag{3.23}$$

Specifically, the N roots of the numerator polynomial are at

$$z_k = ae^{j(2\pi k/N)}, \qquad k = 0, 1, \ldots, N-1. \tag{3.24}$$

(Note that these values satisfy the equation $z^N = a^N$, and when $a = 1$, these complex values are the Nth roots of unity.) The zero at $k = 0$ cancels the pole at $z = a$.

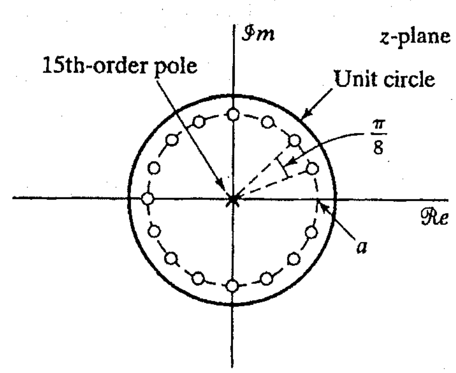


TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

3.2 PROPERTIES OF THE REGION OF CONVERGENCE FOR THE z-TRANSFORM

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

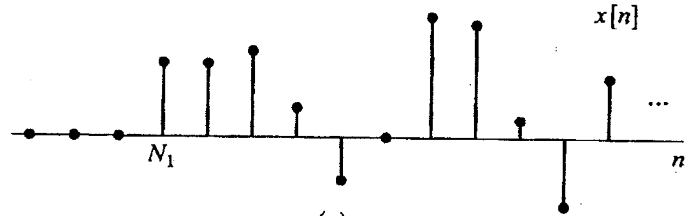
PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

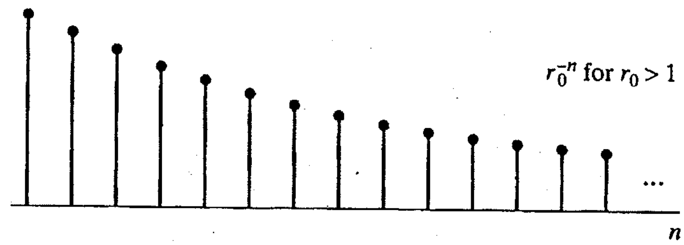
PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

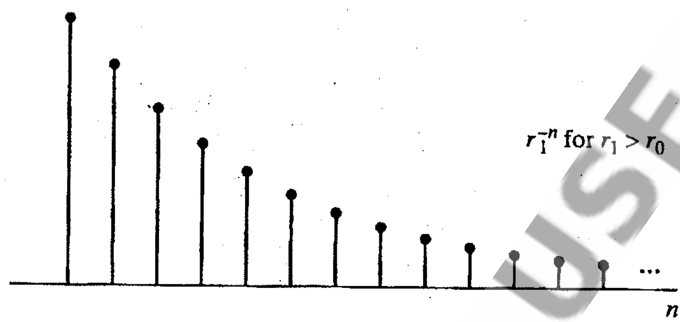
PROPERTY 8: The ROC must be a connected region.



(a)



(b)

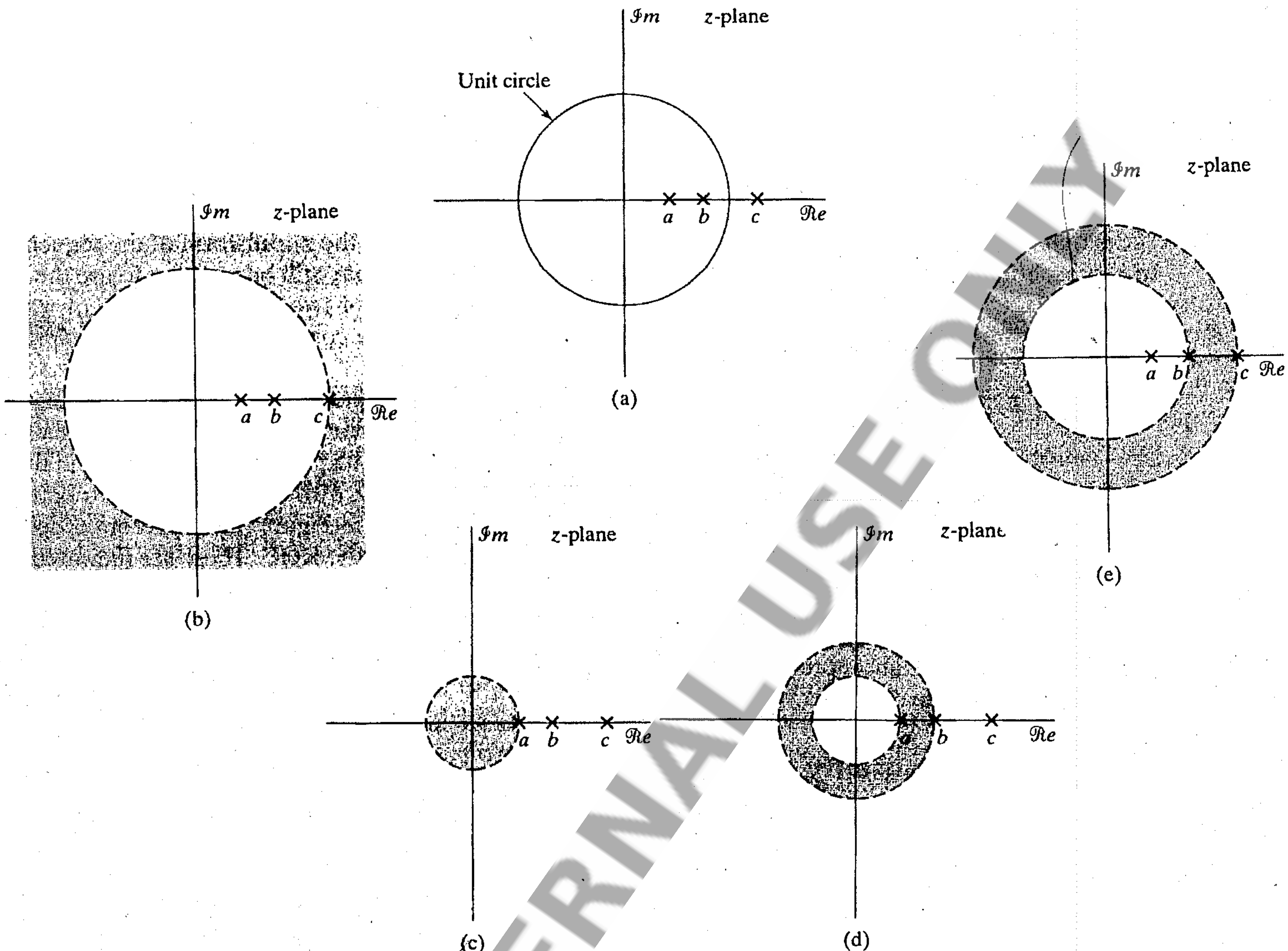


(c)

Property 5 can be interpreted in a somewhat similar manner. Figure illustrates a right-sided sequence and the exponential sequence r^{-n} for two different values of r. A right-sided sequence is zero prior to some value of n, say, N_1. If the circle $|z| = r_0$ is in the ROC, then $x[n]r_0^{-n}$ is absolutely summable, or equivalently, the Fourier transform of $x[n]r_0^{-n}$ converges. Since $x[n]$ is right sided, the sequence $x[n]r_1^{-n}$ will also be absolutely summable if r_1^{-n} decays faster than r_0^{-n}. Specifically, as illustrated in Figure , this more rapid exponential decay will further attenuate sequence values for positive values of n and cannot cause sequence values for negative values of n to become unbounded, since $x[n]z^{-n} = 0$ for $n < N_1$. Based on this property, we can conclude that, for a right-sided sequence, the ROC extends outward from some circle in the z-plane, concentric with the origin.

For right-sided sequences, the ROC is dictated by the exponential weighting required to have all exponential terms decay to zero for increasing n; for left-sided sequences, the exponential weighting must be such that all exponential terms decay to zero for decreasing n. For two-sided sequences, the exponential weighting needs to be balanced, since if it decays too fast for increasing n, it may grow too quickly for decreasing n and vice versa. More specifically, for two-sided sequences, some of the poles contribute only for $n > 0$ and the rest only for $n < 0$. The region of convergence is bounded on the inside by the pole with the largest magnitude that contributes for $n > 0$ and on the outside by the pole with the smallest magnitude that contributes for $n < 0$.

Examples of four z- transforms with the same pole–zero locations, illustrating the different possibilities for the region of convergence. Each ROC corresponds to a different sequence: (b) to a right-sided sequence, (c) to a left-sided sequence, (d) to a two-sided sequence, and (e) to a two-sided sequence.

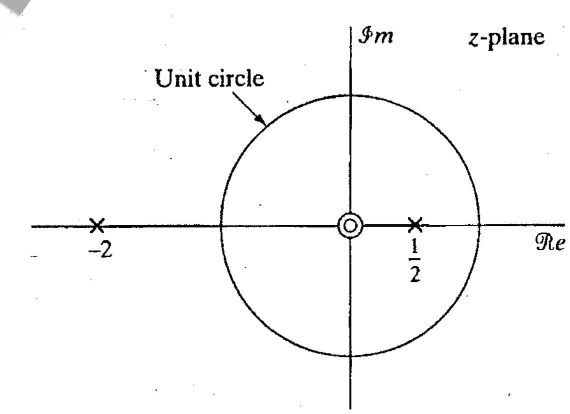


If the sequence is nonzero only in the interval $N_1 \le n \le N_2$, the z-transform

$$X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n} \tag{3.22}$$

has no problems of convergence, as long as each of the terms $|x[n]z^{-n}|$ is finite.

Stability, Causality, and the ROC



There are three possible ROC's consistent with properties 1–8 that can be associated with this pole-zero plot. However, if we state in addition that the system is stable (or equivalently, that $h[n]$ is absolutely summable and therefore has a Fourier transform), then the ROC must include the unit circle. Thus, stability of the system and properties 1–8 imply that the ROC is the region $\frac{1}{2} < |z| < 2$. Note that as a consequence, $h[n]$ is two sided, and therefore, the system is not causal.

If we state instead that the system is causal, and therefore that $h[n]$ is right sided, then property 5 would require that the ROC be the region $|z| > 2$. Under this condition, the system would not be stable; i.e., for this specific pole–zero plot, there is no ROC that would imply that the system is both stable and causal.

3.3 THE INVERSE z-TRANSFORM

Partial Fraction Expansion

To see how to obtain a partial fraction expansion, let us assume that $X(z)$ is expressed as a ratio of polynomials in z^{-1}; i.e.,

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{3.37}$$

Such z-transforms arise frequently in the study of linear time-invariant systems. An equivalent expression is

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}}. \tag{3.38}$$

Equation (3.38) explicitly shows that for such functions, there will be M zeros and N poles at nonzero locations in the z-plane. In addition, there will be either $M-N$ poles at $z=0$ if $M > N$ or $N-M$ zeros at $z=0$ if $N > M$. In other words, z-transforms of the form of Eq. (3.37) always have the same number of poles and zeros in the finite z-plane, and there are no poles or zeros at $z=\infty$. To obtain the partial fraction expansion of $X(z)$ in Eq. (3.37), it is most convenient to note that $X(z)$ could be expressed in the form

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}, \tag{3.39}$$

If $M < N$ and the poles are all first order, then $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}. \tag{3.40}$$

A_k, can be found from

$$A_k = (1 - d_k z^{-1}) X(z)\big|_{z=d_k}. \tag{3.41}$$

Consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}, \qquad |z| > \frac{1}{2}. \tag{3.42}$$

$$X(z) = \frac{A_1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{A_2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

From Eq. (3.41),

$$A_1 = \left(1 - \tfrac{1}{4}z^{-1}\right) X(z)\big|_{z=1/4} = -1,$$

$$A_2 = \left(1 - \tfrac{1}{2}z^{-1}\right) X(z)\big|_{z=1/2} = 2.$$

Therefore,

$$X(z) = \frac{-1}{\left(1 - \frac{1}{4}z^{-1}\right)} + \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)}.$$

Since $x[n]$ is right sided, the ROC for each term extends outward from the outermost pole. From Table 3.1 and the linearity of the z-transform, it then follows that

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n].$$

Clearly, the numerator that would result from adding the terms in Eq. (3.40) would be at most of degree $(N-1)$ in the variable z^{-1}. If $M \geq N$, then a polynomial must be added to the right-hand side of Eq. (3.40), the order of which is $(M-N)$. Thus, for $M \geq N$, the complete partial fraction expansion would have the form

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}. \tag{3.43}$$

If $X(z)$ has multiple-order poles and $M \geq N$, Eq. (3.43) must be further modified. In particular, if $X(z)$ has a pole of order s at $z = d_i$ and all the other poles are first-order, then Eq. (3.43) becomes

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}. \tag{3.44}$$

The coefficients A_k and B_r are obtained as before. The coefficients C_m are obtained from the equation

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} [(1 - d_i w)^s X(w^{-1})] \right\}_{w=d_i^{-1}}. \tag{3.45}$$

The terms $B_r z^{-r}$ correspond to shifted and scaled impulse sequences, i.e., terms of the form $B_r \delta[n-r]$. The fractional terms correspond to exponential sequences. To decide whether a term

$$\frac{A_k}{1 - d_k z^{-1}}$$

corresponds to $(d_k)^n u[n]$ or $-(d_k)^n u[-n-1]$, we must use the properties of the region of convergence that were discussed in Section 3.2. From that discussion, it follows that if $X(z)$ has only simple poles and the ROC is of the form $r_R < |z| < r_L$, then a given pole d_k will correspond to a right-sided exponential $(d_k)^n u[n]$ if $|d_k| < r_R$, and it will correspond to a left-sided exponential if $|d_k| > r_L$. Thus, the region of convergence can be used to sort the poles. Multiple-order poles also are divided into left-sided and right-sided contributions in the same way.

Finite-Length Sequence

Suppose $X(z)$ is given in the form

$$X(z) = z^2\left(1 - \tfrac{1}{2}z^{-1}\right)(1 + z^{-1})(1 - z^{-1}). \qquad (3.50)$$

However, by multiplying the factors of Eq. (3.50), we can express $X(z)$ as

$$X(z) = z^2 - \tfrac{1}{2}z - 1 + \tfrac{1}{2}z^{-1}.$$

Therefore, by inspection, $x[n]$ is seen to be

$$x[n] = \begin{cases} 1, & n = -2, \\ -\frac{1}{2}, & n = -1, \\ -1, & n = 0, \\ \frac{1}{2}, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$x[n] = \delta[n+2] - \tfrac{1}{2}\delta[n+1] - \delta[n] + \tfrac{1}{2}\delta[n-1].$$

Inverse Transform by Power Series Expansion

Consider the z-transform

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|.$$

Using the power series expansion for $\log(1+x)$, with $|x| < 1$, we obtain

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}. \quad \Rightarrow \quad x[n] = \begin{cases} (-1)^{n+1}\frac{a^n}{n}, & n \geq 1, \\ 0, & n \leq 0. \end{cases}$$

Sequence	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_{x_1}
$x_2[n]$	$X_2(z)$	R_{x_2}
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
$x^*[n]$	$X^*(z^*)$	R_x
$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
Initial-value theorem:		
$x[n] = 0, \quad n < 0$	$\lim_{z\to\infty} X(z) = x[0]$	

Shifted Exponential Sequence

Consider the z-transform

$$X(z) = \frac{1}{z - \frac{1}{4}}, \qquad |z| > \tfrac{1}{4}.$$

From the ROC, we identify this as corresponding to a right-sided sequence. We can first rewrite $X(z)$ in the form

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \tfrac{1}{4}. \tag{3.55}$$

$X(z)$ can be written as

$$X(z) = z^{-1}\left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right), \qquad |z| > \tfrac{1}{4}. \tag{3.58}$$

From the time-shifting property, we recognize the factor z^{-1} in Eq. (3.58) as being associated with a time shift of one sample to the right of the sequence $\left(\frac{1}{4}\right)^n u[n]$; i.e.,

$$x[n] = \left(\tfrac{1}{4}\right)^{n-1} u[n-1]. \tag{3.59}$$

Exponential Multiplication

Starting with the transform pair

$$u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \qquad |z| > 1, \tag{3.60}$$

we can use the exponential multiplication property to determine the z-transform of

$$x[n] = r^n \cos(\omega_0 n) u[n]. \tag{3.61}$$

First, $x[n]$ is expressed as

$$x[n] = \tfrac{1}{2}(re^{j\omega_0})^n u[n] + \tfrac{1}{2}(re^{-j\omega_0})^n u[n].$$

Then, using Eq. (3.60) and the exponential multiplication property, we see that

$$\tfrac{1}{2}(re^{j\omega_0})^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{\frac{1}{2}}{1 - re^{j\omega_0}z^{-1}}, \qquad |z| > r,$$

$$\tfrac{1}{2}(re^{-j\omega_0})^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{\frac{1}{2}}{1 - re^{-j\omega_0}z^{-1}}, \qquad |z| > r.$$

From the linearity property, it follows that

$$\begin{aligned} X(z) &= \frac{\frac{1}{2}}{1 - re^{j\omega_0}z^{-1}} + \frac{\frac{1}{2}}{1 - re^{-j\omega_0}z^{-1}}, \qquad |z| > r \\ &= \frac{(1 - r\cos\omega_0 z^{-1})}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}, \qquad |z| > r. \end{aligned} \tag{3.62}$$

Inverse of Non-Rational z-Transform

In this example, we use the differentiation property together with the time-shifting property to determine the inverse z-transform

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|,$$

we first differentiate to obtain a rational expression:

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}}.$$

From the differentiation property,

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}}, \qquad |z| > |a|. \tag{3.63}$$

Specifically, we can express $nx[n]$ as

$$nx[n] = a(-a)^{n-1}u[n-1].$$

Therefore,

$$x[n] = (-1)^{n+1}\frac{a^n}{n}u[n-1] \overset{z}{\longleftrightarrow} \log(1+az^{-1}), \qquad |z| > |a|.$$

Time-Reversed Exponential Sequence

If the sequence $x[n]$ is real or we do not conjugate a complex sequence, the result becomes

$$x[-n] \overset{z}{\longleftrightarrow} X(1/z), \qquad \text{ROC} = \frac{1}{R_x}.$$

As an example of the use of the property of time reversal, consider the sequence

$$x[n] = a^{-n}u[-n],$$

which is a time-reversed version of $a^n u[n]$. From the time-reversal property, it follows that

$$X(z) = \frac{1}{1-az} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, \qquad |z| < |a^{-1}|.$$

Evaluating a Convolution Using the z-Transform

Let $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$. The corresponding z-transforms are

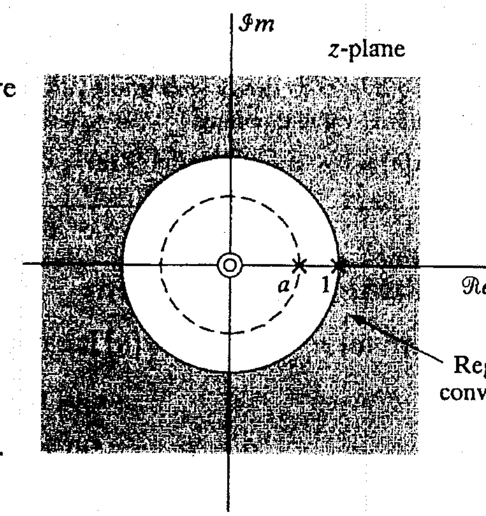
$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}, \qquad |z| > |a|,$$

and

$$X_2(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}, \qquad |z| > 1.$$

If $|a| < 1$, the z-transform of the convolution of $x_1[n]$ with $x_2[n]$ is then

$$Y(z) = \frac{1}{(1-az^{-1})(1-z^{-1})} = \frac{z^2}{(z-a)(z-1)}, \qquad |z| > 1.$$



Expanding $Y(z)$ in Eq. (3.64) in a partial fraction expansion, we get

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right), \qquad |z| > 1.$$

Therefore,

$$y[n] = \frac{1}{1-a}(u[n] - a^{n+1}u[n]).$$