Level 5 Module 2
exemplar with comments

2019-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2019

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)
Question-Answer Book

8:30 am – 11:00 am (2½ hours)
This paper must be answered in English

Please stick the barcode label here.

Candidate Number									

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11 and 13.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

A032E001

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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SECTION A (50 marks)

Answers written in the margins will not be marked.

1. Let $f(x) = \frac{10x}{7+3x^2}$. Prove that $f(1+h) - f(1) = \frac{4h-3h^2}{10+6h+3h^2}$. Hence, find $f'(1)$ from first principles. (4 marks)

$$f(1+h) - f(1) = \frac{10(1+h)}{7+3(1+h)^2} - \frac{10(1)}{7+3(1)^2}$$

$$= \frac{10+10h}{3h^2+6h+10} - 1$$

$$= \frac{10+10h-3h^2-6h-10}{3h^2+6h+10}$$

$$= \frac{-3h^2+4h}{3h^2+6h+10}$$

$$= \frac{4h-3h^2}{10+6h+3h^2} \quad //$$

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{4h-3h^2}{10+6h+3h^2}\right)$$

$$= \lim_{h\to 0}\left(\frac{4-3h}{10+6h+3h^2}\right)$$

$$= \frac{4}{10}$$

$$= \frac{2}{5} \quad //$$

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2. Let $P(x)=\begin{vmatrix} x+\lambda & 1 & 2 \\ 0 & (x+\lambda)^2 & 3 \\ 4 & 5 & (x+\lambda)^3 \end{vmatrix}$, where $\lambda \in \mathbf{R}$. It is given that the coefficient of x^3 in the expansion of $P(x)$ is 160. Find

(a) λ,

(b) $P'(0)$.

(5 marks)

a) $\begin{vmatrix} x+\lambda & 1 & 2 \\ 0 & (x+\lambda)^2 & 3 \\ 4 & 5 & (x+\lambda)^3 \end{vmatrix}$

$= (x+\lambda)^6 + 12 - 8(x+\lambda)^2 - 15(x+\lambda)$

$= (x^6 + 6\lambda x^5 + 15\lambda^2 x^4 + 20\lambda^3 x^3 + 15\lambda^4 x^2 + 6\lambda^5 x + \lambda^6) + 12 - 8(x^2 + 2\lambda x + \lambda^2)$
$- 15x - 15\lambda$

$= x^6 + 6\lambda x^5 + 15\lambda^2 x^4 + 20\lambda^3 x^3 + (15\lambda^4 - 8)x^2 + (6\lambda^5 - 16\lambda - 15)x + \lambda^6 - 8\lambda^2 - 15\lambda$
$+ 12$

$20\lambda^3 = 160$

$\lambda^3 = 8$

$\lambda = 2$ //

b) $P'(0) = \begin{vmatrix} \lambda & 1 & 2 \\ 0 & \lambda^2 & 3 \\ 4 & 5 & \lambda^3 \end{vmatrix}$

$= \lambda^6 + 12 - 8\lambda^2 - 15\lambda$

$= (2)^6 - 8(2)^2 - 15(2) + 12$

$= 14$ //

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3. A researcher performs an experiment to study the rate of change of the volume of liquid X in a vessel. The experiment lasts for 24 hours. At the start of the experiment, the vessel contains 580 cm^3 of liquid X. The researcher finds that during the experiment, $\frac{dV}{dt} = -2t$, where $V \text{ cm}^3$ is the volume of liquid X in the vessel and t is the number of hours elapsed since the start of the experiment.

(a) The researcher claims that the vessel contains some liquid X at the end of the experiment. Is the claim correct? Explain your answer.

(b) It is given that $V = h^2 + 24h$, where $h \text{ cm}$ is the depth of liquid X in the vessel. Find the value of $\frac{dh}{dt}$ when $t = 18$.

(6 marks)

a) Volume of water (V):

$\int -2t \, dt$

$= \frac{-2 \cdot t^2}{2} + C$

$= -t^2 + C$

when $t = 0$

$C = 580$

$\therefore V = -t^2 + 580$

when $t = 24$.

$V = -24^2 + 580 = 4 \text{ cm}^3$

$> 0 \text{ cm}^3$

$\therefore$ Yes, the claim is correct

b) $\left.\frac{dV}{dt}\right|_{t=18} = -2 \times 18 = -36$

$\because V = -t^2 + 580$

$h^2 + 24h = -(18^2) + 580$

$h^2 + 24h - 256 = 0$

$h = 8$ or -32 (rej.)

$\frac{dV}{dt} = (2h + 24) \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{-36}{2 \times 8 + 24} = -\frac{9}{10} \text{ cm/s}$ //

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4. Define $g(x)=\dfrac{\ln x}{\sqrt{x}}$ for all $x \in (0,99)$. Denote the graph of $y=g(x)$ by G .

(a) Prove that G has only one maximum point.

(b) Let R be the region bounded by G , the x-axis and the vertical line passing through the maximum point of G . Find the volume of the solid of revolution generated by revolving R about the x-axis.

(6 marks)

a) $g'(x) = \dfrac{\sqrt{x}\cdot\frac{1}{x} - \ln x\,(\frac{1}{2}x^{-\frac{1}{2}})}{x}$

$= \dfrac{1}{x}\left(\dfrac{\sqrt{x}}{x} - \dfrac{\ln x}{2\sqrt{x}}\right)$

$= \dfrac{1}{x}\left(\dfrac{2\sqrt{x}-\sqrt{x}\ln x}{2x}\right)$

$= \dfrac{\sqrt{x}(2-\ln x)}{2x^2}$

$g'(x)=0$

$\dfrac{\sqrt{x}(2-\ln x)}{2x^2}=0$

$\sqrt{x}=0$ or $2-\ln x=0$

$x=0$ (rej.) $\quad \ln x = 2$

$x=e^2$

x	$x<e^2$	$x=e^2$	$x>e^2$
$g'(x)$	$+$	0	$-$
	$\nearrow$		$\searrow$

$\therefore$ Only one maximum point which is $(e^2, \frac{2}{e})$

b) $\pi\displaystyle\int_{e^2}^{99}\left(\frac{\ln x}{\sqrt{x}}\right)^2 dx = \pi\int_{e^2}^{99}\frac{(\ln x)^2}{x}\,dx$

$= \pi\displaystyle\int_{e^2}^{99}(\ln x)^2\, d(\ln x)$

$= \pi\left[\dfrac{(\ln x)^3}{3}\right]_{e^2}^{99}$

$= \dfrac{\pi}{3}\left((\ln 99)^3 - 8\right)$ cubic unit //

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5. (a) Using mathematical induction, prove that $\sum_{k=n}^{2n} \frac{1}{k(k+1)} = \frac{n+1}{n(2n+1)}$ for all positive integers n.

(b) Using (a), evaluate $\sum_{k=50}^{200} \frac{1}{k(k+1)}$.

(7 marks)

(a) let P(n) be $\sum_{k=n}^{2n} \frac{1}{k(k+1)} = \frac{n+1}{n(2n+1)}$ where n is ~~an~~ positive integer

when $n=1$,

$LHS = \sum_{k=1}^{2} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} + \frac{1}{2(2+1)} = \frac{2}{3}$

$RHS = \frac{1+1}{1(2+1)} = \frac{2}{3}$

$\therefore$ P(1) is true.

Assume P(n) is true for some positive integers m

i.e. $\sum_{k=m}^{2m} \frac{1}{k(k+1)} = \frac{m+1}{m(2m+1)}$

when $n = m+1$,

$LHS = \sum_{k=m+1}^{2(m+1)} \frac{1}{k(k+1)}$

$= \frac{m+1}{m(2m+1)} + \frac{1}{(2m+1)(2m+2)} + \frac{1}{(2m+2)(2m+3)} - \frac{1}{m(m+1)}$

$= \frac{2(m+1)^2+m}{2m(2m+1)(m+1)} + \frac{m-2(2m+3)}{2m(m+1)(2m+3)}$

$= \frac{2m^2+5m+2}{2m(2m+1)(m+1)} - \frac{3(m+2)}{2m(m+1)(2m+3)}$

$= \frac{m+2}{2m(m+1)} - \frac{3(m+2)}{2m(m+1)(2m+3)}$

$= \frac{(m+2)(2m+3-3)}{2m(m+1)(2m+3)}$

$= \frac{m+2}{(m+1)(2m+3)}$

$= \frac{(m+1)+1}{(m+1)(2(m+1)+1)}$

$= RHS$

$\therefore$ P(m+1) is also true

$\therefore$ By the principle of MI, ~~T~~ P(n) is true for all positive integers n. i.e. $\sum_{k=n}^{2n} \frac{1}{k(k+1)} = \frac{n+1}{n(2n+1)}$

Answers written in the margins will not be marked.

b) $\displaystyle\sum_{k=50}^{200} \frac{1}{k(k+1)}$

$$= \sum_{k=100}^{200} \frac{1}{k(k+1)} + \sum_{k=50}^{100} \frac{1}{k(k+1)}$$

$$= \frac{100+1}{100(200+1)} + \frac{50+1}{50(100+1)}$$

$$= \frac{101}{20100} + \frac{51}{5050}$$

$$= \frac{30703}{2030100} \quad //$$

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6. Consider the system of linear equations in real variables x, y, z

$$(E):\begin{cases} x - 2y - 2z = \beta \\ 5x + \alpha y + \alpha z = 5\beta \\ 7x + (\alpha-3)y + (2\alpha+1)z = 8\beta \end{cases}, \text{ where } \alpha, \beta \in \mathbf{R} .$$

(a) Assume that (E) has a unique solution.

(i) Find the range of values of α .

(ii) Express y in terms of α and β.

(b) Assume that $\alpha = -4$. If (E) is inconsistent, find the range of values of β .

(7 marks)

(a) i) $\begin{vmatrix} 1 & -2 & -2 \\ 5 & \alpha & \alpha \\ 7 & \alpha-3 & 2\alpha+1 \end{vmatrix}$

$= \alpha(2\alpha+1) - 14\alpha - 10(\alpha-3) + 14\alpha - \alpha(\alpha-3) + 10(2\alpha+1)$

$= 2\alpha^2 + \alpha - 14\alpha - 10\alpha + 30 + 14\alpha - \alpha^2 + 3\alpha + 20\alpha + 10$

$= \alpha^2 + 14\alpha + 40$

$\begin{vmatrix} 1 & -2 & -2 \\ 5 & \alpha & \alpha \\ 7 & \alpha-3 & 2\alpha+1 \end{vmatrix} \neq 0$

$(\alpha^2 + 14\alpha + 40) \neq 0$

$(\alpha+4)(\alpha+10) \neq 0$

$\therefore \alpha \neq -4$ and $\alpha \neq -10$ //

ii) $y = \dfrac{\begin{vmatrix} 1 & \beta & -2 \\ 5 & 5\beta & \alpha \\ 7 & 8\beta & 2\alpha+1 \end{vmatrix}}{(\alpha+4)(\alpha+10)}$

$= \dfrac{5\beta(2\alpha+1) + 7\alpha\beta - 80\beta + 70\beta - 8\alpha\beta - 5\beta(2\alpha+1)}{(\alpha+4)(\alpha+10)}$

$= \dfrac{-\alpha\beta - 10\beta}{(\alpha+4)(\alpha+10)}$

$= \dfrac{-\beta}{\alpha+4}$ //

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b) $\left[\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 5 & -4 & -4 & 5\beta \\ 7 & -7 & -7 & 8\beta \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & 6 & 6 & 0 \\ 0 & 7 & 7 & \beta \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & -2 & -2 & \beta \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \beta \end{array}\right]$

$\therefore \beta \neq 0$

$\therefore \beta < 0$ or $\beta > 0$ //

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7. (a) Using integration by parts, find $\int e^x \sin \pi x \, dx$.

(b) Using integration by substitution, evaluate $\int_0^3 e^{3-x} \sin \pi x \, dx$.

(7 marks)

a) $\int e^x \sin\pi x \, dx$

$= \int \sin\pi x \, de^x$

$= e^x \sin\pi x - \int e^x \, d\sin\pi x$

$= e^x \sin\pi x - \pi \int e^x \cos\pi x \, dx$

$= e^x \sin\pi x - \pi \int \cos\pi x \, de^x$

$= e^x \sin\pi x - \pi e^x \cos\pi x + \pi \int e^x \, d\cos\pi x$

$= e^x \sin\pi x - \pi e^x \cos\pi x - \pi^2 \int e^x \sin\pi x \, dx$

$(1+\pi^2) \int e^x \sin\pi x \, dx = e^x \sin\pi x - \pi e^x \cos\pi x + C$

$\int e^x \sin\pi x \, dx = \dfrac{e^x \sin\pi x - \pi e^x \cos\pi x}{1+\pi^2} + C$ // , where C is a constant

b) Let $u = 3-x$

$du = -dx$

when $x=3$, $u=0$ when $x=0$, $u=3$

$\int_0^3 e^{3-x} \sin\pi x \, dx = -\int_3^0 e^u \sin\pi(3-u) \, du$

$= \int_0^3 e^u \sin(3\pi - \pi u) \, du$

$= \int_0^3 e^u [\sin 3\pi \cos\pi u - \cos 3\pi \sin\pi u) \, du$

$= \int_0^3 e^u \sin\pi u \, du$

$= \left[\dfrac{e^u \sin\pi u - \pi e^u \cos\pi u}{1+\pi^2} \right]_0^3$

$= \dfrac{\pi e^3}{1+\pi^2} + \dfrac{\pi e^3}{1+\pi^2}$

$= \dfrac{2\pi e^3}{1+\pi^2}$ //

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8. Let $\mathrm{h}(x)$ be a continuous function defined on $\mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. It is given that $\mathrm{h}'(x) = \dfrac{2x^2-7x+8}{x}$ for all $x > 0$.

(a) Is $\mathrm{h}(x)$ an increasing function? Explain your answer.

(b) Denote the curve $y = \mathrm{h}(x)$ by H. It is given that H passes through the point $(1, 3)$. Find

(i) the equation of H,

(ii) the point(s) of inflexion of H.

(8 marks)

a) $h'(x) = \dfrac{2x^2-7x+8}{x} = \dfrac{2(x-\frac{7}{4})^2+\frac{15}{8}}{x}$

$\therefore h'(x) > 0$

$\therefore$ Yes, it is an increasing function

b) i) $y = \displaystyle\int \frac{2x^2-7x+8}{x}\,dx$

$= \displaystyle\int (2x - 7 + \frac{8}{x})dx$

$= \dfrac{2x^2}{2} - 7x + 8\ln|x| + c$

$= x^2 - 7x + 8\ln|x| + c$

Sub (1,3)

$3 = 1 - 7 + 8\ln 1 + c$

$c = 9$

$\therefore$ Equation of H is $y = x^2 - 7x + 8\ln|x| + 9$ //

ii) $h''(x) = \dfrac{x(4x-7) - (2x^2-7x+8)}{x^2}$

$= \dfrac{2x^2-8}{x^2}$

$= \dfrac{2(x-2)(x+2)}{x^2}$

$h''(x) = 0$

$\dfrac{2(x-2)(x+2)}{x^2} = 0$

$x = -2$ or 2

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x	$x<-2$	$x=-2$	$-2<x<2$	$x=2$	$2<x$
$h''(x)$	$+$	0	$-$	0	$+$
y		/		$8\ln 2-10$	

$\therefore$ Point of inflexion is $(2, 8\ln 2-10)$ //

SECTION B (50 marks)

9. Consider the curve Γ: $y=\frac{1}{3}\sqrt{12-x^2}$, where $0<x<2\sqrt{3}$. Denote the tangent to Γ at $x=3$ by L.

(a) Find the equation of L. (3 marks)

(b) Let C be the curve $y=\sqrt{4-x^2}$, where $0<x<2$. It is given that L is a tangent to C. Find

(i) the point(s) of contact of L and C;

(ii) the point(s) of intersection of C and Γ;

(iii) the area of the region bounded by L, C and Γ.

(9 marks)

a) $\frac{dy}{dx} = \frac{1}{3}\left(\frac{1}{2}\right)(12-x^2)^{-\frac{1}{2}}(-2x)$

$= \frac{-x}{3\sqrt{12-x^2}}$

$\left.\frac{dy}{dx}\right|_{x=3} = \frac{-3}{3\sqrt{12-3^2}}$

$= \frac{-1}{\sqrt{3}} \quad = \frac{-\sqrt{3}}{3}$

$y = \frac{1}{3}\sqrt{12-3^2} = \frac{\sqrt{3}}{3}$

Equation of L:

$y-\frac{\sqrt{3}}{3} = \frac{-\sqrt{3}}{3}(x-3)$

$y-\frac{\sqrt{3}}{3} = \frac{-\sqrt{3}}{3}x+\sqrt{3}$

$\frac{\sqrt{3}}{3}x+y-\frac{4\sqrt{3}}{3}=0$

$x+\sqrt{3}y-4=0$ //

b) i) $\begin{cases} x+\sqrt{3}y-4=0 \\ y=\sqrt{4-x^2} \end{cases}$

$x+\sqrt{3}\cdot\sqrt{4-x^2}-4=0$.

$x+\sqrt{12-3x^2}-4=0$

$4-x = \sqrt{12-3x^2}$

$(4-x)^2 = 12-3x^2$

$12-3x^2 = x^2-8x+16$

$4x^2-8x+4 = 0$

Answers written in the margins will not be marked.

$$x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0$$

$$x = 1$$

$$y = \sqrt{4-1} = \sqrt{3}$$

$\therefore$ point of contact is $(1, \sqrt{3})$

ii) $\begin{cases} y = \sqrt{4-x^2} \\ y = \frac{1}{3}\sqrt{12-x^2} \end{cases}$

$$\sqrt{4-x^2} = \frac{1}{3}\sqrt{12-x^2}$$

$$4 - x^2 = \frac{1}{9}(12-x^2)$$

$$36 - 9x^2 = 12 - x^2$$

$$8x^2 = 24$$

$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

when $x = \sqrt{3}$, $y = \sqrt{4-3} = 1$

when $x = -\sqrt{3}$, $y = \sqrt{4-3} = 1$

$\therefore$ Point of intersection are $(\sqrt{3}, 1)$ and $(-\sqrt{3}, 1)$

iii) Area of region =

$$\int_{-\sqrt{3}}^{\sqrt{3}} (\sqrt{4-x^2}) - (\tfrac{1}{3}\sqrt{12-x^2})\, dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2}\, dx - \frac{1}{3}\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{12-x^2}\, dx$$

let $x = 2\cos\theta$

$$\frac{dx}{d\theta} = -2\sin\theta$$

when $x = \sqrt{3}$, $\theta = \frac{\pi}{6}$

when $x = -\sqrt{3}$, $\theta = \frac{5\pi}{6}$

Answers written in the margins will not be marked.

$$\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2}\, dx$$

$$= \int_{\frac{5\pi}{6}}^{\frac{\pi}{6}} \sqrt{4\sin^2\theta} \cdot -2\sin\theta \, d\theta$$

$$= \int_{\frac{5\pi}{6}}^{\frac{\pi}{6}} -d\theta$$

$$= -[\theta]_{\frac{5\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{4\pi}{6}$$

let $x = 2\sqrt{3}\cos\theta$

$\frac{dx}{d\theta} = -2\sqrt{3}\sin\theta$

when $x = \sqrt{3}$, $\theta = \frac{\pi}{3}$

when $x = -\sqrt{3}$, $\theta = \frac{2\pi}{3}$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{12-x^2}\, dx$$

$$= -\int_{\frac{2\pi}{3}}^{\frac{\pi}{3}} \sqrt{12\sin^2\theta} \cdot 2\sqrt{3}\sin\theta \cdot d\theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} d\theta$$

$$= \frac{\pi}{3}$$

Area of region:

$$\int_{-\sqrt{3}}^{\sqrt{3}} \left[\left(\sqrt{4-x^2}\right) - \frac{1}{3}\sqrt{12-x^2}\right] dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2}\, dx - \frac{1}{3}\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{12-x^2}\, dx$$

$$= \frac{4\pi}{6} - \frac{\pi}{9}$$

$$= \frac{5\pi}{9} \text{ sq. units}_{//}$$

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10. (a) Let $0 \le x \le \frac{\pi}{4}$. Prove that $\frac{1}{2+\cos 2x} = \frac{\sec^2 x}{2+\sec^2 x}$. (1 mark)

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2x} \, dx$. (3 marks)

(c) Let $f(x)$ be a continuous function defined on $\mathbf{R}$ such that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.

Prove that $\int_{-a}^{a} f(x) \ln(1+e^x) \, dx = \int_0^a x f(x) \, dx$ for any $a \in \mathbf{R}$. (4 marks)

(d) Evaluate $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2+\cos 2x)^2} \ln(1+e^x) \, dx$. (5 marks)

a) $\frac{1}{2+\cos 2x} = \frac{1}{2+2\cos^2 x - 1}$

$= \frac{1}{2\cos^2 x + 1}$

$= \frac{\frac{1}{\cos^2 x}}{\frac{2\cos^2 x}{\cos^2 x} + \frac{1}{\cos^2 x}}$

$= \frac{\sec^2 x}{2+\sec^2 x}$ //

b) $\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2x} \, dx$

$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{2+\sec^2 x} \, dx$

$= \int_0^{\frac{\pi}{4}} \frac{1}{2+\sec^2 x} \, d\tan x$

$= \int_0^{\frac{\pi}{4}} \frac{1}{2+\tan^2 x+1} \, d\tan x$

$= \int_0^{\frac{\pi}{4}} \frac{1}{\tan^2 x+3} \, d\tan x$

Let $t = \tan x$ and $t = \sqrt{3}\tan\theta$.

$\frac{dt}{d\theta} = \sqrt{3}\sec^2\theta$.

When $x = \frac{\pi}{4}$, $\theta = \frac{\pi}{6}$

When $x = 0$, $\theta = 0$

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$$\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2x}\,dx = \int_0^1 \frac{1}{t^2+3}\,dt$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{3\sec^2\theta}\cdot\sqrt{3}\sec^2\theta\cdot d\theta$$

$$= \frac{1}{\sqrt{3}}\int_0^{\frac{\pi}{6}} d\theta$$

$$= \frac{\sqrt{3}}{3}\cdot\frac{\pi}{6}$$

$$= \frac{\pi\sqrt{3}}{18} \;//$$

c) let $u=-x$.

$du=-dx$

when $x=a$, $u=-a$

when $x=-a$, $u=a$

$$\int_{-a}^{a} f(x)\ln(1+e^x)\,dx$$

$$= -\int_a^{-a} f(-u)\ln(1+e^{-u})\,du$$

$$= \int_{-a}^{a} f(u)\ln\left(\frac{e^u+1}{e^u}\right)du$$

$$= -\int_{-a}^{a} f(u)\ln(1+e^u)\,du + \int_{-a}^{a} f(u)\ln e^u\,du$$

$$= -\int_{-a}^{a} f(x)\ln(1+e^x)\,dx + \int_{-a}^{a} f(x)\,x\,dx.$$

$$2\int_{-a}^{a} f(x)\ln(1+e^x)\,dx = \int_{-a}^{a} x f(x)\,dx.$$

$$\int_{-a}^{a} f(x)\ln(1+e^x)\,dx = \frac{1}{2}\int_{-a}^{a} x f(x)\,dx$$

$$= \int_0^a x f(x)\,dx \;//$$

d) let $f(x) = \frac{\sin 2x}{(2+\cos 2x)^2}$, $a=\frac{\pi}{4}$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2+\cos 2x)^2}\ln(1+e^x)\,dx.$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cancel{x\sin 2x}}{(2+\cos 2x)^2}\,dx$$

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11. Let $M = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Find a pair of real numbers a and b such that $M^2 = aM + bI$. (3 marks)

(b) Prove that $6M^n = (1-(-5)^n)M + (5+(-5)^n)I$ for all positive integers n. (4 marks)

(c) Does there exist a pair of 2×2 real matrices A and B such that $(M^n)^{-1} = A + \frac{1}{(-5)^n}B$ for all positive integers n? Explain your answer. (5 marks)

a) $M^2 = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$

$= \begin{pmatrix} -3 & -28 \\ 4 & 29 \end{pmatrix}$

$M^2 = aM + bI$

$\begin{pmatrix} -3 & -28 \\ 4 & 29 \end{pmatrix} = a\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$7a = -28$

$a = -4$ //

$2a + b = -3$

$b = 5$ //

b) let $P(n)$ be $6M^n = (1-(-5)^n)M + (5+(-5)^n)I$

when $n=1$,

LHS $= 6M$

RHS $= (1-(-5))M + (5+(-5))I$

$= 6M$

$\therefore P(1)$ is true

Assume $P(n)$ is true for some positive integers k

i.e. $6M^k = (1-(-5)^k)M + (5+(-5)^k)I$

when $n=k+1$,

LHS $= 6M^{k+1}$

$= \{[1-(-5)^k]M + [5+(-5)^k]I\}M$

$= [1-(-5)^k]M^2 + [5+(-5)^k]IM$

Answers written in the margins will not be marked.

$= [1-(-5)^k](-4M+5I) + [5+(-5)^k]M$

$= [-4+4(-5)^k+5+(-5)^k]M + [5+(-5)^{k+1}]I$

$= [1-(-5)^{k+1}]M + [5+(-5)^{k+1}]I$

$=$ RHS

$\therefore P(k+1)$ is also true

$\therefore$ By the principle of MI, $P(n)$ is true for all positive integers.

(c) $6M^n = (1-(-5)^n)M + (5+(-5)^n)I$

$M^n = \frac{(1-(-5)^n)}{6}M + \frac{(5+(-5)^n)}{6}I$

$(M^n)^{-1} = \frac{1-(-5)^n}{6}M^{-1} + \frac{5+(-5)^n}{6}I$

$= \left(\frac{1}{6}M^{-1} + \frac{5}{6}I\right) + \frac{(-5)^n}{6}(I - M^{-1})$

$\det M = -5$

$\operatorname{cof} M = \begin{pmatrix} -6 & 1 \\ -7 & 2 \end{pmatrix}$

$\frac{-1}{30}\begin{pmatrix} -6 & -7 \\ 1 & 2 \end{pmatrix} + \frac{5}{6}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} \frac{31}{30} & \frac{7}{30} \\ \frac{-1}{30} & \frac{23}{30} \end{pmatrix}$

$= \frac{1}{30}\begin{pmatrix} 31 & 7 \\ -1 & 23 \end{pmatrix}$

$\frac{(-5)^n}{6}(I - M^{-1})$

$= \frac{(-5)^n}{6}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{31}{30} & \frac{7}{30} \\ \frac{-1}{30} & \frac{23}{30} \end{pmatrix}\right]$

$= \frac{(-5)^n}{6}\begin{pmatrix} \frac{-1}{30} & \frac{-7}{30} \\ \frac{1}{30} & \frac{7}{30} \end{pmatrix}$

$= \frac{(-5)^n}{6}\begin{pmatrix} -1 & -7 \\ 1 & 7 \end{pmatrix}$

$\therefore$ No.

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12. Let $\overrightarrow{OA} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{OB} = -5\mathbf{i} - 4\mathbf{j} + 8\mathbf{k}$ and $\overrightarrow{OC} = -5\mathbf{i} - 12\mathbf{j} + t\mathbf{k}$, where O is the origin and t is a constant. It is given that $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BC}\right|$.

(a) Find t . (3 marks)

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. (2 marks)

(c) Find the volume of the pyramid $OABC$. (2 marks)

(d) Denote the plane which contains A , B and C by Π . It is given that P , Q and R are points lying on Π such that $\overrightarrow{OP} = p\mathbf{i}$, $\overrightarrow{OQ} = q\mathbf{j}$ and $\overrightarrow{OR} = r\mathbf{k}$. Let D be the projection of O on Π .

(i) Prove that $pqr \neq 0$.

(ii) Find $\overrightarrow{OD}$.

(iii) Let E be a point such that $\overrightarrow{OE} = \frac{1}{p}\mathbf{i} + \frac{1}{q}\mathbf{j} + \frac{1}{r}\mathbf{k}$. Describe the geometric relationship between D , E and O . Explain your answer.

(6 marks)

a) $\overrightarrow{AC} = -6\vec{i} - 8\vec{j} + (t-2)\vec{k}$ $\quad \overrightarrow{BC} = -8\vec{j} + (t-8)\vec{k}$

$|\overrightarrow{AC}| = |\overrightarrow{BC}|$

$\sqrt{6^2+8^2+(t-2)^2} = \sqrt{8^2+(t-8)^2}$

$(t-2)^2+36 = (t-8)^2$

$t^2-4t+40 = t^2-16t+64$

$12t = 24$

$t = 2$ //

b) $\overrightarrow{AB} = -6\vec{i} + 6\vec{k}$ $\quad \overrightarrow{AC} = -6\vec{i} - 8\vec{j}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -6 & 0 & 6 \\ -6 & -8 & 0 \end{vmatrix}$

$= 48\vec{i} - 36\vec{j} + 48\vec{k}$ //

Answers written in the margins will not be marked.

c) Volume of the pyramid OABC =

$\frac{1}{3}\left|(\overrightarrow{OA}\times\overrightarrow{OB})\cdot\overrightarrow{OC}\right|$

$= \frac{1}{3}\left|\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -4 & 2\\ -5 & -4 & 8\end{vmatrix}\cdot(-5\vec{i}-12\vec{j}+2\vec{k})\right|$

$= \frac{1}{3}\left|(-24\vec{i}-18\vec{j}+16\vec{k})\cdot(-5\vec{i}-12\vec{j}+2\vec{k})\right|$

$= \frac{1}{3}\left|(-2)(-5)+(-18)(-12)+(16)(2)\right|$

$= 86$ cubic units

d) i) $\because$ Π does not contain O which is the origin

$\therefore$ $p\neq 0$, $q\neq 0$, $r\neq 0$

$\therefore$ $pqr\neq 0$.

ii) Let $\vec{n}$ be unit vector $\perp$ to Π

$\vec{n} = \dfrac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|}$

$= \frac{1}{12\sqrt{41}}(48\vec{i}-36\vec{j}+48\vec{k})$

$= \frac{1}{\sqrt{41}}(4\vec{i}-3\vec{j}+4\vec{k})$

$\overrightarrow{OD} = (\overrightarrow{OA}\cdot\vec{n})\cdot\vec{n}$

$= \frac{1}{\sqrt{41}}(4+12+8)\left(\frac{1}{\sqrt{41}}\right)(4\vec{i}-3\vec{j}+4\vec{k})$

$= \frac{24}{41}(4\vec{i}-3\vec{j}+4\vec{k})$

Answers written in the margins will not be marked.

iii) $\overrightarrow{OE} = \frac{1}{pqr}(qr\vec{i} + pr\vec{j} + pq\vec{k})$

$\overrightarrow{OE} \cdot \overrightarrow{OD} = \frac{1}{pqr}(qr\vec{i} + pr\vec{j} + pq\vec{k}) \cdot \frac{24}{41}(4\vec{i} - 3\vec{j} + 4\vec{k})$

$= \frac{24}{41pqr}(4qr - 3pr + 4pq)$

END OF PAPER

Answers written in the margins will not be marked.

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Comments

The candidate demonstrates comprehensive knowledge and understanding of the concepts underpinning algebra and calculus in the curriculum by applying them successfully at a sophisticated level to a wide range of unfamiliar situations in Questions 9, 10, 11 and 12.

He/She is able to communicate and express views and arguments precisely and logically using mathematical language, notations and diagrams, such as in Questions 1, 3, 5, 6, 7, 8, 9, 10, 11 and 12.

He/She also provides complex mathematical proofs in a logical, rigorous and concise manner in Questions 11(b) and 12(d)(i).

It can be concluded that the candidate has the ability to integrate knowledge and skills from different areas of the curriculum in handling complex tasks using a variety of strategies.