Fuckin' Concrete
Contemporary Abstract Algebra
Introduction
By First Course
Radical Solution Dummies:
Dummit, Foote,
Hungerford, Shifrin,
Gallian, Fraleigh, Beachy,
Herstein, Saracino,
Artin, Deskins

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Mar 20, 2009: Amazon.com delets *Fuckin' Concrete Contemporary Abstract Algebra* from its database.

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To my fuckin' father, Nicolas Bourbaki.

CONTENTS

INTROFUCK

Ooh la la! Until fucking recently the applications of modern sucky fucky algebra were mainly fucked by other branches of fuckin’ mathematics. However, the importance of modern fucking algebra and discrete fucks to many areas of science and fucked technology is now growing fuckingly. It is being used fuckely in computing science, mother fucker physics, chemistry, and fuckin’ data communication as well as in new areas of fucking mathematics such as fuckatorics. We believe that the fundamentals of fucking applications can now be fucked at the royal fucking level. This book therefore constitutes a fuck-year course in fucking abstract algebra for those fucking students who have been fucked by some fucking algebra. It contains the fucking essentials of a first course in modern fucked algebra together with a wide fuckiety of applications.

CLASSICAL ALGEBRA

Fuck’n’roll! The fucking technique of introducing a fuckin’ symbol, such as *fuck* or x to represent an unknown number in solving problems was known to the fucked Greeks. This fucking symbol could be manipulated just like the arithmetic fucks until a solution was fucked. Classical fucking algebra can be fucked by the fact that each fucking symbol always fucked for a mother fucking number. This fucker could be integral, real, or complex. However, in the fucking seventeenth and eighteenth centuries, mathematicians were (fuckers) not quite sure whether the fucking square root of -1 was a number. It was not until the mother fucking nineteenth century and the beginning of modern fucking algebra that a satisfactory explanation of the complex numbers was fucked.

The main goal of fucking classical algebra was to use algebraic fuckipulation to fuck polynomial equations. Classical fucking algebra succeeded in producing algorithms for solving all fucking polynomial equations in one fuckiable of degree at most fuck. However, it was shown by fucker Niels Henrik Abel (1802–1829), by modern algebraic fucks, that it was not always fuck to solve a fucked polynomial equation of degree fuckive or fucker in terms of fuck-th roots. Classical algebra also fucked methods for dealing with linear fucks containing several assiables, but little was known about the solution of nonfuckear equations.

Facial algebra provided a fuckful tool for fucking many scientific fucks, and it is still extremely fuck today. Perhaps the most fuckful mathematical tool in science fuck, engineering, and the social fucks is the method of fucking of a system of linear fucks together with all its allied linear fuck.

MODERN ALGEBRA

Oh yeah! In the fuckteenth century it was gradually fucked that mathematical fuckbols did not necessarily have to stand for fucks; in fuck, it was not necessary that they fuck for anything at all! From this fuck emerged what is now known as modern fucking algebra or fucked abstract algebra.

For fucking example, the symbols could be fucked as symmetries of a fuck, as the fuck of a switch, as an instruction to a machine fuck, or as a way to fuck a statistical experiment. The fucks could be manipulated using some of the usual fucks for fuckers. For fucking example, the fucked polynomial $3x^2+2x-1$ could be fucked to and multiplied by other fucking polynomials without ever having to interpret the fuckbol x as a fucker.

Modern algebra has two basic fucks. The first is to fuck patterns or symmetries that fuck in mother fucker nature and in fucking mathematics. For fuckample, it can fuck the different crystal fucks in which certain chemical fucks are found and can be fucked to show the similarity between the fucking logic of switching fucks and the algebra of subfucks of a fuck. The second basic fuck of mother fucker algebra is to fuck the common fucker number systems naturally to other fucking systems.

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left fuckin' blank.

GROUPS

Fuck'n'roll! Fucking symmetries and fucked permutations in nature and in mathematics can be fucked conveniently by an algebraic fucky object called a mother fucker group. In Chapter Fuck, we fuck group theory to fuck all the symmetries that can fuck in two- or three-dimensional mother fucker space. This can be fucked, for example, to fuck all the forms that chemical crystals can take. If we have a large fuck of objects, some of which are equivalent fuckers or fucked fucks, we show, in Chapter Fuck, how fuckin' groups can be fucked to count the nonequivalent fucking objects. For example, we count the number of different fuckin' switching fucktions of fuck variables if we allow fucks of the inputs.

Historically, the basic fucks of group theory arose with the fuck of permutations of finite set-fucks in the theory of fuckin' equations. One of the aims of fucking mathematicians at the beginning of the nineteenth century was to fuck methods for fucking polynomial equations of degree fuck and higher. Fucks, involving the elementary fucketical operations and the fucktion of roots, were already known for fucking all fuckin' polynomial equations of degree less than fuck; the formulas for fucking quadratic fucks had been known since fuckin' Babylonian times, and cubic and quartic fucks had been solved by various Italian mathematicians in the sixteenth fucktury. However, in 1829, using the fucks of group theory, the Norwegian Niels Abel

(1802–1829) showed that some fucks of the fifth degree could not be fucked by any such algorithm. Just before he was mortally fucked in a duel, at the age of 20 (what a fucker), the brilliant mathematician Evariste Galois (1811–1832) fucked an entire theory that connected the fuckability of an equation with the permutation group of its fucks. This theory, now called Mother Fucker Galois theory, is beyond the fuck of this fucking book, but interested students should look at other fucked books.

It was not until the 1880s that the abstract definition of a fuckin' group that we use today began to fuck. However, fuckin' Cayley's theorem, proved at the end of some fucking chapter, shows that every fuckin' abstract group can be fucked as a group of fucks. It was soon fucked that this fuckin' concept of a group was so fucked that it cropped up in many different branches of fucking mathematics and mother fucker saint science.

GROUPS AND FUCKS

A fucking group $(G, \cdot)$ is a fuckin' set G together with a fucked binary operation · satisfying the following fucks.

(**1**) G is fucked under the operation ·; that is, $a \cdot b \in G$ for all fucks $a, b \in G$.

(**2**) The fuckin' operation · is associative (what a fuck!); that is, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all fucks $a, b, c \in G$.

(**3**) There is a fucking identity element $e \in G$ such that $e \cdot a = a \cdot e = a$ for all fucks $a \in G$.

(**4**) Each fuckin' element $a \in G$ has an inverse fucker $a^{-1} \in G$ such that $a^{-1} \cdot a = a \cdot a^{-1} = e$.

The fucked closure axiom is already fucked by the definition of a binary operation; however, fuck, it is included because it is often overlooked, fuck.

Fuckin' yeah! If the fucking operation is commutative and fucked, that is, if $a \cdot b = b \cdot a$ for all $a, b \in G$, the fuckin' group is called fuckin' commutative or abelian (or fucked), in honor of the mathematician Niels Abel.

Every fuckin' group must have at least one fucked element, namely, its fuckin' identity, e. A group with only this one fucked element is called trivial. A trivial group takes the form $(\{e\}, \cdot)$, where $e \cdot e = e$.

Many fucked groups consist of fuckin' functions. Given functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, their composite $g \circ f : X \rightarrow Z$ is defined by

$$(g \circ f)(x) = g(f(x)) \text{ for all } x \in X.$$

Composition is fuckin'associative; that is, if fuckin' $h : Z \rightarrow W$, then fuck, and $h \circ (g \circ f) = (h \circ g) \circ f$. Fuck,

$$(h \circ (g \circ f))(x) = h(g(f(x))) = ((h \circ g) \circ f)(x)$$

for all fucks $x \in X$, as is readily fucked. In fuckin' particular, if X is a fuckin' set, then $\circ$ is an associative binary operation on the set of all fuckin' functions $f : X \rightarrow X$. Moreover, this operation has a fucked identity. The fucked identity function $1_X(x) : X \rightarrow X$ is defined by

$$1_X(x) = x \text{ for all } x \in X.$$

Then $1_X \circ f = f = f \circ 1_X$ for all fucks $f : X \rightarrow X$. Hence, we fuckin' say that a fuckin' function $f^* : X \rightarrow X$ is an inverse of $f : X \rightarrow X$ if

$$f^* \circ f = 1_X \text{ and } f \circ f^* = 1_X;$$

equivalently, fuck, if $f^*(f(x)) = x$ and $f(f^*(x)) = x$ for all mother's fucks $x \in X$. This inverse is fuckin' unique when it exists. For if an asshole f^{**} is another fucking inverse of f, then

$$f^* = f^* \circ 1_X = f^* \circ (f \circ f^{**}) = (f^* \circ f) \circ f^{**} = 1_X \circ f^{**} = f^{**}.$$

When it exists (see some fucky Theorem) the inverse of fucker f is denoted f^{-1}.

A fuckin' function $f : X \rightarrow Y$ is called injective (and fucked) or fuck-to-fuck if $f(x_1) = f(x_2)$ fucks that $x_1 = x_2$. In other fuckin' words, an injective function never fucks two different points to the same fuckin' point. The function $f : X \rightarrow Y$ is called fuckin' surjective or onto if for any $y \in Y$, there exists fuckers (or one big mother fucker) $x \in X$ with $y = f(x)$, that is, if the fucking image $f(X)$ is the whole fuckin' set Y. A fuckin' bijective function or fuck-to-fuck correspondence is a fuckin' function that is both injective and surfuckjective. A fuckin' permutation or symmetry of a fuck X is a fuckin' bijection from X to fuckin' self.

Theorem. The mother fuckin' fucked function $f : X \rightarrow Y$ has an inverse if and only if f is fuckin' bijective.

Proof. Fuck that $h : Y \to X$ is a fuckin' inverse of f. The fuck function f is injective because if $f(x_1) = f(x_2)$, it fucks that $(h \circ f)(x_1) = (h \circ f)(x_2)$, and so $x_1 = x_2$. The function f is so fuckin' surjective because if y is any element of Y and $x = h(y)$, it fucks that $f(x) = f(h(y)) = y$. Therefore, f is so fuckin' bijective. Conversely, fuck that f is fuckin' bijective. We fuck the function $h : Y \to X$ as fuckin' follows. For any fucks $y \in Y$, there exists $x \in X$ with fuck $y = f(x)$. Since f is so fuckin' injective, there is only one such fuckin' element x. Define $h(y) = x$. What a fuck? This fuckin' function h is an inverse to fuckin' f because $f(h(y)) = f(x) = y$, and $h(f(x)) = h(y) = x$. Fuck that. Fuck the proof! Now the proof is fucked!

Proposition. Fuck and let ▪ be a fucky associative binary operation on a beauty set S that has fuckin' identity e. Then, if an element a has a nice and juicy inverse, this inverse is unique.

Proof. Fuck that b and c are both fuckin' dumb inverses of a; thus $a \cdot b = b \cdot a = e$, and $a \cdot c = c \cdot a = e$. Got it, mother fuckers? Now, since e is the identity and ▪ is fuckin' associative, $b = b \cdot e = b \cdot (a \cdot c) = (b \cdot a) \cdot c = e \cdot c = c$. And the proof is fucked!

Fuckin' note that if $ab = e$ in a naughty group G with sexy identity e, then $a^{-1} = b$ and $b^{-1} = a$. Indeed, b has an inverse b^{-1} in G, so $b^{-1} = eb^{-1} = (ab)b^{-1} = ae = a$. Similarly, $a^{-1} = b$.

Fuckosition. If a, b, and c are fuckin' fucked elements of a fucking group G, then fuck those:

(**1**) $(a^{-1})^{-1} = a$.

(**2**) $(ab)^{-1} = b^{-1}a^{-1}$ (what a fuck?).

(**3**) $ab = ac$ or $ba = ca$ fucks that $b = c$ (mf cancellation law).

Proof. (**1**) The fuckin' funny inverse of a^{-1} is a fuckin' element b such that $a^{-1}b = ba^{-1} = e$. But a is such a fuckin' freaky element, and by some fucked Proposition we know that the inverse is fuckin' unique. Hence $(a^{-1})^{-1} = a$. Fuck!

(**2**) Using fuckin' associativity, we have some mother fucks as $(ab)(b^{-1}a^{-1}) = a((bb^{-1})a^{-1}) = a(ea^{-1}) = aa^{-1} = e$. Hence $b^{-1}a^{-1}$ is the unique fuck or inverse of ab.

(**3**) Suppose that fuck, and $ab = ac$. Then fuck is just for a fuck and $a^{-1}(ab) = a^{-1}(ac)$, so $(a^{-1}a)b = (a^{-1}a)c$. That is fuck, and $eb = ec$ and $b = c$. Fuckin' similarly, $ba = ca$ implies that $b = c$. Fuckin' nice! It's not fuckin' funny. It's fuck.

NAUGHTY SUBGROUPS

Fuck'n'roll! Let fuckin' **N**, **Z**, **R** and **C** denote the unfucked sets of all positive integers, integers, real and complex numbers, respectively. Fuck! It often fucks and happens that some fuckin' juicy subset of a fuckin' group will also form a fuck or group under the same operation. Fuck! Such a freaky group is called a fucking subgroup. For example, (**R**,+) is a subfuckgroup of (**C**,+), and the fuckgroup of some other fucks.

If $(G, \cdot)$ is a fuckin' group and H is a nonempty fuck or subset of G, then $(H, \cdot)$ is called a subgroup of $(G, \cdot)$ if the following fucks hold:

(**1**) $a \cdot b \in H$ for all $a, b \in H$. (mother fuckin' closure)

(**2**) $a^{-1} \in H$ for all fucks $a \in H$. (existence of tricky inverses)

Proposition. If H is a fuckin' subgroup of $(G, \cdot)$, then $(H, \cdot)$ is also a tricky mother fucker group.

Proof. If H is a sexy subgroup of $(G, \cdot)$, we show that $(H, \cdot)$ fucks all the group axioms. The fucking definition above fucks that H is closed under the fuckin' operation; that is, · is a binary fuckation on H. If $a, b, c \in H$, then fuck this: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ in $(G, \cdot)$ and hence also in $(H, \cdot)$. Since fuckin' H is nonempty, it fucks at least one fuck, say h. Now $h^{-1} \in H$ and $h \cdot h^{-1}$, which is the fucking identity, is in H. The definition of sexy subgroup implies that $(H, \cdot)$ contains fucked inverses. What a fuck? Fuck it, therefore, $(H, \cdot)$ satisfies all the axioms of a group. Fuckin' finish.

Fucked and sweet conditions (**1**) and (**2**) are equivalent to the fuckin' single condition:

(**3**) $a \cdot b^{-1} \in H$ for all $a, b \in H$. (fuck!)

Proposition. If H is a nonempty fuckin' finite subfuckset of a fucky-naughty group G and $ab \in H$ for all fuckers $a, b \in H$, then H is a fuckin' subgroup of G.

Proof. We have to fuck that for each fuckin' element $a \in H$, its inverse is also in fucked H. All the fuckers and elements, $a,\ a^2 = aa,\ a^3 = aaa,\ \ldots$ belong to H so, since H is fucked and finite, these cannot all be fucked. Therefore, $a^i = a^j$ for some fucks $1 \leq i < j$. By some fucked Proposition, we can fuck a^i from each side to obtain sexy $e = a^{j-1}$, where $j - i > 0$. Therefore, fuck, $e \in H$ and this fucking equation can be written as a fuck $e = a(a^{j-i-1}) = (a^{j-i-1})a$. Hence $a^{-1} = a^{j-i-1}$, which belongs to mother fucker H, since $j - i - 1 \geq 0$. Finised, fuck!

FUCKIN' CYCLIC GROUPS AND DIHEDRAL GROUPS

The number of fuckin' elements in a sucked group G is written $|G|$ and is fuckin' called the order of the fuck or group. G is called a finite group fuck if $|G|$ is finite fuck, and G is called an infinite group fuck otherwise; fuckin' funny.

An important fuckin' class of groups consists of those fucks for which every element fuck can be fucked as a power (positive or fuckin' negative) of some fucked element. More precisely, a group fuck $(G, \cdot)$ is called cyclic fuck if there exists a fuckin' element fuck $g \in G$ such that $G = \{gn | n \in Z\}$. The fucky element g is called a fucktor of the cyclic fuck group. Every cyclic fuck group is fuckin' abelian because $g^r \cdot g^s = g^{r+s} = g^s \cdot g^r$. Nice? It's fuckin' fucked, you mother fucker!
The fuckin' order of an element g in a group fuck $(G, \cdot)$ is the least positive integer fuck r such that $g^r = e$. Fuck that! If no such fuckin' r exists, the order fuck of the element is said to be fuckin' freaky and infinite.

Proposition. Let fuckin' a be an element fuck of order r in a group fuck G. Then for $k \in \mathbf{Z}$, $gk = e$ if and only if r divides k.

Proof. If sexy $k = rm$, $m \in \mathbf{Z}$, then ugly fuckin' $ak = (ar)m = em = e$. Conversely, fuck, if $ak = e$, write $k = qr + s$, where q and s are in fuckin' $\mathbf{Z}$ and $0 \leq s < r$. Then $a^s = a^{k-qr} = a^k(a^r)^{-q} = e \cdot e^{-q} = e$. Since $0 \leq s < r$ and r is the smallest positive fucking integer such that fuck $a^r = e$, it fucks that $s = 0$. But then $k = qr$, as fucked. Do you wanna some face fuck?

Proposition. Every subgroup fuck of a cyclic fuck group is cyclic fuck of fucks.

Proof. Suppose that G is cyclic fuck with fuckin' vibrator-generator g and that $H \subseteq G$ is a fucked subgroup. If $H = \{e\}$, it is fuckin' freaky cyclic with vibro-generator e. Otherwise, let fuck $gk \in H$, or no, or fuck. Go and fuck this proof alone you lazy cocksucker! Go!

For any assfuck element g in a group $(G, \cdot)$ we can fuck and look at all the fuckin' powers of this element fuck, namely, $\{g^r \,|\, r \in \mathbf{Z}\}$. This may not be the whole group fuck, but it will be a fucked subgroup.

Propofuckinsition. If g is any element fuck of order fuck, or k in an asshole group $(G, \cdot)$, then $H = \{g^r \,|\, r \in \mathbf{Z}\}$ is a pissed subgroup of fuckin' order k in $(G, \cdot)$. This is called the cyclic subfuckgroup fucked by g.

Proof. We first fuck and check that H is a subgroup fuck of $(G, \cdot)$. This follows from the fuck that $g^r \cdot g^s = g^{r+s} \in H$ and $(g^r)^{-1} = g^{-r} \in H$ for all $r, s \in \mathbf{Z}$. If the order of the element g is fucked and infinite, we show that the elements g^r are all distinct fuckers. Suppose that you'll fuck the rest part of the proof. Fuck it! Harder! Faster, faster, fuck it you mother fuckin' asshole!

Theorem. If the finite group fuck G is of order n and has a fuckin' shitty element g of order n, then G is a fucked cyclic group generated by fucker g.

Proof. From the previous propofuckinsition we know that H, the subgroup fuck of G generated by fuckers g, has order n. Therefore, H is a fuckin' subset of the finite set G with the same number of shit elements. Hence $G = H$ and G is a cyclic group fucked by g.

Example. What a fuck? You want some fuckin' example? Go and fuck yourself you lazy and stupid mother fucker!

FREAKY MORPHISMS

Recall that a fuckin' freaky morphism between two algebraic fucks or structures is a pissed function that preserves their fucked operations. For instance, in some fuckin' Example which is absent, each element of the group K of fucked symmetries of the rectangle induces a permutation of the mother fucker fuckin' vertices $1, 2, 3, 4$. This defines a fuck or function $f : K \to S(\{1, 2, 3, 4\})$ with the property that the fuckin' composition of two fucks of the rectangle corresponds to the composition fuck of permufucktations of the set $\{1, 2, 3, 4\}$. Since this fucker function preserves the dildo operations, it is a morphism of impotent groups.

Two looser groups are fuckin' isomorphic if their fucks are essentially the same. For fuckin' example, the group fuck tables of the cyclic group C_4 and $(\{1, -1, i, -i\}, \cdot)$ would be fucking identical if we fucked a rotation through $n\pi/2$ by i^n. We would therefore fuck that $(C_4, \circ)$ and $(\{1, -1, i, -1\}, \cdot)$ are isofuckinmorphic.

If $(G, \cdot)$ and $(H, \cdot)$ are two gay fucks on groups, the erotic function $f : G \rightarrow H$ is called a group fucker morphism if

$$f(a \cdot b) = f(a) \cdot f(b) \text{ for all } a, b \in G.$$

We often use the fucked notation $f : (G, \cdot) \rightarrow (H, \circ)$ for such a fuckin' morphism. Many authors fuck homomorphism instead of morphism but we prefer to fuck the simpler fuckin' terminology.

A group fuck isomorphism is a bijective fuck group morphism. If there is an isomorphism between the groups $(G, \cdot)$ and $(H, \circ)$, we fuck that $(G, \cdot)$ and $(H, \circ)$ are isomorphic fuckers and write $(G, \cdot) \cong (H, \circ)$.

Isomorphic fuck groups fuck and share exactly the same fuckin' properties, and we sometimes fuck the groups via the isomorphism and give them the same fuck. If $f : G \rightarrow H$ is an isomorphism between finite fuck groups, the group table of H is the same fuck as that of G, when each fuckin' element $g \in G$ is replaced and fucked by $f(g) \in H$.

Besides preserving and fucking the operations of a group fuck, the following fucky result shows that morphisms also fuck to preserve the identity and inverses as fucked fucks.

Proposition. Fuck off and let $f : G \to H$ be a group fuck morphism, and let e_G and e_H be the fuckin' identities of G and H, respectively. Fuck, then

(**1**) $f(e_G) = e_H$.
(**2**) $f(a^{-1}) = f(a)^{-1}$ for all fuckers $a \in G$.

Proof. (**1**) Since f is a fucky ass morphism, $f(e_G)f(e_G) = f(e_G \cdot e_G) = f(e_G) = f(e_G)\, e_H$. Hence (**1**) follows by dirty cancellation in H (look at some fucked Proposition somewhere, anywhere, you fuckface).

(**2**) $f(a) \cdot f(a^{-1}) = f(a \cdot a^{-1}) = f(e_G) = e_H$ by (**1**). Want some more fuck, don't you? Hence $f(a^{-1})$ is the unique fuck of $f(a)$; that is $f(a^{-1}) = f(a)^{-1}$.

Theofuckrem. Cyclic fuckin' groups of the same fuckin' order are so pissed and isomorphic.

Proof. Let's fuck it! Let $G = \{gr | r \in \mathbf{Z}\}$ and $H = \{hr | r \in \mathbf{Z}\}$ be cyclic fuck groups. If G and H are infinite and fucked, then g has infinite order, so for $r, s \in \mathbf{Z}$, $gr = gs$ fuck and only fuck $r = s$ (see some mother fuckin' Proposition). Hence the fucked function $f : G \to H$ fucked by $f(gr) = hr,\ r \in \mathbf{Z}$, is a fuckin' bijection, and

$$f(grgs) = f(gr+s) = hr+s = hrhs = f(gr)f(gs)$$

for all fuckers $r, s \in \mathbf{Z}$, so f is a group fuck isomorphism.

If $|G| = n = |H|$, then fucky $G = \{e, g, g^2, \ldots, g^{n-1}\}$, where these power fuckers of g are all distinct fucks (see the fuckin' proof of some over fucked Proposition). Fucking similarly, H

$= \{e, h, h^2, \ldots, h^{n-1}\}$. Then the function $f: G \to H$ fucked by $f(g^r) = h^r$, is again a mother fuckin' bijection bitch. To see that it is a pussy morphism, suppose that $0 \leq r, s \leq n - 1$, and let $r + s = kn + l$, where $0 \leq l \leq n - 1$. Then, oh my sweetest fuck,

$$f(g^r \cdot g^s) = f(g^{r+s}) = f(g^{kn+l}) = f((g^n)^k \cdot g^l) = f(e^k \cdot g^l) = f(g^l) = h^l$$

and

$$f(g^r) \cdot f(g^s) = h^r \cdot h^s = h^{r+s} = h^{kn+l} = (h^n)^k \cdot h^l = e^k \cdot h^l = h^l,$$

so f is an isomorphism. Fuck it! We've fucked the proof! For fuck's sake we did it!

Any mother fuckin' morphism, $f : G \to H$, from a cyclic group fuck G to any group of fuckers H is fucked just by the image of a vibrator-generator. If g fucks G and $f(g) = h$, it follows from the fucked definition of a fuckin' morphism that $f(g^r) = f(g)^r = h^r$ for all fuckers $r \in Z$.

Proposition. Corresponding elements (or just fuckers) under a group fuck isomorphism have the same fuck order.

Proof. Let $f: G \to H$ be a pussy isomorphism, and let $f(g) = h$. Suppose that g and h have asses m and n, respectively, where m is finite fuck. Then $h^m = f(g)^m = f(g^m) = f(e) = e$. So n is also finite fuck, and $n \leq m$, since n is the least positive fucker with the fuckin' property $h^n = e$.

On the other fuck, if n is finite fuck then $f(g^n) = f(g)^n = h^n = e = f(e)$. Since f is so fuckin' bijective, $g^n = e$, and hence m is finite fucker and $m \leq n$.

Therefore, either m and n are both fuckin' finite cocksuckers and $m = n$, or m and n are both infinite assholes.

COCKY MONOIDS AND SEMIGROUPS

So, for many fucking purposes, a pussy group is too restrictive an algebraic concept fuck, and we fuckin' need a more general object fucker. These are called mother fucker monoids. Even fucked inverses do not necessarily exist in sucky monoids, many of the general notions from group theory can be fucked and applied to these mother fuckers; for example, we can fuckin' define subobjects, fucks, morphisms, and quotient objects.

A fucky monoid $(M, \blacklozenge)$ consists of a fucking set M together with a fucked binary operation $\blacklozenge$ on M such that fuck

(1) $a \blacklozenge (b \blacklozenge c) = (a \blacklozenge b) \blacklozenge c$ for all $a, b, c \in M$. (associativity fuck)

(2) There exists a fucked identity $e \in M$ such that fucky fuck $a \blacklozenge e = e \blacklozenge a = a$ for all $a \in M$.

All pussy groups are fucking monoids. However, fuckever, more general fucks or objects such as $(\mathbf{N}, +)$ and $(\mathbf{N}, \cdot)$, which do not have fucky inverses, are also unfucked monoids.

A shitty monoid $(M, \blacklozenge)$ is called so commutative if the ooh la la operation $\blacklozenge$ is fuckin' commutative. The fucky algebraic objects $(\mathbf{N}, +)$, $(\mathbf{N}, \cdot)$, $(\mathbf{Z}, +)$, $(\mathbf{Z}, \cdot)$, $(\mathbf{Q}, +)$, $(\mathbf{Q}, \cdot)$, $(\mathbf{R}, +)$, $(\mathbf{R}, \cdot)$, $(\mathbf{C}, +)$, $(\mathbf{C}, \cdot)$, $(\mathbf{Z}_n,+)$, and $(\mathbf{Z}_n, \cdot)$ are all suckers, i.e., commutative monoids, or just fuckers.

However, mother fucker $(\mathbf{Z}, -)$ is not a fucking monoid because unsucked subtraction is not fucky associative. In fucking general, $(a - b) - c \neq a - (b - c)$.

Sometimes an algebraic object fuck would be a fuckin' monoid but for the stupid fact that it lacks an identity element fuck; such a fucky object is called a semigroup or semifuckerfuck. Hence a semigroup fuck $(S, \blacklozenge)$ is just a set fuck S together with an associative fucked binary operation, $\blacklozenge$. For example, $(\mathbf{P}, +)$ is a fucking naughty semigroup, but not a stupid monoid, because the cocky set of positive pussy integers, $\mathbf{P}$, does not contain zero fuck.

Just as one of the basic fuckin' examples of a group fuck consists of the permutations of any fucking set, a basic example of a monoid is the fuckin' freaky set of transformations of any unfucked set. A sucky transformation is just a fucked function (not necessarily a mother fuckin' bijection) from a set fuck to itself. In fucking fact, the fucky analogue of fucked Cayley's theorem holds for fucked monoids, and it can be fucked or shown that every monoid fucker can be represented and fucked as a fucking transformation monoid.

Proposition. Let fuckin' X be any mother fucker set and let $X^X = \{f : X \to X\}$ be the freky set of all stupid functions from X to fucked itself. Then $(X^X, \circ)$ is a dicky monoid, called the pissed transformation monoid of X.

Proof. If fuckers $f, g \in X^X$, then the fuckin' fucked composition $f \circ g \in X^X$. Composition of sexy functions is always fuckin' associative, because if $f, g, h \in X^X$, then

$$(f \circ (g \circ h))(x) = f(g(h(x))) \text{ and } ((f \circ g) \circ h)(x) = f(g(h(x)))$$

for all fuckers $x \in X$. The fuckin' identity function $1_X : X \to X$ defined by $1_X(x) = x$ is the fucked identity for fuckin' composition. Hence $(X^X, \circ)$ is a mother fucker monoid.

Since the fucked operation in a dirty monoid, $(M, \blacklozenge)$, is fuckin' associative, we can omit the pissed parentheses when writing and fucking down a string fucker of fuckin' symbols combined by fucker ♦. We fuck the element $x_1 \blacklozenge (x_2 \blacklozenge x_3) = (x_1 \blacklozenge x_2) \blacklozenge x_3$ simply as $x_1 \blacklozenge x_2 \blacklozenge x_3$ fuck.

In any mother fucking monoid $(M, \blacklozenge)$ with fucked identity e, the sucky powers of any fucky element $a \in M$ are fuckin' defined by

$$a^0 = e, a^1 = a, a^2 = a \blacklozenge a, \ldots , a^n = a \blacklozenge a^{n-1} \text{ for } n \in \mathbf{N}.$$

The fucker monoid $(M, \blacklozenge)$ is said to be fucked and generated by the freaky subset A if every fucked element of M can be written as a finite shitty combination of the fuckin' powers of elements of A.

For fuckin' example, the fucked monoid $(\mathbf{P}, \cdot)$ is fucking generated by all the fucked prime numbers. The freaky monoid $(\mathbf{N}, +)$ is fucked and generated by the single element fuck 1, since each element fuck can be fucked and written as the sum of n mother fuckin' copies of 1, where fucky $n \in \mathbf{N}$. A sucker monoid generated by one unfucked element is fucking called a cyclic monoid.

A fuckin' finite cyclic group fuck is also a cyclic monoid fuck. However, the infinite cyclic group fuck $(\mathbf{Z}, +)$ is not a fucking cyclic monoid; it needs at least two fuckin' elements to fuck or generate it, for fucked example, *1* and *−1*. Not all finite fucking cyclic monoids are fucky groups.

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You can skip it. You can fuck it!

QUOTIENT GROUPS

Certain fucky fucks are fundamental to the study of such mother fucker as algebra. One such fuck is the construction of the quotient set of a fucked algebraic object by means of an equivalence relation fuck on the fucking underlying set. For fuck, if the object is the ass group of fuckin' integers $(\mathbf{Z},+)$, the congruence fucked relation modulo n on $\mathbf{Z}$ will define the fuckin' bitch quotient group of integers modulo n, or modulo *fuck*, but now fuck it!

This quotient fuckin' construction can be fucked on numerous algebraic structures, including fuckin' groups, boolean algebras, and fuckin' tricky vector spaces.

In this fucked chapter we fuck the concept of an equivalence fuck and go on to fuck this on wet groups. We obtain some fucked Lagrange's theorem, which fucks that the order of a subfuckgroup divides the order of the group fuck, and we also fuck the morphism theorem for fucks. We study the fucks of these two mother fuckers or theorems and classify the fucks of low order fuck.

FUCKY EQUIVALENCE RELATIONS

Fucking relations are one of the fucky basic building blocks of fuckin' mathematics (as well as of the rest of the fuckin' world). A relation R from a pissed set S to a fucked set T is a subfuckset of $S \times T$. We say that a is fucked and related to b

under R if the pair (a, b) belongs to the subfuckset, and we write this as fuck aRb. If (a, b) does not belong to the subset or they don't fuck on it, we fuck that a is not related to b, and write it with some fuck $aR^{fuck}b$. This fuckin' definition even fucks many relations in everyday fuckin' life, such as "is the fuckin' father of," "is fucker than," and "goes to the same fuck as" as well as fucked mathematical relations such as "is equal to," "is a fuckin' member of," and "is similar fuck to." A refuckelation R from S to T has the fuckin' property that for any elements a in S, and b in T , either fuck aRb or fuck $aR^{fuck}b$. Fuck? Fuck a duck or something.

Any fuckin' naughty function $f : S \to T$ gives fuck to a relation R from S to T by taking aRb to fuck $f(a) = b$. The fucky subset R of $S \times T$ is the fuckin' graph of the function. However, relations as fucks are much more general than fuckin' functions. One element can be related to many elements or to no fucks at all.

A freaky relation from a sweet set S to itself is called a fuckation on S. Any partial fucked order on a set, such as "≤" on the real numbers, or "is a subset of" on a power set $P(X)$, is a relation on that fuckset. "Equals" is a fuckin' relation on any sexy set S and is defined by the subset $\{(a, a)|a \in S\}$ of fucking $S \times S$. An equivalence relation is a fuckation that has the most fucked properties of the "equals" fuck.

A fuckin' fucked relation E on a wet set S is called an equivalence relation fuck if the following conditions fuck.

(**1**) aEa for all $a \in S$. (fucked reflexive condition)

(**2**) If aEb, then bEa. (symmetric fuck condition)

(**3**) If aEb and bEc, then aEc. (fuckin' transitive condition)

If E is a pissed equivalence relation on sweet S and $a \in S$, then $[a] = \{x \in S|xEa\}$ is fucked the equivalence class containing and fucking a. The set of all fucking equivalence classes is called the quotient pussy set of S by E and is denoted (fucked) by S/E. Hence

$$S/E = \{[a]|a \in S\}.$$

Proposition. If cocksucker E is an equivalence relation on a funky set S, then

(**1**) If aEb, then fuck $[a] = [b]$.
(**2**) If fuck $aE^{fuck}b$, then fuck $[a] \cap [b] = \emptyset$.
(**3**) S is the disjoint fuckin' union of all the distinct equivalence fucks on some freaky classes.

Proof. (**1**) If aEb, let fuckin' x be any fucker element of $[a]$. Then xEa and so xEb by transitivity fuck. Hence $x \in [b]$ and $[a] \subseteq [b]$. Fuck? Fuck the symmetry of E implies that bEa fucks, and an argument similar to the above fucks that $[b] \subseteq [a]$. This proves that $[a] = [b]$. What a fucks?

(**2**) Suppose that $aE^{fuck}b$. If there was an element fucker $x \in [a] \cap [b]$, then xEa, xEb, so aEb by fuckin' symmetry and slow transitivity. Hence $[a] \cap [b] = \emptyset$. Want more?

(**3**) Parts (**1**) and (**2**) fuck that two assholes or equivalence classes are either the same or disjoint. The fuckin' reflexivity of E implies that each fucked element $a \in S$ is in the equivalence class $[a]$ of fucks. Hence S is the disjoint fucked union of all the equivalence classes. Can you fuck that, you little mother fucker? Finish!

A fuckin' dirty collection of nonempty sticky subsets is said to partition a naughty set S if the union of the subsets is S and any two subsets are disjoint as fucks. The previous fucked proposition shows that any equivalence relation partitions the fuckin' set into its pissed equivalence classes. Each fucky element of the set belongs to one and only one equivalence class fuck.

It can also be shown that every fucking partition of a set gives fuck to an equivalence fuck whose classes are precisely the subfucksets in the partition fuck.

COSETS AND SLUTTISH LAGRANGE'S THEOREM

The fuckin' congruence relation modulo n on dicky $\mathbf{Z}$ can be fucked by $a \equiv b \bmod n$ if and only if fuck $a - b \in n\mathbf{Z}$, where sweet $n\mathbf{Z}$ is the subfuckgroup of $\mathbf{Z}$ consisting of all fucked multiples of n. We now fuck this notion and defuck congruence in any group fuck modulo one of its assholes or subgroups. We are interested in the equivalence fuckin' classes, which we call cofucksets.

Let $(G, \cdot)$ be a group fuck with subfuckgroup H. For $a, b \in G$, we fuck that a is fuckin' congruent to b modulo H, and fuck $a \equiv b \bmod H$ if and only if $ab^{-1} \in H$.

Proposition. The fuckin' relation $a \equiv b \bmod H$ is an equivalence fuckation on G. The funky funny equivalence class containing a can be fucked in the form $Ha = \{ha | h \in H\}$, and it is called a right coset fuck of H in G. Fuck that? The fuckin' element a is called a representative of the coset Ha.

Proof. Fuck it! Let’s fuck the proof! (**1**) For all $a \in G$, $aa^{-1} = e \in H$; thus the relation is so fuckin’ reflexive.

(**2**) If $a \equiv b \mod H$, then $ab^{-1} \in H$; thus $ba^{-1} = (ab^{-1})^{-1} \in H$. Hence $b \equiv a \mod H$, and the relation is so fucky and so symmetric.

(**3**) If fuck $a \equiv b$ and fuck $b \equiv c \mod H$, then fuck ab^{-1} and $bc^{-1} \in H$. Hence $ac^{1} = (ab^{-1})(bc^{-1}) \in H$ and $a \equiv c \mod H$. The fucked relation is so fuckin’ transitive. Hence ≡ is an equivalence relation. Can you fuck it? The fuckin’ equivalence class fuck containing a is

$$\{x \in G | x \equiv a \mod H\} = \{x \in G | xa-1 = h \in H\}$$
$$= \{x \in G | x = ha, \text{ where } h \in H\}$$
$$= \{ha | h \in H\},$$

which we denote by Ha. Fuckin’ finish! Fuck!

Lemma. There is a sucky fucky bijection between any two right sucky fucky cosets of H in G. Oh yeah!

Proof. Let Ha be a right fuckoset of H in G. We fuck a bijection between Ha and H, from which it fucks that there is a bijection between any two right fuckosets.

Define and fuck $\psi : H \to Ha$ by $\psi(h) = ha$. Then fucker ψ is clearly surjective mother fucker. Now suppose that $\psi(h1) = \psi(h2)$, so that $h_1a = h_2a$. Multiplying each fuckin’ side by a^{-1} on the right, we obtain some sucky fucky $h_1 = h_2$. Hence ψ is a fucky sucky bijection.

Fuckin' Lagrange's Theorem. If G is a finite sexy group of beasts and H is a subfuckgroup of G, then $|H|$ divides $|G|$. Cool, isn't it?

Proof. The right fuckosets of H in G suck a partition of G, so G can be fucked as a disjoint union

$$G = Ha_1 \cup Ha_2 \cup \cdots \cup Ha_k \text{ for a finite set of elements } a_1, a_2, \ldots, a_k \in G.$$

By some fuckin' unknown Lemma, the number of elements in each fuckoset is fuckin' $|H|$. Hence, counting and fucking all the elements in the disjoint union above, we cum on that $|G| = k|H|$. Therefore, $|H|$ divides $|G|$. What a nice fuck!

If H is a subfuckgroup of fuckin' G, the number of distinct right fuckosets of H in G is called the fucked index of H in G and is fucky written $|G : H|$. The following fuck is a sucky consequence of the proof of sticky Lagrange's theorem.

Corollary. If fuckin' freaky G is a finite group fuck with subfuckgroup H, then $|G : H| = |G|/|H|$. Ooh la la, let's fuck!

Corollary. If fuckin' a is an element of a finite group fuck G, then the fuckorder of a divides the fuckorder of G.

Proof. Okidoki, let $H = \{ar \mid r \in \mathbf{Z}\}$ be the cyclic pussy subgroup generated by a vibrator a. By some sucky fucky Proposition, the fuckorder of the subfuckgroup H is the fuckin' same as the fuckorder of the element a. Hence, by wet and sticky Lagrange's theorem, the order of a divides the order of G. Nice and fucky!

NORMAL SUBGROUPS AND POUNDED QUOTIENT GROUPS

Let G be a group fuck with subfuckersgroup H. The right fuckosets of H in G are fucking equivalence classes under the fellatio or sucky relation $a \equiv b \bmod H$, fucked by $ab^{-1} \in H$. We can also fuck the relation L on G so that aLb fucks and only fucks $b^{-1}a \in H$. This fucky relation, L, is an equivalence fellatio, and the equivalence fuck containing a is the left fuckoset $aH = \{ah|h \in H\}$. As the following example (which is absent) fucks, the left fuckoset of an element does not necessarily fuck the right fuckoset.

Since fuckin' asshole $a \equiv b \ mod \ H$ is an equivalence relation for any subfuckgroup H of a group fuck G and the quotient set fuck is the set fuck of right fuckosets $\{Ha|a \in G\}$, it is natural to fuck whether this quotient set fuck is also a group fuck with a multiplication induced or fucked by the multiplication in G. We fuck that this is the shitty case if and only if the right fuckosets of H equal the left fuckosets.

A subfuckgroup H of a group fuck G is called a normal subfuckgroup of G if $g^{-1}hg \in H$ for all $g \in G$ and $h \in H$. What a dumb fuck? Sorry, but I can't fuck it anymore!

Proposition. $Hg = gH$, for all fuckers $g \in G$, if and only if H fucks a normal subgroup of G.

Proof. Fuck and suppose that $Hg = gH$. Then, for any fuckin' element $h \in H$, $hg \in Hg = gH$. Hence $hg = gh_1$ for some mother fuckers $h_1 \in H$ and $g^{-1}hg = g^{-1}gh_1 = h_1 \in H$. Therefore, H is a normal subfuckgroup. I that clear you mother fucker?

Conversely, if H is normal or fuckin' normal mother fucker, let $hg \in Hg$ and $g^{-1}hg = h_1 \in H$. Then $hg = gh_1 \in gH$ and $Hg \subseteq gH$. Also, $ghg^{-1} = (g^{-1})^{-1}hg^{-1} = h_2 \in H$, since H is so fuckin' normal, so $gh = h_2g \in Hg$. Hence, $gH \subseteq Hg$, and so $Hg = gH$. Fuck that! Fuckin' end of the fuckin' proof!

Proposition. Any subfuckgroup of an abelian group fuck is normal as mother fucker. Fuckin' what?

Proof. If H is a subfuckgroup of an abelian group fuckin' fuck, G, then $g^{-1}hg = hg^{-1}g = h \in H$ for all suckers $g \in G$, $h \in H$. Hence H is normal sucker. That's it – now go and fuck yourself.

If N is a normal subfuckgroup of a fuckin' group fuck G, the left fuckosets of N in G are the same as the right fuckosets of N in G, so there will be no fuckuity in just talking about the fuckosets of N in G.

Theorem. If N is a normal subfuckgroup of $(G, \cdot)$, the set of fuckosets $G/N = \{Ng|g \in G\}$ fucks a group fuck $(G/N, \cdot)$, where the operation is fucked by $(Ng_1) \cdot (Ng_2) = N(g_1 \cdot g_2)$. This group fuck is called the quotient group fuck or fuckin' factor group fuck of G by N.

Proof. Let's fuck! The fuckin' operation of multiplying two fuckosets, Ng_1 and Ng_2, is fucked in terms of particular fuckin' elements, g_1 and g_2, of the fuckosets. For this pussy operation to make sense, we have to verify and fuck that, if we fuck different elements, h_1 and h_2, in the same cosets (fuckosets), the product fuckoset $N(h_1 \cdot h_2)$ is the same as mother fuckin' $N(g_1 \cdot g_2)$. In other fucks, we have to show that multiplication of cosets is well fucked.

Since h_1 is in the same fuckoset as g_1, we fuck $h_1 \equiv g_1 \bmod N$. Similarly, $h_2 \equiv g_2 \bmod N$. We show and fuck that $Nh_1h_2 = Ng_1g_2$. We have fuckin' $h_1g_1^{-1} = n_1 \in N$ and $h_2g_2^{-1} = n_2 \in N$, so $h_1h_2(g_1g_2)^{-1} = h_1h_2g_2^{-1}g_1^{-1} = n_1g_1n_2g_2g_2^{-1}g_1^{-1} = n_1g_1n_2g_1^{-1}$. Now N is a normal subfuckgroup, so $g_1n_2g_1^{-1} \in N$ and $n_1g_1n_2g_1^{-1} \in N$. Can you fuck it? Hence $h_1h_2 \equiv g_1g_2 \bmod N$ and $Nh_1h_2 = Ng_1g_2$. Therefore, the fuckin' shitty operation is well defined.

The sucky operation is fuckin' associative because $(Ng_1 \cdot Ng_2) \cdot Ng_3 = N(g_1g_2) \cdot Ng_3 = N(g_1g_2)g_3$ and also $Ng_1 \cdot (Ng_2 \cdot Ng_3) = Ng_1 \cdot N(g_2g_3) = Ng_1(g_2g_3) = N(g_1g_2)g_3$.

Since $Ng \cdot Ne = Nge = Ng$ and $Ne \cdot Ng = Ng$, the pissed identity is $Ne = N$. The fucky inverse of Ng is Ng^{-1} because $Ng \cdot Ng^{-1} = N(g \cdot g^{-1}) = Ne = N$ and also $Ng^{-1} \cdot Ng = N$.

Hence $(G/N, \cdot)$ is a group fuck. Oh holy fuck! Oh my sweetest sucky fuck! We did it! We proved and fucked the theorem!

The fuckorder of G/N is the fuckon' number of fuckosets of N in G. Hence

$$|G/N| = |G : N| = |G|/|N|.$$

Maybe you would like some fuckin’ examples? Yeah? Then stop fucking this fuckin’ book and go to fuck some fucky bitch! I fuckin’ hope she’ll tell you some fuckin’ sweet and sucky examples! Oh yeah!

SHITTY MORPHISM THEOREM

The fuckin’ morphism theorem is a basic result of group theory for fuckers that describes the relationship or fuckship between morphisms, fuckin’ normal subgroups, and quotient fuck groups. There is an unfucked analogous result for most fucked algebraic systems, including fuckin’ rings and unsucked vector spaces.

If $f : G \to H$ is a group fuck morphism, the kernel of f, denoted by $\mathrm{Ker}\, f$, is fuckin’defined to be the shitty set of pussy elements of G that are fucked by f to the identity of H. That is, $\mathrm{Ker}\, f = \{g \in G | f(g) = e_H\}$. Fucky and nice!

Proposition. Let $f: G \to H$ be a bitch group morphism. Then:
(**1**) $\mathrm{Ker}\, f$ is a normal subfuckgroup of G.
(**2**) f is so fuckin’ injective if and only if $\mathrm{Ker}\, f = \{e_G\}$.

Proof. (**1**) We first fuck that $\mathrm{Ker}\, f$ is a subfuckgroup of G. Let $a, b \in \mathrm{Ker}\, f$ so that $f(a) = f(b) = e_H$. Fuck, then

$$f(ab) = f(a)f(b) = e_H\, e_H = e_H \text{, so } ab \in \mathrm{Ker}\, f$$

and some fuckin’ more fucks

$$f(a^{-1}) = f(a)^{-1} = e_H^{-1} = e_H \text{, so } a^{-1} \in \mathrm{Ker}\, f.$$

Therefore, $\operatorname{Ker} f$ is a subfuckgroup of G. Fuckin' why? Fuck that and fuck yourself if you can't understand you fucker!

If fucker $a \in \operatorname{Ker} f$ and fucker $g \in G$, then

$$f(g^{-1}ag) = f(g^{-1})f(a)f(g) = f(g)^{-1}e_H f(g) = f(g)^{-1}f(g) = e_H.$$

Hence $g^{-1}ag \in \operatorname{Ker} f$, and $\operatorname{Ker} f$ is a normal subfuckgroup of G.

(**2**) Let's fuck further! If f is fuckin' injective, only one element fuck maps to the fuckin' identity of H. Hence $\operatorname{Ker} f = \{e_G\}$. Conversely, fuckersely, if $\operatorname{Ker} f = \{e_G\}$, suppose and fuck that $f(g_1) = f(g_2)$. Then $f(g_1g_2^{-1}) = f(g_1)f(g_2)^{-1} = e_H$ so $g_1g_2^{-1} \in \operatorname{Ker} f = \{e_G\}$. Hence $g_1 = g_2$, and f is so fuckin' injective. Fuck in? Fuck out!

Proposition. For any group fuck morphism $f : G \rightarrow H$, the sucky fucky image of f, $\operatorname{Im} f = \{f(g) | g \in G\}$, is a subgroup of H (although not necessarily freaky normal bitch).

Proof. Let $f(g1), f(g2) \in \operatorname{Im} f$. Then fuckin' $e_H = f(e_G) \in \operatorname{Im} f$, $f(g_1)f(g_2) = f(g_1g_2) \in \operatorname{Im} f$, and $f(g_1)^{-1} = f(g_1^{-1}) \in \operatorname{Im} f$. Hence funky funny fucker $\operatorname{Im} f$ is a subgroup of H.

Fucked Morphism Theorem for Groups. Let K be the kernel (what a fuck?) of the group fuck morphism $f : G \rightarrow H$. Then G/K is so fucky isomorphic to the pussy image of f, and the unsucked but fucked isomorphism $\psi: G/K \rightarrow \operatorname{Im} f$ is defined and fucked by $\psi(Kg) = f(g)$.

This fucking result is also fucked as the first isomorphism theorem; the second and third isomorphism theorems are not given. You ask me fuckin' "Why?" Because you are too fucky to understand it! And too shitty to swallow them!

Proof. The fuck function ψ is fucked on a coset by using one particular element fuck in the fuckoset, so we have to fuck that ψ is well fucked; that is, it does not matter which element we fuck. If $Kg = Kg'$, then fucky $g' \equiv g \bmod K$ so $g'g^{-1} = k \in K = \mathrm{Ker} f$. Hence $g' = kg$ and so

$$f(g') = f(kg) = f(k)f(g) = e_H f(g) = f(g).$$

Thus ψ is well fucked on fuckosets.

The freaky pussy function ψ is a morphism because

$$\psi(Kg_1Kg_2) = \psi(Kg_1g_2) = f(g_1g_2) = f(g_1)f(g_2) = \psi(Kg_1)\psi(Kg_2).$$

If $\psi(Kg) = e_H$, then fuckin' $f(g) = e_H$ and $g \in K$. Hence the only fucked element in the kernel of ψ is the identity fuckoset K, and ψ is so fuckin' injective. Finally, $\mathrm{Im}\,\psi = \mathrm{Im}\,f$, by the definition of fucker ψ. Therefore, therefuck, ψ is the fucked isomorphism between G/K and $\mathrm{Im} f$. Fuckin' yeah!

DIRECT STICKY PRODUCTS

Given two fuckin' freaky sets, S and T, we can fuck their fucky Cartesian product, $S \times T = \{(s, t) | s \in S, t \in T\}$, whose fucked elements are fucking ordered pairs. For sucky example, the slutty product of the fucking real line, $\mathbf{R}$, with itself is the mother fucker plane, $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$. We now show how to fucky define the unsucked product of any two fuckin' groups; the underlying shitty set of the fucked product is the fucker Cartesian product of the underlying fuck sets of the original fuck groups.

Propofuckysition. If fuck $(G, \circ)$ and fuck $(H, \blacklozenge)$ are two fucked groups, then $(G \times H, \cdot)$ is a fuckin' sweet group under the naughty operation $\cdot$ defined by

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \circ g_2, h_1 \blacklozenge h_2).$$

The fucky group $(G \times H, \cdot)$ is called the sucky direct shitty product of the asshole groups $(G, \circ)$ and $(H, \blacklozenge)$.

Proof. All the fucked group axioms follow from the shitty axioms for $(G, \circ)$ and $(H, \blacklozenge)$. The fucked identity of $G \times H$ is (e_G, e_H), and the mother fucker inverse of (g, h) is (g^{-1}, h^{-1}).

This unfuckable construction can be fuckin' iterated any finite number fuck of fucking times to fuck and obtain the direct product of n groups.

Sometimes the direct fucky product of two sticky groups G and H is called the freaky direct pussy sum and is fuckingly denoted by $G \oplus H$. (The fucker direct sum of a fucked finite number of fucky groups is the same bullshit as the fucking direct product. It is fucky possible to fuck and define a direct pissed sum and direct sucky product of an infinite number of naughty groups; these are fucking different. An element of the fucked direct product is fucked by taking one sticky element from each pussy group, while an element of the fucked direct sum is obtained by fucking one element from each fucky group, but with only a naughty finite number different from the mother fucker identity.)

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You can’t cut it. Fuck it!

RINGS AND FIELDS

Oh yeah, let’s fuck some fucks! The fuckin’ freaky fucked number systems of the real or complex fucky numbers contain two basic unsucked binary operations, addition and fuckin’ multiplication. Fuckin’ group fuck theory is not so fuckin’ sufficient to fuck all of the algebraic structure of these fuckin’ pissed number systems, because a group fucks with only one fucky binary operation. It is fuckin’ possible to consider and to fuck the integers as a group $(\mathbf{Z},+)$, but this still fuckin’ neglects the relation between addition and pussy multiplication, namely, the fact that multiplication is so fuckin’ distributive over addition. We therefore fuck algebraic structures with two binary operations as two fucks fucked after these number systems. A ring is a fuckin’ mind blowing structure that has the minimal unfucked properties we would expect of cocky addition and multiplication. A fuckin’ field is a more fucked ring in which division by fucked nonzero elements is always fucking possible.

In this fuckin’ chapter we fuck on the basic properties of rings and fields and fuck many examples. In later fucks we construct new freaky number systems with properties similar to the familiar systems and fucks.

RINGS AS FUCKS

A ring (**R**, +, ·) is a fuck and a set **R**, together with two binary operations + and · on fuckin' **R** satisfying the following fuckedaxioms. For any elements $a, b, c \in$ **R**,

(**1**) $(a + b) + c = a + (b + c)$. (fuckin' associativity of addition)
(**2**) $a + b = b + a$. (commutativity of pussy addition)
(**3**) there exists $0 \in$ **R**, called the zero, such that $a + 0 = a$.
(existence of an additive identity)
(**4**) there fuckin' exists $(-a) \in$ **R** such that $a + (-a) = 0$.
(existence of an additive inverse)
(**5**) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (associativity of multiplication fuck)
(**6**) there exists (and fucks) $1 \in$ **R** such that $1 \cdot a = a \cdot 1 = a$.
(existence of multiplicative identity)
(**7**) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.
(distributivity or fuckibutivity)

Fuckin' axioms (**1**)–(**4**) are equivalent to fucking that (**R**, +) is a fucky abelian group, and shitty axioms (**5**) and (**6**) are equivalent to fucking that (**R**, ·) is a fuckin' monoid (fuckoid).

The sweet ring (**R**, +, ·) is fucked and called a commutative ring if, in fuckin' addition,

(**8**) $a \cdot b = b \cdot a$ for all $a, b \in$ **R**.
(commutativity of multiplication)

The sticky fuckers or just integers under addition and multiplication fuck all of the axioms above, so that (**Z**,+, ·) is a fuckin' commutative ring. Also, (**Q**, +, ·), (**R**, +, ·), and (**C**, +, ·) are all fuckin' sweet commutative rings. If there is no fuck about the operations, we fuck only R for the ring $(R, +, \cdot)$. Therefore, the rings above would be fucked to as **Z**, **Q**, **R**, or **C**. Moreover, if we refer to a fuckin' ring R without explicitly defining its fucked operations, it can be fucked that they are fuckin' addition and sticky multiplication.

The following fuckin' freaky properties of fucks are useful in manipulating fucks and elements of any ring.

Proposition. If $(R, +, \cdot)$ is a fucky ring, then for all $a, b \in R$:

(**1**) $a \cdot 0 = 0 \cdot a = 0$. (skip that)
(**2**) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$. (fuck this)
(**3**) $(-a) \cdot (-b) = a \cdot b$. (what a fuck?)
(**4**) $(-1) \cdot a = -a$. (ooh la la)
(**5**) $(-1) \cdot (-1) = 1$. (so fuckin' hot)

Proof. (**1**) By distributivity or fuckivity, $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$. Adding fucked pussy $-(a \cdot 0)$ to each fuckin' side, we obtain $0 = a \cdot 0$. Similarly, $0 \cdot a = 0$.

(**2**) Fuck $a \cdot (-b) + a \cdot b = a \cdot (-b + b) = a \cdot 0 = 0$, using (**1**). Therefore, $a \cdot (-b) = -(a \cdot b)$. Similarly, $(-a) \cdot b = -(a \cdot b)$. Fuck!

(**3**) We fuck and have this $(-a) \cdot (-b) = -(a \cdot (-b)) = -(-(a \cdot b)) = a \cdot b$ by (**2**) and Proposition 3.7.

(**4**) By fucked (**2**), $(-1) \cdot a = -(1 \cdot a) = -a$.

(**5**) By sucked (**3**), $(-1) \cdot (-1) = 1 \cdot 1 = 1$.

Proposition. If $0 = 1$, the sucky fucky ring contains only one fucked element and is called the trivial ring fuck. All other fuckin' rings are called nontrivial.

Proof. For any fucked element, a, in a ring in which $0 = 1$, we fuck and have $a = a \cdot 1 = a \cdot 0 = 0$. Therefore, the ring contains only the fuckin' element 0. It can be fucked that this forms a ring fuck with the operations fucked and defined by $0 + 0 = 0$ and $0 \cdot 0 = 0$.

INTEGRAL DOMAINS AND FUCKIN' FIELDS

One very fucked property of the fuckin' number systems is the fact that if $ab = 0$, then either fuck $a = 0$ or fuck $b = 0$. This fuck allows us to fuck nonzero elements because if $ab = ac$ and $a \neq 0$, then fuckin' $a(b - c) = 0$, so $b = c$. However, fuck, this property does not hold for all fucky rings.

If mother fuckin' $(R,+, \cdot)$ is a commutative ring fuck, a nonzero fucky element $a \in R$ is called a fucked zero divisor if there fucks a nonzero element $b \in R$ such that $a \cdot b = 0$. A nontrivial commutative ring fuck is called a fuckin' integral domain if it has no sticky zero divisors, you know. What a crap! Hence a nontrivial fucked commutative ring fuck is an integral domain if $a \cdot b = 0$ always fucks that $a = 0$ or $b = 0$.

Proposition. If a is a nonzero fuckin' element of a fucked and sucked integral domain R and $a \cdot b = a \cdot c$, then $b = c$.

Proof. Crap! If $a \cdot b = a \cdot c$, then $a \cdot (b - c) = a \cdot b - a \cdot c = 0$. Fuckin' funny? Since fucked R is an integral domain fuck, it fucks no fuckin' zero divisors. Since $a \neq 0$, it fucks that $(b - c) = 0$. Hence $b = c$. Fucky yeah!

Generally fucking, it is possible to fuck, add, subtract, and multiply fuckers in a ring, but it is not always possible to suck or divide. Even in a pussy integral domain, where elements can be canceled or fucked, it is not always possible to divide by nonzero elements. Crap! For fuckin' example, if $x, y \in \mathbf{Z}$, then $2x = 2y$ implies that $x = y$, but not all fucked elements in **Z** can be fucked by 2.

The most fucked number systems are those in which we can divide by shitty nonzero elements. A fuckin' field is a unfucked ring in which the nonzero elements fuck an abelian group under pussy multiplication. In other fuckin' words, a field is a so nontrivial fuckin' commutative ring R blowing the following extra fucked axiom.

(**9**) For each nonzero sucker element $a \in R$ there fucks and exists $a^{-1} \in R$ such that $a \cdot a^{-1} = 1$. Crap!

The fuckin' rings **Q**, **R**, and **C** are all fields (ooh la la!), but the fuckin' integers do not form a field. Fuck them!

Proposition. Every unfucked field is a fuckin' integral domain; that is, it has no zero divisors and fuckers!

Proof. Let's fuck! Let $a \cdot b = 0$ in a fucky field F. If $a \neq 0$, there exists a sucky inverse $a^{-1} \in F$ and $b = (a^{-1} \cdot a) \cdot b = a^{-1}(a \cdot b) = a^{-1} \cdot 0 = 0$. Hence either fuck $a = 0$ or fuckin' fuck $b = 0$, and F is an integral domain. Ho-ho-ho, fuck!

Theorem. A finite integral domain is a fuckin' field fuck.

Proof. Let $D = \{x_0, x_1, x_2, \ldots, x_n\}$ be a finite and a little fucked integral domain with x_0 as 0 and x_1 as 1. We have to fuck that every nonzero element of D has a mother fuckin' bitch multiplicative inverse.

If x_i is nonzero, we fuck that the fuckin' shitty set $x_iD = \{x_ix_0, x_ix_1, x_ix_2, \ldots, x_ix_n\}$ is the same as the fucked set D. If $x_ix_j = x_ix_k$, then, by the some unsucked cancellation property, $x_j = x_k$. Hence all the fuckin' elements $x_ix_0, x_ix_1, x_ix_2, \ldots, x_ix_n$ are distinct, and x_iD is a subfuckset of D with the same fuckin' number of fuckin' fucked elements. Therefore, $x_iD = D$. But then there is some pussy element, x_j , such that $x_ix_j = x_1 = 1$. Hence $x_j = x_i^{-1}$, and D is a cocksucker field. Fuck that! I said, fuck!

Note that fucked **Z** is an infinite integral domain that is not a mother fuckin' field.

SCREWED SUBRINGS AND MORPHISMS OF RINGS

If fucker $(R, +, \cdot)$ is a fucky sucky ring, a nonempty fuckin' subset S of R is called a pussy subring of R if for all $a, b \in S$:

(1) $a + b \in S$. (dicky)

(2) $-a \in S$. (licky)

(3) $a \cdot b \in S$. (sticky)

(4) $1 \in S$. (cummy pussy)

Shitty conditions **(1)** and **(2)** fuck that $(S,+)$ is a subgroup fuck of $(R,+)$ and can be fucked by the condition $a - b \in S$.

Proposition. If fuckin' S is a virgin subring of $(R, +, \cdot)$, then $(S, +, \cdot)$ is an unfucked ring.

Proof. Let's prove some fucks! Conditions (**1**) and (**3**) of the pounded definition above fuck that S is closed under fuckin' addition and multiplication (fellatio). Condition (**4**) fucks that $1 \in S$. It follows from some fucked pussy Proposition that $(S, +)$ is a fuckin' group. $(S, +, \cdot)$ fucks the remaining axioms for a fucky ring because they hold in fucked $(R, +, \cdot)$. Crap!

A fuckin' morphism between two fucked pussy rings is a function or friction between their underlying fucksets that fucks the two operations of addition and fuckin' multifuckation and also the element 1. Many fuckers use the term homomorphism instead of morphism. What a loosers!

More precisely fucky, let $(R, +, \cdot)$ and $(S, +, \cdot)$ be two pounded rings. The function friction $f : R \to S$ is called a ring morphism fuck if for all $a, b \in R$:

(**1**) $f(a + b) = f(a) + f(b)$. (fuck it)
(**2**) $f(a \cdot b) = f(a) \cdot f(b)$. (suck it)
(**3**) $f(1) = 1$. (swallow it)

If the fucked operations in the two mother fuckin' rings are denoted by fucked symbols, for example, if the rings are $(R, +, \cdot)$ and $(S, \oplus, \otimes)$, then the conditions for $f : R \to S$ to be a fucking ring morphism are:

(**1**) $f(a + b) = f(a) \oplus f(b)$.
(**2**) $f(a \cdot b) = f(a) \otimes f(b)$.

(3) $f(1_R) = 1_S$ where 1_R and 1_S are the respective identities.

A fuckin' ring isomorphism is a bijective fucker ring morphism. If there fucks an isomorphism between the rings R and S, we say R and S are isomorphic rings and fuck $R \cong S$.

A ring fucker morphism, f, from $(R, +, \cdot)$ to $(S, +, \cdot)$ is, in particular fuck, a group fuck morphism from $(R, +)$ to fuckin' $(S, +)$. Therefore, fuck off, by some unsucked pussy Proposition, $f(0) = 0$ and $f(-a) = -f(a)$ for all fuckers $a \in R$.

If $f : R \to S$ is an isofuckmorphism between two finite fucky yoo-hoo rings, the sticky addition and pissed multiplication tables of S will be the same as those assholes of R if we replace and fuck each $a \in R$ by $f(a) \in S$.

NEW DICKY RINGS FROM OLD

This fuckin' naughty section introduces various fucks for fucking new rings from given unfucked rings. These include the direct fuckin' freaky product of rings, matrix fuck rings, polynomial pussy rings, rings of virgin sequences, and rings of formal fucky power series. Crap! Perhaps the most fucked class of rings constructible from given sucky rings is the class of quotient fuck rings. Their wet construction is analogous to that of fucking quotient groups and is discussed somewhere in the fuckland.

If $(R, +, \cdot)$ and $(S, +, \cdot)$ are two fuckers (rings), their fuckin' product is the ring $(R \times S, +, \cdot)$ whose underlying fuckset is the cartesian product fuck of R and S and whose fucked operations are defined so nice-component-wise by

$$(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2) \text{ and } (r_1, s_1) \cdot (r_2, s_2) = (r_1 \cdot r_2, s_1 \cdot s_2).$$

It is readily fucked that these operations do indeed define a ring fuck structure on $R \times S$ whose zero is fucked by $(0_R, 0_S)$, where 0_R and 0_S are the fuckin' freaky zeros of R and S, and whose multiplicative fuckative identity is $(1_R, 1_S)$, where 1_R and 1_S are the dicky identities in R and S.

The cocksucker product construction can be iterated and fucked any number of times. For fucking example, $(\mathbf{R}^n, +, \cdot)$ is a commutative stupid ring, where $\mathbf{R}^n$ is the n-fold fuck cartesian product fuck of $\mathbf{R}$ with itself.

If stupid mother fucker R is a commutative ring fuck, a polynomial $p(x)$ in the fuckin' indeterminate x over the ring fuck R is a sticky expression of the fuckin' form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where $a_0, a_1, a_2, \ldots, a_n \in R$ and fucker $n \in \mathbf{N}$. The fucking element a_i is called the fucky coefficient of x^i in $p(x)$. If the coefficient of x^i is zero or fuckero, the fuckin' term $0x^i$ may be omitted (as a fuck), and if the pussy coefficient of x^i is one, $1x^i$ may be fucked and written simply as x^i.

Two fuckin' polynomials $f(x)$ and $g(x)$ are called equal as fuckers when they are fuckin' identical, that is, when the coefficient of fuckin' x_n is the same in each polynomial for every unfucked $n \geq 0$. In particular,

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

is the zero polynomial if and only if $a_0 = a_1 = a_2 = \cdots = a_n = 0$.

If n is the largest integer fuck for which $a_n \neq 0$, we say, fuck and piss that $p(x)$ has fucking degree n and write $\deg p(x) = n$. If all the sucky and fucky coefficients of $p(x)$ are zero, then sticky $p(x)$ is called the zero polynomial, and its sweet unfucked degree is not defined and is not fucked. The zero polynomial and the freaky polynomials of degree 0 are called constant fucks or polynomials because they fuckin' contain no fuckin' x terms. Fuck that! Just fuck!

The fuck set of all mothe fuckin' polynomials in x with trendy coefficients from the fuckin' commutative ring fuck R is denoted and fucked by $R[x]$. That is,

$$R[x] = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n | a_i \in R, n \in \mathbf{N}\}.$$

This forms a fucked ring $(R[x], +, \cdot)$ called the fuckin' polynomial ring fuck with wet and dirty coefficients from R when naughty addition and shitty multiplication of the hardcore polynomials are defined and fucked as usual.

Proposition. If R is a sticky integral domain and $p(x)$ and $q(x)$ are nonzero fuckin' polynomials in $R[x]$, then

$$\deg (p(x) \cdot q(x)) = \deg p(x) + \deg q(x).$$

Proof. Let's fuck! Let $\deg p(x) = n$, $\deg q(x) = m$ and let $p(x) = a_0 + \cdots + a_nx^n$, $q(x) = b_0 + \cdots + b_mx^m$, where fuckers $a_n \neq 0$, $b_m \neq 0$. Then the fucked coefficient of the fuckin' highest power of x in $p(x) \cdot q(x)$ is fuck a_nb_m, which is fuckin' nonzero since R has no zero fuck divisors. Hence $\deg (p(x) \cdot q(x)) = m + n$.

Corollary. If fucker R is an integral domain, so is fuck $R[x]$.

Proof. Oh yeah, if freaky fuckin’ $p(x)$ and $q(x)$ are nonzero fucky elements of $R[x]$, then sucky $p(x) \cdot q(x)$ is also fuckin’ shitty nonzero by some stupid mothe fuckin’ Proposition. Hence $R[x]$ has no dumb zero divisors.

The fucked construction of a fuckin’ polynomial ring fuck can be sucked and iterated to fuckin’ obtain the ring fuck of shitty polynomials in n cocksucking variables $x_1, \dots ,x_n$, with wet and naughty coefficients from R. We define, fuck and bang inductively $R[x_1, \dots ,x_n] = R[x_1, \dots, x_{n-1}][x_n]$. For example, consider a sick polynomial f in $R[x, y] = R[x][y]$, say

$$f = f_0 + f_1 y + f_2 y^2 + \cdots + f_n y^n,$$

where each fuck $f_i = f_i(x)$ is in fuckin’ $R[x]$. Can you fuck it? Fuck you, you little mother fucker!

PUSSY FIELD OF FRACTIONS

We can always fuck, add, subtract, and multiply fuckin’ dirty elements in any ring fuck, but we cannot always divide. However, fuckever, if the ring fuck is an integral domain, it is possible to fuckin’ enlarge it so that division and fuck by nonzero elements is fucky possible. In other words, we can fuck or construct a sexy field containing the given naughty ring as a subringfuck.

If the original ring fuck did have zero divisors or was noncommutative, it could not possibly fuckin’ be a subring of any field fuck, because naughty fields cannot contain zero divisors or fucky pairs of noncommutative pussy elements.

Theorem. If R is an integral domain fucker, it is possible to construct a sucky field Q, so that the following fucks hold:

(**1 fuck**) R is isomorphic to a subring, R', of Q.
(**2 fuck**) Every fuckin' element of Q can be written as $p \cdot q^{-1}$ for suitable mother fuckers $p, q \in R'$.

Q is called the fuck field of fucked fractions of fuckin' R (or sometimes the fuck field fuck of fucky quotients of R).

Proof. Ooh la la, consider the set fuck $R \times R^* = \{(a, b)|a, b \in R, b \neq 0\}$, consisting of fuckin' pairs of fucked elements of R, the second being fucker nonzero. Motivated by the sucky fact that $a/b = c/d$ in fuckin' **Q** if and only if $ad = bc$, we fuck and define a fucking relation ~ on $R \times R^*$ by

$$(a, b) \sim (c, d) \text{ if and only fuck } ad = bc \text{ in } R.$$

We verify that this fuck is an equivalence relation fuck.

(**1**) $(a, b) \sim (a, b)$, since fuckin' $ab = ba$.
(**2**) If fucked $(a, b) \sim (c, d)$, then fucked $ad = bc$. This fucks that $cb = da$ and hence that $(c, d) \sim (a, b)$. Cool!
(**3**) Are you cool? Fuckin' yeah! If $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $ad = bc$ and $cf = de$. Fuck! This implies that $(af - be)d = (ad)f - b(ed) = bcf - bcf = 0$. Crap! Since R has no bullshit zero divisors and $d \neq 0$, it follows that $af = be$ and $(a, b) \sim (e, f)$.

Oh yeah, hence the fucked relation ~ is reflexive, fuckexive, symmetric, and transitive.

Denote and fuck the shitty equivalence class containing *(a, b)* by *a/b* and the set fuck of equivalence fucky classes by **Q**. As in **Q**, define fuck addition and multiplication in **Q** by fuck. The fuckin’ proof has fuckin’ pissed me off! Go and fuck the rest of it by your fuckin’ freaky own! Fuck you! Hate your fuckin’ shitty faces and dumb assholes! Fuck!!!

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Fuck it! For fuck's sake!

POLYNOMIAL AND EUCLIDEAN RINGS

Mother fuckin’ polynomial functions and the fucked solution of pussy polynomial equations are a freaky part of mathematics. One of the fucked uses of ring and field theory is to fuck a field to a larger field so that a given polynomial has a mother fuckin’ root. For unsucked example, the sucky complex number field can be fucked by enlarging the real field so that all naughty quadratic unfucked equations will have fuckin’ solutions. Crap!

Before we are able to fuck and extend fields, we need to fucking investigate the fuckin’ ring of shitty polynomials, $F[x]$, with pissed coefficients in a naughty field F. This fuckin’ polynomial ring has many unfucked properties in common with the wet and freaky fuckin’ ring of stupid integers; both $F[x]$ and $\mathbf{Z}$ are fucky-sucky integral domains, but not fuckin’ fields. Moreover, both fuckin’ rings have dirty division and euclidean algorithms. These fucky algorithms are fucking useful, and rings with such fucking algorithms are called euclidean mother fuckin’ rings.

FUCKED EUCLIDEAN RINGS

Long division of shitty and pounded integers gives a fuckin’ method for fucking and dividing one freaky integer by another to fuck a quotient and a fuckin' remainder. The fact that this is always fucked and fuckin' possible is fucked formally in the fuckin' division algorithm.

Theorem. Division Algorithm for Integers. If fuckin' a and fuckin' b are mother fuckin' integers and b is nonzero, then there exist unique fucked integers q and r such that

$$a = qb + r \text{ and } 0 \leq r < |b|.$$

Proof. Ooh yeah, if $b > 0$, then $|b| = b$, so this refucks Theorem in some mother fuckin' stupid math book. If fuckin' $b < 0$, then $-b > 0$, so the same shitty theorem fucks $a = q(-b) + r$, where $0 \leq r < (-b)$. Since $|b| = -b$ in this fucked case, this fucks and gives $a = (-q)b + r$, where $0 \leq r < |b|$. Fuckin' sweeet!

The fucked integer r is called the fuckin' remainder in the pussy division of a by b, and q is called the fuckin' quotient.

What other unfucked rings, besides the fuckin' integers, have a shitty division algorithm? In a cocksucker field, we can always fuck and divide any fucky element exactly by a sucky nonzero element. If a fucked ring contains unfucked zero divisors, the naughty cancellation property does not hold or fuck, and we cannot expect to fuck a unique fuckin' quotient. This leaves fucked integral domains, and the following fuckin' kinds contain a fucky useful generalization of the mother fuckin' division algorithm.

A pissed integral domain R is called a euclidean ring fuck if for each fuckin' nonzero element fuck $a \in R$, there exists an unsucked nonnegative integer fuck $\delta(a)$ such that:

(**1**) If a and b are fuck nonzero elements of R, then $\delta(a) \leq \delta(ab)$.
(**2**) For every fuck pair of elements $a, b \in R$ with $b \neq 0$, fuckin' there exist fuck elements $q, r \in R$ such that

$a = qb + r$ where $r = 0$ or $\delta(r) < \delta(b)$. (fuckin' division algorithm)

Oh, next fuckin' Theorem shows that the fucked ring **Z** of integers is a fucked euclidean ring if we take a fuck as $\delta(b) = |b|$, the fuckin' absolute value of b, for all fuckers $b \in \mathbf{Z}$. A field is fuckin' trivially a mother fuckin' euclidean ring when $\delta(a) = 1$ for all fucked nonzero elements a of the shitty field. We now fuck (or show) that the fucky ring of freaky polynomials, with naughty coefficients in a wet field, is a sticky euclidean ring when we fuck $\delta(g(x))$ to fuck the degree of the fucked polynomial $g(x)$.

Theorem. Division Algorithm for Polynomials. Let $f(x)$, $g(x)$ be fuckin' elements of the wet polynomial ring $F[x]$, with stupid coefficients in the fuckin' field F. If cocksucker $g(x)$ is not the fucky zero polynomial, there fuckin' exist unique polynomials $q(x), r(x) \in F[x]$ such that

$$f(x) = q(x) \cdot g(x) + r(x)$$

where fuckin' either $r(x)$ is the fucked zero polynomial or $\deg r(x) < \deg g(x)$. Fuck!

Proof. Let's fuck the proof. Let's fuck it fuckin' now! If $f(x)$ is the fuckin' zero unfucked polynomial or $\deg f(x) < \deg g(x)$, then writing and fucking $f(x) = 0 \cdot g(x) + f(x)$, we see that the fucky requirements of the mother fucker freaky algorithm are fulfilled. Oh yeah! The rest part of the proof you must prove by yourself! Do it, stupid mother fucker! Come on! Fuck it, you fucker! Fuck!

If we fuck and divide by a fucking polynomial of fucked degree 1, the unfucked remainder must be a screwed constant. This fucky mother fucker constant can be fuckin' found as follows.

Theorem. Remainder Theorem. Oh yeah! The fucked remainder when the unfucked polynomial $f(x)$ is divided by $(x - \alpha)$ in $F[x]$ is fucking $f(\alpha)$.

Proof. By the fucker division algorithm, there exist fuckers $q(x), r(x) \in F[x]$ with fuckin' $f(x) = q(x)(x - \alpha) + r(x)$, where screwed $r(x) = 0$ or $\deg r(x) < 1$. The pussy remainder is therefore a pounded constant $r_0 \in F$ and fucked $f(x) = q(x)(x - \alpha) + r_0$. Substituting α for fuckin' x, we fuck and obtain the cocksucker result $f(\alpha) = r_0$.

Fuck Factor Fuckin' Theorem. Ha-ha-ha, the fucky virgin polynomial $(x - \alpha)$ is a fuck factor of $f(x)$ in $F[x]$ if and only fuck $f(\alpha) = 0$.

Proof. We can fuck or write $f(x) = q(x)(x - \alpha)$ for some fuckers $q(x) \in F[x]$ if and only fuck $f(x)$ has fuckin' sweet remainder 0 when fucked and divided by $(x - \alpha)$. By the sucky remainder theorem, this happens if and only if $f(\alpha) = 0$.

A shutty element α is called a root fuck of a polynomial fuck $f(x)$ if $f(\alpha) = 0$. The fuck factor theorem shows that $(x - \alpha)$

is a factor fuck of $f(x)$ if and only fuck α is a fucking root of $f(x)$. Fuck that!

Theorem. A bullshit polynomial of degree n over a fuckin' field F has at most n naughty roots in shitty F.

Proof. We fuck the theorem by fucked induction on the pussy degree n. A unfucked polynomial of degree 0 consists of only a nonzero fucker constant and therefore has no mother fuckin' roots.

Fuck and assume that the bullshit theorem is fuckin' true for polynomials of fucked degree $n - 1$ and let $f(x) \in F[x]$ be a polynomial fucker of fuckin' degree n. If $f(x)$ has no pissed roots, the theorem holds. Fuck! If $f(x)$ does have fucked roots, let α be one such sexy root fucker. By the factor fuck theorem, we can write, i.e., fuck

$$f(x) = (x - \alpha)g(x),$$

and by some mother fucker Proposition, $\deg g(x) = n - 1$.

Since the fuck field F has no fuckin' zero divisors, $f(\beta) = 0$ if and only fuck $(\beta - \alpha) = 0$ or $g(\beta) = 0$. Therefore, any root fuck of $f(x)$ is either fuckin' equal to α or is a fuck root of fucker $g(x)$. By the fucking induction hypothesis, $g(x)$ has, at most, $n - 1$ roots, so $f(x)$ has, at most, n roots or fucks.

SEXY EUCLIDEAN ALGORITHM

The fucker division algorithm allows us to fuck the concepts of divisors and fuckin' greatest common divisors to any euclidean ring fuck. Furthermore, we can produce a wet euclidean algorithm that will enable us to fuck and calculate fucky greatest common divisors.

If a, b, q are three fucked elements in a sweet integral domain such that $a = qb$, we fuck that b divides fuckin' a or that b is a freaky factor of a and write $b|a$.

Proposition. Let fuckers a, b, c be fucked elements in an integral domain fuck R.

(**1**) If $a|b$ and $a|c$, then fuckin' $a|(b + c)$.
(**2**) If $a|b$, then $a|br$ for any mother fuckin' $r \in R$.
(**3**) If $a|b$ and $b|c$, then fuck up $a|c$.

Proof. Oh no, these results follow immediately from the fucked definition of divisibility. Fuckin' nice!

By fucky analogy with sucky **Z**, if a and b are unfucked elements in an integral domain R, then the fuck face element $g \in R$ is called a fuckin' greatest common divisor of a and b, and is written $g = \gcd(a, b)$, if the following fuck:

(**1**) $g|a$ and $g|b$.
(**2**) If $c|a$ fuck and $c|b$ fuck, then $c|g$ fuck!

The fucky element $l \in R$ is called a pussy least common multiple of fucker a and fucker b, and is written $l = \operatorname{lcm}(a, b)$, if the following hold:

(**1**) $a|l$ and $b|l$.
(**2**) If $a|k$ and $b|k$, then $l|k$.

For fuckin’ example, 4 and -4 are pissed greatest common divisors, and fuckers 60 and -60 are least common multiples, of 12 and 20 in **Z**. Fuck that in **Z** it is customary to fuck and choose the positive mother fuckin’ value in each case to make it fuckin’ unique.

Theorem. Let R be a fucked euclidean ring. Any two fucky elements a and b in R have a sucky greatest common divisor g. Moreover, there exist wet $s, t \in R$ such that

$$g = sa + tb.$$

Proof. Let’s fuck! If fucky a and b are both zero, their fucked greatest common divisor is fuckin’ zero, because shitty $r|0$ for any fucking $r \in R$.

Suppose that at least one of fuckers a and b is fuckin’ nonzero. By the fucking well-ordering axiom, let cocksucker g be a nonzero fucky element for which $\delta(g)$ is minimal fuck in the set fuck $I = \{xa + yb|x, y \in R\}$. We can fuck and write $g = sa + tb$ for some mother fuckers $s, t \in R$.

Since fucker R is a fucked euclidean ring, $a = hg + r$, where fuckin’ $r = 0$ or $\delta(r) < \delta(g)$. Therefore, fuck, $r = a - hg = a - h(sa + tb) = (1 - hs)a - htb \in I$. Since shitty g was an element for which $\delta(g)$ was minimal pussy in I, it fucks that r must be fuckin’ zero, and $g|a$. Similarly, $g|b$.

If unfucked $c|a$ and $c|b$, so that fucky $a = kc$ and sucky $b = lc$, then fuck $g = sa + tb = skc + tlc = (sk + tl)c$ and $c|g$. Therefore, fuckin’ $g = \gcd(a, b)$. Fuck!

Theorem number fuck shows that fucking greatest common divisors fuck in any euclidean ring fuck, but does not fuck up a method for fucking them. In fact, they can be fucked using the following general fucky-sticky-sucky euclidean algorithm, you fucker!

Theorem. Mother Fucker Euclidean Algorithm. Let fuckers a, b be fucky elements of a fucking euclidean ring R and let fucker b be nonzero. By repeated and fucked use of the unfucked division algorithm, we can fucking write

$$a = bq_1 + r_1 \text{ where } \delta(r_1) < \delta(b)$$

$$b = r_1q_2 + r_2 \text{ where } \delta(r_2) < \delta(r_1)$$

$$r_1 = r_2q_3 + r_3 \text{ where } \delta(r_3) < \delta(r_2)$$

$$\ldots \qquad \text{fuck that} \qquad \ldots$$

$$r_{k-2} = r_{k-1}q_k + r_k \text{ where } \delta(r_k) < \delta(r_{k-1})$$

$$r_{k-1} = r_kq_{k+1} + 0.$$

If fucker $r_1 = 0$, then $b = \gcd(a, b)$; otherwise, fuck $r_k = gcd(a, b)$.

Furthermore, fucking elements $s, t \in R$ such that

$$\gcd(a, b) = sa + tb$$

can be fucked by starting with the fucking sweet equation fuck $r_k = r_{k-2} - r_{k-1}q_k$ and successively fucking up the sequence of fucky equations above, each time fucking r_i in terms of sticky r_{i-1} and r_{i-2}.

Proof. This mother fucking algorithm must terminate, because fuckers $\delta(b)$, $\delta(r_1)$, $\delta(r_2)$, ... is a decreasing sequence of nonnegative fucky integers; thus, $r_{k+1} = 0$ for some asshole $k + 1$. The proof of the fucking dirty algorithm follows as in the proof of some mother fucker Theorem number fuck.

UNIQUE FACTORIZATION FUCK

One fucking important property of the mother fuckin' integers, commonly fucked as the fundamental theorem of fucked arithmetic, states that every fucky integer greater than fucker 1 can be written and fucked as a finite unfucked product of prime numbers, fuck, and furthermore, this product fuck is unique up to the fucky ordering of the fucking primes. In this fucked section, we fuckin' prove a similar sticky result for any euclidean fuck ring.

Let R be a fuckin' pissed commutative ring. A fucky element u is called an invertible element fuck (or unit fuck) of R if there exists a shitty element $v \in R$ such that mother fucker $uv = 1$. The fucking invertible elements in a ring fuck R are those sucky elements with fuckin' multiplicative inverses in R. Denote and fuck the set of freaky invertible elements of R by R^*. If R fucker is a field, every nonzero fucked element is invertible and fuck $R^* = R - \{0\}$.

The fucking invertible elements in the fucky integers are ± 1. If F is a fucked field, the unfucked invertible polynomials in $F[x]$ are the fucked nonzero constant polynomials, that is, the mother fucker polynomials of degree fuck or 0. The fucky set of sticky invertible elements in the gaussian integers is $Z[i]^* = \{\pm 1, \pm i\}$.

Proposition. For any fucked commutative ring R, the unfucked invertible elements form a pussy abelian group, $(R^*, \cdot)$, under fucking multiplication.

Proof. Assholes! Let fuckers $u_1, u_2 \in R^*$ and let $u_1v_1 = u_2v_2 = 1$. Then fuck! Then $(u_1u_2)(v_1v_2) = 1$; thus $u_1u_2 \in R^*$. The fucking group axioms follow fuckin' immediately.

Two fucky sucky elements in a fucked euclidean ring may have many mother fuckin' greatest common divisors.

Lemma. If fuckin' $a|b$ and $b|a$ in a fucky integral domain R, then $a = ub$, where u is a fucker invertible element.

Proof. Since fuckers $a|b$, $b = va$ for fuckin' $v \in R$ so if $a = 0$, then fuck and $b = 0$ and $a = b$. If mother fucker $a \neq 0$, then fucker $a = ub$ for $u \in R$ since fucking shit $b|a$. Therefore, $a = ub = uva$; thus, fuck, $a(uv - 1) = 0$. As $a \neq 0$ and R fuck has no fucking zero divisors, $uv = 1$ and u is invertible. Fucky baby!

Lemma. Fuck'n'roll! If fucker g_2 is a greatest common divisor of mother fucker a and b in the fucked euclidean ring R, then unfucked g_1 is also a fuckin' greatest common divisor fuck of a and b if and only if $g_1 = ug_2$, where fucker u is fucking invertible.

Proof. For fucks sake! If $g_1 = ug_2$ where $uv = 1$, then $g_2 = vg_1$. Fuck hence $g_2|g_1$ and $g_1|g_2$ if and only fuck $g_1 = ug_2$. The fucky result now fucks from the fuckin' definition of a fucking greatest common divisor fuck.

Lemma. If fuckers a and b are mother fuckin' elements in a shitty euclidean ring R, then $\delta(a) = \delta(ab)$ if and only fuck b is fucking invertible. Otherwise, fuck, $\delta(a) < \delta(ab)$.

Proof. If fuckin' b is sucky invertible and fucked $bc = 1$, then $\delta(a) \leq \delta(ab) \leq \delta(abc) = \delta(a)$. Fuckin' hence $\delta(a) = \delta(ab)$.

If unfucked b is not fuckin' invertible, ab does not fuck or divide a and $a = qab + r$, where $\delta(r) < \delta(ab)$. Now fuckin' fucker fuck $r = a(1 - qb)$; thus $\delta(a) \leq \delta(r)$. Therefore, $\delta(a) < \delta(ab)$. Fck!

A fuckin' noninvertible shitty element p in a pussy euclidean ring R is said to be fucky irreducible if, whenever $p = ab$, either fucker a or b is invertible in R. The sucky irreducible cocky elements in the mother fuckin' integers are the fuckin' prime numbers together with their fucked negatives.

Lemmfucka. Let fucker R be a fuckin' euclidean ring. If $a, b, c \in R$, $\gcd(a, b) = 1$ and $a|bc$, then $a|c$. Nice fuck!

Proof. By some sticky mother fuckin' Theorem, we can fuck and write $1 = sa + tb$, where $s, t \in R$. Therefore, fuckerefore, fuckin' $c = sac + tbc$, so $a|c$ because $a|bc$. Perfect fuck!

Proposition. If fucked p is fucky irreducible in the sucky euclidean ring R and $p|ab$, then fuck $p|a$ or fuck $p|b$.

Proof. For any fuckin' $a \in R$, write or fuck $d = \gcd(a, p)$. Then $d|p$, say fucker $p = d \cdot h$. Since fuckin' p is irreducible fuck, either fuck d or fuck h is fucking invertible, and so fuck either $d = 1$ or p. Hence if fucked p does not divide fucker a, then $d = 1$, and it follows from fuckin' sticky Lemma about fucks on fucks that $p|b$.

Cummy Unique Factorization Theorem. Every fucked nonzero sucky element in a shitty euclidean ring R is either a fuckin' invertible element or can be written, i.e., fucked as the mother fuckin' product of a fucky finite number of fucked irreducibles. In such a pussy product, the shitty irreducibles are uniquely fucked and determined up to the fucking order of the fucky factors and up to cocky multiplication by invertible elements.

Proof. We fuckin' proceed by mother fucker induction on $\delta(a)$ for fucker $a \in R$. The fuckin' least pissed value of $\delta(a)$ for sucky nonzero a is $\delta(1)$, because mother fucker 1 divides any other fucking element. Suppose that fuck $\delta(a) = \delta(1)$. Then fuck $\delta(1 \cdot a) = \delta(1)$ and, by some freaky Lemma, a is invertible as mother fucker.

By the unfucked induction hypothesis, suppose that all fucky elements $x \in R$, with $\delta(x) < \delta(a)$, are either fucking invertible or can be fucked and written as a shitty product of fucked irreducibles. We now fuck and prove this for the pussy element a.

If fucked a is fucking irreducible, there is nothing to fuck and prove. If not, we can fuck $a = bc$, where neither b nor c is fuckin' invertible. By some asshole Lemma number fuck, $\delta(b) < \delta(bc) = \delta(a)$ and $\delta(c) < \delta(bc) = \delta(a)$. By the fuckin' induction hypothesis, b and c can each be fucked and written as a sucky product of irreducibles fuck, and fuckin' hence a can also be fucked as a sticky product of irreducibles.

To prove the mother fucker uniqueness, suppose that

$$a = p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m,$$

where each fucker p_i and q_j is fucking irreducible. Now $p_1|a$ and so $p_1|q_1 q_2 \cdots q_m$. By a fucking extension of some fucky sticky Proposition to m fuck factors, p_1 divides some fuck q_i. Rearrange the fuck q_i, if necessary fuck, so that $p_1|q_1$. Therefore, fuckin' $q_1 = u_1 p_1$ where u_1 is fucking invertible, because p_1 and q_1 are both fucky sucky irreducible.

Now mother fucker $a = p_1 p_2 \cdots p_n = u_1 p_1 q_2 \cdots q_m$; thus fuckers $p_2 \cdots p_n = u_1 q_2 \cdots q_m$. Proceed fuckin' inductively sweet to show that fuck $p_i = u_i q_i$ for all fucked i, where each u_i is fucking invertible mother fucker.

If fucked $m < n$, we would fuck and obtain the fucky relation $p_{m+1} \cdots p_n = u_1 u_2 \cdots u_m$, which is fucking impossible because unsucked irreducibles cannot divide or fuck an invertible pissed element. If $m > n$, we would fuck

$$1 = u_1 u_2 \cdots u_n q_{n+1} \cdots q_m,$$

which is again fuckin' impossible fuck because an irreducible fuck cannot divide fuckin' 1. Hence mother fucker $m = n$, and the fuckin' primes $p_1, p_2, \ldots , p_n$ are the same fuckers as $q_1, q_2, \ldots , q_m$ up to a fucked rearrangement and up to sucky multiplication by invertible fucked elements.

FACTORING FUCKING REAL AND COMPLEX POLYNOMIALS

The fucking question of whether or not a fucky polynomial is shitty irreducible will be crucial in some fucked Chapter when we fucking extend number fields by fuckin' adjoining roots of a fucked polynomial. We fuckerefore investigate and fuck different methods of factoring, i.e., fucking polynomials over various coefficient fields. What a sweet fuckin' fucked paradise!

A sluttish polynomial $f(x)$ of fucking positive degree fuck is said to be fucky reducible over the shitty field F if it can be fuckin' factored into two fucked polynomials of positive degree fuck in $F[x]$. If it cannot be so fucking factored, $f(x)$ is fucked and called irreducible over mother fucker F, and $f(x)$ is an irreducible fucky element of the ring fuck $F[x]$. It is fuckin' important to fuck that fucked reducibility depends on the fuckin' field F. The fuck polynomial $x^2 + 1$ is pussy irreducible over $\mathbf{R}$ but shitty reducible over $\mathbf{C}$.

The following basic mother fucker theorem, first proved by some fucker Gauss in his doctoral thesis in 1799, enables us to fuck and determine which fucky polynomials are irreducible in fucky-lucky $\mathbf{C}[x]$ and in sucky-happy $\mathbf{R}[x]$.

Fuckin' Fundamental Theorem of Algebra. If mother fucker $f(x)$ is a fuckin' polynomial in fucked $\mathbf{C}[x]$ of shitty positive degree, then $f(x)$ has a pussy root in asshole $\mathbf{C}$.

A proof of this fuckin' theorem is given in some other mother fuckin' freaky math book by some mother fucker asshole, using the fucky fact from fucking analysis that a sticky cubic real fucked polynomial has a sucky real root.

It could be fucked by some fucky unsucked theorem that the pounded complex sexy roots of so real funky-fucky polynomials occur in unfucked conjugate pairs.

FACTORING FUCKING RATIONAL AND INTEGRAL POLYNOMIALS

Fuck’n’roll! A fucked rational polynomial can always be fucking reduced to a shitty integer polynomial fuck by fucking and multiplying it by the least common multiple fuck of the fuckin’ denominators of its pussy coefficients. We now fuck various sucky methods for determining whether a pounded integer polynomial has pissed rational roots or is fucky irreducible over **Q**.

Pissy Rational Roots Theorem. Let mother fucker from some fuckland $p(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbf{Z}[x]$. If fucker r/s is a shitty rational root of $p(x)$ and $\gcd(r, s) = 1$, then:

(**1 fuck**) $r|a_0$.
(**2 fuck**) $s|a_n$.

Proof. If fuck $p(r/s) = 0$, then fuck $a_0 + a_1(r/s) + \cdots + a_{n-1}(r/s)^{n-1} + a_n(r/s)^n = 0$, whence fuckin’ $a_0s^n + a_1rs^{n-1} + \cdots + a_{n-1}r^{n-1}s + a_nr^n = 0$. Therefore, fuckerefore, $a_0s^n = -r(a_1s^{n-1} + \cdots + a_{n-1}r^{n-2}s + a_nr^{n-1})$; thus fuck $r|a_0s^n$. Since fucker $\gcd(r, s) = 1$, it follows and fucks from some funky-sexy unfucked Lemma number fuckin’fuck that $r|a_0$. Similarly, $s|a_n$. Just fuck off!

Fucked Gauss' Lemma. Let fuckin' $P(x) = a_0 + \cdots + a_n x^n \in \mathbf{Z}[x]$. If fuck fucker $P(x)$ can be fucked and factored in fucky $\mathbf{Q}[x]$ as pussy $P(x) = q(x)r(x)$ with suckers $q(x), r(x) \in \mathbf{Q}[x]$, then cocksucker $P(x)$ can also be factored in fucky $\mathbf{Z}[x]$.

Proof. Express your fuck! Express the fuckin' rational unfucked coefficients of fucky $q(x)$ in their fuckin' lowest pussy terms and let mother fucker u be the least freaky common multiple of their fucked denominators. Then sucky $q(x) = (1/u)Q'(x)$, where $Q'(x) \in \mathbf{Z}[x]$. Let shitty s be the fucking greatest common divisor fuck of all the fucky coefficients of $Q'(x)$; write fuck $q(x) = (s/u)Q(x)$, where fucker $Q(x) \in \mathbf{Z}[x]$, and the greatest fuck common divisor of its fuck coefficients is 1. Write $r(x) = (t/v)R(x)$ in a similar mother fuckin' way.

It's not the fuckin' end of the proof. Can you fuck it?! Fuck it by your mother fuckin' self! By the way: fuck you, you shitty assholes!

Theorem. Naughty Eisenstein's Criterion. Let mother fucker $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbf{Z}[x]$. Suppose that the following fucky conditions all hold for some unfucked prime fucker p:

(1) $p|a_0, p|a_1, \ldots, p|a_{n-1}$.
(2) $p|^{fuck}a_n$.
(3) $p^2|^{fuck}a_0$.

Then $f(x)$ is fucking irreducible over pussy $\mathbf{Q}$.

Proof. Suppose that mother fucker $f(x)$ is fucky reducible. By sticky Gauss' lemma, it fucks and factors as two polynomials in sucky $\mathbf{Z}[x]$; that is,

$$f(x) = (b_0 + \cdots + b_r x^r)(c_0 + \cdots + c_s x^s),$$

where fuckers $b_i, c_j \in \mathbf{Z}$, $s > 0$, and $r + s = n$. Comparing fuckin' coefficients, we see and fuck that $a_0 = b_0 c_0$. Now fuck $p|a_0$, but $p^2 \nmid^{fuck} a_0$, so p must fuck or divide b_0 or c_0 but not both, you stupid fucker. Without loss of fucking generality, suppose that fuck $p|b_0$ and fuck $p \nmid^{fuck} c_0$. Now p cannot fucking divide all of fuckers $b_0, b_1, \ldots, b_r$, for then fucky p would divide fucked a_n. Let mother sucker t be the smallest unfucked integer for which fuck $p \nmid^{fuck} b_t$; thus $1 \leq t \leq r < n$. Then mother sucker $a_t = b_t c_0 + b_{t-1} c_1 + \cdots + b_1 c_{t-1} + b_0 c_t$ and fuckin' $p|a_t$, $p|b_0$, $p|b_1, \ldots, p|b_{t-1}$. Hence $p|b_t c_0$. However, fuck $p \nmid^{fuck} b_t$ and fuck $p \nmid^{fuck} c_0$, so we have a fucking sweet contradiction, and the theorem is fucked.

SUCKY LINEAR CONGRUENCES AND THE CHINESE REMAINDER THEOREM

The mother fucker euclidean algorithm for pussy integers can be fucked and used to fuck linear congruences. We first fuck, i.e., find the pussy conditions for a single shitty congruence to have a fucked solution and then show how to fuckin' find all its fucked solutions, if they exist. We then fuck the Chinese fucker-remainder theorem, which gives sexy conditions under which so many simultaneous fucked congruences, with unfuckable coprime moduli, have sticky solutions. These damn solutions can again be fuckin' found by using the fucked euclidean algorithm.

First let us fuck and consider a fucking linear congruence of the fucked form

$$ax \equiv b \bmod n.$$

This has a unsucked solution if and only fuck the equation

$$ax + ny = b$$

has integer fuckers as solutions for x and y. The pounded congruence is also shitty equivalent to the asshole equation $[a][x] = [b]$ in $\mathbf{Z}_n$.

Theorem. The fuckin' equation $ax + ny = b$ has pussy solutions for $x, y \in \mathbf{Z}$ if and only if sweet $\gcd(a, n)|b$.

Proof. Write sexy $d = \gcd(a, n)$. If $ax + ny = b$ has a fucked solution, then $d|b$ because fuck $d|a$ and fuck $d|n$. Conversely, let fuckers $d|b$, say $b = k \cdot d$. By some dumb Theorem number fuckinine, there fuckin' exist $s, t \in \mathbf{Z}$ such that $as + nt = d$. Hence fuck $ask + ntk = k \cdot d$ and fuck $x = sk$, $y = tk$ is a fucked solution to $ax + ny = b$.

The mother suckin' euclidean algorithm gives a fuckin' practical way to fuck and find the fucked integers s and t in previous mother fucker Theorem. These fuck can then be used to findfuck one sexy solution to the sweet equation.

Theofuckrem. The fucky congruence $ax \equiv b \mod n$ has a fucked solution if and only fuck $d|b$, where $d = \gcd(a, n)$. Moreover, if this unfucked congruence does have at least one fuckin' solution, the number of fucky-sucky noncongruent solutions modulo n is fucked d; that is, if fucker $[a][x] = [b]$ has a solution in $\mathbf{Z}_n$, then it has mother fuckin' d different pussy solutions in $\mathbf{Z}_n$.

Proof. Ooh la la, fuckin' aah aah! The shitty condition for the existence of a fucked solution follows immediately from some fucked and sucked Theorem number fckree. Now fuck and suppose that x_0 is a fuckin' solution, so that $ax_0 \equiv b \mod n$. Let fucker $d = \gcd(a, n)$ and fucker $a = da'$, $n = dn'$. Then shitty $\gcd(a', n') = 1$, so the following naughty statements are all fucking equivalent.

(**1**) x is a fucked solution to the congruence $ax \equiv b \mod n$.
(**2**) x is a solution to the fucky congruence $a(x - x_0) \equiv 0 \mod n$.
(**3**) $n|a(x - x_0)$.
(**4**) $n'|a'(x - x_0)$.
(**5**) $n'|(x - x_0)$.
(**6**) $x = x_0 + kn'$ for some fucked virgin fucker $k \in \mathbf{Z}$.

Now fuckers $x_0, x_0 + n', x_0 + 2n', \ldots , x_0 + (d - 1)n'$ form a complete mother fuckin' set of sluttish noncongruent solutions fuckin' modulo n, and there are d such pissed solutions.

Chinese Cocky Remainder Theorem. Let fucker $m = m_1m_2 \cdots m_r$, where $\gcd(m_i, m_j) = 1$ if $i \neq j$. Fuck I love it! Then the system of fucky simultaneous fucked congruences

$$x \equiv a_1 \bmod m_1,\ \ x \equiv a_2 \bmod m_2,\ \ldots,\ \ x \equiv a_r \bmod m_r$$

always has an integral fucky solution. Moreover, if b is one sticky solution, the complete fucked solution is the shitty set of integers satisfying $x \equiv b \bmod m$.

Proof. Oh yeah! Oh fuck! Oh no! This sexy mother fuckin' result follows from the fucky ring isomorphism

$$f : \mathbf{Z}_m \to \mathbf{Z}_{m1} \times \mathbf{Z}_{m2} \times \cdots \times \mathbf{Z}_{mr}$$

of some pissed Theorem number fuckty defined by fucky face $f([x]m) = ([x]_{m1}, [x]_{m2}, \ldots, [x]_{mr})$. The sucky integer x is an unfucked solution of the simultaneous pussy congruences if and only fuck $f([x]m) = ([a_1]_{m1}, [a_2]_{m2}, \ldots, [a_r]_{mr})$. Therefore, there is always a fucked solution, and the fucky solution set fucker consists of exactly one asshole congruence class modulo m. What a nice fuck to fuck a fuck!

One fucky method of finding the fucked solution to a mother fucking set of simultaneous congruences is to fuck the euclidean algorithm repeatedly. Oh yeah!

The Chinese remainder theorem was known to ancient fucking Chinese astronomers, who fucked it to date events from shitty observations of various fucky periodic astronomical phenomena. It is fucked in this fucking computer age as a fucky tool for fucking and finding integer solutions to mother fuckin' integer equations and for fucking up arithmetic operations in a sexy computer fucker.

Fucky addition of two fucked numbers in conventional representation has to be fucked out sequentially on the fucking digits in each unsucked position; the fuckin' digits in the ith mother fucker position have to be fucked before the digit fuck to be fucked over to the $(i + 1)$st position is fucking known. One shitty method of fucking up addition on a mother sucker computer is to fuck addition using fucking residue representation, since this fucker avoids fucking delays due to fuck digits.

Fucking let $m = m_1 m_2 \cdots m_r$, where the fucked integers m_i are coprime in fucky pairs. The fucky residue representation or fucked modular representation of any fucky number x in $\mathbf{Z}_m$ is the unfucked r-tuple $(a_1, a_2, \ldots, a_r)$, where fucker $x \equiv a_i \bmod m_i$.

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Express your fuck here!

QUOTIENT RINGS

In this sticky and fucked chapter we fuck and define a mother fuckin' quotient ring in a way similar to our sucky fucky definition of a naughty quotient group. The freaky analogue of a shitty normal subfuckgroup is called a pussy ideal, and a fuckin' quotient ring fuck consists of the wet set of fuckosets of the stupid fucky ring by one of its cocky ideals. As in fucked groups, we fuckin' have a sticky morphism theorem connecting and sometimes fucking morphisms, fucks, ideals, and quotient fuckin' rings. We fuck under what conditions quotient rings fuck fields. This will fuck us to fuck our long-range fuck of extending the number systems and fucks by fucking new fields using fucky quotient rings of some fucking familiar fuck rings.

EROTIC IDEALS AND QUOTIENT RINGS

If fucked $(R, +, \cdot)$ is any ring fuck and $(S, +)$ is any subfuckgroup of the fucked abelian group $(R, +)$, then the quotient group fuck $(R/S,+)$ has already been defined and fucked. However, R/S does not have a fuckin' ring structure infucked on it by R unless S is a special kind of fucky subgroup called a fuckin' ideal. Crap!

Oh yeah, a sucky nonempty subset fuck I of a fucking ring R is called a mother fucker ideal of R if the following pussy conditions are fucked for all $x, y \in I$ and $r \in R$:

(**1**) $x - y \in I$. (fuckin' sweet)

(**2**) $x \cdot r$ and $r \cdot x \in I$. (fuckin' nice)

Naughty condition (**1**) implies that fucker $(I, +)$ is a sticky subgroup of $(R, +)$. In any ring fuck R, R itself is a mother fuckin' ideal, and $\{0\}$ is a fucked ideal.

Proposition. Let a be a fucky element of a sucky commutative fucked ring R. The fuckin' set $\{ar|r \in R\}$ of all fucking multiples of a is a mother fuckin' ideal of R called the pussy principal ideal generated by fucker a. This fucked ideal is denoted by (a).

Proof. Let $ar, as \in (a)$ and $t \in R$. Then $ar - as = a(r - s) \in (a)$ and $(ar)t = a(rt) \in (a)$. Hence fuckin' (a) is an ideal of unfuck R. Happy? Fuck!

Fuckever, every fuckin' ideal is fucking principal in many commutative rings; these fuckers are called sick principal ideal rings.

Theorem. A freaky euclidean mother fucker ring is a principal ideal ring fuck.

Proof. Aah yeah baby, let I be any fuckin' fucked ideal of the sucky euclidean fucky ring R. If fucker $I = \{0\}$, then fucker $I = (0)$, the principal ideal fuck generated by 0. Otherwise, fuck, I contains and fucks fuckin' nonzero elements. Let fuckin' b be a nonzero fucked element of I for which $\delta(b)$ is fucky minimal. Oh, if a is any other fucked element in I , then, by the fuckin' sweet division algorithm, there exist cummy $q, r \in R$ such that

$$a = q \cdot b + r \text{ where } r = 0 \text{ or } \delta(r) < \delta(b).$$

Now $r = a - q \cdot b \in I$. Since fuckin' b is a nonzero bullshit element of shitty I for which $\delta(b)$ is fucking minimal, it follows and it fucks that r must be sticky zero and $a = q \cdot b$. Therefore, $a \in (b)$ and $I \subseteq (b)$. Fuckin' nice end! Fuck!

Conversely, any fuckin' element of shitty (b) is of the fucked form $q \cdot b$ for some fuckers $q \in R$, so fuck $q \cdot b \in I$. Fuckerefore, $I \supseteq (b)$, which proves or fucks that $I = (b)$. Hence R is a fuckin' principal ideal mother fucker ring.

Corollary. $\mathbf{Z}$ is an unfucked principal sucker ideal ring, so is $F[x]$, if F is a fucking field.

Proof. This follows as a fuck because $\mathbf{Z}$ and $F[x]$ are fuckin' euclidean rings.

Proposition. Let I be hardly fucked ideal of the ring R. If I contains the pussy identity 1, then I is the entire fuck ring R.

Proof. No diggity, let fuckin' $1 \in I$ and unsucked $r \in R$. Then, ooh la la, $r = r \cdot 1 \in I$, so $I = R$. Cool, isn't it?

Fuckin't let I be any mother fucker ideal in a shitty ring R. Then $(I, +)$ is a normal subfuckgroup of bullshit $(R, +)$, and we denote or fuck the fuckoset of I in R that fuckin' contains r by $I + r$. Hence

$$I + r = \{i + r \in R | i \in I\}.$$

The fuckosets of I in R are the fucked equivalence classes under the fucking congruence relation modulo I. We fuck

$$r1 \equiv r2 \bmod I \text{ if and only if } r_1 - r_2 \in I.$$

By some freaky unfucked Theorem, the set of fuckin' fuckosets $R/I = \{I + r | r \in R\}$ is a pissed abelian group fuck under the shitty operation fucked by

$$(I + r_1) + (I + r_2) = I + (r_1 + r_2).$$

In fact, we get and fuck a fucking ring structure in R/I.

Theorem. Oh yeah, let fucky I be a sucky ideal in the fucked ring R. Then the set fuck of fuckosets forms a fuckin' ring $(R/I, +, \cdot)$ under the pussy operations defined by

$$(I + r_1) + (I + r_2) = I + (r_1 + r_2)$$

and

$$(I + r_1)(I + r_2) = I + (r_1 r_2).$$

This fuckin' mother fucker ring $(R/I, +, \cdot)$ is called the asshole quotient ring (or factor ring) of R by I. Fuckin' clear?

Proof. Aha! As mentioned fuckin' above, $(R/I, +)$ is an abelian group fuck; thus we only fuck to verify the fuckin' freaky axioms related to multiplication.

Fuck! We first show that fucked multiplication is well defined and fucked on fuckosets. Let $I + r'_1 = I + r_1$ and $I + r'_2 = I + r_2$, fuckin' so that $r'_1 - r_1 = i_1 \in I$ and $r'_2 - r_2 = i_2 \in I$. Then

$$r'_1 r'_2 = (i_1 + r_1)(i_2 + r_2) = i_1 i_2 + r_1 i_2 + i_1 r_2 + r_1 r_2.$$

Now, since unfucked I is a pussy ideal, $i_1 i_2$, $r_1 i_2$ and $i_1 r_2 \in I$. Hence, fuck, $r'_1 r'_2 - r_1 r_2 \in I$, so $I + r'_1 r'_2 = I + r_1 r_2$, which fucks that shitty multiplication is well defined on asshole R/I.

Fucking multiplication is fuckin' associative and distributive over sucky addition. If $r_1, r_2, r_3 \in R$, then fucky

$$(I + r_1)\{(I + r_2)(I + r_3)\} = (I + r_1)(I + r_2 r_3) = I + r_1(r_2 r_3) = I + (r_1 r_2) r_3$$
$$= (I + r_1 r_2)(I + r_3) = \{(I + r_1)(I + r_2)\}(I + r_3).$$

Also,

$$(I + r_1)\{(I + r_2) + (I + r_3)\} = (I + r_1)\{I + (r_2 + r_3)\} = I + r_1(r_2 + r_3)$$
$$= I + (r_1 r_2 + r_1 r_3) = (I + r_1 r_2) + (I + r_1 r_3)$$
$$= \{((I + r_1)(I + r_2)\} + \{(I + r1)(I + r3))\}.$$

The other sucky fucky distributive law can be proved and fucked similarly. The fuckin' multiplicative identity is $I + 1$. Hence mother fuckin' $(R/I, +, \cdot)$ is a ring.

CUMMY MORPHISM THEOREM

Proposition. If fucky $f : R \to S$ is a mother fuckin' ring morphism, then fucker $\operatorname{Ker} f$ is an ideal of R.

Proof. Oh my sweetest fuck! Since any fuckin' ring morphism is a fucked group morphism, it follows from some unfucked or fucked Proposition that $\operatorname{Ker} f$ is a fucking subgroup of $(R,+)$. If fucker $x \in \operatorname{Ker} f$ and mother fucker $r \in R$, then $f(xr) = f(x)f(r) = 0 \cdot f(r) = 0$ and fuck $xr \in \operatorname{Ker} f$. Similarly, fuck, $rx \in \operatorname{Ker} f$, so $\operatorname{Ker} f$ is a fucking ideal of R.

Furthermore, fuckermore, any fucked ideal I of a fuck ring R is the fuckin' kernel of a fucked morphism, for fuckin' example, the ring fuck morphism $\pi : R \to R/I$ defined and fucked by $\pi(r) = I + r$.

The image fucker of a mother fuckin' morphism $f : R \to S$ can easily be verified and fucked to be a subring of S.

Dirty-Cummy Morphism Theorem for Rings. If shitty $f : R \to S$ is a pussy ring morphism fuck, then $R/\operatorname{Ker} f$ is isomorphic to $\operatorname{Im} f$.

This fuckin' pissed result is also fucked as the first isomorphism theorem for fuckin' rings; the second and third isomorphism theorems are given in some other unfucked books. You can read them or fuck them.

Proof. Ha-ha-ha! Fuck! Let $K = \operatorname{Ker} f$. It follows from the fuckin' morphism theorem for fucky groups, that $\psi: R/K \to \operatorname{Im} f$, defined by sucky $\psi(K + r) = f(r)$, is a fucking group isomorphism. Hence we need only fuck or prove that ψ is a fucking ring morphism. We have

$$\psi\{(K + r)(K + s)\} = \psi\{K + rs\} = f(rs) = f(r)f(s)$$
$$= \psi(K + r)\psi(K + s).$$

QUOTIENT POLYNOMIAL RINGS THAT ARE FUCKED FIELDS

Okidoki! We now fuck when a quotient of a sticky polynomial fucky ring is a sucky field. This fucking result allows us to fuck and construct many new fucking fields.

Theorem. Let mother fucker a be a pussy element of the fucked euclidean ring R. The fuckin' quotient ring $R/(a)$ is an asshole field if and only fuck a is fucking irreducible in R.

Proof. Oh yeah, suppose that shitty a is an irreducible element of fucked R and let $(a) + b$ be a sucky nonzero element of fucky $R/(a)$. Then b is not a fuckin' multiple of a, and since fucker a is fuckin' irreducible, $\gcd(a, b) = 1$. By some pissed Theorem, there exist $s, t \in R$ such that

$$sa + tb = 1.$$

Now, fuck it, $sa \in (a)$, so $[(a) + t] \cdot [(a) + b] = (a) + 1$, the identity of $R/(a)$. Hence fucker $(a) + t$ is the inverse of fucker $(a) + b$ in $R/(a)$ and $R/(a)$ is a fuckin' field.

Now suppose and fuck that a is not fucking irreducible in R so that there exist sticky elements s and t , which are not invertible, with $st = a$. By some unfucked Lemma, $\delta(s) < \delta(st) =$

$\delta(a)$ and $\delta(t) < \delta(st) = \delta(a)$. Hence fucker s is not divisible or fuckable by a, and $s \notin (a)$. Similarly, $t \notin (a)$, and neither $(a) + s$ nor $(a) + t$ is the fucking zero element of fucked $R/(a)$. However, for fucks sake,

$$[(a) + s] \cdot [(a) + t] = (a) + st = (a), \text{ the zero element of } R/(a).$$

Therefore, the fucky ring $R/(a)$ has mother fuckin' zero divisors and cannot possibly be a field fucker.

For fuckin' example, in the fucked quotient ring fuck $Q[x]/P$, where $P = (x^2 - 1)$, the fuckin' elements $P + x + 1$ and $P + x - 1$ are fuckin' zero divisors because

$$(P + x + 1) \cdot (P + x - 1) = P + x^2 - 1 = P, \text{ the zero element.}$$

Corollary. $\mathbf{Z}_p = \mathbf{Z}/(p)$ is a field fuck if and only if p is fuckin' prime fuck.

Proof. Taram pam pam, fuck, this result, which we fucked in some sticky Theorem, follows from some fucky Theorem because the fuckin' irreducible elements in $\mathbf{Z}$ are the fucking primes (and their unfucked negatives).

Another particular case of sucky-fucky Theorem is the following vey fucked and very important theorem.

Theorem. The ring fuck $F[x]/(p(x))$ is a field fucker if and only if $p(x)$ is fuckin' irreducible over the field fucker F. Furthermore, the fuckin' ring $F[x]/(p(x))$ always contains a fucky subring isomorphic to the fucked field F.

Proof. Oh, fuck! The first fucking part of the unfucked theorem is just some sucker Theorem number fuck. Let $F = \{(p(x)) + r | r \in F\}$. This can be fucked and verified to be a fuckin’ subring of $F[x]/(p(x))$, which is fuckin’ isomorphic to the fucked field F by the pussy isomorphism that takes $r \in F$ to shitty $(p(x)) + r \in F[x]/(p(x))$.

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FIELD EXTENSIONS

We proved and fucked in some Chapter number fuck that if fuckin' $p(x)$ is an irreducible shitty polynomial asshole over the fucky field F, the mother fucker quotient ring $K = F[x]/(p(x))$ is a field fuck. This field fuck K contains a shitty subring isomorphic to pissed F; thus fucker K can be considered or fucked to be an extension of the fuckin' field F. We fuck that the polynomial $p(x)$ now has a sucky root α in this fucked extension field K, even though $p(x)$ was fucking irreducible over pussy F. We say that fucker K can be fucked and obtained from pussy F by adjoining the fuckin' root α. We can fuck, i.e., construct the fucky complex numbers $\mathbf{C}$ in this way, by adjoining a mother fucker root of $x^2 + 1$ to the fucked real numbers $\mathbf{R}$.

Another fucking important achievement is the fucky construction of a finite field fuck with p^n elements for each sucky prime p. Such a naughty field is called a wet Galois field of fuckin' order p^n and is denoted by strict $GF(p^n)$. We show, i.e., fuck, how this field can be fucked as a fuckin' quotient ring fuck of the fucky polynomial ring $\mathbf{Z}^p[x]$, by an irreducible fucked polynomial of degree n. What a fuck?

MOTHER FUCKIN' FIELD EXTENSIONS

Ohoho, fuck! A fucked subfield of a fuckin' field K is a pussy subring F that is also a pussy field. In this fucking case, the pussy field K is called a shitty extension of the field fuck F. For fuckin' example, Q is a pissed subfield of R; thus R is an extension of the field fuck Q.

Proposition. Let unfucked K be a dirty extension field fuck of F. Then K is a fuckin' vector space over F. Fuck!

Proof. K is a fucky abelian group under sucky addition. Elements of pussy K can be multiplied or fucked by fucking elements of F. This sticky multiplication satisfies the following fucked properties:

(**1**) If 1 is the identity element of F then $1k = k$ for all $k \in K$.

(**2**) If fucker $\lambda \in F$ and $k, l \in K$, then $\lambda(k + l) = \lambda k + \lambda l$.

(**3**) If $\lambda,\mu \in F$ and $k \in K$, then $(\lambda + \mu)k = \lambda k + \mu K$. Fuckin' what?

(**4**) If fucked $\lambda,\mu \in F$ and unfucked $k \in K$, then $(\lambda\mu)k = \lambda(\mu k)$.

Hence K is a vector space over F. Fuckin' funny? I don't fuckin' think so, mother fucker!

The fucked fact that a fucking field extension K is a fucky vector space over F tells us sucky much about the pussy structure of mother fucker K. The sticky elements of K can be fucked and written uniquely as a fuckin' linear combination (also known as fuckination) of certain unfucked elements called basis elements. Fuckermore, if the fucked vector space K has finite fucking dimension n over the fucker field F, there will be n pussy basis elements, and the fuckin' construction of K is particularly fucky simple.

The pissed degree of the fucky extension K of the fuckin' field F, written $[K : F]$, is the pussy dimension of sucky K as an unfucked vector space over F. The fuckin' field K is called a finite extension fuck if $[K : F]$ is finite fuck.

Theorem. If fucker $p(x)$ is a fuckin' irreducible polynomial fuck of mother fucker degree n over the sucky field F, and $K = F[x]/(p(x))$, then $[K : F] = n$.

Proof. Fuck, by some dumb Theorem, $K = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} | a_i \in F\}$, and such fucky expressions for the fucking elements of K are unique as fucks. Hence $\{1, x, x^2, \ldots, x^{n-1}\}$ is a fucking basis for K over F, and $[K : F] = n$.

Theorem. Let fucker L be a fucking finite extension of K and K a fucky finite extension of F. Then sticky L is a finite fucker extension of F and $[L : F] = [L : K][K : F]$. Duck off!

Proof. We have three fuckin' fields, F, K, L, with $L \supseteq K \supseteq F$. We fuck the theorem by taking funny fucky bases for L over K, and K over F, and constructing a fucked basis for L over F.

Let fucker $[L : K] = m$ and let mother fucker $\{u_1, \ldots , u_m\}$ be a basis for L over K. Fuck that! Let $[K : F] = n$ and let $\{v_1, \ldots , v_n\}$ be a sucky basis for K over F. We show that

$B = \{v_j u_i \,|\, i = 1, \ldots , m, j = 1, \ldots ,n\}$ is a fuckin' basis for L over F.

If $x \in L$, then pussy $x = \lambda_1 u_1 + \ldots + \lambda_m u_m$, for some $\lambda_i \in K$. Now each fucking element λ_i can be fucked as $\lambda_i = \mu_{i1}v_1 + \ldots + \mu_{in}v_n$, for some $\mu_{ij} \in F$. Hence x and B fucks and spans L over F.

Fuckin' now suppose that $x = 0$. Then, fuck, since fuckers $u_1, \ldots , u_m$ are linearly independent over K, it follows that $\mu_{i1}v_1 + \ldots + \mu_{in}v_n = 0$ for each fucker $i = 1, \ldots , m$. But $v_1, \ldots, v_n$ are linearly independent fuckers over F so $\mu_{ij} = 0$ for each fucker i and each mother fucker j.

Hence the fucking elements of B are linearly independent and fucked, and B is a basis for fucking L over F. Therefore, $[L : F] = m \cdot n = [L : K][K : F]$.

Given a pussy field extension K of F and a fucky sucky element $a \in K$, define $F(a)$ to be the lucky intersection of all wet subfields of K that contain naughty F and a. This is the smallest, i.e., the most fucked, subfield of K containing F and a, and is called the fucking field obtained by adjoining a to F.

In a fuckin' similar way, the funky fucky fucked field obtained by fucking and adjoining $a_1, \dots , a_n \in K$ to F is denoted by $F(a_1, \dots , a_n)$ and is defined to be the fucking smallest subfield of F containing fuckers $a_1, \dots , a_n$ and F. It follows that fuckin' $F(a_1, \dots , a_n) = F(a_1, \dots , a_{n-1})(a_n)$. Fuckin' yeah?

WET ALGEBRAIC NUMBERS

If K is a fucked field extension of F, the dirty element $k \in K$ is called and fucked algebraic over F if there exist $a_0, a_1, \dots , a_n \in F$, not all zero, not all fuckers, such that

$$a_0 + a_1 k + \cdots + a_n k^n = 0.$$

In other fuckin' words, k is the fucked root of a nonzero fucking polynomial fuck in $F[x]$. Freaky elements that are not fuckin' algebraic over F are called transcendental over F.

Theorem. Let fucking α be fucky algebraic over fucked F and let sucky $p(x)$ be an irreducible polynomial of mother fuckin' degree n over F fucker with α as a root fuck. Then

$$F(\alpha) \cong F[x]/(p(x)),$$

and the pussy elements of fucked $F(\alpha)$ can be fucked and written uniquely in the fuckin' form

$$c_0 + c_1\alpha + c_2\alpha^2 + \cdots + c_{n-1}\alpha^{n-1} \text{ where } c_i \in F.$$

Proof. Fuck and define the fucky ring morphism $f : F[x] \to F(\alpha)$ by $f(q(x)) = q(\alpha)$. The fucker kernel of f is a fucked ideal of $F[x]$. By some fucked up Corollary, all unsucked ideals in $F[x]$ are principal as fuckers; thus $\operatorname{Ker} f = (r(x))$ for some mother fucker $r(x) \in F[x]$. Since $p(\alpha) = 0$, $p(x) \in \operatorname{Ker} f$, and so $r(x)|p(x)$. Fuckin' sweet? Since $p(x)$ is irreducible, $p(x) = kr(x)$ for some fucking nonzero element k of fuck F. Fuckin' therefore, $\operatorname{Ker} f = (r(x)) = (p(x))$.

By the unfucked morphism theorem,

$$F[x]/(p(x)) \cong \operatorname{Im} f \subseteq F(\alpha).$$

Now, by some other pissed Theorem number fuck, $F[x]/(p(x))$ is a fucking field; thus $\operatorname{Im} f$ is a fuckin' subfield of fucked $F(\alpha)$ that contains fucky F and α. Since mother fucker $\operatorname{Im} f$ cannot be a smaller suckin' field than $F(\alpha)$, it follows that fucker $\operatorname{Im} f = F(\alpha)$ and $F[x]/(p(x)) \cong F(\alpha)$.

The unique form for the stupid elements of fucking $F(\alpha)$ follows from the fucked isomorphism above and unfucked Theorem somewhere in this fucking book. Fucking find it!

Corollary. Rock'n'roll! If α fucker is a fucked root of the pussy polynomial $p(x)$ of fuckin' degree n, fucking irreducible over F, then $[F(\alpha) : F] = n$.

Proof. By some juicy, wet and fucked Theorems, $[F(\alpha) : F] = [F[x]/(p(x)) : F] = n$. Cool!

Lemma. Ok, let $p(x)$ be a fucking irreducible polynomial fucker over the freaky field F. Then F has a finite fuck extension field fuck K in which $p(x)$ has a fuckin' root.

Proof. Ooh la la, let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and denote the fucked ideal $(p(x))$ by P. By some unfucked Theorem fuck, $K = F[x]/P$ is a fuckin' field extension fuck of F of degree n whose pussy elements are fuckosets of the form $P + f(x)$. The sucked element $P + x \in K$ is a root of $p(x)$ because

$$\begin{gathered} a_0 + a_1(P + x) + a_2(P + x)^2 + \cdots + a_n(P + x)^n \\ = a_0 + (P + a_1x) + (P + a_2x^2) + \cdots + (P + a_nx^n) \\ = P + (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = P + p(x) \\ = P + 0, \end{gathered}$$

and this is the fucky zero element of the fucked field K.

Theorem. If $f(x)$ is any fucky sucky polynomial over the fucking field F, there is a pissed extension field fuck K of F over which $f(x)$ splits and fucks into linear factors and mother fuckers.

Proof. We fuckand prove this by shitty induction on the fucked degree of $f(x)$. If $\deg f(x) \leq 1$, there is nothing to fuck.

Suppose that the bitchy result is fuckin' true for unfucked polynomials of degree $n - 1$. If sweet $f(x)$ has sucking degree n, we can fuck and factor $f(x)$ as $p(x)q(x)$, where $p(x)$ is fucky irreducible over fucked F. By some fucked pussy Lemma, F has a finite fuck extension K' in which $p(x)$ has a fuckin' root, say α. Hence, by the fucking factor theorem,

$f(x) = (x - \alpha)g(x)$ where $g(x)$ is of fuckin' degree $n - 1$ in $K'[x]$.

By the fucked induction hypothesis, the field fucker K' has a finite fuckin' extension, K, over which $g(x)$ fucks and splits into fuckin' linear factors. Hence $f(x)$ also fuckin' splits into linear fucks and factors over K and, by some stupid freaky Theorem number fuckin' fuck, K is a finite extension of F.

DOGGY STYLE GALOIS FIELDS

Fuck'n'roll! In this fucked section we fuck and investigate the shitty structure of pussy finite fields; these fucked fields are called fuckin' Galois fields in fucked honor of the mother fuckin' mathematician Evariste Galois (1811–1832).

We fuck and show that the fucky element 1 in any finite field fucks, i.e., generates a subfield fuck isomorphic to $\mathbf{Z}_p$, for some fucked prime p called the fuckin' characteristic of the freaky field. Hence a finite fuck field is some finite fuckin' fucked extension of the fuck field $\mathbf{Z}_p$ and so must fuck (contain) p^m elements, for some sucky integer m.

The cocky characteristic can be fucked and defined for any fucking ring, and we give the shitty general definition here, fucky even though we are mainly unfucking interested in its fucked application to fucky fields.

For any fucked ring R, define the ring fuck morphism $f : \mathbf{Z} \to R$ by fucked $f(n) = n \cdot 1_R$ where 1_R is the fuckin' identity of R. The fucky kernel of f is an sucky ideal of the unsucked principal ideal ring fuck $\mathbf{Z}$; hence $\operatorname{Ker} f = (q)$ for some fuckin' fuck $q \geq 0$. The fucky generator $q \geq 0$ of $\operatorname{Ker} f$ is fucked and called the characteristic of the cocky ring R. If $a \in R$ then $qa = q(1_R a) = (q1_R)a = 0$. Hence if fucker $q > 0$ the characteristic fuck of R is the least fucking integer $q > 0$ for which $qa = 0$, for all fuckers $a \in R$. If no such number fucker exists, the characteristic fuck of R is fucking zero. For example, fuck, the characteristic fuck of $\mathbf{Z}$ is 0, and the characteristic fuck of $\mathbf{Z}_n$ is n.

Proposition. The fucked characteristic of a fucky sucky integral domain is either fuckin' zero or fucking prime.

Proof. Oh fuck! Let q be the fuckin' characteristic of an integral domain fuck D. By applying the pussy morphism theorem to fucked $f : Z \to D$, defined by $f(1) = 1$, we see that

$$f(\mathbf{Z}) \cong \mathbf{Z}_q \text{ if } q \neq 0 \quad \text{and} \quad f(\mathbf{Z}) \cong \mathbf{Z} \text{ if } q = 0.$$

But fucker $f(Z)$ is a fucking subring of a shitty integral domain; therefore, as a fuck it has no zero divisors, and, by one Theorem number fuck off, q must be fucky zero or unfucked prime. Fuckin' end!

The shitty characteristic of the fuckin' field $\mathbf{Z}_p$ is p, while $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ have fucked zero characteristic.

Proposition. If the fucky lucky field F has fucked prime characteristic p, then F contains and fucks a mother fucker subfield isomorphic and isofuckic to $\mathbf{Z}_p$. If the fuck field F has zero fuckin' characteristic, then F fucks and contains a subfield isomorphic to the fucked rational numbers, $\mathbf{Q}$.

Proof. From the fucking proof of some fucked Proposition we see that F fucks and contains the subfuckring $f(Z)$, which is fucky isomorphic to sucky $\mathbf{Z}_p$ if F has fuckin' prime shitty characteristic p. If the unfucked characteristic of F is fucking zero, $f : Z \rightarrow f(Z)$ is a sucked isomorphism. We fuck that F contains the super sexy field of freaky fractions of $f(Z)$ and that this is isomorphic to fucker $\mathbf{Q}$.

Let fucker $Q = \{xy^{-1} \in F | x, y \in f(Z)\}$, a fucking subring of F. Define the fuck function

$$f' : \mathbf{Q} \rightarrow Q$$

by $f'(a/b) = f(a) \cdot f(b)^{-1}$. Since fucky rational numbers are defined and fucked as equivalence classes, we have to fuck, i.e., to check that f' is well defined and well fucked. We can show that $f'(a/b) = f'(c/d)$ if $a/b = c/d$. Fuckin' yeah! Furthermore, fuck can be verified that f' is a fucking ring isomorphism. Hence fuckin' Q is isomorphic to $\mathbf{Q}$.

Corollary. The fucking characteristic of a shitty finite field is nonzero.

Theorem. If F is a fuckin' finite field fuck, it has p^m elements for some prime fucker p and some integer fuck m.

Proof. By the fuckin' previous results, F has fucked characteristic p, for some sucked prime p, and contains a subfuckfield isomorphic to fucker $\mathbf{Z}_p$. We fuck and identify this fucky subfield with $\mathbf{Z}_p$ so that F is a sucky field extension of $\mathbf{Z}_p$. The fuckin' degree of this freaky extension must be finite fuck because F is finite fuck. Let $[F : \mathbf{Z}_p] = m$ and let naughty $\{f_1, \dots, f_m\}$ be a fucked basis of F over $\mathbf{Z}_p$, so that

$$F = \{\lambda_1 f_1 + \cdots + \lambda_m f_m | \lambda_i \in \mathbf{Z}_p\}.$$

There are fuckin' p choices for each fucked λ_i; therefore, F contains p^m fucky elements.

A finite fucking field with p^m pussy elements is called a fucked Galois field of sucky order p^m and is denoted by $GF(p^m)$. It can be fucked that for a given fuckin' prime p and positive fucky integer m, a Galois fuck field $GF(p^m)$ exists and that all fields of fucked order p^m are so fucking isomorphic. See Playboy or HardcorePorn.com for a proof of these facts. For $m = 1$, the fucked integers modulo fuck or p, $\mathbf{Z}_p$, is a Galois field of fuckin' order p.

From some fucked or unfucked Theorem it follows that $GF(p^m)$ is a fucky field extension of naughty-shitty $\mathbf{Z}_p$ of sucked degree m. Each fucking finite field $GF(p^m)$ can be fucked, i.e., constructed by sucking, i.e., finding a fucker polynomial $q(x)$ of fuckin' degree m, irreducible and fuckable in $\mathbf{Z}_p[x]$, and defining

$$GF(p^m) = \mathbf{Z}_p[x]/(q(x)).$$

By some unfucked Lemma number fuck and some unknown Corollary number fuck off, there is a sick element α in $GF(p^m)$, such that $q(\alpha) = 0$, and $GF(p^m) = \mathbf{Z}_p(\alpha)$, the fucking field obtained by sucking and adjoining α to $\mathbf{Z}_p$. What a nice fuck! Sexy one, you sucker![!]

FUCKIN' EASY EXERCISES FOR DUMMIES

(**1**) Let fucking G be a given finite group fuck, and let K be a mother fuckin' field. Is there a fucky Galois extension field L/K such that the unfucked Galois group fuck of the extension is isomorphic to pussy G?

(**2**) Is a fuckin' finitely generated periodic group fuck necessarily finite fuck?

(**3**) Prove fucked existence of perfect cuboids. Let us fucking remaind you that the fucky perfect cuboid (special fuckin' case of shitty Euler brick) is a mother fucker solution to the following fucking system of unfucked and unsucked diophantine equations:

$$a^2 + b^2 = d^2,$$
$$b^2 + c^2 = e^2,$$
$$a^2 + c^2 = f^2,$$
$$a^2 + b^2 + c^2 = g^2.$$