Interlace Patterns in Islamic and Moorish Art

Branko Grünbaum and G. C. Shephard

ABSTRACT

Repetitive interlace patterns are one of the hallmarks of Islamic and Moorish art. Through the study of various collections of such patterns, it is easy to verify that, despite the considerable complexity of the designs, most of the interlaces are formed by strands of a small number of shapes—often just a single shape stretching over many repeats of the design. This observation is described and documented by the authors, who present a simple explanation for this phenomenon.

Interlace patterns occur frequently in the ornaments of various cultures, especially in Islamic and Moorish art. By *interlace pattern* we mean a periodic pattern that appears to consist of interwoven *strands* (see Figs 1 and 2). (The exact meaning of the term 'periodic pattern' will be explained later.) The largest published collection of interlace patterns is that of J. Bourgoin [1]; more precisely, the majority of Bourgoin's diagrams can be interpreted as representing such patterns, and it appears that they were, in fact, created with interlace patterns as originals. Other examples can be found in works by d'Avennes, El Said and Parman, Wade, and Humbert [2–5]. A number of authors have relied on the Bourgoin drawings in their research; for example, they have been used by Makovicky and Makovicky in their analysis of the frequencies with which various classes of symmetry groups are represented in Islamic patterns [6].

This article has three main goals. First, we will observe that most of the interlace patterns shown by Bourgoin consist either of identical (congruent) strands or of two shapes of strands. This is unexpected because in many cases the strand patterns are very complicated—some of them are so intricately interwoven that a repeat is unlikely to have been visible in any actual ornament.

The complications of an interlace pattern can be measured in several ways—for example, by the number of different shapes of strands or by the number of crossings in a repeat. Our second aim is to relate these measures to the symmetry properties of the pattern. The results that we obtain lead to our third goal: a plausible explanation of the frequent appearance of interlace patterns consisting of strands of a single shape or of just two different shapes in Islamic art.

BACKGROUND EXPLANATIONS

Two of Bourgoin's diagrams appear in Figs 1a and 2a. It can be observed in each of these diagrams that pairs of lines appear to *cross* at various points, but in no case do three or more lines pass through the same point. In the following discussion we shall restrict our attention to those drawings for which this condition holds—as it does in nearly 85% of Bourgoin's diagrams. By interpreting the lines as strands of small but positive width, we can obtain an interlace pattern by assigning an ordering of the strands at each crossing point—in other words by specifying which strand is *uppermost* or 'passes over' the other strand. (In the type of interlace pattern known as a 'fabric', multiple crossings are

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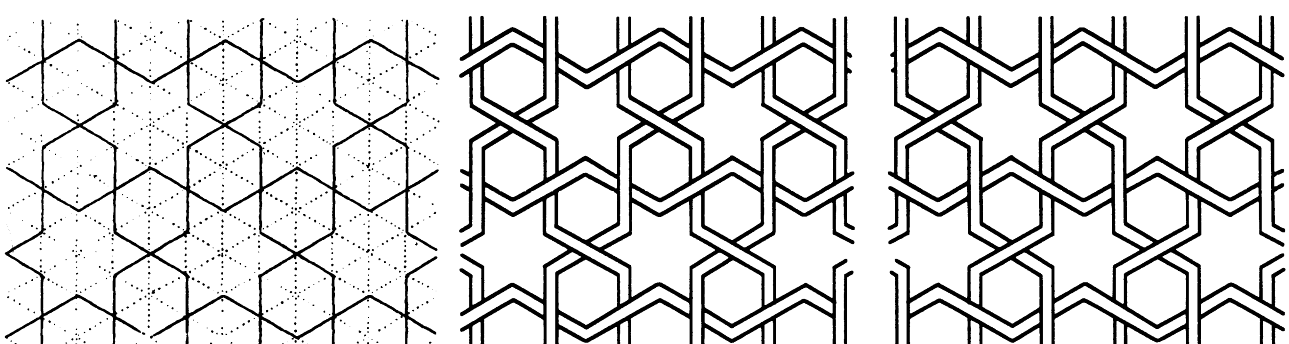
Fig. 1. (a, left) An often-used Islamic ornamental pattern, taken from Bourgoin [18]. It is the design from which the two interlace patterns shown in (b, center) and (c, right) can be obtained. The strands in these interlace patterns are unbounded, and all are congruent. The two interlace patterns differ only in the way the strands cross; at corresponding crossings, the strand that is on top in one pattern is on bottom in the other. In each of the two interlace patterns, all strands play equivalent roles under symmetries of the interlace pattern.



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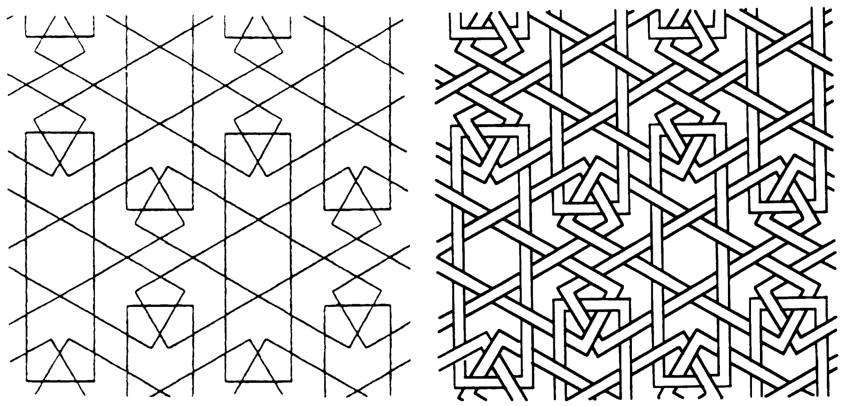


Fig. 2. (a, left) Another frequent Islamic ornamental pattern, from Bourgoin [19]. (b, right) One of the two interlace patterns that can be obtained from the design in (a). Here the strands are bounded (that is, they are closed loops); all strands are equivalent under symmetries of the interlace pattern.

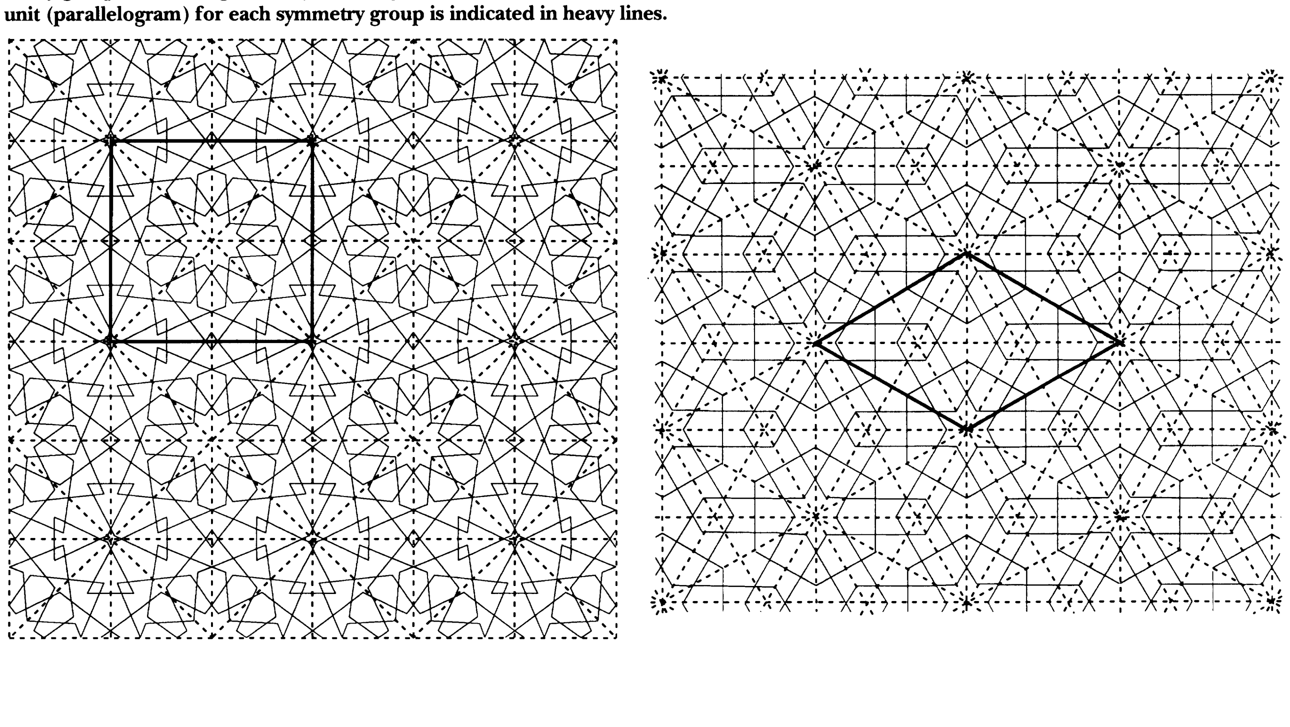
sometimes admitted [7], but we shall not consider this possibility here.)

In the overwhelming majority of interlace patterns occurring in actual ornaments, the strands weave alternately under and over each other [8]. Therefore we shall also restrict our attention to interlace patterns with this property. It follows that the specification for which strand is uppermost at just one crossing point determines the ordering of the two strands at all others. Thus a diagram such as Fig. 1a represents two interlace patterns according to the choice at one crossing point, as shown in Figs 1b and 1c. A simple argument shows that, for the kinds of patterns under consideration, once the ordering at one crossing point has been chosen, the ordering at every other crossing point can be assigned in a consistent manner, that is, so that every strand weaves alternately over and under each of the strands it meets. One consequence is that Bourgoin's diagrams, though not directly showing interlace patterns, determine them uniquely, apart from the possibility of interchanging all the under- and over-crossings. Any such diagram will be called a *design* for the interlace pattern. Note that all the properties of an interlace pattern can be deduced from its design; from now on we shall be concerned almost exclusively with investigating the designs rather than with the interlace patterns themselves. Since the strands in interlace patterns do not have ends, they must be either unbounded (as in Fig. 1) or closed loops (as in Fig. 2), or a mixture of both.

Now imagine the design for an interlace pattern as being unbounded, that is, extending arbitrarily far in every direction. By a *symmetry* of a design we mean any rigid motion that maps the design onto itself. Every rigid motion, other than the identity, is of one of four kinds: a translation, rotation, reflection or glide-reflection. Each of the patterns, and therefore each of the designs that we are considering, is *periodic*. This means that the design possesses at least two independent (nonparallel) translations as symmetries. It also means that a part of the design in the shape of a parallelogram is repeated over and over again, these parallelogram-shaped patches fitting together edge to edge so as to cover the whole plane. For the designs shown in Fig. 3, the parallelograms are indicated by heavy lines. If such a parallelogram is chosen to be as small as possible, it is referred to as a *translational repeating unit* for the design under consideration [9].

The set of all symmetries of a periodic design, with the usual method of composition, forms what is known as a *crystallographic* or *wallpaper group;* these are of 17 kinds [10,11]. It is easily verified that many of the designs of interlace patterns illustrated in Bourgoin's book have wallpaper groups of one of two types, usually designated by their crystallographic symbols *p4m* and *p6m*. In Fig. 3a,b we illustrate these two kinds of symmetry groups. It is clearly impossible to indicate all the symmetries in these figures, but we show all the lines of reflection (mirrors) that, as can be seen in Fig. 3, cut the plane into congruent triangles. The representation is adequate since every symmetry in either group can be expressed as a product (composition) of a finite

Fig. 3. Two designs of interlace patterns, redrawn after Bourgoin [20]. In each we show by dashed lines mirrors for reflections in the symmetry group of the design. The symmetry group of the design in (a, left) is *p4m* and of that in (b, right) is *p6m*. A translational repeating unit (parallelogram) for each symmetry group is indicated in heavy lines.



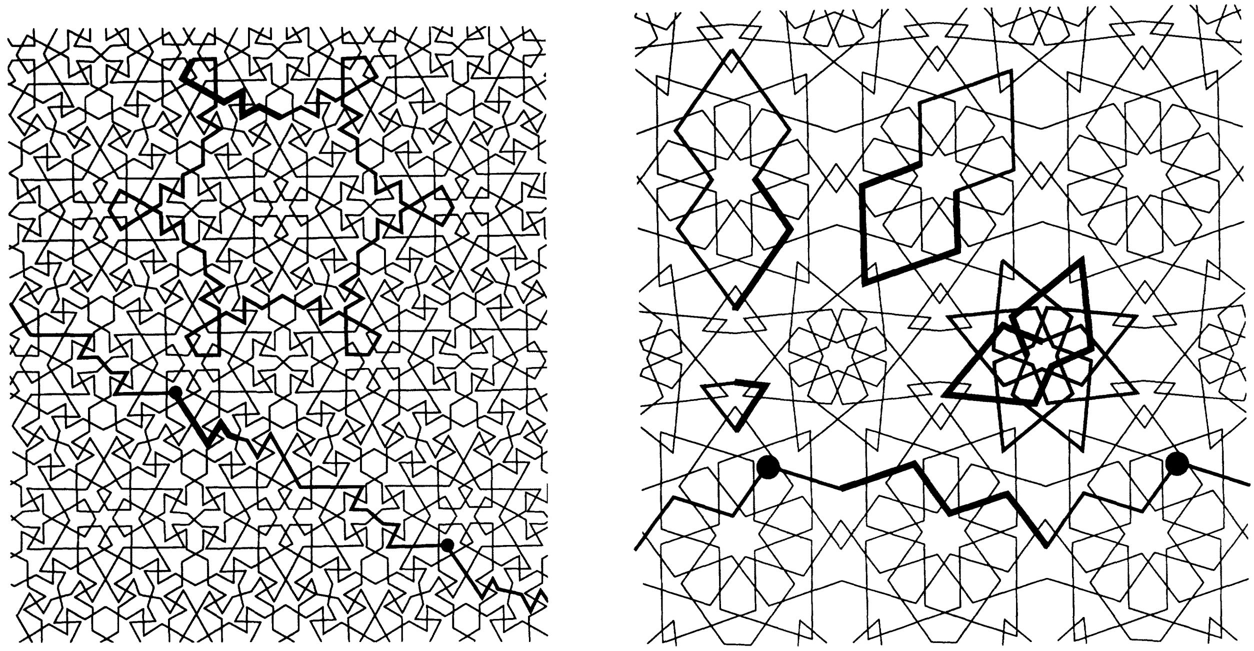


Fig. 4. Designs of interlace patterns, redrawn after Bourgoin [21]. The heavily drawn lines emphasize various strands. The thickened parts show a fundamental region for each of the strands, with respect to the induced group. In the case of unbounded strands, the portion between the heavy dots is a transitional repeating unit.

number of reflections. For this reason, the groups *p4m* and *p6m* are said to be *generated by reflections.* Moreover, each such triangle is a *fundamental region* for the group, which simply means that, once the part of the design in one triangle is known, then the whole design can be constructed by simply copying this into each of the other triangles in an appropriate manner, as in a kaleidoscope. This is clearly shown in both parts of Fig. 3, where it will also be seen that a translational repeating unit can be built up from the triangles, eight in the case of *p4m* and 12 in the case of *p6m.*

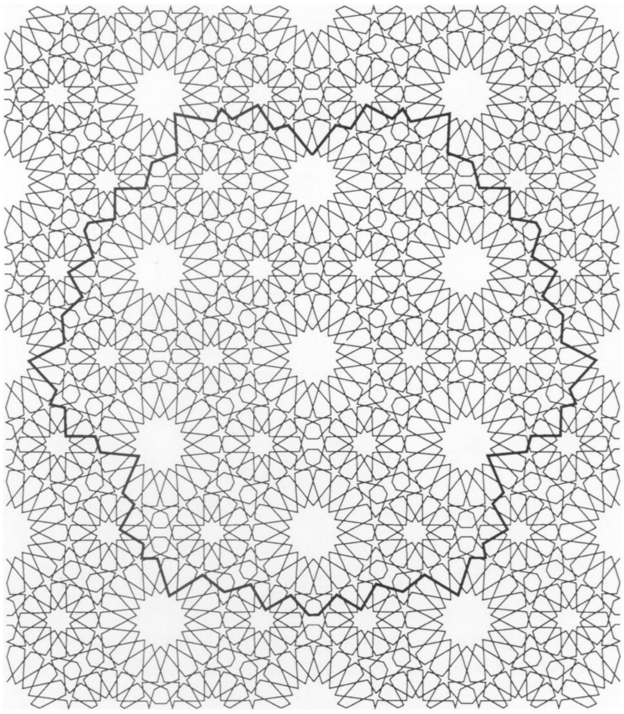
It is possible to construct fundamental regions for each of the other 15 kinds of crystallographic groups (although not all of them can be chosen to be triangles), but the details are more complicated, and as the groups *p4m* and *p6m* are by far the most common (about 75% of Bourgoin's designs have these symmetry groups) we shall restrict attention to them here. The reader is invited to investigate the other cases by suitable modifications of the steps explained below.

We shall also be concerned with the symmetries of a design that map a strand *S* (or, more precisely, the representation of *S* in the design) onto itself. These form what is known as the *induced group* of the strand *S.* If a strand is bounded, then its induced group must either be a *cyclic group cn* of order *n,* or a *dihedral group dn* of order 2*n,* where *n* = 1, 2, 3, 4 or 6. If a strand is unbounded, then its induced group must be one of seven kinds (known as *strip* or *frieze groups*) [12]. The distinction between the induced group of a strand and its symmetry group is illustrated in Fig. 4b. The 'rosette' by itself has symmetry group *d8,* but its induced group is *d2.* Of the two heavily drawn congruent octagonal strands in the upper part of Fig. 4b, one has induced group *d2,* and the other *c2.* Concerning strands, in each case there are direct analogues of the terms 'translational repeating unit' and 'fundamental region'. A *fundamental region* of the strand is the smallest part of a strand that can be used to build up the whole strand by application of operations in the induced group. In Fig. 4 the fundamental regions of a number of strands are indicated by thickened lines. A *translational repeating unit* of a strand is the smallest part of a strand that can be used to build up the whole strand using only translations in the induced group. In Fig. 4, limits of translational repeating units are indicated by large dots. For a bounded strand, the translational repeating unit is the strand itself.

Systematically tracing the strands in the design of an interlace pattern and examining their shapes is often not easy. In many cases Bourgoin's figures do not show a sufficiently large 'patch' of the design for a complete bounded strand, or a translational repeating unit of an unbounded strand, to be shown. It is of course possible to make copies of each of Bourgoin's diagrams and carefully paste several of them together to obtain a sufficiently large patch of the design. If this is done, then we can see the complicated shapes in which a strand can occur (see Fig. 5), and also the large number of crossings involved—as many as 138 for the looped strands (Fig. 5a) and 120 for each translational repeating unit of an unbounded strand (Fig. 5b). These numbers also give a measure of the intricacy of the patterns.

In addition to the symmetry groups of the designs, another method of classification is by the number of *shapes* in which the strands occur, as well as the induced group of each shape. We shall say that two strands are of the *same shape* if there exists a symmetry of the design which maps one onto the other. If the strands are of *n* different shapes then we call it an *n-strand pattern.* (In cases such as the octagonal strands in Fig. 4b, or Bourgoin's Figs 134 and 135, two strands may be congruent, but if there is no symmetry of the design that maps one onto the other, then we consider them as being of different shapes.) It is a surprise to observe that, in spite of the complicated structure of many of the designs, in most of them the number of different shapes of strands is very small. Out of the 169 diagrams shown in Bourgoin that can be interpreted as designs of interlace patterns, 44 are 1-strand patterns, 61 are 2-strand patterns, and only 40 are *n*-strand patterns with $n \geq 3$.

Below we shall explain how (in the two cases under consideration, namely where the symmetry group of the design is *p4m* or *p6m*) it is possible to determine the number of different shapes of strands, their groups and the number of crossings by looking at the triangular fundamental region for the symmetry group of the design. Similar considera-



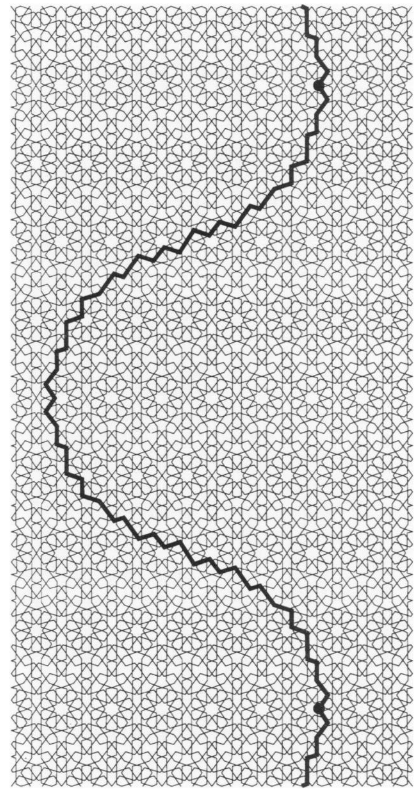


Fig. 5. Designs of 1-strand interlace patterns, redrawn after Bourgoin [22]; one strand in each is emphasized by heavy line. The looped strand in (a, left) crosses other strands 138 times, and each translational repeating unit in (b, right) (the portion of the strand between the heavy dots) crosses other strands 120 times. The large number of crossings indicates the complexity of these interlace patterns.

tions apply to the other 15 symmetry groups, but we shall not discuss these here. The reader is again invited to investigate them independently.

The Groups *p4m* and *p6m* and Their Cayley Diagrams

Before we explain the procedures involved, it is necessary to recall some of the properties of the groups *p4m* and *p6m*. In Fig. 6a we show the arrangement of mirrors in the group *p4m* (compare to Fig. 3a); a fundamental region is shaded [13]. The mirrors forming the sides of this fundamental region are labelled *a, b, c,* and we shall use these same letters, with no danger of confusion, for reflections in the corresponding mirrors. As each reflection is of period 2 we have

$$a^2 = b^2 = c^2 = e \qquad (1)$$

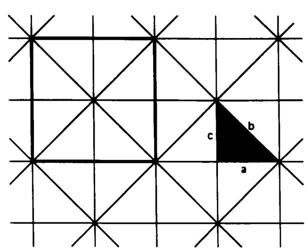
where *e* is the group identity. The product *ab* is a rotation about the vertex *C* through angle $\pi/4$, the product *bc* is a rotation about *A* through angle $\pi/4$, and the product *ca* is a rotation about *B* through angle $\pi/2$. We deduce that:

$$(ab)^4 = (bc)^4 = (ac)^2 = e. \qquad (2)$$

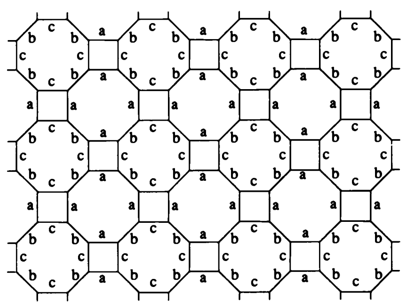
It is well known that (1) and (2) form a complete set of relations for the group *p4m* in the sense that every algebraic relation between the elements *a, b* and *c* can be deduced from these six. Although this is theoretically true, it is not always easy to perform the necessary algebraic manipulations, as we can show by an example. We wonder how long it would take the reader to deduce, using relations (1) and (2) only, that

$$(c\,a\,b\,a\,b\,c\,b\,c\,b\,a\,b\,a)^4 \qquad (3)$$

is equal to the identity! We shall now show that manipulations of this kind can be performed very easily using a *Cayley diagram* for the group. For the group *p4m*, part of the Cayley diagram is shown in Fig. 6b. It appears as a tiling of the plane by squares and regular octagons; however, it is the edges that chiefly interest us. These have been labelled *a, b* and *c*, and



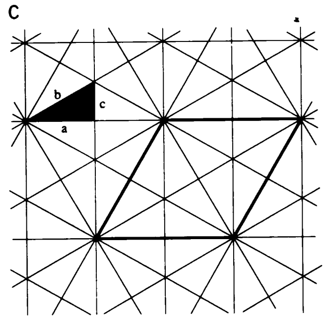
a



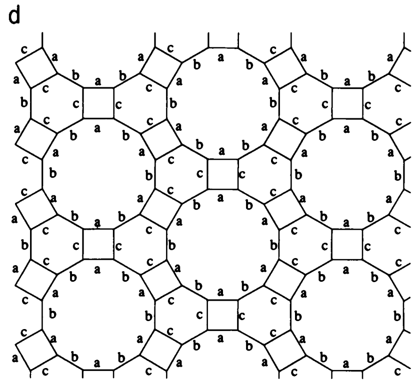
b

Fig. 6. Diagrams illustrating [in (a) and (c)] the concepts of fundamental region (shaded) and translational repeating unit (heavily drawn parallelogram), and [in (b) and (d)] the concept of Cayley diagram; parts (a) and (b) concern the group *p4m*, parts (c) and (d) the group *p6m*.

c



d



it will be observed that one edge of each type is incident with each vertex. Moreover, if we go round the sides of one of the squares we get $acac = (ac)^2$, which, by (2), is the identity e. As we go round each of the two kinds of octagons we get, in a similar manner, $abababab = (ab)^4$ and $bcbcbcbc = (bc)^4$, both of which are equal to the identity. Thus the algebraic identities (2) are represented in the Cayley diagram by circuits (closed paths). In fact, every circuit in the Cayley diagram represents a product of reflections a, b, c, which is equal to the identity, that is, it can be reduced to the identity using relations (1) and (2).

Now consider the vertices in the Cayley diagram. If we choose one vertex to represent the identity e, then each vertex represents a distinct element of the group. Various paths leading from the vertex e to a given vertex x, say, correspond to different products of reflections that are equal to the element x. Thus the Cayley diagram represents the group in a remarkably simple and convenient way, and we shall make extensive use of this representation in the following discussion.

In Figs 6c and 6d we show, in a similar manner, the arrangement of mirrors and the Cayley diagram for the group *p6m*. Here the relations are

$$a^2 = b^2 = c^2 = (ab)^6 = (bc)^3 = (ac)^2 = e, \qquad (4)$$

and the Cayley diagram is a tiling using squares, regular hexagons and regular 12-gons. Exact analogues of the equations stated above hold here also, and we shall make use of these later.

The Uses of Symmetries

The procedure for determining the types of strands, and other data, from the designs will be illustrated by means of four examples. These will clarify the meaning of the various assertions, for which formal proofs would be inappropriate in this framework. We chose them to show various combinations of symmetry groups and induced groups.

Example 1.

Consider the design of Fig. 7a with group *p4m*. The fundamental region of this design is shown in Fig. 7b. There is a path $P\ Q\ R\ S\ldots$ that we may think of as the track of a billiard ball on a triangular table with cushions a, b and c, as shown. Starting from P on cushion c, the ball goes to Q where it 'bounces' on cushion a, then via R to S where it bounces on cushion b. The ball then bounces on a at T, passes through R, bounces at U on b, at V on c, and then arrives at W. Here it bounces back on b and retraces the track going via $V\ U\ R\ T\ S\ R\ Q$ back to P. We notice that every part of the design in the fundamental region has been covered by this one track,

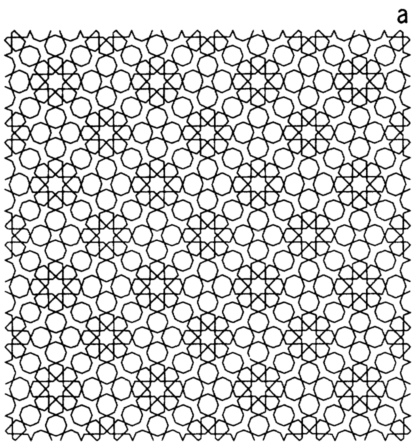
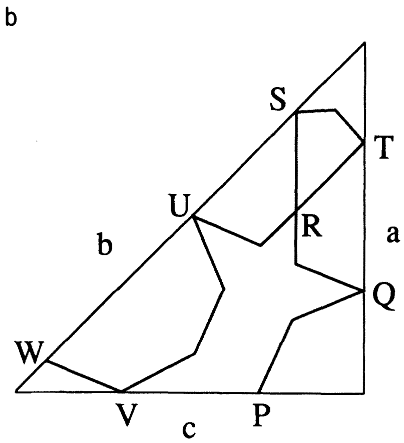
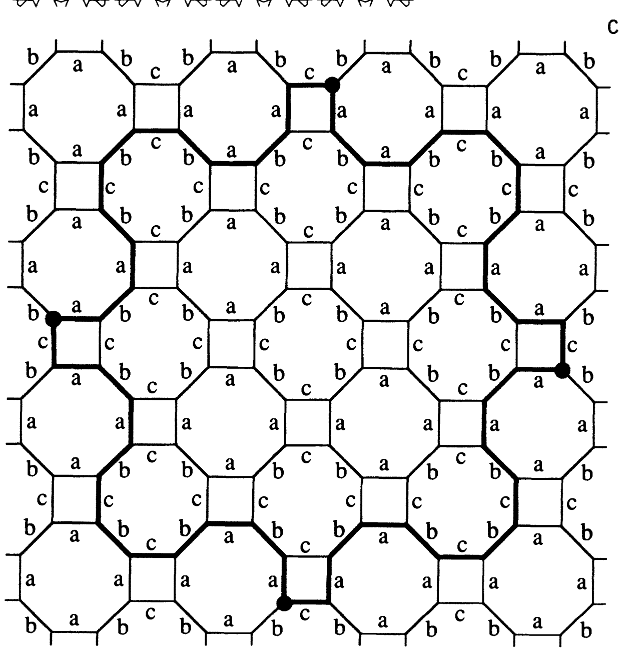
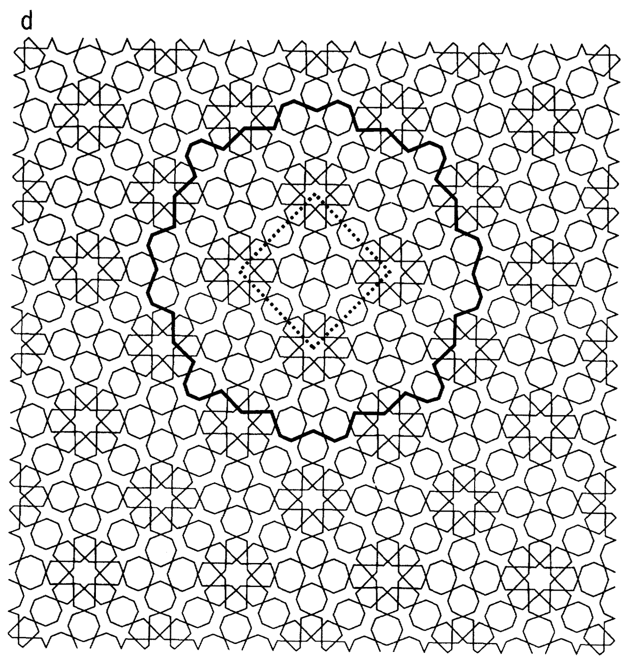


Fig. 7. Illustration of the analysis of an interlace pattern, as discussed in Example 1. (a) A design of the interlace pattern, redrawn from Bourgoin [23]. (b) A fundamental region for the design in (a), with a track for the design. (c) The element of the symmetry group that corresponds to the word representing the track in (b) is the portion between adjacent solid dots on the heavily drawn loop in the Cayley diagram of the group. The loop consists of four repetitions of that element. (d) A strand in the design that corresponds to the track in (b) and the loop in (c) is heavily drawn; a translational repeating unit of the design is indicated by the dotted square.







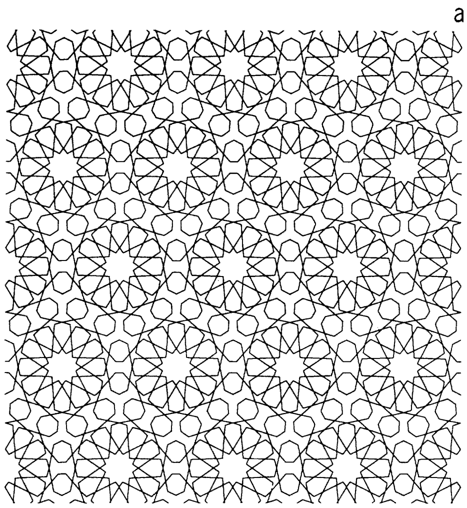
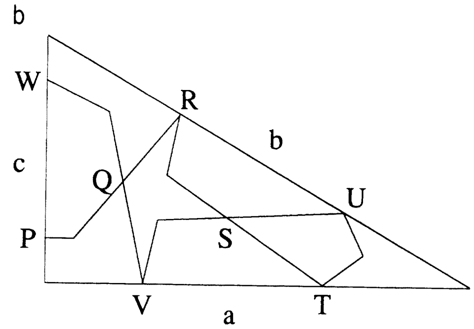
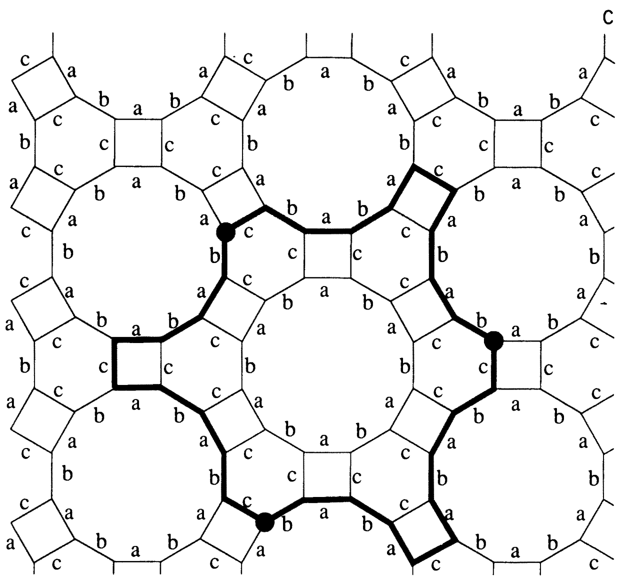
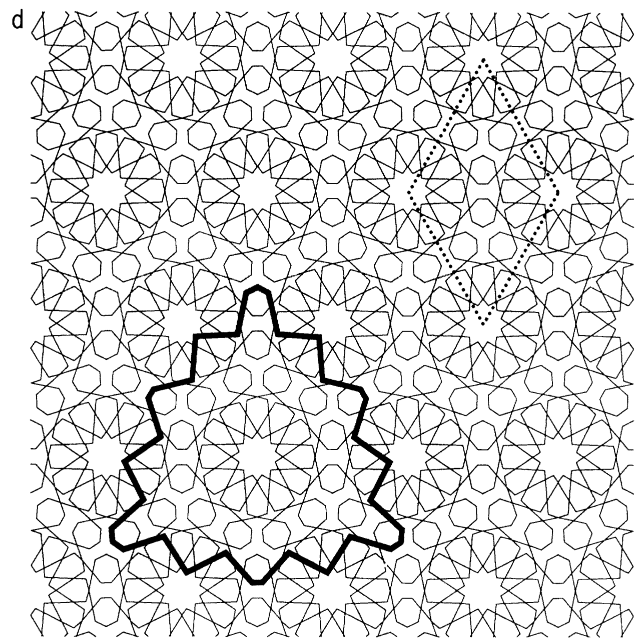


Fig. 8. Illustration of the analysis of another 1-strand interlace pattern, as discussed in Example 2. The four parts correspond to the parts of Fig. 7. (a) A design, redrawn from Bourgoin [24]. (b) A fundamental region, and a track for the design. (c) The element corresponding to the track in (b), in the Cayley diagram of the symmetry group. (d) A strand in the design, and a translational repeating unit.







and so we deduce that this is a 1-strand pattern. More generally, we arrive at Proposition 1.

Proposition 1.

If the number of tracks in a fundamental region of a design D is n, then D represents an n-strand pattern.

Examples with $n > 1$ will be given later (Examples 3 and 4). (Of course, one may complain that a billiard ball does not behave in such a bizarre fashion, with crooked paths across the 'billiard table', and with bounces occurring with no regard to the laws of dynamics! One should regard this wording as merely a convenient fiction, remembering the Mikado in the Gilbert and Sullivan musical of that name, who set up games of billiards "on a cloth untrue with a twisted cue, and elliptical billiard balls" [14].)

The track described above can be represented by

$$P\,Q\,R\,S\,T\,R\,U\,V\,W\,V\,U\,R\,T\,S\,R\,Q, \tag{5}$$

repeated indefinitely each time the billiard ball goes round it or bounces on the cushions like this:

$$c\;a\;b\;a\;b\;c\;b\;c\;b\;a\;b\;a. \tag{6}$$

Clearly the 'word' (6) represents an element of the group *p4m*, and properties of the strand can be determined from this. To do so, we represent this element on a Cayley diagram of the group (see Fig. 7c). We start at one of the marked vertices, and then go along the edges specified in (6), which will bring us to the next marked vertex. Thus the path in the Cayley diagram between two marked vertices corresponds to the word (6). We repeat this operation and find that after three repetitions we arrive back at the vertex from which we started. The complete path thus obtained is called the extended path corresponding to the word. We deduce that element (6) is of finite period (four in this case); in the general case we shall denote the period of the word corresponding to a strand S by $p(S)$. This illustrates our next proposition.

Proposition 2.

If the group element corresponding to a strand is of finite period, then the corresponding strand is finite, that is, a loop. We can say more.

Proposition 3.

The strand has the same induced symmetry group as the extended path in the Cayley diagram corresponding to the associated word.

In our example, Fig. 7c shows that the looped strand has, as induced group, the dihedral group *d4* of order 8, as can be verified by Fig. 7d.

Using (5) we can discover the number of strands that the given strand S crosses. We count (i) bounces on the sides of the fundamental region except where the track turns back

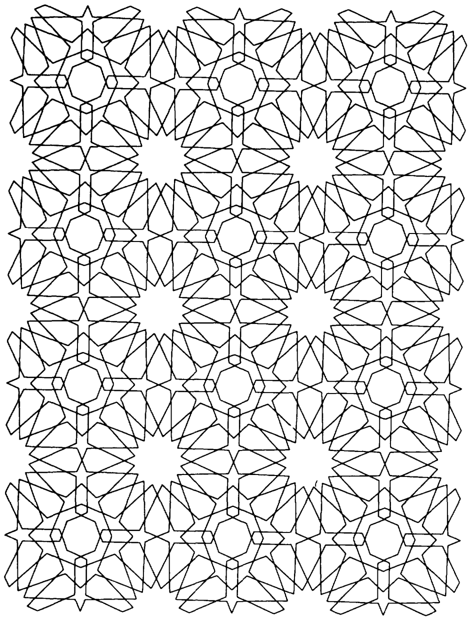
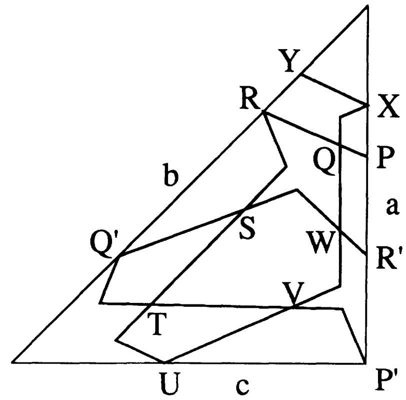
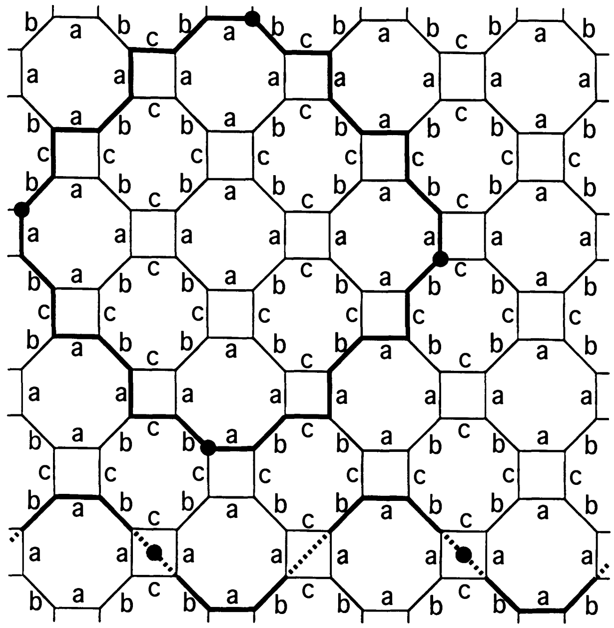
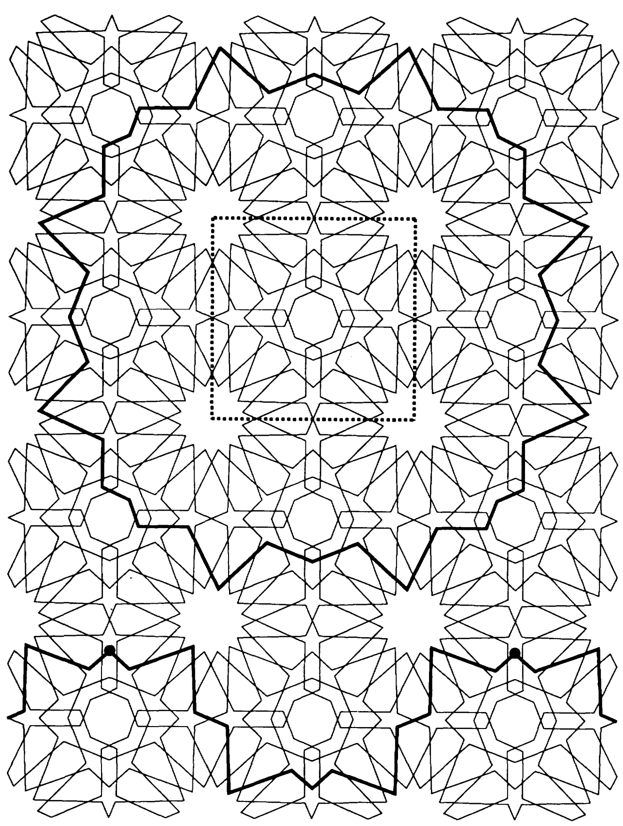


Fig. 9. Illustration of the analysis of a 2-strand interlace pattern, as discussed in Example 3. The four parts correspond to the parts of Fig. 7. (a, left) A design, redrawn from Bourgoin [25]. (b, right) A fundamental region, and a track of the design. (c, center right) The element corresponding to the track in (b), in the Cayley diagram of the symmetry group. (d, bottom right) A strand in the design, and a translational repeating unit.







on itself and (ii) crossings within the fundamental region, as we go once round the track corresponding to the given strand. In our example we see there are two bounces at each of *Q, S, T, U* and *V*, and four crossings at *R* (since we go through *R* twice when traversing the track in each direction), making a total of 14. Let us denote this total by $c(S)$; then we come to Proposition 4.

Proposition 4.

A finite strand *S* crosses a total of $c(S) \times p(S)$ times.

In our example, $c(S) = 14$ and $p(S) = 4$, so *S* has 56 crossings, which can be easily verified from Fig. 7d. Later we shall give the analogue of Proposition 4 for unbounded strands.

We can also deduce the number of crossings in a period parallelogram for the symmetry group of the design.

Proposition 5.

The number of crossings in a translational repeating unit of a design with group *p4m* corresponding to a 1-strand pattern is $2c(S)$. If the group is *p6m* the number of crossings is $3c(S)$. If there is more than one strand, we replace $c(S)$ in these formulae by the sum of the crossing numbers for each of the strands in the fundamental region.

In the case of our example, this yields 2 × 14 crossings, as can be easily verified from the period parallelogram marked in Fig. 7d. (Notice that crossings occurring on the common boundary of two or more fundamental regions are shared, and each region is assigned only the appropriate fraction of such crossings.)

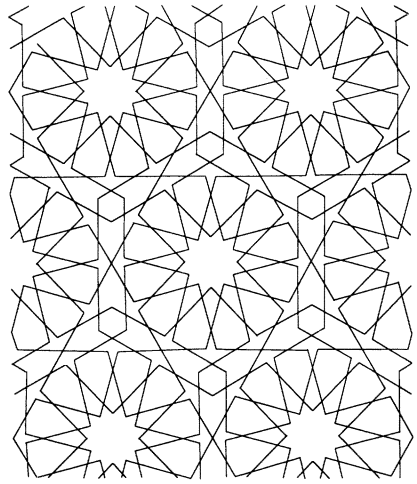
Example 2.

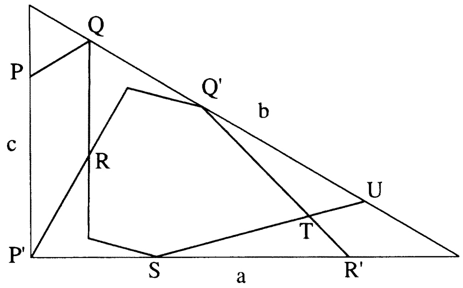
This is similar to Example 1 except that here the group is *p6m* (see Fig. 8a–d). The design has one track, namely

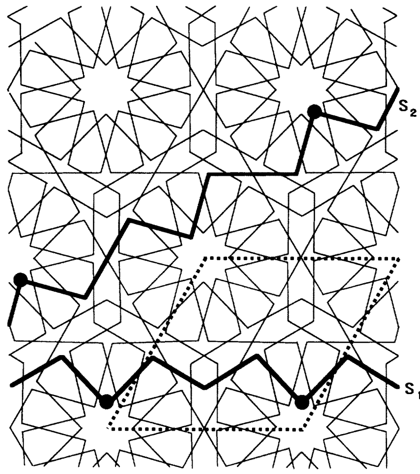
P Q R S T U S V Q W Q V S U T S R Q

and so, by Proposition 1, this is the design of a 1-strand pattern. Denote the strand by *S*. The corresponding group element is

c b a b a c a b a b,







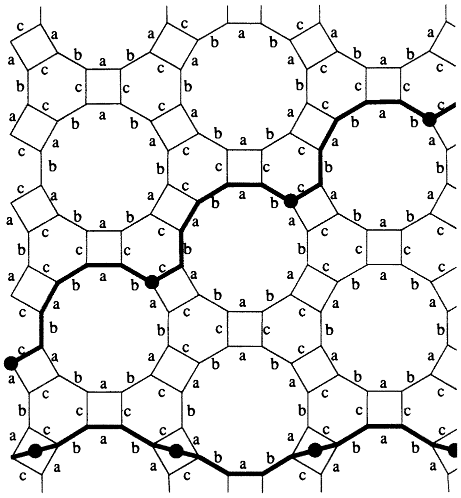


Fig. 10. Illustration of the analysis of another 2-strand interlace pattern, as discussed in Example 4. The four parts correspond to the parts of Fig. 7. (a, far left) A design, redrawn from Bourgoin [26]. (b, left) A fundamental region, and a track for the design. (c, below left) The element corresponding to the track in (b), in the Cayley diagram of the symmetry group. (d, below right) A strand in the design, and a translational repeating unit.

and, as can be seen from Fig. 8c, it is of period 3, so $p(S) = 3$. By Proposition 2 the strand is a loop, and by Proposition 3 its induced group is *d3*. A count of the number of crossings gives $c(S) = 16$, so by Proposition 4 the strand S crosses other strands $16 \times 3 = 48$ times and, by Proposition 5, the number of crossings in a period parallelogram is $3 \times 16 = 48$. These numbers can be verified from Fig. 8d.

Example 3.

The design in Fig. 9a is a 2-strand design. This is clear from Proposition 1 and from the fact that the fundamental region shown in Fig. 9b contains two tracks. The first, corresponding to strand S_1 is

P Q R S T U V W Q X Y X Q W V U T S R Q

(of which *Q, S, T, V* and *W* are crossings, and *P, R, U, X* and *Y* are bounces) so that $c(S_1) = 18$. The second, corresponding to S_2, is

P′ V T Q′ S W R′ W S Q′ T V

of which *P′*, *Q′* and *R′* are bounces and $c(S_2) = 11$. The word corresponding to strand S_1 is

a b c a b a c b

and, looking at the Cayley diagram (Fig. 9c), we see that this strand is a loop (Proposition 2) with induced group *d4* (Proposition 3) (see Fig. 9d). The other track introduces a new feature since it starts from *P′*, which is a corner of the fundamental region. This clearly counts as a bounce against both *a* and *c*, and the question 'which does it bounce against first?'. As there is no answer to this question, we write the corresponding word

$$(ac)\ b\ a\ b \tag{7}$$

and then employ the artifice shown in the lower part of Fig. 9c, namely, we make the path go diagonally through the *acac* square as shown; also we count this as a crossing in the calculation of $c(S_2)$. The path in the Cayley diagram does not close up, so we deduce from Proposition 2 that the strand is unbounded, and from Proposition 3 that the induced group of this strand is *pma2*. Notice that under the restrictions imposed at the beginning, it is only possible for a track to go through the intersection of two mirrors if those mirrors are perpendicular to one another.

Since $c(S_1) = 18$, the strand S_1 crosses other strands $18 \times 4 = 72$ times (see Fig. 9d). As a translational repeating unit of the extended path corresponding to the strand S_2 in the Cayley diagram needs two applications of the word (7), we deduce that the number of crossings in a translational repeating unit of the strand S_2 is $11 \times 2 = 22$ (see Fig. 9d).

In all, the fundamental region contains $c(S_1) + c(S_2) = 18 + 11 = 29$ crossings. As the group is *p4m*, the number of crossings in a translational repeating unit is $29 \times 2 = 58$ (see Fig. 9d).

Example 4.

The design in Fig. 10a has two unbounded strands S_1 and S_2, corresponding to the tracks *P Q R S T U T S R Q* and *P′ R*

$Q'\ T R'\ T Q'\ R$ in Fig. 10b. The corresponding paths in the Cayley diagram are shown in Fig. 10c, from which it will be observed that the first has symmetry group *pma2,* and the second has the group *pm11.* In the second the track goes through the intersection of mirrors *a* and *c,* and we deal with this situation as in the previous example. Here $c(S_1) = 8$ and $c(S_2) = 7$, hence the two strands have 8 and $7 \times 2 = 14$ crossings in a translational repeating unit, respectively. The number of crossings in a fundamental region is $8 + 7 = 15$, so the number of crossings in a translational repeating unit of the design is $15 \times 3 = 45$, as can be verified by inspection of Fig. 10d.

Why So Few Shapes of Strands?

An explanation of the empirical fact that many Islamic interlace patterns contain only one or two shapes of strands is now at hand. Since it is likely [15] that early artisans employed stencils in drawing these patterns, and since, for practical reasons, these stencils were probably made as small as possible for a given pattern, they may well have consisted of just one translational repeat unit. That unit itself was probably obtained by repeating a geometric construction that was carried out in a fundamental region, or by some equivalent procedure. If so, by our Proposition 1, the number of strands in a pattern would be the same as the number of tracks used in the fundamental region—and it is easy to understand why this number was often chosen to be small. Once it was realised that visually interesting and attractive patterns can arise from a fundamental domain that is not very complicated, the traditional approach of training by means of apprenticeships would naturally tend to preserve both the patterns and their methods of generation. The possibility of producing different patterns by slight changes in the fundamental regions (and therefore the stencils) may also have had an influence on the choice of motifs. Thus, we feel, the mystery of how the very complicated interlaces were designed has a reasonably simple explanation. Creation of such ornaments relies only on geometry and technology that were readily available to medieval artists; the fact that 1-strand or 2-strand interlaces result is an accidental consequence, probably neither noticed by the designers nor relevant to them.

The reader may find it interesting to experiment in various ways with the procedures described above. On the one hand, the designs shown in Figs 1–5 can be analysed as to their symmetry groups, fundamental domain and so on. It should be noted that the designs in Figs 4b and 5b have symmetry groups different from the two studied here, so that some independent work is necessary. On the other hand, creation of new designs—whether simple or complicated—can be easily done by the methods shown, or by easy variations of them. Moreover, the use of color greatly enhances the possibilities for experimentation; see Color Plate A No. 1 [16] for a colored version of one of the two interlace patterns that arise from the design shown in Fig. 5a [17].

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References and Notes

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7. B. Grünbaum and G. C. Shephard, "Isonemal Fabrics", *American Mathematical Monthly* **95** (1988) pp. 5–30.

8. See Bourgoin [1] plate 7, and the upper parts of plates 6 and 8; see also the illustrations in d'Avennes [2]; El-Said and Parman [3]; Wade [4]; Humbert [5]; and Makovicky and Makovicky [6].

9. For discussions of the concepts of symmetry, periodic patterns, and translational repeating units, see G. E. Martin, *Transformation Geometry: An Introduction to Symmetry* (New York: Springer-Verlag, 1982).

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12. For descriptions and illustrations of all of these groups, see Grünbaum and Shephard [10].

13. For more on the various concepts and facts discussed in this section, see H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups* (Berlin: Springer-Verlag, 1980).

14. W. S. Gilbert, *The Mikado and Other Plays* (New York: Boni and Liveright, 1917) p. 43, lines 14–16.

15. W. K. Chorbachi, "In the Tower of Babel: Beyond Symmetry in Islamic Design", *Computers Math. Appl.* **17** (1989) pp. 751–789. This article was also published in I. Hargittai, ed., *Symmetry 2: Unifying Human Understanding* (New York: Pergamon, 1989) pp. 751–789.

16. A detailed discussion of perfect and other systematic colorings of various kinds of patterns is given in chapter 8 of Grünbaum and Shephard [10].

17. An example of a 2-colored interlace pattern from the Alhambra is shown in B. Grünbaum, Z. Grünbaum and G. C. Shephard, "Symmetry in Moorish and Other Ornaments", *Computers Math. Appl.* **12B** (1986) pp. 641–653. This article was also published in I. Hargittai, ed., *Symmetry: Unifying Human Understanding* (New York: Pergamon, 1986) pp. 641–653.

18. Bourgoin [1] Fig. 1.

19. Bourgoin [1] Fig. 11.

20. Bourgoin [1] Figs 71 and 6.

21. Bourgoin [1] Figs 39 and 109.

22. Bourgoin [1] Figs 137 and 187b.

23. Bourgoin [1] Fig. 151.

24. Bourgoin [1] Fig. 74.

25. Bourgoin [1] Fig. 57.

26. Bourgoin [1] Fig. 76.