A even more perplexing polynomial problem (or is it?)

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Yesterday one of my colleagues asked a question in tea time: Given a polynomial $f(x)$ with integer coefficients, its degree is unknown. You can query its value $f(n)$ at any integer points. How many queries at least do you need to identify the polynomial?

When the coefficient a_j is assumed to be non negative integers, this has a very nice answer: just two. Query $f(1)$ will give you an upper bound A on its coefficients, and then query $f(A)$ will give you the a_j's. The reason is that you can have a unique A-adic presentation of the value $f(A)$. In fact, as long as you can have this upper bound A of the a_j's, you can use such trick. Query sth big, say $f(2A)$, will give you a unique $2A$-adic presentation, even if some a_j are negative. However in general it is hard to get such a bound.

In the general case, the coefficients of $f(x)$ can be any integer. Then if you don't know the degree a priori, this question has no answer. Ask your friend to think of a polynomial. After an arbitrary finite list of queries $f(n_i)$, I can always has a polynomial $f(x) + \Pi_i(x - f(n_i))$ produce the same evaluation at n_i but whose degree is going up as more queries are asked. So you can not identify a polynomial in this case. Now if we put more restriction, say only one of the coefficients is allowed to be negative. Then does such question has a finite strategy?

In the following we allow a_j to take value in real number, but only one is allowed to be negative. Obviously, for a polynomial with degree unknown, if the only negative coefficient is at most at the a_{n_0}'s place, then by monotonicity it can have at most $n_0 + 1$ zeros. (I am not saying this bound is sharp.) Now ask your friend to think of polynomial, say its degree is n_0. Now after an **arbitrary** list of $n_0 + 2$ queries, you will be able to identify the polynomial among the collection of polynomials that has only one negative coefficients. Let's emphasis here this is an arbitrary list of queries, i.e. no strategy is applied yet. If for each degree n_0, there exist polynomials satisfies our condition but does require Cn_0 queries to identify, where C is a positive constant, then for an arbitrary query we won't be able to tell how many questions should be asked before we are sure of the answer, i.e. we can't solve the problem. I don't know if we can pick a clever strategy though.

References

[BR] On a perplexing polynomial puzzle, Bettina Richmond, The College Mathematis Journal, Vol.41.No.5, Nov 2010.