General Relativity Beyond (3+1)

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Abstract

In this paper we examine general relativity in dimensions other than 3 spatial and 1 temporal dimension. Initially we examine general relativity in higher temporal dimensions and present simple arguments as to why these cases can be eliminated. The remaining part of the paper is dedicated to discussing (N+1) dimensional spacetimes and observe how singularity theorems, stability, and black holes behave and depend upon dimension.

Introduction and History

Every day experience tells us that there are three spatial dimensions and one time dimensions. Despite appearing to the modern physicist as an obvious idea, the notion of a four dimensional universe was a revolutionary idea when first proposed. Indeed, for much of human history, there were only three dimensions (x, y, z). However, in 1905, Minkowski, building upon the work of Einstein's special relativity, introduced the notion of time as a fourth dimension thereby merging space and time famously stating that "henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality"[1].

The natural questions arises though: is (3+1) space-time privileged? Is there any fundamental reason that we live in a (3+1) universe? Could it be possible we in fact live in an (N+1) dimensional universe but can only see $N = 3$ spatial dimensions of it? Or even why are we privileged to only one temporal dimension? Why can't consider an (N+M) dimensional universe? As we shall show, there are theoretical reasons for believing we live in a universe with a single temporal dimension but when it comes to extra spatial dimensions, they have not been ruled out and are currently being pursued by a variety of researchers. Even then, the question still remains as to whether there is a deep fundamental reason we live in a 4 dimensional universe other then it simply is that way.

To this day there are active research programmes into providing a unified theory of physics that makes use of extra dimensions. However, the idea of using extra dimensions in order to unify physics dates back to work in early 1910s by Nordstrom [2], albeit it did not make the exact predictions as observed in nature. The earliest approach that was consistent with Einstein's general relativity, was the theories of Kaluza-Klein [3][4]. The idea, first proposed by Nordstrom but developed by Klein and fully by Kaluza, was to consider a five dimensional metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ with $\mu, \nu = 0, 1, \ldots, 4$ subject to the constraint $\partial_4 g_{\mu\nu} = 0$, the "cylinder condition"[1], and rescaling such that $g_{44} = 1$. The idea is now to take the remaining components of the metric g_{4i} and identify those with the potential of electromagnetism, A^μ. If one makes such an identification, it can be shown that the resulting 5 dimensional Ricci scalar is

$$R^{(5)} = R^{(4)} + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

with $F_{ik} = A_{i,k} - A_{k,i}$. As can be seen, this agrees very closely with how one deals with electromagnetism in general relativity. Klein also hoped that he would be able to unify the new quantum mechanics with this theory and demonstrated that if the fifth dimension were closed, its period would be $l = 8 \times 10^{-33}$m, much smaller than any known object. It was this smallness that caused Klein to postulate that this was closely related to quantum mechanics.

Despite the appeal of the theory, even Einstein worked on it for a few years [5], it gradually fell out of favour and ultimately abandoned and so too did the ideas of higher dimensional theories for some time. However with the revival of string theory and braneworld theories (which postulate higher dimensions) in the last twenty years, the interest in higher

[1] This name can be seen by looking at the line element for a cylinder $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$

dimensional theories continues to be a topic of strong research interest. There is hopes that nature makes use of other hidden higher dimensions that allow quantum mechanics and gravity to be unified.

In this paper we shall consider some of the consequences of higher dimensional analogues of classical general relativity such as higher time dimensions, lower spatial dimensions and the consequences of the simpler mathematical structure, and higher spatial geometry and its consequences. An understanding of how general relativity operates in other dimensions potentially will give insight into why the universe appears to be (3+1). As an example we will show that there do exist results related to black holes that are entirely dimensionally dependent.

One nice benefit of the mathematics of general relativity, differential geometry, is the full generality it allows one to formulate the mathematics of the theory. A priori, we do not need to select a psuedo-Riemannian manifold with signature (-,+,+,+) and indeed most of the results of differential geometry (definitions of various tensorial quantities and identities for example) are formulated in arbitrary dimensions.

Higher Time Dimensions

First we consider the problem of higher temporal dimensions. In our universe, as far as our perception and experiments go, our universe has only one temporal dimension. However, the question arises, can there exist more than one temporal dimension and if they do exist, why don't we observe them? Unlike higher spatial dimensions which can be physically pictured easily, higher time dimensions do not provide an easy physical picture.

There are two different arguments that can be used to argue that time is simply one dimensional. The first [6] is arguing based off the simple structure of the Minkowski geometry while the other [7] is based off mathematical properties of partial differential equations.

The argument considered by Dorling [6] is very simple, short, and elegant taking about a page. The idea is as follows. Consider an $(M + N)$ dimensional Minkowski spacetime. Now draw a spacelike or timelike straight line between two points. Suppose we take the timelike line and, by bending it in a timelike direction, we can make it longer and similarly if we bent it in a spacelike direction we could make it shorter. Similar reasoning applies for the spacelike line except when we bend it in a spacelike direction we make it longer and when we bend it in a timelike direction we make it shorter. Thus in arbitrary dimensions spacelike or timelike geodesics are not maximal and hence are symmetric in this respect. Now consider only a single time dimension. It is no longer possible to bend the line in a timelike direction - since such a direction does not exist - thus bending it in a spacelike direction will only make it shorter, hence the fact that timelike geodesics are maximal curves. This is a well known property.

Now use fact to consider the decaying of a particle. An important property of decay is conservation of energy-momentum which states that the initial energy-momentum will be the same as the decay energy-momentum. From [6], "in other words it states that we can draw a polygon in spacetime, whose sides are parallel to the velocity vectors of the original

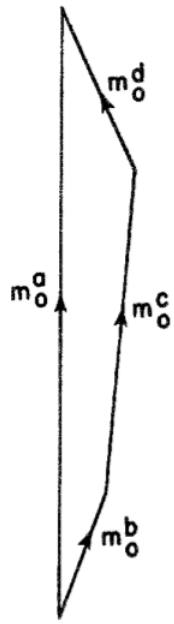

FIG. 1. Energy-momentum conservation diagram for decay process.

Figure 1: Conservation of energy-momentum of a decay process from [6]

particle and its decay products, and the lengths of the particle are proportional to the rest masses of the particle and its decay products," as illustrated in Figure 1. Now let us exploit the fact that timelike geodesics are maximal. Since m_0^a is simply a particle at rest (and hence a geodesic) it must be maximal and thus its length must be greater than the decay line, i.e. the sums of $m_0^b + m_0^c + m_0^d$. Thus the types of decays are restricted due to this requirement and in turn, are an important requirement for the stability of protons and electrons. This requirement on the rest masses is entirely dependent upon the dimensionality of time and if we remove this requirement of one temporal time dimension we could observe protons decaying into neutrons or neutrinos. A similar argument also holds for photons. We do not observe this in nature and hence strongly suggests the universe only makes use of one time dimension.

These types of arguments also provide insight into the stability of tachyons. Since space-like curves are not maximal, there is no such requirement on the rest masses of tachyons and they would be very unstable. Supposing tachyons were stable, we could not eliminate the possibility of photons decaying unstably into a tachyon and anti-tachyon pair. Additionally, Dorling concludes that the relationship between higher dimensionality time and faster than light are closely linked and since we don't observe backwards in time causation there is more evidence pointing to no higher temporal dimensions.

The second argument due to Tegmark [7] is based on the analysis of the PDEs underlying general relativity. An important aspect of relativity is the ability to predict the future of some field ψ at a future point based off its values within past light cone as some point, local causality. Now consider the classification of second order linear PDEs which can be generally written as

$$(A_{ij}\partial_i\partial_j + b_i\partial_i + c)\psi = 0$$

We only need to be concerned with linear PDEs as we can always choose a small neighbourhood (such as using a Riemann normal co-ordinate system) where we can linearlise the equations of motion. It is possible classify the above PDE as follows: hyperbolic the eigenvalues of A are either all positive except one or vice versa, elliptic is all the eigenvalues are of the same sign, and ultrahyperbolic otherwise. The prototypical of a PDE that is relevant in much of physics is Klein-Gordon equation

$$\psi_{;\mu\nu}g^{\mu\nu} + m^2\psi = 0$$

Depending on what type of PDE we are dealing with we obtain very different behaviour. For example, hyperbolic problems are well posed initial value problems in the sense that if we know the initial conditions we can predict the evolution of the equation. Clearly, if we have a standard metric of (3+1) the equation is hyperbolic. Elliptic equations however are well posed boundary value problems and are very sensitive to initial conditions[2], hence if one wants to predict the evolution of the equation, one must know the solution everywhere along the manifold at any instant in time. We can write the Klein-Gordon equation as an elliptic equation if we had a signature with the same sign.

As stated, hyperbolic equations are well posed if the initial condition along a surface is known. In general relativity, this is the Cauchy surface which is identical to a manifold being globally hyperbolic (hence the name). However the important caveat is that the Cauchy surface being a spacelike hypersurface and if the hypersuface is instead timelike, the problem becomes ill-posed.

There exists an interesting theorem [7] that states if we instead specify the initial data on a cylinder (which is clearly not spacelike but instead spacelike and timelike) we can still obtain the solution in some region bounded by two lightcones. Even more remarkable is that we can take the radius of this circle to be arbitrarily small, effectively making the cylinder a line, and still be able to obtain a solution. Unfortunately this results in an ill-posed problem and hence an unpredictable solution.

This theorem also applies to the case of ultrahyperbolic equations. If we lived in a universe with multiple temporal and spatial dimensions and we tried to specify the initial data on a hypersurface (which is, by definition a manifold of degree $n-1$) we would have a hypersurface that is spacelike and timelike. Thus our problem is going to be ill-posed and hence we would no longer be able to predict the future.

An interesting consequence of using this type of reasoning with PDE theory is that if we swap the dimensions of space and time (which is just mathematically flipping the signs in the equations) we would end up with a tachyon universe in which the minimal speed is c[7]. These results are summarised nicely in Figure 2.

[2] The classical example being Laplace's equation with the solution decaying to 0 at infinity. Depending on whether or not there is a point charge located somewhere, the solution is very different.

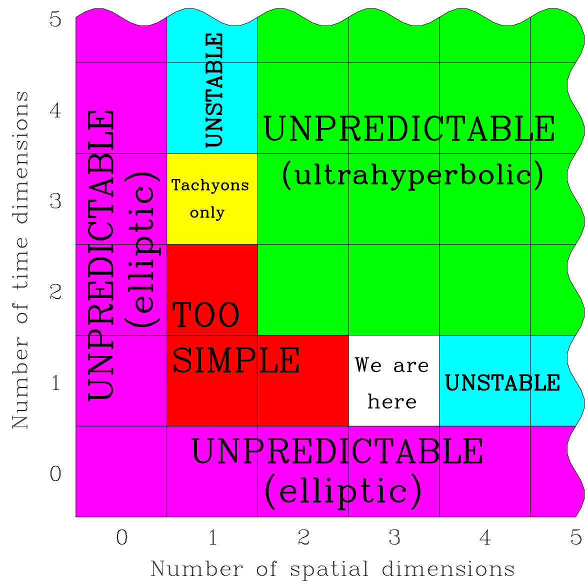

Figure 2: A summary of the predictability of general relativity in various dimensions, from [7]

Overall there is good reason to believe that the universe does not make use of extra temporal dimensions as demonstrated by two arguments above. It also appears, however that higher dimensional general relativity is also unstable, a result we will show below. One workaround, as suggested in the introduction, is that the extra dimensions are very small and hence the scales over which the unstability occurs is so small that is undetectable.

Lower Spatial Dimensions

In this section we look at general relativity in lower dimensions. While these types of theories are clearly not physical they provide a testing ground for quantum gravities and are, typically, easier to solve. Additionally, the dynamics of classical general relativity are radically different in lower dimensions then in the other dimensions. It is worth pointing out immediately that (0+1) or (1+0) theories have no meaning as there does not exist either a timelike or a spacelike dimension so there is no general relativity.

When discussing temporal dimensions no reference was really made to the Einstein field equations themselves, rather the underlying structure (hyperbolicity and Minkowski geometry) itself. The question arises, what do the Einstein field equations look like in arbitrary dimensions? Work in the early 20th century done by Weyl, Cartan, and Vermeil demonstrated that the following dimensional independent argument holds[8].

Arbitrary Dimension Field Equations

Suppose we want a theory that is of the form (indices vary from $0, 1, \ldots, n-1$)

$$f^{\alpha\beta}(g_{\gamma\delta}) = \kappa T^{\alpha\beta}$$

where $f^{\alpha\beta}$ is some tensor function that depends upon the metric and a finitely many number of its derivatives. $T^{\alpha\beta}$ is the energy-momentum tensor that describes the matter in the dimension chosen, and κ is dimensional coupling constant. We make the following three assumptions

1. $f^{\alpha\beta}$ depends at most on the second order derivatives of the metric
2. $f^{\alpha\beta}$ is linear in the second derivatives of the metric
3. Diffeomorphism invariance of $T^{\alpha\beta}$ is satisfied, i.e. $f^{\alpha\beta}{}_{;\beta} = 0$

Then Weyl et al showed that the resulting field equations are uniquely given by

$$f^{\alpha\beta} = G^{\alpha\beta} + \Lambda g^{\alpha\beta}$$

where $G^{\alpha\beta}$ is the Einstein tensor in n dimensions. Thus the Einstein field equations written in arbitrary dimension are given by

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa T_{\mu\nu}$$

We can reabsorb the cosmological constant into the definition of the energy-momentum tensor and we thus obtain

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \kappa T_{\mu\nu}$$

which is identical to equations Einstein derived from other arguments. We shall show later that a generalisation of Sakharov's argument will also yield the same equation.

Two and Three Dimensional Gravity

Let us now consider (1+1) and (2+1) dimensional gravity. However we immediately run into problems. Recalling a course in the differential geometry of surfaces in 2 and 3 dimensions elucidates the potential problem. The Riemann curvature tensor, which generalises the idea of curvature to higher dimensions, becomes a much simpler equation. Indeed, for some surfaces, it depends solely on the determinant of the metric, up to a constant. Thus, we look closer at the number of independent components amongst the various tensors to fully investigate what is going on.

Recall that through the various symmetries and Bianchi identities the number of independent components of the Riemann curvature tensor is $n^2(n^2-1)/12$, which, when $n = 2$, yields only one component. Similarly, since the Ricci tensor and the Ricci scalar are contractions of the Riemann curvature tensor, they will only have one term as well. Thus, in two dimensions, Ricci scalar = Ricci tensor = Riemann. The explicit dependence is given by [8]

$$R_{\alpha\beta\delta\gamma} = \frac{1}{2}R(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta})$$

let us now take the trace

$$\begin{aligned} R_{\beta\gamma} = R^{\alpha}{}_{\beta\alpha\gamma} &= \frac{1}{2}R(\delta^{\alpha}_{\alpha}g_{\beta\gamma} - \delta^{\alpha}_{\gamma}g_{\beta\alpha}) \\ &= \frac{1}{2}R(2g_{\beta\gamma} - g_{\beta\gamma}) \\ &= \frac{1}{2}Rg_{\beta\gamma} \Rightarrow R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = G_{\alpha\beta} = 0 \end{aligned}$$

thus, the Einstein tensor is 2 dimensions vanish identically! Thus, by the very definition of the field equations, the energy-momentum tensor must be zero and hence things simply fall apart. Thus Einstein's formulation of gravity does not exist in two dimensions and must be replaced with something else that is not general relativity. See [9] for an explanation of how this can be achieved using a constant curvature model.

Thus, we must consider at least a 3 dimensional space. However, we again run into problems. How many independent components of the Ricci scalar are there? Since it is a symmetric tensor, there are $n + (n-1) + \ldots + 2 + 1 = n(n+1)/2$ independent components. What happens in $n = 3$? The number of independent components is now 6. What about the Riemann curvature tensor? $3^2(3^2 - 1)/12 = 72/12 = 6!$ Thus in three dimensions the Riemann curvature tensor is completely determined by the Ricci tensor. Recalling that the Riemann curvature tensor can be decomposed into the traceless Weyl tensor and the Ricci tensor, we confirm the well known fact that the Weyl tensor vanishes identically in 3 dimensions.

It is also not hard to show that the number of components of the Einstein tensor is identical to the Ricci tensor. It follows immediately as

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$$

whereupon taking the trace yields

$$G = \frac{(2-n)}{2}R \Rightarrow R_{\alpha\beta} = G_{\alpha\beta} - \frac{1}{n-2}Gg_{\alpha\beta}$$

except in $n = 2$ as we showed above. However, this immediately tells us something very surprising about (2+1) Einstein gravity. Recall that that the Riemann tensor is directly proportional to the Ricci tensor which is directly proportional to the Einstein tensor which is proportional, via the field equations to the matter distribution

$$R_{\alpha\beta\delta\gamma} \propto R_{\alpha\beta} \propto G_{\alpha\beta} \propto T_{\alpha\beta}$$

thus the Riemann tensor depends directly upon the energy-momentum tensor[3] and if the

[3] The exact relationship[8] is $R_{\alpha\beta\delta\gamma} = \kappa[(g_{\alpha\gamma}T_{\beta\delta} + g_{\beta\delta}T_{\alpha\gamma} - g_{\alpha\delta}T_{\beta\gamma} - g_{\beta\gamma}T_{\alpha\delta}) + T(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta})]$

energy-momentum tensor vanishes, so does the Riemann tensor. Thus whenever $T_{\alpha\beta} = 0, R_{\alpha\beta\delta\gamma} = 0$ identically. So whenever there is no energy-momentum, the spacetime is perfectly flat.

Another interesting property of three dimensional gravity is that it has no Newtonian limit [8]. It seems, a priori, that the argument used to obtain the (3+1) Newtonian limit of general relativity should not depend on dimension, but in the case of (2+1) gravity it no longer holds. The problem arises in that when one considers the small perturbations of the metric $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, |h| \ll 1$ one obtains the following Poisson equation (identifying $h_{00} \approx -2\phi$ from the geodesic equation)

$$\nabla^2\phi = \kappa\frac{n-3}{2(n-2)}\rho$$

which in the case of $n = 3$ the equation becomes completely independent of the density. Thus there does not exist an equivalent Newtonian limit for a (2+1) dimensional theory. A related result that can be derived through similar calculations is what gravitational waves look like in a (2+1) theory. Without doing the calculation, it is quite clear that if the Weyl tensor vanishes and Riemann tensor is flat outside matter, then one should not expect gravitational waves and this is exactly what occurs [8]. This raises the question whether the lack of a Newtonian limit is related to the flatness of spacetime outside matter. Unfortunately this is not the case as a simple (1+1) dimensional electromagnetic theory produces a Newtonian limit but does not produce electromagnetic radiation. Thus one is led to the conclusion that this relationship is simply a property of (2+1) dimensional spacetimes.

Thus lower dimensional versions of general relativity are radically different from the standard (3+1) dimensional with the (1+1) case not existing and the (2+1) case have very bizarre dynamics. Despite this, if modifications are made to the formulation of general relativity in these dimensions, with a theory of quantum gravity in mind, it is possible to reformulate these equations into a form which does not yield trivial dynamics, however such a discussion is beyond the scope of this paper.

Higher Spatial Dimensions

We now turn to higher spatial dimensional theories. Unlike the situation in lower dimensions, higher spatial dimensions do not encounter such difficulties and allow for a much richer classes of phenomena.

Stability

In the section on higher temporal dimensions, it was mentioned that higher dimensional spatial dimensions, in classical general relativity, are unstable. The unstability of such theories can be argued using Poisson-type equations. Since Newton's law of gravitation decays as an inverse square and the surface area of a sphere grows as r^2 the two effects perfectly cancel out in Gauss' law. However if we have any other type of law the two do not cancel out and we will not obtain stability. To see why, we follow a similar argument due to [10] although first due to Ehrenfest in a less rigorous form. From quantum field theory we have that when an exchange of a massless particle between two particle occurs, the propagator depends upon $1/\mathbf{k}^2$ which follows from the interacting terms in the Lagrangian. Since the potential is given by the Fourier transform we have that

$$V(r) \propto \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} \frac{1}{\mathbf{k}^2} \sim \frac{1}{r}$$

In higher dimensions, (l+3+1) we have the modification that

$$V(r) \propto \int d^{3+l}k e^{i\mathbf{k}\cdot\mathbf{x}} \frac{1}{\mathbf{k}^2} \sim \frac{1}{r^{1+l}}$$

which gives a different force law so, without modification, higher dimensional general relativity is unstable.

Sakharov's Idea

In the 1960s Sakharov [11] developed a method of deriving general relativity from an effective field theory. To see what happens in higher dimensions, consider the following effective field theory action

$$S_{eff} = \int_M d^n x \sqrt{g}(L^{matter} + L^{matter}_{quantum} + c_1 + c_2 R + c_3 O(R^2))$$

This integral will diverge unless we introduce a minimum cutoff length l_c. By dimensional analysis we can determine the values of the constants recalling that $[R] \sim l^{-2}$

$$c_1 = \lambda_1 l_c^{-n} \qquad c_2 = \lambda_2 l_c^{-n+2} \qquad c_3 = \lambda_3 l_c^{-n+4}$$

since the minimum cut-off length is very small we have that $c_1 \gg c_2 \gg c_3 \gg \ldots$. If now consider the variation of the action we obtain

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{c_1}{2c_2} g_{\mu\nu} = \frac{1}{2c_2} T_{\mu\nu}$$

which is identical to the action obtained by generalising the Einstein-Hilbert action. However, we can immediately see that the constants will be different in higher dimensions by simple dimensional arguments. Additionally, we cannot compare the constant c_2 with an experimental value because we will obtain a different set of equations for Newton's laws which are not realised in nature so without modification they would not be the same. However, the important point is that a higher dimensional analogue of the effective field theory will yield an action similar to the Einstein-Hilbert action up to some dimensional coupling constant.

Singularity Theorems

To derive the singularity theorems, we assume a globally hyperbolic spacetime. As discussed above in terms of temporal dimensions, these types of conditions are important to ensuring that we have proper casual structure on the manifold. We now consider the derivation of singularity theorems in higher dimensional general relativity. We have that we can decompose any tensor as follows

$$B_{\mu\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + t_{\mu\nu}$$

where $\omega_{\mu\nu}$ is an anti-symmetric tensor, $\sigma_{\mu\nu}$ is a symmetric and traceless tensor, and $t_{\mu\nu}$ is everything else. It is quite clear that $2\omega_{\mu\nu} = B_{\mu\nu} - B_{\nu\mu}$ from anti-symmetry. We now define a common quantity known as the projection tensor

$$h_{\mu\nu} = g_{\mu\nu} + \xi_\mu \xi_\nu \qquad g(\xi, \xi) = -1 \qquad \nabla_\xi \xi = 0$$

for ξ along a geodesic. It is also important to note that we have

$$\mathrm{Tr}(h_{\nu\mu}) = g^{\mu\nu}(g_{\mu\nu} + \xi_\mu \xi_\nu) = n - 1$$

We now define a quantity known as the expansion

$$\theta = B^{\mu\nu} h_{\mu\nu}$$

with the property that $\mathrm{Tr}(B) = \theta$ which follows immediately from the above description. Thus we now define the remaining two tensors as

$$\sigma_{\mu\nu} = \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu}) - B_{\mu\nu}$$

and since we define this to be traceless

$$t_{\mu\nu} = \frac{1}{n-1} \theta h_{\mu\nu}$$

these three tensors are known as, $\omega_{\mu\nu}$ the twist tensor, $\sigma_{\mu\nu}$ the shear tensor, $t_{\mu\nu}$ the expansion tensor.

Since θ tells us the evolution of geodesics, we now derive the higher dimensional version of the Raychaudhuri equation. Following [12][13] we that

$$\xi^\mu B_{\alpha\beta;\mu} = -B^\mu{}_\beta B_{\alpha\mu} + R_{\alpha\beta\mu\gamma} \xi^\mu \xi^\gamma$$

where, upon taking the trace and recalling that $\xi = d/d\tau$ we will obtain an evolution equation. Taking the trace of the right hand side of the above results in a Ricci tensor which will be given by the Einstein field equations while the first term involves slightly more attention. We have that the term $Tr(B^2)$ is given by

$$\mathrm{Tr}(B^2) = g^{\mu\nu} B^\alpha{}_\mu B_{\nu\alpha} = \omega_{\nu\mu}\omega^{\nu\mu} + \sigma_{\mu\nu}\sigma^{\mu\nu} + \frac{1}{(n-1)^2}\theta^2 h_{\mu\nu} h^{\mu\nu}$$

We now need to calculate the trace of the projection tensor

$$h^{\mu\nu}h_{\mu\nu} = (g^{\mu\nu} + \xi^\mu\xi^\nu)(g_{\mu\nu} + \xi_\mu\xi_\nu) = n - 1$$

Now we appeal to the Einstein field equations (derived above)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$$

In higher dimensions we have that

$$R = \frac{2\kappa}{2-n}$$

so the Einstein field equations now read

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{n-2}T\right)$$

Putting this all together we obtain

$$\frac{d}{d\tau}\theta = \underbrace{-\frac{1}{n-1}\theta^2 - \sigma^2 - \omega^2}_{\text{negative}} - \underbrace{R_{\mu\nu}\xi^\mu\xi^\nu}_{\text{pos or neg?}}$$

we want the last term to be positive, thus we require that

$$T_{\mu\nu}\xi^\mu\xi^\nu + \frac{1}{n-2}T \geq 0$$

thus we can define the strong energy condition to be

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq -\frac{T}{n-2} \qquad \xi^\nu\xi_\nu < 0 \tag{1}$$

when $n = 2$ we obtain the regular strong energy condition. Now assuming the strong energy condition, we can re-write the Raychaudhuri equation to obtain

$$\frac{d}{d\tau}\theta^{-1} \geq \frac{1}{n-1} \tag{2}$$

thus even in higher dimensions we would still expect to obtain Big Bang/Big Crunch cosmological singularities.

Even those this simple result seems to suggest that we can generalise singularity theorems to higher dimensions, however we must trend carefully. Since these results are essentially topological results, a result holding in one dimension does not necessarily mean it will hold in others and, even when it does, the result might hold for very different reasons. For some guidance, consider the Poincairé conjecture which, while it is now known to be true in higher dimensions, the cases of $n = 3, 4$ took substantially more work then the result for $n > 4$.

Thus should we expect the singularity theorems to generalise to higher dimensions? The subject of singularity theorems on manifolds becomes a mathematically detailed topic with different topological considerations on manifolds leading to different results. Fortunately, many results do generalise. Consider the following theorem of Gannon [14][15][16]

Theorem 1 (Gannon)**.** *Let M^n be a spacetime which satisfies the null energy condition $R_{\mu\nu}v^{\mu\nu} \geq 0$ for all timelike and null v^ν, and admits a smooth spacelike Cauchy hypersurface $\Sigma \subseteq M$ which is regular near infinity. If Σ is non-simply connected then M^n is null geodesically incomplete.*

Following Wald this theorem shows that a spacetime that is asymptotically flat and has initial non-trivially topology, the spacetime will eventually obtain a singularity. This result was first proved only for $n = 4$ however the result can be generalised to higher dimensions[16]. However, a result that does not generalise to higher dimensions is a result due to Hawking about topological properties of asymptotically flat static black holes which was shown recently to be false in higher dimensions[16].

Results of this type lead to interesting results about potential cosmic censorship, and in this case, topological censorship[17] in which observers are prohibited from learning too much about the topological structure of the spacetime. The ideas of a topological censorship is certainly an interesting idea because even though general relativity predicts very bizarre topological behaviour, it hides such behaviour from any observer. Again, as with the cosmic censorship hypothesis there is hope that a quantum gravity would solve such problems. Such topics are an on-going area of research and many interesting results are being obtained yearly - the counterexample to Hawking's result was found in 2006 and generalisations of Gannon's theorem were obtained in 2010, so new results are being obtained yearly.

Black holes

It would be negligent to not discuss generalisations of black holes to higher spacetimes although they were not covered in the course. Much research has gone into discovering higher dimensional analogues of black holes in higher dimensional spacetimes.

To illustrate how the simpler results of black holes generalise, considered the higher dimensional analogue of the Reissner-Nordstrom black holes. Unlike the singularity theorems which do not necessarily generalise nicely, black hole results do. Choosing appropriately scaled units, we can write down the action for both cases as [18]

$$S = \int d^n x \sqrt{-g} \left(R + 4\pi F_{\alpha\beta} F^{\alpha\beta} \right)$$

which applying a variational principle yields

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -8\pi (F_\alpha{}^\lambda F_{\beta\lambda} - \frac{1}{4} g_{\alpha\beta} F_{\lambda\tau} F^{\lambda\tau})$$

where both results are exactly what we would expect from regular general relativity and electromagnetism. A technicality is that in higher dimensions the electromagnetic potential is slightly modified

$$A_0 = \frac{q}{nr^n}$$

as mentioned above. Plugging these in and grinding through the calculations one obtains a solution of the form

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2m}{(n-3)r^{(n-3)}} + \frac{8\pi q^2}{(n-3)(n-2)r^{2(n-3)}}\right) dt^2 \\ &+ \left(1 - \frac{2m}{(n-3)r^{n-3}} + \frac{8\pi q^2}{(n-3)(n-2)r^{2(n-3)}}\right)^{-1} dr^2 \\ &+ r^2 d\theta_1^2 + r^2 \sin^2\theta_1 d\theta_2^2 + \ldots \end{aligned}$$

where we would continue on with the proper n-2 dimensional metric on a sphere. As we can see, the result generalises nicely the result of charged non-rotating black holes in 4 dimensions to higher dimensions. As can be seen, the structure of the equations is very similar but the horizons are going to be located at slightly different locations which could, in principle, we used to test higher dimensional theories.

The most famous result of higher dimensional black holes is due to Myers-Perry [19] which generalises the results of the Kerr black holes to arbitrary dimensions. Such a discovery was important for string theory which uses higher dimensional analogues of general relativity, albeit in a different formulation. Much work has also been done in five dimensional black holes where very interesting phenomena can occur, such as so-called black ring solutions which have no four dimensional analogue and are the solutions that violate Hawkings result from above. Since black holes contain some of the richest applications of general relativity, understanding how they behave in higher dimensions is crucial to understanding how possible quantum gravities that exploit higher dimensions behave [20].

Conclusions

The question of what the dimensions of the universe we live is currently an active area of research in theoretical physics. As was discussed, we have strong reason to believe that there is no good evidence that the universe exploits a higher time dimension but instead just uses a single time dimension. This is an interesting observation given the asymmetry between the spatial and temporal dimensions. Of course, there is no reason why the universe couldn't have chosen higher temporal dimensions and simply had more exotic particle decays. Perhaps there is a deep fundamental reason for this.

Along those lines, the symmetry of the Einstein field equations in 4 dimensions is also unknown if it has deep significance. In mathematics, the four dimensional nature of manifolds is very different from lower and higher dimensions with results being simple to prove in $n < 4$ and $n > 4$ but difficult for $n = 4$ such as the aforementioned Poincaré conjecture among others[4]. So perhaps from a mathematical perspective there is a reason that the universe chose four dimensions.

However, it also the case that the physical results do not exhibit any particular behaviour in any dimension. Take, for example, singularity theorems. As shown above, the existence

[4]Wikipedia has a good listing on the 4-manifold page

of a cosmological singularity doesn't depend on dimension and will exist in any dimension. Similarly, the result of Gannon does not really depend too much on dimension. Even in black hole physics, where interesting black hole behaviour does occur, many of the results tend to be well behaved in all dimensions and that fourth dimension is not picked out for any particular reason.

In the section on higher dimensional general relativity it was shown that the theory is unstable without modification. However, in the introduction, the ideas of Kaluza-Klein were mentioned which posit that we can have a five dimensional world if the fifth dimension is very tiny. If this dimension is very tiny compared with the others, the issues of stability are not as big of a problem since these instabilities are constrained to very tiny length scales which will not be observed in our everyday (3+1) world.

These ideas were revived in 1999 by Randall and Sundrum to solve the hierarchy problem. They postulated that if gravity lived in a fifth dimension but the other three forces were constrained to a four dimensional world, then one would be able to explain why gravity is so weak; it doesn't entirely live on our (3+1) universe. In effect, they postulate that gravity is weak simply because we are only seeing a slice of it permeating through our four dimensional world and it is fact much stronger but we are not able to see this strength at all. Additionally, the bounds on the sizes of such dimensions are no longer on the level of $10^{-30}m$ but become as large as 1 mm! Thus there exist "large" extra dimensions we just don't see them. One advantage of this is that we are able to probe Newton's laws of gravity at such length scales albeit with very precise and detailed experiments which, while possible, are difficult to do. However it is much better than having dimensions so small we cannot even fathom how to test experimentally.

Of course, with the Randall Sundrum theory, which happens to be a popular higher dimensional theory, it is not without its very detractors. Perhaps the most obvious criticism is that there is simply no reason as to why gravity is able to probe the extra dimensions while the other forces are constrained to our universe. One could always bring up anthropic principle, as is often done, but that line of reasoning is hardly intellectually satisfying.

So where is the future of general relativity in higher dimensions? As demonstrated above, we cannot simply allow the Einstein field equations to exist in higher dimensions without modification. The way of modifying such theories are wide and varied with braneworld type theories being the most popular. Alternative formulations include $f(R)$ theories , scalar-tensor theories, Gauss-Bonnet theories, and Lovelock gravity. One can, for example, formulate higher dimensional actions based off the Cartan formalism[21] and deduce certain properties of the action that would be more susceptible to quantum gravities. By far the most important thing holding back the higher dimensional theories is that of experimental verification. As we probe higher energies and smaller regimes, we put more bounds on the possible theories on nature which, for or for worse, has so far but restrictions on the more popular idea of theoretical physicists. However, from these experimental tests, we should be able to gain insight into whether the universe actually exploits higher dimensions.

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