Then $\mathrm{alg}\{h, \iota h\}$ $(\iota(t) = t)$ is dense in $C_0(\mathbb{R})$, by Stone-Weierstrass. Finally, if f is continuous, $|f(t)| \leq C|t|$ $\forall t \in \mathbb{R}$, then hf, $\iota h f \in C_0(\mathbb{R}) \subseteq A_\beta$, and also $\iota^2 h f = \iota(\iota h f) \in A$. Thus $f = hf + \iota^2 hf$ so $f \in A$. ∎

Corollary: If $N \in B(\mathcal{H})$ is normal, $f \in C_\beta(\mathbb{R})$, $g = g^*$ in $L^\infty(\sigma(N), \mathcal{B})$, then $f(g(N)) = f \circ g(N)$.

Proof.
$$g = \operatorname*{s-lim}_{n\to\infty} g_n, \quad g_n = g_n^* \text{ in } C(\sigma(N))$$
$$\left[\begin{array}{l} g = \mathrm{Re}\, g \text{ then } g_n \xrightarrow[s]{n\to\infty} g \\ \mathrm{Re}\, g_n \xrightarrow[s]{} g, \quad \mathrm{Im}\, g_n \longrightarrow 0 \end{array}\right]$$
Then $f \circ g_n \xrightarrow[s]{n\to\infty} f \circ g$, and we use lemma. ∎

<u>von Neumann algebras</u>

A von Neumann algebra is a subalgebra $\mathcal{W} \subseteq B(\mathcal{H})$ which:
- is self-adjoint;
- contains I;
- is WOT-closed.

We have that
$$\text{WOT} \subsetneq \text{SOT} \subsetneq \text{norm topology}.$$

Example: $U \in B(\ell^2(\mathbb{N}))$, $U(x_1, \dots) = (0, x_1, x_2, \dots)$
$U^*(x_1, \dots) = (x_2, \dots)$
$\Rightarrow (U^*)^n \xrightarrow[SOT]{n\to\infty} 0 \quad \Rightarrow (U^*)^n \xrightarrow[WOT]{n\to\infty} 0 \qquad$ But $\|U^{*n}\| = 1$.
$U^n \xrightarrow[WOT]{n\to\infty} 0 \qquad U^n$ ~~$\xrightarrow[SOT]{}$~~ isometry so does not converge SOT

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<u>Ex</u> (generic construction of von Neumann algebras)
$I \in A \subseteq B(\mathcal{H})$ any $*$-subalgebra. Then
$$\mathcal{W} = \overline{A}^{WOT}$$
is a von Neumann algebra.

Facts: • $A_\lambda \xrightarrow[\lambda\in\Lambda]{WOT} W$ in $B(\mathcal{H})$, $A_\lambda \in A$

Check $A_\lambda^* \xrightarrow[\lambda\in\Lambda]{WOT} W^*$ (not true with SOT)

• $A_\lambda \xrightarrow[\lambda\in\Lambda]{WOT} W$, $B_\mu \xrightarrow[\mu\in M]{WOT} V$, check that $\lim_{\lambda\in\Lambda} \lim_{\mu\in M} A_\lambda B_\mu = WV$ in WOT. $\square$

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Proposition: Let $f: B(\mathcal{H}) \to \mathbb{C}$ be linear. Then
f is WOT-continuous if and only if f is SOT-continuous.
In this case, there are $x_1, \dots, x_n, y_1, \dots, y_n \in \mathcal{H}$ st
$$f(T) = \sum_{i=1}^n \langle Tx_i, y_i \rangle.$$

Proof: Since WOT $\leq$ SOT, f WOT-continuous implies that f is SOT-continuous.
Now assume f is SOT-continuous. Then
$$\underbrace{f^{-1}(\mathbb{D})}_{\text{SOT-open}} \supseteq \bigcap_{i=1}^n \{T \in B(\mathcal{H}) ; \|Tx_i\| \leq 1\},$$
$x_1, \dots, x_n \in \mathcal{H}$,
$$\supseteq \{T \in B(\mathcal{H}) ; \sum_{i=1}^n \|Tx_i\|^2 < 1\}$$
Hence if $T \in f^{-1}(\mathbb{D})$, we have

(*) $$|f(T)| \leq |f(T)|^{1/2} \leq \|(Tx_1, \dots, Tx_n)\|_2 .$$

Let
$$\mathcal{H}^n = \mathcal{H} \oplus_2 \cdots \oplus_2 \mathcal{H} \quad (n \text{ times})$$
and consider the subspace
$$\mathcal{M} = \{(Tx_1, \dots, Tx_n) ; T \in B(\mathcal{H})\}.$$

$g(Tx_1, \dots) = f(T)$

Then $g: \mathcal{M} \to \mathbb{C}$ has norm at most 1, by (*), so we consider any Hahn-Banach extension $\tilde{g}: \mathcal{H}^n \to \mathbb{C}$, with $\|\tilde{g}\| \leq 1$.
By Riesz Representation theorem,
$$\tilde{g}(z_1, \dots, z_n) = \langle (z_1, \dots, z_n), (y_1, \dots, y_n) \rangle = \sum_{i=1}^n \langle z_i, y_i \rangle$$
for some $(y_1, \dots, y_n) \in \mathcal{H}^n$. Hence we see that
$$f(T) = g(Tx_1, \dots, Tx_n) = \sum_{i=1}^n \langle Tx_i, y_i \rangle. \quad \square$$

"real nine - line"

Corollary: If $C \subseteq B(\mathcal{H})$ is convex, then $\overline{C}^{WOT} = \overline{C}^{SOT}$.

Proof: Hahn-Banach separation theorem. ▪

Proposition: If $A \in B(\mathcal{H})$, $A \geq 0 \iff \langle Ax, x\rangle \geq 0$ $\forall x$ in $\mathcal{H}$.

Proof: ($\Rightarrow$) $A = B^*B$, $\langle Ax, x\rangle = \|Bx\|^2 \geq 0$
($\Leftarrow$) $\langle Ax, x\rangle \geq 0 \Rightarrow \langle A^*x, x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle} = \langle Ax, x\rangle$ and hence $A = A^*$, so $\sigma(A) \subseteq \mathbb{R}$, ie $\sigma(A) = \partial\sigma(A) \subseteq \sigma_{ap}(A)$.
Thus for $\lambda \in \sigma(A) = \sigma_{ap}(A)$, $\exists \|x_n\| = 1$ st $(\lambda I - A)x_n \xrightarrow{n\to\infty} 0$ in $\mathcal{H}$. Hence

$$0 = \lim_{n\to\infty} \langle (\lambda I - A)x_n, x_n\rangle$$

$$= \lim_{n\to\infty} (\lambda - \langle Ax_n, x_n\rangle)$$

so $\lambda = \lim_{n\to\infty} \langle Ax_n, x_n\rangle \geq 0$. ▪

Remark: If $\mathcal{W} \subseteq B(\mathcal{H})$ is a von-Neumann algebra. Then the following convex sets are WOT-, ie SOT-closed.
- $b(\mathcal{W})$, $(s_\lambda) \subset b(\mathcal{W})$, $s = \text{SOT-}\lim_\lambda s_\lambda \in b(\mathcal{W})$ (check)
- $\mathcal{W}_h$, $s \mapsto s^*$ on $B(\mathcal{H})$ is WOT-WOT cts (check)
- $\mathcal{W}_+$, use last prop

Kaplansky's Density Theorem.
Let $\mathcal{A} \subseteq B(\mathcal{H})$ be a non-degenerate (ie $\overline{\operatorname{span} \mathcal{A}\mathcal{H}} = \mathcal{H}$) C^*-subalgebra, and let $\mathcal{W} = \overline{\mathcal{A}}^{WOT}$. Then
(i) $\overline{b(\mathcal{A}_h)}^{WOT} = b(\mathcal{W}_h)$
(ii) $\overline{b(\mathcal{A}_+)}^{WOT} = b(\mathcal{W}_h)$
(iii) $\overline{b(\mathcal{A})}^{WOT} = b(\mathcal{W})$

Proof: Each "$\subseteq$" follows from the remark above. Further, since each set on the left of (i), (ii), (iii) above is convex, SOT-closure = WOT-closure. Let $H \in b(\mathcal{W}_h)$, $(A_\lambda) \subseteq \mathcal{A}$ be so that $\text{WOT-}\lim_\lambda A_\lambda = H$.
Then

$$\text{WOT-}\lim_\lambda \overset{\operatorname{Im}(A_\lambda)}{\cancel{A_\lambda}} = \text{WOT-}\lim_\lambda \frac{1}{2}(A_\lambda + A_\lambda^*)$$

so

$$H = \text{WOT-}\lim_\lambda \operatorname{Re} A_\lambda.$$

So

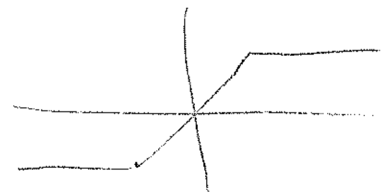
$$H \in \overline{A_h}^{WOT} = \overline{A_h}^{SOT},$$

so

$$H = \text{SOT-}\lim_\lambda H_\lambda, \quad H_\lambda \in A_h.$$

Let $g : \mathbb{R} \to \mathbb{R}$,

$$g(t) = \max\{\min\{1, t\}, -1\}$$



so $g : \mathbb{R} \to [-1,1]$, and g, being continuous, is SOT-continuous, by the lemma last class. Hence

$$H = g(H) = \text{SOT-}\lim_\lambda g(H_\lambda), \quad g(H_\lambda) \in b(A_h)$$

(continuous functional calculus). Moreover, if $H \in b(\mathcal{W}_+)$, we replace g by

$$h(t) = \max\{\min\{1, t\}, 0\}$$

Hence (i), (ii) are proved. To get (iii), we work as follows:

$$\mathcal{B}(\mathcal{H}^2) \cong M_2(\mathcal{B}(\mathcal{H})) \qquad T \mapsto \begin{bmatrix} P_1 T|_{\mathcal{H}_1} & P_1 T|_{\mathcal{H}_2} \\ P_2 T|_{\mathcal{H}_1} & P_2 T|_{\mathcal{H}_2} \end{bmatrix}$$

$$P_1 : \mathcal{H}^2 \to \mathcal{H}, \quad P_1(x_1, x_2) = x_1,$$
$$P_2 : \mathcal{H}^2 \to \mathcal{H}, \quad P_2(x_1, x_2) = x_2,$$
$$\mathcal{H}_1 = \mathcal{H} \oplus 0, \quad \mathcal{H}_2 = 0 \oplus \mathcal{H}.$$

If

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_2(\mathcal{B}(\mathcal{H})), \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}.$$

Also

$$S \in b(\mathcal{B}(\mathcal{H})) \iff \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix} \in b(M_2(\mathcal{H})_h)$$

And
$$\overline{M_2(\mathcal{A})}^{WOT} = M_2(\mathcal{W})$$

Using (i), on $M_2(\mathcal{B}(\mathcal{H}))$, if $S \in b(B(\mathcal{H}))$ then
$$\begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix} = \text{WOT-}\lim_{\lambda} \begin{bmatrix} A_\lambda & C_\lambda^* \\ C_\lambda & B_\lambda \end{bmatrix}, \quad \begin{bmatrix} A_\lambda & C_\lambda^* \\ C_\lambda & B_\lambda \end{bmatrix} \in b(M_2(B(\mathcal{H}))_h)$$
But then $S = \text{WOT-}\lim_{\lambda} C_\lambda$ with
$$\|C_\lambda\| = \left\|\begin{bmatrix} 0 & 0 \\ C_\lambda & 0 \end{bmatrix}\right\| = \Big\| \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}}_{\|\cdot\| \le 1} \underbrace{\begin{bmatrix} A_\lambda & C_\lambda^* \\ C_\lambda & B_\lambda \end{bmatrix}}_{\|\cdot\| \le 1} \underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{\|\cdot\| \le 1} \Big\| \le 1 \quad \blacksquare$$

Notation: $\emptyset \neq \mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ Commutant:
$$\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) ;\ TS = ST \quad \forall S \in \mathcal{S}\}$$

Notes:
- $\mathcal{S} \subseteq \mathcal{T} \Rightarrow \mathcal{S}' \supseteq \mathcal{T}'$
- $\mathcal{S}'$ always a WOT-closed subalgebra of $\mathcal{B}(\mathcal{H})$ (check!)
- $\mathcal{S}^* = \{S^* ; S \in \mathcal{S}\} = \mathcal{S} \Rightarrow \mathcal{S}'$ is a von Neumann algebra (check!)

von Neumann's Double Commutant Theorem:
Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a non-degenerate C^*-algebra. Then
$$\overline{\mathcal{A}}^{WOT} = (\mathcal{A}')'$$