Let's see if
I can spell
today

then

$$\pi(e_\lambda)y = \sum_{i=1}^{n} \pi(e_\lambda a_i)x_i \xrightarrow{\lambda} y,$$

and by density of such elements y in $\mathcal{H}$, we are done.

(ii) Let

$$\Xi = \{\{x_i\}_{i\in I} ;\ \|x_i\|=1,\ \pi(A)x_i \perp \pi(A)x_j,\ i\neq j\}.$$

We can assign a partial order by $\subseteq$. If $\Gamma \subseteq \Xi$ is a chain then $\bigcup\Gamma$ is clearly an upper bound. By Zorn's lemma, there is ~~an~~ maximal element $F = \{x_i\}_{i\in I}$ of Ξ. Let

$$\mathcal{M}_i = \overline{\pi(A)x_i}.$$

If

$$\mathcal{M} = \ell^2\text{-}\bigoplus_{i\in I} \mathcal{M}_i \subsetneq \mathcal{H},$$

then we could find $x \in \mathcal{M}^\perp$, $\|x\| = 1$. But, ~~since~~ since $*$-representations are reducing, $\{x_i\}_{i\in I} \cup \{x\} \supsetneq F$ and is an element of Ξ. This contradicts maximality. ∎

Borel Functional Calculus

Let $\mathcal{H}$ be a Hilbert space. The weak operator topology (WOT) is the initial topology/locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by functionals

$$S \mapsto \langle Sx, y\rangle, \quad x, y \in \mathcal{H}.$$

The strong operator topology (SOT) is the initial topology/locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by

$$S \mapsto Sx : \mathcal{B}(\mathcal{H}) \to \mathcal{H}, \quad x \in \mathcal{H}.$$

Corollary (to the Riesz Representation Theorem):
If $(x,y) \mapsto [x,y] : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ is sesquilinear and $|[x,y]| \le C\|x\|\|y\|$ for some $C>0$, then there is some $S \in \mathcal{B}(\mathcal{H})$ such that $[x,y] = \langle Sx, y\rangle$.

Proof. If $f : \mathcal{H} \to \mathbb{C}$ is conjugate linear and bounded ($|f(x)| \le M\|x\|$), then $y \mapsto \overline{f(y)}$ is linear and bounded, so $\overline{f(y)} = \langle y, x_f\rangle$ for some x_f in $\mathcal{H}$, $f(y) = \langle x_f, y\rangle$. Now, for x in $\mathcal{H}$, let $Sx \in \mathcal{H}$, be given by $y \mapsto [x,y] = \langle Sx, y\rangle$. It is easy to verify that S is linear and bounded. ∎

2015 03 27

Notation: Let X be a set and $\ell^\infty(X) = \{f : X \to \mathbb{C} ;\ \|f\|_\infty = \sup_{x\in X} |f(x)| < \infty\}$. The <u>strong topology</u> generated by seminorms

$$f \mapsto \sum_{n=1}^{\infty} \frac{1}{2^n} |f(x_n)| \text{ , where } \{x_n\}_{n=1}^\infty \subset X.$$

Hence if $(f_n)_{n=1}^\infty \subset \ell^\infty(X)$, $f \in \ell^\infty(X)$ then

$$\operatorname*{s-lim}_{n\to\infty} f_n = f \iff \lim_{n\to\infty} f_n(x) = x\ \forall x, \text{ and } \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty.$$

(If $(f_n)_{n=1}^\infty$ is not bounded, by going to subsequence may devise x_n so $|f_n(x_n)| \geq 2^{2n}$, and then $f \mapsto \sum_{n=1}^\infty |f_n(x)|$ is unbounded on $(f_n)_{n=1}^\infty$.)

Lemma: Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{C}$ and

$$L^\infty(\mathbb{C},\mathcal{B}) = \{f \in \ell^\infty(\mathbb{C}) ;\ f \text{ is } \mathcal{B}\text{-measurable}\}.$$

Then $L^\infty(\mathbb{C},\mathcal{B})$ is the smallest subspace of $\ell^\infty(X)$ containing $C_\beta(\mathbb{C})$ and closed under strong convergence of ~~series~~ sequences. ?

Proof: If $(f_n)_{n=1}^\infty \subset L^\infty(\mathbb{C},\mathcal{B})$ converges strongly to f in $\ell^\infty(\mathbb{C})$ then $f \in L^\infty(\mathbb{C},\mathcal{B})$ (measure theory). Hence $L^\infty(\mathbb{C},\mathcal{B})$ contains all strong limits of sequences $(f_n)_{n=1}^\infty \subset C_\beta(\mathbb{C})$.
If $f, g \in L^\infty(\mathbb{C},\mathcal{B})$, $f = \operatorname*{s-lim}_{n\to\infty} f_n$, $g = \operatorname*{s-lim}_{n\to\infty} g_n$, $f_n, g_n \in C_\beta(\mathbb{C})$, then

$$f+g = \operatorname*{s-lim}_{n\to\infty} (f_n + g_n)$$

and

$$fg = \operatorname*{s-lim}_{n\to\infty} f_n g_n,$$

so the space of such limits is algebra of functions. Let $U \subseteq \mathbb{C}$ be open, so

$$U = \bigcup_{n=1}^{\infty} K_n \text{ , } K_n \text{ compact, } K_n \subseteq K_{n+1}.$$

Then we find $f_n \in C_\beta(\mathbb{C})$ such that $\operatorname{supp}(f_n) \subset U$, $f_n|_{K_n} = 1$, and $0 \leq f_n \leq 1$.
Then $1_U = \operatorname*{s-lim}_{n\to\infty} f_n$.
If $\{B_k\}_{k=1}^\infty \subset \mathcal{B}$, then

$$1_{\bigcap_{k=1}^\infty B_k} = \operatorname*{s-lim}_{n\to\infty} 1_{\bigcap_{k=1}^n B_k} = \operatorname*{s-lim}_{n\to\infty} \prod_{k=1}^n 1_{B_k}.$$

Also, if $B \in \mathcal{B}$, $1_B = 1 - 1_{\mathbb{C}\setminus B}$, and we can use DeMorgan's law to see

$$1_{\bigcup_{k=1}^\infty B_k} = \operatorname*{s-lim}_{n\to\infty} \prod_{k=1}^n (1 - 1_{B_k}).$$

Hence it follows that the space $\mathcal{A}$ of all strong limits of sequences from $C_\beta(X)$ contains each 1_B, $B \in \mathcal{B}$. [ie $\{B \subseteq \mathbb{C} ;\ 1_B \in \mathcal{A}\}$, all open B in here, and it is closed under complement, countable union, hence $\mathbb{B}$.]

Any $f \in L^\infty(\mathbb{C}, \mathbb{B})$ is a strong limit of Borel simple functions, and hence a strong limit of elements of $C_\beta(\mathbb{C})$. $\blacksquare$

Corollary: If $F \subseteq \mathbb{C}$ is any closed set, then $L^\infty(F, \mathbb{B})$ is the space of strong limits of elements of $C_\mathbb{B}(F)$ ($C(F)$ if F is compact).

Construction. Let $N \in \mathcal{B}(\mathcal{H})$ be normal ($NN^* = N^*N$). We recall that $C_e^*(N) \cong C(\sigma(N))$. If $x, y \in \mathcal{H}$, then define a $\mathbb{C}$-measure$^{\text{Borel}}$ $\mu_{x,y} = \mu^N_{x,y}$ by

$$\int_{\sigma(N)} f \, d\mu_{x,y} = \langle f(N)x, y \rangle \qquad \text{for } f \in C(\sigma(N)).$$

[Riesz Rep'n Thm: $C(\sigma(N))^* \cong M(\sigma(N))$.]

If $f \in L^\infty(\sigma(N), \mathbb{B})$, $f = \operatorname*{s\text{-}lim}_{n\to\infty} f_n$, $f_n \in C(\sigma(N))$, we define for x, y in $\mathcal{H}$

$$[x,y] = \int_{\sigma(N)} f \, d\mu_{x,y} \underset{\substack{\text{indep of}\\ \text{choice of } (f_n)_n}}{\overset{\text{LDCT}}{=}} \lim_{n\to\infty} \int_{\sigma(N)} f_n \, d\mu_{x,y} = \lim_{n\to\infty} \langle f_n(N)x, y \rangle.$$

Note

$$|[x,y]| \le \limsup_{n\to\infty} |\langle f_n(N)x, y\rangle| \le \sup_{n\in\mathbb{N}} \|f_n\|_\infty \|x\|\|y\|$$

So $(x,y) \mapsto [x,y]$ is sesquilinear and bounded, hence there is an operator, denoted $f(N)$, such that $\langle f(N)x, y\rangle = [x,y]$.

Remark: If in $L^\infty(\sigma(N), \mathbb{B})$ we have $f = \operatorname*{s\text{-}lim}_{n\to\infty} f_n$, then

$$f(N) = \operatorname*{WOT\text{-}lim}_{n\to\infty} f_n(N).$$

Indeed, we ~~???~~ use LDCT: if $x, y \in \mathcal{H}$

$$\langle f(N)x, y\rangle = \int_{\sigma(N)} f \, d\mu_{x,y} = \lim_{n\to\infty} \int_{\sigma(N)} f_n \, d\mu_{x,y} = \lim_{n\to\infty} \langle f_n(N)x, y\rangle.$$

Theorem (Borel Functional Calculus):

Let $N \in \mathcal{B}(\mathcal{H})$ be normal. Then there is a $*$-homomorphism

$$f \mapsto f(N) : L^\infty(\sigma(N), \mathbb{B}) \to \mathcal{B}(\mathcal{H})$$

which

(0) extends the continuous functional calculus $C(\sigma(N)) \to C_e^*(N) \subseteq \mathcal{B}(\mathcal{H})$;

(1) $\|f(N)\| \le \|f\|_\infty$, hence if $f = \operatorname*{u\text{-}lim}_{n\to\infty} f_n$ then $\lim_{n\to\infty} \|f(N) - f_n(N)\| = 0$;

2015 03 27(2

(ii) If $f = \operatorname*{s-lim}_{n\to\infty} f_n$ in $L^\infty(\sigma(N), \mathcal{B})$, then $f(N) = \operatorname*{SOT-lim}_{n\to\infty} f_n(N)$;
(iii) If $\lambda \in \sigma_p(N)$ then for $f \in L^\infty(\sigma(N), \mathcal{B})$ and $x \in \ker(\lambda I - N)$, $f(N)x = f(\lambda)x$;
(iv) If $NA = AN$ $(A \in \mathcal{B}(\mathcal{H}))$, $f \in L^\infty(\sigma(N), \mathcal{B})$ then $f(N)A = Af(N)$;
(v) For $f \in L^\infty(\sigma(N), \mathcal{B})$, $f(N) = 0$ if and only if

$$f \in \bigcap_{x\in\mathcal{H}} \mathcal{N}_{\mu_{x,x}}$$

where $\mathcal{N}_{\mu_{x,x}} = \{g \in L^\infty(\sigma(N), \mathcal{B});\ \mu_{x,x}(g^{-1}(\mathbb{C}\setminus\{0\})) = 0\}$.

Remark: $f \in C(\sigma(N))$, $f \geq 0$

$$\int_{\sigma(N)} f \, d\mu_{x,x} = \langle \underbrace{f(N)}_{\geq 0} x, x\rangle \geq 0 \implies \mu_{x,x} \text{ pos. meas.}$$

2015 03 31

Proof: Let $f, g \in L^\infty(\sigma(N), \mathcal{B})$, let $f = \operatorname*{s-lim}_{n\to\infty} f_n$, $g = \operatorname*{s-lim}_{n\to\infty} g_n$, $f_n, g_n \in C(\sigma(N))$. Then

$$f^* = \operatorname*{s-lim}_{n\to\infty} f_n^*, \quad fg = \operatorname*{s-lim}_{n\to\infty} f_n g_n$$

and hence for x, y in $\mathcal{H}$

$$\langle f^*(N)x, y\rangle = \lim_{n\to\infty} \langle f_n^*(N)x, y\rangle \overset{f_n^*(N) = f_n(N)^*}{\underset{\text{cts func. calc.}}{=}} \lim_{n\to\infty} \langle x, f_n(N)y\rangle = \langle x, f(N)y\rangle = \langle f(N)^* x, y\rangle,$$

$$\begin{aligned}
\langle f(N)g(N)x, y\rangle &= \langle g(N)x, f(N)^* y\rangle \\
&= \lim_{n\to\infty} \langle g_n(N)x, f(N)^* y\rangle \\
&= \lim_{n\to\infty} \langle f(N)g_n(N)x, y\rangle \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \langle \underbrace{f_m(N)g_n(N)}_{f_m g_n(N)} x, y\rangle \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \int_{\sigma(N)} f_m g_n \, d\mu_{x,y} \\
&\overset{\text{LDCT}}{=} \lim_{n\to\infty} \int_{\sigma(N)} f g_n \, d\mu_{x,y} \\
&\overset{\text{LDCT}}{=} \int_{\sigma(N)} fg \, d\mu_{x,y} = \langle fg(N)x, y\rangle,
\end{aligned}$$

so $f^*(N) = f(N)^*$, $f(N)g(N) = fg(N)$. Likewise $(f+g)(N) = f(N) + g(N)$, so

$f \mapsto f(N)$ is indeed a $*$-homomorphism.

(o): By construction.

(i) Automatic continuity of $*$-homomorphisms. Note that $L^\infty(\sigma(N), \mathcal{B}) \subseteq \ell^\infty(\sigma(N))$ is C^*-subalgebra (measure theory)

(ii) If $x \in \mathcal{H}$,

$$\begin{aligned}\|(f_n(N)-f(N))x\|^2 &= \langle (f_n(N)-f(N))^*(f_n(N)-f(N))x, x\rangle \\ &= \langle |f_n-f|^2(N)x, x\rangle \\ &= \int_{\sigma(N)} |f_n - f|^2 \, d\mu_{x,x} \xrightarrow[n\to\infty]{LDCT} 0\end{aligned}$$

(iii) $f = \operatorname{s-lim}_{n\to\infty} f_n$, $f_n \in C(\sigma(N))$. Then

$$f(N)x \underset{by (ii)}{=} \lim_{n\to\infty} f_n(N)x = \lim_{n\to\infty} f_n(\lambda)x = f(\lambda)x$$

(iv) If $f = \operatorname{s-lim}_{n\to\infty} f_n$, $f_n \in C(\sigma(N))$, then for x, y in $\mathcal{H}$

$$\begin{aligned}\langle f(N)Ax, y\rangle &= \lim_{n\to\infty} \langle \underbrace{f_n(N)}_{\text{uniformly approx by poly's}} Ax, y\rangle \\ &= \lim_{n\to\infty} \langle Af_n(N)x, y\rangle \\ &= \lim_{n\to\infty} \langle f_n(N)x, A^*y\rangle \\ &= \langle f(N)x, A^*y\rangle = \langle Af(N)x, y\rangle.\end{aligned}$$

(v) Let

$$\mathcal{N}_N := \{f \in L^\infty(\sigma(N), \mathcal{B}) ; f(N) = 0\},$$

which is a closed ideal of $L^\infty(\sigma(N), \mathcal{B})$, by (i).

$\operatorname{sgn} z = \begin{cases} \frac{z}{|z|} & z \neq 0 \\ 0 & z = 0\end{cases}$

If $f \in \bigcap_{x\in\mathcal{H}} \mathcal{N}_{\mu_{x,x}}$ then $\langle f(N)x, x\rangle = \int_{\sigma(N)} f \, d\mu_{x,x} = 0$ for every $x \in \mathcal{H}$.
Hence $f(N) = 0$.

If $f \in \mathcal{N}_N$, $|f| = (\operatorname{sgn} f)^* \cdot f$. Hence if

$$|f|(N) = \operatorname{sgn} f(N)^* \underbrace{f(N)}_{=0} = 0$$

so for $x \in \mathcal{H}$,

$$\int_{\sigma(N)} |f| \, d\mu_{x,x} = \langle |f|(N)x, x\rangle = 0.$$

Notice that for $h \in C(\sigma(N))_+$ we have $(h(N) \geq 0 \Rightarrow h(N) = B^*B)$

2015 03 31(

$$\int_{\sigma(N)} h\,d\mu_{x,x} = \langle h(N)x, x\rangle = \langle Bx, Bx\rangle = \|Bx\|^2 \geq 0$$

so it follows (from Riesz rep'n for $C(\sigma(N))^*$) that $\mu_{x,x} \geq 0$. Thus, for f as before

$$\int_{\sigma(N)} f\,d\mu_{x,x} = 0 \implies f \in \mathcal{N}_{\mu_{x,x}} \text{ (measure theory)}.$$ ∎

Hence let

$$L^\infty(N) = L^\infty(\sigma(N), \mathcal{B}) \Big/ \underbrace{\bigcap_{x\in\mathcal{H}} \mathcal{N}_{\mu_{x,x}}}_{=\mathcal{N}_N}$$

and the map $f + \mathcal{N}_N \mapsto f(N) : L^\infty(N) \to \mathcal{B}(\mathcal{H})$, is an injective, hence isometric, $*$-homomorphism.

Remark. If $f \in L^\infty(\sigma(N), \mathcal{B})$, then

$$\sigma_{L^\infty(N)}(f + \mathcal{N}_N) = \operatorname{essran}_N f := \bigcap_{x\in\mathcal{H}} \bigcap_{\substack{S\in\mathcal{B} \\ \mu_{x,x}(S)=0}} \overset{\text{closure}}{\overline{f(\sigma(N)\setminus S)}}$$

Hence this gives $\sigma_{\mathcal{B}(\mathcal{H})}(f(N))$.

Proposition: (Spectral Measure)

Let $N \in \mathcal{B}(\mathcal{H})$ be normal. Then there is a function $E_N : \underset{\text{Borel sets on } \sigma(N)}{\mathcal{B}_{\sigma(N)}} \to \mathcal{B}(\mathcal{H})$ which satisfies

(i) $E_N(\emptyset) = 0$, $E_N(\sigma(N)) = I$;

(ii) $E_N(X)^* = E_N(X)$, $X \in \mathcal{B}_{\sigma(N)}$;

(ii') $E_N(X)E_N(Y) = E_N(X\cap Y)$, $X, Y \in \mathcal{B}_{\sigma(N)}$;

$\Big\}\Rightarrow$ each $E_N(X)$ is a projection

(iii) If $B = \bigsqcup_{n=1}^\infty B_n$, $B_n \in \mathcal{B}_{\sigma(N)}$, then

$$E_N(B) = \text{SOT-}\sum_{n=1}^\infty E_N(B_n), \text{ ie } \text{SOT-}\lim_{k\to\infty} \sum_{n=1}^k E_N(B_n)$$

Proof: Let $E_N(B) = 1_B(N)$. Then (i), (ii), (ii') are immediate from Borel func. calc. Also, observe that in (iii),

$$1_B = \text{s-}\lim_{n\to\infty} 1_{\bigsqcup_{k=1}^n B_k} = \text{s-}\lim_{n\to\infty} \sum_{k=1}^n 1_{B_k} = \text{s-}\sum_{k=1}^\infty 1_{B_k}$$

We use (ii) of Borel functional calculus. ∎

Note: $B \subseteq C$ in $\mathrm{Bor}(N) \Rightarrow E_N(B) \leq E_N(C)$ as $E_N(C) - E_N(B) = E_N(C \setminus B) \geq 0$ in $\mathcal{B}(\mathcal{H})$

Remark: If $\lambda \in \sigma_p(N)$, then $E_N(\{\lambda\}) = P_{\ker(\lambda I - N)}$. This follows from (iii) of Borel functional calculus.

Hence

$$E_N(\sigma(N) \setminus \{0\}) = I - E_N(\{0\}) = P_S, \quad S = (\ker(N))^{\perp}.$$

Corollary (Spectral Theorem)

If $H = H^*$ in $\mathcal{B}(\mathcal{H})$ ~~then~~ and $(\{I_{n,1}, \dots, I_{n,m_n}\})_{n=1}^{\infty}$ is a sequence of partitions of $[-\|H\|, \|H\|]$ by subintervals, and $t_{n,j} \in I_{n,j}$ and

$$\lim_{n\to\infty} \max_{j \in [m_n]} m(I_{n,j}) = 0$$

then

$$H = \lim_{n\to\infty} \sum_{j=1}^{m_n} t_{n,j} \underbrace{E_H(I_{n,j})}_{= E_H(I_{n,j} \cap \sigma(H))} =: \int_{-\|H\|}^{\|H\|} t \, dE_H(t).$$

"Riemann integral"

Proof. $\iota(t) = t$, then

$$\iota = \underset{n\to\infty}{u\text{-}\lim} \sum_{j=1}^{m_n} t_{n,j} 1_{I_{n,j}}.$$

$\blacksquare$

helps with A4

ideals in $\mathcal{B}(\mathcal{H})$

Corollary: If $A \geq 0$ in $\mathcal{B}(\mathcal{H})$, then $A \in \mathcal{K}(\mathcal{H}) \Leftrightarrow E_A((\varepsilon, \|A\|])$ is finite-rank for $0 < \varepsilon \leq \|A\|$.

Proof:

A≥0 not cmpt then build proj not finite rank and look at ideal gen. by this and get I in here?

$$A = \lim_{n\to\infty} \left[0 \cdot E_A\left(\left[0, \tfrac{\|A\|}{n}\right]\right) + \sum_{j=2}^{n} \frac{j}{n} \|A\| \, E_A\left(\left(\tfrac{j-1}{n}\|A\|, \tfrac{j}{n}\|A\|\right]\right)\right)$$ $\blacksquare$

2015 04 0

Def] Let $f: \mathbb{R} \to \mathbb{C}$ be a Borel measurable function. We call f SOT-continuous if for any $(H_\lambda)_{\lambda \in \Lambda} \subset \mathcal{B}(\mathcal{H})_h$ such that

$$H = \underset{\lambda}{\text{SOT-lim}}\, H_\lambda \in \mathcal{B}(\mathcal{H})_h,$$

then

$$\underset{\lambda}{\text{SOT-lim}}\, f(H_\lambda) = f(H).$$

Lemma: If $f:\mathbb{R}\to\mathbb{C}$ is continuous and

$$\sup_{|t|\geq 1}\left|\frac{f(t)}{t}\right| < \infty,$$

then f is SOT-continuous.

Proof: Let $\mathcal{A}$ be the set of SOT-continuous functions on $\mathbb{R}$. Clearly, $\mathcal{A}$ is a vector space under pointwise ~~limit~~ operations.

Let $\mathcal{A}_\beta$ denote the subspace of uniformly bounded elements of $\mathcal{A}$.

If $f\in\mathcal{A}_\beta$, $g\in\mathcal{A}$, then for

$$H = \text{SOT-}\lim_\lambda H_\lambda \in \mathcal{B}(\mathcal{H})_h,$$

then for x in $\mathcal{H}$

$$\begin{aligned}\|((fg)(H_\lambda) - (fg)(H))x\| &= \|(f(H_\lambda)g(H_\lambda) - f(H)g(H))x\| \\ &\leq \underbrace{\|f(H_\lambda)\|}_{\leq \|f\|_\infty}\cdot\|(g(H_\lambda)-g(H))x\| + \|(f(H_\lambda)-f(H))g(H)x\| \xrightarrow{\lambda\in\Lambda} 0\end{aligned}$$

so $fg\in\mathcal{A}$. In particular, $\mathcal{A}_\beta$ is an algebra.

If $f_n \xrightarrow{n\to\infty} f$ uniformly in $\mathcal{A}_\beta$, then for

$$H = \text{SOT-}\lim_\lambda H_\lambda \text{ in } \mathcal{B}(\mathcal{H})_h$$

we have for x in $\mathcal{H}$, and n st $\|f_n - f\|_\infty < \varepsilon$, $\varepsilon>0$

$$\begin{aligned}\|(f(H_\lambda)-f(H))x\| &\leq \|(f(H_\lambda)-f_n(H_\lambda))x\| + \|(f_n(H_\lambda)-f_n(H))x\| + \|(f_n(H)-f(H))x\| \\ &\leq 2\varepsilon\|x\| + \|(f_n(H_\lambda)-f_n(H))x\| \xrightarrow{\lambda\in\Lambda} 2\varepsilon\|x\|.\end{aligned}$$

Thus we see that

$$\lim_\lambda \|(f(H_\lambda)-f(H))x\| = 0$$

Let's verify, first, that $C_0(\mathbb{R})\subseteq\mathcal{A}_\beta$. Let

$$h(t) = \frac{1}{1+t^2}$$

Then for

$$H = \text{SOT-}\lim_\lambda H_\lambda \text{ in } \mathcal{B}(\mathcal{H})_h,$$

for $x\in\mathcal{H}$,

$$\begin{aligned}\|(h(H)-h(H_\lambda))x\| &= \|h(H_\lambda)(I+H^2-I-H_\lambda^2)h(H)x\| \\ &= \|h(H_\lambda)(H_\lambda(H-H_\lambda)+(H-H_\lambda)H)h(H)x\| \\ &\leq \underbrace{\|h(H_\lambda)H_\lambda\|}_{\leq \sup_t\left|\frac{t}{1+t^2}\right|}\|(H-H_\lambda)h(H)x\| + \underbrace{\|h(H_\lambda)\|}_{\leq 1}\cdot\|(H-H_\lambda)Hh(H)x\| \xrightarrow{\lambda\in\Lambda} 0\end{aligned}$$

Then $\mathrm{alg}\{h, \iota h\}$ $(\iota(t)=t)$ is dense in $C_0(\mathbb{R})$, by Stone-Weierstrass. Finally, if f is continuous, $|f(t)| \leq C|t|$ $\forall t \in \mathbb{R}$, then $hf, \iota hf \in C_0(\mathbb{R}) \subseteq \mathcal{A}_\beta$, and also $\iota^2 h f = \iota(\iota h f) \in \mathcal{A}$. Thus $f = hf + \iota^2 hf$ so $f \in \mathcal{A}$. ∎

Corollary: If $N \in \mathcal{B}(\mathcal{H})$ is normal, $f \in C_\beta(\mathbb{R})$, $g = g^*$ in $L^\infty(\sigma(N), \mathcal{B})$, then $f(g(N)) = f \circ g(N)$.

Proof:
$$g = \underset{n\to\infty}{s\text{-}\lim}\, g_n, \quad g_n = g_n^* \text{ in } C(\sigma(N))$$
$$\left[\begin{array}{l} g = \mathrm{Re}\, g \text{ then } g_n \xrightarrow[s]{n\to\infty} g \\ \mathrm{Re}\, g_n \xrightarrow[s]{} g, \quad \mathrm{Im}\, g_n \longrightarrow 0 \end{array}\right]$$
Then $f \circ g_n \xrightarrow[s]{n\to\infty} f \circ g$, and we use lemma. ∎

<u>von Neumann algebras</u>

A von Neumann algebra is a subalgebra $\mathcal{W} \subseteq \mathcal{B}(\mathcal{H})$ which:
- is self-adjoint;
- contains I;
- is WOT-closed.

We have that
$$\text{WOT} \subsetneq \text{SOT} \subsetneq \text{norm topology}.$$

Example: $U \in \mathcal{B}(\ell^2(\mathbb{N}))$, $U(x_1, \dots) = (0, x_1, x_2, \dots)$
$U^*(x_1, \dots) = (x_2, \dots)$
$\Longrightarrow (U^*)^n \xrightarrow[\text{SOT}]{n\to\infty} 0 \implies (U^*)^n \xrightarrow[\text{WOT}]{n\to\infty} 0$ But $\|U^{*n}\| = 1$.
$U^n \xrightarrow[\text{WOT}]{n\to\infty} 0$ $\quad U^n$ ~~$\xrightarrow[\text{SOT}]{}$~~ isometry so does not converge SOT