# $C^*$-algebras

Let $A$ be a $\mathbb{C}$-algebra. An involution on $A$ is a map $a \mapsto a^*$ on $A$ such that for $a,b \in A$, $\alpha \in \mathbb{C}$,

$$(a+\alpha b)^* = a^* + \bar{\alpha} b^* \quad \text{(conjugate linearity)}$$
$$(ab)^* = b^* a^* \quad \text{(anti-multiplicativity)}$$
$$(a^*)^* = a \quad \text{(self-inverse}$$

If $A$ is a Banach algebra, we will always insist that
$$\|a^*\| = \|a\| \quad \text{(isometry)}.$$

A Banach algebra $A$ with involution is called a $C^*$-algebra if for all $a$ in $A$,
$$\|a^* a\| = \|a\|^2.$$

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$*$-algebra is the lazy version of involutive algebra

Remark: Notice in a Banach $*$-algebra,
$$\|a^* a\| \leq \|a^*\| \, \|a\| = \|a\|^2$$
is automatic. Hence, checking the $C^*$-condition amounts to observing
$$\|a\|^2 \leq \|a^* a\|.$$

Eg

(i) $X$ l.c.H. space, $A = C_0(X)$. Define $f^*(x) = \overline{f(x)}$, $f \in C_0(X)$.
Check that this is an involution. Also $f^* f = |f|^2$ and hence
$$\|f^* f\|_\infty = \sup_{x \in X} |f(x)|^2 = \|f\|_\infty^2$$
(also $\|f^*\|_\infty = \|f\|_\infty$).

(ii) $G$-group, $A = \ell^1(G)$. Let $f^*(s) = \overline{f(s^{-1})}$. Notice
$$\delta_t(s) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$$
satisfies $\delta_t^* = \delta_{t^{-1}}$. Check $\|f^*\|_1 = \|f\|_1$, $(f * g)^* = g^* * f^*$, etc.
This is not a $C^*$-algebra. Let $G = \mathbb{Z}_2$. Let $\alpha \in \mathbb{C}$, $f = \delta_0 + \alpha \delta_1$. We have
$$f^* * f = (1 + |\alpha|^2)\delta_0 + 2\operatorname{Re}\alpha \, \delta_1$$

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but
$$\|f\|_1^2 = (1+|\alpha|)^2 = 1+2|\alpha|+|\alpha|^2$$
which are different if $\alpha \notin \mathbb{R}$.

(iii) On $C(\mathbb{T})$, define $f^\circ(z) = \overline{f(\bar{z})}$. Then $f \mapsto f^\circ$ is an algebra involution.
Also, $z \mapsto \bar{z}$ is a homeomorphism on $\mathbb{T}$. We see that
$$\|f^\circ\|_\infty = \sup_{z\in\mathbb{T}} |\overline{f(\bar{z})}| = \sup_{z\in\mathbb{T}} |f(\bar{z})| = \sup_{z\in\mathbb{T}} |f(z)| = \|f\|_\infty.$$
Notice $A(\mathbb{D})$ is invariant under $f \mapsto f^\circ$. We shall see that $(C(\mathbb{T}), f\mapsto f^\circ)$ is <u>not</u> a $C^*$-algebra.

(iv) (Operators on Hilbert space)
Riesz Representation Theorem: If $\mathcal{H}$ is a Hilbert space, $f \in \mathcal{H}^*$, there is a unique $y$ in $\mathcal{H}$ such that $f(x) = \langle x, y\rangle$. Further, $\|f\| = \|y\|$.
Note: $x \in \mathcal{H}$, $\|x\| = \sup_{\|y\|\le 1} |\langle x, y\rangle| = \langle x, \frac{1}{\|x\|} x\rangle$, $x \neq 0$.
If $T \in \mathcal{B}(\mathcal{H})$, $y \in \mathcal{H}$, then
$$f_{T,y}(x) = \langle Tx, y\rangle$$
defines a bounded linear functional, $\|f_{T,y}\| \le \|T\|\,\|y\|$. Hence there is a unique $T^*y$ in $\mathcal{H}$ such that
$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$
(check that $y \mapsto T^*y$ is linear.) (check also that $T=S$ in $\mathcal{B}(\mathcal{H}) \iff \langle Tx, y\rangle = \langle Sx, y\rangle$, $\langle x, T^*y\rangle = \overline{\langle T^*y, x\rangle}$)

If $S, T \in \mathcal{B}(\mathcal{H})$, $\alpha \in \mathbb{C}$, for $x, y$ in $\mathcal{H}$ we have
$$\begin{aligned}\langle x, (T+\alpha S)^*y\rangle &= \langle (T+\alpha S)x, y\rangle \\ &= \langle Tx + \alpha Sx, y\rangle \\ &= \langle Tx, y\rangle + \alpha\langle Sx, y\rangle \\ &= \langle x, T^*y\rangle + \alpha\langle x, S^*y\rangle \\ &= \langle x, T^*y\rangle + \langle x, \bar{\alpha}S^*y\rangle \\ &= \langle x, (T^* + \bar{\alpha}S^*)y\rangle.\end{aligned}$$
It follows that $(T+\alpha S)^* = T^* + \bar{\alpha}S^*$. Likewise one can show that
$$(ST)^* = T^*S^* \quad \& \quad (T^*)^* = T.$$
Now
$$\|T^*\| = \sup_{\|y\|\le 1} \|T^*y\| = \sup_{\|x\|,\|y\|\le 1} \underbrace{|\langle T^*y, x\rangle|}_{|\langle x, T^*y\rangle|} = \sup_{\|x\|,\|y\|\le 1} |\langle Tx, y\rangle| = \sup_{\|x\|\le 1} \|Tx\| = \|T\|.$$
$$\|T^*T\| = \sup_{\|x\|,\|y\|\le 1} |\langle T^*Tx, y\rangle| \ge \sup_{\|x\|\le 1} |\langle T^*Tx, x\rangle| = \sup_{\|x\|\le 1} |\langle Tx, Tx\rangle| = \sup_{\|x\|\le 1} \|Tx\|^2 = \|T\|^2.$$
From before, $\|T^*T\| \le \|T\|^2$ already holds

(iv') Any closed subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$ such that $a^* \in A$ whenever $a \in A$, is also a $C^*$-algebra. We call this a <u>$C^*$-subalgebra</u> of $\mathcal{B}(\mathcal{H})$.

Proposition (Unitization):
Let $A$ be a $C^*$-algebra without identity. Then the unitization $\tilde{A} = A \oplus \mathbb{C}$ is a $C^*$-algebra with

$$(a,\alpha)^* = (a^*, \bar{\alpha})$$

and

$$\|(a,\alpha)\| = \sup_{\|b\| \le 1} \|ab + \alpha b\|.$$

With $\|(a,0)\| = \|a\|$, so $A \subseteq \tilde{A}$ is a $C^*$-subalgebra.

Proof: Clearly, $(a,\alpha) \mapsto (a,\alpha)^*$ is an algebra involution. Now

$$\begin{aligned}
\|(a,\alpha)\|^2 &= \sup_{\|b\|\le 1} \|ab + \alpha b\|^2 \\
&= \sup_{\|b\|\le 1} \|(ab+\alpha b)^*(ab+\alpha b)\| \\
&= \sup_{\|b\|\le 1} \|b^* a^* ab + \alpha b^* a^* b + \bar{\alpha} b^* ab + |\alpha|^2 b^* b\| \\
&\le \sup_{\|b\|\le 1} \underbrace{\|b^*\|}_{\le 1} \|a^* ab + \alpha a^* b + \bar{\alpha} ab + |\alpha|^2 b\| \\
&= \|(a^* a + \alpha a^* + \bar{\alpha} a, |\alpha|^2)\| \\
&= \|(a^*, \bar{\alpha})(a,\alpha)\| \\
&\le \|(a,\alpha)^*\| \, \|(a,\alpha)\|
\end{aligned}$$

$\|(a,\alpha)\| = \|L_a + \alpha I\|_{B(A)}$

Hence $\|(a,\alpha)\| \le \|(a,\alpha)^*\|$. By self-inversion of the involution,

$$\|(a,\alpha)^*\| \le \|(a,\alpha)\|.$$

Hence $\|(a,\alpha)^*\| = \|(a,\alpha)\|$ and $\|(a,\alpha)\|^2 = \|(a,\alpha)^*(a,\alpha)\|$. Finally, if $a \ne 0$,

$$\|(a,0)\| = \sup_{\|b\|\le 1} \|ab\| \ge \left\|a \tfrac{1}{\|a^*\|} a^*\right\| = \tfrac{1}{\|a^*\|}\|aa^*\| = \tfrac{1}{\|a^*\|}\|(a^*)^* a^*\| = \|a^*\| = \|a\|. \quad \blacksquare$$

Remark: If $A$ is a $*$-algebra, with identity $e$, then for $a \in A$,

$$e^* a = (a^* e)^* = (a^*)^* = a, \quad ae^* = (ea^*)^* = (a^*)^* = a,$$

so $e = e^*$. Hence if $A$ is a unital $C^*$-algebra then

$$\|e\|^2 = \|e^* e\| = \|ee\| = \|e\|$$

so $\|e\| \in \{0,1\}$. Clearly $\|e\| = 1$, if $A \ne \{0\}$.

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Proposition: If $A$ is a unital Banach $*$-algebra, then for $a \in A$,

$$a \in GL(A) \iff a^* \in GL(A)$$

Hence $\sigma(a^*) = \overline{\sigma(a)}$.

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Proof: $a \in GL(A)$. $\Rightarrow aa^{-1} = e = a^{-1}a$
$$\Rightarrow (a^{-1})^* a^* = e^* = e = a^*(a^{-1})^*$$
$$\Rightarrow (a^*)^{-1} = (a^{-1})^* \in GL(A)$$
Symetrically, $a^* \in GL(A) \Rightarrow a = a^{**} \in GL(A)$.
Hence for $z \in \mathbb{C}$, $ze - a \in GL(A) \iff \bar{z}e - a^* \in GL(A)$. ∎

Proposition: Let $A$ be a $C^*$-algebra.
(i) Suppose $A$ is unital and $u \in A$ is <u>unitary</u>: $u^*u = e = uu^*$. Then $\sigma(u) \subseteq \mathbb{T}$.
(ii) Suppose $h \in A$ is <u>hermitian</u>: $h^* = h$. Then $\sigma(h) \subseteq \mathbb{R}$.

Proof:
(i) $\|u^*\|^2 = \|uu^*\| = \|e\| = 1$ so $\sigma(u^*) \subseteq \overline{\mathbb{D}}$ (closed disk).
But, symmetrically $\|u\| = 1$ so $\sigma(u) \subseteq \overline{\mathbb{D}}$. Hence
$$\sigma(u) = \overline{\sigma(u^*)} = \overline{\sigma(u^{-1})} \subseteq \overline{\mathbb{D}} \cap \underbrace{\overline{\mathbb{D}}^{-1}}_{\{z \in \mathbb{C};\ \frac{1}{z} \in \overline{\mathbb{D}}\}} = \mathbb{T}$$
(ii) Replace $A$ with $\tilde{A}$ if $A$ is non-unital. Hence let us suppose $A$ is unital. If $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, then $|e^{iz}| = |e^{ix}e^{-y}| = e^{-y}$. So $|e^{iz}| = 1 \iff z \in \mathbb{R}$. Let
$$u = \exp(ih) = \sum_{n=0}^{\infty} \frac{i^n}{n!} h^n \in A.$$
We see that
$$u^* = \exp(ih)^* = \exp(-ih) = \exp(ih)^{-1} = u^{-1}.$$
By spectral mapping theorem,
$$\exp(i\sigma(h)) = \sigma(\exp(ih)) \subseteq \mathbb{T}$$
by (i). And hence $\sigma(h) \subseteq \mathbb{R}$. ∎

Proposition: If $A$ is a $C^*$-algebra and $h = h^*$ in $A$. Then
$$r(h) = \|h\|.$$

Proof: $\|h^2\| = \|h^*h\| = \|h\|^2$. By induction, $\|h^{2^n}\| = \|h\|^{2^n}$ and hence
$$r(h) = \lim_{n\to\infty} \|h^{2^n}\|^{1/2^n} = \|h\|$$ ∎

Proposition: Let $A$ be a unital $C^*$-algebra and $C \subseteq A$ be a $C^*$-subalgebra which contains the identity $e$. Then for $a \in C$ we have
$$\sigma_C(a) = \sigma_A(a).$$

Proof: First, suppose $a = a^*$ in $C$. Then

$$\sigma_C(a) \subseteq \mathbb{R} \subseteq \mathbb{C},$$

so

$$\sigma_C(a) = \partial\sigma_C(a) \subseteq \sigma_A(a) \subseteq \sigma_C(a)$$

so $\sigma_C(a) = \sigma_A(a)$.

Now if $a \in C_i$, then

$$a \in C \cap GL(A) \Rightarrow aa^*, a^*a \in GL(C)$$
$$\Rightarrow a \in GL(C).$$

Indeed, $a \in C \cap GL(A) \Rightarrow aa^*, a^*a$ are hermitian and in $C \cap GL(A)$

$\Rightarrow$ hence $aa^*, a^*a \in GL(C)$, as above

$\Rightarrow$ hence $\exists b, c \in C$ for which $b = (a^*a)^{-1}$, $c = (aa^*)^{-1}$.

$\Rightarrow ba^*a = e = aa^*c$

$\Rightarrow ba^* = ba^*aa^*c = ea^*c = a^*c = a^{-1} \in C$ ∎

Gelfand-Naimark Theorem (for commutative $C^*$-algebras):

Let $A$ be a commutative $C^*$-algebra. Then

(i) for $\gamma \in \Gamma_A$, then $\gamma(a^*) = \overline{\gamma(a)}$ for $a \in A$; and

(ii) $a \mapsto \hat{a} : A \to C_0(\Gamma_A)$ is a surjective isometry with $\widehat{a^*} = \hat{a}^*$. isometry $\Rightarrow$ injective

Proof.

(i) If $h \in A$ is hermitian then for $\gamma \in \Gamma_A$ we have $\gamma(h) \in \sigma(h) \subseteq \mathbb{R}$. For $a \in A$, let

$$\mathrm{Re}\, a = \tfrac{1}{2}(a + a^*), \quad \mathrm{Im}\, a = \tfrac{1}{2i}(a - a^*).$$

So $(\mathrm{Re}\, a)^* = \mathrm{Re}\, a$, $(\mathrm{Im}\, a)^* = \mathrm{Im}\, a$ and $a = \mathrm{Re}\, a + i\,\mathrm{Im}\, a$. Thus

$$\gamma(a^*) = \gamma(\mathrm{Re}\, a - i\,\mathrm{Im}\, a) = \gamma(\mathrm{Re}\, a) - i\gamma(\mathrm{Im}\, a) = \overline{\gamma(a)}.$$

• (ii) The algebra $\hat{A} = \{\hat{a} ; a \in A\} \subseteq C_0(\Gamma_A)$ satisfies

- point separating (easy)
- conjugate closed ($\widehat{a^*} = \hat{a}^*$)
- separates each point from $0$ ($\gamma \in \Gamma_A$ $\exists a \in A$ $\hat{a}(\gamma) \neq 0$)

[• if $A$ is unital, $\hat{e} = 1$]

Hence by Stone-Weierstrass theorem, $\hat{A}$ is dense in $C_0(\Gamma_A)$.

Now, if $a \in A$ we have

$$\|a\|^2 = \|a^*a\| \overset{a^*a \text{ hermitian}}{=} r(a^*a) \underset{\substack{\text{since } \Gamma_A(a^*a) \\ = \sigma(a^*a)}}{=} \|\widehat{a^*a}\|_\infty = \|\widehat{a^*}\hat{a}\|_\infty = \|\hat{a}^*\hat{a}\|_\infty = \||\hat{a}|^2\|_\infty = \|\hat{a}\|_\infty^2$$

$\Rightarrow \|a\| = \|\hat{a}\|_\infty$.

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my prejudice against
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Since $a \mapsto \hat{a}$ is an isometry it has closed range, hence is all of $C_0(\Gamma_A)$. ∎

Remarks:

(i) Recall on $C(\mathbb{T})$ $f^*(z) = \overline{f(\bar{z})}$. If $z \in \mathbb{T} \setminus \{\pm 1\}$ and the identity function $\iota$, then

$$\delta_z(\iota^*) = \iota^*(z) = \overline{\iota(\bar{z})} = z.$$

Hence

$$\delta_z(\iota^*) \neq \overline{\delta_z(\iota)}.$$

This violates (i), above, so $(C(\mathbb{T}), f \mapsto f^*)$ is not a $C^*$-algebra.

(ii) $G$ abelian group, consider

$$\gamma \in \Gamma_{\ell^1(G)} \cong \hat{G} = \{\chi : G \to \mathbb{T} \,;\, \chi \text{ is multiplicative}\}.$$

$\gamma = \gamma_\chi$, $\quad \gamma_\chi(f) = \sum_{s \in G} f(s)\chi(s)$.

Then

$$\gamma_\chi(f^*) = \sum_{s \in G} \overline{f(s^{-1})}\,\chi(s) = \sum_{s \in G} \overline{f(s)}\,\underbrace{\chi(s^{-1})}_{\overline{\chi(s)}} = \overline{\gamma_\chi(f)}.$$

Even though $\ell^1(G)$ is not a $C^*$-algebra, we still have $\gamma(f^*) = \overline{\gamma(f)}$ for $\gamma \in \Gamma_{\ell^1(G)}$.

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Corollary (Continuous functional calculus):
Let $A$ be a unital $C^*$-algebra and $a \in A$ be normal: $a^*a = aa^*$. Let

$$C^*_e(a) = \langle a, a^* \rangle_{e_A} \text{ and } C^*(a) = \langle a, a^* \rangle$$

Then there is an isometric isomorphism

$$f \mapsto f(a) : C(\sigma(a)) \to C^*_e(a)$$

which restricts to a bijective isometry

$$f \mapsto f(a) : C_0(\sigma(a) \setminus \{0\}) \to C^*(a).$$

Furthermore, $\iota^n(a) = a^n$, $n \in \{0\} \cup \mathbb{N}$, $f^*(a) = f(a)^*$, (Spectral Mapping) $\sigma(f(a)) = f(\sigma(a))$, if $g \in C(f(\sigma(a)))$, then $g \circ f(a) = g(f(a))$.

Proof: Recall $\sigma_{C^*_e(a)}(a) = \sigma_A(a)$, by spectral permanence. We have that the map

$$\gamma \mapsto \gamma(a) : \Gamma_{C^*_e(a)} \to \sigma(a)$$

is a continuous bijection, hence a homeomorphism. To see that this is injective, we note that if $\gamma, \gamma'$ in $\Gamma_{C^*_e(a)}$ with $\gamma(a) = \gamma'(a)$, then

$$\gamma(a^*) = \overline{\gamma(a)} = \overline{\gamma'(a)} = \gamma(a^*)$$

and hence it follows that $\gamma = \gamma'$ on all of $\mathrm{alg}(a, a^*, e)$, thus by continuity, $\gamma = \gamma'$ on $C^*_e(a)$. Hence by Gelfand-Naimark Theorem,

$$b \mapsto \hat{b} : C^*_e(a) \to C(\sigma(a))$$

is an isometric bijection. Let $\Phi: C(\sigma(a)) \to C^*_e(a)$ denote the inverse of ~~the~~ $b \mapsto \hat{b}$. We have, with $f(a) := \Phi(f)$,

- $\Phi(\iota^n) = a^n$, since $\hat{a} = \iota$ so $\widehat{a^n} = \hat{a}^n = \iota^n$
- $\Phi(f^*) = \Phi(f)^*$, since the same is true of $\Phi^{-1}$.
- $\sigma(\Phi(f)) \underset{\substack{\uparrow \\ \Phi \text{ isomorphism}}}{=} \sigma_{C(\sigma(a))}(f) = f(\sigma(a))$

  i.e. $\sigma(f(a)) = f(\sigma(a))$

(f) • note that $g \mapsto g \circ f : C(f(\sigma(a))) \to C(\sigma(a))$ is an isometry.

(ff) Now $C(f(\sigma(a))) = C(\sigma(f(a))) \cong \underset{\substack{\uparrow \\ f(a) \in C^*_e(a)}}{C^*_e(f(a)) \subseteq} C^*_e(a) \cong C(\sigma(a))$

and the sequence of identifications (ff) gives the isometric map given by (f). Hence $g \circ f(a)$, in $C^*_e(a)$, is the same as $g(f(a))$, in $C^*_e(f(a))$.

If $0 \in \sigma(a)$, then $\delta_0 \in \Gamma_{C(\sigma(a))}$ corresponds to a character $\gamma_0 \in \Gamma_{C^*_e(a)}$ for which $a, a^* \in \ker \gamma_0$. Then

$$C^*(a) = \ker(\gamma_0) \cong \ker(\delta_0) = C_0(\sigma(a) \setminus \{0\}),$$

ie $f \mapsto f(a)$ restricts to $C_0(\sigma(a) \setminus \{0\})$, as advertized.

If $0 \notin \sigma(a)$, then $C_0(\sigma(a) \setminus \{0\}) = C_0(\sigma(a)) = C(\sigma(a))$. By Stone-Weierstrass, there are two-variable polynomials $(p_n)_{n=1}^{\infty}$ such that $p_n(z, \bar{z}) \xrightarrow{n \to \infty} z^{-1}$ uniformly for $z$ in $\sigma(a)$, and $p_n(0,0) = 0$. Hence, in $C^*_e(a) \cong C(\sigma(a))$, we have $\underbrace{p_n(a, a^*)}_{\in C^*(a)} \xrightarrow{n \to \infty} a^{-1}$ in norm, so $a^{-1} \in C^*(a)$, hence $e = aa^{-1} \in C^*(a)$, so $C^*(a) = C^*_e(a)$ ∎

<u>Def</u> An element $b$ of a $C^*$-algebra $A$ is <u>positive</u> provided $b = b^*$ and $\sigma(b) \subseteq [0, \infty)$. Write $b \geq 0$.

Corollary (Square roots) Let $b \geq 0$ in a $C^*$-algebra $A$. Then there exists a unique element $c \in A$ such that $c^2 = b$ and $c^* = c$.

Proof. Since $\sigma(b) \subseteq [0, \infty)$ and $b^* = b$, by functional calculus, we may let $c = b^{1/2}$. Notice $(t \mapsto \sqrt{t}) \in C_0(\sigma(b))$. Also $c^* = c$ since $\sigma(b^{1/2}) = \{\sqrt{t};\ t \in \sigma(b)\}$

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by spectral mapping. Now if $a^* = a$ and $a^2 = b$ then in $C^*(a)$ we have $(a^2)^{1/2} = a$, while in $C^*(b) \subseteq C(a)$ we have $(a^2)^{1/2} = b^{1/2} = c$. Hence $a = c$. ∎

Corollary (positive and negative parts):
If $h = h^*$ in a $C^*$-algebra $A$, then there exist $h_+, h_- \geq 0$ such that
$$h = h_+ - h_- \quad \text{and} \quad h_+ h_- = 0.$$

Proof: Recall $\sigma(h) \subseteq \mathbb{R}$. Let $\iota_+ \in C(\sigma(h))$, $\iota_+(t) = \max\{t, 0\}$, $\iota_- \in C(\sigma(h))$, $\iota_-(t) = \max\{-t, 0\}$. So $\iota = \iota_+ - \iota_-$ and $\iota_+\iota_- = 0$. Let $h_+ = \iota_+(h)$, $h_- = \iota_-(h)$. ∎

Lemma: If $a = a^*$ in a unital $C^*$-algebra then the following are equivalent:
(i) $a \geq 0$
(ii) $a = b^2$ for some $b = b^*$
(iii) $\|\alpha e - a\| \leq \alpha \quad \forall \alpha \geq \|a\|$
(iv) $\|\alpha e - a\| \leq \alpha$ for some $\alpha \geq \|a\|$

Proof: (i)⇒(ii) Let $b = a^{1/2}$ (functional calculus)
(ii)⇒(iii) $a = b^2 = bb^*$ so $\|a\| = \|b\|^2$ ($C^*$ identity). Hence for $t \in \sigma(b) \subseteq [-\|b\|, \|b\|] \subseteq \mathbb{R}$,
$$0 \leq t^2 \leq \|b\|^2 = \|a\| \leq \alpha, \quad \text{ie } 0 \leq \iota^2 \leq \|a\| 1 \leq \alpha 1 \text{ in } C(\sigma(b))$$
$$\Rightarrow 0 \leq \alpha 1 - \iota^2 \leq \alpha 1$$
$$\Rightarrow \|\alpha e - a\| = \|\alpha 1 - \iota^2\|_\infty \leq \alpha$$

(iii)⇒(iv) ✓
(iv)⇒(i) $\alpha \geq \|\alpha e - a\| = \|\alpha 1 - \iota\|_\infty$ in $C(\sigma(a))$
ie $\alpha \geq |\alpha - t| \quad \forall t \in \sigma(a) \subseteq \mathbb{R}$ $(a = a^*)$
$\Rightarrow t \geq 0 \quad \forall t \in \sigma(a)$ ie $\sigma(a) \subseteq [0, \infty)$ ∎

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Corollary: If $A$ is a unital $C^*$-alg., $a, b \in A$, $a \geq 0$, $b \geq 0$, then $a + b \geq 0$ too.

Proof: Fix $\alpha \geq \|a\|$, $\beta \geq \|b\|$, and we have
$$\|(\alpha + \beta)e - (a + b)\| \leq \|\alpha e - a\| + \|\beta e - b\| \leq \alpha + \beta$$
by (iii above). By (iv) above, $a + b \geq 0$. ∎

Proposition: Let $A$ be a unital Banach Algebra, $a, b \in A$. Then
$$\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}.$$

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**Proof.** If $z \in \mathbb{C} \setminus (\sigma(ab) \cup \{0\})$, then
$$(ze - ba)\tfrac{1}{z}\left(e + b(ze-ab)^{-1}a\right) = e - \tfrac{1}{z}ba + \tfrac{1}{z}b(ze-ab)(ze-ab)^{-1}a = e$$
Likewise the opposite multiplication holds. Hence $\sigma(ba) \subseteq \sigma(ab) \cup \{0\}$. ∎

**Theorem.** If $b \in A$, $A$ a $C^*$-algebra, then
$$b \geq 0 \iff b = a^*a \text{ for some } a \in A.$$

**Proof.** $(\Rightarrow)$ $a = b^{1/2}$ (functional calculus)
$(\Leftarrow)$: Fix $a$ in $A$, let $b = a^*a$. Notice $b^* = (a^*a)^* = a^*a^{**} = a^*a = b$.
Decompose $b = b_+ - b_-$, $b_+, b_- \geq 0$, $b_+b_- = 0$. Let $c = ab_-^{1/2}$.
Then
$$c^*c = b_-^{1/2}a^*ab_-^{1/2} = b_-^{1/2}(b_+ - b_-)b_-^{1/2} = -b_-^2$$
So
$$\sigma(c^*c) = \sigma(-b_-^2) = -\sigma(b_-)^2 \subseteq (-\infty, 0].$$
Let $x = \operatorname{Re} c$, $y = \operatorname{Im} c$, so
$$c^*c + cc^* = 2x^2 + 2y^2 \Rightarrow cc^* = \underbrace{2(x^2+y^2)}_{\geq 0} \underbrace{- c^*c}_{\geq 0} \geq 0$$
Hence $\sigma(cc^*) \cup \{0\} = \sigma(c^*c) \cup \{0\}$
$$\subseteq [0,\infty) \cap (-\infty, 0] = \{0\}$$
But $\|c^*c\| = r(c^*c) = 0$. Hence
$$b_-^2 = -c^*c = 0$$
so $b_- = (b_-^2)^{1/2} = 0$, ie $b = b_+$. ∎

$\sqrt{t} = \lim_{n\to\infty} p_n(t)$ uniformly on compact sets of $[0,\infty)$

☆ Remark. $\mathcal{H}$ is always a Hilbert space.

**Proposition** (orthogonal projections).
Let $P = P^2$ in $B(\mathcal{H})$. Then the following are equivalent:
(i) $\|P\| \leq 1$
(ii) $P^* = P$
(ii') $P \geq 0$
(iii) $\|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$ $\forall x$ in $\mathcal{H}$.

**Proof.**
(i) $\Rightarrow$ (ii) $\|P^*\| = \|P\| \leq 1$ and for $x$ in $\mathcal{H}$,
$$\langle Px, P^*x \rangle = \langle P^2x, x \rangle = \langle Px, x \rangle$$

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Hence

$$\|Px - P^*x\|^2 = \langle Px - P^*x, Px - P^*x\rangle$$
$$= \|Px\|^2 + \|P^*x\|^2 - 2\mathrm{Re}\langle Px, P^*x\rangle$$
$$\le \|Px\|^2 + \|x\|^2 - 2\mathrm{Re}\langle Px, x\rangle$$
$$= \|Px - x\|^2$$

Let $x = Py$, and we see $\|Py - P^*Py\|^2 = 0$ so $P = P^*P$.
We get (ii'), hence (ii).
(ii') $\Leftrightarrow$ (ii) obvious
(ii) $\Rightarrow$ (iii) $\|Px\|^2 + \|(I-P)x\|^2 = \langle Px, x\rangle + \langle (I-P)x, x\rangle$
$= \langle x, x\rangle = \|x\|^2$.
(iii) $\Rightarrow$ (i) $\|Px\|^2 \le \|x\|^2$. $\blacksquare$

Remarks:
(i) If $\mathcal{R} = \mathrm{ran}\, P$ ($P$ as above). Then $P(\mathcal{R}) = \mathcal{R}$, $(I-P)(\mathcal{R}) = \{0\}$ and $\mathcal{R} \perp \mathrm{ran}(I-P)$ by (iii) above. In particular,
$$\mathcal{R}^\perp = \mathrm{ran}(I-P) \quad \text{(check)}.$$
(ii) If $\mathcal{M} \subseteq \mathcal{H}$ is closed, $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{M}$. Define

$$P_{\mathcal{M}}x = \sum_{i\in I} \langle x, e_i\rangle e_i.$$

Then $P_{\mathcal{M}} = P_{\mathcal{M}}^2$, $\|P_{\mathcal{M}}\| \le 1$ (Bessel's inequality), $\mathrm{ran}\, P_{\mathcal{M}} = \mathcal{M}$.
(iii) If, also $Q = Q^2 = Q^*$, and $\mathrm{ran}\, P = \mathrm{ran}\, Q$. Then $QP = P$ and $PQ = Q$. Then
$$P = P^* = (QP)^* = PQ = Q.$$
Hence $P_{\mathcal{M}}$ in (ii) is unique being an orthogonal projection with range $\mathcal{M}$.
(iv) $P, Q$ as above
$$\mathrm{ran}\, P \perp \mathrm{ran}\, Q \Leftrightarrow PQ = 0 \Leftrightarrow QP = 0$$
$$\Leftrightarrow (P+Q)^2 = P+Q$$
Indeed, $0 = \langle Px, Qy\rangle = \langle PQx, y\rangle$
$\Leftrightarrow PQ = 0 \Leftrightarrow P = (PQ)^* = 0$
$\Rightarrow (P+Q)^2 = P + Q$
If $(P+Q)^2 = P+Q$, this implies $PQ = -QP$ so $PQ = PQ^2 = -QPQ$
so $(PQ)^* = PQ$. Then $QP = (PQ)^* = PQ$

You just put the pieces together
It's like doing a puzzle with
your grandmother

(v) If $\{P_i\}_{i=1}^{\infty} \subset \mathcal{B}(\mathcal{H})$, $P_i^2 = P_i^* = P_i$, $P_iP_j = 0$, $i \neq j$ ("mutually orthogonal") Then

$$Px := \sum_{i=1}^{\infty} P_i x$$

converges. Indeed, if $m < n$ we have

$$\left\| \sum_{i=m}^{n} P_i x \right\|^2 \underset{(iii)}{=} \sum_{i=m}^{n} \|P_i x\|^2$$

and for $m=1$,

$$\left\| \sum_{i=1}^{n} P_i x \right\|^2 \leq \|x\|^2 \quad \text{(Bessel's ineq)}$$

Hence

$$\left( \sum_{i=1}^{n} P_i x \right)_{n=1}^{\infty}$$

is Cauchy in $\mathcal{H}$.
Check that $x \mapsto Px$ defines an operator with $P^2 = P = P^*$

Corollary (to the normal functional calculus)
Let $K \in \mathcal{K}(\mathcal{H})$ be normal. Then $\sigma(K) = \sigma_p(K)$ (ie $0 \in \sigma_p(K)$) and there is, for each $\lambda \in \sigma_p(K) \setminus \{0\}$, an orthogonal projection $P_\lambda$ so $P_\lambda K = P_\lambda K$, $P_\lambda = P_{\ker(\lambda I - K)}$, and

switch?

$$K = \sum_{i=1,2,\ldots} \lambda_i P_{\lambda_i}$$

Proof. $C^*(K) \cong C_0(\sigma(K) \setminus \{0\}) = C_0(\{\lambda_1, \ldots\}) = \overline{\text{span}}\{1_{\{\lambda_i\}};\ i = 1, 2, \ldots\}$.
Let $P_{\lambda_j} = 1_{\{\lambda_j\}}(K)$. ∎

Polarization Law: (complex!)

$x, y \in \mathcal{H}$: $\langle x, y \rangle = \sum_{k=0}^{3} i^k \langle x + i^k y, x + i^k y \rangle$.

Thus if $H = H^*$ in $\mathcal{B}(\mathcal{H})$

$$\langle Hx, y \rangle = \sum_{k=0}^{3} i^k \langle H(x + i^k y), x + i^k y \rangle.$$

Hence, $S, T \in \mathcal{B}(\mathcal{H})$, then

$S = T \iff \langle Sx, x \rangle = \langle Tx, x \rangle$ for all $x$ in $\mathcal{H}$

A3 Q4 Defn of $\mathcal{U}$: $E \in \mathcal{U} \Rightarrow m(E) > 0$, $E, F \in \mathcal{U} \Rightarrow E \cap F \in \mathcal{U}$, etc $\Big\}$ replace $\updownarrow$ $m(E \cap F) > 0$    see website



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Idea: $S = \operatorname{Re} S + i \operatorname{Im} S$, same with $T$,
$\langle Sx, x \rangle = \langle Tx, x \rangle$
$\Rightarrow \langle \operatorname{Re} S x, x \rangle = \langle \operatorname{Re} T x, x \rangle$, etc.

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Proposition (Partial Isometries):
Let $U \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:
(i) if $S = (\ker U)^\perp$, then $U|_S$ is an isometry;
(ii) $(U^*U)^2 = U^*U$ (hence orthogonal projection);
(iii) if $R = \operatorname{ran} U$, then $R$ is closed and $\cancel{U|_{R^*}}$ $U^*|_R$ is an isometry;
(iv) $(UU^*)^2 = UU^*$.
Moreover, $U^*U = P_S$, $UU^* = P_R$.

Proof:
(i)$\Rightarrow$(ii): By definition of $S$,
$$U(I - P_S) = UP_{S^\perp} = 0$$
so for $x$ in $\mathcal{H}$, $x = P_S x + (I - P_S)x$
$$\|Ux\| = \|UP_S x\| = \|P_S x\|, \quad \text{(assumption on } U\text{)}$$
Hence for all $x$
$$\langle U^*Ux, x \rangle = \|Ux\|^2 = \|P_S x\|^2 = \langle P_S x, x \rangle$$
$$\Rightarrow U^*U = P_S.$$
(ii)$\Rightarrow$(i) Let $PU^* = U$ so by assumption $P^2 = P = P^*$. Then for $x$ in $\mathcal{H}$,
$$\|UPx\|^2 = \langle U^*UPx, Px \rangle = \langle Px, Px \rangle = \|Px\|^2$$
so $U|_{\operatorname{ran} P}$ is isometric. Also
$$\|U(I-P)x\|^2 = \langle \underbrace{U^*U}_{P}(I-P)x, (I-P)x \rangle = 0$$
$P$ orth. proj.
so $\operatorname{ran}(I - P) = (\operatorname{ran} P)^\perp \subseteq \ker U$. However as $\mathcal{H} = P\mathcal{H} + (I-P)\mathcal{H}$
we see $(\operatorname{ran} P)^\perp = \ker(U)$.
[(i) & (ii)] $\Rightarrow$ (iii) We have that $R = \operatorname{ran} U$ is closed as $U|_S$ is an isometry (check!). By kernel-annihilator formula $(\ker U^*)^\perp = \overline{\operatorname{ran} U} = R$.
Also for $x$ in $\mathcal{H}$
$$\|U^*\underbrace{Ux}\|^2 = \langle U^*U x, \underbrace{U^*U}_{\text{idempotent}} x \rangle = \langle U^*Ux, x \rangle = \|\underbrace{Ux}\|^2$$
So $U^*|_R$ is an isometry.
(iii)$\Leftrightarrow$(iv) similar to (i)$\Leftrightarrow$(ii)
[(iii) & (iv)] $\Rightarrow$ (i) as above ∎

you should find
this minorly creepy

**Theorem** (polar decomposition).
Let $T \in \mathcal{B}(\mathcal{H})$. Define

$$|T| = (T^*T)^{1/2}.$$

Then there is a partial isometry $U$ on $\mathcal{H}$ such that

$$T = U|T|; \quad U^*U = P_S, \quad S = (\ker T)^\perp; \quad UU^* = P_R, \quad R = \overline{\operatorname{ran} T}$$

Furthermore,

$$T^* = U^*|T^*|.$$

*Proof:* For $x \in \mathcal{H}$, observe that

$$\| |T|x \|^2 = \langle |T|x, |T|x \rangle = \langle |T|^2 x, x \rangle = \langle T^*Tx, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2.$$

Hence, $\ker|T| = \ker T$ and hence (as $|T|^* = |T|$)

$$\overline{\operatorname{ran} T} = (\ker|T|)^\perp = (\ker T)^\perp = S.$$

Define

$$U_0 : \operatorname{ran}|T| \to \mathcal{H}$$
$$U_0|T|x = Tx.$$

so

$$\|U_0|T|x\| = \|Tx\| = \||T|x\|$$

and

$$U_0(|T|x + \alpha|T|y) = U_0|T|(x + \alpha y) = T(x + \alpha y) = Tx + \alpha Ty$$

so $U_0$ is linear and well-defined. Further $U_0$ is an isometry, so it admits a unique extension to an isometry $\overline{U_0} : \overline{\operatorname{ran}|T|} \to \mathcal{H}$. Define $U$ on $\mathcal{H}$, $Ux = \overline{U_0}P_S x$, so $U$ is well-defined and linear. Then $\ker U = S^\perp$. If $x \in \mathcal{H}$ we have

$$U|T|x = U_0|T|x = Tx$$

so $U|T| = T$. Now $TU^* = U|T|U^*$ so

$$UT^* = (TU^*)^* = (U|T|U^*)^* = U|T|U^* = TU^*$$

Thus

$$\underbrace{UT^*}\,\underbrace{TU^*} = TU^*UT^* = \underbrace{TP_S}T^* = TT^*$$

so

$$U(T^*T)^2U^* = \underline{UT^*}\,TT^*TU^* = TU^*TT^*UT^* = TT^*UU^*TT^* = (TT^*$$

so by induction we can get $U(T^*T)^nU^* = (TT^*)^n$.
We approximate $\sqrt{\cdot}$ on $\sigma(T^*T) \cup \sigma(TT^*) \subseteq [0,\infty)$ uniformly by polynomials $p_n$, $p_n(0) = 0$. We see that

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$$U|T|U^* = \lim_{n\to\infty} U p_n(T^*T) U^*$$
$$= \lim_{n\to\infty} p_n(TT^*) = (TT^*)^{1/2} = |T^*|$$

Thus

$$T^* = (U|T|)^* = |T|U^* = U^*U|T|U^* = U^*|T^*|. \quad \blacksquare$$

Notation.

$$\mathcal{A}_h = \{a \in \mathcal{A} ; a = a^*\}, \quad \mathcal{A}_+ = \{a \in \mathcal{A} ; a \geq 0\}.$$

If $a, b \in \mathcal{A}_h$, we write $a \leq b$ if $b - a \geq 0$. Recall, $a, b \in \mathcal{A}_+ \Rightarrow a + b \in \mathcal{A}_+$. Thus if $a \leq b$, $b \leq c$ then

$$c - a = (c-b) + (b-a) \in \mathcal{A}_+$$

so $a \leq c$. Thus we have a partial order. Also, $a \in \mathcal{A}$, $b \in \mathcal{A}_+$, then $a^*ba \in \mathcal{A}_+$. Indeed, $b = c^*c$, $a^*ba = (ac)^*ac \geq 0$.

There are $C^*$-algebras without identity.

Eg $C_0(\mathbb{R})$.

Let $e_n \in C_0(\mathbb{R})$ be given by

$e_n \geq 0$

$\|e_n\| = 1$


If $f \in C_c(\mathbb{R})$, then $\mathrm{supp}(f) \subseteq [-n_0, n_0]$, so $e_n f - f = 0$ for $n \geq n_0$.

If $f \in C_0(\mathbb{R})$, $\varepsilon > 0$, find $g$ in $C_c(\mathbb{R})$ so $\|f - g\|_\infty < \varepsilon$. Then

$$\|f - e_n f\|_\infty \leq \|f - g\|_\infty + \|g - e_n g\|_\infty + \underbrace{\|e_n g - e_n f\|_\infty}_{\leq \underbrace{\|e_n\|_\infty}_{=1} \|g - f\|_\infty} \xrightarrow{n\to\infty} \leq \varepsilon + 0 = \varepsilon$$

So

$$\limsup_{n\to\infty} \|f - e_n f\| \leq \varepsilon$$

for any $\varepsilon > 0$. Thus $\lim_{n\to\infty} e_n f = f$.

Exercise: Make a contractive positive approximate identity for $K(\ell^2(\mathbb{N}))$.

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Also note that: ($\mathcal{A}$ unital)

$$0 \leq a \leq b \Rightarrow b \geq 0 \;\&\; b \leq \|b\|e \quad \text{(functional calc on } b\text{)}$$
$$\Rightarrow 0 \leq a \leq \|b\|e.$$
$$\Rightarrow \|a\| \leq \|b\| \quad \text{(functional calc on } a\text{)}$$

Proposition: (Contractive positive approximate identity – c.p.a.i.)
Let $A$ be a non-unital $C^*$-alg. Then there is a net $(e_\lambda)_\Lambda \subseteq A$ st $e_\lambda \geq 0$, $\|e_\lambda\| \leq 1$ for all $\lambda$ and

$$\|ae_\lambda - a\|, \|e_\lambda a - a\| \xrightarrow{\lambda} 0.$$

Proof: Let $D \subseteq A$ be any dense subset. Let

$$\Lambda = \{\lambda \subseteq D;\ \lambda \text{ is finite}\},$$

$\lambda_1 \leq \lambda_2$ iff $\lambda_1 \subseteq \lambda_2$. Also let $f, h : [0,\infty) \to [0,\infty)$

$$f(t) = \frac{t}{1+t} \quad \text{and} \quad h(t) = \frac{t}{(1+t)^2}$$

Note $f(0) = 0 = h(0)$, $0 \leq f(t) \leq 1$, $0 \leq h(t) \leq \alpha$ (some $\alpha > 0$), $t \in [0,\infty)$

$$1 - f(t) = \frac{1}{1+t} \quad \text{and} \quad (1-f(t))\,t\,(1-f(t)) = h(t)$$

Let for $\lambda$ in $\Lambda$

$$s_\lambda = |\lambda| \sum_{d\in\lambda} d^*d \quad \text{and} \quad e_\lambda = f(s_\lambda).$$

Hence, $s_\lambda \geq 0$ so by func calc, $0 \leq e_\lambda = f(s_\lambda)$ and, working in the unitization $\tilde{A}$, $\|e_\lambda\| = \|f(s_\lambda)\| \leq \sup_{t\in[0,\infty)} |f(t)| = 1$.
In $\tilde{A}$, let us compute for $b$ in $D$ with $\lambda \ni b$

$$\begin{aligned}(\tilde{e} - e_\lambda) b^*b (\tilde{e} - e_\lambda) &\leq \tfrac{1}{|\lambda|} (\tilde{e} - e_\lambda) |\lambda| \textstyle\sum_{d\in\lambda} d^*d\, (\tilde{e} - e_\lambda) \\ &= \tfrac{1}{|\lambda|} (\tilde{e} - f(s_\lambda)) s_\lambda (\tilde{e} - f(s_\lambda)) \\ &= |\lambda|^{-1} h(s_\lambda).\end{aligned}$$

Hence

$$\begin{aligned}\|b - be_\lambda\|^2 &= \|(b - be_\lambda)^*(b - be_\lambda)\| \\ &= \|(\tilde{e} - e_\lambda)b^*b(\tilde{e} - e_\lambda)\| \\ &\leq |\lambda|^{-1} \|h(s_\lambda)\| \leq \tfrac{\alpha}{|\lambda|} \xrightarrow{\lambda} 0\end{aligned}$$

And, similarly, if $\lambda \ni b^*$, we have $\|b - e_\lambda b\|^2 \xrightarrow{\lambda} 0$
If $a \in A$ and $\varepsilon > 0$, we find $b$ in $D$ so $\|a - b\| < \varepsilon$. We have

$$\begin{aligned}\|a - ae_\lambda\| &\leq \|a - b\| + \|b - be_\lambda\| + \|(b-a)e_\lambda\| \\ &\leq \varepsilon + \|b - be_\lambda\| + \varepsilon\|e_\lambda\|.\end{aligned}$$

So $\limsup_{\lambda\in\Lambda} \|a - ae_\lambda\| \leq 2\varepsilon$, so as $\varepsilon > 0$ is arbitrary,

$$\lim_{\lambda\in\Lambda} \|a - ae_\lambda\| = 0.$$

Similarly for $a - e_\lambda a$. ∎

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Remark: If $A$ is separable, we have that each
$$\Lambda_n = \{\lambda \in \Lambda ; |\lambda| = n\}$$
is countable and
$$\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n .$$
We can extract a sequence $(e_n)_{n=1}^{\infty} \subset A$ which is a c.p.a.i.

Proposition: Let $A$ be a $C^*$-algebra and $J$ be a closed ideal in $A$. Then $J$ is self-adjoint: $x \in J \Rightarrow x^* \in J$.

Proof: Let $\tilde{J} = \{x^* ; x \in J\}$. Then $\tilde{J}$ is also a closed ideal (check). Thus $J \cap \tilde{J}$ is a closed ideal which is self adjoint, and contains

both $J\tilde{J}$ and $\tilde{J}J$

$$\tilde{J}\,J = \overline{\operatorname{span}}\{x^* y ; x, y \in J\}$$

Let $(e_\lambda)_{\lambda\in\Lambda}$ be a c.p.a.i for $J \cap \tilde{J}$. If $x \in J$ we compute
$$\|e_\lambda x^* - x^*\|^2 = \|(x - x e_\lambda)(e_\lambda x^* - x^*)\|$$
$$= \|e_\lambda x^* x - x^* x + e_\lambda (x^* x e_\lambda - x^* x)\| \xrightarrow{\lambda} 0$$
as $x^* x \in \tilde{J} J \subseteq J \cap \tilde{J}$. Hence
$$x^* = \lim_{\lambda\in\Lambda} e_\lambda x^* \in (J \cap \tilde{J})\tilde{J} \subseteq J\tilde{J} \subseteq J \cap \tilde{J} \subseteq J. \qquad \blacksquare$$

Theorem: If $A$ is a $C^*$-alg, $J$ is a closed ideal, then $A/J$ is also a $C^*$-alg.

Proof: In $A/J$, let $(a+J)^* = a^* + J$; notice that
$$\|a^* + J\| = \inf_{x\in J} \|a^* - x\| = \inf_{x\in J} \|a - x^*\| = \|a + J\|.$$
so $a + J \mapsto a^* + J$ is isometric and well-defined. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a c.p.a.i for $J$. Given $a$ in $A$, $\varepsilon > 0$, let $x$ in $J$ be so $\|a - x\| < \|a + J\| + \varepsilon$. Then if $\lambda_0 \in \Lambda$ is st for $\lambda \geq \lambda_0$, $\|x - x e_\lambda\| < \varepsilon$, then
$$\|a - a e_\lambda\| \leq \|a - x\| + \|x - x e_\lambda\| + \|(x - a) e_\lambda\| \leq \|a + J\| + 2\varepsilon.$$
We conclude that
$$\|a + J\| = \lim_{\lambda\in\Lambda} \|a - a e_\lambda\|.$$
Hence, working in the unitization if necessary, we have

$$
\begin{aligned}
\|(a^* + J)(a+J)\| &= \|a^*a + J\| \\
&= \lim_{\lambda\in\Lambda} \|a^*a - a^*a e_\lambda\| \\
&= \lim_{\lambda\in\Lambda} \|a^*a(e-e_\lambda)\| \\
&\geq \lim_{\lambda\in\Lambda} \|(e-e_\lambda)a^*a(e-e_\lambda)\| \\
&= \lim_{\lambda\in\Lambda} \|a - ae_\lambda\|^2 \\
&= \|a+J\|^2 .
\end{aligned}
$$

since $\|e-e_\lambda\| \leq 1$ as $0 \leq e_\lambda$, $\|e_\lambda\| \leq 1$ and we use functional calculus

∎

Proposition. Let $A$, $B$ be Ban algs, $\pi: A \to B$ a homomorphism. Then for $a$ in $A$

$$\sigma_A(a) \cup \{0\} \supseteq \sigma_B(\pi(a)).$$

Proof: WLOG $B$ is unital. If $A$ is unital, $\pi(e_A)$ is an idempotent in $B$.

$$\pi(GL(A)) \subseteq GL_{\pi(e_A)}(B)$$

(as $aa^{-1} = e \Rightarrow \pi(a)\pi(a^{-1}) = \pi(e_A)$ etc.)

Hence

$$\sigma_B(\pi(a)) \subseteq \sigma_{\pi(e_A)}(\pi(a)) \cup \{0\} \subseteq \sigma_A(a) \cup \{0\}.$$

If $A$ is not unital, then define $\tilde{\pi}: \tilde{A} \to \tilde{B}$ by

$$\tilde{\pi}(\dot{a}, \alpha) = \pi(a) + \alpha\hat{e}_B \qquad \hat{e}_B = (0,1)$$

~~This is a~~ It is straightforward to ~~check~~ verify that this is a homomorphism.

∎

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Theorem: Let $A$, $B$ be $C^*$-algebras, and $\pi: A \to B$ be a $*$-homomorphism; ie $\pi$ is a linear homomorphism and $\pi(a^*) = \pi(a)^*$. Then $\pi(A) \cong A/\ker(\pi)$, isometrically. [In Banach space terminology: $*$-homomorphisms of $C^*$-algebras are quotient maps.]

20150319c

Proof: If $a \in A$ then $\sigma(a) \cup \{0\} \supseteq \sigma(\pi(a))$. Hence for $a$ in $A$,

$$\|a\|^2 = \|a^*a\| = r(a^*a) \geq r(\pi(a^*)\pi(a)) = r(\pi(a)^*\pi(a)) = \|\pi(a)\|^2$$

So $\|\pi\| \leq 1$.

Let us suppose that $\pi$ is injective. If $a \in A_h \setminus \{0\}$ satisfies $\|\pi(a)\| < \|a\|$, then $r(\pi(a)) = \|\pi(a)\| < \|a\| = r(a)$ (as $\pi(a)^* = \pi(a)$ and $a^* = a$), so there would be $f \in C_0(\sigma(a)\setminus\{0\})\setminus\{0\}$ such that $f|_{\sigma(\pi(a))} = 0$. We can find polynomials $p_n$, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} p_n = f$ uniformly on $[-\|a\|, \|a\|]$ (Stone-Weierstrass), $p_n(0) = 0$ and hence

$$\pi(f(a)) = \lim_{n\to\infty} \pi(p_n(a)) = \lim_{n\to\infty} p_n(\pi(a)) = f(\pi(a)) = 0$$

while $f(a) \neq 0$, contradicting assumption of injectivity of $\pi$. Hence, for any $a$ in $A$,

$$\|\pi(a)\|^2 = \|\pi(a^*a)\| = \|a^*a\|^2 = \|a\|^2.$$

If $\pi$ is not injective, then $\dot{\pi} : A/\ker\pi \to B$, $\dot{\pi}(a + \ker(\pi)) = \pi(a)$ is easily checked to be a $*$-homomorphism of $C^*$-algebras. ☒

<u>Def</u>] Let $A$ be a unital $C^*$-algebra. A linear functional $\omega: A \to \mathbb{C}$ is <u>positive</u> if $\omega(a) \geq 0$ whenever $a \geq 0$.

Notes:

(i) If $a \in A_h$, $a = a_+ - a_-$, $a_+, a_- \geq 0$ so $\omega(a) = \omega(a_+) - \omega(a_-) \in \mathbb{R}$.
If $a \in A$, generally, $a = \mathrm{Re}\,a + i\,\mathrm{Im}\,a$, $\omega^*(a) = \omega(\mathrm{Re}\,a - i\,\mathrm{Im}\,a) = \overline{\omega(a)}$

(ii) Define for $a, b$ in $A$

$$[a,b] = \omega(b^*a).$$

This is a sesquilinear form (biadditive, linear in $1^{st}$ coordinate, conj lin in $2^{nd}$), which is Hermitian ($[a,b] = \overline{[b,a]}$) and positive semi-definite ($[a,a] \geq 0$). Hence by Cauchy-Schwarz inequality

$$|[a,b]| \leq [a,a]^{1/2}[b,b]^{1/2}$$

Hence $|\omega(b^*a)|^2 \leq \omega(b^*b)\omega(a^*a)$.

Proposition (Automatic continuity). If $\omega$ is a positive linear functional on $A$, then $\omega$ is bounded and

$$\|\omega\| = \lim_{\lambda \in \Lambda} \omega(e_\lambda), \quad (e_\lambda)_{\lambda\in\Lambda} \text{ c.p.a.i.}.$$

Proof: If $A$ is unital, for $a$ in $A$, $\|a\|^2 e - a^*a = \|a^*a\| e - a^*a \geq 0$ (func. calc.). So

$|\omega(a)|^2 = |\omega(ea)|^2 \leq \omega(e)\omega(a^*a) \leq \omega(e)\|a\|^2\omega(e)$

$\Rightarrow \|\omega\| = \sup_{\|a\| \leq 1} |\omega(a)| \leq \omega(e), \leq \|\omega\|.$

If $A$ is not unital, let us first establish boundedness of $\omega$. If $\omega$ were not bounded, then for each $n$ we could find $a_n$ in $A_+$, $\|a_n\| \le 1$ st $\omega(a_n) > 2^n$. Let

$$a = \sum_{n=0}^{\infty} \frac{1}{2^n} a_n \in A_+ \quad \text{(lower semi-continuity of spectrum)}$$

and

$$a \ge \sum_{n=0}^{k} \frac{1}{2^n} a_n .$$

Hence

$$\infty > \omega(a) \ge \omega(\quad) > k \qquad \forall k.$$

Hence $\omega$ is bounded on $A_+$. Thus if $a \in A$ we have

$$a = \underbrace{\mathrm{Re}\,a + i\,\mathrm{Im}\,a}_{\|\cdot\| \le \|a\|} = (a_1 - a_2) + i(a_3 - a_4), \quad a_j \ge 0,\ \|a_j\| \le \|a\|.$$

Thus

$$\begin{aligned} |\omega(a)| &= |\omega(a_1) - \omega(a_2) + i(\omega(a_3) - \omega(a_4))| \\ &\le \textstyle\sum_{j=1}^{4} |\omega(a_j)| \\ &\le 4M\|a\| \end{aligned}$$

where

$$M = \sup_{\substack{b \in A_+ \\ \|b\| \le 1}} \omega(b).$$

Let $a \in A$, and $(e_{\lambda(\nu)})_{\nu \in N}$ be a subnet of $(e_\lambda)_{\lambda \in \Lambda}$ for which

$$\lim_{\nu \in N} \omega(e_{\lambda(\nu)}) \underset{\in ?}{\subseteq} [0\ M].$$

Then

$$\begin{aligned} |\omega(a)|^2 &= \lim_{\nu \in N} |\omega(e_{\lambda(\nu)}^{1/2} e_{\lambda(\nu)}^{1/2} a)|^2 \\ &\le \lim_{\nu \in N} \omega(e_{\lambda(\nu)})\,\omega(a^* e_{\lambda(\nu)} a) \\ &= \lim_{\nu \in N} \omega(e_{\lambda(\nu)})\, \underbrace{\omega(a^* a)}_{\le \|\omega\|\,\|a^* a\| = \|\omega\|\,\|a\|^2} \end{aligned}$$

So

$$\|\omega\|^2 = \sup_{\|a\| \le 1} |\omega(a)|^2 \le \lim_{\nu \in N} \overbrace{\omega(e_{\lambda(\nu)})}^{\|\,\| \le 1} \|\omega\| \le \|\omega\|^2$$

Hence the only cluster point of $(\omega(e_\lambda))_{\lambda \in \Lambda}$ is $\|\omega\|$, and thus this is the limit.

2015 03 1[illegible]

Notation. If $A$ is a $C^*$-alg, let $A^*_+$ denote the set of positive linear functionals. If $\omega \in A^*_+$ and $1 = \|\omega\| = \lim_{\lambda \in \Lambda} \omega(e_\lambda)$ ($(e_\lambda)_{\lambda\in\Lambda}$ c.p.a.i) we call $\omega$ a state. Write $\omega \in S(A)$ (state space).

Gelfand-Naimark-Segal Construction: Let $A$ be a $C^*$-algebra and $\omega \in S(A)$. Then there exists a $*$-representation $\pi_\omega : A \to B(\mathcal{H}_\omega)$ and $x_\omega \in \mathcal{H}_\omega$, a cyclic vector: $\overline{\pi_\omega(A)x_\omega} = \mathcal{H}_\omega$, s.t. $\|x_\omega\| = 1$, $\omega(a) = \langle \pi_\omega(a)x_\omega, x_\omega\rangle$, $a \in A$.

2015 03 2[illegible]

Proof: Let

$$N = \{n \in A ; \omega(n^*n) = 0\}.$$

Then for $a \in A$ we have

$$|\omega(an)|^2 \le \omega(aa^*)\omega(n^*n) = 0$$

so

$$N = \{n \in A ; \forall a \in A, \omega(an) = 0\},$$

and hence $N$ is a subspace of $A$. Also

$$\omega((an)^*an) = \omega(n^*a^*an) = \omega((a^*an)^*n) = 0$$

so $N$ is a left ideal. Define on $A/N$ an inner product

$$\langle a + N, b + N\rangle = \omega(b^*a),$$

which is well-defined, sesquilinear, positive definite, and hermitian. We let

$$\mathcal{H}_\omega = \overline{A/N}^{\|\cdot\|_2},$$

$\|\cdot\|_2$ norm from inner product. Define $\pi_0 : A \to \mathcal{L}(A/N)$ (linear operators) by

$$\pi_0(a)(b+N) = ab + N.$$

This is well-defined as $N$ is a left ideal. Notice that if $a \ne 0$

$$\langle \pi_0(a)(b+N), c+N\rangle = \langle ab+N, c+N\rangle = \omega(b^*a^*c) = \langle b+N, \pi_0(a^*)(c+N)\rangle$$

$$\begin{aligned}\|\pi_0(a)(b+N)\|_2^2 &= \langle ab+N, ab+N\rangle = \omega(b^*a^*ab) \\ &= \|a^*a\|\,\omega\Big(b^*\underbrace{\tfrac{1}{\|a^*a\|}a^*a}_{\le 1}b\Big) \le \|a^*a\| \lim_{\lambda\in\Lambda} \omega(b^*e_\lambda b) \\ &= \|a^*a\|\,\omega(b^*b) \\ &= \|a\|^2\|b+N\|_2^2\end{aligned}$$

$c \mapsto \omega(b^*cb)$ is a pos. func.

so $\|\pi_0(a)\| \le \|a\|$ and hence $\|\pi_0\| \le 1$, and thus extends to $\pi_\omega(a)$ on $\mathcal{H}_\omega$, and $\pi_\omega(a^*) = \pi_\omega(a)^*$. Also, we have $\pi_\omega(ab) = \pi_\omega(a)\pi_\omega(b)$ (easy check).

To find $x_\omega$, let us first assume $A$ is unital. Let $x_\omega = e + N$. Then

$$\pi_\omega(a)x_\omega = a + N$$

so

$$\overline{\pi_\omega(A)x_\omega}^{\|\cdot\|_2} = \mathcal{H}_\omega.$$

nice positive and small,
which I like

Also

$$\langle \pi_\omega(a) x_\omega, x_\omega\rangle = \langle a + N, e + N\rangle = \omega(e^* a) = \omega(a).$$

If $A$ is not unital, we require a c.p.a.i. $(e_\lambda)_{\lambda\in\Lambda}$. Notice for $a$ in $A$,

$$\|ae_\lambda^2 - a\| \le \|(ae_\lambda - a)\underbrace{e_\lambda}_{\le 1}\| + \|ae_\lambda - a\| \le 2\|ae_\lambda - a\| \xrightarrow{\lambda} 0,$$

and similarly $\|e_\lambda^2 a - a\| \xrightarrow{\lambda} 0$, so $(e_\lambda^2)_{\lambda\in\Lambda}$ is a c.p.a.i. Let $\varepsilon > 0$, $\lambda_0$ in $\Lambda$ be so that for $\lambda \ge \lambda_0$, $1 = \|\omega\| \ge \omega(e_\lambda)$, $\omega(e_\lambda^2) \ge \|\omega\| - \varepsilon = 1 - \varepsilon$. Then, let $\lambda \ge \lambda_0$ be such that

(*) $\qquad \|e_\lambda e_{\lambda_0} - e_{\lambda_0}\| < \varepsilon, \quad \|e_{\lambda_0} e_\lambda - e_{\lambda_0}\| < \varepsilon$

$\omega(e_\lambda e_{\lambda_0}) = \overline{\omega((e_\lambda e_{\lambda_0})^*)} = \overline{\omega(e_{\lambda_0} e_\lambda)}$

and we have

$$\begin{aligned}\|(e_\lambda + N) - (e_{\lambda_0} + N)\|_2^2 &= \omega((e_\lambda - e_{\lambda_0})(e_\lambda - e_{\lambda_0}))\\ &= \omega(e_\lambda^2) - \omega(e_{\lambda_0} e_\lambda) - \omega(e_\lambda e_{\lambda_0}) + \omega(e_{\lambda_0}^2)\\ &\le 2 - 2\omega(e_{\lambda_0}) + |\omega(e_{\lambda_0} - e_\lambda e_{\lambda_0})| + |\omega(e_{\lambda_0} - e_{\lambda_0} e_\lambda)|\\ &\le 2 - 2(1-\varepsilon) + \varepsilon + \varepsilon = 4\varepsilon.\end{aligned}$$

Also, if $\lambda' \ge \lambda_0$ is such that (*) holds, then

$$\begin{aligned}\|(e_\lambda + N) - (e_{\lambda'} + N)\| &\le \|(e_\lambda - e_{\lambda_0}) + N\| + \|(e_{\lambda_0} - e_{\lambda'}) + N\|\\ &\le 2\sqrt{\varepsilon} + 2\sqrt{\varepsilon} = 4\sqrt{\varepsilon}.\end{aligned}$$

Hence $(e_\lambda + N)_{\lambda\in\Lambda}$ is Cauchy in $H_\omega$ and hence admits a limit $x_\omega$. Now

$$\|x_\omega\|_2^2 = \lim_{\lambda\in\Lambda} \|e_\lambda + N\|_2^2 = \lim_{\lambda\in\Lambda} \omega(e_\lambda^2) = \|\omega\| = 1.$$

And if $a \in A$,

$$\langle \pi_\omega(a) x_\omega, x_\omega\rangle = \lim_{\lambda\in\Lambda} \langle \pi_\omega(a)(e_\lambda + N), e_\lambda + N\rangle = \lim_{\lambda\in\Lambda} \omega(e_\lambda a e_\lambda) = \omega(a) \qquad \blacksquare$$

We call $(\pi_\omega, H_\omega, x_\omega)$ the <u>GNS-triple</u> associated to $\omega$.

<u>Remark</u>: If $\pi : A \to B(\mathcal{H})$ is a $*$-representation of a $C^*$-algebra, $x \in B(\mathcal{H})$, then

$$\omega(a) = \langle \pi(a)x, x\rangle, \quad \omega \in A_+^*.$$

Indeed

$$\omega(a^*a) = \langle \pi(a^*a)x, x\rangle = \langle \pi(a)x, \pi(a)x\rangle = \|\pi(a)x\|^2 \ge 0.$$

Corollary: If $A$ is a non-unital $C^*$-algebra and $\omega \in S(A)$, then there is a unique $\tilde\omega \in S(\tilde A)$ such that $\tilde\omega|_A = \omega$.

Proof: Consider $(\pi, \mathcal{H}, x) = (\pi_\omega, \mathcal{H}_\omega, x_\omega)$, the G-N-S triple. Let $\tilde{\mathcal{H}} = \mathcal{H} \oplus_2 \mathbb{C}$,

$$\langle (x,\alpha), (y,\beta)\rangle = \langle x, y\rangle + \alpha\bar\beta.$$

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We notice $\pi: A \to B(\mathcal{H}) \subseteq B(\tilde{\mathcal{H}})$, $\pi(a)(x,\beta) = (\pi(a)x, 0)$. so $\pi: A \to B(\tilde{\mathcal{H}})$ is a $*$-homomorphism, with $\pi(A) \not\ni \tilde{I}$, identity operator on $\tilde{\mathcal{H}}$. Define $\tilde{\pi}: \tilde{A} \to B(\tilde{\mathcal{H}})$ by

(f)
$$\tilde{\pi}(a,\alpha)(b,\beta) = (\pi(a) + \alpha\tilde{I})(x,\beta) = (\pi(a)x + \alpha x, \alpha\beta).$$

Using (f), it is easy to see that $\tilde{\pi}$ is a $*$-representation, with $\tilde{\pi}(0,1) = \tilde{I}$.
We let
$$\tilde{\omega}(a,\alpha) = \langle \tilde{\pi}(a,\alpha)(x_\omega, 0), (x_\omega, 0)\rangle.$$
We notice that $\tilde{\omega} \in \tilde{A}^*_+$ and
$$1 = \|\omega\| = \|\tilde{\omega}|A\| \leq \|\tilde{\omega}\| \leq \|\tilde{\pi}\| \|(x_\omega, 0)\|^2 \leq \|x_\omega\|^2 \leq 1,$$
so $\|\tilde{\omega}\| = 1$. We see that $\tilde{\omega}(0,1) = 1$ is the unique available choice. ∎

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Eg Let $A = C_0(\mathbb{R})$, $\mu: A \to B(L^2[0,1])$ (Lebesgue measure)
$$\mu(f)h(s) = f(s)h(s) \quad \text{a.e. } s \text{ in } [0,1],\ f \in A,\ h \in L^2[0,1]$$
Notice
$$\omega(f) = \int_{[0,1]} f \, dm$$
is a state and $\mu = \pi_\omega$ (A4). $f|_{[0,1]} = 1$, $\mu(f) = I$

Lemma: Let $\pi: A \to B(\mathcal{H})$ be a $*$-representation. If $\mathcal{M} \subseteq \mathcal{H}$ is a closed, $\pi$-invariant subspace (ie $\pi(a)\mathcal{M} \subseteq \mathcal{M}\ \forall a \in A$) then $\mathcal{M}^\perp$ is also $\pi$-invariant. (We say that $\mathcal{M}$ is <u>orthogonally reducing</u>.)

Proof: Let $P = P_\mathcal{M}$. Then by $\pi$-invariance we have
$$\pi(a)P = P\pi(a)P \quad \forall a \in A.$$
Hence $P\pi(a) = (\pi(a^*)P)^* = (P\pi(a^*)P)^* = P\pi(a)P = \pi(a)P$. That is, $\pi(a)(I-P) = (I-P)\pi(a)(I-P)$, so $\mathcal{M}^\perp = (I-P)\mathcal{H}$ is $\pi$-invariant. ∎

<u>Def</u> We say a $*$-representation $\pi: A \to B(\mathcal{H})$ of a $C^*$-algebra $A$ is irreducible if there is no $\pi$-invariant $\mathcal{M}$ with $\{0\} \subset \mathcal{M} \subset \mathcal{H}$.

Corollary: A $*$-representation $\pi: A \to B(\mathcal{H})$ of a $C^*$-algebra $A$, is irreducible if and only if the only $P = P^2 = P^*$ in $B(\mathcal{H})$ such that $P\pi(a) = \pi(a)P$ $\forall a \in A$, are $0$ and $I$.

Proof: ($\Rightarrow$) From lemma above.
($\Leftarrow$) $\operatorname{ran} P = P\mathcal{H}$ is $\pi$-invariant. ■

<u>Def</u>] Let $A$ be a $C^*$-algebra. A <u>pure state</u> is an $\omega$ in $S(A)$ which is an extreme point for $S(A)$.

Remark: If $A$ is unital, then $S(A)$ is convex and $\omega^*$-compact so $\operatorname{ext}(S(A)) \neq \emptyset$ by Krein-Milman.

pure state

<u>Def</u>] Let $PS(A) = \operatorname{ext} S(A)$.

Theorem: If $\omega \in PS(A)$, $A$ a $C^*$-algebra, then the GNS representation $\pi_\omega : A \to \mathcal{B}(\mathcal{H}_\omega)$ is irreducible.

Remark: The converse is true.

Proof: Let $P = P^2 = P^*$ in $\mathcal{B}(\mathcal{H})$ be so that $P\pi(a) = \pi(a)P$ for all $a \in A$. Let $x_\omega$ denote the GNS cyclic vector. Then if $Px_\omega \neq 0$ and $(I-P)x_\omega \neq 0$, we have for $a$ in $A$,

$$\begin{aligned}
\omega(a) &= \langle \pi_\omega(a)x_\omega, x_\omega\rangle \\
&= \langle \pi_\omega(a)x_\omega, Px_\omega\rangle + \langle \pi_\omega(a)x_\omega, (I-P)x_\omega\rangle \\
&= \|Px_\omega\|^2 \left\langle \pi_\omega(a)\underbrace{\frac{1}{\|Px_\omega\|}Px_\omega}_{x_1}, \frac{1}{\|Px_\omega\|}Px_\omega\right\rangle \\
&\quad + \|(I-P)x_\omega\|^2 \left\langle \pi_\omega(a)\underbrace{\frac{1}{\|(I-P)x_\omega\|}(I-P)x_\omega}_{x_2}, \frac{1}{\|(I-P)x_\omega\|}(I-P)x_\omega\right\rangle \\
&= t\omega_1(a) + (1-t)\omega_2(a)
\end{aligned}$$

where $t = \|Px_\omega\|^2$, $\omega_j(a) = \langle \pi_\omega(a)x_j, x_j\rangle$. Notice

$$1 = \|x_\omega\|^2 = \|Px_\omega\|^2 + \|(I-P)x_\omega\|^2$$

so $0 < t < 1$. Also

$$\omega_j(a^*a) = \langle \pi_\omega(a)x_\omega, \pi_\omega(a)x_\omega\rangle = \|\pi_\omega(a)x_\omega\|^2 \geq 0,$$
$$\|\omega_j\| \leq \|\pi_\omega\| \cdot \|x_j\|^2 = 1.$$

Now $\omega \in PS(A)$, $\omega = t\omega_1 + (1-t)\omega_2$, $\omega_1, \omega_2 \in A^*_+$, $\|\omega_1\|, \|\omega_2\| \leq 1$, and hence it follows that $\omega_1, \omega_2 \in S(A)$ and $\omega_1 = \omega = \omega_2$. Hence for all $a$ in $A$,

$$\langle \pi_\omega(a)x_\omega, x_\omega\rangle = \omega(a) = \omega_1(a) = \langle \pi_\omega(a)x_\omega, \|Px_\omega\|^{-1}Px_\omega\rangle.$$

Since $\overline{\pi_\omega(A)x_\omega} = \mathcal{H}_\omega$,

$$x_\omega = \|Px_\omega\|^{-2} Px_\omega$$
$$\Rightarrow Px_\omega = \|Px_\omega\|^{-2} Px_\omega$$
$$\Rightarrow \|Px_\omega\|^2 = 1$$
$$\Rightarrow (I-P)x_\omega = 0 \quad \text{and } Px_\omega \neq 0$$

which contradicts assumptions. Hence $P\pi_\omega(a) = \pi_\omega(a)P$ for all $a$, then $Px_\omega = x_\omega$. Hence

$$\mathcal{H}_\omega = \overline{\pi_\omega(A)x_\omega} = \overline{\pi_\omega(A)Px_\omega} = P\overline{\pi_\omega(A)x_\omega} = P\mathcal{H}_\omega$$

so $P = I$. If we assume $(I-P)x_\omega \neq 0$, we find $P = 0$. We appeal to the last corollary. ∎

Proposition (norm characterization of states):
Let $A$ be a unital $C^*$-algebra, and $\omega \in A^*$. Then

$$\omega \in S(A) \iff \|\omega\| = 1 = \omega(e).$$
$$[\ \omega \in A^*_+ \iff \|\omega\| = \omega(e)\ ]$$

Proof: $(\Rightarrow)$ earlier proposition.
$(\Leftarrow)$ The maps

$$t \mapsto 2t-1 : [0,1] \to [-1,1]$$
$$t \mapsto \tfrac{1}{2}t + 1 : [-1,1] \to [0,1]$$

are mutually inverse. Hence if $a, b \in A_h$ we have

$$a \geq 0,\ \|a\| \leq 1 \iff \sigma(a) \subseteq [0,1] \iff \sigma(2a - e) \subseteq [-1,1]$$
$$\|b\| \leq 1 \iff \sigma(b) \subseteq [-1,1] \iff \sigma(\tfrac{1}{2}b + e) \subseteq [0,1].$$

Thus for $\omega \in A^*$,

$$\omega \in S(A) \iff \omega(a) \in [0,1] \text{ for all } a \geq 0,\ \|a\| \leq 1$$
$$\iff \omega(b) \subseteq [-1,1] \text{ for all } b = b^*,\ \|b\| \leq 1$$

Observe that

$$(*) \qquad [-1,1] = \bigcap_{t \in \mathbb{R}} \left(it + \sqrt{1+t^2}\,\overline{\mathbb{D}}\right).$$

Hence for $b^* = b$ in $A$

$$\sigma(b) \subseteq [-1,1] \iff \text{for all } t \in \mathbb{R},\ \|b + ite\| = \sup_{s \in \sigma(b)} |s + it| = \sup_{s \in \sigma(b)} \sqrt{s^2 + t^2} \leq \sqrt{1+t^2}$$

If $\|\omega\| = \omega(e) = 1$ then for $b = b^*$ with $\|b\| \leq 1$ we have for all $t \in \mathbb{R}$

$$|\omega(b) + it| = |\omega(b + ite)| \leq \|b + ite\| \leq \sqrt{1+t^2}.$$

Hence by $(*)$, $\omega(b) \in [-1,1]$. ∎

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Proposition (sufficiently many states): or $S(\mathcal{A})$

Let $\mathcal{A}$ be a $C^*$-algebra and $a \in \mathcal{A}_h$, then there is $\omega \in PS(\mathcal{A})$ such that

$$\|a\| = |\omega(a)|.$$

Proof: By unique extension of states, we may assume that $\mathcal{A}$ is unital. We recall that $C^*_e(a) \cong C(\sigma(a))$, $\sigma(a) \subseteq \mathbb{R}$, and $r(a) = \|a\|$ so there is $\gamma \in \Gamma_{C^*_e(a)}$ such that $|\gamma(a)| = \|a\|$.

Let $\omega$ be any Hahn-Banach extension of $\gamma$ to $\mathcal{A}$ with $\|\omega\| = \|\gamma\| = 1$. Then $\omega(e) = \gamma(e) = 1$ so $\omega \in S(\mathcal{A})$.

Now let

$$\mathcal{F}_a = \{\omega \in S(\mathcal{A}) ; |\omega(a)| = \|a\|\}.$$

Then $\mathcal{F}_a$ is a face of $S(\mathcal{A})$, ie, if $0<t<1$, $\omega_1, \omega_2 \in S(\mathcal{A})$ such that

$$(1-t)\omega_1 + t\omega_2 \in \mathcal{F}_a$$

then $\omega_1, \omega_2 \in \mathcal{F}_a$. Indeed, we have for $\omega \in \mathcal{F}_a$, $\omega_1, \omega_2$ as above,

$$\|a\| = |\omega(a)| \leq (1-t)|\omega_1(a)| + t|\omega_2(a)|$$

and $|\omega_1(a)|, |\omega_2(a)| \leq \|a\|$, which necessitates that $|\omega_1(a)| = \|a\| = |\omega_2(a)|$.

From the proof of Kein-Milman, $\mathrm{ext}(\mathcal{F}_a) \subseteq \mathrm{ext}\, S(\mathcal{A}) = PS(\mathcal{A})$. ∎

Notation: If $\mathcal{A}$ is a $C^*$-algebra, $\pi_i : \mathcal{A} \to B(\mathcal{H}_i)$, $*$-representations, $i \in I$. We let

$$\pi = \bigoplus_{i \in I} \pi : \mathcal{A} \to \mathcal{B}\Big(\underbrace{\ell^2\text{-}\bigoplus_{i \in I} \mathcal{H}_i}_{\mathcal{H}}\Big), \quad \pi(a)(x_i)_{i \in I} = (\pi_i(a)x_i)_{i \in I}$$

$$(x_i)_{i \in I}, (y_i)_{i \in I} \in \mathcal{H}, \quad \langle (x_i)_{i \in I}, (y_i)_{i \in I} \rangle = \sum_{i \in I} \langle x_i, y_i \rangle$$

Verify: $\pi$ a $*$-rep'n a $\|\pi(a)\| = \sup_{i \in I} \|\pi_i(a$

be the <u>direct sum</u> of representations.

is it direct product though?

A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is called <u>completely reducible</u> if there is a family $(\mathcal{M}_i)_{i \in I}$ of mutually orthogonal, closed, $\pi$-invariant subspaces such that $P_{\mathcal{M}_i}\pi(\cdot)|_{\mathcal{M}_i}$ are irreducible, $\mathcal{H} \cong \ell^2\text{-}\bigoplus_{i \in I} \mathcal{M}_i$, and $\pi \cong \bigoplus_{i \in I} P_{\mathcal{M}_i}\pi(\cdot)|_{\mathcal{M}_i}$.

## <u>GNS Theorem</u>

Let $\mathcal{A}$ be a $C^*$-algebra. Then there exists an injective, hence isometric, $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Moreover, we may arrange $\pi$ to be completely reducible; and, if $\mathcal{A}$ is separable, then we can arrange $\mathcal{H}$ to be separable.

Proof. Let $D \subset \mathcal{A}$ be a dense subset. For each $d \in D$, let $\omega_d \in PS(\mathcal{A})$ such that $\omega_d(d^*d) = \|d^*d\|$. Let it's not strictly necessary to go this hard



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$$\pi = \bigoplus_{d\in D} \pi_{\omega_d} \quad \text{on} \quad \mathcal{H} = \ell^2\text{-}\bigoplus_{d\in D} \mathcal{H}_{\omega_d}.$$

Since

$$\|a^*a - d^*d\| \le \|a^*a - a^*d\| + \|a^*d - d^*d\| \le (\|a\|+\|d\|)\|a-d\|$$

we see that

$$|\omega_d(a^*a) - \omega_d(d^*d)| \le (\|a\|+\|d\|)\|a-d\|. \qquad \text{———} \qquad (f)$$

$$\|\pi(a)\|^2 = \|\pi(a^*a)\|$$

$$= \sup_{d\in D} \|\pi_{\omega_d}(a^*a)\|$$

$$\underset{\text{GNS consh}}{\ge} \sup_{d\in D} \langle \pi_{\omega_d}(a^*a)x_{\omega_d}, x_{\omega_d}\rangle$$

$$= \sup_{d\in D} \omega_d(a^*a) > 0 \text{ by } (f).$$

Hence $\pi$ is injective. Also $\pi$ is completely reducible.
If $A$ is separable, we can arrange $D$ to be countable. Each

$$\mathcal{H}_{\omega_d} = \overline{\pi_{\omega_d}(A)x_{\omega_d}}, \quad a \mapsto \pi_{\omega_d}(a)x_\omega : A \to \mathcal{H}_{\omega_d}$$

is contractive.

Note: We call

\varpi

$$\varpi = \bigoplus_{\omega\in S(A)} \underbrace{\pi_\omega}_{\text{cyclic}} \quad \text{on} \quad \ell^2\text{-}\bigoplus_{\omega\in S(A)} \mathcal{H}_\omega$$

the universal representation.

Lemma (cyclic decomposition):
Let $\pi: A\to B(\mathcal{H})$ be a non-degenerate representation: $\overline{\operatorname{span}\pi(A)\mathcal{H}} = \mathcal{H}$. Then
(i) $\lim_{\lambda\in\Lambda} \pi(e_\lambda)x = x$ for all $x$ in $\mathcal{H}$, $(e_\lambda)_{\lambda\in\Lambda}$ c.p.a.i.
(ii) there exists a family $(M_i)_{i\in I}$ of closed, mutually orthogonal $\pi$-invariant cyclic subspaces: $M_i = \overline{\pi(A)x_i}$ for some $x_i$ in $M_i$, such that $\mathcal{H} = \ell^2\text{-}\bigoplus_{i\in I} M_i$.

Proof:
(i) If

$$y = \sum_{i=1}^n \pi(a_i)x_i, \quad a_1,\dots,a_n\in A, \ x_1,\dots,x_n\in\mathcal{H},$$

Let's see if
I can spell
today

then

$$\pi(e_\lambda)y = \sum_{i=1}^{n} \pi(e_\lambda a_i)x_i \xrightarrow{\lambda} y,$$

and by density of such elements $y$ in $\mathcal{H}$, we are done.

(ii) Let

$$\Xi = \{\{x_i\}_{i\in I} ;\ \|x_i\|=1,\ \pi(A)x_i \perp \pi(A)x_j,\ i\neq j\}.$$

We can assign a partial order by $\subseteq$. If $\Gamma \subseteq \Xi$ is a chain then $\bigcup\Gamma$ is clearly an upper bound. By Zorn's lemma, there is ~~an~~ maximal element $F = \{x_i\}_{i\in I}$ of $\Xi$. Let

$$\mathcal{M}_i = \overline{\pi(A)x_i}.$$

If

$$\mathcal{M} = \ell^2\text{-}\bigoplus_{i\in I} \mathcal{M}_i \subsetneq \mathcal{H},$$

then we could find $x \in \mathcal{M}^\perp$, $\|x\| = 1$. But, ~~some~~ since $*$-representations are reducing, $\{x_i\}_{i\in I} \cup \{x\} \supsetneq F$ and is an element of $\Xi$. This contradicts maximality. ∎

## Borel Functional Calculus

Let $\mathcal{H}$ be a Hilbert space. The <u>weak operator topology</u> (WOT) is the initial topology/locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by functionals

$$S \mapsto \langle Sx, y\rangle, \quad x, y \in \mathcal{H}.$$

The <u>strong operator topology</u> (SOT) is the initial topology/locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by

$$S \mapsto Sx : \mathcal{B}(\mathcal{H}) \to \mathcal{H}, \quad x \in \mathcal{H}.$$

Corollary (to the Riesz Representation Theorem):
If $(x,y) \mapsto [x,y] : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ is sesquilinear and $|[x,y]| \leq C\|x\|\|y\|$ for some $C>0$, then there is some $S \in \mathcal{B}(\mathcal{H})$ such that $[x,y] = \langle Sx, y\rangle$.

Proof. If $f : \mathcal{H} \to \mathbb{C}$ is conjugate linear and bounded ($|f(x)| \leq M\|x\|$), then $y \mapsto \overline{f(y)}$ is linear and bounded, so $\overline{f(y)} = \langle y, x_f\rangle$ for some $x_f$ in $\mathcal{H}$, $f(y) = \langle x_f, y\rangle$. Now, for $x$ in $\mathcal{H}$, let $Sx \in \mathcal{H}$, be given by $y \mapsto [x,y] = \langle Sx, y\rangle$. It is easy to verify that $S$ is linear and bounded. ∎