K K K

Remark: If for "sufficiently many" z

$$(ze-a)^{-1} \in \langle a \rangle_e \quad \text{(closed alg.)}$$

Then for any $f \in \mathrm{Hol}(\sigma(a))$ and any suitable contour system T

$$f(a) = \frac{1}{2\pi i} \int_T f(z) \underbrace{(ze\cdot a)^{-1}}_{\in \langle a \rangle_e.} dz \in \langle a \rangle_e$$

Compact & Fredholm Operators

Defn Let X be a Ban space.

$$\mathcal{K}(X) = \{K \in \mathcal{B}(X);\ \overline{K(b_1(X))} \text{ is compact in } X\}.$$

We call elements of $K(X)$ compact operators.

Remark: For $K \in \mathcal{B}(X)$, TFAE

(i) $K \in \mathcal{K}(X)$

(ii) $K(b_1(X))$ is totally bounded: given $\varepsilon > 0$ $\exists \{x_1, \dots, x_n\} \in b_1(X)$ st

$$K(b_1(X)) \subset \bigcup_{j=1}^{n} (Kx_j + \varepsilon b_1(X))$$

(iii) Any sequence $(x_n)_{n=1}^{\infty} \subset b_1(X)$ has that $(Kx_n)_{n=1}^{\infty}$ admits a converging subsequence (to a point in $\overline{K(b_1(X))}$).

Proposition: $\mathcal{K}(X)$ is a closed ideal in $\mathcal{B}(X)$.

Proof: Let us see that $\mathcal{K}(X)$ is norm-closed. Suppose $(K_n)_{n=1}^{\infty} \subset \mathcal{K}(X)$, $\lim_{n\to\infty} K_n = K$ in $\mathcal{B}(X)$. Given $\varepsilon > 0$, let n be so $\|K_n - K\| < \varepsilon/3$. Find $x_1, \dots, x_m \in b_1(X)$ st

$$\bigcup_{j=1}^{m} \left(K_n x_j + \tfrac{\varepsilon}{3} b_1(X)\right) \supset K_n(b_1(X))$$

If $x \in b_1(X)$, then there is x_j ($j \in \{1, \dots, m\}$) so that $\|K_n x - K_n x_j\| < \varepsilon/3$

Hence

$$\|Kx - Kx_j\| \le \underbrace{\|Kx - K_n x\|}_{\le \|K - K_n\| < \varepsilon/3} + \underbrace{\|K_n x - K_n x_j\|}_{< \varepsilon/3} + \underbrace{\|K_n x_j - K x_j\|}_{\le \|K - K_n\| < \varepsilon/3} < \varepsilon.$$

ie $K(b_1(X)) \subset \bigcup_{j=1}^{m} (Kx_j + \varepsilon b_1(X))$.

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If $K, L \in \mathcal{K}(X)$, $\alpha \in \mathbb{C}$, $(x_n)_{n=1}^{\infty} \subset b_1(X)$, then there is a subsequence of $(Kx_{n_k})_{k=1}^{\infty}$, $(Lx_{n_k})_{k=1}^{\infty}$ converge. Hence $((K+\alpha L)(x_{n_k}))_{k=1}^{\infty}$ converges.
Finally, if $S \in \mathcal{B}(X)$,

$$SK(b_1(X)) \subseteq S(\underbrace{\overline{K(b_1(X))}}_{\text{compact}})$$

is compact as S continuous.

$$KS(b_1(X)) \subseteq K(\|S\|\, b_1(X)) = \|S\| \underbrace{K(b_1(X))}_{\text{pre-compact}} \qquad \blacksquare$$

($\mathcal{X}$ Ban sp)

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Remarks:

(i) If ~~$\mathcal{X}$~~ $\mathcal{Y}$ is another Ban space, $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, $S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ then for $K \in \mathcal{K}(\mathcal{X})$ we have $TKS \in \mathcal{K}($~~$\mathcal{X}$~~ $\mathcal{Y})$.

(ii) Let

$$\mathcal{F}(\mathcal{X}) = \{F \in \mathcal{B}(\mathcal{X});\ \operatorname{rank} F = \dim F(\mathcal{X}) \text{ is finite}\}$$

It is clear that
- $\mathcal{F}(\mathcal{X})$ is an ideal in $\mathcal{B}(\mathcal{X})$
- $\mathcal{F}(\mathcal{X}) \subseteq \mathcal{K}(X)$ (Heine-Borel)

Let $\mathcal{A}(\mathcal{X}) = \overline{\mathcal{F}(\mathcal{X})} \subseteq \mathcal{B}(\mathcal{X})$. So $\mathcal{A}(\mathcal{X})$ is a closed ideal and $\mathcal{A}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X})$. We call elements of $\mathcal{A}(\mathcal{X})$ <u>approximable operators</u>.

Dark Fact: Sometimes $\mathcal{A}(\mathcal{X}) \subsetneq \mathcal{K}(\mathcal{X})$

$\mathcal{X} = \mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ Hilbert) does not admit $\mathcal{A}(\mathcal{X}) = \mathcal{K}(\mathcal{X})$.

<u>Def</u> We say that a Banach space $\mathcal{X}$ has <u>approximation property</u> if there exists a net $(F_\nu)_{\nu \in N} \subset \mathcal{F}(\mathcal{X})$ such that for any compact $\Omega \subset \mathcal{X}$,

$$\lim_{\nu \in N} \sup_{x \in \Omega} \|F_\nu x - x\| = 0.$$

Proposition: If $\mathcal{X}$ has approx. property then $\mathcal{A}(\mathcal{X}) = \mathcal{K}(\mathcal{X})$.

Proof: Let $(F_\nu)_{\nu \in N}$ be as above. Then if $K \in \mathcal{K}(\mathcal{X})$, we have

$$\|F_\nu K - K\| = \sup_{x \in b_1(\mathcal{X})} \|(F_\nu K - K)x\| \leq \sup_{y \in \overline{K(b_1(\mathcal{X}))}} \|F_\nu y - y\| \xrightarrow[\nu \in N]{} 0$$

Thus

$$K = \lim_{v\in N} T_v K \in \mathcal{A}(\mathcal{H}).$$

We already saw that $A(\mathcal{X}) \subseteq K(\mathcal{X})$. ▨

Lemma: If we have a net $(T_v)_{v\in N} \subset \mathcal{B}(\mathcal{X})$, and T in $\mathcal{B}(\mathcal{X})$ st.

(a) • $M = \sup_{v\in N} \|T_v\| < \infty$,

(b) • for $x \in \mathcal{X}$, $\lim_{v\in N} \|(T_v - T)x\| = 0$,

then for any compact $\Omega \subseteq \mathcal{X}$,

$$\lim_{v\in N} \sup_{x\in\Omega} \|(T_v - T)x\| = 0.$$

Remark: If $N = \mathbb{N}$, then by uniform bdd principle shows (b) $\Rightarrow$ (a).

Proof: Let for each $v \in N$, $f_v \in C(\Omega)$, $f_v(x) = \|(T_v - T)x\|$. Then

• $\|f_v\|_\infty = \sup_{x\in\Omega} f_v(x) = \sup_{x\in\Omega} \|(T_v - T)x\| \leq (M + \|T\|) \sup_{x\in\Omega} \|x\| < \infty$

(boundedness in $C(\Omega)$)

• if $x, y \in \Omega$

$$|f_v(x) - f_v(y)| \leq \|(T_v - T)x - (T_v - T)y\| \leq (M + \|T\|)\|x - y\|$$

(equi-Lipschitz $\Rightarrow$ equi-continuity)

Hence, Arzela-Ascoli theorem tells us that $C = \overline{\{f_v\}_{v\in N}} \subset C(\Omega)$ is compact. Let π denote the locally convex topology given by semi-norms $\{f \mapsto |f(y)| ; y \in \Omega\}$. (Topology of point-wise convergence)

We know that π is Hausdorff, $\pi \subseteq \tau$, thus $\pi|_C = \tau|_C$ (τ norm topology}, $\tau|_C = \{U \cap C ; U \in \tau\}$ "relativised topology")

Hence

$$\lim_{v\in N} \sup_{x\in\Omega} |f_v(x)| = \lim_{v\in N} \|f_v - 0\|_\infty = 0$$

as $\pi\text{-}\lim_{v\in N} f_v = 0$, i.e. $|f_v(x) - 0| \xrightarrow{v\in N} 0$. ▨

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Corollary: If $\mathcal{X}$ admits a net $(F_\nu)_{\nu\in N} \subset \mathcal{F}(\mathcal{X})$ such that

- $\sup_{\nu\in N} \|F_\nu\| < \infty$, and
- $\lim_{\nu\in N} \|F_\nu x - x\| = 0$,

then $\mathcal{X}$ has approx. property. Hence $\mathcal{A}(\mathcal{X}) = \mathcal{K}(\mathcal{X})$.

Examples

(i) ℓ^p $(1\le p<\infty)$, c_0

$$P_n(x_1, x_2, \ldots) = (x_1, \ldots, x_n, 0, 0, \ldots)$$

so $P_n \in \mathcal{F}(\mathcal{X})$, $\mathcal{X} = \ell^p$ or c_0

Check that: $\|P_n x - x\| \xrightarrow{n\to\infty} 0$.

(ii) $\ell^p(I)$ $(1\le p<\infty)$, $|I| > \aleph_0$

$$\overset{\shortparallel}{\{f: I \to \mathbb{C} ; \sum_{i\in I} |f(i)|^p < \infty\}}, \quad \sum_{i\in I} |f(i)|^p = \sup_{F\subset I \text{ finite}} \sum_{i\in F} |f(i)|^p.$$

Hence we consider for finite $F \subseteq I$,

$$P_F f(i) = \begin{cases} f(i) & i\in F \\ 0 & \text{else} \end{cases}$$

Let $\mathcal{J} = \{F \subseteq I ; \text{finite}\}$, $F_1 \le F_2 \iff F_1 \subseteq F_2$

and $(P_F)_{F\in\mathcal{J}}$ shows that $\ell^p(I)$ has approximation property.

Theorem Let $\mathcal{X}$ be a Banach space, $K \in \mathcal{B}(\mathcal{X})$. Then TFAE

(i) $K \in \mathcal{K}(\mathcal{X})$

(ii) $K^*|_{\overline{b_1(\mathcal{X}^*)}}$ is weak*-norm continuous

(iii) $K^* \in \mathcal{K}(\mathcal{X}^*)$

Proof: (i)$\Rightarrow$(ii) Let $(f_\alpha) \subset \overline{b_1(\mathcal{X}^*)}$ be a net with

$$f = w^*\lim_\alpha f_\alpha \in \overline{b_1(\mathcal{X}^*)}$$

(Banach-Alaoglu tells us ~~that~~ $\overline{b_1(\mathcal{X}^*)}$ compact, hence closed.)

Given $\varepsilon>0$, let $\{x_1, \ldots, x_m\} \subset b_1(\mathcal{X})$ st

$$K(b_1(\mathcal{X})) \subseteq \bigcup_{j=1}^m \left(x_j + \frac{\varepsilon}{2} b_1(\mathcal{X})\right).$$

Thus is $x \in b_1(\mathcal{X})$ there is x_j so $\|Kx - Kx_j\| < \varepsilon/2$. Hence for each α

$$
\begin{aligned}
&|f_\alpha(Kx) - f(Kx)| \\
&\leq |f_\alpha(Kx) - f_\alpha(Kx_j)| + |f_\alpha(Kx_j) - f(Kx_j)| + |f(Kx_j) - f(Kx)| \\
&\leq \frac{\varepsilon}{3} + (\|K\|+1)\left|(f_\alpha - f)\left(\frac{1}{\|K\|+1} Kx_j\right)\right| + \frac{\varepsilon}{2} \xrightarrow{\alpha} \varepsilon
\end{aligned}
$$

ε's?

Thus

$$\|K^* f_\alpha - K^* f\| = \sup_{x \in b_1(\mathcal{X})} |f_\alpha(Kx) - f(Kx)| \xrightarrow[\alpha]{\limsup} \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we see that $\lim_\alpha \|K^* f_\alpha - K^* f\| = 0$.

(ii)$\Rightarrow$(iii): $\overline{K^*(b_1(\mathcal{X}^*))} \subseteq \overline{K^*(\underbrace{\overline{b_1(\mathcal{X}^*)}}_{w^*\text{-compact}})}$

(iii)$\Rightarrow$(i): Let $\kappa : \mathcal{X} \to \mathcal{X}^{**}$ be the canonical injection. Notice

$$K^{**}\kappa(x)(f) = \kappa(x)(K^* f) = K^* f(x) = f(Kx) = \kappa(Kx)(f)$$

so $K^{*}\circ\kappa = \kappa \circ K$ so $K^{**}\kappa(\mathcal{X}) \subseteq \kappa(\mathcal{X})$. Hence $K = \kappa^{-1}\circ K^{**}\circ\kappa$
so K is compact, as K^{**} by (i)$\Rightarrow$(ii)$\Rightarrow$(iii) applied to K^*. ■

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Riesz lemma: Let $\mathcal{X}$ be a Banach space, $\mathcal{Y} \subsetneq \mathcal{X}$ a closed subspace, then if $0 < \varepsilon < 1$ there is $x_0 \in b_1(\mathcal{X})$ such that

$$\operatorname{dist}(x_0, \mathcal{Y}) > 1 - \varepsilon.$$

Proof: Let $x \in \mathcal{X} \setminus \mathcal{Y}$, define $f : \mathcal{Y} + \mathbb{C}x \to \mathcal{Y}$ by $f(y + \alpha x) = \alpha$
Then f is linear and $\ker f = \mathcal{Y}$ is closed so f is continuous.
Hence there is a Hahn-Banach extension $\hat{f} \in \mathcal{X}^*$ of f, in particular $\hat{f} \neq 0$ and $\mathcal{Y} \subseteq \ker(\hat{f})$. Let x_0 in $b_1(\mathcal{X})$ such that

$$|\hat{f}(x_0)| > (1-\varepsilon)\|\hat{f}\|.$$

Then for y in $\mathcal{Y}$

$$\|y - x_0\| \geq \left|\frac{1}{\|\hat{f}\|}\hat{f}(y - x)\right| > \frac{1}{\|\hat{f}\|}(1-\varepsilon)\|\hat{f}\| = (1-\varepsilon).$$ ■

Corollary: $\overline{b_1(\mathcal{X})}$ is compact if and only if $\mathcal{X}$ is finite dim
Hence $I \in \mathcal{K}(\mathcal{X})$ if and only if $\mathcal{X}$ is finite dimensional.

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Proof: (⇐) Heine-Borel
(⇒) Induction: find family $\{x_n\}_{n=1}^{\infty} \subset b_1(\mathcal{X})$ such that $\|x_n - x_m\| \geq \frac{1}{2}$ for $n \neq m$. ■

Key Lemma (for structure of compact operators):
Let $\mathcal{X}$ be a Banach space and $K \in \mathcal{K}(\mathcal{X})$. Suppose that there exist closed subspaces

$$\mathcal{Y}_0 \subsetneq \mathcal{Y}_1 \subsetneq \mathcal{Y}_2 \subsetneq \cdots$$

and scalars $(\alpha_n)_{n=1}^{\infty}$ such that

$$(\alpha_n I - K)\mathcal{Y}_n \subseteq \mathcal{Y}_{n-1}, \quad n \geq 1.$$

Then

$$\lim_{n \to \infty} \alpha_n = 0.$$

Proof: If not, then by dropping to a subsequence we may assume

$$\inf_{n \in \mathbb{N}} |\alpha_n| = \varepsilon > 0.$$

We use the Riesz Lemma to find, for each n, $x_n \in \mathcal{Y}_n$ st

(*) — $$\operatorname{dist}(x_n, \mathcal{Y}_{n-1}) \geq \tfrac{1}{2}$$

Then

$$y_n = (\alpha_n I - K)x_n \in \mathcal{Y}_{n-1} \subseteq \mathcal{Y}_n$$

so $Kx_n = \alpha_n x_n - y_n \in \mathcal{Y}_n$. If $m < n$ we have

$$\begin{aligned}\|Kx_n - Kx_m\| &= \|\alpha_n x_n - y_n - Kx_m\| \\ &= |\alpha_n| \Big\| x_n - \frac{1}{\alpha_n}\big(\underbrace{\underbrace{y_n}_{\in \mathcal{Y}_{n-1}} + \underbrace{Kx_m}_{\in \mathcal{Y}_m \subseteq \mathcal{Y}_{n-1}}}_{\in \mathcal{Y}_{n-1}}\big)\Big\| \\ &\overset{(*)}{\geq} \frac{|\alpha_n|}{2} \geq \frac{\varepsilon}{2} > 0\end{aligned}$$

Hence $(Kx_n)_{n=1}^{\infty}$ admits no Cauchy subsequence, violating assumption that $K \in \mathcal{K}(\mathcal{X})$. ■

Remarks:

(i) If $\Omega \subseteq \mathbb{C}$ is compact with $\Omega^\circ \neq \emptyset$. Then $|\partial\Omega| = c$. Indeed, if $z \in \Omega^\circ$ let for $\theta \in [0, 2\pi]$,

$$r_\theta = \inf\{r > 0;\ z + re^{i\theta} \notin \Omega\}.$$

Then $\{z + r_\theta e^{i\theta};\ \theta \in [0, 2\pi]\} \subseteq \partial\Omega$.

(ii) If $T \in \mathcal{B}(\mathcal{X})$, $\{\lambda_1, \dots, \lambda_n\} \subseteq \sigma_p(T)$, $\lambda_i \neq \lambda_j$, $i \neq j$, and

$$x_i \in \ker(\lambda_i I - T) \setminus \{0\},$$

then $\{x_1, \dots, x_n\}$ is linearly independent.

Theorem (Structure of compact operators):
Let $\mathcal{X}$ be an infinite dimensional Banach space, $K \in \mathcal{K}(\mathcal{X})$.

(i) If $\lambda \in \mathbb{C} \setminus \{0\}$, then each "generalized eigenspace"

$$\ker((\lambda I - K)^n), \quad n \geq 1$$

is finite dimensional and there exists n_λ such that

$$\ker(\lambda I - K)^n = \ker(\lambda I - K)^{n_\lambda}, \quad n \geq n_\lambda.$$

(ii) $\sigma(K) = \sigma_p(K) \cup \{0\}$. Furthermore, $\sigma_p(K) = \{\lambda_1, \dots\}$ (finite or infinite, if $\neq \emptyset$) with

$$\lim_{n \to \infty} \lambda_n = 0$$

if infinite.

Proof: First, observe that

(★) — $$\ker(\lambda I - K) \subseteq \ker(\lambda I - K)^2 \subseteq \cdots$$

Let $n_1 = 1$, and for $k \in \mathbb{N}$ let

$$n_{k+1} = \min\{n \in \mathbb{N};\ \ker(\lambda I - K)^{n_k} \subsetneqq \ker(\lambda I - K)^n\}.$$

(If (★) stablizes at a finite stage we stop, if not we define n_k.)
Let $\mathcal{Y}_0 = \{0\}$, $\mathcal{Y}_k = \ker(\lambda I - K)^{n_k}$, $k \in \mathbb{N}$ so

$$\mathcal{Y}_0 \subsetneqq \mathcal{Y}_1 \subsetneqq \cdots \quad \text{and} \quad (\lambda I - K)\mathcal{Y}_k \subseteq \mathcal{Y}_{k-1}.$$

According to the Key Lemma, $\lim_{n \to \infty} \lambda = 0$. But $\lambda \neq 0$. So we get n_λ as promised.
Now, if $\dim(\lambda I - K)^{n_\lambda} = \infty$, then using n_k's and $\mathcal{Y}_k$'s as above, there is k so that $\dim \mathcal{Y}_k = \infty$ while $\dim \mathcal{Y}_{k-1} < \infty$. Let $\{x_j\}_{j=1}^{\infty} \subset \mathcal{Y}_k \setminus \mathcal{Y}_{k-1}$ be linearly independent. Let $V_0 = \mathcal{Y}_{k-1}$, $V_1 = \mathcal{Y}_{k-1} + (x_1, \dots$
$V_n = \mathcal{Y}_{k-1} + \mathrm{span}\{x_1, \dots, x_n\}$. So

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$$V_0 \subsetneq V_1 \subsetneq \cdots \quad \text{and} \quad (\lambda I - K)V_n \subseteq (\lambda I - K)\mathcal{Y}_k$$
$$\subseteq \mathcal{Y}_{k-1} = V_0 \subseteq V_{n-1} \quad \forall n \in \mathbb{N}$$

Again, $\lim_{n\to\infty} \lambda = 0$, by Key lemma, contradicting that $\lambda \neq 0$.

(ii) Let us first show that $\sigma_{ap}(K) \subseteq \sigma_p(K) \cup \{0\}$. If $\lambda \in \sigma_{ap}(T) \setminus \{0\}$ then there is a sequence $(x_n)_{n=1}^{\infty} \subset \mathcal{H}$, $\|x_n\| = 1$ such that

$$(\lambda I - K)x_n \xrightarrow{n\to\infty} 0.$$

Since $K \in \mathcal{K}(\mathcal{H})$, by dropping to a subsequence, we may assume that $\lim_{n\to\infty} Kx_n = y$ exists. Then

$$\|\lambda x_n - y\| \leq \|(\lambda I - K)x_n\| + \|Kx_n - y\| \xrightarrow{n\to\infty} 0$$

so

$$K(\tfrac{1}{\lambda}y) = \lim_{n\to\infty} Kx_n = y,$$

that is,

$$K(y) = \lambda y.$$

Thus $\lambda \in \sigma_p(K)$.

Now suppose

$$\{\lambda_n\}_{n=1}^{\infty} \subseteq \partial\sigma(K) \setminus \{0\} \subseteq \sigma_{ap}(K) \setminus \{0\} \subseteq \sigma_p(K),$$

with $\lambda_k \neq \lambda_l$, $k \neq l$. Let $\mathcal{Y}_0 = \{0\}$,

$$\mathcal{Y}_n = \sum_{k=1}^{n} \ker(\lambda_k I - K)$$

(finite dimensional, hence closed). By remark (ii),

$$\mathcal{Y}_0 \subsetneq \mathcal{Y}_1 \subsetneq \cdots$$

and

$$(\lambda_n I - K)\mathcal{Y}_n \subseteq \mathcal{Y}_{n-1} \quad \text{for } n \in \mathbb{N}$$

Then the Key lemma tells us that $\lim_{n\to\infty} \lambda_n = 0$. Hence

$$\partial\sigma(K) \subseteq \{0\} \cup \bigcup_{n=1}^{\infty} \underbrace{\left(\partial\sigma(K) \setminus \tfrac{1}{n}\mathbb{D}\right)}_{\text{finite}} \quad \text{countable}$$

Thus by remark (i), $\sigma(K)^{\circ} = \emptyset$. So $\sigma(K) = \partial\sigma(K)$. ■

Corollary: (Fredholm Alternative) $K \in \mathcal{K}(\mathcal{H})$

If $\mathcal{H}$ is a Banach space, $\lambda \in \mathbb{C} \setminus \{0\}$ then TFAE:

(i) $\lambda I - K \in GL(\mathcal{H})$

(ii) $\ker(\lambda I - K) = \{0\}$

(iii) $\lambda I - K$ is surjective $\overset{\text{O.M.T.}}{\Longleftrightarrow}$ $\lambda I - K$ bdd below

(Exercise)

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Example:

(i) If $K \in \mathcal{K}(\mathcal{X})$, we do not necessarily expect $0 \in \sigma_p(K)$.

$$K : \ell^p \to \ell^p \quad (1 \le p < \infty)$$

$$K(x_1, \ldots) = (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \ldots)$$

$$K_n(x_1, \ldots) = (x_1, \ldots, \tfrac{1}{n}x_n, 0, 0, \ldots)$$

$$\|K_n - K\| \le \tfrac{1}{n} \xrightarrow{n\to\infty} 0, \quad K_n \in \mathcal{F}(\ell^p)$$

So $K \in \mathcal{K}(\ell^p)$.

$$\sigma(K) = \{\tfrac{1}{n}\}_{n=1}^{\infty} \cup \{0\}$$

and $\|Kx\|_p > 0$, $x \in \ell^p \setminus \{0\}$ so $0 \notin \sigma_p(K)$.

(ii) $K : \ell^p \to \ell^p \quad (1 \le p < \infty)$

$$K(x_1, \ldots) = (0, x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \ldots)$$

Similarly as above, $K \in \mathcal{K}(\ell^p)$.

$$K^n(x_1, \ldots) = (0, \ldots, 0, \tfrac{1}{n!}x_1, \ldots, \tfrac{(k-1)!}{(n+k-1)!}x_k, \ldots)$$

$$\frac{1}{n!} \ge \frac{(k-1)!}{(n+k-1)!} \text{ (check)} \quad \text{so } \|K^n\| \le \frac{1}{n!} \text{ (check)}$$

$$r(K) = \lim_{n\to\infty} \|K^n\|^{1/n} \le \lim_{n\to\infty} \frac{1}{(n!)^{1/n}} = 0$$

and hence $\sigma(K) = \{0\}$. Notice $\|Kx\|_p > 0$ if $x \ne 0$. Hence $\sigma_p(K) = \emptyset$.

On idempotents & invariant subspaces

Let $\mathcal{X}$ be a Banach space, $E = E^2 \in \mathcal{B}(\mathcal{X})$, $\mathcal{E} = \operatorname{ran} E$.

(i) $\mathcal{E}$ is closed. Indeed, if $x = \lim_{n\to\infty} Ex_n$ then

$$Ex = \lim_{n\to\infty} E^2 x_n = \lim_{n\to\infty} Ex_n = x$$

so $x = Ex \in \mathcal{E}$.

(ii) $\mathcal{E}$ is <u>invariant</u> for T in $\mathcal{B}(\mathcal{X})$, if $T(\mathcal{E}) \subseteq \mathcal{E}$.

Notice: $\mathcal{E}$ is invariant for T if and only if $TE = ETE$.

(iii) $\mathcal{E}$ is <u>reducing</u> for T in $\mathcal{B}(\mathcal{X})$ if there exists a closed complement: $\mathcal{E}' \subseteq \mathcal{X}$ closed so $\mathcal{X} = \mathcal{E} + \mathcal{E}'$ and $\mathcal{E} \cap \mathcal{E}' = \emptyset$, for which $T(\mathcal{E}) \subseteq \mathcal{E}$ and $T(\mathcal{E}') \subseteq \mathcal{E}'$

$E = E^2$ is <u>reducing</u> for T if $TE = ETE$ and $T(I-E) = (I-E)T(I-E)$.

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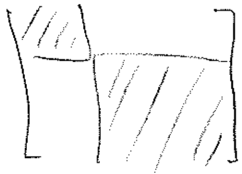
Notice:
$$\begin{array}{ccc} T(I-E) & = & (I-E)T(I-E) \\ \| & & \| \\ \cancel{T} - \cancel{TE} & & T - ET - \cancel{TE} + ETE \end{array}$$
$\rightarrow ET = ETE$

Hence E is reducing for T if and only if $ET = TE$.

Picture: In matrices

invariance



reducing

Recall: $a \in \mathcal{A}$ (unital Ban alg)

$\sigma(a) = \sigma_1 \cup \sigma_2$ (disconnection)

$U \supset \sigma_1$ open, $\sigma_2 \subset (\mathbb{C} \setminus U)^\circ$

Riesz idempotent $e_1 = \mathbb{1}_U(a)$

$\sigma(e_1 a) = \sigma_1 \cup \{0\}$

Relative spectrum at $e_1 = e_1^2$

$\sigma_{e_1}(e_1 a) = \sigma_1$

Proposition. If $K \in \mathcal{K}(\mathcal{X})$, $\lambda \in \sigma(K) \setminus \{0\} \subseteq \sigma_p(K)$, then $\ker(\lambda I - K)^{n_\lambda}$ is reducing for K. In fact $\ker(\lambda I - K)^{n_\lambda} = \operatorname{ran} E_\lambda$, E_λ Riesz idempotent associated to $\{\lambda\}$ in $\sigma(K)$.

???

Proof: First $E_\lambda K = K E_\lambda$. Let $\mathcal{E}_\lambda = \operatorname{ran} E_\lambda$. First
$$\sigma(E_\lambda K|_{\mathcal{E}_\lambda}) = \sigma_{\mathcal{E}_\lambda}(E_\lambda K) = \{\lambda\}$$
Also, since $K \in \mathcal{K}(\mathcal{X})$, $E_\lambda K|_{\mathcal{E}_\lambda}$ is compact and hence on finite-dim subspace space (0 not in spectrum).
$$E_\lambda K|_{\mathcal{E}_\lambda} = \lambda I_{\mathcal{E}_\lambda} + N, \quad N^{\dim \mathcal{E}_\lambda} = 0 \qquad \text{(Jordan form)}$$
In fact,
$$N = (E_\lambda - E_\lambda K)|_{\mathcal{E}_\lambda}$$

so $E_\lambda K = \lambda E_\lambda - (\lambda E_\lambda - E_\lambda K)$ where

$$(\lambda E_\lambda - E_\lambda K)^{\dim \mathcal{E}_\lambda} \overset{?}{=} N^{\dim \mathcal{E}_\lambda} = 0.$$

If $n \geq \dim \mathcal{E}_\lambda$, then

$$(\lambda I - K)^n E_\lambda = (\lambda E_\lambda - E_\lambda K)^n = 0$$

Thus $\operatorname{ran} E_\lambda \supseteq \ker (\lambda I - K)^{n_\lambda}$.

Now, let $\mathcal{E}_\lambda' = \operatorname{ran}(I - E_\lambda)$. Since

$$\sigma\left((I - E_\lambda)K\big|_{\mathcal{E}_\lambda'}\right) \subseteq \sigma((I - E_\lambda)K) = \sigma(K) \setminus \{\lambda\}.$$

We have $\lambda I_{\mathcal{E}_\lambda'} - \underbrace{K_\lambda'}_{(I - E_\lambda)K|_{\mathcal{E}_\lambda'}} \in GL(\mathcal{E}_\lambda')$

Hence

$$\ker(\lambda I - K)^{n_\lambda} \cap \mathcal{E}_\lambda' = \{0\}$$

Since $\mathcal{E}_\lambda \cap \mathcal{E}_\lambda'$ $(= \phi ?)$,

$$\mathcal{E}_\lambda + \mathcal{E}_\lambda' = E(X)(I - E_\lambda)(X) = X$$

We see that $\ker(\lambda I - K)^{n_\lambda} = \mathcal{E}_\lambda$. ■

Corollary: In the notation above, $(\lambda I - K)(\mathcal{E}_\lambda')$ is closed.

Recall: A finite dimensional subspace $\mathcal{F} \subseteq \mathcal{X}$ (Ban. space) automatically has a closed complement.

i.e. $E \in \mathcal{B}(\mathcal{X})$, $E = E^2$, $\mathcal{F} = \operatorname{ran} E$

Lemma: Let $\mathcal{X}$ be a Banach space, $\mathcal{X}_0$ be a closed subspace.

(i) If $\mathcal{F}$ is a finite dimensional, then $\mathcal{F} + \mathcal{X}_0$ is closed

(ii) If $\mathcal{X}/\mathcal{X}_0$ is finite dimensional, then there exists finite dimensional $\mathcal{F} \subseteq \mathcal{X}$ such that $\mathcal{X} = \mathcal{X}_0 + \mathcal{F}$.

(iii) $(\mathcal{X}/\mathcal{X}_0)^* \cong \mathcal{X}_0^\perp$ isometrically.

Proof: We let $Q : \mathcal{X} \to \mathcal{X}/\mathcal{X}_0$ be the quotient map $Qx = x + \mathcal{X}_0$.

Note $\|Q\| \leq 1$.

(i) $Q(\mathcal{F}) \subseteq \mathcal{X}/\mathcal{X}_0$ is finite dimensional, hence closed and

$$\mathcal{F} + \mathcal{X}_0 = Q'(Q\mathcal{F}) \subseteq \mathcal{X}.$$

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(ii) Let $\{x_1+\mathcal{X}_0, \dots, x_n+\mathcal{X}_0\}$ be a basis for $\mathcal{X}/\mathcal{X}_0$, and
$\mathcal{F} = \operatorname{span}\{x_1, \dots, x_n\}$.
$\mathcal{F}+\mathcal{X}_0 = Q^{-1}(\mathcal{X}/\mathcal{X}_0) = \mathcal{X}$, and $\mathcal{F}\cap\mathcal{X}_0 = \{0\}$, by the fact we used a basis.

(iii) $Q(b_1(\mathcal{X})) = b_1(\mathcal{X}/\mathcal{X}_0)$ (check)
If $f \in (\mathcal{X}/\mathcal{X}_0)^*$ we have

$$\|Q^*f\| = \sup_{x\in b_1(\mathcal{X})} \|f(Qx)\|$$

$$= \sup_{x+\mathcal{X}_0 \in b_1(\mathcal{X}/\mathcal{X}_0)} \|f(x+\mathcal{X}_0)\| = \|f\|$$

so $Q^* : (\mathcal{X}/\mathcal{X}_0)^* \to \mathcal{X}$ is an isometry, hence with closed range.
By kernel-annihilator formulas

$$\operatorname{ran} Q^* = (\ker Q)^\perp = \mathcal{X}_0^\perp. \qquad \blacksquare$$

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<u>Def</u> Let $\mathcal{X}$ be an infinite dimensional Banach space. An operator $T \in \mathcal{B}(\mathcal{X})$ is called <u>Fredholm</u>, written $T \in \mathrm{Fred}(\mathcal{X})$, if

- $\mathrm{null}(T) = \dim\ker T < \infty$;
- $\operatorname{ran} T$ closed;
- $\dim(\mathcal{X}/\operatorname{ran}T) < \infty$.

Notice

$$\dim(\mathcal{X}/\operatorname{ran}T) = \dim((\mathcal{X}/\operatorname{ran}T)^*) = \dim((\operatorname{ran}T)^\perp) = \dim(\ker T^*) = \mathrm{null}(T$$

Notes:
(i) $S \in GL(\mathcal{X}) \Rightarrow S^* \in GL(\mathcal{X}^*)$. Hence $\mathrm{null}(S) = 0 = \mathrm{null}(S^*)$. Also $\operatorname{ran} S = \mathcal{X}$ is closed. Hence $GL(\mathcal{X}) \subseteq \mathrm{Fred}(\mathcal{X})$

(ii) $S \in GL(\mathcal{X})$, $T \in \mathrm{Fred}(\mathcal{X})$. Then $ST, TS \in \mathrm{Fred}(\mathcal{X})$. This is because S is a homeomorphism on $\mathcal{X}$.

(iii) If $T \in \mathcal{B}(\mathcal{X})$ for which $\operatorname{ran}T$ is closed then $T : \mathcal{X} \to \operatorname{ran}T$ is open (open mapping theorem). In particular, if $K \in \mathcal{K}(\mathcal{X})$ with $\operatorname{ran}K$ closed, then $K \in \mathcal{F}(\mathcal{X})$, ie f.d. rage. However, then $\mathrm{null}(K) = \infty$ (see for f.d. subspaces $\mathcal{E}$ of $\mathcal{X}$, $\dim(\mathcal{E}\cap\ker K) \leq \dim\operatorname{ran}K$ and we may use arbitrarily high dimension $\mathcal{E}$)
Hence $\mathcal{K}(\mathcal{X}) \cap \mathrm{Fred}(\mathcal{X}) = \emptyset$ (if $\mathcal{X}$ is infinite dimensional).

Proposition: Let $K \in \mathcal{K}(\mathcal{X})$, $\mathcal{X}$ Banach space.

(i) If $\lambda \in \mathbb{C} \setminus \{0\}$, then $\lambda I - K \in \mathrm{Fred}(\mathcal{X})$

(ii) If $S \in GL(\mathcal{X})$, then $\underbrace{S+K}_{\text{compact perterbation of an invertible.}} \in \mathrm{Fred}(\mathcal{X})$

Proof: (i) If $\lambda \notin \sigma(K)$ then $\lambda I - K \in GL(\mathcal{X}) \subseteq \mathrm{Fred}(\mathcal{X})$. If $\lambda \in \sigma(K)$, hence $\lambda \in \sigma_p(K)$, then $\mathrm{null}(\lambda I - K) < \infty$, by structure theorem for compact operators. Also, $K^* \in \mathcal{K}(\mathcal{X}^*)$, so $\mathrm{null}((\lambda I - K)^*) = \mathrm{null}(\lambda I_{\mathcal{X}^*} - K^*) < \infty$. It remains to show that $\mathrm{ran}(\lambda I - K)$ is closed.

Let $\mathcal{E}_\lambda = \ker(\lambda I - K)^{n_\lambda}$ ((largest) generalized eigenspace. By a prop'n, $\mathcal{E}_\lambda$ is a reducing subspace, hence admits K-invariant complement $\mathcal{E}_\lambda'$. Let $\mathcal{K}_\lambda' \subseteq \mathcal{E}_\lambda$ denote any linear complement to $\ker(\lambda I - K)$.

We have

$$\mathcal{X} = \mathcal{E}_\lambda + \mathcal{E}_\lambda', \quad \mathcal{E}_\lambda \cap \mathcal{E}_\lambda' = \{0\},$$

and

$$\mathcal{E}_\lambda = \ker(\lambda I - K) + \mathcal{K}_\lambda', \quad \ker(\lambda I - K) \cap \mathcal{K}_\lambda' = \{0\}.$$

Then

$$\mathrm{ran}(\lambda I - K) = \underbrace{(\lambda I - K)(\mathcal{K}_\lambda')}_{\text{f.d.}} + \underbrace{(\lambda I - K)(\mathcal{E}_\lambda')}_{= \mathrm{ran}\, \underbrace{\mathcal{E}_\lambda(\lambda I - K)|_{\mathcal{E}_\lambda'}}_{\in GL(\mathcal{E}_\lambda')} \to \text{closed}}$$

is closed.

(ii) $$S+K = \underbrace{-S\underbrace{(I - \underbrace{S^{-1}K}_{\text{cmpt}})}_{\text{Fred}}}_{\text{Fred}}$$ ∎

We define the Fredholm index of $T \in \mathrm{Fred}(\mathcal{X})$ by

$$\mathrm{ind}_F T = \mathrm{null}(T) - \mathrm{null}(T^*). \in \mathbb{Z}$$

Theorem: If $S \in GL(\mathcal{X})$, $K \in \mathcal{K}(\mathcal{X})$, then $\mathrm{ind}_F(S+K) = 0$.

Proof: Let us consider $\lambda I - K$, $\lambda \in \mathbb{C} \setminus \{0\}$. Let $\mathcal{K}_\lambda = \ker(\lambda I - K)$ which is finite dimensional, and let $\mathcal{F}$ denote any linear complement of

$\operatorname{ran}(\lambda I - K)$. We let

$$d = \min\{\dim K_\lambda, \dim \mathcal{F}\} = \min\{\operatorname{null}(\lambda I - K), \operatorname{null}(\lambda I_{\mathcal{X}^*} - K^*)\}$$

Let $f_1, \ldots, f_d$ be a linearly independent family in K^*_λ, and take H-B extensions $\tilde{f}_1, \ldots, \tilde{f}_d$. Let $e_1, \ldots, e_d$ be a linearly independent subset of $\mathcal{F}$. Let

$$F = \sum_{i=1}^{d} \tilde{f}_i(\cdot)\, e_i \in \mathcal{F}(\mathcal{X}) \subseteq \mathcal{K}(\mathcal{X}).$$

Then

$$(\lambda I - K) + F = \lambda I - \underbrace{(K - F)}_{\in \mathcal{K}(\mathcal{X})}$$

is surjective if and only if it is injective, by the Fredholm Alternative. It is injective if and only if $d = \dim K_\lambda$. It is surjective if and only if $d = \dim \mathcal{F}$. Hence $\dim K_\lambda = \dim \mathcal{F}$, ie

$$\operatorname{null}(\lambda I - K) = \operatorname{null}(\lambda I_{\mathcal{X}^*} - K^*).$$

Observe that if $S \in GL(\mathcal{X})$, $T \in \operatorname{Fred}(\mathcal{X})$, then

$$\operatorname{ind}_F(TS) = \operatorname{ind}_F(ST) = \operatorname{ind}_F(T).$$

Hence $S + K = S(I - S^{-1}K)$ has index 0. ■

<u>Ex</u>: Let $S \in \mathcal{B}(\ell^p)$ $(1 \le p < \infty)$ be the unilateral shift

$$S(x_1, \ldots) = (0, x_1, x_2, \ldots).$$

Then $S^* \in \mathcal{B}(\ell^{p'})$ $\left(\frac{1}{p'} + \frac{1}{p} = 1\right)$ is given by

$$S^*(y_1, \ldots) = (y_2, y_3, \ldots).$$

S isometry so $\operatorname{ran} S$ is closed and $\operatorname{null}(S) = \{0\}$, $\operatorname{null}(S^*) = 1$ (check). $S \in \operatorname{Fred}(\ell^p)$ and $\operatorname{ind}_F S = -1 \neq 0$. Hence $\nexists K \in \mathcal{K}(\ell^p)$ such that $S + K \in GL(\ell^p)$.

Note: $\operatorname{ind}_F S^n = -n$ (check)