$h(\psi(I)) = \phi \Rightarrow$ by part ①
$C_c(X\setminus E) \subseteq \psi(I) \Rightarrow \psi(I) = C_0(X\setminus E)$
$\Rightarrow I = \ker(E)$

Let F and E are two distinct closed subsets of X.
By Urysohn's lemma, $\ker(E) \neq \ker(F)$. ∎

Corollary: X l.c.H. space $E \subseteq X$ closed subset, then E is a set of spectral synthesis.

Proof: $\ker(E)$ is the unique closed ideal of $C_0(X)$ which has E as its hull ∎

<u>Vector-valued Riemann Integrals</u>

$[a,b] \subseteq \mathbb{R}$, X: Banach space

$$S([a,b], X) = \text{span}\{1_I x;\ I \subseteq [a,b] \text{ an interval},\ x \in X\}$$

$$S([a,b], X) \subseteq \ell^\infty([a,b], X) = \{f: [a,b] \to X;\ \sup\{\|f(t)\|;\ t \in [a,b]\} < \infty\}$$

Every $\varphi \in S([a,b], X)$ can be represented as

$$\varphi = \sum_{i=1}^n x_i 1_{I_i} \quad \text{with} \quad \bigsqcup_{i=1}^n I_i = [a,b]$$

Define

$$\int_a^b \varphi = \sum_{i=1}^n \ell(I_i) x_i$$

Note. (a) If

$$\varphi = \sum_{i=1}^n a_i 1_{I_i} \qquad \bigsqcup_{i=1}^n I_i = [a,b]$$

and

$$\varphi = \sum_{j=1}^m b_j 1_{I_j'} \qquad \bigsqcup_{j=1}^m I_j' = [a,b]$$

then

$$\sum_{i=1}^n a_i \ell(I_i) = \sum_{j=1}^m b_j \ell(I_j').$$

(b) $\int(\varphi+c\psi) = \int\varphi + c\int\psi \qquad c\in\mathbb{C}$

(c) If $\varphi = \sum_{i=1}^{n} x_i 1_{I_i} \qquad \bigsqcup_{i=1}^{n} I_i = [a,b]$

then

$$\left\|\int\varphi\right\| = \left\|\sum_{i=1}^{n} x_i\, \ell(I_i)\right\| \le \sum_{i=1}^{n} \|x_i\|\, \ell(I_i) \le \|\varphi\|_\infty (b-a)$$

because

$$\|\varphi\|_\infty = \max\{\|x_i\|;\ i\in\{1,\dots,n\}\}.$$

(d) If $T\in\mathcal{B}(X,Y)$, Y another Banach space then

$$T\int\varphi = \int T\circ\varphi.$$

(Exercises)

Proposition: Let $\bar{S} = \overline{S([a,b],X)}$ denote the closure inside $\ell^\infty([a,b],X)$.
(1) If $\tau\in\bar{S}$, and $\varphi_n\to\tau$ (uniformly) and $\psi_n\to\tau$ (uniformly) with $\varphi_n,\psi_n\in S([a,b],X)$, then $\lim_n\int\varphi_n$ and $\lim_n\int\psi_n$ exist and are equal.

Define

$$\int\tau = \lim_{n\to\infty}\int\varphi_n.$$

(2) $\tau\mapsto\int\tau$ is linear, also

$$\left\|\int_a^b\tau\right\| \le \|\tau\|_\infty (b-a)$$

(3) $C([a,b],X)\subseteq\bar{S}$.

Proof: (1) $\{\varphi_n\}$ is Cauchy $\overset{\text{above}}{\underset{(c)}{\Longrightarrow}}$ $\{\int\varphi_n\}$ Cauchy

$\{\int\psi_n\}$ Cauchy

$\lim\int\varphi_n = \lim\int\psi_n$

because $\|\varphi_n-\psi_n\|_\infty\to 0$

(2) follows from (b) and (1), and norm estimate follows from (c)

(3) let $\tau\in C([a,b],X)$.

$$\tau = \|\cdot\|_\infty\text{-}\lim_{n\to\infty}\left(\sum_{i=0}^{n(b-a)}\tau\left(a+\tfrac{i}{n}\right) 1_{[a+i/n,\,a+\frac{i+1}{n}]}\right)$$

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Iterated integrals

Let $[a,b],[c,d]\subseteq\mathbb{R}$, X Ban space. Let

$$S = S([a,b]\times[c,d],X) = \mathrm{span}\{1_{I\times J}(\cdot,\cdot)x;\ I\subseteq[a,b],\ J\subseteq[c,d] \text{ sub-intervals},\ x\in X\}.$$

(i) If $\varphi\in S$ we may write

$$\varphi=\sum_{i=1}^{n}\sum_{j=1}^{m}1_{I_j\times J_i}(\cdot,\cdot)x_{ij},\quad x_{ij}\in X,\ [a,b]=\bigsqcup_{j=1}^{n}I_j,\ [c,d]=\bigsqcup_{i=1}^{m}J_i$$ indices

(ii) If $\varphi\in S$, as in (i),

$$\int_a^b\varphi(s,\cdot)\,ds=\sum_{j=1}^{n}\sum_{i=1}^{m}l(I_j)1_{J_i}(\cdot)x_{ij}$$

$$\int_c^d\varphi(\cdot,t)\,dt=\sum_{j=1}^{n}\sum_{i=1}^{m}l(J_i)1_{I_j}(\cdot)x_{ij}$$

are each step functions, themselves. Then

$$\int_c^d\int_a^b\varphi(s,t)\,ds\,dt=\sum_{j=1}^{n}\sum_{i=1}^{m}l(I_j)l(J_i)x_{ij}=\int_a^b\int_c^d\varphi(s,t)\,dt\,ds$$

and moreover, this independent of the form from (i).

(iii) $\varphi\in S$,

$$\left\|\int_a^b\int_c^d\varphi\right\|\le(b-a)(d-c)\|\varphi\|_\infty$$

(iv) If $\tau\in\overline{S}\subseteq l^\infty([a,b]\times[c,d],X)$, $\tau=\lim_{n\to\infty}\varphi_n$, $\varphi_n\in S$, then

$$\int_a^b\int_c^d\tau=\lim_{n\to\infty}\int_a^b\int_c^d\varphi_n$$

exists and, is independent of choice of $(\varphi_n)_{n=1}^\infty\subset S$. Moreover, this limit equals

$$\int_c^d\int_a^b\tau.$$

(v) $C([a,b]\times[c,d],X)\subseteq\overline{S}$.

Vector-valued contour integrals

Let $[a,b]\subseteq\mathbb{R}$, X Ban space, $\tau\in C^1[a,b]$ ($\mathbb{C}$-valued). Let $\tau^*=\tau([a,b])$.
If $F:\tau^*\to X$ is continuous define

$$\int_\tau F(z)\,dz=\int_a^b F(\tau(t))\tau'(t)\,dt.$$

A $\tau \in C[a,b]$ is called <u>rectifiable</u> if there exists $a=c_0<c_1\cdots<c_n=b$ such that $\tau_j = \tau|_{[c_{j-1},c_j]} \in C^1[c_{j-1},c_j]$ $\forall j\in\{1,\dots,n\}$. We then let

$$\int_\tau F(z)dz = \sum_{j=1}^{n}\int_{\tau_j} F(z)dz.$$

We call $\tau\in C[a,b]$ <u>closed</u> if $\tau(a)=\tau(b)$.

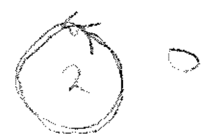
A <u>contour system</u> is a family $T=\{\tau_j\}_{j=1}^{m}$ where each τ_j is closed, rectifiable 'contour', i.e. $\tau\in C[a_j,b_j]$. Let $T^* = \tau_1^*\cup\cdots\cup\tau_m^*$.
If $F: T^*\to X$ is continuous, define

$$\int_T F(z)dz = \sum_{j=1}^{n}\int_{\tau_j} F(z)dz.$$

If $z_0\in\mathbb{C}\setminus T^*$ define the <u>index</u> by

$$\operatorname{ind}_T(z_0) = \frac{1}{2\pi i}\int_T \frac{dz}{z-z_0}.$$

Fact: $\operatorname{ind}_T : \mathbb{C}\setminus T^* \to \mathbb{Z}$ and is continuous.



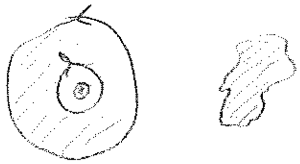
We let

$$\operatorname{ins} T = \{z\in\mathbb{C}\setminus T^*;\ \operatorname{ind}_T(z)=1\} \quad (\text{"simple" inside})$$
$$\operatorname{out} T = \{z\in\mathbb{C}\setminus T^*;\ \operatorname{ind}_T(z)=0\} \quad (\text{outside})$$

<u>'Typical' picture</u>

$\operatorname{ind}_T(\mathbb{C}\setminus T^*) \subseteq \{0,1\}$

"homologically simple"



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Cauchy's Theorem (homology version)

Let $\emptyset\neq U\subseteq\mathbb{C}$ be open, $f\in \operatorname{Hol}(U)=\{g: U\to\mathbb{C};\ g \text{ holomorphic}\}$, and T be a contour system such that $\mathbb{C}\setminus U\subseteq \operatorname{out}(T)$. Then

$$\int_T f(z)dz = 0.$$

Consequence: In assumptions above, if $z_0\in U\setminus T^*$,

$$2\pi i\, \operatorname{ind}_T(z_0) f(z_0) = \int_T \frac{f(z)}{z-z_0}dz.$$

2015 02 10c

Fact: If $\emptyset \neq K \subseteq U \subseteq \mathbb{C}$, K compact, U open, then there exists a contour system T with
- $T^* \subset U \setminus K$
- $K \subseteq \operatorname{ins} T$
- $\mathbb{C} \setminus U \subseteq \operatorname{out} T$

We shall say "T is suitable for K and U"

Idea: Cover K with finitely many open disks, each of whose closures is in U. Consider the boundary of the union of ~~union~~ the disks oriented "positively".

Lemma: Let A be a unital Ban. algebra, $a \in A$. If $\sigma(a) \subset U$, U open, and S, T are contour systems, each suitable to $\sigma(a)$ and U, and if $f \in \operatorname{Hol}(U)$, then

$$\int_T f(z)(ze-a)^{-1}\,dz = \int_S f(z)(ze-a)^{-1}\,dz \qquad \text{in } A.$$

Proof: If $\sigma \in C[a,b]$, let $\dot{-}\sigma(t) = \sigma(a+b-t)$. If $S = \{\sigma_i\}_{i=1}^m$, let $\dot{-}S = \{\dot{-}\sigma_i\}_{i=1}^m$. If $T = \{\tau_j\}_{j=1}^n$ we let $T \dot{-} S = \{\tau_j, \dot{-}\sigma_i\}_{j=1, i=1}^{n \quad m}$.
Now for $z \in \sigma(a) \cup (\mathbb{C} \setminus U)$. Then

$$\operatorname{ind}_{T\dot{-}S}(z) = \frac{1}{2\pi i}\left(\int_T \frac{d\xi}{\xi - z} + \underbrace{\int_{\dot{-}S} \frac{d\xi}{\xi - z}}_{-\int_S \frac{d\xi}{\xi - z}} \right)$$

$$= \operatorname{ind}_T(z) - \operatorname{ind}_S(z)$$

$$= \begin{cases} 1-1 & z \in \sigma(a) \\ 0-0 & z \in \mathbb{C} \setminus U \end{cases}$$

$$= 0.$$

Let $\mu \in A^*$, and let

$$F_\mu(z) = \mu\left(f(z)(ze-a)^{-1}\right) = f(z)\,\mu\left((ze-a)^{-1}\right)$$

so F_μ is holomorphic. Thus, for $z \in \mathbb{C} \setminus (U \setminus \sigma(a)) = \sigma(a) \cup (\mathbb{C} \setminus U)$ we have, by Cauchy's theorem (holom. version)

$$0 = \int_{T\dot{-}S} F_\mu(z)\,dz = \mu\left(\int_T f(z)(ze-a)^{-1}dz - \int_S f(z)(ze-a)^{-1}dz \right)$$

and hence by H-B thm, we are done. $\blacksquare$

2015 02 1

Notation: If $\emptyset \neq K \subset \mathbb{C}$ compact let

$$\mathrm{Hol}(K) = \bigcap_{\substack{U \supset K \\ \text{open}}} \mathrm{Hol}(U)$$

<u>Def</u>] Let A be a unital Banach algebra, $a \in A$. If $f \in \mathrm{Hol}(\sigma(a))$ so $f \in \mathrm{Hol}(U)$, $U \supset \sigma(a)$, U open we let

$$f(a) = \frac{1}{2\pi i} \int_T f(z)(ze-a)^{-1} dz$$

where T is any contour system suitable for $\sigma(a)$ and U.

The last lemma shows that $f(a)$ is well-defined, ie independent of suitable T.

Theorem: Let A be a unital Ban alg, $a \in A$ Then

(i) $f \mapsto f(a) : \mathrm{Hol}(\sigma(a)) \longrightarrow A$ is a homomorphism (we have a functional calculus, this is often called Riesz Functional Calculus)

(ii) if $k \in \mathbb{N}_0$, $\iota^k \in \mathrm{Hol}(\mathbb{C})$, $\iota^k(z) = z^k$ then $\iota^k(a) = a^k$

(iii) if $\sigma(a) \subset U$, U open, $g_n, g \in \mathrm{Hol}(U)$, $\lim_{n\to\infty} g_n = g$ uniformly on compact sets, then

$$\lim_{n\to\infty} \|g_n(a) - g(a)\| = 0.$$

(iv) If $r > r(a)$, $f \in \mathrm{Hol}(r\mathbb{D})$ then

$$f(a) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} a^k$$

Proof.

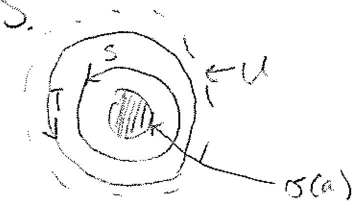
(i) If $f, g \in \mathrm{Hol}(\sigma(a))$, there are open sets $V, W \supset \sigma(a)$ so $f \in \mathrm{Hol}(V)$, $g \in \mathrm{Hol}(W)$, thus $f, g \in \mathrm{Hol}(U)$, $U = V \cap W$.

If T is a contour system, suitable for $\sigma(a)$ and U, then for $\alpha \in \mathbb{C}$,

$$(f+\alpha g)(a) = \frac{1}{2\pi i} \int_T (f(z) + \alpha g(z))(ze-a)^{-1} dz = f(a) + \alpha g(a)$$ by lin of integral

2015 02 12

Let S be a contour system with $T^k \subset \operatorname{out} S$, $\sigma(a) \subset \operatorname{ins} S$.



Also recall for $z, \xi \in \mathbb{C}\setminus\sigma(a)$,

$$(ze-a)^{-1} - (\xi e-a)^{-1} = (ze-a)^{-1}\left((\xi e-a)-(ze-a)\right)(\xi e-a)^{-1}$$

and hence

$$(ze-a)^{-1}(\xi e-a)^{-1} = \frac{1}{\xi - z}\left((ze-a)^{-1} - (\xi e-a)^{-1}\right)$$

Then

$$f(a)g(a) = \frac{-1}{4\pi^2}\int_T\int_S f(z)g(\xi)(ze-a)^{-1}(\xi e-a)^{-1}\,d\xi\,dz$$

$$= -\frac{1}{4\pi^2}\int_T f(z)\int_S \frac{g(\xi)}{\xi - z}\,d\xi\,(ze-a)^{-1}\,dz$$

$$+\frac{1}{4\pi^2}\int_S g(\xi)\underbrace{\int_T \frac{f(z)}{\xi - z}\,dz}_{=0 \text{ by Cauchy's thm}}(\xi e-a)^{-1}\,d\xi$$

$$= \frac{1}{2\pi i}\int_T f(z)g(z)(ze-a)^{-1}\,dz = fg(a)$$

(ii) Let $r > \|a\|$, $\tau: [0,2\pi] \to \mathbb{C}\setminus\sigma(a)$, $\tau(t) = re^{it}$. Then

$$\zeta^k(a) = \frac{1}{2\pi i}\int_\tau z^k(ze-a)^{-1}\,dz = \frac{1}{2\pi i}\int_\tau z^{k-1}\left(e - \tfrac{1}{z}a\right)^{-1}dz \qquad |z| = r > \|a\|,\ z\in\tau$$

$$= \frac{1}{2\pi i}\int_\tau z^{k-1}\underbrace{\sum_{n=0}^{\infty}\frac{1}{z^n}a^n}_{\text{converge uniformly for } z\in\tau^k}dz$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_\tau z^{k-n-1}\,dz\,a^n$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_0^{2\pi}(re^{it})^{k-n-1}\,ri e^{it}\,dt\,a^n$$

$$= \frac{1}{2\pi}\sum_{n=0}^{\infty} r^{k-n}\int_0^{2\pi}e^{i(k-n)t}\,dt\,a^n \qquad 0 \text{ unless } (k-n)=0$$

$$= a^k.$$

(iii) Let $T = \{\tau_j\}_{j=1}^m$, $\tau_j \in C[a_j,b_j]$ closed, rectifiable contours, and $a_j = c_{0j} < \cdots < c_{n_j j} = b_j$ st $\tau_j|_{[c_{i-1,j},c_{ij}]} \in C^1[c_{i-1j}, c_{ij}]$. Then

$$\|g_n(a) - g(a)\| = \left\|\frac{1}{2\pi i}\int_T (g_n(z)-g(z))(ze-a)^{-1}\,dz\right\|$$

$$\le \frac{1}{2\pi}\sum_{j=1}^{n}\sum_{i=1}^{n_j}\left\|\int_{c_{i-1j}}^{c_{ij}}(g_n-g)(\tau_j(t))(ze-a)^{-1}\tau'(t)\,dt\right\|$$

$\|\int_a^b F(t)dt\| \le \int_a^b \|F(t)\|dt$

$$\le \frac{1}{2\pi}\sum_{j=1}^{r}\sum_{i=1}^{n_j} \max_{z\in T^*}\left[|g_n(z)-g(z)|\cdot\|(ze-a)^{-1}\|\right]\int_{t_{i-1}}^{t_j}|\tau_j'(t)|dt$$

$$= \frac{1}{2\pi}\max_{z\in T^*}\Big[\underbrace{\qquad\qquad}_{\xrightarrow[n\to\infty]{} 0}\qquad\qquad\Big]\, length(T) \xrightarrow[n\to\infty]{} 0$$

(iv) Combine (ii) and (iii). Use curve $r > r' > r(a)$,
$\tau : [0,2\pi] \to r\mathbb{D}\setminus\sigma(a)$
$\tau(t) = r'e^{it}$ ▨

<u>Spectral Mapping Theorem</u>: Let A be a unital Ban alg, $a\in A$. If $f\in Hol(\sigma(a))$, then
$$\sigma(f(a)) = f(\sigma(a)).$$

Proof: Let M be a maximal commutative subalgebra of A with $a\in M$. Thus $\sigma_M(a) = \sigma_A(a)$.
If $\gamma\in\Gamma_M$, then for a suitable contour system T,
$$\gamma(f(a)) = \frac{1}{2\pi i}\int_T f(z)\gamma((ze-a)^{-1})dz$$
$$= \frac{1}{2\pi i}\int_T f(z)(ze-\gamma(a))^{-1}dz = f(\gamma(a)).$$
Recall, $\sigma(a) = \hat{a}(\Gamma_M) = \{\gamma(a);\ \gamma\in M\}$. Clearly $f(a)\in M$. (functional calc. fi: if) so the same is true of $\sigma(f(a))$. ▨

Proposition (Riesz ~~projection~~ idempotent): If A is a unital Ban alg, $a\in A$, $\sigma(a)$ is disconnected, $\sigma(a) = \sigma_1\cup\sigma_2$, σ_1,σ_2 closed in $\sigma(a)$. Then there is an idempotent e st $e^2 = e$, ~~and~~ $ea = ae$, and if $\sigma(ae) = \cancel{\sigma_1}\ \sigma_1\cup\{0\}$, $\sigma(a(e_A - e)) = \sigma_2\cup\{0\}$.

word not supposed to be there?

Proof: Let U, V be open, $\sigma_1\subset U$, $\sigma_2\subset V$, $U\cap V = \emptyset$. $1_U\in Hol(\sigma(a))$.
Let $e = 1_U(a)$. Use spectral mapping thm and functional calculus. ▨

Q3e: hints: -learn what it means for f to be <u>homotopic</u> to g in $GL(C(\mathbb{T}))$ — 2015 02 13
-identify $C(\mathbb{T}) = CC[0,2\pi] = \{f\in C[0,2\pi];\ f(0) = f(2\pi)\}$
- $CC'[0,2\pi]$ dense in $CC[0,2\pi]$, $ind_{fg}(0) = ind_f(0)ind_g(0)$ etc $f,g\in GL(CC[0,2\pi])\cap CC'[0,2\pi]$

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Remark: A a Ban alg with identity $e\in A$. If $w\in\mathbb{C}\setminus\sigma(a)$, $f\in Hol(\sigma(a))$

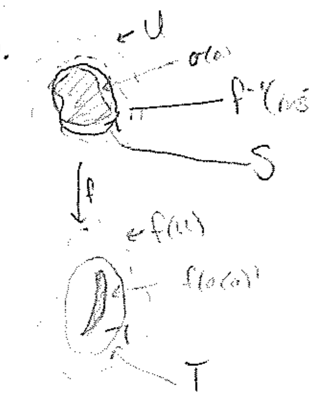
$$\frac{1}{w1-f}\in Hol(\sigma(a))$$

$$\frac{1}{w1-f}(a)=(we-a)^{-1}$$

by functional calculus.

Proposition: Let A be a unital Ban alg, $a\in A$, $f\in Hol(\sigma(a))$ which is constant on no nhd of any point in $\sigma(a)$, and $g\in Hol(f(\sigma(a)))$. Then $g\circ f\in Hol(\sigma(a))$ and $g\circ f(a)=g(f(a))$.

Proof: Let $U\supset\sigma(a)$ be open with f constant on no connected component of U. By the open mapping theorem for non-constant holomorphic functions, $f(U)$ is an open neighbourhood of $f(\sigma(a))=\sigma(f(a))$. Let T be a contour system which is suitable for $f(\sigma(a))$ and $f(U)$. Notice that $\sigma(a)\subset f^{-1}(\mathrm{ins}\,T)$. Hence we can find a contour system suitable for $\sigma(a)$ and $f^{-1}(\mathrm{ins}\,T)$. We thus compute



$$g(f(a))=\frac{1}{2\pi i}\int_T g(w)\underbrace{(we-f(a))^{-1}}_{\text{Remark}}dw$$

$$=-\frac{1}{4\pi^2}\int_T\int_S\frac{g(w)}{w-f(z)}(ze-a)^{-1}dz\,dw$$

$$=\frac{1}{2\pi i}\int_S\underbrace{\left(\frac{1}{2\pi i}\int_T\frac{g(w)}{w-f(z)}dw\right)}_{g(f(z))=g\circ f(z)}(ze-a)^{-1}dz=g\circ f(a)\quad\blacksquare$$

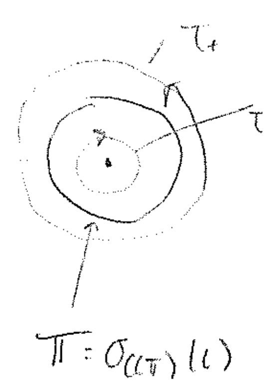
Remark: We wish to be cautious about the "context" in which we do functional calculus.

$$\iota:\mathbb{C}\to\mathbb{C},\quad \iota(z)=z$$

$\iota\in C(\mathbb{T})$ (qua element of $C(\mathbb{T})$)

$\iota^{-1}\in Hol(\mathbb{C}\setminus\{0\})$ (qua holomorphic function)

$T=\{\tau_+,\tau_-\}$, $\tau_+(t)=2e^{it}$, $\tau_-(t)=\frac{1}{2}e^{-it}$, $t\in[0,2\pi]$



$$\frac{1}{\iota}=\frac{1}{2\pi i}\int_T\frac{1}{z}(z1-\iota)^{-1}dz\qquad(\dagger)$$

$\langle\iota\rangle_e=A(\mathbb{D})$, $\sigma_{A(\mathbb{D})}(\iota)=\overline{\mathbb{D}}$, $\frac{1}{\iota}\notin Hol(\overline{\mathbb{D}})$. hence (†) makes no sense in $A(\overline{\mathbb{D}})=\langle\iota\rangle_e$

K K K

Remark: If for "sufficiently many" z
$$(ze-a)^{-1} \in \langle a \rangle_e \quad (\text{closed alg.})$$
then for any $f \in \mathrm{Hol}(\sigma(a))$ and any suitable contour system Γ
$$f(a) = \frac{1}{2\pi i}\int_\Gamma f(z)\underbrace{(ze-a)^{-1}}_{\in \langle a\rangle_e} dz \in \langle a \rangle_e$$

Compact & Fredholm Operators

Defn Let X be a Ban space.
$$\mathcal{K}(X) = \{K \in \mathcal{B}(X);\ \overline{K(b_1(X))} \text{ is compact in } X\}.$$
We call elements of $\mathcal{K}(X)$ compact operators.

Remark: For $K \in \mathcal{B}(X)$, TFAE
(i) $K \in \mathcal{K}(X)$
(ii) $K(b_1(X))$ is totally bounded: given $\varepsilon > 0$ $\exists \{x_1, \ldots, x_n\} \in b_1(X)$ st
$$K(b_1(X)) \subset \bigcup_{j=1}^{n} (Kx_j + \varepsilon b_1(X))$$
(iii) Any sequence $(x_n)_{n=1}^\infty \subset b_1(X)$ has that $(Kx_n)_{n=1}^\infty$ admits a converging subsequence (to a point in $\overline{K(b_1(X))}$).

Proposition: $\mathcal{K}(X)$ is a closed ideal in $\mathcal{B}(X)$.

Proof: Let us see that $\mathcal{K}(X)$ is norm-closed. Suppose $(K_n)_{n=1}^\infty \subset \mathcal{K}(X)$, $\lim_{n\to\infty} K_n = K$ in $\mathcal{B}(X)$. Given $\varepsilon > 0$, let n be so $\|K_n - K\| < \varepsilon/3$. Find $x_1, \ldots, x_m \in b_1(X)$ st
$$\bigcup_{j=1}^{m} \left(K_n x_j + \tfrac{\varepsilon}{3} b_1(X)\right) \supset K_n(b_1(X))$$
If $x \in b_1(X)$, then there is x_j ($j \in \{1, \ldots, m\}$) so that $\|K_n x - K_n x_j\| < \varepsilon/3$.
Hence
$$\|Kx - Kx_j\| \le \underbrace{\|Kx - K_n x\|}_{\le \|K-K_n\| < \varepsilon/3} + \underbrace{\|K_n x - K_n x_j\|}_{<\varepsilon/3} + \underbrace{\|K_n x_j - K x_j\|}_{\le \|K-K_n\| < \varepsilon/3} < \varepsilon.$$
ie $K(b_1(X)) \subset \bigcup_{j=1}^{m} (Kx_j + \varepsilon b_1(X))$.