where $e = ee_A = e[\quad * \quad] = (\lambda e - a)eb$,
and likewise $eb(\lambda e - a) = e$. ■

2015 01 2.

Character Theory (of commutative Ban. alg's)

Let A be $\mathbb{C}$-algebra. Let

$$\Gamma_A = \{\gamma : A \to \mathbb{C} ; \gamma \text{ is linear, multiplicative, non-zero}\}.$$

Notice, if $\gamma \in \Gamma_A$, then $\gamma(ab - ba) = 0$, ie γ annihilates all commutators. Hence, determining Γ_A will only be interesting if A is commutative.

Ex (Warning, $\Gamma_A \neq \emptyset$)?

(i) $A = M_n$. Let $\{E_{ij}\}_{i,j=1}^n$ be a matrix unit:

$a \in M_n$, $a = \sum_{i,j=1}^n a_{ij} E_{ij}$ (ie $a = [a_{ij}]$)

Notice if i, j, l are distinct then $E_{ij} = E_{il} E_{lj} - E_{lj} \cdot E_{il}$

So if $\gamma : M_n \to \mathbb{C}$ is lin and mult. then $\gamma(E_{ij}) = 0$. Hence $\gamma = 0$ on M_n

(ii) Let X be a $\mathbb{C}$-vector space (Banach space (if you want?))

Define $xy = 0$ x, y in X (This is a Ban alg if X is a Ban space.) If $\gamma : X \to \mathbb{C}$ is lin and mult. Then $\gamma(x)^2 = \gamma(x)\gamma(x) = \gamma(x^2) = \gamma(0) = 0$ so $\gamma(x) = 0$ ie $\gamma = 0$

Proposition. Let A be a $\mathbb{C}$-alg, $\tilde{A} = A \oplus \mathbb{C}$ its unitization. Then, any $\gamma \in \Gamma_A$ extends uniquely to $\tilde{\gamma} \in \Gamma_{\tilde{A}}$, $\tilde{\gamma}(a, \alpha) = \gamma(a) + \alpha$. Hence

$$\Gamma_{\tilde{A}} = \tilde{\Gamma}_A \cup \{\gamma_\infty\}, \quad \gamma_\infty(a, \alpha) = \alpha.$$

$\gamma_\infty \neq 0$

Proof. It is obvious that if $\gamma \in \Gamma_A$ then $\tilde{\gamma} \in \Gamma_{\tilde{A}}$. Note that if $\tilde{\tilde{\gamma}}$ is any extension of $\gamma \in \Gamma_A$ to $\tilde{A}$ then if $\gamma(a) \neq 0$, we have $\tilde{\tilde{\gamma}}(a, 0) = \tilde{\tilde{\gamma}}((a,0)(0,1)) = \tilde{\tilde{\gamma}}(a,0)\tilde{\tilde{\gamma}}(0,1) = \gamma(a)\tilde{\tilde{\gamma}}(0,1)$

Hence $\tilde{\tilde{\gamma}}(0,1) = 1 = \tilde{\gamma}(0,1)$, and $\tilde{\tilde{\gamma}}|_A = \tilde{\gamma}|_A$, so $\tilde{\tilde{\gamma}} = \tilde{\gamma}$.

It is obvious that $\Gamma_{\tilde{A}}$ is as advertized. ■

2015 01 2

Proposition: Let A be a Banach algebra and $\gamma \in \Gamma_A$. Then for a in A,
$$|\gamma(a)| \le r(a) \le \|a\|.$$
Hence $\Gamma_A \subset b_1(A^*)$.

Proof: We may assume, wlog, that A is unital. Since for $\gamma \in \Gamma_A$, $\gamma \neq 0$, we find that $\gamma(e) = 1$. Hence if $a \in GL(A)$, then $\gamma(a)\gamma(a^{-1}) = 1$ so $\gamma(a) \neq 0$.
Thus, if $a \in A$,
$$\gamma(\gamma(a)e - a) = \gamma(a) - \gamma(a) = 0$$
so $\gamma(a)e - a \notin GL(A)$, ie $\gamma(a) \in \sigma(a)$. ■

Theorem: (Gelfand-Naimark, for comm Ban. alg.s)
Let A be a comm. Ban. alg. Then
(a) if A is unital, $\Gamma_A \subseteq b_1(A^*)$ is w^*-compact;
(b) if A is not unital, $\Gamma_A \subseteq b_1(A^*)$ is w^*-locally compact.
(Furthermore) the map $a \mapsto \hat{a} : A \to C_0(\Gamma_A)$, $\hat{a}(\gamma) = \gamma(a)$, is a contractive homomorphism.
This map is called the <u>Gelfand Transform</u> of A, and Γ_A as the <u>Gelfand spectrum</u> (or <u>character space</u>)

Proof: (a) By the Banach-Alaoglu Thm, $b_1(A^*)$ is weak*-compact. Hence, we require to see that Γ_A is w^*-closed. Let $(\gamma_\alpha) \subset \Gamma_A$ be a net with $w^*\text{-}\lim_\alpha \gamma_\alpha = \chi \in b_1(A^*)$. If $a, b \in A$ we have
$$\chi(ab) = \lim_\alpha \gamma_\alpha(ab) = \lim_\alpha \gamma_\alpha(a)\gamma_\alpha(b) = \lim_\alpha \gamma_\alpha(a) \lim_\alpha \gamma_\alpha(b) = \chi(a)\chi(b).$$
Also,
$$\chi(e) = \lim_\alpha \gamma_\alpha(e) = \lim_\alpha 1 = 1 \neq 0.$$
Hence $\chi \neq 0$ so $\chi \in \Gamma_A$.

(b) The map $\mu \mapsto \mu|_A : \tilde{A}^* \to A^*$ is w^*-w^* cts. Since $\Gamma_{\tilde{A}} = \tilde{\Gamma}_A \cup \{\gamma_0\}$ is w^*-compact we have
$$\Gamma_A \cup \{0\} = \tilde{\Gamma}_A|_A \cup \{0\} = \Gamma_{\tilde{A}}|_A \subseteq A^*$$
So $\Gamma_A \cup \{0\} \subseteq A^*$ is w^*-cmpt. Thus Γ_A is w^*-locally compact.
Now consider the canonical map $K : A \to A^{**}$, $K(a)(\mu) = \mu(a)$. Then $\|K(a)\| = \|a\|$ (by H.B. thm)
$$\hat{a} = K(a)|_{\Gamma_A}$$
and we have

$$\|\hat{a}\|_\infty = \sup_{\gamma\in\Gamma_A} |\hat{a}(\gamma)| = \sup_{\gamma\in\Gamma_A} |\gamma(a)| \leq \sup_{\mu\in b(A^*)} |\mu(a)| \overset{\text{H.B. thm}}{=} \|a\|$$

ie $\|\hat{a}\|_\infty \leq \|a\|$.
Also if $a,b\in A$

$$\hat{ab}(\gamma) = \gamma(ab) = \gamma(a)\gamma(b) = \hat{a}(\gamma)\hat{b}(\gamma) = \hat{a}\hat{b}(\gamma)$$

and linearity is similar. ☐

Notation: Let <u>A be a (unital) Ban alg</u>. If $a\in A$ we let

$$\langle a\rangle_e = \{p(a) ; p\in\mathbb{C}[t]\} \subseteq A$$

<u>(algebra generated by a and e)</u>

$$\langle a\rangle = \{p(a) ; p\in\mathbb{C}[t], p(0)=0\} \subseteq A$$

(algebra generated by a)

Proposition. $\Gamma_{A(\mathbb{D})} = \left\{ \delta_z ; z\in\overline{\mathbb{D}} \right\}$, $z\mapsto\delta_z$ homeo

$\delta_z(f) = f(z)$ $\qquad (A(\overline{\mathbb{D}}) \subseteq C(\overline{\mathbb{D}}))$

Proof: It is obvious that $\supseteq$ holds.
Let $\iota(z) = z$ on $\overline{\mathbb{D}}$. Note that $A(\mathbb{D}) = \langle\iota\rangle_e$, since polynomials are dense. Also for z, z' in $\overline{\mathbb{D}}$, we have $\delta_z(\iota) = \delta_{z'}(\iota) \Leftrightarrow z=z'$. and hence if $z\neq z'$ we have $\delta_z \neq \delta_{z'}$. Thus $z\mapsto\delta_z : \overline{\mathbb{D}}\to \Gamma_{A(\mathbb{D})}$ is injective, and cts (w^*-top on $\Gamma_{A(\mathbb{D})}$) and hence is a homeomorphism onto its range.

review

Now if $\gamma\in\Gamma_{A(\mathbb{D})}$, let $z = \gamma(\iota)\in\mathbb{C}$. Then

$$|z| = |\gamma(\iota)| \leq \|\iota\|_\infty = 1,$$

and ~~Furthermore~~ for a poly $p = \sum_{k=0}^{n} a_k\iota^k$, we find

$$\gamma(p) = \sum_{k=0}^{n} a_k \underbrace{\gamma(\iota)}_{=z}{}^k = p(z)$$

and hence $\gamma = \delta_z$. ☐

Remark: For any simply gen. comm. Ban. alg., we have a recipe for computing the Gelfand spectrum.

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Prop: Let S be an abelian semigroup, and
$$\hat{S} = \{\chi : S \to \overline{\mathbb{D}} ; \chi \neq 0, \chi(st) = \chi(s)\chi(t), s,t \in S\}$$
(contractive characters on S).
Then $\Gamma_{\ell^1(S)} \cong \hat{S}$ where $\hat{S} \subseteq \ell^\infty(S) \cong \ell^1(S)^*$.
Hence, $\hat{S} \subseteq \ell^\infty(S)$ is $w*$-compact if $\ell^1(S)$ is unital, and w^*-loc. compact otherwise.

Proof: Let $\gamma \in \Gamma_{\ell^1(S)}$, then $\exists\ \chi \in \ell^\infty(S)$ st for $f \in \ell^1(S)$
$$\gamma(f) = \sum_{s \in S} f(s)\chi(s).$$
Notice that $\chi(s) = \gamma(\delta_s)$, so
$$\chi(st) = \gamma(\delta_{st}) = \gamma(\delta_s * \delta_t) = \gamma(\delta_s)\gamma(\delta_t) = \chi(s)\chi(t).$$
Also $\|\chi\|_\infty = \|\gamma\| \leq 1$, so $\chi \in \hat{S}$.
(Since $\gamma \neq 0$, $\chi \neq 0$ too.)
Conversely, if $\chi \in \hat{S}$, then for $f, g \in \ell^1(S)$
$$\left(f \in \ell^1(S), \varphi \in \ell^\infty(S), \langle f, \varphi \rangle := \sum_{s \in S} f(s)\varphi(s) \right) \quad \text{dual pairing}$$
$$\langle f * g, \chi \rangle = \sum_{s \in S} f * g(s)\chi(s) =$$
$$= \sum_{s \in S} \sum_{\substack{(t,u) \in S^2 \\ tu = s}} f(t)g(u)\chi(tu)$$
$$= \sum_{s} \sum_{t,u} f(t)\chi(t)\, g(u)\chi(u)$$
$$= \left(\sum_{t \in S} f(t)\chi(t) \right)\left(\sum_{u \in S} g(u)\chi(u) \right) = \langle f, \chi \rangle \langle g, \chi \rangle$$
Hence $\gamma(f) = \langle f, \chi \rangle$ defines a chm. of $\Gamma_{\ell^1(S)}$. ■

Examples

(i) $S = (\mathbb{N}_0, +) = \langle 1 \rangle_0$

$\chi \in \hat{S}$, $z = \chi(1) \in \overline{\mathbb{D}}$

$n \mapsto z^n : \mathbb{N}_0 \to \overline{\mathbb{D}}$ is χ $(0^0 = 1)$

$\widehat{(\mathbb{N}_0, +)} \cong \overline{\mathbb{D}}$ (exercise: justify that this is a homeo)

(ii) $S = (\mathbb{N}, +)$, $\widehat{(\mathbb{N}, +)} \cong \underbrace{\overline{\mathbb{D}} \setminus \{0\}}_{\text{locally compact, not compact}}$

(iii) $S = (\mathbb{Z}, +)$, $\chi \in \hat{\mathbb{Z}}$

let $z = \chi(1)$. Then since $\chi(0) = 1$ (ie $\chi \neq 0$) we see

$$1 = \chi(0) = \chi(-1)\chi(1)$$

so $\chi(-1) = z^{-1}$

In particular, since $\operatorname{ran} \chi \subseteq \overline{\mathbb{D}}$, we see that

$$z \in \mathbb{T} = \{z \in \mathbb{C} ; |z| = 1\}.$$

$$\hat{\mathbb{Z}} = \{n \mapsto z^n, z \in \mathbb{T}\} \cong \mathbb{T}$$

(iv) S, T unital abelian semigroups then

$$\widehat{S \times T} \cong \hat{S} \times \hat{T} \quad \text{(exercise)}$$

$$(s,t) \mapsto \chi(s)\gamma(t), \quad \chi \in \hat{S}, t \in \hat{T}$$

Theorem: $\Gamma_{L^1(\mathbb{R})} = \{\gamma_y ; y \in \mathbb{R}\}$

where

$$\gamma_y(f) = \int_{\mathbb{R}} f(x) e^{iyx} dx, \quad f \in L^1(\mathbb{R})$$

Furthermore, $y \mapsto \gamma_y : \mathbb{R} \to \Gamma_{L^1(\mathbb{R})}$ is a homeomorphism

Proof: See website. ☐

Relationship between characters & ideals

Proposition (Mazur's Theorem):

If D is a Ban alg which is a division algebra, ie $D = GL(D) \cup \{0\}$, then $D \cong \mathbb{C}$.

Proof: If $d \in D$, then $\sigma(d) \neq \emptyset$, so there is $\lambda \in \sigma(d)$. Then $\lambda e - d \in D \setminus GL(D) = \{0\}$ so $\lambda e = d$. Hence $\lambda \mapsto \lambda e$ $: \mathbb{C} \to D$ is a cts alg ~~hom~~ iso. ☐

Def) Let A be a $\mathbb{C}$ alg. Let I be an ideal of A. Then

- I is maximal if $\{0\} \subsetneq I \subsetneq A$ and for any ideal J with $I \subseteq J \subseteq A$ we have $J \in \{I, A\}$.
- I is modular (or regular) if there is $u \in A \setminus I$ such that $ux - x, xu - x \in I \quad \forall x \in A$

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Note:
(i) I is modular $\iff A/I$ is unital (identity $u+I$)
(ii) If is unital, then each proper ideal is modular.

Lemma: Let A be a comm Ban ~~~~ alg.
(i) If I is a modular ideal of A, then so ~~~~ too is its closure $\bar{I}$.
(ii) Any modular ideal I, is contained in a maximal ideal, which is also modular
(iii) Each maximal modular ideal M of A is closed and of co-dimension 1 (dim of A/M).

Proof (i) Let $u \in A \setminus I$ st $ua - a \in I$ $\forall a \in A$. If $a \in I$ allows that $\|u-a\| < 1$ then in $\tilde{A}$, we have that $(0,1)+(a-u,0)$ is invertible. Hence $a-u$ admits an adverse b in A.

$$0 = (a-u)b + b + a - u$$
$$= \underbrace{ab}_{\in I} - \underbrace{(ub - I)}_{\in I} \underbrace{+a-u}_{\in I}$$

so $u \in I$ ~~~~ but then $I = A$ contradicting assumptions.
Thus $\mathrm{dist}(u, I) \geq 1$ hence $u \notin \bar{I}$, so $\bar{I} \subsetneq A$. By cty of multiplication in A, $\bar{I}$ is also an ideal.
(ii) Let $\Xi = \{J ; I \subseteq J \subsetneq A$ an ideal$\}$
Notice, if $J \in \Xi$ then $u \notin J$. Indeed, if otherwise, we would have for a in A, $ua - a \in I \subseteq J$, in which case $a \in ua + J \subseteq J + J \subseteq J$, so $A = J$
In particular, J is modular.
Now if $\Gamma \subseteq \Xi$ is a chain wrt $\subseteq$, then $K = \bigcup_{J \in \Gamma} J$ is clearly an ideal, and $u \notin K$ so $K \in \Xi$. Therefore Γ has an upper bound.
Thus, maximal elements of Ξ exist. These are maximal ideals.
(iii) By (i), if M is a max. mod. ideal of A, then so too is $\bar{M}$, so $M = \bar{M}$. Thus $D = A/M$ is a unital Ban alg. Now D is a division ~~~~ algebra. Indeed, if not, we would have $d \neq 0$. $d \notin GL(D)$ so dD is a proper ideal in D, so $K = \{a \in A ; a + M \in dD\}$ would be a proper ideal in A, with K ~~~~ $\supseteq M$. Hence $K = M$ ~~so~~ so $d = a + M$, $a \in M$, contradicting that $d \neq 0$.
By Mazur's thm, $D \cong \mathbb{C}$, ie M is codim 1 $\square$

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Theorem: Let A be a commutative Ban alg. Then there is a bijection
$$\gamma \mapsto \ker\gamma : \Gamma_A \to \mathrm{Max}(A)$$
where
$$\mathrm{Max}(A) = \{I \subseteq A \,;\, I \text{ is a maximal modular ideal}\}$$
Proof: If $\gamma \in \Gamma_A$ then $\ker\gamma$, being codim 1, is a maximal ideal, which is modular since $A/\ker(\gamma) \cong \operatorname{ran}\gamma = \mathbb{C}$ is unital.
If $\gamma, \chi \in \Gamma_A$, $\ker(\gamma) = \ker(\chi)$ then $\gamma = \lambda\chi$ for some $\lambda \in \mathbb{C}$.
If $u \in A$ satisfies $\chi(u) = 1$, So
$$\lambda = \lambda\chi(u)^2 = \lambda\chi(u^2) = \gamma(u^2) = \gamma(u)\gamma(u)$$
$$= \lambda\chi(u)\lambda\chi(u) = \lambda^2\chi(u)^2 = \lambda^2$$
so $\lambda = \lambda^2 \Rightarrow \lambda \in \{0,1\} \Rightarrow \lambda = 1$ as $\gamma \neq 0$.
Hence $\gamma \mapsto \ker(\gamma) : \Gamma_A \to \mathrm{Max}(A)$ is injective.
Now, if $I \in \mathrm{Max}(A)$, then by Lemma (iii), $A/I \cong \mathbb{C}$. So $\gamma : A \to A/I \cong \mathbb{C}$ be the quotient map, $\gamma \in \Gamma_A$ with $I = \ker\gamma$. Hence $\gamma \mapsto \ker(\gamma) : \Gamma_A \to \mathrm{Max}(A)$ is surjective. ∎

Corollary: Let A be a comm. Ban. alg., $a \in A$. Then
(i) if A is unital, then
$$a \in GL(A) \Leftrightarrow \underbrace{\hat{a}(\Gamma_A) = \{\gamma(a) ; \gamma \in \Gamma_A\}}_{\text{this is true}} \underbrace{\not\ni 0}_{\text{this is the condition}}$$
(ii) $\sigma(a) = \hat{a}(\Gamma_A)$ if A is unital, and $\sigma(a) = \hat{a}(\Gamma_A) \cup \{0\}$ if A is non-unital.
Proof (i): Since A is unital, all proper ideals are modular
$$a \notin GL(A) \Leftrightarrow aA \text{ proper ideal}$$
$$\underset{\text{Thm + Lem (ii)}}{\Leftrightarrow} a \in \ker(\gamma),\ \gamma \in \Gamma_A \Leftrightarrow \hat{a}(\Gamma_A) \ni 0$$

(ii): If A is unital, this is immediate from (i). Indeed, for $\lambda \in \mathbb{C}$, $\lambda \in \sigma(a) \Leftrightarrow \lambda e - a \in \ker(\gamma)$ some $\gamma \in \Gamma_A \Leftrightarrow \lambda = \gamma(a)$.
If A is non-unital, then, recall
$$\sigma_A(a) = \sigma_{\tilde{A}}(a) \cup \{0\} = \hat{a}(\Gamma_{\tilde{A}}) \cup \{0\} = \hat{a}(\Gamma_A) \cup \{0\}$$
since $\Gamma_{\tilde{A}} = \tilde{\Gamma}_A \cup \{\gamma_\infty\}$ (notation of earlier proposition).

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Corollary: (Abstract characterization of uniform algebras)
If A is a comm. Ban. alg for which $\|a^2\| = \|a\|^2$, then the Gelfand transform $a \mapsto \hat{a} : A \to C_0(\Gamma_A)$ is an isometry.

Proof: Inductively, $\|a^{2^n}\| = \|a\|^{2^n}$. From (ii) in corollary above,

$$\|\hat{a}\|_\infty = r(a).$$

By Beurlings spec. rad. formula,

$$r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|$$

Hence $\|\hat{a}\|_\infty = \|a\|$. ∎

Corollary: (i) If $A = \langle a \rangle_e$, then $\gamma \mapsto \gamma(a) : \Gamma_A \to \sigma(a)$ is a homeomorphism.
(ii) If $A = \langle a \rangle$, then $\gamma \mapsto \gamma(a) : \Gamma_A \to \sigma(a) \setminus \{0\}$ is a " .

Proof: ~~∎~~
(i) If $\gamma, \chi \in \Gamma_A$, $\gamma(a) = \chi(a)$, then $\gamma(p(a)) = p(\gamma(a)) = p(\chi(a)) = \chi(p(a))$ for any poly p. Hence $\chi = \gamma$ on $\{p(a) ; p \text{ poly}\}$ so $\gamma = \chi$ on $\langle a \rangle_e$ (γ, χ is cts). Thus

$$\gamma \mapsto \gamma(a) : \Gamma_A \to \underbrace{\sigma(a)}_{\text{range thx to 2 corollaries above}}$$

is bijective, and is continuous (w^*-top), and thus a homeo
(ii) Similar to (i). Show

$$\gamma \mapsto \gamma(a) : \Gamma_A \cup \{0\} \to \sigma(a) \cup \{0\}$$

is a homeo. Thus

$$\gamma \mapsto \gamma(a) : \Gamma_A \to \sigma(a) \setminus \{0\}$$

is a homeo of (loc.) compact spaces. ∎

Remark: Let $\iota : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, $\iota(z) = z$. We saw that $A(\mathbb{D}) = \langle \iota \rangle_e$ has spectrum $\cong \overline{\mathbb{D}}$. Hence

$$\sigma_{A(\mathbb{D})}(\iota) = \overline{\mathbb{D}}$$

However, $\iota \in C(\mathbb{T}) \supseteq A(\overline{\mathbb{D}})$,

$$\sigma_{C(\mathbb{T})}(\iota) = \iota(\mathbb{T}) = \mathbb{T}$$

Proposition: Let C be a unital ~~comm.~~ Ban. alg., $A \subseteq C$ be a subalg. containing the identity. Then for $a \in A$,

not used

$$\partial\sigma_A(a) \subseteq \sigma_C(a) \subseteq \sigma_A(a).$$

Proof: We have $GL(A) \subseteq GL(C)$, which gives 2^{nd} inclusion.
To see 1^{st} inclusion, recall

$$a \in GL(A) \Longleftrightarrow L_a \in GL(B(A)) \qquad (L_a b = ab)$$

Hence if $\lambda \in \partial\sigma_A(a) = \partial\sigma_{B(A)}(L_a) \subseteq \sigma_{app}(L_a)$ (prop earlier)
Thus $\exists (a_n)_{n=1}^{\infty} \subset A$, $\|a_n\| = 1$ so

$$(\lambda e - a)a_n = (\lambda I - L_a)a_n \xrightarrow[n\to\infty]{} 0$$

But, then $\lambda \in \sigma_{app, B(C)}(L_a) \subseteq \sigma_{B(C)}(L_a) = \sigma_C(a)$. ∎

Proposition: If A is a unital Ban. alg., $M \subseteq A$ is a maximal comm subalg, (if $a \in A \setminus M$, then a does not commute with all elements of M.) Then M is closed and contains identity, and for $a \in M$,

$$\sigma_M(a) = \sigma_A(a)$$

Proof: M is closed by cty on mult. on A. ~~Clearly~~ Obviously $e \in M$.
If $b \in GL(A) \cap M$ then for $a \in M$,

$$ab = ba \Rightarrow b^{-1}a = ab^{-1}$$

so $GL(A) \cap M = GL(M)$. Thus $\sigma_M(a) = \sigma_A(a)$ for a in M. ∎

Remark: If A is a Ban. alg, $a \in A$, then there exists a maximal comm. ~~abelian~~ subalg. $M \ni a$.
(Use a Zorn's lemma argument.)

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Proposition: Let K be a compact Hausdorff space. Then

(†) $$\mathrm{Max}(C(K)) = \{k(\{x\});\ x \in K\}.$$

Hence

$$\Gamma_{C(K)} = \{\delta_x;\ x \in K\}$$

where $\delta_x(f) = f(x)$. Moreover, $x \mapsto \delta_x : K \to \Gamma_{C(K)}$ is a homeomorphism.

Proof: Once (†) is established, the structure of $\Gamma_{C(K)}$ follows from the previous theorem. Furthermore, $x \mapsto \delta_x : K \to \Gamma_{C(K)}$ is cts (w^*-top on $\Gamma_{C(K)}$) and is a bijection (injective by Urysohn's lemma), and hence is a homeomorphism.
Let us show (†). Let $I \subseteq C(K)$ be an ideal. Suppose for each x in K, there is $f_x \in I$ st $f_x(x) \neq 0$. Let

$$U_x = f^{-1}(\mathbb{C} \setminus \{0\}),$$

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so U_x is open in K, and

$$K \subseteq \bigcup_{x\in K} U_x$$

by supposition. Hence there is $\{U_{x_0}, \ldots, U_{x_n}\}$ covering K. Thus if

$$f = \sum_{i=1}^{n} |f_{x_i}|^2 = \sum_{i=1}^{n} \overline{f_{x_i}} \underbrace{f_{x_i}}_{\in I} \in I$$

then $f(K) \not\ni 0$. Hence $\frac{1}{f} \in C(K)$, so $1 = \frac{1}{f} f \in I$, so $I = C(K)$. Thus, if $I \subsetneq C(K)$, there must be x in K so $f(x) = 0$ for all f in I, ie $I \subseteq k(\{x\})$. It is clear that $\ker(\{x\}) = \ker(\delta_x)$ is a maximal ideal ■

Remark: The same proof shows that

$$\Gamma_{C[0,1]} = \{\delta_t \,;\, t \in [0,1]\} \underset{\text{homeomorphic}}{\cong} [0,1].$$

Corollary: If X is a l.c. Hausdorff space, then

$$\Gamma_{C_0(X)} = \{\delta_x \,;\, x \in X\} \cong X$$

Proof: $\widetilde{C_0(X)} \cong C(X_0)$. Hence

$$\Gamma_{C_0(X)} \cong \Gamma_{\widetilde{C_0(X)}} = \Gamma_{C(X_0)} \setminus \{\delta_\infty\} \cong X_\infty \setminus \{\infty\} \cong X. ■$$

Semi-simplicity

Def) Let A be a comm. Ban. alg. We call A semi-simple if the Gelfand map $a \mapsto \hat{a} : A \to C_0(\Gamma_A)$ is injective, ie if $a \in A$ has $\gamma(a) = 0 \ \forall \gamma \in \Gamma_A$ then $a = 0$.

Examples (i) X l.c.H. space, $C_0(X)$ is semisimple
(i') any uniform alg is semi-simple

(ii) Let for a Ban sp X, $xy=0$ $\forall x,y$ in X. This is not semi-simple since $\Gamma_X = \phi$.
(iii) $A = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} ; a, b \in \mathbb{C} \right\} \subseteq M_2$

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}^2 = 0 \quad \text{so for } \gamma \in \Gamma_A , \ \gamma \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = 0$$

Thus $\left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, b \in \mathbb{C} \right\} \subseteq$ kernel of the Gelfand transform.

Theorem (Automatic Continuity) with B semi-simple
If A, B are comm. Ban. alg. and $\Phi : A \to B$ a homomorphism, then Φ is automatically bounded.

Proof: We shall the closed graph theorem. Let $(a_n)_{n=1}^{\infty} \subset A$ be st $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} \Phi(a_n) = b$ in B.
If $\gamma \in \Gamma_B$, then $\gamma \circ \Phi \in \Gamma_A$ so $\|\gamma \circ \Phi\| \le 1$. Since $\|\gamma\| \le 1$ we have

$$\gamma(b) = \lim_n \gamma \circ \Phi (a_n) = 0.$$

Hence by semi-simplicity of B, $b = 0$. ☒

Corollary: (uniqueness of norm)
If $(A, \|\cdot\|)$ is a comm semi-simple Ban. alg and $\|\cdot\|'$ is any norm by which $(A, \|\cdot\|')$ is a Banach alg, then $\|\cdot\|' \sim \|\cdot\|$ (eq. of norms).

Proof: Consider $id : (A, \|\cdot\|') \to (A, \|\cdot\|)$ which is a homomorphism.
By the theorem above, $\|\cdot\| \le \|id\| \|\cdot\|'$.
Since id is a bijection, the open mapping theorem tells us that id is bounded below. ☒

Corollary. Let X, Y be locally compact Hausdorff spaces Then TFAE
(i) $X \cong Y$ (homeomorphic)
(ii) $C_0(X) \cong C_0(Y)$ (isometrically isomorphic)
(iii) $C_0(X) \cong C_0(Y)$ (isomorphic as algebras)

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Proof: (i)⇒(ii): Suppose $\varphi: Y \to X$ is a homeomorphism. Notice that $\varphi(K)$ is compact in X for compact $K \subseteq Y$. Hence
$$f \mapsto f\circ\varphi : C_c(X) \to C_c(Y)$$
is a homomorphism and $\|f\circ\varphi\|_\infty = \|f\|_\infty$. Hence, by unique extension
$$f \mapsto f\circ\varphi : C_0(Y) \to C_0(X)$$
is an isometric isomorphism.

(ii)⇒(iii): Obvious

(iii)⇒(i): If $\Phi: C_0(X) \to C_0(Y)$ is an alg iso., then Φ is bdd, by the last thm. Furthermore $\Phi^*(\Gamma_{C_0(Y)}) \subseteq \Gamma_{C_0(X)}$ ie $\Phi^*\gamma = \gamma\circ\Phi$ is a character if γ is a character. Check that $\Phi^*(\Gamma_{C_0(Y)}) = \Gamma_{C_0(X)}$, Φ^* is w^*-w^* cts, and hence induces
$$\varphi: Y \to X,$$
which is a homeomorphism ($\Phi(\delta_y) = \delta_{\varphi(y)}$). ∎

Corollary: The alg. $C^\infty[0,1]$ of infinitely differentiable functions on $[0,1]$ admits no norm making it a Ban alg

Proof: Suppose $(C^\infty[0,1], \|\cdot\|)$ is a Ban alg. Then since $C[0,1]$ is semi-simple, the formal identity
$$\iota: (C^\infty[0,1], \|\cdot\|) \to (C[0,1], \|\cdot\|_\infty)$$
is cont., for $f \in C^\infty[0,1]$,
$$\|f\|_\infty \le \|\iota\|\,\|f\|.$$
Consider the operator $f \mapsto f'$ on $C^\infty[0,1]$, which we will show must be bounded.

Say $(f_n)_{n=1}^\infty \subset C^\infty[0,1]$ is st
$$\|f_n\| \xrightarrow{n\to\infty} 0$$
$$\|f_n' - g\| \xrightarrow{n\to\infty} 0$$
Notice that
$$\|f_n\|_\infty \xrightarrow{n\to\infty} 0$$
$$\|f_n' - g\|_\infty \xrightarrow{n\to\infty} 0$$
If $x, y \in [0,1]$
$$\left|\int_y^x g\right| \le \left|\int_y^x f_n'\right| + \left|\int_y^x (g - f_n')\right|$$
$$\underset{\text{FTC}}{\le} |f_n(x) - f_n(y)| + |x-y|\,\|g - f_n'\|_\infty$$

$$\leq 2\|f_n\|_\infty + |x-y|\,\|g - f_n'\|_\infty \longrightarrow 0$$

Hence $g = 0$ (as it is cts). Thus by closed graph theorem, $\|f'\| \leq m\|f\|$, for $f \in C^\infty[0,1]$.
Let ~~f~~

$$f(t) = e^{2mt} \quad t \in [0,1].$$

So $f \in C^\infty[0,1]$, but

$$\|f'\| = \|2mf\| = 2m\|f\| > m\|f\|,$$

a contradiction. ∎

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Def) Let A be a commutative Ban alg. Then A is called regular if for each ~~compact~~ (closed) $K \subseteq \Gamma_A$, and each $\gamma \in \Gamma_A \setminus K$ there exists a in A $(\gamma(a) =)\ \hat{a}(\gamma) \neq 0$, $\hat{a}|_K = 0$.

Eg X l.c. H. space, $C_0(X)$ is regular (Uryzohn's lemma)

Theorem: The Ban. alg. $\ell^1(\mathbb{Z})$ is regular.

Proof: Recall

$$\Gamma_{\ell^1(\mathbb{Z})} \cong \hat{\mathbb{Z}} \cong \mathbb{T}$$

Let

$$A(\mathbb{T}) = \widehat{\ell^1(\mathbb{Z})} \subseteq C(\mathbb{T})$$

Let

$$\pi : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \longrightarrow \ell^1(\mathbb{Z})$$
$$\pi(g, h) = g\bar{h}.$$

Notice

$$\|\pi(g,h)\|_1 \leq \|g\|_2 \|h\|_2$$

(by C.-S. or Hölder). [Also π is surjective: $f = \pi(\operatorname{sgn} f\, |f|^{1/2}, |f|^{1/2})$.]
We note in $L^2(\mathbb{T})$ - here Lebesgue measure is given

$$\int_{\mathbb{T}} f(z)\,dm(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,dt$$

- that $\beta = (z \mapsto z^n)_{n\in\mathbb{Z}}$ is an o.n. basis [first span $\beta \subset C(\mathbb{T})$

is dense by Stone-Weierstrass, hence $\|\cdot\|_2$-dense in $L^2(\mathbb{T})$, and hence maximal o.n. set] Hence we have the Plancherel unitary,

$$U : \ell^2(\mathbb{Z}) \longrightarrow L^2(\mathbb{T})$$
$$(U(g))(z) = \textstyle\sum_{n\in\mathbb{Z}} g(n) z^n \quad \text{a.e. } z \text{ in } \mathbb{T}$$

Now define

$$\lambda : \mathbb{T} \longrightarrow B(L^2(\mathbb{T}))$$
$$\lambda(z) g(w) = g(\bar{z}w) \quad z\in\mathbb{T} \text{ a.e. } w\in\mathbb{T} \quad (\text{left regular rep.})$$
$$\hat{\lambda} : \mathbb{T} \longrightarrow B(\ell^2(\mathbb{Z}))$$
$$\hat{\lambda}(z) g(n) = \bar{z}^n g(n) \quad n\in\mathbb{Z} \quad z\in\mathbb{T}$$

We have for g in $\ell^2(\mathbb{Z})$, z in $\mathbb{T}$, a.e. w in $\mathbb{T}$

$$(\lambda(z) U g)(w) = Ug(\bar{z}w) = \sum_{n\in\mathbb{Z}} g(n)(\bar{z}w)^n$$

$$= (U \hat{\lambda}(z) g)(w)$$

so $U^* \lambda(z) U = \hat{\lambda}(z)$, (this $*$ is in the context of Hilbert spaces, for the first time.)

Now let $K, F \subseteq \mathbb{T}$ be compact, $K \cap F = \emptyset$, $\varepsilon > 0$. Let

$$K_\varepsilon = \{z\in\mathbb{T} ;\ \mathrm{dist}(K,z) < \varepsilon\} \text{ and}$$
$$V_\varepsilon = \{z\in\mathbb{T} ;\ |z-1| < \varepsilon\}$$

These are open, hence of positive Lebesgue measure. Let $g,h \in \ell^2(\mathbb{Z})$ st

$$Ug = 1_{K_\varepsilon} \qquad Uh = \tfrac{1}{2\varepsilon} 1_{V_\varepsilon}$$

Now if $f = \pi(g,h)$. We have $\hat{f} \in A(\mathbb{T})$ that for z in $\mathbb{T}$

$$\hat{f}(z) = \sum_{n\in\mathbb{Z}} f(n) z^n = \sum_{n\in\mathbb{Z}} g(n) \overline{\bar{z}^n h(n)}$$

$$= \langle g \mid \hat{\lambda}(z) h \rangle_{\ell^2(\mathbb{Z})} \quad (\text{inner prod})$$

$$= \langle Ug \mid U\hat{\lambda}(z) h \rangle_{L^2(\mathbb{T})} \quad (\text{property of unitary})$$

$$= \langle \underbrace{Ug}_{1_{K_\varepsilon}} \mid \lambda(z) \underbrace{Uh}_{\frac{1}{2\varepsilon} 1_{V_\varepsilon}} \rangle_{L^2(\mathbb{T})}$$

$$= \frac{1}{2\varepsilon} \int_{\mathbb{T}} 1_{K_\varepsilon}(w) \overline{\lambda(z) 1_{V_\varepsilon}(w)}\, dm(w)$$

→ is real so bar disappears

$$= \frac{1}{2\varepsilon} \int_{\mathbb{T}} 1_{K_\varepsilon}(w) 1_{zV_\varepsilon}(w)\, dm(w)$$

$$= \frac{1}{2\varepsilon}\int_{\mathbb{T}} 1_{K_\varepsilon \cap zV_\varepsilon}(\omega)\, dm(\omega)$$

$$= \frac{m(K_\varepsilon \cap zV_\varepsilon)}{2\varepsilon},$$

notice that this is

$$= \begin{cases} 1 & \text{if } z \in K \quad (\text{i.e. } zV_\varepsilon \subseteq K_\varepsilon) \\ 0 & \text{if } z \notin K_{2\varepsilon} \quad (\text{check this}) \end{cases}$$

Thus if $\varepsilon < \frac{1}{2}\operatorname{dist}(K,F)$, then $\hat{f}|_K = 1$, $\hat{f}|_F = 0$. ∎

Remark: $\ell^1(\mathbb{Z})$ is semi-simple. Prove later, with aid $\mathbb{C}^*$-algebras

<u>Hull-kernel topology</u>

Let S be a set. A <u>(Kuratowski) closure</u> operation is a map

$$E \longmapsto \overline{E} : \mathcal{P}(S) \to \mathcal{P}(S)$$

which satisfies

- $\overline{\emptyset} = \emptyset$
- $E \subseteq \overline{E}$
- $\overline{\overline{E}} = \overline{E}$
- $\overline{E \cup F} = \overline{E} \cup \overline{F}$

Then

$$\tau = \{S \setminus \overline{E} \,;\, E \in \mathcal{P}(S)\} \subseteq \mathcal{P}(S)$$

is a topology.

Now let A be a comm. Ban. alg. If $E \subseteq \Gamma_A$ we let its <u>kernel</u> be given by

$$k(E) = \bigcap_{\gamma \in E} \ker(\gamma) \qquad (\text{closed ideal in } A) \qquad (\text{let } \cap\emptyset = A)$$

If I is an ideal in A, we let its hull be given by

$$h(I) = \{\gamma \in \Gamma_A' \,;\, I \subseteq \ker(\gamma)\}$$

Note, I is modular then $h(I) \neq \emptyset$.

If $E \subseteq \Gamma_A$ we let its <u>hull-kernel closure</u> be given by

$$\overline{E}^{hk} = h(k(E))$$

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Proposition:

(o) $\overline{E}^{hk}$ is w^*-closed in Γ_A

(i) $\overline{\phi}^{hk} = \phi$

(ii) $E \subseteq \overline{E}^{hk}$

(iii) $\overline{\overline{E}^{hk}}^{hk} = \overline{E}^{hk}$

(iv) $\overline{E \cup F}^{hk} = \overline{E}^{hk} \cup \overline{F}^{hk}$

Proof: (o), (i), (ii) easy exercises

(iii): Observe that

$$\gamma \in \overline{E}^{hk} \iff \ker(\gamma) \supseteq \bigcap_{\chi \in E} \ker(\chi)$$

Thus

$$k(E) = \bigcap_{\chi \in E} \ker(\chi) = \bigcap_{\chi \in \overline{E}^{hk}} \ker(\chi) = k(\overline{E}^{hk})$$

and hence

$$\underset{\overline{E}^{hk}}{\overset{}{h(k(E))}} = h(k(\overline{E}^{hk})) = \overline{\overline{E}^{hk}}^{hk}$$

(iv) This follows from the elementary facts that

$$k(E \cup F) = k(E) \cap k(F),$$

$$h(I) \cap h(J).$$

Proposition: A reg. comm. Ban. alg. then for any $E \subseteq \Gamma_A$, $\overline{E}^{hk} = \overline{E}^{w^*}$. Hence hull-kernel top. coincides with usual w^*-topology on Γ_A.

Proof: We have $\supseteq$ by (o) above

~~Let us consider~~

If $\gamma \in \Gamma_A \setminus \overline{E}^{w^*}$ then regularity provides a in A with $\hat{a}|_E = 0$ (hence $\hat{a}|_{\overline{E}^{w^*}} = 0$) and $\hat{a}(\gamma) \neq 0$. Thus $a \in k(E)$, but $\ker(\gamma) \not\ni a$, so $\ker(\gamma) \not\supseteq k(E)$. ∎ 2015 02 03

Remark: In fact, if hull-kernel top agrees with w^*-top on Γ_A, then A is necessarily regular. (Proof is harder.)

Ex: Consider $A(\mathbb{D})$. If $C \subseteq \overline{\mathbb{D}} \cong \Gamma_{A(\mathbb{D})}$ admits a cluster point inside $\mathbb{D}$ then $k(C) = \{0\}$ since any holomorphic function on $\mathbb{D}$ which vanishes on C vanishes on all of $\mathbb{D}$. Hence

$$\overline{C}^{hk} = h(k(C)) \underset{\uparrow}{=} h(\{0\}) = \overline{\mathbb{D}}$$

(slight abuse of notation)

even if C is closed, $C \subsetneq \overline{\mathbb{D}}$.

Consequences:

(i) $A(\mathbb{D})$ is not regular.

(ii) $\ell^1(\mathbb{N}_0)$ is not regular

[$f \longmapsto \sum_{n=0}^{\infty} f(n) \iota^n$ $(\iota(z) = z)$ is a contractive homomorphism from $\ell^1(\mathbb{N}_0)$ into $A(\mathbb{D})$, with dense range. Also $\Gamma_{\ell^1(\mathbb{N}_0)} \cong \overline{\mathbb{D}}$. If $\ell^1(\mathbb{N}_0)$ were regular, this would immediately imply regularity of $A(\mathbb{D})$.]

Def) Let A be a comm. Ban. alg., a closed (w^*) set $E \subseteq \Gamma_A$ is said to be of spectral synthesis for A if the only closed ideal I of A for which $h(I) = E$ is $I = k(E)$.

Notice $k(E)$ is the unique ideal with hull E, hence E can be unambiguously "synthesized" from A.

Ex: Consider in $C^1[0,1]$ the ideal

$$I = \{ f \in C^1[0,1] ; f(\tfrac{1}{2}) = 0 = f'(\tfrac{1}{2}) \}$$

Note that I is an ideal: $f \in I$, $g \in C^1[0,1]$

$$(fg)(\tfrac{1}{2}) = 0, \quad (fg)'(\tfrac{1}{2}) = f'g(\tfrac{1}{2}) + fg'(\tfrac{1}{2}) = 0$$

and I is evidently a subspace. Also, I is closed: $(f_n)_{n=1}^{\infty} \subset I$, $\lim_{n\to\infty} f_n = f$ in $C^1[0,1]$, then

$$0 = \lim_n \|f_n - f\|_{C^1} = \lim_n (\|f_n - f\|_\infty + \|f_n' - f'\|_\infty)$$

and hence

$$f(\tfrac{1}{2}) = \lim_n f_n(\tfrac{1}{2}) = 0, \quad f'(\tfrac{1}{2}) = \lim_n f_n'(\tfrac{1}{2}) = 0.$$

Recall that $\Gamma_{C^1[0,1]} \cong [0,1]$. We observe that $h(I) = \{\tfrac{1}{2}\}$. Indeed, $g(t) = (t - \tfrac{1}{2})^2$, then $g \in I$, and $g(t) = 0$ only if $t = \tfrac{1}{2}$.

On the other hand, $k(\{\tfrac{1}{2}\}) \supsetneq I$ since $h(t) = t - \tfrac{1}{2}$ then $h \in k(\{\tfrac{1}{2}\}) \setminus I$

Conclusion: $\{\tfrac{1}{2}\}$ is not of spectral synthesis for $C^1[0,1]$. ∎

2015 02 03

Goal: In $C_0(X)$, every closed set is of spectral synthesis.

<u>Tietze Extension Theorem:</u>
Let X be a l.c.H space, $L \subseteq X$ compact and $f \in C(L)$. Then there exists $\tilde{f} \in C_c(X)$ such that $\tilde{f}|_L = f$ and $\|\tilde{f}\|_\infty = \|f\|_\infty$.

Proof: First, $f = \mathrm{Re} f + i \,\mathrm{Im} f$. Hence we may assume that f is $\mathbb{R}$-valued. Also, since f is bounded, by scaling, we may assume $f : X \to [-1,1]$. ← X should be L?
<u>(I)</u> If $f : L \to [-a, a]$ $(a > 0)$ is cont., then there is $g : X \to [-a/3, a/3]$ which is cont., $\mathrm{supp}(g)$ is compact and so $\|f - g|_L\|_\infty \leq \frac{2}{3} a$.
Indeed, let

$$L_+ = f^{-1}([a/3, a]) \quad \text{and} \quad L_- = f^{-1}([-a, -a/3]),$$

so L_-, L_+ are compact in L, hence in X, and disjoint. Thus, by Uryzohn's lemma, there is $h \in C_c(X)$ such that $h|_{L_+} = 1$, $h|_{L_-} = 0$ and $h(X) \subseteq [0,1]$. Let $u \in C_c(X)$ be such that $u|_L = 1_L$ (Uryzohn's lemma). Let

$$g = \tfrac{1}{3}(2ah - a 1_X) u \in C_c(X).$$

Check that g works.
<u>(II)</u> We suppose $f : L \to [-1,1]$ and one of $-1, 1 \in f(L)$. Let g_1 be as in (I), $a = 1$. Let g_2 be as in (I) to $f - g_1$ (in place of f), with $a = \frac{2}{3}$. ... Let g_n be as in (I) to $f - (g_1 + \cdots + g_{n-1})$ (in place of f) with $a = (2/3)^n$. Let

$$\tilde{f} = \sum_{n=1}^{\infty} g_n \in C_0(X)$$

(Weierstrass M-test) as $\|g_n\|_\infty \leq (1/3)^n$. Also

$$\Big\| f - \sum_{n=1}^{N} g_n|_L \Big\|_\infty \leq \left(\frac{2}{3}\right)^N \xrightarrow{N \to \infty} 0. \qquad \blacksquare$$

Corollary: Let X be a l.c.H. space, $E \subseteq X$ is closed and not compact.
(i) If $f \in C_0(E)$, there exists $\tilde{f} \in C_0(X)$, $\tilde{f}|_E = f$ and $\|\tilde{f}\|_\infty = \|f\|_\infty$.
(ii) $C_0(X)/k(E) \cong C_0(E)$.

Proof:
(i) Plainly, $E_\infty = E \cup \{\infty\} \subseteq X_\infty$. If $f \in C_0(E) \subseteq C(E_\infty)$ then $f(\infty) = 0$. We let $\tilde{f} \in C(X_\infty)$ satisfy $\tilde{f}|_{E_\infty} = f$ and $\|\tilde{f}\|_\infty = \|f\|_\infty$. Notice that $\tilde{f}(\infty) = f(\infty) = 0$, so $\tilde{f} \in C_0(X)$.

(ii) Let $R_E : C_0(X) \to C_0(E)$ be the restriction map, $R_E f = f|_E$. This is clearly a contractive homomorphism, which is surjective by (i), with, if $g \in C_0(E)$, $g = R_E \tilde{g}$ with $\|\tilde{g}\|_\infty = \|g\|_\infty$. We note that $k(E) = \ker(R_E)$. Use the Banach space first isomorphism theorem. ∎

05 cont'd

2015 02 06
TA lecture

Theorem: X l.c.H. space
(i) Let $I \lhd C_0(X)$ be such that $h(I) = \emptyset$. Then $C_c(X) \subseteq I$.
(2) Every closed ideal of $C_0(X)$ is of the form $\ker(E)$ for some closed subset $E \subseteq X$.
Moreover, if $E \neq F$ closed subsets of X, then $\ker(E) \neq \ker(F)$.

Proof: (i) $h(I) = \emptyset \Rightarrow \forall x\ \exists f_x \in I$ st $f_x(x) \neq 0$.

$x \in f_x^{-1}(\mathbb{C}\setminus\{0\}) \Rightarrow \{f_x^{-1}(\mathbb{C}\setminus\{0\})\}$ open cover for X.

Let K be a compact subset of X. $\exists x_1, \ldots, x_n$ st

$$K \subseteq \bigcup_{i=1}^{n} f_{x_i}^{-1}(\mathbb{C}\setminus\{0\})$$

Let

$$f = \sum_{i=1}^{n} |f_{x_i}|^2 = \sum_{i=1}^{n} f_{x_i}\overline{f_{x_i}} \in I.$$

$\forall x \in K$, $\exists j$ st $f_{x_j}(x) \neq 0 \Rightarrow f(x) \neq 0$. So $f|_K$ never assumes 0.

$$g = \frac{1}{f|_K} \in C(K)$$

$$\xRightarrow{\text{Tietze's thm}} \exists\, \tilde{g} \in C_0(X) \text{ with } \tilde{g}|_K = g.$$

Look at $fg \in I$ and $fg|_K \equiv 1$.
For any $h \in C_c(X)$ with $\mathrm{supp}(h) \subseteq K$ then

$$h = hfg \in I \Rightarrow C_c(X) \subseteq I.$$

(ii) Let $I \lhd C_0(X)$ be a closed ideal. Let $h(I) = E$. We consider the map:
For any closed subset E of X, define

$$\ker(E) \xrightarrow{\psi} C_0(X\setminus E)$$
$$f \longmapsto f|_{X\setminus E}$$

isometric isomorphism (exercise).
going back to I:

$h(I) = E \Rightarrow I \subseteq \ker(E)$

$\psi(I) \subseteq C_0(X\setminus E)$ closed ideal

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$h(\psi(I)) = \phi \Rightarrow$ by part ①
$C_c(X \setminus E) \subseteq \psi(I) \Rightarrow \psi(I) = C_0(X \setminus E)$
$\Rightarrow I = \ker(E)$

Let F and E are two distinct closed subsets of X.
By Urysohn's lemma, $\ker(E) \neq \ker(F)$. ∎

Corollary: X l.c.H. space $E \subseteq X$ closed subset, then E is a set of spectral synthesis.

Proof: $\ker(E)$ is the unique closed ideal of $C_0(X)$ which has E as its hull ∎

Vector-valued Riemann Integrals

$[a,b] \subseteq \mathbb{R}$, X: Banach space

$$S([a,b], X) = \text{span}\{1_I x;\ I \subseteq [a,b] \text{ an interval}, x \in X\}$$

$$S([a,b], X) \subseteq \ell^\infty([a,b], X) = \{f : [a,b] \to X ; \sup\{\|f(t)\| ; t \in [a,b]\} < \infty\}$$

Every $\varphi \in S([a,b], X)$ can be represented as

$$\varphi = \sum_{i=1}^n x_i 1_{I_i} \quad \text{with} \quad \bigsqcup_{i=1}^n I_i = [a,b]$$

Define

$$\int_a^b \varphi = \sum_{i=1}^n \ell(I_i) x_i$$

Note. (a) If

$$\varphi = \sum_{i=1}^n a_i 1_{I_i} \qquad \bigsqcup_{i=1}^n I_i = [a,b]$$

and $$\varphi = \sum_{j=1}^m b_j 1_{I_j'} \qquad \bigsqcup_{j=1}^m I_j' = [a,b]$$

then

$$\sum_{i=1}^n a_i \ell(I_i) = \sum_{j=1}^m b_j \ell(I_j').$$