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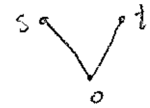
Invertibility & Spectrum

A Banach algebra is unital if it admits an identity. Depending on context, we write e, e_A, 1, or I. The identity is unique ($e' = ee' = e$).

Remarks: Since $\|a\| = \|ae\| \le \|a\|\|e\|$, we have $\|e\| \ge 1$.

(i) It is possible that $\|e\| > 1$.

Let $S = \{s, t, o\}$ with relations:



$$x^2 = x,\ x \in S,\ ox = o = xo;\ st = o = ts$$

In $\ell^1(S)$ we have $e = \delta_s + \delta_t - \delta_o$, $\|e\|_1 = 3$

(ii) It is always possible to renorm a unital Banach algebra A so that $\|e\|_{new} = 1$.

We define for $a \in A$, $L_a \in \mathcal{B}(A)$, by $L_a(b) = ab$. Then $a \mapsto L_a : A \to \mathcal{B}(A)$ is a contractive homomorphism ($\|L_a\| \le \|a\|$, $L_{ab} = L_a L_b$) with $\|L_e\| = 1$ since $L_e = I$.

For $a \in A$ let $\|a\|_L = \|L_a\|$.

We saw $\|a\|_L \le \|a\|$. Conversely

$$\frac{1}{\|e\|}\|a\| = \left\|L_a\left(\frac{1}{\|e\|}e\right)\right\| \le \|L_a\| = \|a\|_L .$$

An element a in a unital Banach algebra A is invertible if there is an element a^{-1} in A st $aa^{-1} = e = a^{-1}a$. Inverses are unique ($a' = a'aa^{-1} = a^{-1}$). We let

$$GL(A) = \{a \in A ;\ a \text{ is invertible}\}$$

Note that $GL(A)$ is a group with identity e.

Remark: If an element admits a one-sided inverse, it is not necessarily invertible.

ex $A = \mathcal{B}(\ell^p)$, $(1 \le p \le \infty)$

Let $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ (unilateral shift)

If $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ then $TS = I \ne ST$ (check)

If $P(x_1, x_2, \ldots) = (x_1, 0, 0, \ldots)$ then $T(S+P) = I$, so 'right inverse' isn't unique.

Proposition: Let A be a unital Banach algebra. Then for $a \in A$ we have $a \in GL(A)$ if and only if $L_a \in GL(\mathcal{B}(A))$.
Moreover, $L_a^{-1} = L_{a^{-1}}$.
Proof: ($\Rightarrow$) obvious.
($\Leftarrow$): Let for $a \in A$, $R_a(b) = ba$. Warning: $a \mapsto R_a : A \to \mathcal{B}(A)$ is not necessarily a homomorphism, $R_a R_b = R_{ba}$. By associativity, $L_a R_b = R_b L_a$, $a,b \in A$. Hence, if $L_a \in GL(\mathcal{B}(A))$, we have $R_b L_a^{-1} = L_a^{-1} R_b$. Thus, if $a' = L_a^{-1}(e)$, we have

$$aa' = L_a L_a^{-1}(e) = e$$
$$a'a = R_a a' = R_a L_a^{-1}(e) = L_a^{-1} R_a(e) = L_a^{-1}(a) = L_a^{-1} L_a(e) = e. \ \blacksquare$$

Invertibility Theorem: Let A be a unital Banach algebra (with $\|e\|=1$).
(i) If $a \in A$, with $\|a\|<1$, then $e-a \in GL(A)$ with

$$(e-a)^{-1} = \sum_{k=0}^{\infty} a^k. \qquad (a^0 = e)$$

(ii) If $a \in GL(A)$, and $b \in A$, with $\|b-a\| < \|a^{-1}\|^{-1}$ then $b \in GL(A)$ with

(*) $$b^{-1} - a^{-1} = \sum_{n=1}^{\infty} \left(a^{-1}(a-b)\right)^n a^{-1}$$

In particular, $GL(A)$ is open in A and $a \mapsto a^{-1} : GL(A) \to GL(A)$ is continuous.

Proof: (i) We have $\|a^k\| \le \|a\|^k$ (induction) and since $\|a\|<1$ we have

$$\sum_{k=0}^{\infty} \|a^k\| \le \sum_{k=0}^{\infty} \|a\|^k < \infty$$

so it follows that

$$\sum_{k=0}^{\infty} a^k = \lim_{n\to\infty} \sum_{k=0}^{n} a^k \in A$$

We have

$$(e-a)\sum_{k=0}^{\infty} a^k = \sum_{k=0}^{\infty}(a^k - a^{k+1}) = e - a + a - a^2 + a^2 - a^3 + \cdots = e - \underbrace{a^n}_{\|\cdot\| \xrightarrow{n\to\infty} 0} + \underbrace{\sum_{k=n}^{\infty}(a^k - a^{k+1})}_{\|\cdot\| \xrightarrow{n\to\infty} 0 \text{ (check)}} = e$$

Likewise, $\left(\sum_{k=0}^{\infty} a^k\right)(e-a) = e$ too.
(ii) We write

$$b = a - (a-b) = \underbrace{a}_{\in GL(A)} \underbrace{(e - \underbrace{a^{-1}(a-b)}_{\|\cdot\|<1})}_{\in GL(A)} \in GL(A)$$

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Also,

$$b^{-1} = \sum_{n=0}^{\infty} \left(a^{-1}(a-b)\right)^n a^{-1}$$

from above, giving (*). Hence

$$\|b^{-1} - a^{-1}\| \leq \sum_{n=1}^{\infty} \|a^{-1}\|^n \|b-a\|^n \|a^{-1}\|^n$$

$$= \frac{\|a^{-1}\| \cdot \|a^{-1}\| \cdot \|b-a\|}{1 - \|a^{-1}\| \cdot \|b-a\|} \xrightarrow{b \to a} 0$$ ▨

Corollary: If $(a_n)_{n=1}^{\infty} \subset GL(A)$, $a = \lim_{n\to\infty} a_n$ and $\sup_{n \in \mathbb{N}} \|a_n^{-1}\| < \infty$. Then $a \in GL(A)$.

Proof: Let $M = \sup_{n\in\mathbb{N}} \|a_n^{-1}\|$. Then for large enough n, we have

$$\|a_n - a\| < \frac{1}{M} \leq \frac{1}{\|a_n^{-1}\|}$$

and so $a \in GL(A)$. ▨

Def] If A is a unital Banach algebra, $a \in A$, then the <u>spectrum</u> of a in A is

$$\sigma_A(a) = \sigma(a) = \{\lambda \in \mathbb{C} ; \lambda e - a \notin GL(A)\}.$$

ex (I) Let $A = M_n \cong B(\mathbb{C}^n)$, $\mathbb{C}^n$ admits any norm

Then for $a \in A$, $\sigma(a) = \{\lambda \in \mathbb{C} ; \ker(\lambda I - a) \supsetneq \{0\}\}$. ie eigenvalues

(II) $A = C(K)$, K compact Hausdorff space. We note that

$$GL(A) = \{f \in C(K) ; 0 \notin f(K)\}.$$

ie $f(K)$ is compact, so $\min_{x\in K} |f(x)| > 0$ so $\frac{1}{f} \in C(K)$ if $f(K) \not\ni 0$.

Thus we find for $f \in C(K)$, $\sigma(f) = f(K)$.

(III) Fix a Banach space X, and consider $A = B(X)$.

Recall: (Open Mapping Theorem) If $S \in B(X)$ admits an inverse operator $S^{-1} : X \to X$ then S^{-1} is linear and bounded.

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(III) $\mathcal{B}(X)$, X Banach space

<u>Def</u>/ If $Y \subseteq X$ is a subspace we define its <u>annihilator</u>
$$Y^{\perp} = \{f \in X^*;\ f|_Y = 0\}.$$
If $Z \subseteq X^*$ is a subspace, define its <u>pre-annihilator</u> by
$$Z_{\perp} = \{x \in X;\ f(x) = 0\ \forall f \in Z\}.$$

Notes: If $\kappa: X \to X^{**}$ denotes the canonical embedding by evaluation functionals, then
$$Y^{\perp} = \bigcap_{y \in Y} \underbrace{\ker(\kappa(y))}_{w^*\text{-closed}}$$
so is a weak*-closed subspace of X^*. Using the geometric form of the Hahn-Banach theorem
$$(Y^{\perp})_{\perp} = \overline{Y}, \quad (Z_{\perp})^{\perp} = \overline{Z}^{w^*} \qquad (*)$$

(Kernel-annihilator formulas)
If $T \in \mathcal{B}(X)$ then
$$\begin{aligned}\ker(T) &= \{x \in X;\ Tx = 0\} \\ &= \{x \in X;\ T^*g(x) = g(Tx) = 0\ \forall g \in X^*\} \\ &= (\operatorname{ran} T^*)_{\perp} \\ \ker(T^*) &= \{f \in X^*;\ f(Tx) = T^*f(x) = 0 \quad \forall x \in X\} \\ &= (\operatorname{ran} T)^{\perp}\end{aligned}$$
Hence using $(*)$, we see that
$$\begin{aligned}\ker T^* = \{0_{X^*}\} &\iff (\operatorname{ran} T)^{\perp} = \{0_{X^*}\} \\ &\iff \overline{\operatorname{ran} T} = \{0_{X^*}\}_{\perp} = X\end{aligned}$$
and likewise
$$\ker(T) = \{0_X\} \iff \overline{\operatorname{ran} T^*}^{w^*} = X^*$$

Theorem. If $T \in \mathcal{B}(X)$ then the following are equivalent
(i) T is invertible
(ii) $T^* \in \mathcal{B}(X^*)$ is invertible
(iii) $\overline{\operatorname{ran} T} = X$ and $\inf\{\|Tx\|,\ x \in X,\ \|x\| = 1\} > 0$
(We call the latter condition <u>bounded below</u>).

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Proof

(i)$\Rightarrow$(ii): Recall for $T,S\in\mathcal{B}(X)$, then $(ST)^* = T^*S^*$.
Check that $(T^{-1})^*$ serves as $(T^*)^{-1}$. (~~...~~ ker-ran form)

(ii)$\Rightarrow$(iii): We have $(\operatorname{ran}T)^\perp = \ker T^* = \{0\}$ thus $\overline{\operatorname{ran}T} = X$.
If $\|x\|=1$ in X, find $f\in X^*$ st $|f(x)|=1$, $\|f\|=1$ (H.B. thm)
Then

$$1=|f(x)| = \left|\left(T^*(T^*)^{-1}f\right)(x)\right| = \left|\left((T^*)^{-1}f\right)(Tx)\right| \le \|(T^*)^{-1}f\|\,\|Tx\|$$
$$\le \|(T^*)^{-1}\|\cdot\|Tx\|$$

Hence $\|Tx\| \ge \frac{1}{\|(T^*)^{-1}\|}$.

(iii)$\Rightarrow$(i): Since T is bdd below, $\ker(T)=\{0\}$. Let us see that $\operatorname{ran}T$ is closed. If $y=\lim_{n\to\infty}Tx_n$, $(x_n)_{n=1}^\infty \subset X$, and if we let

$$c=\inf\{\|Tx\|;\ x\in X;\ \|x\|=1\},$$

we have for $n,m\in\mathbb{N}$

$$\|x_n-x_m\|\le c^{-1}\|T(x_n-x_m)\| = \|Tx_n-Tx_m\|$$

so $(x_n)_{n=1}^\infty$ is Cauchy. Hence $x=\lim_{n\to\infty}x_n$ exists and

$$y=\lim_{n\to\infty}Tx_n = T\lim_{n\to\infty}x_n = Tx\in\operatorname{ran}T$$

Thus T is bijective. Result follows from Open Mapping Theorem. ∎

Recall: $\sigma(T)=\{\lambda\in\mathbb{C};\ \lambda I-T\notin GL(X)\}$

We observe that if $T\in\mathcal{B}(X)\setminus GL(X)$ then at least one of the following holds:

(a) $\{0\}\subsetneq\ker(T)$

(b) T is not bounded below

(c) $\operatorname{ran}T\subsetneq X$, $\overline{\operatorname{ran}T}\subsetneq X$

Thus we get components of $\sigma(T)$:

(a') <u>point spectrum</u>: $\sigma_p(T)=\{\lambda\in\mathbb{C};\ \ker(\lambda I-T)\supsetneq\{0\}\}$ (eigenvalues)

(b') <u>approximate point spectrum</u>: $\sigma_{ap}(T)=\{\lambda\in\mathbb{C};\ \lambda I-T$ is not bdd below$\}$
Hence $\exists(x_n)_{n=1}^\infty\subset S(X)=\{x\in X;\ \|x\|=1\}$, $\|(\lambda I-T)x_n\|\xrightarrow{n\to\infty}0$

(c') <u>residue spectrum</u>: $\sigma_{res}(T)=\{\lambda\in\mathbb{C};\ \overline{\operatorname{ran}(\lambda I-T)}\subsetneq X\}\setminus\sigma_p(T)$

(d') <u>continuous spectrum</u>: $\sigma_c(T)=\sigma_{ap}(T)\setminus(\sigma_{res}(T)\cup\sigma_p(T))$

Let A be a unital Banach algebra. Recall $a \mapsto L_a : A \to \mathcal{B}(A)$ preserves invertibility, ie $\sigma_A(a) = \sigma_{\mathcal{B}(A)}(L_a)$. Hence $a \in A \setminus GL(A)$, if at least one of the following holds.

(a'') $ab = 0$ for some $b \in A \setminus \{0\}$ (a is a zero divisor)

(b'') $\exists$ sequence $(b_n)_{n=1}^{\infty} \subset S(A)$ st $\|ab_n\| \xrightarrow{n\to\infty} 0$ (a is an approximate z.d.)

(c'') $\overline{aA} \subsetneq A$, ie $\overline{aA}$ is a proper right ideal.

Corollary: If $T \in \mathcal{B}(X)$ then $\sigma_{\mathcal{B}(X)}(T) = \sigma_{\mathcal{B}(X^*)}(T^*)$.

Proof: $(\lambda I_X - T)^* = \lambda I_{X^*} - T^*$ ∎

Ex. Let $S \in \mathcal{B}(\ell^p)$ $(1 \le p < \infty)$ be given by $S(x_1, x_2, \dots) = (0, x_1, \dots)$

(i) $\sigma_p(S) = \emptyset$. Indeed if for $\lambda \in \mathbb{C}$ we have $(\lambda x_1, \dots) = (0, x_1, \dots)$ then $\lambda x_1 = 0$.

First, notice that $\ker(S) = \{0\}$ so $\lambda \neq 0$. Hence we would find $x_1 = 0$, hence $x_2 = 0$, etc. ie $x = (0, 0, \dots)$

(ii) We have $\sigma(S) = \overline{\mathbb{D}}$. Let us just show $\sigma(S) \supset \mathbb{D}$, other details will be obvious, later.

Indeed, we have $\sigma(S) = \sigma(S^*)$, where $S^* \in \mathcal{B}(\ell^{p'})$ $(\frac{1}{p'} + \frac{1}{p} = 1)$ is given by

$$S^*(y_1, \dots) = (y_2, y_3, \dots)$$

Observe that if $\lambda \in \mathbb{D}$, ie $|\lambda| < 1$, then $y_\lambda = (\lambda, \lambda^2, \dots) \in \ell^{p'}$, with $S^* y_\lambda = \lambda y_\lambda$. Thus $\mathbb{D} \subseteq \sigma_p(S^*) \subseteq \sigma(S^*) = \sigma(S)$.

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Remark about assignment: for A a comm. Ban. alg

$$\Gamma_A = \{\gamma : A \to \mathbb{C} ; \gamma \text{ is linear \& multiplicative}, \gamma \neq 0\}.$$

Proposition: Let $T \in \mathcal{B}(X)$, then $\partial\sigma(T) \subseteq \sigma_{ap}(T)$.

Proof: If $\lambda \in \partial\sigma(T)$ then $\exists (\lambda_n)_{n=1}^{\infty} \subset \mathbb{C} \setminus \sigma(T)$ st $\lambda = \lim_{n\to\infty} \lambda_n$. Hence

$$\|(\lambda I - T) - (\lambda_n I - T)\| = \|(\lambda - \lambda_n) I\| = |\lambda - \lambda_n| \xrightarrow{n\to\infty} 0.$$

So since $\lambda I - T \notin GL(X)$, we must have

$$\limsup_{n\to\infty} \|(\lambda_n I - T)^{-1}\| = \infty.$$

(cor. to the invertibility theorem) We may drop to subsequence, so

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$$\lim_{n\to\infty} \|(\lambda_n I - T)^{-1}\| = \infty$$

Fix, for each n, $\|x_n\|=1$ in X. So

$$\alpha_n = \|(\lambda_n I - T)^{-1} x_n\| > \|(\lambda_n I - T)^{-1}\| - \frac{1}{n}.$$

Notice that $\alpha_n \xrightarrow{n\to\infty} \infty$. Then

$$y_n = \frac{1}{\alpha_n}(\lambda_n I - T)^{-1} x_n$$

satisfies $\|y_n\| = 1$, and

$$\begin{aligned}(\lambda I - T) y_n &= (\lambda_n I - T) y_n + (\lambda - \lambda_n) y_n \\ &= \tfrac{1}{\alpha_n} x_n + (\lambda - \lambda_n) y_n \xrightarrow{n\to\infty} 0.\end{aligned}$$ ∎

Question: What does $\sigma(a)$, $a \in A$ (unital Banach alg.) look like? Is $\sigma(a) \neq \emptyset$?

Def] Fix a unital Banach algebra A, $a \in A$. Let

$$R : \mathbb{C} \setminus \sigma(a) \to A$$
$$R(z) = (ze - a)^{-1}$$

Notice that R is cts on $\mathbb{C}\setminus\sigma(a)$. We call R the resolvent function of a.

Def] Let X be a Banach space, $U \subseteq \mathbb{C}$ be open. A function $F : U \to X$ is is holomorphic (analytic) if

$$\lim_{h\to 0} \frac{1}{h}\big(f(z+h) - f(z)\big)$$

exists for each $z \in U$.

Notice, if $T \in B(X,Y)$ then $T \circ F$ is also holomorphic since T is continuous.

Remark: It is standard to see that holomorphic F is cts on its domain.

Lemma (i) (Liouville's thm)

If $F : \mathbb{C} \to X$ is holomorphic, and bounded, then F is constant.

(ii) (Laurent series)

If $r > 0$, $F : \mathbb{C} \setminus r\overline{\mathbb{D}} \to X$ is holomorphic, then F admits a Laurent series

$$F(z) = \sum_{k=-\infty}^{\infty} z^k x_k, \quad \lim_{n\to\infty} \|x_{-n}\|^{1/n} \le r$$

Proof:

(i) Let $\mu \in X^*$, then $\mu \circ F : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, hence constant. Thus

$$0 = \mu \circ F(z) - \mu \circ F(0) = \mu(F(z) - F(0))$$

for all z. By H.B. Thm, we have that $F(z) - F(0) = 0$.

(ii) Let $\rho > r$, and for $n \in \mathbb{Z}$ let

$$x_n = \frac{1}{2\pi} \int_0^{2\pi} \rho e^{-int} F(e^{it})\, dt \qquad \text{(vector-valued Riemann integral)}$$

Now, for $\mu \in X^*$, we have

$$(f) \qquad \mu(x_n) = \frac{1}{2\pi} \int_0^{2\pi} \rho e^{-int} \mu \circ F(e^{it})\, dt$$

By Cauchy's Theorem (holonomy form) we have that $\mu \circ F : \mathbb{C} \setminus r\mathbb{D} \to \mathbb{C}$ has Laurent series

$$\mu \circ F(z) = \sum_{n=-\infty}^{\infty} \mu_n z^n, \quad |z| > r$$

where $\mu_n = \mu(x_n)$ by (f). Thus by the root test

$$(*) \qquad \lim_{n \to \infty} |\mu_{-n}|^{1/n} \le r.$$

Suppose

$$\limsup_{n \to \infty} \|x_{-n}\|^{1/n} > r,$$

and hence, for some $\varepsilon > 0$ and some subsequence

$$\liminf_{k \to \infty} \|x_{-n_k}\|^{1/n} > r + \varepsilon > r.$$

But then

$$\frac{1}{(r+\varepsilon)^{n_k}} x_{-n_k}$$

would be unbounded in k. Thus, by uniform bdd principle (Banach-Steinhaus) there would exist μ in X^* for which

$$\frac{1}{(r+\varepsilon)^{n_k}} \mu_{-n_k} = \mu\left(\frac{1}{(r+\varepsilon)^{n_k}} x_{-n_k}\right)$$

is unbounded. But this contradicts (*). $\square$

Theorem: If A is a unital Ban. alg., $a \in A$, then $\sigma(a)$ is non-empty and compact.

Proof: If $|\lambda| > \|a\|$ then $\|\frac{1}{\lambda} a\| < 1$ and hence

$$\lambda e - a = \lambda(e - \tfrac{1}{\lambda}a) \in GL(A)$$

(invertibility thm). Hence $\sigma(a) \subseteq \|a\| \overline{\mathbb{D}}$. is bounded. Let $H(z) = ze - a : \mathbb{C} \to A$. Then $\mathbb{C} \setminus \sigma(a) = H^{-1}(\underbrace{GL(A)}_{\text{open}})$, and hence is open. Hence by Heine-Borel thm, $\sigma(a)$ is compact.

Now we consider $R(z) = (ze - a)^{-1}$, $R : \mathbb{C} \setminus \sigma(a) \to A$. We have for $z, z_0 \in \mathbb{C} \setminus \sigma(a)$

$$\begin{aligned} R(z) - R(z_0) &= (ze-a)^{-1} - (z_0 e - a)^{-1} \\ &= (ze-a)^{-1}\big((z_0 - a) - (ze - a)\big)(z_0 - a)^{-1} \\ &= R(z)\big((z_0 - z)e\big)R(z_0) \end{aligned}$$

so

$$\frac{1}{z - z_0}\big(R(z) - R(z_0)\big) = -R(z)R(z_0) \xrightarrow{z \to z_0} -R(z_0)^2$$

Hence $R : \mathbb{C} \setminus \sigma(a) \to A$ is holomorphic. Also, if $|z| > \|a\|$,

$$R(z) = \frac{1}{z}\left(e - \frac{1}{z}a\right)^{-1} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{1}{z^k} a^k$$

So

$$\|Rz\| \le \frac{1}{|z|}\sum_{k=0}^{\infty} \frac{1}{|z|^k}\|a\|^k = \frac{1}{|z|}\frac{1}{1 - \frac{1}{|z|}\|a\|} = \frac{1}{|z| - \|a\|}$$

~~Hence~~ and $\|R(z)\| \xrightarrow{|z| \to \infty} 0$.

If $\sigma(a) = \emptyset$, then $R : \mathbb{C} \to A$ with $\lim_{|z| \to \infty} \|R(z)\| = 0$, so would be bounded. By Liouville's thm, R is constant and, in fact, $R = 0$. But $0 \notin GL(A)$. Hence we find that $\sigma(a) \neq \emptyset$. ∎

Def: We let the <u>spectral radius</u> of $a \in A$ (unital Ban. alg.) is given by

$$r(a) = \max\{|\lambda| ; \lambda \in \sigma(a)\}.$$

Proposition. (Spectral Mapping Theorem)
Let A be a unital Ban. alg., $a \in A$ and $p \in \mathbb{C}[t]$ (polynomials). Then

$$\sigma(p(a)) = p(\sigma(a)).$$

Proof: If p is constant then this is obvious. Otherwise, given λ in $\mathbb{C}$, we factor

$$\lambda - p(t) = \alpha \prod_{k=1}^{n} (\lambda_k - t)^{m_k}$$

with $\alpha, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $m_1, \ldots, m_n \in \mathbb{N}$. Then

$$\lambda e - p(a) = \alpha \prod_{k=1}^{n} (\lambda_k e - a)^{m_k}.$$

Hence

$$\lambda e - p(a) \notin GL(A) \iff (\lambda_k e - a) \notin GL(A), \text{ some } k,$$

so

$$\lambda \in \sigma(p(a)) \iff \lambda_k \in \sigma(a), \text{ some } k$$

Hence

$$\mu \in \sigma(a) \iff \lambda - p(\mu) = 0 \text{ some } \lambda \text{ in } \sigma(p(a)).$$ ∎

Corollary: $\sigma(a^k) = \{\lambda^k ; \lambda \in \sigma(a)\}$.

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<u>Beurling's Spectral Radius Formula</u>

If A is a unital Banach algebra, $a \in A$, then

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$

Proof: If $z \in \mathbb{C}$, $|z| > \|a\|$, then $\|\frac{1}{z} a\| < 1$, and hence

$$ze - a = z(e - \tfrac{1}{z}a) \in GL(A)$$

So

$$R(z) = (ez - a)^{-1} = \frac{1}{z}\left(e - \frac{1}{z}a\right)^{-1} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{1}{z^k} a^k$$

?

Hence the Laurent series for $R(z)$ on $\mathbb{C} \setminus r(a)\overline{\mathbb{D}}$ is given as above, and hence

$$\limsup_{k\to\infty} \|a^k\|^{1/k} \le r(a)$$

Conversely, we already saw $r(a) \le \|a\|$, and hence,

$$r(a^n) \le \|a^n\|.$$

By spectral mapping theorem, $r(a^n) = r(a)^n$, and thus

$$r(a) \le \|a^n\|^{1/n}.$$

We obtain

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$$r(a) \le \liminf_{n\to\infty} \|a^n\|^{1/n}.$$

Hence

$$\limsup_{n\to\infty} \|a^n\|^{1/n} = \liminf_{n\to\infty} \|a^n\|^{1/n} = r(a). \qquad \blacksquare$$

Unitization

Many Banach algebras have no identity:
- $C_0(X)$ (X l.c. but not compact),
- $L^1(\mathbb{R})$
- some $\ell^1(S)$ (S some non-unital semigroup)
- some $\mathrm{alg}(S)$, $S \subset \mathcal{B}(X)$ (sometimes when S not bdd below)

Let for a (non-unital) Banach algebra A, $\tilde{A} = A \oplus \mathbb{C}$, with multiplication

$$(a,\alpha)(b,\beta) = (ab + \beta a + \alpha b, \alpha\beta).$$

Notice, if $\tilde{e} = (0,1)$, then $\tilde{e}$ is the identity for $\tilde{A}$, and if $\iota : A \hookrightarrow \tilde{A}$ (first coordinate embedding)

$$(\iota(a) + \alpha\tilde{e})(\iota(b) + \beta\tilde{e}) = \iota(ab + \beta a + \alpha b) + \alpha\beta\tilde{e}.$$

Let for $(a,\alpha) \in \tilde{A}$,

$$\|(a,\alpha)\|_1 = \|a\| + |\alpha|.$$

Notice

$$\begin{aligned}\|(a,\alpha)(b,\beta)\| &= \|(ab + \beta a + \alpha b, \alpha\beta)\|_1 \\ &= \|ab + \beta a + \alpha b\| + |\alpha\beta| \\ &\le \|a\|\|b\| + |\beta|\|a\| + |\alpha|\|b\| + |\alpha||\beta| \\ &= \|(a,\alpha)\|_1 \|(b,\beta)\|_1.\end{aligned}$$

Other unitization norms

Let B be a Banach algebra with identity e_B, $\|e_B\| = 1$. Suppose we have an isometric homomorphism $\Phi : A \to B$, ie
- $\|\Phi(a)\| = \|a\|$
- Φ linear, multiplicative

with $e_B \notin \Phi(A)$

Then for $(a,\alpha) \in \tilde{A}$ define

$$\|(a,\alpha)\|_\Phi = \|\Phi(a) + \alpha e_B\|$$

It is easy to see that $\|\cdot\|_\Phi$ is a norm which makes $\tilde{A}$ a Banach algebra. Notice

$$\|(a,\alpha)\|_\Phi \le \|\Phi(a)\| + \|\alpha e_B\| = \|a\| + |\alpha| = \|(a,\alpha)\|_1$$

Hence the identity map

$$id : (\tilde{A}, \|\cdot\|_\Phi) \to (\tilde{A}, \|\cdot\|_1)$$

is bounded, and bijective, hence bounded below (open mapping theorem). Thus $\|\cdot\|_\Phi \sim \|\cdot\|_1$ on $\tilde{A}$.

<u>Ex</u> X l.c. non-compact space (eg $X = \mathbb{R}$)

$\Phi : C_0(X) \hookrightarrow C(X_\infty)$,

$$\Phi f(x) = \begin{cases} f(x) & x \in X \\ 0 & x = \infty \end{cases}$$

Then Φ is an isometric homomorphism, $1_{X_\infty} \notin \Phi(C_0(X))$. Notice that

$$\|(f,\alpha)\|_\Phi = \|f + \alpha 1_\infty\|_\infty = \sup_{x \in X_\infty} |f(x) + \alpha| = \sup_{x \in X} |f(x) + \alpha|$$

(as X is dense in X_∞).

Notes:

(i) $A \lhd \tilde{A}$ (ideal), $\tilde{A}/A \cong \mathbb{C}$

Let $\gamma_\infty : \tilde{A} \to \mathbb{C}$ be given by

$$\gamma_\infty(a,\alpha) = \alpha,$$

then $\gamma_\infty \in \tilde{A}^*$, $\gamma_\infty(\tilde{a}\tilde{b}) = \gamma_\infty(\tilde{a})\gamma_\infty(\tilde{b})$, $\tilde{a}, \tilde{b} \in \tilde{A}$ and $A = \ker(\gamma_\infty)$.

(ii) If A has identity e_A, then $(e_A, 0)$ is an idempotent in $\tilde{A}$, ie $(e_A,0)^2 = (e_A,0)$.

<u>Def</u> If A is a non-unital Banach algebra, then for $a \in A$, define

$$\sigma(a) = \sigma_A(a) := \sigma_{\tilde{A}}(a,0)$$

To study this concept further, it is useful to define the concept of <u>adverse</u>.

If $a \in A$, its <u>adverse</u> is an element $\hat{a} \in A$ such that
$$a\hat{a} + a + \hat{a} = 0 = \hat{a}a + a + \hat{a}$$
Note $(\hat{a}, 1) = (a, 1)^{-1}$ in $\tilde{A}$, if $\hat{a}$ exists. Notice that, if they exist, adverses are unique, and in this case we say that a is <u>advertible</u>.

Remarks: A non-unital, $a \in A$.

(i) $0 \in \sigma(a)$

Indeed, $(-a, 0) = 0(0,1) - (a, 0)$ cannot be invertible in $\tilde{A}$:
$$(-a,0)(b,\beta) = (-ab - \beta a, 0) \neq (0,1)$$
(ii) $\lambda \in \sigma(a) \setminus \{0\} \iff -\frac{1}{\lambda} a$ is not advertible

Easy exercise

Hence for $a \in A$,
$$\sigma(a) = \{0\} \cup \{\lambda \in \mathbb{C} \setminus \{0\} ; -\tfrac{1}{\lambda} a \text{ not advertible}\}$$

Def/ Suppose a Banach algebra A admits an idempotent e, ie $e^2 = e$. An element a of A is <u>e-invertible</u> if
$$ae = a = ea \quad \text{"e local identity at a"}$$
$$\exists a^e \in A \text{ st } a^e e = a^e = e a^e \text{ and } a^e a = e = a a^e$$

Proposition (i) If A is a unital Banach algebra and $a \in A$, then $\sigma_{\tilde{A}}(a, 0) = \sigma_A(a) \cup \{0\}$.
(ii) If A is unital and admits an idempotent e, $e \neq e_A$, then if $a \in A$, $ea = a = ae$, then
$$\{\lambda \in \mathbb{C} ; \lambda e - a \text{ not e-invertible}\} \cup \{0\} = \sigma_A(a).$$
Proof: (i) follows from (ii) with evident changes of notation

(ii): If $b = ac$, $c \in A$ then $eb = b$ so $b \neq e_A$. Hence $0 \in \sigma(a)$.
Now, if $\lambda \in \mathbb{C} \setminus \{0\}$, then
$$\lambda e_A - a = \lambda(e_A - e) + (\lambda e - a)$$
is invertible $\iff$ $\lambda e - a$ is e-invertible. Indeed, if $\lambda e - a$ is e-inv. then
$$\tfrac{1}{\lambda}(e_A - e) + (\lambda e - a)^e = (\lambda e - a)^{-1}.$$
Conversely, if $b = (\lambda e_A - a)^{-1}$ exists then
$$e_A = (\lambda e_A - a) b$$
$$= \underline{\lambda(e_A - e) b} + \underline{(\lambda e - a) e b}$$
$*$

where $e = ee_A = e[\quad * \quad] = (\lambda e - a)eb$,
and likewise $eb(\lambda e - a) = e$. ∎

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Character Theory (of commutative Ban. alg's)

Let A be $\mathbb{C}$-algebra. Let
$$\Gamma_A = \{\gamma : A \to \mathbb{C} \,;\, \gamma \text{ is linear, multiplicative, non-zero}\}.$$
Notice, if $\gamma \in \Gamma_A$, then $\gamma(ab - ba) = 0$, ie γ annihilates all commutators.
Hence, determining Γ_A will only be interesting if A is commutative.

Ex (Warning, $\Gamma_A \neq \emptyset$)?
(i) $A = M_n$. Let $\{E_{ij}\}_{i,j=1}^n$ be a matrix unit:
$a \in M_n$, $a = \sum_{i,j=1} a_{ij} E_{ij}$ (ie $a = [a_{ij}]$)
Notice if i,j,l are distinct then $E_{ij} = E_{il}E_{lj} - E_{lj} \cdot E_{il}$
So if $\gamma : M_n \to \mathbb{C}$ is lin and mult. then $\gamma(E_{ij}) = 0$. Hence $\gamma = 0$ on M_n
(ii) Let X be a $\mathbb{C}$-vector space (Banach space (if you want?))
Define $xy = 0$ x,y in X (This is a Ban alg if X is a Ban space.) If $\gamma : X \to \mathbb{C}$ is lin and mult. then $\gamma(x)^2 = \gamma(x)\gamma(x) = \gamma(x^2) = \gamma(0) = 0$ so $\gamma(x) = 0$ ie $\gamma = 0$

Proposition. Let A be a $\mathbb{C}$-alg, $\tilde{A} = A \oplus \mathbb{C}$ its unitization. Then, any $\gamma \in \Gamma_A$ extends uniquely to $\tilde{\gamma} \in \Gamma_{\tilde{A}}$, $\tilde{\gamma}(a, \alpha) = \gamma(a) + \alpha$.
Hence
$$\Gamma_{\tilde{A}} = \tilde{\Gamma}_A \cup \{\gamma_\infty\}, \quad \gamma_\infty(a, \alpha) = \alpha.$$
($\gamma_\infty = \tilde{0}$)

Proof. It is obvious that if $\gamma \in \Gamma_A$ then $\tilde{\gamma} \in \Gamma_{\tilde{A}}$. Note that if $\tilde{\tilde{\gamma}}$ is any extension of $\gamma \in \Gamma_A$ to $\tilde{A}$ then if $\gamma(a) \neq 0$, we have
$\tilde{\tilde{\gamma}}(a,0) = \tilde{\tilde{\gamma}}((a,0)(0,1)) = \tilde{\tilde{\gamma}}(a,0)\tilde{\tilde{\gamma}}(0,1) = \gamma(a)\tilde{\tilde{\gamma}}(0,1)$
Hence $\tilde{\tilde{\gamma}}(0,1) = 1 = \tilde{\gamma}(0,1)$, and $\tilde{\tilde{\gamma}}|_A = \tilde{\gamma}|_A$, so $\tilde{\tilde{\gamma}} = \tilde{\gamma}$.
It is obvious that $\Gamma_{\tilde{A}}$ is as advertized. ∎