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PMATH 810 - Banach Algebras and Operator Theory

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Standing Assumption: All Banach spaces are over $\mathbb{C}$.

Def] A Banach Algebra is a Banach space A which admits an associative, bilinear multiplication

$$(a,b) \mapsto ab \qquad a,b \in A$$
$$(ab)c = a(bc) \qquad \alpha \in \mathbb{C}$$
$$(\alpha a)b = \alpha(ab)$$

for which

$$\|ab\| \leq \|a\|\|b\|.$$

Note: Hence if $a \in A$, we get linear maps

$$L_a : A \to A \quad L_a(b) = ab$$
$$R_a : A \to A \quad R_a(b) = ba$$

with $\|L_a\| \leq \|a\|$, $\|R_a\| \leq \|a\|$.

Remark: If A is a Banach ~~space~~ algebra and $B \subseteq A$ is a closed subalgebra, then B is a Banach algebra.

Def] A left ideal, in a $\mathbb{C}$-algebra A, is a subspace for which $ax \in I$ for any $a \in A$, $x \in I$.
We likewise define right ideals, and two-sided ideals (which we shall call "ideals").

Note: (right/left) ideals are subalgebras.

Proposition: Let A be a Banach algebra and $I \subseteq A$ a closed ideal. Then A/I with multiplication

$$(a+I)(b+I) = ab + I$$

is a Banach algebra.

Proof: Recall that if X is a Banach space, $Y \subseteq X$ a closed subspace, then $X/Y = \{x+Y ; x \in X\}$ is a Banach space with

$$\|x+Y\| = \inf\{\|x-y\| ; y \in Y\} =: \mathrm{dist}(x, Y).$$

Check: $x+Y \mapsto \|x+Y\|$ is a norm.

Recall that a normed space $(Z, \|\cdot\|)$ is a Banach space $\Leftrightarrow$ $\sum_{n=1}^{\infty} z_n$ defines an element of Z whenever $\sum_{n=1}^{\infty} \|z_n\| < \infty$.

Let us use this to check completeness of $(X/Y, \|\cdot\|)$.

Suppose $(x_n)_{n=1}^{\infty} \subset X$ is st

$$\sum_{n=1}^{\infty} \|x_n + Y\| < \infty.$$

Let $\varepsilon > 0$. We may suppose that each x_n is chosen st

$$\|x_n\| < \|x_n + Y\| + \frac{\varepsilon}{2^n}.$$

But then

$$\sum_{n=1}^{\infty} \|x_n\| < \infty.$$

Check that

$$\sum_{n=1}^{\infty} (x_n + Y) = \lim_{N \to \infty} \sum_{n=1}^{N} (x_n + Y) = \left(\sum_{n=1}^{\infty} x_n \right) + Y.$$

Now we check the submultiplicativity condition of the norm.

$$\begin{aligned}
\|ab + I\| &= \inf_{y \in Y} \|ab - y\| \\
&\leq \inf_{x, y \in Y} \|ab - ax - by + xy\| \\
&= \inf_{x, y \in Y} \|(a - y)(b - x)\| \\
&\leq \inf_{x, y \in Y} \|a - y\| \cdot \|b - x\| \\
&= \|a + I\| \|b + I\|
\end{aligned}$$

∎

<u>Examples</u>

(I) Let K be a compact Hausdorff space. Let

$$C(K) = \{ f : K \to \mathbb{C} \mid f \text{ is continuous} \}.$$

For any $f \in C(K)$,

$$\|f\|_\infty = \max_{x \in K} |f(x)| < \infty.$$

Then $\|\cdot\|_\infty$ is a norm on $C(K)$ and $(C(K), \|\cdot\|_\infty)$ is complete, ie $C(K)$ is a Banach space.

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If $f,g \in C(K)$, then
$$fg : K \to \mathbb{C}, \quad fg(x) = f(x)g(x), \quad x \in K$$
is also an element of $C(K)$.
Further,
$$\begin{aligned} \|fg\|_\infty &= \max_{x\in K} |f(x)g(x)| \\ &\leq \max_{x,y\in K} |f(x)g(y)| \\ &= \max_{x\in K} |f(x)| \max_{y\in K} |g(y)| \\ &= \|f\|_\infty \|g\|_\infty . \end{aligned}$$
Hence $C(K)$ is a Banach algebra.

Fact: Let $Y \subseteq K$, and let
$$k(Y) = \{ f \in C(K) \mid f|_Y = 0 \}.$$
Check that $k(Y)$ is a closed ideal of $C(K)$ and $k(Y) = k(\bar{Y})$.

Proposition: Let X be a Hausdorff space. The following are equivalent:
(i) every x in X admits a neighbourhood U st $\bar{U}$ is compact
(ii) the space $X_\infty = X \sqcup \{\infty\}$ equiped with topology
(*) $\{U ; U \text{ open in } X\} \cup \{\{\infty\} \sqcup (X \setminus K) ; K \subseteq X \text{ compact}\}$
is a compact Hausdorff space.

We say that such X is <u>locally compact</u>. The space X_∞ is called the <u>one-point compactification</u> (or <u>Alexandrov</u> compactification)

Proof: (i)$\Rightarrow$(ii): Check that (*) is a topology.
Now suppose $(U_\alpha)_{\alpha \in A}$ is a cover of X_∞ of sets of the form (*).
Then $\infty \in U_{\alpha_\infty} = (X\setminus K) \sqcup \{\infty\}$, K compact. Thus $(U_\alpha)_{\alpha \in A \setminus \{\alpha_\infty\}}$ covers K and hence admits a finite subcover. Add U_{α_∞} and done.
(ii)$\Rightarrow$(i): easy. ∎

X - l.c.H. space 2015 01 08

We define

$$C_0(X) = \{f \in C(X_\infty) ; f(\{\infty\}) = 0\}|_X$$
$$= \{f \in \underbrace{C_b(X)}_{\text{cts, bdd}} ; \forall \varepsilon > 0, \overline{\{x \in X ; f(x) > \varepsilon\}} \text{ compact}\}$$

Let

$$C_c(X) = \{f \in C_0(X) ; \overline{\{x \in X ; f(x) \neq 0\}} \text{ compact}\}.$$

Uryzohn's Lemma:

Let X be a l.c.H. space, K compact in X and F closed in X with $K \cap F = \emptyset$. There exists $f \in C_c(X)$ such that

also get $0 < f < 1$

$$f|_K = 1, \quad f|_F = 0.$$

Sketch of proof:

(1) Metric case, metric d

$$\text{dist}(K,F) = \inf\{d(x,y) ; x \in K, y \in F\}$$
$$\text{dist}(x,F) = \inf\{d(x,y) ; y \in F\}$$

Check that $x \mapsto \text{dist}(x,F)$ is continuous, hence $\text{dist}(K,F) > 0$. Let

$$f(x) = \min\left\{\frac{\text{dist}(x,F)}{\text{dist}(K,F) + d(x,K)}, 1\right\}$$

does he mean normality?

(2) General case

In locally compact Hausdorff spaces we have <u>regularity</u>: if K, F are disjoint closed sets, K compact, then there is a nhd $U_0 \supset K$ st $\overline{U_0} \cap F = \emptyset$. Moreover, we can choose U_0 so $\overline{U_0}$ is compact. (Exercise.)

$$K \subset U_0 \subset \overline{U_0} \subset X \setminus F$$

Next, find open $U_{1/2}$ st

$$K \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_0$$

Next, find $U_{3/4}, U_{1/4}$ st

$$K \subset U_{3/4} \subset \overline{U_{3/4}} \subset U_{1/2} \subset \overline{U_{1/2}} \subset U_{1/4} \subset \overline{U_{1/4}} \subset U_0.$$

Keep going until we have sets U_q, $q \in (0,1)$ is a dyadic rational, so that

$$K \subset U_q \subset \overline{U_q} \subset U_p \subset \overline{U_p} \subset U_0$$

whenever $p < q$.

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Define $f_n : X \to \mathbb{C}$ by

$$f_n(x) = \begin{cases} 1 & x \in K \\ q - \frac{1}{2^n} & \text{for } q = \sum_{k=1}^{n} \frac{\varepsilon_k}{2^k},\ \varepsilon_k \in \{0,1\},\ x \in U_q \setminus \overline{U_{q-}} \\ 0 & x \in X \setminus \overline{U_0} \end{cases}$$

One can prove that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists and f defines a cts function. ∎

Corollary: $C_c(X)$, hence $C_0(X)$ separates points of X.
i.e. $\forall$ $x \neq y$ in X, $\exists f \in C_c(X)$ $f(x) \neq f(y)$

Remark: $\overline{C_c(X)}^{\|\cdot\|} = C_0(X)$

(I') Any closed subalgebra $\mathcal{A} \subseteq C_0(X)$ (X l.c.H sp) is a Banach algebra. We shall call such algebras <u>uniform algebras</u>. If $\mathcal{A}$ separates points, we will call it <u>natural</u> on X.

Ex. Let $\Omega \subseteq \mathbb{C}$ be non-empty, open, bdd. Let

$$\mathcal{A}(\Omega) = \{ f \in C(\bar{\Omega})\ ;\ f|_{\Omega} \text{ is holomorphic} \}$$

Theorem:
(i) If $f \in \mathcal{A}(\Omega)$, then

$$\underbrace{\|f\|_\infty}_{= \sup_{z\in\bar\Omega} |f(z)| = \sup_{z\in\Omega} |f(z)|} = \|f|_{\partial\Omega}\|_\infty$$

Hence $\mathcal{A}(\Omega)|_{\partial\Omega} \subseteq C(\partial\Omega)$ is a subalgebra.
(ii) $\mathcal{A}(\Omega)|_{\partial\Omega}$ is closed in $C(\partial\Omega)$.
Proof: "Given $0<r<1$, let

$$\Omega_r = \{ z \in \Omega\ ;\ \text{dist}(z, \partial\Omega) \leq 1-r \}, \quad \text{closed}$$

Notice

$$\bigcup_{0<r<1} \Omega_r = \Omega,$$

and

$$\mathrm{dist}(\partial\Omega_r, \partial\Omega) \xrightarrow{r \nearrow 1} 0.$$

Let $f \in A(\Omega)$, since $f|_\Omega$ is holomorphic, the maximum modulus principle tells us that

$$\|f|_{\Omega_r}\|_\infty = \|f|_{\partial\Omega_r}\|_\infty$$

Hence, since f is continuous on $\overline{\Omega}$, we find

$$\|f\|_\infty = \sup_{0<r<1} \|f|_{\Omega_r}\|_\infty = \sup_{0<r<1} \|f|_{\partial\Omega_r}\|_\infty$$

$$= \lim_{r \nearrow 1} \|f|_{\partial\Omega_r}\|_\infty = \|f|_{\partial\Omega}\|_\infty$$

$\Omega_r \subset \Omega_{r'}$, $r \leq r'$

In particular, for $f, g \in A(\Omega)$, $f|_\Omega = g|_\Omega$ if and only if $f|_{\partial\Omega} = g|_{\partial\Omega}$.

(ii) Let $(f_n)_{n=1}^\infty \subset A(\Omega)$ st there is $f \in C(\overline{\Omega})$ st

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0$$

Then for each closed rectifiable curve in Ω, which is contractable in Ω, Cauchy's thm then provides that for each n

$$\int_\gamma f_n(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt = 0.$$

If $\mu(f) = \int_\gamma f(z)dz$, $f \in C(\overline{\Omega})$ we have

$$|\mu(f)| \leq \int_\gamma |f(z)|dz \leq \|f\|_\infty \|\gamma'\|_\infty$$

so $\|\mu\| \leq \|\gamma'\|_\infty$. Hence

$$\int_\gamma f(z)d\gamma = \mu(f) = \lim_{n\to\infty} \mu(f_n) = 0.$$

This holds for any such curves, so by Morera's thm, $f|_\Omega$ is holomorphic. ▨

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Let

$$\mathbb{D} = \{z \in \mathbb{C} \,;\, |z| < 1\}, \quad \mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C} \,;\, |z| = 1\}.$$

The algebra $\mathcal{A}(\mathbb{D})$ is called the disc algebra.

Proposition:

(i) polynomial functions $z \mapsto \sum_{k=0}^{n} a_k z^k$ form a dense subalgebra of $\mathcal{A}(\mathbb{D})$

(ii) $\mathcal{A}(\mathbb{D})|_{\mathbb{T}} \subsetneq C(\mathbb{T})$

Proof:

(i) Let $f \in \mathcal{A}(\mathbb{D})$. Let, for $0 < r < 1$, $f_r(z) = f(rz)$, so f_r is holomorphic on $\frac{1}{r}\mathbb{D}$. Thus f_r admits a McLaurin series

$$f_r(z) = \sum_{k=0}^{\infty} \frac{f_r^{(k)}(0)}{k!} z^k$$

which converges uniformly on

$$\mathbb{D} \subseteq \tfrac{1}{r}\mathbb{D}.$$

But also

$$\lim_{r \nearrow 1} \|f_r - f\|_\infty = 0.$$

(ii) Consider the functional on $C(\mathbb{T})$,

$$\mu(f) = \frac{1}{2\pi} \int_0^{1} f(e^{it}) e^{+it}\, dt \;=\; \oint_{\mathbb{T}} f(z)\, dz.$$

So $\|\mu\| \le 1$. If $g(z) = \bar{z}$ then $\mu(g) = 1$. But $\mu(p) = 0$ for any poly p. By H.B. thm, $\mathcal{A}(\mathbb{D}) \subsetneq C(\mathbb{T})$. ∎

Remark: $\mathbb{A} = \mathbb{A}(0, 1/2, 1) = \{z \in \mathbb{C} \,;\, \frac{1}{2} < |z| < 1\}$.
Check that polynomials are not dense in $\mathcal{A}(\mathbb{A})$.
It can be shown (lots of complex analysis) that

$$\mathcal{A}(\mathbb{A})|_{\partial\mathbb{A}} \subsetneq C\left(\tfrac{1}{2}\mathbb{T} \cup \mathbb{T}\right).$$

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(II) X - set

$$\ell^\infty(X) = \{f: X \to \mathbb{C} \,;\, \|f\|_\infty = \sup_{x \in X} |f(x)| < \infty\}$$

Standard: This is a Banach space.

Easy: this is a Banach alg with pointwise mult.

If X admits a topology, we let

$$C_b(X) = \{f \in \ell^\infty(X) \,;\, f \text{ cts}\}$$

This is a closed subalgebra.

Now suppose $\mathcal{B} \subseteq \mathcal{P}(X)$ is a σ-algebra. Let

$$L^\infty(X, \mathcal{B}) = \{f \in \ell^\infty(X) \,;\, f \text{ is } \sigma\text{-measurable}\}$$

Then $L^\infty(X, \mathcal{B})$ is a closed subalgebra of $\ell^\infty(X)$.

Now suppose $\mu: \mathcal{B} \to [0, \infty]$ is a measure.

Let

$$\mathcal{N}(\mu) = \{f \in L^\infty(X, \mathcal{B}) \,;\, \mu(f^{-1}(\mathbb{C} \setminus \{0\})) = 0\}$$

It is easy to see that $\mathcal{N}(\mu)$ is a closed ideal of $L^\infty(X, \mathcal{B})$ (in fact, if $(\mu, \mathcal{B})$ is complete, then $\mathcal{N}(\mu)$ is even an ideal in $\ell^\infty(X)$).

We let

$$L^\infty(X, \mu) = L^\infty(X, \mathcal{B}) / \mathcal{N}(\mu)$$

Check that this agrees with the "usual" defn of $L^\infty(X, \mu)$.

(III) Let

$$C^1[0,1] = \{f : [0,1] \to \mathbb{C} \,;\, f' \text{ exists and is cts}\}$$

Let

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$$

Suppose $(f_n)_{n=1}^\infty \subset C^1[0,1]$ is Cauchy. Then there exists $g, h \in C[0,1]$ st $f_n \xrightarrow{n \to \infty} g$, $f_n' \xrightarrow{n \to \infty} h$. By FTC,

$$f_n(x) = \int_0^x f_n' + f_n(0)$$

Then since

$$\lim_{n \to \infty} \|f_n' - h\|_\infty = 0,$$

$$f_n(x) \xrightarrow{n \to \infty} \int_0^x h + g(0)$$

$$f_n(x) \xrightarrow{n \to \infty} g(x)$$

Again, by FTC, we have $g' = h$. Also,

$$\begin{aligned} \|fg\|_{C^1} &= \|fg\|_\infty + \|(fg)'\|_\infty \\ &\le \|f\|_\infty \|g\|_\infty + \|f\|_\infty \|g'\|_\infty + \|f'\|_\infty \|g\|_\infty \\ &\le \|f\|_{C^1} \|g\|_{C^1} \end{aligned}$$

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(IV) A <u>semigroup</u> S is a set with a binary operation $(s,t) \mapsto st : S\times S \to S$ which is associative: $(st)u = s(tu)$. Let

$$\ell^1(S) = \{f : S \to \mathbb{C} \,;\, \underbrace{\textstyle\sum_{s\in S} |f(s)|}_{\|f\|_1 := \;=\; \sup_{F\subseteq S \text{ finite}} \sum_{s\in F} |f(s)|} < \infty\}$$

Fact: $(\ell^1(S), \|\cdot\|_1)$ is a Banach space.

We define convolution by

$$f*g(u) = \sum_{(s,t)\in S^2,\, st=u} f(s)g(t)$$

$$\|f*g\|_1 = \sum_{u\in S} \Big|\sum_{st=u} f(s)g(t)\Big| \le \sum_u \sum_{st=u} |f(s)|\cdot|g(t)|$$

$$= \sum_s |f(s)| \cdot \sum_t |g(t)| \le \|f\|_1 \|g\|_1$$

indicator function

Hence $f*g(u)$ makes sense, and $f*g \in \ell^1(S)$. We let $\delta_s(t) = 1_{\{s\}}(t)$.

We have

$$\delta_s * \delta_t(u) = \textstyle\sum_{s't'=u} \delta_s(s')\delta_t(t') = \delta_{st}(u).$$

Moreover, each $f \in \ell^1(S)$ satisfies

$$f \xlongequal{\|\cdot\|_1\text{-limit}} \lim_{\substack{F\nearrow S\\ \text{finite}}} \sum_{s\in F} f(s)\delta_s = \sum_{s\in S} f(s)\delta_s$$

We obtain alternative form

$$f*g = \textstyle\sum_s \sum_t f(s)g(t)\,\delta_{st}.$$

The associativity now follows from that of S.

If $S = \Gamma$ is a group then cancellation allows

$$f*g(u) = \sum_{s\in\Gamma} f(s)g(s^{-1}u) = \sum_{t\in\Gamma} f(ut^{-1})g(t).$$

Fact: We have that $f*g = g*f$ $\forall f,g \in \ell^1(S) \Leftrightarrow st = ts$ $\forall s,t \in S$.

(V) Let X be a Banach space, $B(X)$ the space of bdd linear operators on X, with

$$\|T\| = \sup\{\|Tx\| \,;\, x \in b(X)\} = \sup\{\|Tx\|/\|x\| \,;\, x \in X\setminus\{0\}\}.$$

This is a Banach space. This norm satisfies $\|Tx\| \le \|T\|\|x\|$ and

$$\|T\| = \max\{c \in [0,\infty) \,;\, \|Tx\| \le c\|x\| \;\; \forall x \in X\}.$$

Also

$$\|ST\| = \sup_{x\in b(X)} \|STx\| \le \|S\| \sup_{x\in b(X)} \|Tx\| = \|S\|\cdot\|T\|.$$

Examples of closed subalgebras:

(i) $\Lambda = \{L_\alpha\}_{\alpha\in A}$, each L_α a closed subsp of X.

$$\mathrm{Alg}(\Lambda) = \{T \in B(X) ; T(L_\alpha) \subseteq L_\alpha, \ \alpha \in A\}$$

Check that $\mathrm{Alg}(\Lambda)$ is a subalgebra of $\mathcal{B}(X)$. Also, if $(T_n)_{n=1}^{\infty} \subset \mathrm{Alg}(\Lambda)$ with $T = \lim_{n\to\infty} T_n \in \mathcal{B}(X)$, then $\forall x \in X$, $T_n x \xrightarrow{\text{norm}} Tx$, hence if $x \in L_\alpha$, we see that $Tx \in L_\alpha$.

(ii) An operator K is compact if $\overline{K(b(X))}$ is norm-compact. We know that ~~$K(X)$~~

$$\mathcal{K}(X) = \{K \in B(X) ; K \text{ is compact}\}$$

is a closed subspace. If $S \in \mathcal{B}(X)$, $K \in \mathcal{K}(X)$, then

$$\overline{SK(b(X))} \subseteq S(\overline{K(b(X))}) \quad \text{(check)}$$

$$\overline{KS(b(X))} \subseteq \|S\| \overline{K(b(X))} \quad \text{(easy)}$$

Thus $\mathcal{K}(X)$ is a closed ideal. We call the quotient

$$\mathcal{Q}(X) = \mathcal{B}(X)/\mathcal{K}(X)$$

is called the <u>Calkin algebra</u>.

(VI) Consider $L^1(\mathbb{R}) = L^1(\mathbb{R}, m)$ — Lebesgue measure. If $f, g \in L^1(\mathbb{R})$ then the a.e. defined "function"

$$(x,y) \mapsto f(x)g(y-x)$$

is integrable:

$$\int_{\mathbb{R}^2} |f(x)g(y-x)|\, d(x,y) \overset{\text{Tonelli}}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)||g(y-x)|\, dy\, dx$$

$$\overset{\text{trans inv}}{=} \int_{\mathbb{R}} |f(x)|\,dx \int_{\mathbb{R}} |g(y)|\,dy < \infty$$

Hence, by Fubini's Thm, it is integrable. Thus for almost each y we can do the margin integral

$$f*g(y) = \int_{\mathbb{R}} f(x)\, g(y-x)\, dx$$

so $f*g \in L^1(\mathbb{R})$, $\|f*g\|_1 \le \|f\|_1 \|g\|_1$.

We stealthily use trans. inv. and some rearrangements to see

$$(f*g)*h = f*(g*h).$$

This is known as the <u>$(L^1$-$)$ group algebra</u> of $(\mathbb{R}, +)$.

$$s*f(x) = f(x-s)$$

$$s \mapsto s*f : \mathbb{R} \to L^1(\mathbb{R}) \text{ cts.}$$