(last) new topic

Let G be a finite group, $H \leq G$ any subgroup, and $f : H \to \mathbb{C}$ a class function. Define

$$\left(\mathrm{Ind}_H^G(f)\right)(g) = \frac{1}{|H|} \sum_{t \in G;\, t^{-1}gt \in H} f(t^{-1}gt).$$

It is a class function on G. Note also that $\underbrace{\mathrm{Ind}_H^G(\chi_\rho)}_{\text{class function on } H} = \underbrace{\chi_{\mathrm{Ind}_H^G(\rho)}}_{\text{class function on } G}$.

Theorem [Frobenius Reciprocity]: Let ψ be a class function on H, and let φ be a class function on G. Then

$$\langle \psi, \phi|_H \rangle_H = \langle \mathrm{Ind}_H^G(\psi), \varphi \rangle_G.$$

by linearity

Proof: We may assume that ψ and φ are irred. chars, corresponding to the irred. rep-s τ and ρ respectively. Then $\langle \psi, \phi|_H \rangle_H$ is the # of copies of τ in the irred. decomp. of ~~$\phi|_H$~~ the representation associated to $\phi|_H$, and similarly for $\langle \mathrm{Ind}_H^G(\psi), \varphi \rangle_G$. Notice that if $\mu \cong a_1\rho_1 \oplus \cdots \oplus a_r\rho_r$ for irred reps ρ_i with $\rho_i \not\cong \rho_j$, then $\dim(\mathrm{Hom}(\rho_i, \mu)) = a_i$ as we saw earlier (via modules). So

$$\langle \psi, \phi|_H \rangle_H = \dim(\mathrm{Hom}_{\mathbb{C}[H]}(V, W)),$$
$$\langle \mathrm{Ind}_H^G \psi, \phi \rangle_G = \dim(\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V, W)).$$

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Define $\Lambda : \mathrm{Hom}_{\mathbb{C}[H]}(V, W) \to \mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V, W)$ by

$$\Lambda(T) = \left(g \otimes v \longmapsto (\rho(g))(T(v))\right).$$

It is easy to check that Λ is well-defined, linear, and injective. Finally, to see surjectivity, note that $\forall f : \mathbb{C}[G] \otimes V \to W$ hom of $\mathbb{C}[G]$-modules,

$$f = \Lambda(f|_{1 \otimes V}).$$

Thus Λ is an isomorphism so the 2 dimensions are equal as desired. ∎

When is ~~the~~ $\mathrm{Ind}_H^G \rho$ irreducible?
Say H, K are subgroups of a finite group G, $\rho: H \to GL(V)$ a representation of H. What is $\mathrm{Ind}_H^G \rho|_K$?

Def) A double coset of (H,K) in G is a set of the form KgH for some $g \in G$.

Remark: For any $g_1, g_2 \in G$, either $Kg_1H = Kg_2H$ or $Kg_1H \cap Kg_2H = \emptyset$.

Let $\{g_1, \dots, g_m\}$ be a set of double coset representatives for (H,K) in G and let

$$H_i = g_i H g_i^{-1} \cap K.$$

Define $\rho_i : H_i \to GL(V)$ by

$$\rho_i(x) = \rho(g_i^{-1} x g_i).$$

Theorem: $$\left(\mathrm{Ind}_H^G \rho\right)\Big|_K \cong \bigoplus_{i=1}^{m} \mathrm{Ind}_{H_i}^{K} \rho_i$$

Proof: Note

$$\left(\mathrm{Ind}_H^G \rho\right)\Big|_K = \left(\mathrm{Ind}_H^G \rho \text{ as a } \mathbb{C}[K]\text{-module}\right)$$

$$= \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$$

considered as a $\mathbb{C}[K]$-module. We want to show that $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \cong \bigoplus_i \left(\mathbb{C}[K] \otimes_{\mathbb{C}[H_i]} V\right)$ as $\mathbb{C}[K]$-modules.
Define $\phi : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \to \bigoplus_i \mathbb{C}[K] \otimes_{\mathbb{C}[H_i]} V$ by

$$\phi(kg_ih \otimes v) = (0, \dots, 0, k \otimes (\rho(h))(v), 0, \dots, 0),$$

where the non-zero entry is in the i^{th} coordinate. To show that ϕ is well-defined, we need to show that if $k_1 g_i h_1 = k_2 g_i k_2$ then $k_1 \otimes (\rho(h_1))(v) = k_2 \otimes (\rho(h_2))(v)$ $\forall v \in V$. 2013 11 27

Well, $k_1 = k_2 (g_i h_2 h_1^{-1} g_i^{-1})$, so

$$k_1 \otimes (\rho(h_1))(v) = k_2(g_ih_2h_1^{-1}g_i^{-1}) \otimes (\rho(h_1))(v) = k_2 \otimes \underbrace{\rho_i(g_ih_2h_1^{-1}g_i^{-1})(\rho(h_1))}_{\rho(h_2h_1^{-1})\rho(h_1) = \rho(h_2)}(v)$$

as desired

Thus ϕ is well-defined.

Now define $\psi : \bigoplus_i \mathbb{C}[K] \otimes_{\mathbb{C}[H_i]} V \to \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ by

$$\psi(k_1 \otimes v_1, \dots, k_n \otimes v_n) = \sum_i k_i g_i \otimes v_i.$$

We must show that if $g_i h g_i^{-1} \in H_i$ then

$$\psi(k g_i h g_i^{-1} \otimes v) = \psi(k \otimes (\rho_i(g_i h g_i^{-1}))(v)).$$

But

$$\psi(0, \dots, k g_i h g_i^{-1} \otimes v, \dots, 0) = k g_i h \otimes v = k g_i \otimes (\rho(h))(v)$$

and

$$\psi(k \otimes (\rho_i(g_i h g_i^{-1}))(v)) = k g_i \otimes (\rho(h))(v)$$

so yay. Finally note

$$\psi(\phi(k g_i h \otimes v)) = \psi((0, \dots, 0, k \otimes (\rho(h))(v), 0, \dots, 0)) = k g_i \otimes (\rho(h))(v)$$
$$= k g_i h \otimes v.$$

Hence ψ and ϕ are mutually inverse, so ϕ is an isomorphism. ■

Theorem [Mackey's Irreducibility Criterion]:
Let G be a finite group, $H \leq G$ a subgroup, $\rho : H \to GL(V)$ a representation. Then $\mathrm{Ind}_H^G \rho$ is irreducible if and only if, ρ is irreducible and $\forall g \in G \setminus H$, $\langle \chi_{\rho^g}, \chi_\rho|_{H_g} \rangle = 0$ where $H_g = gHg^{-1} \cap H$ and $\rho^g : H_g \to GL(V)$ is given by $\rho^g(t) = \rho(g^{-1} t g)$.

$g_1, \dots, g_n$ double coset reps for (H, H) and $g_1 = 1$

Proof: $\mathrm{Ind}_H^G \rho$ is irreducible if and only if $\langle \mathrm{Ind}_H^G \chi_\rho, \mathrm{Ind}_H^G \chi_\rho \rangle_G = 1$, 2013 11 2
if and only if $\langle \chi_\rho, \mathrm{Res}_H \mathrm{Ind}_H^G \chi_\rho \rangle_H = 1$ if and only if
$\langle \chi_\rho, \sum \mathrm{Ind}_{H_i}^H \chi_{\rho_i} \rangle_H = 1$ if and only if $\sum \langle \mathrm{Res}_{H_i} \chi_\rho, \chi_{\rho_i} \rangle_{H_i} = 1$,
if and only if $\langle \mathrm{Res}_H \chi_\rho, \chi_\rho \rangle_H = 1$ and $\langle \mathrm{Res}_{H_i} \chi_\rho, \chi_{\rho_i} \rangle_{H_i} = 0$ $\forall i \neq 1$
if and only if ρ is irreducible and $\langle \mathrm{Res}_{H_g} \chi_\rho, \chi_{\rho^g} \rangle_{H_g} = 0$ $\forall g \in G \setminus H$. ■

Corollary If $H \trianglelefteq G$ is a normal subgroup, $\rho: H \to GL(V)$ a representation, then $\operatorname{Ind}_H^G \rho$ is an irreducible representation of G if and only if

(i) ρ is irreducible, and

(ii) $\langle \chi_{\rho^g}, \chi_\rho \rangle = 0 \quad \forall g \in G \setminus H$.

where $\rho^g: H \to GL(V)$, $\rho^g(h) = \rho(g^{-1}hg)$

ex $D_4 = \langle x, y \mid x^2 = y^4 = 1,\ xy = y^{-1}x \rangle$, $\rho: \langle y \rangle \to GL(\mathbb{C})$, $\rho(y^a) = i^a$.

Is $\operatorname{Ind}_{\langle y \rangle}^{D_4} \rho$ irreducible?

Since $\dim(\rho) = 1$, ρ is irreducible. Any $g \in D_4 \setminus \langle y \rangle$ is of the form xy^a so $g\langle y \rangle g^{-1} = x\langle y \rangle x$, so $\langle y \rangle_g = \langle y \rangle_x$ and $\rho^g = \rho^x$. And $\rho^x(y^a) = \rho(xy^ax) = \rho(y^{-a}) = i^{-a}$, so $\rho^x \neq \rho$. Thus, since ρ^x is also irreducible, $\langle \chi_{\rho^x}, \chi_\rho \rangle = 0$. So $\operatorname{Ind}_{\langle y \rangle}^{D_4} \rho$ is irreducible by MIC.

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all directly about assignment material (except Young tableaux)

6 straightforward
- y/n
- rep calculations
- tensor prod. reps from sept.
- induced reps (modules or old fash)
- modules (not nec. about rep theory)
- integral elements

4 more