So we can find, by similar computation,

$$\tau(y) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \tau(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

From here we get $\chi_{\tau} = (2, 0, -2, 0, 0, 0, 0, 0)$. This is consistent with the theorem we had. Yay.

Redefining $\mathbb{C}[G]$

Def Let G be a group. The <u>complex group ring</u> of G is $\mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C}g$, where $(ag)(bh) = ab(gh)$ for $g, h \in G$ and $a, b \in \mathbb{C}$.

Note: $\mathbb{C}[G]$ is a ring, and is commutative $\Leftrightarrow$ G is abelian.

new topic: modules

Def Let R be a ring with identity. A left R-module is an abelian group M together with an operation $\cdot : R \times M \to M$ satisfying: 2013 10 30

i) $r_1(r_2 m) = (r_1 r_2)m$

ii) $1m = m$

iii) $(r_1 + r_2)m = r_1 m + r_2 m$

iv) $r(m_1 + m_2) = rm_1 + rm_2$

for all $r, r_1, r_2 \in R$, $m, m_1, m_2 \in M$.

Remark: If R is a field, then an R-module is the same thing as an R-vector space.

ex $\mathbb{Z}^n$ is a $\mathbb{Z}$-module

ex $\mathbb{Z}_n$ is a $\mathbb{Z}$-module

ex any abelian group is a $\mathbb{Z}$-module, and vice versa

ex Let R be any ring an $I \leq R$ a left ideal. Then I is an R-module. In fact, a left ideal R is exactly a left R-submodule of R.

Say $\rho : G \to GL(V)$ is a representation of G. We can make V into a left $\mathbb{C}[G]$-module by

$$\left(\sum_{i=1}^{m} a_i g_i\right)(v) = \sum_{i=1}^{m} a_i (\rho(g_i))(v).$$

since we can multiply by $z \in \mathbb{C}$ via $(ze) \cdot v$, $e \in G$

Conversely, let V be a left $\mathbb{C}[G]$-module. Then V is a $\mathbb{C}$-vector space, and we can define $(\rho(g))(v) = gv$ for $g \in G$, $v \in V$. It is not hard to check that $\rho(g)$ is a linear transformation for all $g \in G$, and that ρ is a homomorphism $G \to GL(V)$.

ex $\rho: \mathbb{Z}_2 \to GL(\mathbb{C})$, $\rho(1) = -1$, $\rho(0) = 1$. Then $\mathbb{C}[G] = \mathbb{C} \cdot 0 + \mathbb{C} \cdot 1$ and $\mathbb{C}$ is a module over $\mathbb{C}[G]$, via $(a(0) + b(1))z = a(\rho(0))(z) + b(\rho(1))(z) = az + b(-z)$
$= z(a-b)$

Notice that $\phi(p(t)) = p(1) \in \mathbb{C}[G]$ is a ring hom from $\mathbb{C}[t] \to \mathbb{C}[G]$. ϕ is onto since $\phi(at+b) = b \cdot (0) + a \cdot (1)$. The kernel of ϕ is $\langle t^2 - 1 \rangle$. Since $\mathbb{C}[t]/\langle t^2-1 \rangle$ is 2-dimensional as a $\mathbb{C}$-vector space, $\mathbb{C}[G]$ is 2-dim. as a $\mathbb{C}$-vector space, and since $\hat{\phi}: \mathbb{C}[t]/\langle t^2-1 \rangle \to \mathbb{C}[G]$ is onto, we conclude $\hat{\phi}$ is an isomorphism. So $\mathbb{C}[t]/\langle t^2-1 \rangle \cong \mathbb{C}[G]$. By the chinese remainder theorem, $\mathbb{C}[t]/\langle t^2-1 \rangle \cong \mathbb{C} \oplus \mathbb{C}$.

2013 11 01

Def Let R be a ring (with identity). Let M, N be left R-modules. An R-module homomorphism from M to N is a function $f: M \to N$ such that $\underbrace{f(m_1 + m_2) = f(m_1) + f(m_2)}_{(1)}$ and $\underbrace{f(\lambda m) = \lambda f(m)}_{(2)}$ for all $m, m_1, m_2 \in M$, $\lambda \in R$.

Note: If $R = \mathbb{C}[G]$, then an R-module homomorphism is exactly a morphism of representations. (as (1) is linearity and (2) is $f((\rho(\lambda))(m)) = \rho(\lambda)(f(m))$)

Def An isomorphism of R-modules is a homomorphism of R-modules that has an inverse homomorphism.

Def Let R be a ring, M an R-module. Then $\mathrm{End}_R(M)$ is called the endomorphism ring of M over R, and is the set of R-module homomorphisms from M to M, equipped with pointwise function addition and composition. (An endomorphism of M is a homomorphism $M \to M$.)

Note that Schur's lemma says that $\mathrm{End}_{\mathbb{C}[G]}(V) \cong \mathbb{C}$ if V is an irreducible representation of G.

<u>Def</u>] Let M, N be R-modules. Then $\mathrm{Hom}_R(M,N)$ is the set of R-module homomorphism from M to N.

Note that $\mathrm{Hom}_R(M,N)$ is an R-module via $(f_1+f_2)(m) = f_1(m)+f_2(m)$, $(rf)(m) = rf(m)$.

$\mathbb{C}[G]$-modules, irred as reps:

Recall also that if $V_1 \not\cong V_2$ with V_1, V_2 irreducible, then $\mathrm{Hom}_{\mathbb{C}[G]}(V_1,V_2) = 0$.

i.e. simple

If M, N are R-modules, then $M \oplus N$ is also an R-module (in the obvious way as pairs). So <u>direct sum of $\mathbb{C}[G]$-modules corresponds to direct sum of representations.</u>

<u>Def</u>] An R-module M is simple if its only R-submodules are 0 and M.

Notice that simple $\mathbb{C}[G]$-modules correspond precisely to irreducible representations of G. Thus every $\mathbb{C}[G]$-module that is a finite dimensional $\mathbb{C}$-vector space is isomorphic to a direct sum of simple $\mathbb{C}[G]$-modules.

<u>Def</u>] An R-module M is called semi-simple if for every submodule $N \subseteq M$, there is a submodule $N^\perp \subseteq M$ st $M \cong N \oplus N^\perp$.

Theorem (Maschke): Every $\mathbb{C}[G]$-module is semi-simple if G is finite.

Note: We have already proven this if the $\mathbb{C}[G]$-module is a finite dimensional $\mathbb{C}$-vector space.

Proof: Let V be a $\mathbb{C}[G]$-module, $W \subseteq V$ a submodule. We want to find a submodule $W' \subseteq V$ such that $V \cong W \oplus W'$ as $\mathbb{C}[G]$-modules. Let $f: V \to W$ be any $\mathbb{C}$-linear projection onto W (so $f|_W = \mathrm{id}$). If f were a $\mathbb{C}[G]$-module homomorphism, then $W' = \ker(f)$ would work. Unfortunately, this may not be the case. So define $h: V \to W$ by 2013 11 04

$$h(v) = \frac{1}{|G|}\sum_{g\in G} g^{-1} f(gv).$$

Since W is a submodule, it is "G-invariant", so $h(v)\in W$, as claimed.
Since $f|_W = id$, $f(w) = \frac{1}{|G|}\sum_{g\in G} g^{-1}gw = w$ $\forall w \in W$. Hence h is a projection onto W.
To see that h is a $\mathbb{C}[G]$-module homomorphism,

$$\begin{aligned} h\left(\left(\sum_{i=1}^{m} a_i g_i\right)v\right) &= \sum_{i=1}^{m} a_i h(g_i v) \\ &= \sum_{i=1}^{m} a_i \frac{1}{|G|}\sum_{g\in G} g^{-1} f(gg_i v) \\ &= \sum_{i=1}^{m} a_i \frac{1}{|G|}\sum_{h_i\in G} g_i h_i^{-1} f(h_i v) \\ &= \left(\sum_{i=1}^{m} a_i g_i\right)\frac{1}{|G|}\sum_{h_i\in G} h_i^{-1} f(h_i v) = \left(\sum_{i=1}^{m} a_i g_i\right) h(v), \end{aligned}$$

"$h_i = gg_i$" $\Rightarrow h_i^{-1} = g_i^{-1} g^{-1}$

as desired.
So $V \cong W \oplus \ker(h)$ as vector spaces. Since W and $\ker(h)$ are both $\mathbb{C}[G]$-modules, we have $V \cong W \oplus \ker(h)$ as $\mathbb{C}[G]$-modules. ∎

Our next goal is to understand the ring $\mathbb{C}[G]$ better.

As a left $\mathbb{C}[G]$-module, $\mathbb{C}[G]$ corresponds to the left-regular representation. So as a $\mathbb{C}[G]$-module, we have

$$\mathbb{C}[G] \cong n_1 V_1 \oplus \cdots \oplus n_r V_r$$

where $V_1, \dots, V_r$ are the simple $\mathbb{C}[G]$-modules (corresponding to irreducible representations), with $n_i = \dim(V_i)$ as $\mathbb{C}$-vector spaces.

Claim: $\mathbb{C}[G] \cong \mathrm{End}(\mathbb{C}[G])$ as rings, if we make $\mathbb{C}[G]$ into a $\mathbb{C}[G]$-module by left mult
Proof: Define $\phi: \mathbb{C}[G] \to \mathrm{End}\,\mathbb{C}[G]$ by

$$\phi\left(\sum a_i g_i\right) = \left(\sum b_i g_i \mapsto \left(\sum b_i g_i\right)\left(\sum a_i g_i^{-1}\right)\right).$$

It is easy to see that ϕ is well-defined and $\mathbb{C}[G]$-linear. It's also clearly injective, so we need only check surjectivity. To this end, let $f \in \mathrm{End}\,\mathbb{C}[G]$.
Note that $\forall b \in \mathbb{C}[G]$ we have $f(b) = bf(1)$, so $f = \phi(f(1))$. ∎

As a left $\mathbb{C}[G]$-module, 2013 11 06

$$\mathbb{C}[G] \cong d_1V_1 \oplus \cdots \oplus d_rV_r$$

where $V_1, \dots, V_r$ are the irreducible representations of G (up to isomorphism). Since Shur's lemma implies that $\mathrm{Hom}(i,j) = 0$ if $i \neq j$, we see that any endomorphism of $\mathbb{C}[G]$ must be the sum of endomorphisms of d_iV_i (sum over i).

Theorem: Let V be a simple $\mathbb{C}[G]$ module (ie a module corresponding to an irreducible representation), and n any positive integer. Then $\mathrm{End}(nV) \cong M_n(\mathbb{C})$

Proof: Define $\phi: M_n(\mathbb{C}) \to \mathrm{End}(nV)$ by

$$\phi(M) = \left((v_1, \dots, v_n) \mapsto M\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\right).$$

It is easy to check that ϕ is a well-defined ring homomorphism whose kernel is zero (so ϕ is injective). It remains to show that ϕ is surjective.

Let $f \in \mathrm{End}(nV)$. Let $f_i = \pi_i \circ f$. So each f_i is a $\mathbb{C}[G]$ module homomorphism from nV to V. By restricting f_i to the j^{th} coordinate, we get a $\mathbb{C}[G]$ module homomorphism $f_{i,j}: V \to V$. By Shur's lemma, $f_{i,j} = a_{i,j} I$ for some $a_{i,j} \in \mathbb{C}$. So

$$\begin{aligned} f(v_1, \dots, v_n) &= (f_1(v_1, \dots, v_n), \dots, f_n(v_1, \dots, v_n)) \\ &= (f_{11}(v_1) + \cdots + f_{1n}(v_n), \dots, f_{n1}(v_1) + \cdots + f_{nn}(v_n)) \\ &= (a_{11}v_1 + \cdots + a_{1n}v_n, \dots, a_{n1}v_1 + \cdots + a_{nn}v_n) \\ &= \left[a_{ij}\right]_{i,j} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = Mv = \phi(M). \end{aligned}$$

Therefore ϕ is surjective as required, and thus an isomorphism of rings. ∎

So we see

ring isomorphisms

$$\begin{aligned} \mathbb{C}[G] &\cong \mathrm{End}(\mathbb{C}[G]) \cong \mathrm{End}(d_1V_1 \oplus \cdots \oplus d_rV_r) \cong \mathrm{End}(d_1V_1) \oplus \cdots \oplus \mathrm{End}(d_rV_r) \\ &\cong M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_r}(\mathbb{C}). \end{aligned}$$

What is the isomorphism? (the following is not the one we theorized above)

Let $\rho_i: G \to GL(V_i)$ be the representation corresponding to V_i.

Define $\rho: \mathbb{C}[G] \to M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_r}(\mathbb{C})$

$$\rho(g_i) = (\rho_1(g_i), \dots, \rho_r(g_i))$$ and extend to a ring homomorphism.

It is clear that ρ is a ring homomorphism. It is injective because if $\rho(\sum a_i g_i) = 0$ with not all $a_i = 0$ then the representations ρ_i would be linearly indep, meaning that their characters are linearly independent, but they are orthonormal. Since we already know that

$$\mathbb{C}[G] \cong M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_r}(\mathbb{C})$$

as $\mathbb{C}$-vector spaces, we conclude that ρ is surjective.

Notice that this means that the i^{th} component $\rho_i: \mathbb{C}[G] \to M_{d_i}(\mathbb{C})$ 2013 11 02
of ρ is surjective. That is, every linear transformation $f: \mathbb{C}^{d_i} \to \mathbb{C}^{d_i}$ can be realized as a linear combination of the matrices $\rho_i(g)$, $g \in G$.

The centre of $\mathbb{C}[G]$ is isomorphic to the centre of $M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_r}(\mathbb{C})$, which is $\mathbb{C}I \oplus \cdots \oplus \mathbb{C}I \cong \mathbb{C} \oplus \cdots \oplus \mathbb{C}$.
As a subring of $\mathbb{C}[G]$, the centre is a complex vector space of dimension r = # of conjugacy classes of G.
Say $(\sum a_i g_i) g = g(\sum a_i g_i)$ $\forall g \in G$. This is the same as $\sum a_i g_i g = \sum a_i g g_i$. Thus, if $\sum a_i g_i$ is in the centre of $\mathbb{C}[G]$, then for every g, i, j st $g g_i = g_j g$, we must have $a_i = a_j$. But $g g_i = g_j g \Leftrightarrow g^{-1} g_j g = g_i$, so in order for $\sum a_i g_i$ to be in the centre of $\mathbb{C}[G]$, we need $a_i = a_j$ if g_i and g_j are in the same conjugacy class.
Thus every element of the centre of $\mathbb{C}[G]$ is in the span of the elements $\sum_{g \in C} g$ for C a conjugacy class of G. This span has dimension r, so it must be equal to the centre of G

This only works for comm. rings

Let R be a ~~not necessarily~~ commutative ring (with identity). Let M, N be left R-modules.
Let B be the free R-module on the set $\{m \otimes n;\ (m,n) \in M \times N\}$. So $B = \{\sum a_i (m_i \otimes n_i);\ (m_i, n_i) \in M \times N,\ a_i \in R\}$. Let Z be the set of R-linear combinations of elements of B of the following form:

$$(m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n$$
$$m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2$$
$$(rm) \otimes n - r(m \otimes n)$$
$$m \otimes (rn) - r(m \otimes n)$$

Define $M \otimes_R N = B/Z$, where B/Z is the R-module that is the abelian group B/Z with the R-action $r(b+Z) = rb + Z$.

ex $M \otimes_R R \cong M$, as every element is of the form $m \otimes 1$

ex $R^n \otimes_R R^m \cong R^{nm}$

ex $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}] \cong \mathbb{Q}(\sqrt{2})$

Hilroy

ex (cont.) Define $\phi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}] \to \mathbb{Q}(\sqrt{2})$ by 2013 11 11

$$\phi\left(\sum_{i=1}^{n} q_i \otimes (a_i + b_i\sqrt{2})\right) = \sum_{i=1}^{n} q_i a_i + q_i b_i \sqrt{2}.$$

Then ϕ is a homomorphism of $\mathbb{Z}$-modules. To see that ϕ is surjective, well this is easy. To see injectivity, we will show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ and $\mathbb{Q}(\sqrt{2})$ are both 2-dim $\mathbb{Q}$-vector spaces and that ϕ is a $\mathbb{Q}$-linear map. All of these are easy to see except $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]) = 2$. But it is easy to check that $1 \otimes 1$ and $1 \otimes \sqrt{2}$ span the space. So it is at most two dimensional. But we have a surjective linear map to a 2-dim. space, so it is at least 2-dimensional.

Let R be a (not necessarily commutative) ring with identity. Let T be a ring with R as a subring, and let M be an left R-module. Define

can take T to be a right R-module

$$B = \{ \textstyle\sum_i t_i \otimes m_i \ ; \ t_i \in T, \ m_i \in M \}$$

Let Z be all integer-linear combinations of all elements of B of the form

$$(t_1 + t_2) \otimes m - t_1 \otimes m - t_2 \otimes m$$
$$t \otimes (m_1 + m_2) - t \otimes m_1 - t \otimes m_2$$
$$(tr) \otimes m - t \otimes (rm)$$

Define $T \otimes_R M = B/Z$ as abelian groups with a left T-module structure by $t(\sum t_i \otimes m_i) = \sum (tt_i) \otimes m_i$

ex let V be $\mathbb{C}$ with the ~~trivial~~ trivial-group action.
Consider $\mathbb{C}[\mathbb{Z}_2] \otimes_{\mathbb{C}[\{0\}]} V$. It is a $\mathbb{C}$-vector space, and

$$\sum (a_i \cdot 0 + b_i \cdot 1) \otimes z_i = \sum (a_i z_i \cdot 0 + b_i z_i \cdot 1) \otimes 1 \in \operatorname{span}\{0 \otimes 1, 1 \otimes 1\}$$

So $\mathbb{C}[\mathbb{Z}_2] \otimes_{\mathbb{C}[\{0\}]} V$ is a $\mathbb{C}[\mathbb{Z}_2]$-module, and is easily checked to correspond to the (2-dim) left regular rep of $\mathbb{Z}_2$. 2013 11 13

ex. Consider $\rho : \mathbb{Z}_4 \to GL(\mathbb{C})$, $\rho(n) = i^n$. Find $W = \mathbb{C}[D_4] \otimes_{\mathbb{C}[\mathbb{Z}_4]} \mathbb{C}$
An arbitrary elm of

$$\cancel{\sum a_i g_i \otimes v_i = \sum a_i g \otimes z_i + \sum b_i g \otimes z_i = \sum a_i \otimes z_i}$$

W is of the form

$$\sum_i \Big(\sum_j a_{ij} g_j \Big) \otimes v_i = \sum_i \Big(\sum_j A_{ij} y^j \Big) \otimes z_i + \sum_i \Big(\sum_j B_{ij} x y^j \Big) \otimes z_i'$$

where $B_{ij} = a_{ij}$ for j corresponding to xy^j and $A_{ij} = a_{ij}$ for j corresponding to y^j.

$$= \sum_i \Big(\sum_j A_{ij} \otimes \rho(y^j) z_i \Big) + \sum_i \Big(\sum_j B_{ij} \otimes \rho(y^j) z_i' \Big)$$

$$= A(1 \otimes 1) + B(x \otimes 1)$$

where $A, B \in \mathbb{C}$.

Let $\tau : D_4 \to GL(V)$ be the rep. associated to W.

Then

$$(\tau(y^n))(1 \otimes 1) = i^n (1 \otimes 1)$$
$$(\tau(y^n))(x \otimes 1) = i^{-n} (x \otimes 1)$$
$$(\tau(xy^n))(1 \otimes 1) = i^n (x \otimes 1)$$
$$(\tau(xy^n))(x \otimes 1) = i^{-n} (1 \otimes 1).$$

Theorem: Let G be a finite group, $H \leq G$ a subgroup, $\rho: H \to GL(V)$ a representation, and $\tau: G \to GL(V)$ ~~as~~ the induced representation. 2013 11 15
Then τ corresponds to the $\mathbb{C}[G]$-module $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$.

Proof: To construct τ, we choose $g_1, \ldots, g_r \in G$ so that $g_1 H, \ldots, g_r H$ is a complete and irredundant list of left H-cosets in G. Let $W = g_1 V \oplus \cdots + g_r V$ with
$(\tau(g))(g_i v) = g_j (\rho(h))(v)$, where $g g_i = g_j h$ for some $j \in [r]$, $h \in H$.
Define $W' = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ with $\mathbb{C}[G]$-module structure induced by multiplication on the left.
Define $\phi: W \to W'$ by $\phi(\sum g_i v_i) = \sum g_i \otimes v_i$. We will show that ϕ is an isomorphism of $\mathbb{C}[G]$-modules.
It is straight forward to show that ϕ is a homomorphism of $\mathbb{C}[G]$-modules. It is surjective because W' is spanned by elements of the form $g \otimes v$, and $g_i h \otimes v = g_i \otimes h(v)$.
It remains to show that ϕ is injective. But it suffices to show $\dim(W) = \dim(W')$, because ϕ is a $\mathbb{C}$-lin. transformation.
Well, $\dim(W) = [G:H] \dim(V)$.
To compute $\dim(W')$, define the linear map $T: \mathbb{C}[G] \otimes_{\mathbb{C}} V \to \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ by $T(\alpha \otimes v) = \alpha \otimes v$ and extending linearly. It is not hard to see that T is a linear transformation and that it is surjective. The kernel of T is the span of elements of the form $gh \otimes v - g \otimes hv$. The kernel of T is ~~also~~ thus also spanned by elements of the form
$$g_i h \otimes v_j - g_i \otimes h v_j,$$
where $v_1, \ldots, v_n$ is a basis for V. In this list, we may further neglect terms with $h = 1$. So $\ker(T)$ is now spanned by the following collection of elements:
$$g_i h \otimes v_j - g_i \otimes h v_j; \quad h \in H \setminus \{1\}.$$
There are $[G:H](\#H - 1)\dim(V)$ such elements, so $\ker(T)$ has at most this dimension. By rank-nullity, $\dim(W') = \dim(\mathrm{im}(T)) \geq \dim(V)\dim(\mathbb{C}[G]) - \dim(\ker(T)) \geq \cdots = \dim(V)[G:H]$.
Since ϕ maps W onto W', we conclude by counting dimensions that ϕ is an isomorphism. ■