new topic

Recall that a group G is abelian if and only if all of its irreducible representations are 1-dimensional. 2013 10 21

Let G be a group, and $H \leq G$ a normal subgroup such that G/H is abelian. This means that there is a surjective homomorphism $q: G \to A$ where A is an abelian group and $\ker(q) = H$.

$\underbrace{\mathrm{im}(q)}_{A} \cong G/\ker(q)$

Note

$$q(xy) = q(x)q(y) = q(y)q(x) = q(yx)$$

for any $x, y \in G$. In particular,

$$q(xyx^{-1}y^{-1}) = 1.$$

The element $xyx^{-1}y^{-1}$ is called the commutator of x and y, written as $[x,y]$ sometimes.

Let N be the subgroup generated by all commutators of G. Then N is a normal subgroup of G, and is called the commutator subgroup of G, sometimes denoted $[G,G]$.

The preceding argument and its unwritten converse show that G/H is abelian if and only if H contains the commutator subgroup of G.

Every one-dimensional representation $\rho: \overbrace{G/N}^{=:A} \to GL(\mathbb{C})$ gives rise to a one-dimensional representation $\rho \circ q: G \to GL(\mathbb{C})$.

Conversely, if $\tau: G \to GL(\mathbb{C})$ is a one-dimensional representation, then since $GL(\mathbb{C})$ is abelian, we have $N \subseteq \ker(\tau)$. By the Universal Property of Quotients, τ induces a homomorphism $\tilde{\tau}: G/N \to GL(\mathbb{C})$ which is a representation of G/N.

We have therefore established the following Theorem.

Theorem: The one-dimensional representations of a finite group G are in one-to-one correspondence with the one-dimensional representations of G/N, where N is the commutator subgroup.

Remark: G/N is called the abelianization of G.

Theorem: Let G be a finite group, and let $A \subseteq G$ be an abelian subgroup. Then any irreducible representation of G has dimension at most $|G|/|A|$.

Proof: Let $\rho: G \to GL(V)$ be an irreducible representation of G. Let $\tau = \rho|_A$. Let $W \subseteq V$ be an A-invariant subspace of V. Then $\dim(W) = 1$, say $\{w\}$ is a basis for W. Let $W' = \text{span}\{(\rho(g))(w);\ g \in G\}$. Then W' is G-invariant. So either $W' = \{0\}$ or $W' = V$. Thus $V = W'$. But if $g_1 g_2^{-1} \in A$ then $g_1(w) = (g_2 a)(w) = g_2(\lambda w) = \lambda g_2(w)$ (some $\lambda \in \mathbb{C}$). Hence $|W'| \leq$ the number of cosets. ∎

Since D_n has an abelian (cyclic, even) subgroup of order n, it follows that every irreducible representation of D_n has dimension 1 or 2.

new topic

Say G_1 and G_2 are finite groups. What are the irreducible representations of $G_1 \times G_2$. 2013 10 23

Let $\rho_i : G_i \to GL(V_i)$ be two representations. Define

$$\rho : G_1 \times G_2 \to GL(V_1 \otimes V_2)$$

$$\rho((g_1, g_2)) = \left(\textstyle\sum_{j=1}^{n} v_j \otimes w_j \mapsto \sum_{j=1}^{n} (\rho_1(g_1))(v_j) \otimes (\rho_2(g_2))(w_j)\right).$$

We usually write $\rho_1 \otimes \rho_2$ for ρ. $\leftarrow$ this notation is an abuse of nature

It is easy to check that ρ is a representation, and that if ρ_1, ρ_2 are irreducible then ρ is irreducible. To see the second, note that

$$\chi_\rho((g_1, g_2)) = \chi_{\rho_1}(g_1)\chi_{\rho_2}(g_2)$$

and so

$$\langle \chi_\rho, \chi_\rho \rangle = \frac{1}{|G_1 \times G_2|} \sum_{(g_1,g_2) \in G_1 \times G_2} \chi_\rho((g_1,g_2)) \overline{\chi_\rho(g_1,g_2)}$$

$$= \frac{1}{|G_1|} \frac{1}{|G_2|} \sum_{g_1 \in G_1} \sum_{g_2 \in G_2} \chi_{\rho_1}(g_1)\chi_{\rho_2}(g_2) \overline{\chi_{\rho_1}(g_1)}\, \overline{\chi_{\rho_2}(g_2)}$$

$$= \langle \chi_{\rho_1}, \chi_{\rho_1} \rangle \langle \chi_{\rho_2}, \chi_{\rho_2} \rangle = 1 \cdot 1 = 1.$$

We know that if G_i has a_i conjugacy classes, then $G_1 \times G_2$ has $a_1 a_2$ conjugacy classes, and hence $a_1 a_2$ irreducible representations. But we just constructed $a_1 a_2$ irreducible representations. To see that no two are isomorphic, a similar calculation shows that if $(\rho_1, \rho_2) \neq (\rho_1', \rho_2')$ then $\langle \chi_\rho, \chi_{\rho'} \rangle = \langle \chi_{\rho_1}, \chi_{\rho_1'} \rangle \langle \chi_{\rho_2}, \chi_{\rho_2'} \rangle = 0$.

new topic

Let G be a finite group, H a subgroup. Let $\rho : H \to GL(V)$ be a representation. We will define $\mathrm{Ind}_H^G \rho$.

Pick $g_1, \ldots, g_n \in G$ so that $g_1H, \ldots, g_nH$ is a complete and distinct list of left cosets of H in G. Define $W = g_1V \oplus \cdots \oplus g_nV$

Define a homomorphism $\tau : G \to GL(W)$ by

$$(\tau(g))(g_1\vec{v}_1 + \cdots + g_n\vec{v}_n) = g_{j_1}(\rho(h_1))(v_1) + \cdots + g_{j_n}(\rho(h_n))(v_n)$$

where $gg_r = g_{j_r}h_r$ for $h_r \in H$ and $j_r \in \{1, \ldots, n\}$. A straightforward check shows that τ is a representation of G. We define $\mathrm{Ind}_H^G \rho = \tau$.

ex Let G be a group and $H \subset G$ trivial, with $\rho : H \to GL(\mathbb{C})$ the trivial representation. Then $\{g_1, \ldots, g_n\} = G$ and $W = \bigoplus_{g \in G} g\mathbb{C}$. For any r we see $gg_r = g_{j_r}$, and so

write as $gg_1 \cdot v_1$ and it looks natu

$$((\mathrm{Ind}_H^G \rho)(g))(g_1v_1 + \cdots + g_nv_n) = \rho(gg_1)v_1 + \cdots + \rho(gg_n)v_n.$$

That is, $\mathrm{Ind}_H^G \rho$ is the left regular representation.

ex $\mathrm{Ind}_G^G \rho \cong \rho$

2013 10 25

ex Let $\rho : \{0,2\} \cong \mathbb{Z}_2 \to GL(\mathbb{C})$ be the trivial representation. Then $\mathrm{Ind}_{\{0,2\}}^{\mathbb{Z}_4} \rho$ is the permutation representation of G acting on left H-cosets by left multiplication.

ex Let $\rho : \{0,2\} \to GL(\mathbb{C})$ be the sign representation. What is $\overset{\tau}{\overset{\shortparallel}{\mathrm{Ind}_{\{0,2\}}^{\mathbb{Z}_4} \rho}}$?

Note: In general we (always) have $\dim(\mathrm{Ind}_H^G \rho) = [G:H]\dim(\rho) = |G|/|H| \dim(\rho)$.

ex (cont.) A basis for W is $\{0(1), 1(1)\}$ (ie $g_1 = 0$, $g_2 = 1$). Now we can compute matrices for τ wrt this matrix. Certainly

$$\tau(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Next note

$$(\tau(1))(0(1)) = (1+0)(1) = 1(\tau(0))(1) = 1(1)$$
$$(\tau(1))(1(1)) = \underbrace{(1+1)}_{= 0+2}(1) = 0(\tau(2))(1) = 0(-1) = -0(1).$$

Therefore

$$\tau(1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since 3 is the inverse of 1, $\tau(3)$ will the inverse of $\tau(1)$:

$$\tau(3) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Finally,

$$(\tau(2))(0(1)) = (2+0)(1) = 0(\rho(2))(1) = 0(-1) = -0(1)$$
$$(\tau(2))(1(1)) = (2+1)(1) = 1(\rho(2))(1) = 1(-1) = -1(1)$$

and so

$$\tau(2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that $\rho \cong \rho_1 \oplus \rho_2$ where ρ_1 and ρ_2 are the two extensions of ρ to $\mathbb{Z}_4$.

Theorem: Let $\rho : H \to GL(V)$ be a representation, $H \leq G$.

$$\chi_{\mathrm{Ind}_H^G \rho}(g) = \sum_{g_i^{-1} g g_i \in H} \chi_\rho(g_i^{-1} g g_i) = \frac{1}{|H|} \sum_{x^{-1} g x \in H} \chi_\rho(x^{-1} g x)$$

Proof: Let $\{g_i v_j\}$ be a basis for $\bigoplus_{i=1}^{n} g_i V$ (so $\{v_j\}$ basis for V). The trace of $\mathrm{Ind}_H^G(g)$ is the sum of all coefficients of $g_i v_j$ in $((\mathrm{Ind}_H^G \rho)(g))(g_i v_j)$. Write $g g_i = g_k h_k$ for a g_k and an $h_k \in H$. If $g_i \neq g_k$ then the coefficient will be 0. Now, $g_i = g_k \iff h_i = g_i^{-1} g g_i$, in which case $((\mathrm{Ind}_H^G \rho)(g))(g_i v_j) = g_i \cdot \rho(h_i)(v_j) = g_i \cdot \rho(g_i^{-1} g g_i)(v_j)$ $= \rho(g_i^{-1} g g_i)(g_i v_j)$, which has coefficient $\rho(g_i^{-1} g g_i)$. The second equality follows from group theory.

$D_4 = \langle x, y ; x^2 = y^4 = 1, xy = y^{-1}x \rangle$

2013 10 28

ex Let $y \in D_4$ have order 4 (a rotation)

Define $\rho : \langle y \rangle \to GL(\mathbb{C})$ by $\rho(y^a) = i^a$.

Note we cannot extend ρ to a representation of G. What is $\tau = \mathrm{Ind}_{\langle y \rangle}^{D_4} \rho$?

$W = \mathbb{C} \oplus x\mathbb{C}$ because x can't send $\mathbb{C}$ back to itself.

That is, $xv := (\tau(x))(v)$ for $v \in \mathbb{C}$.

Then we must have

$$(\tau(y))(xv) = (\tau(y))((\tau(x))(v)) = ((\tau(y))(\tau(x)))(v) = (\tau(yx))(v)$$
$$= (\tau(xy^{-1}))(v) = (\tau(x)\tau(y^{-1}))(v) = (\tau(x))((\tau(y^{-1}))(v)) = (\tau(x))(-iv)$$
$$= -i(\tau(x))(v) = -ixv.$$

So we can find, by similar computation,

$$\tau(y) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \tau(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

From here we get $\chi_{\tau} = (2,0,-2,0,0,0,0,0)$. This is consistent with the theorem we had. Yay.

Redefining $\mathbb{C}[G]$

Def Let G be a group. The <u>complex group ring</u> of G is $\mathbb{C}[G] = \bigoplus_{g\in G} \mathbb{C}g$, where $(ag)(bh) = ab(gh)$ for $g,h \in G$ and $a,b \in \mathbb{C}$.

Note: $\mathbb{C}[G]$ is a ring, and is commutative $\Leftrightarrow$ G is abelian.

new topic: modules

Def Let R be a ring with identity. A left R-module is an abelian group M together with an operation $\cdot : R\times M \to M$ satisfying: 2013 10 30

i) $r_1(r_2 m) = (r_1 r_2)m$

ii) $1m = m$

iii) $(r_1+r_2)m = r_1 m + r_2 m$

iv) $r(m_1+m_2) = rm_1 + rm_2$

for all $r, r_1, r_2 \in R$, $m, m_1, m_2 \in M$.

Remark: If R is a field, then an R-module is the same thing as an R-vector space.

ex $\mathbb{Z}^n$ is a $\mathbb{Z}$-module

ex $\mathbb{Z}_n$ is a $\mathbb{Z}$-module

ex any abelian group is a $\mathbb{Z}$-module, and vice versa

ex Let R be any ring an $I \leq R$ a left ideal. Then I is an R-module. In fact, a left ideal R is exactly a left R-submodule of R.

Say $\rho: G \to GL(V)$ is a representation of G. We can make V into a left $\mathbb{C}[G]$-module by

$$\left(\sum_{i=1}^{m} a_i g_i\right)(v) = \sum_{i=1}^{m} a_i (\rho(g_i))(v).$$