Now we apply this theorem to pt. That is, to look for more (irreducible) characters, we consider pt⊗pt and try to decompose this into irreducible representations. We don't actually know the representation pt, but since we only care about the characters, we can skip past the actual representations and use the previous theorem. So, with $\rho = pt$ in the theorem, we get for χ_{ρ_s}:

$$(1) \mapsto \frac{4^2+4}{2} = 10, \quad (123) \mapsto \frac{1+1}{2} = 1, \quad (12)(34) \mapsto \frac{0+4}{2} = 2, \quad (12345), (13452) \mapsto 0.$$

Note

$$\langle \chi_{\rho_s}, \chi_{\rho_s} \rangle = \frac{1}{60}\left(10^2 + 20 \cdot 1^2 + 15 \cdot 2^2 + 12 \cdot 0^2 + 12 \cdot 0^2\right) = 3,$$

and hence ρ_s is a sum of 3 irreducible representations. Check that $\langle \chi_{\rho_s}, \chi_{triv} \rangle = \langle \chi_{\rho_s}, \chi_{pt} \rangle = 1$. Hence $\rho_s \cong \rho_{triv} \oplus \rho_{pt} \oplus \rho_\eta$ for some irreducible representation ρ_η. So $\chi_{\rho_\eta} = \chi_{\rho_s} - \chi_{triv} - \chi_{pt}$ is added to the table

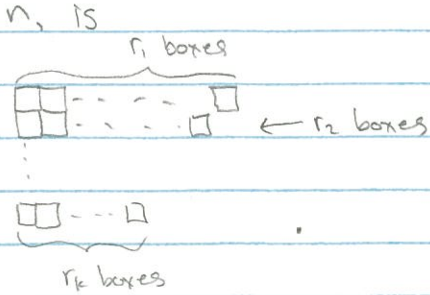
Next looking at χ_{ρ_a}, we find that it contains non of the already know irreducible representations and must be the sum of the last two characters, say $\rho_a \cong \rho_1 \oplus \rho_2$. This can now be solved with linear algebra to get $\chi_{\rho_1}, \chi_{\rho_2}$. Note they each have dimension 3. So we have the character table for A_5.

now look at S_n

Recall: The conjugacy classes of S_n are the sets of permutations with the same cycle structure. 2013 10 11

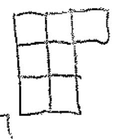
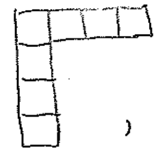
Fact: The conjugacy classes in S_n are in one-to-one correspondence with Young tableaux. The young tableau associated to

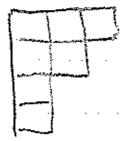
$S_n \ni (a_{1,1} \cdots a_{1,r_1})(a_{2,1} \cdots a_{2,r_2}) \cdots (a_{k,1} \cdots a_{k,r_k})$, where $r_1 \geq \cdots \geq r_k$ and $r_1 + \cdots + r_k = n$, is



.

ex in S_7,

$(1\ 2\ 3)(4\ 5)(6\ 7) \longleftrightarrow$  , $(1\ 2\ 3\ 4) \longleftrightarrow$  ,

$(1\ 2)(3\ 4\ 5) \longleftrightarrow$ 

<u>Def</u>] A <u>numbering</u> of a Young tableau is an injective function from $\{1,\dots,n\}$ to the boxes in the tableau

Remark: S_n acts on the set of numberings of a fixed tableau.

ex for S_7,

$$(1\ 2\ 3)(5\ 6)\cdot \begin{array}{|c|c|c|}\hline 3&4&5\\\hline 6&7&\\\cline{1-2} 1&&\\\cline{1-1} 2&&\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 2&4&6\\\hline 5&7&\\\cline{1-2} 3&&\\\cline{1-1} 1&&\\\cline{1-1}\end{array}$$

<u>Def</u>] A <u>tabloid</u> is an equivalence class of numberings (of some fixed tableau), where two numberings are equivalent if for each row, both numberings have the same set of numbers used.

ex
$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 11&5&7&\\\cline{1-3} 6&9&&\\\cline{1-2} 10&8&&\\\cline{1-2}\end{array} \sim \begin{array}{|c|c|c|c|}\hline 4&1&3&2\\\hline 11&7&5&\\\cline{1-3} 6&9&&\\\cline{1-2} 10&8&&\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|}\hline 1&2\\\hline 2&4\\\hline\end{array} \not\sim \begin{array}{|c|c|}\hline 3&4\\\hline 2&1\\\hline\end{array}$$

Remark: S_n also acts on the set of tabloids of a given fixed shape. — (*)

ex Let s be a fixed shape (Young tableau). Let $M^s = \bigoplus_T \mathbb{C}T$, where T ranges over all tabloids of shape s. That is, M^s is a vector space over $\mathbb{C}$ with the set of tabloids of shape s as a basis. Then there is a representation of S_n in M^s coming from the S_n action on the tabloids.

after next definition

<u>Def</u>] Let $[T]$ be the tabloid associated with T. Define
$$V_T = \sum_{\sigma \in C(T)} (-1)^{\mathrm{sgn}(\sigma)} [\sigma(T)] \in M^s,$$
~~One can check that this is well-defined quite easily.~~

where s is the shape of T.

Def Let T be a numbering of a tableau. Let $\underline{C(T)}$ be the subgroup of S_n of elements $\sigma \in S_n$ that preserve the columns of T (so m and $\sigma(m)$ are in the same column). Let $\underline{R(T)}$ be the subgroup of S_n of elements of S_n that preserve the rows of T.

Theorem: Let $S^s = \text{span}\{v_T ; \underbrace{}_{T \text{ a tableau of shape } s}\}$. Then S^s is an S_n-invariant subspace of M^s.

Proof: Note

$$\tau(v_T) = \tau\Big(\sum_{\sigma \in C(T)} (-1)^{\operatorname{sgn}(\sigma)} [\sigma(T)]\Big) = \sum_{\sigma \in C(T)} (-1)^{\operatorname{sgn}(\sigma)} [(\tau \circ \sigma)(T)]$$

$$= \sum_{\sigma \in C(\tau(T))} (-1)^{\operatorname{sgn}(\sigma)} [(\tau \circ (\tau^{-1} \circ \sigma \circ \tau))(T)]$$

$$= \sum_{\sigma \in C(\tau(T))} (-1)^{\operatorname{sgn}(\sigma)} [\sigma(\tau(T))]$$

$$= v_{\underbrace{\tau(T)}_{\text{tableau of shape } s}} \in S^s \qquad \blacksquare$$

Def S^s is called the Specht module.

Def Let s and t be Young tableaux of the same size. Write $s = (s_1, \dots, s_k)$ and $t = (t_1, \dots, t_k)$ where x_i is the number of boxes in row i of x, and s_i or t_i is zero if it doesn't exist. We say s dominates t if $s_1 + \dots + s_m \geq t_1 + \dots + t_m$ $\forall m$. We say that s strictly dominates t if s dominates t and $s \neq t$. 2013 11 16

ex  dominates ,  is not comparable to 

not too important

Theorem: Let T, T' be numberings of shapes s, s' of the same size, with s not strictly dominating s'. Then either:

(i) there are 2 different numbers in the same row of T' and the same column of T; or

(ii) $s = s'$ and there is some $p' \in R(T')$ and $q \in C(T)$ such that $p'(T') = q(T)$.

Proof. Assume (i) fails. We will show (ii) holds. Choose $q_1 \in C(T)$ such that all the numbers in the first row of T' are in the first row of $q_1(T)$ or higher, which is possible because (i) fails. Keep going until you get $q = q_k \circ \cdots \circ q_1 \in C(T)$ such that for every i, all the numbers in the i^{th} row of T' are in the i^{th} row of $q(T)$ ~~or this~~ or higher. This means that s dominates s'. Therefore $s = s'$ by assumption, and each row of T' has the same set of numbers as the corresponding rows $q(T)$. So there is some $p' \in R(T')$ such that $p'(T') = q(T)$. ☐ sketch

Def] Let

$$b_T = \sum_{\sigma \in C(T)} (-1)^{\mathrm{sgn}(\sigma)} \sigma \in \bigoplus_g \mathbb{C}g$$

$\sigma([T]) = [\sigma(T)]$ check well-defined (it is by (*)) ↑ prev. page

Remark: $V_T = b_T([T])$. Better yet, $b_T \sigma = (-1)^{\mathrm{sgn}(\sigma)} b_T$, and

$$b_T b_T = \sum_{\sigma \in C(T)} \sum_{\tau \in C(T)} (-1)^{\mathrm{sgn}(\sigma) + \mathrm{sgn}(\tau)} \sigma\tau \qquad = \#C(T)\, b_T$$

very important

Theorem: Let T, T', s, s' be as above. If (i) holds then $b_T([T']) = 0$. Otherwise $b_T([T']) \in \{\pm V_T\}$.

Proof: If (i) holds, let τ be the transposition switching the two numbers. 2013 11 18

Then $\tau \in R(T)$ and $\tau \in C(T')$. Then $b_T \cdot \tau = -b_T$ and

$b_T(T') = b_T(\tau(T')) = (b_T \cdot \tau)(T') = -b_T(T')$.

Hence $b_T(T') = 0$.

If (ii) holds, then choose $p' \in R(T')$ and $q \in C(T)$ such that $p'(T') = q(T)$.

Then $b_T(T') = b_T(p'(T')) = b_T(q(T)) = (-1)^{\mathrm{sgn}(q)} b_T(T) = \pm V_T$. ☐

(T still a numbering of s)

We will show that S^s is irreducible with $S^s \not\cong S^t$ for $s \neq t$.

Assume $S^s = V \oplus W$ for some subspaces V, W that are S_n-invariant. Let T be any numbering of s. Then

$$b_T(S^s) = b_T(\mathrm{span}\{V_{T'};\ T' \text{ tableau of shape } s\})$$
$$= \mathrm{span}\{b_T(V_{T'});\ T' \text{ tableau of shape } s\}$$
$$= \mathrm{span}\{V_T\}$$

which is 1-dimensional. So $b_T(V) \oplus b_T(W)$ is 1-dimensional, meaning (WLOG) $b_T(V) = \mathrm{span}\{V_T\}$. But V is S_n-invariant, so $b_T(V) \subseteq V \Rightarrow V_T \in V$. As V is S_n-invariant, $\mathrm{span}\{\sigma(V_T);\ \sigma \in S_n\} \subseteq V$. But $\sigma(V_T) = V_{\sigma(T)}$ so $S^s = \mathrm{span}\{\sigma(V_T)\} \subseteq V$. ☐

Now for $s \neq t$, we have $b_T(S^t) = 0$ but $b_T(S^s) \neq 0$ so $S^s \not\cong S^t$.

ex Compute a character table for S_4. We know 5 permutation representations of S_4:

not the character table

	1	6	8	6	3	
	(1)	$(1\,2)$	$(1\,2\,3)$	$(1\,2\,3\,4)$	$(1\,2)(3\,4)$	
$M^{(4)}$	1	1	1	1	1	1
$M^{(3,1)}$	4	2	1	0	0	3
$M^{(2,2)}$	6	2	0	0	2	2
$M^{(2,1,1)}$	12	2	0	0	0	[illegible]
$M^{(1,1,1,1)}$	24	0	0	0	0	

See A4 for some ideas.