~~$\chi_\rho = \mathrm{Tr}\circ\rho$~~

Def Let $\rho: G \to GL(V)$ be a representation. The <u>character of ρ</u> is the function $\chi_\rho : G \to \mathbb{C}$ given by $\chi_\rho = \mathrm{Tr}\circ\rho$ (trace). If ρ is irreducible then χ_ρ is called an <u>irreducible character</u>.

Remark: If $\dim(\rho) = 1$ then $\chi_\rho \cong \rho$.

Remark: If $\rho \cong \tau$ then $\chi_\rho = \chi_\tau$. This is because trace is invariant under linear isomorphism.

$\chi_\rho(g) = tr(\rho(g)) = tr(\rho(g)\circ T) = tr(T\circ\tau(g)) = tr(\tau(g))$

Remark. $\chi_\rho(1) = \dim(\rho)$, $\chi_\rho(g^{-1}hg) = \chi_\rho(h)$.

$\mathrm{Tr}(\rho(g)^{-1}\rho(h)\rho(g))$
$= \mathrm{Tr}(\rho(g)\cancel{\rho(g)^{-1}}\rho(h))$
$= \mathrm{Tr}(\rho(h))$

Remark: $\chi_{\rho\oplus\tau} = \chi_\rho + \chi_\tau$, $\chi_{\rho\otimes\tau} = \chi_\rho \cdot \chi_\tau$.

Remark $\chi_\rho(g^{-1}) = \sum$ eigenvalues of $(\rho(g^{-1}) = \rho(g)^{-1})$
$= \sum \overline{\text{eigenvalues of } \rho(g)}$
$= \sum \overline{\text{eigenvalues of } \rho(g)}$
$= \overline{\sum \text{eigenvalues of } \rho(g)} = \overline{\chi_\rho(g)}$

Theorem: Let $\rho: G \to GL(V)$ and $\tau: G \to GL(W)$ be representations. If $T: V \to W$ is any linear transformation then $T' = \sum_{g\in G} \tau(g^{-1})\circ T\circ\rho(g)$ is a morphism from ρ to τ. In particular, if ρ, τ are irreducible and $\rho \not\cong \tau$ then $T' = 0$.
Proof: Exercise.

Def Let G be a finite group. Define $\mathbb{C}[G] = \{f: G \to \mathbb{C}\}$ is the <u>complex group ring</u>.

Remark: $\mathbb{C}[G]$ has a natural inner product

$$\langle \varphi, \psi \rangle = \frac{1}{|G|}\sum_{g\in G} \varphi(g)\overline{\psi(g)}.$$

Note that this corresponds to the standard inner product on $\mathbb{C}^{|G|}$, as we can think of elements of $\mathbb{C}[G]$ as $|G|$-tuples, except we have normalized it by $1/|G|$.

literally, they form a subset of $\mathbb{C}[G]$?

2013 09 30

Theorem: Irreducible characters are an orthonormal subset of $\mathbb{C}[G]$.

Proof: We show that if ρ and τ are irreducible representations to V and W respectively, then

$$\langle \chi_\rho, \chi_\tau \rangle = \begin{cases} 1 & \rho \cong \tau, \\ 0 & \rho \not\cong \tau. \end{cases}$$

Pick bases for V and W. For each $g \in G$, write

$$\rho(g) = (r_{ij}(g))_{ij},$$
$$\tau(g) = (t_{ij}(g))_{ij},$$

as matices. Then

$$\langle \chi_\rho, \chi_\tau \rangle = \frac{1}{|G|} \sum_{g\in G} \sum_{i=1}^{?} r_{ii}(g) \sum_{j=1}^{?} \overline{t_{jj}(g)}$$
$$= \frac{1}{|G|} \sum_{g\in G} \sum_{i=1} r_{ii}(g) \sum_{j=1} t_{jj}(g^{-1}).$$

Let T be any linear transformation $V \to W$. Then

$$T' = \sum_{g\in G} \tau(g^{-1}) \circ T \circ \rho(g)$$

is a morphism from ρ to τ.

If $\rho \not\cong \tau$, then $T' = 0$ (by thm last day). So if $T = (T_{ij})_{ij}$, then

$$\sum_{g\in G} \sum_{i=1}^{?} \sum_{j=1}^{?} t_{k,i}(g^{-1})\, T_{i,j}\, r_{j,\ell}(g) = 0, \quad \forall (k,\ell).$$

$T_{ij}(k,\ell)$ So in particular, we can take $T_{ij} = \delta_{(i,j),(k,\ell)}$, to get

$$\sum_{g\in G} t_{kk}(g^{-1})\, r_{\ell\ell}(g) = 0.$$

Summing over all (k,ℓ), we get $|G|\langle \chi_\rho, \chi_\tau \rangle = 0$ as desired.
If $\rho \cong \tau$, then by Shur's lemma we have $T' = \lambda I$, where $\lambda = |G| \operatorname{tr}(T)/\dim(V)$.
Exactly the same argument gives $\langle \chi_\rho, \chi_\tau \rangle = 1$. ∎

Remark: This means that there are only finitely many irreducible representations of G, up to isomorphism, because the corresponding characters are an orthonormal set (and hence linearly independent) in a finite dimensional vector space.

where $m_i\rho_i = \underbrace{\rho_i \oplus \cdots \oplus \rho_i}_{m_i \text{ times}}$

Remark: If $\rho \cong \bigoplus_{i=1}^{r} m_i \rho_i$, for $m_i \in \mathbb{Z}$ and the ρ_i pairwise non isomorphic irreducible representations, then $\langle \chi_\rho, \chi_{\rho_j} \rangle = m_j$, because
$\langle \chi_\rho, \chi_{\rho_j} \rangle = \langle \sum_{i=1}^{r} m_i \chi_{\rho_i}, \chi_{\rho_j} \rangle = \sum_{i=1}^{r} m_i \langle \chi_{\rho_i}, \chi_{\rho_j} \rangle = \sum_{i=1}^{r} m_i \delta_{ij} = m_j$.

Remark: Irreducible decompositions of representations are unique up to isomorphism.

Remark: Non-isomorphic representations have different characters because the character determines the irreducible decomposition.

Remark: We have $\langle \chi_\rho, \chi_\rho \rangle = \sum_{i=1}^{r} m_i^2$. Hence $\langle \chi_\rho, \chi_\rho \rangle = 1$ if and only if ρ is irreducible.

ex. From A1Q5, the traces of $\rho(g)$ are $2, 0, 0, 0, -1, -1$, and

$$\frac{2^2 + 0^2 + 0^2 + 0^2 + (-1)^2 + (-1)^2}{6} = 1$$

$\therefore$ it is irreducible.

ex. Let $\rho : S_3 \to GL(\mathbb{C}^3)$ be the permutation representation. Write ρ as a sum of irreducible representations: 2013 10 02

$$\chi_\rho(1) = 3, \quad \chi_\rho(\text{2-cycle}) = 1, \quad \chi_\rho(\text{3-cycle}) = 0.$$

Hence

$$\langle \chi_\rho, \chi_\rho \rangle = \tfrac{1}{6}(3^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2) = 2.$$

Let $\triangle$ be the representation above. Then

$$\langle \chi_\rho, \chi_\triangle \rangle = \tfrac{1}{6}(3 \cdot 2 + 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 0 + 0 \cdot -1 + 0 \cdot -1) = 1.$$

This means $\rho \cong \triangle \oplus \rho'$ for some ρ' whose irreducible decomposition contains no $\triangle$. Further, note

$$\langle \chi_\rho, \chi_{\text{triv}} \rangle = \tfrac{1}{6}(3 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 + 0 \cdot 1) = 1.$$

Thus $\rho \cong \triangle \oplus \text{triv}$.

ex. Let G be any finite group. Let

$$V = \bigoplus_{g \in G} \mathbb{C} g.$$

Define $\rho : G \to GL(V)$ by $(\rho(g))(h) = gh$, and extend linear (as G is a basis for V).

Then $\chi_\rho(1) = |G|$ and $\chi_\rho(h) = 0$ $\forall h \in G \setminus \{1\}$. So $\langle \chi_\rho, \chi_\rho \rangle = |G|$. $\frac{1}{|G|}(|G|^2 + 0 + \cdots + 0) = |G|$

If τ is any irreducible representation of G, then $\langle \chi_\rho, \chi_\tau \rangle = \frac{1}{|G|}(|G| (\dim(\tau)) + 0 \cdots + 0) = \dim(\tau)$.

So $\rho \cong (d_1\rho_1) \oplus \cdots \oplus (d_r\rho_r)$ where $\rho_1, \ldots, \rho_r$ are all the irreducible representations of G (up to isomorphism) and $d_i = \dim(\rho_i)$.
This means

$$|G| = \langle \chi_\rho, \chi_\rho \rangle = d_1^2 + \cdots + d_r^2.$$

How many irreducible representations does G have? Do they span $\mathbb{C}[G]$? Not usually.
Recall that characters are constant on conjugacy classes.

<u>Def</u>] A class function on a group G is a function $G \to \mathbb{C}$ that is constant on conjugacy classes.

Remark: Class functions are a subspace ω of $\mathbb{C}[G]$.

Do the irreducible characters span ω?

Theorem: Let G be a finite group. The irreducible characters of G are an orthonormal basis of the space of class functions of G.

Remark: This shows that if G has k conjugacy classes then it has k irreducible characters.

Proof: We will show that $W^\perp \cap \omega = \{0\}$ where W is the span of the irreducible characters. Note that $W = \overline{W}$ and $W^\perp = \overline{W^\perp}$, so $f \in W^\perp$ if and only if $\bar{f} \in W^\perp$. Let $f \in W^\perp$
Define

$$T_f^\rho(v) = \sum_{g \in G} f(g)(\rho(g))(v).$$

2013 10 04

It is straightforward to check that T_f^ρ is a morphism $\rho \to \rho$. If ρ is irreducible, then $T_f^\rho = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{C}$. But

$$0 = \langle \chi_\rho, \bar{f} \rangle = \sum_{g \in G} \chi_\rho(g) f(g) = \sum_{g \in G} f(g) \operatorname{Tr}(\rho(g)) = \operatorname{Tr}(T_f^\rho),$$

and so $\lambda = 0$, where ρ is any irreducible representation. By linearity,

$T_f^\rho = 0$ for any representation ρ.
In particular, let ρ be the left regular representation. Then

$$0 = T_f^\rho(h) = \sum_{g\in G} f(g)(\rho(g))(h) = \sum_{g\in G} f(g)gh \qquad \leftarrow \text{take } h=e$$

Since $\{gh;\ g\in G\}$ is linearly independent, $f(g)=0\ \forall g\in G$.
This means $W^\perp \cap \omega = \{0\}$, and so $W = \omega$ ∎

We'll write down a list of all irreducible characters of D_4. This is called a character table for D_4.
Write $D_4 = \langle x, y;\ xy = yx^{-1},\ x^4 = 1,\ y^2 = 1\rangle$, where x represents rotation by $90°$ and y represents a reflection.
The conjugacy classes: $\{1\}, \{x^2\}, \{x, x^3\}, \{xy, x^3y\}, \{y, x^2y\}$

D_4	$\{1\}$	$\{x^2\}$	$\{x,x^3\}$	$\{xy, x^3y\}$	$\{y, x^2y\}$
$\chi_{\rho_{triv}}$	1	1	1	1	1
$\chi_{\rho_{sgn}}$	1	1	1	-1	-1
χ_{ρ_1}	1	1	-1	-1	1
χ_{ρ_2}	1	1	-1	1	-1
$\chi_{\square}$	2	-2	0	0	0

$\rho_{sgn}(g) = (-1)^{\varphi(g)}$ where $\varphi: D_4 \to \mathbb{Z}_2$, $\varphi(x) = 0$, $\varphi(y) = 1$
also $\Psi: D_4 \to \mathbb{Z}_2^2$, $\Psi(x) = (1,0)$, $\Psi(y) = (0,1)$
gives 2 more:

$$\rho_1(g) = (-1)^{\pi_1(\Psi(g))} \qquad \rho_2(g) = (-1)^{(\pi_1+\pi_2)(\Psi(g))}$$

There are lots of ways to find the last character of D_4. The easiest is to use the fact that, with the other 4 characters, it is an orthonormal basis of the space of class functions. Or you can just guess it; it's the realisation of D_4 as the symmetries of a square in $\mathbb{C}^2$. 2013 10 07 ex. Compute a character table for A_5.

	#1 (1)	#20 (1 2 3)	#15 (1 2)(3 4)	#12 (1 2 3 4 5)	#12 (1 3 4 5 2)
χ_{triv}	1	1	1	1	1
χ_{pt}	4	1	0	-1	-1
χ_{pn}	5	-1	1	0	0
χ_{ρ_1}	3	0	-1	ϕ	$-\frac{1}{\phi}$
χ_{ρ_2}	3	0	-1	$-\frac{1}{\phi}$	ϕ

We need 5 representations. We have the permutation representation in $\mathbb{C}^5$: χ_{perm}: (5, 2, 1, 0, 0)

Note $\langle \chi_{perm}, \chi_{perm} \rangle = \frac{1}{60}(5^2 + 20\cdot 2^2 + 15\cdot 1^2 + 0 + 0) = 2$

Note $\langle \chi_{perm}, \chi_{triv} \rangle =$ - - - - - $= 1$

Define $\chi_{pt} = \chi_{perm} - \chi_{triv}$. This is irreducible.

Need 3 more! Try $Sym^2(pt)$ and $Alt^2(pt)$. We need a formula for the characters though.

Theorem: Let $\rho: G \to GL(V)$ be a representation of a finite group. Let ρ_s and ρ_a be the subrepresentations $Sym^2(\rho)$ and $Alt^2(\rho)$ respectively. Then $\chi_{\rho_s}(g) = \frac{1}{2}(\chi_\rho(g)^2 + \chi_\rho(g^2))$, $\chi_{\rho_a}(g) = \frac{1}{2}(\chi_\rho(g)^2 - \chi_\rho(g^2))$. 2013 10 09

Proof: Since G is finite (or since ρ is unitary), $\rho(g)$ is diagonalizable. Let $\{e_1, \dots, e_n\}$ be an eigenbasis for V, with respect to $\rho(g)$, with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. Note $\chi_\rho(g) = \sum_{i=1}^n \lambda_i$ and $\chi(g^2) = \sum_{i=1}^n \lambda_i^2$.

Now,

$$(\rho_s(g))(e_i \otimes e_i) = (\rho_s(g))(e_i) \otimes (\rho_s(g))(e_i) = (\lambda_i e_i) \otimes (\lambda_i e_i) = \lambda_i^2 (e_i \otimes e_i),$$

and similarly

$$(\rho_s(g))(e_i \otimes e_j + e_j \otimes e_i) = \lambda_i \lambda_j (e_i \otimes e_j + e_j \otimes e_i).$$

Hence

$$\chi_{\rho_s}(g) = \sum_{i=1}^n \lambda_i^2 + \sum_{1 \le i < j \le n} \lambda_i \lambda_j = \frac{1}{2}(\chi_\rho(g)^2 + \chi_\rho(g^2)).$$

The result for χ_{ρ_a} follows in a similar fashion. ■

Now we apply this theorem to ρ_t. That is, to look for more (irreducible) characters, we consider $\rho_t \otimes \rho_t$ and try to decompose this into irreducible representations. We don't actually know the representation ρ_t, but since we only care about the characters, we can skip past the actual representations and use the previous theorem. So, with $\rho = \rho_t$ in the theorem, we get for χ_{ρ_s}:

$$(1) \mapsto \frac{4^2+4}{2} = 10, \quad (123) \mapsto \frac{1+1}{2} = 1, \quad (12)(34) \mapsto \frac{0+4}{2} = 2, \quad (12345), (13452) \mapsto 0.$$

Note

$$\langle \chi_{\rho_s}, \chi_{\rho_s} \rangle = \frac{1}{60}\left(10^2 + 20 \cdot 1^2 + 15 \cdot 2^2 + 12 \cdot 0^2 + 12 \cdot 0^2\right) = 3,$$

and hence ρ_s is a sum of 3 irreducible representations. Check that $\langle \chi_{\rho_s}, \chi_{triv} \rangle = \langle \chi_{\rho_s}, \chi_{\rho_t} \rangle = 1$. Hence $\rho_s \cong \rho_{triv} \oplus \rho_{\rho t} \oplus \rho_\eta$ for some irreducible representation ρ_η. So $\chi_{\rho_\eta} = \chi_{\rho_s} - \chi_{triv} - \chi_{\rho t}$ is added to the table

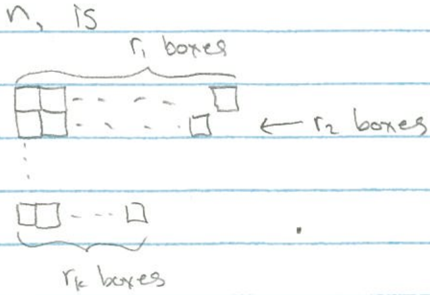
Next looking at χ_{ρ_a}, we find that it contains non of the already know irreducible representations and must be the sum of the last two characters, say $\rho_a \cong \rho_1 \oplus \rho_2$. This can now be solved with linear algebra to get $\chi_{\rho_1}, \chi_{\rho_2}$. Note they each have dimension 3. So we have the character table for A_5.

now look at S_n

Recall: The conjugacy classes of S_n are the sets of permutations with the same cycle structure. 2013 10 11

Fact: The conjugacy classes in S_n are in one-to-one correspondence with Young tableaux. The young tableau associated to

$S_n \ni (a_{1,1} \cdots a_{1,r_1})(a_{2,1} \cdots a_{2,r_2}) \cdots (a_{k,1} \cdots a_{k,r_k})$, where $r_1 \geq \cdots \geq r_k$ and $r_1 + \cdots + r_k = n$, is



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