$$\overset{T}{\widetilde{\rho(g)}}\circ\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\circ\overset{T}{\widetilde{\rho(g)}} \quad \forall h\in G$$

Theorem: Let $G$ be a finite abelian group, and let $\rho: G\to GL(V)$ be an irreducible representation. Then $\dim(V)=1$.

Proof: Since $G$ is abelian, $\rho(g)$ is a morphism for each $g\in G$. So $\rho(g) = \lambda_g \,\mathrm{id}$ for some $\lambda_g\in\mathbb{C}$, for each $g\in G$. But then every subspace is $G$-invariant, so as $\rho$ is irreducible $V$ cannot have any non-trivial subspaces.

$w\in W \Rightarrow (\rho(g))(w) = \lambda_g w \in W \quad \forall g\in G$

works for any field (both spaces over the same "any field")

Def Let $V$ and $W$ be complex vectors spaces. Let $H$ be the vector space whose basis is $\{v\otimes w ; v\in V, w\in W\}$. Let $R\subseteq H$ be the span of all vectors in $H$ of the following forms: 2013 09 20

$$v_1\otimes w + v_2\otimes w - (v_1+v_2)\otimes w\ ;$$
$$v\otimes w_1 + v\otimes w_2 - v\otimes(w_1+w_2)\ ;$$
$$(\lambda v)\otimes w - \lambda(v\otimes w)\ ;$$
$$v\otimes(\lambda w) - \lambda(v\otimes w).$$

Now define $\underline{V\otimes W} = H/R$. (as vector spaces)

ex $\{0\}\otimes\{0\} = \{\underset{0\otimes 0}{0}\}$ since $H = \mathrm{span}\{0\otimes 0\}$ and $\lambda(0\otimes 0) = (\lambda 0)\otimes 0 = 0\otimes 0$.

ex $\{0\}\otimes W = \{0\}$ since $0\otimes w + 0\otimes w = \cdots = 0\otimes w \Rightarrow 0\otimes w = 0 \quad \forall w\in W$

ex $\mathrm{span}\{v\}\otimes\mathrm{span}\{w\} = \mathrm{span}\{v\otimes w\}$ since $(\lambda v)\otimes(\mu w) = (\lambda\mu)(v\otimes w)$

is $v\otimes w = 0\otimes 0$?

Let $q: H\to V\otimes W$ be the quotient map. Then $q$ is onto and $H\neq\{0\}$ so it suffices to show $q$ is non-zero.

Define $T: H\to\mathbb{C}$ by $T\left(\sum_i a_i((\lambda_i v)\otimes(\mu_i w))\right) = \sum_i a_i\lambda_i\mu_i$. This is clearly a linear transformation. $T(H) = \mathbb{C}$ as $T(v\otimes w) = 1$. As $R\subseteq\ker(T)$, ← easy check

$T': V\otimes W\to\mathbb{C}$ is well defined and $T'(v\otimes w) = 1\neq 0$ so $v\otimes w\neq 0$.

Theorem (Universal Property of Quotients): Let $U$ be a vector space, $K\subseteq U$ any subspace, $q: U\to U/K$ the quotient map. Let $T: U\to V$ be a linear transformation. Then there is a linear transformation $\tilde{T}: U/K\to V$ such that $T = \tilde{T}\circ q$ if and only if $K\subseteq\ker(T)$.

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[d, "q"'] & V \\ U/K \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

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Theorem: Let $V$ and $W$ be finite dimensional vector spaces. Let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ be bases for $V$ and $W$ respectively. Then $\{v_i \otimes w_j;\ i \in [n], j \in [m]\}$ is a basis for $V \otimes W$. In particular, $\dim(V \otimes W) = \dim(V)\dim(W)$.

Proof: First we show this set is spanning. Let $\sum_{k=1}^{\ell} x_k \otimes y_k \in V \otimes W$. Then we can write this as

$$\sum_{k=1}^{\ell}\left(\left(\sum_{i=1}^{n} a_{i,k} v_i\right) \otimes \left(\sum_{j=1}^{m} b_{j,k} v_j\right)\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sum_{k=1}^{\ell} a_{i,k} b_{j,k}\right)(v_i \otimes w_j)$$

which is in $\operatorname{span}\{v_i \otimes w_j;\ i \in [n], j \in [m]\}$.

Next we show linear independence. Define a linear transformation $T: H \to \mathbb{C}^{nm}$ by

$$T\left(\sum_{k=1}^{\ell} c_k\left(\left(\sum_{i=1}^{n} a_{i,k} v_i\right) \otimes \left(\sum_{j=1}^{m} b_{j,k} w_j\right)\right)\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sum_{k=1}^{\ell} c_k a_{i,k} b_{j,k}\right)\overbrace{e_{i,j}}^{\text{standard basis}}.$$

As an exercise, check that $R \subseteq \ker(T)$. Note $T(v_i \otimes w_j) = e_{i,j}$, so $T$ is surjective. By the universal property of quotients, $\hat{T}: H/R \to \mathbb{C}^{nm}$ is well-defined and surjective. As $\{v_i \otimes w_j;\ i \in [n], j \in [m]\}$ maps to $\{e_{i,j};\ i \in [n], j \in [m]\}$, we must have that the first is linear independent since the second is linearly independent ∎

Def] Suppose $T: U \to V$ and $S: W \to X$ are linear transformations. We define the linear transform

$$T \otimes S: U \otimes W \to V \otimes X$$
$$u_i \otimes w_j \mapsto T(u_i) \otimes S(w_j)$$

ex if $M_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ wrt $\{e_1, e_2\}$, $M_2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ wrt $(e_1, e_2)$ then $M_1 \otimes M_2 = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix}$ wrt $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ (?) maybe

Def] Let $\rho: G \to GL(V)$ and $\tau: G \to GL(W)$ be representations. Then $\rho \otimes \tau: G \to GL(V \otimes W)$ is given by $(\rho \otimes \tau)(g) = \rho(g) \otimes \tau(g)\ \forall g \in G$. 2013 09 25

Note: Show this is well-defined (exercise).

ex. Consider $\rho: S_3 \to GL(\mathbb{C})$, the trivial representation, and $\tau: S_3 \to GL(\mathbb{C})$, the sign representation. Then $\rho\otimes\tau : S_3 \to GL(\mathbb{C}\otimes\mathbb{C})$ is

$$(\rho\otimes\tau)(g) = \rho(g)\otimes\tau(g) \cong \rho(g)\tau(g) = \tau(g)$$

since $GL(\mathbb{C}\otimes\mathbb{C}) \cong GL(\mathbb{C})$ since $\mathbb{C}\otimes\mathbb{C}\cong\mathbb{C}$ via $1\otimes 1 \mapsto 1$.

ex. If $\rho: G\to GL(V)$ is a one dimensional and $\tau: G \to GL(W)$ is any representation, then $\rho\otimes\tau \cong \rho\tau$.

Remark: If $\rho$ and $\tau$ are irreducible, then $\rho\otimes\tau$ is not necessarily irreducible.

Proposition: Let $\rho: G\to GL(V)$ be an irreducible representation, with $\dim(V)>1$. Then $\rho\otimes\rho$ is reducible.

Proof: Define the linear map $\theta: H \to V\otimes V$ by $\theta(v\otimes w) = w\otimes v$ and extend linearly. A simple check shows $R \subseteq \ker(\theta)$. Hence $\tilde{\theta}: V\otimes V \to V\otimes V$ is well-defined. Note $\theta^2 = \mathrm{Id}$, so $\theta$ is diagonalizable with eigenvalues $\rightarrow$ $\theta$ is a root of $x^2-1$ 1 and -1. Let $\mathrm{Sym}^2 V$ be the 1-eigenspace and let $\mathrm{Alt}^2 V$ be the (-1)-eigenspace. Note then that $\mathrm{Sym}^2 V + \mathrm{Alt}^2(V) = V\otimes V$ and $\mathrm{Sym}^2 V \cap \mathrm{Alt}^2 V = \{0\}$. Thus $\mathrm{Sym}^2 V \oplus \mathrm{Alt}^2 V \cong V\otimes V$ as vector spaces.

Finally, we claim $\mathrm{Sym}^2 V$ and $\mathrm{Alt}^2 V$ are $G$-invariant. Note that $\leftarrow$ exercise $\{v_i\otimes v_i,\ v_i\otimes v_j + v_j\otimes v_i ;\ i\neq j\}$ is a basis for $\mathrm{Sym}^2 V$ and $\{v_i\otimes v_j - v_j \otimes v_i\}$ is a basis for $\mathrm{Alt}^2 V$. So we check $G$-invariance:

where $\{v_i\}$ is a basis for $V$, $i,j\in[n]$

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$$\begin{aligned}\theta\big(((\rho\otimes\rho)(g))(v_i\otimes v_j + v_j\otimes v_i)\big) &= \theta\big((\rho(g))(v_i)\otimes(\rho(g))(v_j)\big) + \theta\big((\rho(g))(v_j)\otimes(\rho(g))(v_i)\big)\\ &= (\rho(g))(v_j)\otimes(\rho(g))(v_i) + (\rho(g))(v_i)\otimes(\rho(g))(v_j).\end{aligned}$$

Hence $((\rho\otimes\rho)(g))(v_i\otimes v_j + v_j \otimes v_i) \in \mathrm{Sym}^2 V$, for any $g\in G$.

The other two cases are similar. Thus $\mathrm{Sym}^2 V$ and $\mathrm{Alt}^2 V$ are $G$-invariant. As they are proper, non-trivial subspaces of $V\otimes V$, we have shown that $\rho\otimes\rho$ is reducible, as required. ∎

Remark If $\dim(V) = n$ then $\dim(V\otimes V) = n^2$. Further, $\dim(\mathrm{Sym}^2 V) = n + \binom{n}{2} = \frac{n(n+1)}{2}$ and $\dim(\mathrm{Alt}^2 V) = \binom{n}{2} = \frac{n(n-1)}{2}$, as we can see by the bases given above.