PMATH 745 - Representations of Finite Groups 2013 09 09

we will use only finite groups

Def Let G be a group. A <u>representation</u> of G in a vector space V is a homomorphism $\rho: G \to GL(V)$.

* Where we always take V to be a finite dimensional vector space over $\mathbb{C}$.

* $GL(V)$ is the group of invertible linear maps $V \to V$.

ex. The trivial homomorphism (in $\mathbb{C}'$) is a representation, called the trivial or unit representation. $(\rho(g) = (v \mapsto v) \ \forall g)$

ex. Recall D_4 is the group of symmetries of the square. There is a representation of D_4 in $\mathbb{C}^2$ corresponding to this description of it.

ex. The permutation representation of S_n is the homomorphism that takes a permutation to the corresponding permutation matrix

Def A representation is <u>faithful</u> if it is one-to-one.

ex Let G be a finite group acting on a finite set X. There is a permutation representation corresponding to this action, on the vector space $\bigoplus_{x \in X} \vec{v}_x$

ex. Every group G acts on itself by left multiplication. If G is finite, the corresponding permutation representation is called the left regular representation

(G-invariant) depends on ρ

Def Let $\rho: G \to GL(V)$ be a representation. A subspace $W \subseteq V$ is G-stable (or <u>G-invariant</u>) if $(\rho(g))(w) \in W \ \forall w \in W, \forall g \in G$. In other words, $\rho(g)|_W \in GL(W) \ \forall g \in G$. 2013 09 11

ex. $\rho: \mathbb{Z}_2 \to GL(\mathbb{C}^2)$, $\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\rho(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

let $W = \operatorname{span}_{\mathbb{C}}\{(1,1)\}$. Then W is G-invariant (so are $\{(0,0)\}$ and $\mathbb{C}^2$)

let $W' = \operatorname{span}_{\mathbb{C}}\{(1,-1)\}$ is also G-invariant

Remark: It is easy to see that to check the G-invariance of W, it suffices to check that $(\rho(g_i))(w_j) \in W$ for generators g_i of G and a basis w_j of W.

Def | Let $\rho: G \to GL(V)$ and $\tau: G \to GL(W)$ be representations. A morphism from ρ to τ is a linear transformation $T: V \to W$ such that $\tau(g) \circ T = T \circ \rho(g)$ for each $g \in G$.

Remark: To check that T is a morphism, it suffices to show that $(\tau(g) \circ T)(v) = (T \circ \rho(g))(v)$ on a basis $v \in \beta \subseteq V$ and a generating set $g \in S \subseteq G$

ex. $\rho: \mathbb{Z}_4 \to GL(\mathbb{C}^1)$, $\rho(n) = i^n$

$\tau: \mathbb{Z}_4 \to GL(\mathbb{C}^2)$, $\tau(n) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^n$

define $T: \mathbb{C} \to \mathbb{C}^2$, $T(z) = (z, iz)$. Then T is a morphism from ρ to τ.

As 1 is a basis for $\mathbb{C}$ and 1 generates $\mathbb{Z}_4$, so we need only check

$$(\tau(1))(T(1)) \stackrel{?}{=} T((\rho(1))(1))$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad T(i)$$

$$\begin{bmatrix} i \\ -1 \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix}$$

Theorem: Let $\rho: G \to GL(V)$, $\tau: G \to GL(W)$ be representations and let $T: V \to W$ be a morphism from ρ to τ. Let $V_1 \subseteq V$ and $W_1 \subseteq W$ be subrepresentations of V and W respectively. Then $T(V_1)$ is a subrepresentation of W and $T^{-1}(W_1)$ is a subrepresentation of V. 2013 09 13

Proof: No. (Exercise) ∎

Def | Let $\rho: G \to GL(V)$ be a representation. Then ρ is irreducible if the only subrepresentations of ρ are 0 and ρ.

ex. Every 1-dimensional representation is irreducible, as the only subspaces are 0 or the whole space.

Def | The direct sum of vector spaces V and W is the vector space $V \oplus W = \{v \oplus w ; v \in V, w \in W\}$ with $v_1 \oplus w_1 + v_2 \oplus w_2 = (v_1 + v_2) \oplus (w_1 + w_2)$ and $\lambda(v \oplus w) = (\lambda v) \oplus (\lambda w)$

Note $\dim(V \oplus W) = \dim(V) + \dim(W)$

Def | Let $\rho: G \to GL(V)$ and $\tau: G \to GL(W)$ be representations of G. Define $\rho \oplus \tau: G \to GL(V \oplus W)$ by $(\rho \oplus \tau)(g) = \rho(g) \oplus \tau(g)$ $\forall g \in G$.

should always get a (2-)block diagonal matrix for $\rho_1 \oplus \rho_2$

ex $\rho: \mathbb{Z}_2 \to \mathbb{C}^*$ trivial, $\tau: G \to \mathbb{C}^*$, $\tau(0)=1$, $\tau(1)=-1$
then $(\rho\oplus\tau)(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $(\rho\oplus\tau)(1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

2013 09 16

Def A <u>complex inner product space</u> is a complex vector space V with a pairing $\langle \cdot, \cdot \rangle$ from $V \times V \to \mathbb{C}$ satisfying:

i) $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$;

ii) $\langle cv, w\rangle = c\langle v, w\rangle$;

iii) $\langle v, w\rangle = \overline{\langle w, v\rangle}$;

iv) $\langle v, v\rangle \in \mathbb{R}$ and $\langle v, v\rangle \geq 0$ with equality if and only if $v = 0$.

ex $\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$ (on $\mathbb{C}^n$)

Def A linear transformation $T: V \to V$ is <u>unitary</u> if ~~and only if~~ $\forall v, w \in V$ we have $\langle Tv, Tw\rangle = \langle v, w\rangle$.

Def Let W be a subspace of a complex inner product space V. The <u>orthogonal complement</u> of W in V is $W^\perp = \{x \in V;\ \langle w, x\rangle = 0\ \forall w \in W\}$

Facts: $W \cap W^\perp = \{0\}$, $\dim(W) + \dim(W^\perp) = \dim(V)$.

This definition goes earlier

this thm holds even if G isn't finite

Theorem: Let V be a (finite dimensional (always)) complex inner product space and let G be a (finite (always)) group. Let $\rho: G \to GL(V)$ be a representation such that $\rho(g)$ is unitary for all $g \in G$. Then there are irreducible representations $\rho_1, \ldots, \rho_n$ of G such that $\rho \cong \rho_1 \oplus \cdots \oplus \rho_n$.

Def A morphism T from ρ to τ is an <u>isomorphism</u> if there is a morphism T' from τ to ρ such that $T \circ T' = id$ and $T' \circ T = id$.

ie if T is inverti[ble] and T^{-1} is a morphism. It follows that if T is inv[ertible] then T^{-1} is a morphism (in this ca[se])

Proof: By a simple induction, it suffices to prove that if V is not irreducible then there are two proper subrepresentations W, W' such that $V = W \oplus W'$. Thus, assume V is reducible. Then there is a proper nontrivial subrepresentation $W \subset V$. If we show $W^\perp$ is G-invariant then we are done. So we want to show $(\rho(g))(w) \in W^\perp$ for $w \in W^\perp$. Let $v \in W$. Then
$\langle v, (\rho(g))(w)\rangle = \langle (\rho(g))((\rho(g^{-1}))(v)), (\rho(g))(w)\rangle = \langle (\rho(g^{-1}))(v), w\rangle$
since $(\rho(g))$ is unitary. But $(\rho(g^{-1}))(v) \in W$ and $w \in W^\perp$ so this is 0, as required. ∎

2013 09

Theorem: Let G be a finite group and let $\rho: G \to GL(V)$ be a finite dimensional representation. Then there exists a complex inner product on V such that $\rho(g)$ is unitary for all $g \in G$.

Proof: Let $\langle \cdot,\cdot \rangle'$ be any inner product on V. Define a new pairing on V:

$g(u) = (\rho(g))(u)$

$$\langle u,v \rangle = \sum_{g\in G} \langle g(u), g(v) \rangle'.$$

Check this is an inner product as an exercise. It remains to show $\rho(g)$ is unitary (wrt $\langle \cdot,\cdot \rangle$) for each $g \in G$. For $u,v \in V$ we have

$$\begin{aligned}
\langle g(u), g(v) \rangle &= \sum_{h\in G} \langle h(g(u)), h(g(v)) \rangle' \\
&= \sum_{h\in G} \langle (hg)(u), (hg)(v) \rangle' \\
&= \sum_{h\in G} \langle h(u), h(v) \rangle' \\
&= \langle u,v \rangle.
\end{aligned}$$

■

Remark: We now see that every finite dimensional representation of a finite group is isomorphic to a direct sum of irreducible representations.

Theorem: Let $\rho: G \to GL(V)$ and $\tau: G \to GL(W)$ be irreducible representations of a group G. Let $T: V \to W$ be morphism from ρ to τ. Then T is either an isomorphism or identically zero.

Proof: The kernel of T is a subrepresentation of V, which by the irreducibility of ρ, must be 0 or V. If $\ker(T)=V$, then $T=0$. If $\ker(T)=0$ then T is injective, so $\operatorname{im}(T) \in \{0, W\}$ as τ is irreducible. Hence we have either $T=0$ or T is surjective, in which case T is an isomorphism. ■

Theorem (Schur's Lemma): Let $T: V \to V$ be a morphism of irreducible representations. Then $T = \lambda \cdot id$ for some $\lambda \in \mathbb{C}$.

Proof: Let λ be an eigenvalue of T. Then $T - \lambda I$ is a morphism (exercise), from V to V. By above, we must have $T - \lambda I = 0$, so $T = \lambda I$. ■

$\overset{T}{\rho(g)}\circ\rho(h) = \rho(gh) = \rho(hg) = \rho(h)\circ\overset{T}{\rho(g)} \quad \forall h\in G$

Theorem: Let G be a finite abelian group, and let $\rho: G\to GL(V)$ be an irreducible representation. Then $\dim(V)=1$.

Proof: Since G is abelian, $\rho(g)$ is a morphism for each $g\in G$. So $\rho(g) = \lambda_g \, id$ for some $\lambda_g\in\mathbb{C}$, for each $g\in G$. But then every subspace is G-invariant, so as ρ is irreducible V cannot have any non-trivial subspaces.

$w\in W \Rightarrow (\rho(g))(w) = \lambda_g w \in W \ \forall g\in G$

2013 09 20

works for any field (both spaces over the same "any field")

Def Let V and W be complex vectors spaces. Let H be the vector space whose basis is $\{v\otimes w;\ v\in V, w\in W\}$. Let $R\subseteq H$ be the span of all vectors in H of the following forms:

$v_1\otimes w + v_2\otimes w - (v_1+v_2)\otimes w$;

$v\otimes w_1 + v\otimes w_2 - v\otimes(w_1+w_2)$;

$(\lambda v)\otimes w - \lambda(v\otimes w)$;

$v\otimes(\lambda w) - \lambda(v\otimes w)$.

Now define $\underline{V\otimes W} = H/R$. (as vector spaces)

ex $\{0\}\otimes\{0\} = \{\underset{0\otimes 0}{0}\}$ since $H = \operatorname{span}\{0\otimes 0\}$ and $\lambda(0\otimes 0) = (\lambda 0)\otimes 0 = 0\otimes 0$.

ex $\{0\}\otimes W = \{0\}$ since $0\otimes w + 0\otimes w = \cdots = 0\otimes w \to 0\otimes w = 0 \ \forall w\in W$

ex $\operatorname{span}\{v\}\otimes\operatorname{span}\{w\} = \operatorname{span}\{v\otimes w\}$ since $(\lambda v)\otimes(\mu w) = (\lambda\mu)(v\otimes w)$

is $v\otimes w = 0\otimes 0$?

Let $q: H\to V\otimes W$ be the quotient map. Then q is onto and $H\neq\{0\}$ so it suffices to show q is non-zero.

Define $T: H\to\mathbb{C}$ by $T\left(\sum_i a_i((\lambda_i v)\otimes(\mu_i w))\right) = \sum_i a_i\lambda_i\mu_i$. This is clearly a linear transformation. $T(H) = \mathbb{C}$ as $T(v\otimes w) = 1$. As $R\subseteq\ker(T)$, ← easy check
$T': V\otimes W\to\mathbb{C}$ is well defined and $T'(v\otimes w) = 1\neq 0$ so $v\otimes w\neq 0$.

Theorem (Universal Property of Quotients): Let U be a vector space, $K\subseteq U$ any subspace, $q: U\to U/K$ the quotient map. Let $T: U\to V$ be a linear transformation. Then there is a linear transformation $\tilde{T}: U/K\to V$ such that $T = \tilde{T}\circ q$ if and only if $K\subseteq\ker(T)$.

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[d, "q"'] & V \\ U/K \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$